Math 111 Shirazi S01

Skye Kychenthal

Week 2: 9/7 $\rightarrow$ 9/9

Lecture One (9/7)

Formal Definition in Terms

ϵ arbitrary distance around a to find limit. Range of y.
δ the distance being checked from a. Range of x.
$\forall$ for all
$\exists$ there exists

if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that all x,
if $0 <\mid x - a \mid< \delta$, then $\mid f(x) - L \mid< \epsilon$

$\forall \epsilon > 0 \rightarrow$ for all epsilon greater than 0
$\exists \delta > 0$ there exists a $\delta > 0$
$\forall x$ for all x values

Re-write as:

$\forall \epsilon > 0 \ \ \exists \delta > 0$ such that if $0 <\mid x - a \mid< \delta$, then $\mid f(x) - L \mid< \epsilon$

Examples:

Example proof n.1

math.libretexts.org Example 1.5.1

Prove:

$$\lim_{x \to 1} (2x + 1) = 3$$

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \quad \text{such that} \quad \text{if } 0 <\mid x - 1 \mid< \delta, \quad \text{then} \quad \mid (2x + 1) - 3 \mid< \epsilon$$

Since the first part of the formal definition of a limit is $\forall \epsilon > 0$ every positive value of ϵ has to make this equation work. The second part is trying to find out what value of δ we should be using. $\exists$ essentially says there has to be an answer, we just have to find it.

Since we want eventually $\mid f(x) - L \mid< \epsilon$, the expression can be substituted for and manipulated:

$$\mid 2x + 1 - 3 \mid< \epsilon$$

$$\mid 2x - 2 \mid< \epsilon$$

$$\mid 2 \mid\mid x - 1 \mid< \epsilon$$

$$\mid x - 1 \mid< \frac{\epsilon}{2}$$

Since epsilon is the y-value range in which the limit exists, it makes sense to make $\delta = \frac{\epsilon}{2}$ so that we check a smaller range around ϵ. δ is more or less arbitrary, but has to be based on ϵ

Now that δ has been chosen, we need to prove that

$$\text{if } 0 <\mid x-a \mid< \delta \text{ then } \mid f(x)-L \mid< \epsilon$$
$$\text{if } 0 <\mid x-1 \mid< \delta \text{ then } \mid x-1 \mid< \frac{\epsilon}{2}$$
$$-\delta < x-1 < \delta \quad \rightarrow \quad \text{delta neighborhood}$$
$$-\frac{\epsilon}{2} < x-1 < \frac{\epsilon}{2} \quad \rightarrow \quad \text{choice of } \delta = \frac{\epsilon}{2}$$
$$-\epsilon < 2x-2 < \epsilon \quad \rightarrow \quad \text{factoring}$$
$$2x-2 < \epsilon$$

So officially there exists ($\exists$) a δ value ($\frac{\epsilon}{2}$) such that for every ϵ over 0 ($\forall \epsilon > 0$), if $\mid x-1 \mid< \frac{\epsilon}{2}$ then $2x+1-3 < \epsilon$.

Which means that, $\lim_{x\to 1}(2x+1) = 3$ is true based on the formal definition of: $\forall \epsilon > 0 \;\; \exists \delta > 0 \quad \text{such that} \quad \text{if } 0 <\mid x-1 \mid< \delta, \quad \text{then} \quad \mid 2x+1-3 \mid< \epsilon$

Example proof n.2

$$\lim_{x\to 2}(x^2+1) = 5$$

$$a = 0$$
$$f(x) = \frac{x^2}{x+1}$$
$$L = 5$$

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \quad \text{such that} \quad \text{if } 0 <\mid x-2 \mid< \delta, \quad \text{then} \quad \mid (x^2+1)-5 \mid< \epsilon$$

$$\mid f(x)-5 \mid=\mid x^2+1-5 \mid=\mid x^2-4 \mid=\mid x+2 \mid\mid x-2 \mid$$

$$1 <\mid x-2 \mid< 3 \;\rightarrow\; 3 < x < 5$$
$$\mid x-2 \mid< \max\{\mid 3 \mid, \mid 5 \mid\}$$

$$\mid x+2 \mid\mid x-2 \mid< \epsilon$$
$$\mid x-2 \mid< \frac{\epsilon}{x+2}$$
$$\delta = \frac{\epsilon}{x+2}$$

$$\mid x-2 \mid< \delta$$
$$\mid x-2 \mid< \frac{\epsilon}{x+2}$$
$$\mid x-2 \mid\mid x+2 \mid< \epsilon$$

So:

$$\forall x > 0$$
$$\exists \delta (\delta = \frac{\epsilon}{x+2})$$

Such that:

$$\text{if, } 0 <\mid x-2 \mid< \frac{\epsilon}{x+2}$$

Then:

$$\mid x^2+1-5 \mid< \epsilon$$

Therefore, based on the formal definition of a limit, the equation $\lim_{x\to 2}(x^2+1) = 5$ is valid (and based)!

Exercises

AAAA

Prove the following:

Theorum 9.

Prove the following:

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$