Решение уравнения теплопроводности. Вариант 11.

Содержание

Введение

Для задачи

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ 0 < t \le T = 0.1$$

$$u(x,0) = 2\sin(2\pi x) - \sin(3\pi x),\ 0 \le x \le 1$$

$$u(0,t) = 0,\ u(1,t) = 0, 0 < t \le T$$

найти решение, используя ряд Фурье и дискретный ряд Фурье.

Построение решения с помощью ряда Фурье

Построим решение нашей задачи, используя ряд Фурье, реализация которого описана в методе fourier_series. Для определенности посчитаем первые 40 слагаемых.

```
syms x phi t;
phi = 2*sin(2*pi*x) - sin(3*pi*x);
uf = fourier_series(phi,40);
```

Выведем на печать таблицу значений нашего решения на крупной сетке.

```
uf_table = generateSolutionTable(uf);
printBigGridTable(uf_table);
```

```
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7027	0.6332	-0.4343	-1.0246	0
3	0.0400	0	0.3649	0.2592	-0.2255	-0.4194	0
4	0.0600	0	0.1734	0.1129	-0.1072	-0.1827	0
5	0.0800	0	0.0801	0.0504	-0.0495	-0.0816	0
6	0.1000	0	0.0366	0.0228	-0.0226	-0.0368	0

Построение решения с помощью дискретного ряда Фурье

Построим решение нашей задачи с помощью дискретного ряда Фурье, используя метод dfs. Рассмотрим решения при трех различных значениях N = 5,10,20.

```
udfs_5=dfs(phi,5);
udfs_10 = dfs(phi,10);
udfs_20 = dfs(phi,20);
udfs_5_table = generateSolutionTable(udfs_5);
udfs_10_table = generateSolutionTable(udfs_10);
udfs_20_table = generateSolutionTable(udfs_20);
```

Сравним значения полученных решений с решением, полученным в первом пункте.

```
N_values = [5;10;20];
dfs_varNames = {'N','||uf - udsf||'};
dfs_errors = [norm(uf_table,udfs_5_table);
    norm(uf_table,udfs_10_table);
    norm(uf_table,udfs_20_table)];
dfs_errors = table(N_values,dfs_errors,'VariableNames',dfs_varNames)
```

```
dfs_errors = 3×2 table
```

	N	||uf - udsf||
1	5	1.0658e-14
2	10	1.1102e-14
3	20	7.9936e-15

Как мы можем заметить, с помощью дискретного ряда Фурье было получено достаточно точное решение.

Схема с весами

Используя дискретный ряд Фурье и схему с весами, построим сеточные решения задачи при различных значениях параметра $\sigma = 1, 0, \frac{1}{2}, \frac{1}{2} - \frac{h^2}{12\tau}$ и различных парах значений (h, τ).

Зам: реализация содержится в методе implicitSchema

```
h = [0.2, 0.1, 0.05, 0.05];
tau=[ 0.02, 0.005, 0.00125, 0.005];
sigma = [0,1,0.5];
spectral_evaluation_test = zeros(4,4);
error_values = zeros(4,4);
for i=1:4
   disp("================================");
   for j=1:3
       disp("(h,tau)=("+h(i)+","+tau(i)+"), sigma = "+sigma(j));
       u_table = implicitSchema(phi,sigma(j),h(i),tau(i));
       error_values(j,i) = norm(uf_table,u_table);
       printBigGridTable(u_table);
       disp("--------------------------------------------------------------------");
   end
   extrasigma = 0.5 -h(i)^2/(12*tau(i));
```

```
    u_table=implicitSchema(phi,extrasigma ,h(i),tau(i));
    error_values(4,i) = norm(uf_table,u_table);
    disp("(h,tau)=("+h(i)+","+tau(i)+"), sigma = "+extrasigma);
    printBigGridTable(u_table);
end
```

```
================================
(h,tau)=(0.2,0.02), sigma = 0
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.8817	0.1816	-0.5449	-0.2939	0
3	0.0400	0	0.0908	0.1684	-0.0561	-0.2725	0
4	0.0600	0	0.0842	0.0173	-0.0520	-0.0281	0
5	0.0800	0	0.0087	0.0161	-0.0054	-0.0260	0
6	0.1000	0	0.0080	0.0017	-0.0050	-0.0027	0

```
--------------------------------------------------------------------
(h,tau)=(0.2,0.02), sigma = 1
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7130	0.9498	-0.4406	-1.5367	0
3	0.0400	0	0.4868	0.5214	-0.3009	-0.8436	0
4	0.0600	0	0.3161	0.2909	-0.1954	-0.4706	0
5	0.0800	0	0.1992	0.1645	-0.1231	-0.2661	0
6	0.1000	0	0.1231	0.0940	-0.0761	-0.1521	0

```
--------------------------------------------------------------------
(h,tau)=(0.2,0.02), sigma = 0.5
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7267	0.6946	-0.4491	-1.1239	0
3	0.0400	0	0.4086	0.3038	-0.2525	-0.4916	0
4	0.0600	0	0.2103	0.1407	-0.1300	-0.2276	0
5	0.0800	0	0.1047	0.0669	-0.0647	-0.1083	0
6	0.1000	0	0.0514	0.0323	-0.0318	-0.0522	0

```
--------------------------------------------------------------------
(h,tau)=(0.2,0.02), sigma = 0.33333
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7495	0.5674	-0.4632	-0.9181	0
3	0.0400	0	0.3581	0.2305	-0.2213	-0.3730	0

	t\x	0	0.2	0.4	0.6	0.8	1
4	0.0600	0	0.1596	0.0994	-0.0986	-0.1609	0
5	0.0800	0	0.0702	0.0435	-0.0434	-0.0703	0
6	0.1000	0	0.0308	0.0190	-0.0190	-0.0308	0

```
================================
(h,tau)=(0.1,0.005), sigma = 0
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7013	0.5738	-0.4334	-0.9284	0
3	0.0400	0	0.3355	0.2241	-0.2074	-0.3626	0
4	0.0600	0	0.1479	0.0934	-0.0914	-0.1511	0
5	0.0800	0	0.0639	0.0397	-0.0395	-0.0642	0
6	0.1000	0	0.0274	0.0170	-0.0169	-0.0275	0

```
--------------------------------------------------------------------
(h,tau)=(0.1,0.005), sigma = 1
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7063	0.7321	-0.4365	-1.1845	0
3	0.0400	0	0.4098	0.3276	-0.2533	-0.5300	0
4	0.0600	0	0.2184	0.1537	-0.1350	-0.2487	0
5	0.0800	0	0.1123	0.0741	-0.0694	-0.1199	0
6	0.1000	0	0.0567	0.0362	-0.0351	-0.0586	0

```
--------------------------------------------------------------------
(h,tau)=(0.1,0.005), sigma = 0.5
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7055	0.6567	-0.4360	-1.0625	0
3	0.0400	0	0.3773	0.2746	-0.2332	-0.4443	0
4	0.0600	0	0.1846	0.1219	-0.1141	-0.1972	0
5	0.0800	0	0.0876	0.0556	-0.0541	-0.0899	0
6	0.1000	0	0.0410	0.0256	-0.0254	-0.0415	0

```
--------------------------------------------------------------------
(h,tau)=(0.1,0.005), sigma = 0.33333
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7047	0.6298	-0.4355	-1.0190	0
3	0.0400	0	0.3645	0.2574	-0.2253	-0.4165	0

	t\x	0	0.2	0.4	0.6	0.8	1
4	0.0600	0	0.1727	0.1120	-0.1067	-0.1812	0
5	0.0800	0	0.0795	0.0500	-0.0491	-0.0809	0
6	0.1000	0	0.0362	0.0225	-0.0224	-0.0364	0

```
================================
(h,tau)=(0.05,0.00125), sigma = 0
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7021	0.6194	-0.4339	-1.0023	0
3	0.0400	0	0.3581	0.2506	-0.2213	-0.4055	0
4	0.0600	0	0.1673	0.1080	-0.1034	-0.1748	0
5	0.0800	0	0.0760	0.0477	-0.0470	-0.0772	0
6	0.1000	0	0.0342	0.0213	-0.0212	-0.0344	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.00125), sigma = 1
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7038	0.6596	-0.4350	-1.0672	0
3	0.0400	0	0.3775	0.2762	-0.2333	-0.4470	0
4	0.0600	0	0.1853	0.1227	-0.1145	-0.1986	0
5	0.0800	0	0.0881	0.0560	-0.0544	-0.0906	0
6	0.1000	0	0.0414	0.0259	-0.0256	-0.0418	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.00125), sigma = 0.5
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7031	0.6397	-0.4345	-1.0351	0
3	0.0400	0	0.3681	0.2633	-0.2275	-0.4261	0
4	0.0600	0	0.1764	0.1153	-0.1090	-0.1865	0
5	0.0800	0	0.0820	0.0518	-0.0507	-0.0838	0
6	0.1000	0	0.0377	0.0235	-0.0233	-0.0380	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.00125), sigma = 0.33333
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7028	0.6330	-0.4344	-1.0242	0
3	0.0400	0	0.3649	0.2591	-0.2255	-0.4192	0

	t\x	0	0.2	0.4	0.6	0.8	1
4	0.0600	0	0.1734	0.1128	-0.1072	-0.1826	0
5	0.0800	0	0.0800	0.0504	-0.0495	-0.0816	0
6	0.1000	0	0.0365	0.0228	-0.0226	-0.0368	0

```
================================
(h,tau)=(0.05,0.005), sigma = 0
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.6994	0.5513	-0.4323	-0.8919	0
3	0.0400	0	0.3231	0.2117	-0.1997	-0.3426	0
4	0.0600	0	0.1383	0.0867	-0.0854	-0.1402	0
5	0.0800	0	0.0896	0.0289	-0.0207	-0.0144	0
6	0.1000	0	60.1200	-3.2733	37.2269	91.4998	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.005), sigma = 1
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7067	0.7132	-0.4367	-1.1540	0
3	0.0400	0	0.4024	0.3137	-0.2487	-0.5076	0
4	0.0600	0	0.2102	0.1452	-0.1299	-0.2349	0
5	0.0800	0	0.1059	0.0691	-0.0655	-0.1118	0
6	0.1000	0	0.0526	0.0333	-0.0325	-0.0539	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.005), sigma = 0.5
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7054	0.6358	-0.4360	-1.0287	0
3	0.0400	0	0.3678	0.2612	-0.2273	-0.4227	0
4	0.0600	0	0.1755	0.1142	-0.1085	-0.1848	0
5	0.0800	0	0.0813	0.0512	-0.0503	-0.0829	0
6	0.1000	0	0.0373	0.0232	-0.0231	-0.0376	0

```
--------------------------------------------------------------------
(h,tau)=(0.05,0.005), sigma = 0.45833
ans = 6×7 table
```

	t\x	0	0.2	0.4	0.6	0.8	1
1	0	0	0.9511	1.7634	-0.5878	-2.8532	0
2	0.0200	0	0.7052	0.6290	-0.4358	-1.0177	0
3	0.0400	0	0.3645	0.2570	-0.2253	-0.4158	0

	t\x	0	0.2	0.4	0.6	0.8	1
4	0.0600	0	0.1725	0.1118	-0.1066	-0.1809	0
5	0.0800	0	0.0793	0.0499	-0.0490	-0.0807	0
6	0.1000	0	0.0361	0.0225	-0.0223	-0.0364	0

Рассмотрим таблицу значений модулей разности между uf и полученными выше сеточными решениями

```
varNames = {'(0.2,0.02)','(0.1,0.005)','(0.05,0.00125)','(0.05,0.005)'};
rowNames = {'sigma = 0','sigma = 1','sigma = 0.5','sigma = 0.5 - h^2/(12 tau)'};
error_table = table(error_values(:,1),error_values(:,2), ...
    error_values(:,3),error_values(:,4),'VariableNames',varNames,'RowNames',rowNames)
```

```
error_table = 4×4 table
```

	(0.2,0.02)	(0.1,0.005)	(0.05,0.00125)	(0.05,0.005)
1 sigma = 0	0.7307	0.0962	0.0223	91.5366
2 sigma = 1	0.5122	0.1599	0.0426	0.1294
3 sigma = 0.5	0.0993	0.0379	0.0105	0.0041
4 sigma = 0.5 - h^2/(12 tau)	0.1064	0.0055	0.0003	0.0069

Используем спектральный признак устойчивости чтобы выяснить, при каких из наборов параметров методы получились неустойчивыми.

```
for i=1:4
    spectral_evaluation_test(1,i) = tau(i)/h(i)^2 <= 0.5;
    spectral_evaluation_test(2,i) = 1;
    spectral_evaluation_test(3,i) = 1;
    spectral_evaluation_test(4,i) = tau(i) <= h(i)^2/(2*(1-2*(0.5 - h(i)^2/(12*tau(i)))));
end
spectral_eval_table = table(spectral_evaluation_test(:,1),spectral_evaluation_test(:,2), ...
    spectral_evaluation_test(:,3), ...
    spectral_evaluation_test(:,4),'VariableNames',varNames,'RowNames',rowNames)
```

```
spectral_eval_table = 4×4 table
```

	(0.2,0.02)	(0.1,0.005)	(0.05,0.00125)	(0.05,0.005)
1 sigma = 0	1	1	1	0
2 sigma = 1	1	1	1	1
3 sigma = 0.5	1	1	1	1
4 sigma = 0.5 - h^2/(12 tau)	1	1	1	1

Как и ожидалось, неустойчивым оказался лишь процесс при $(h, \tau) = (0.05, 0.005),\ \sigma = 0$.

Наиболее точное решение было получено при $(h, \tau) = (0.05, 0.00125),\ \sigma = \frac{1}{2} - \frac{h^2}{12\tau}$.

Вспомогательные функции

Скалярное произведение в $L_2(0,1)$

```
function res = dotprod(phi,psi)
    res = vpaintegral(phi*psi,0,1);
end
```

Модуль разности между двумя решениями

```
function res = norm(u,v)
    res = max(max(abs(u-v)));
end
```

Ряд Фурье

```
function res  = fourier_series(phi,N)
    syms x psi_p t res;
    res = 0;
    for p = 1:N
        psi_p = sqrt(2)*sin(p*pi*x);
        cp = dotprod(phi,psi_p);
        expon = exp(-pi^2*p^2*t);
        res = res + cp *expon * psi_p;
    end
end
```

Дискретный ряд Фурье

```
function res =dfs_coeff(phi,p,h,N)
    res = 0;
    for i=1:N-1
        res = res + subs(phi,i*h)*sin(p*pi*i*h);
    end
    res = res * sqrt(2)*h;
end
function res = dfs(phi,N)
    syms res x t;
    res = 0;
    h = 1/N;
    for p = 1:N-1
        cp = dfs_coeff(phi,p,h,N);
        expon = exp(-pi^2*p^2*t);
        res = res + cp*expon*sin(p*pi*x);
    end
    res = res*sqrt(2);
end
```

Схема с весами

```
function res = find_lambda(p,sigma,h,tau)
    num = 1 - 4*(1-sigma)*tau* (sin(p*pi*h/2))^2/ h^2 ;
    denum = 1 + 4*sigma*tau  * (sin(p*pi*h/2))^2 / h^2;
    res = num / denum;
end
function res = implicitSchema(phi,sigma,h,tau)
    syms u x k;
    u = 0;
    N = 1/h;
    M = 0.1/tau;
    for p = 1:N-1
        cp = dfs_coeff(phi,p,h,N);
        lambda = find_lambda(p,sigma,h,tau);
        u = u + cp* (lambda^k)*sin(p*pi*x);
    end
    u = u*sqrt(2);
    res = zeros(6,6);
    k_val = 0:M/5:M;
    i_val = 0:N/5:N;
    for i=1:6
        for j=2:5
            res(i,j) = subs(u,[x, k],[i_val(j)*h,k_val(i)]);
        end
        res(i,1) = 0;
        res(i,6) = 0;
    end

end
```

Вспомогательные функции

```
function result = generateSolutionTable(u)
    syms x t;
    x_vals = 0:0.2:1;
    t_vals = 0:0.02:0.1;
    values = zeros(size(x_vals,2),size(x_vals,2));
    for i=1:size(t_vals,2)
        for j = 1:size(x_vals,2)
            values(i,j) = subs(u,[x, t],[x_vals(1,j),t_vals(1,i)]);
        end
    end
    result = values;
end
function printBigGridTable(u)
    t_vals = 0:0.02:0.1;
    variableNames = {'t\x','0','0.2','0.4','0.6','0.8','1'};
    table(transpose(t_vals),u(:,1),u(:,2),u(:,3), ...
        u(:,4),u(:,5),u(:,6),'VariableNames',variableNames)
end
```