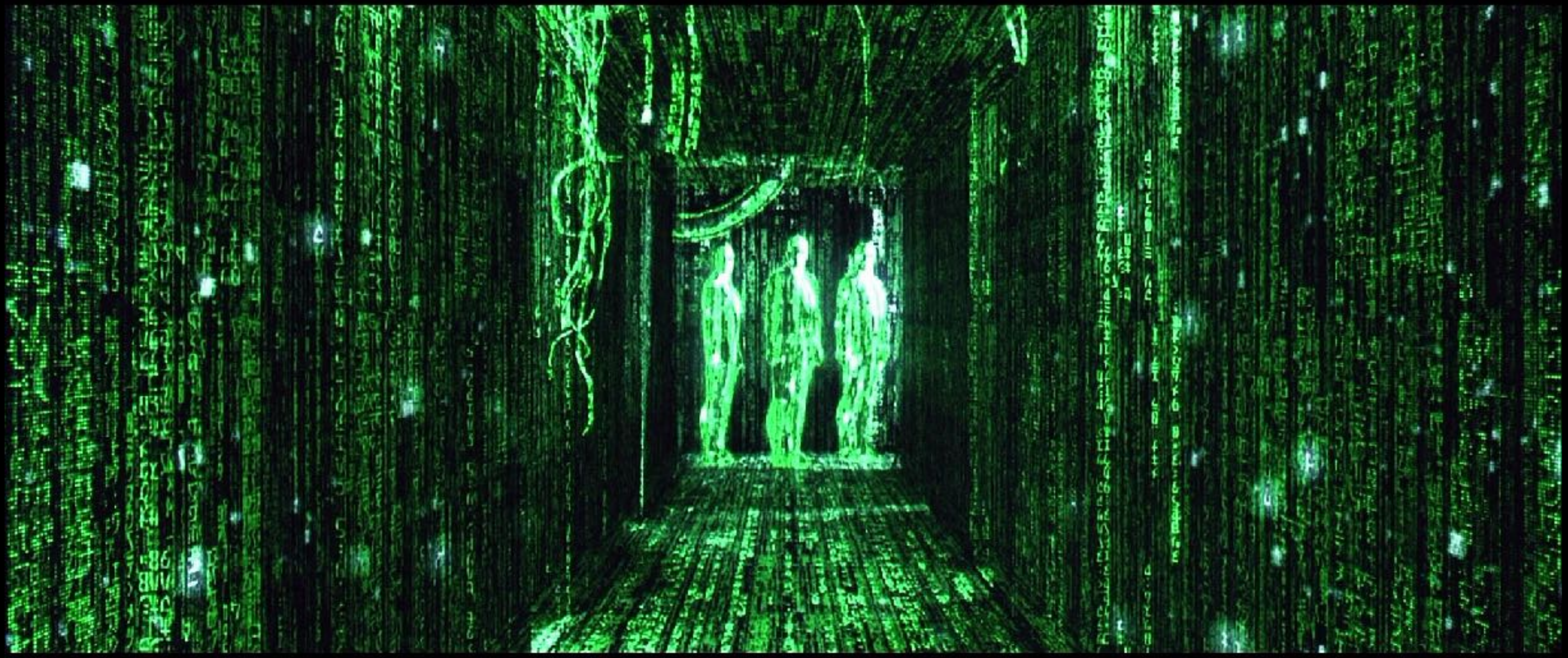
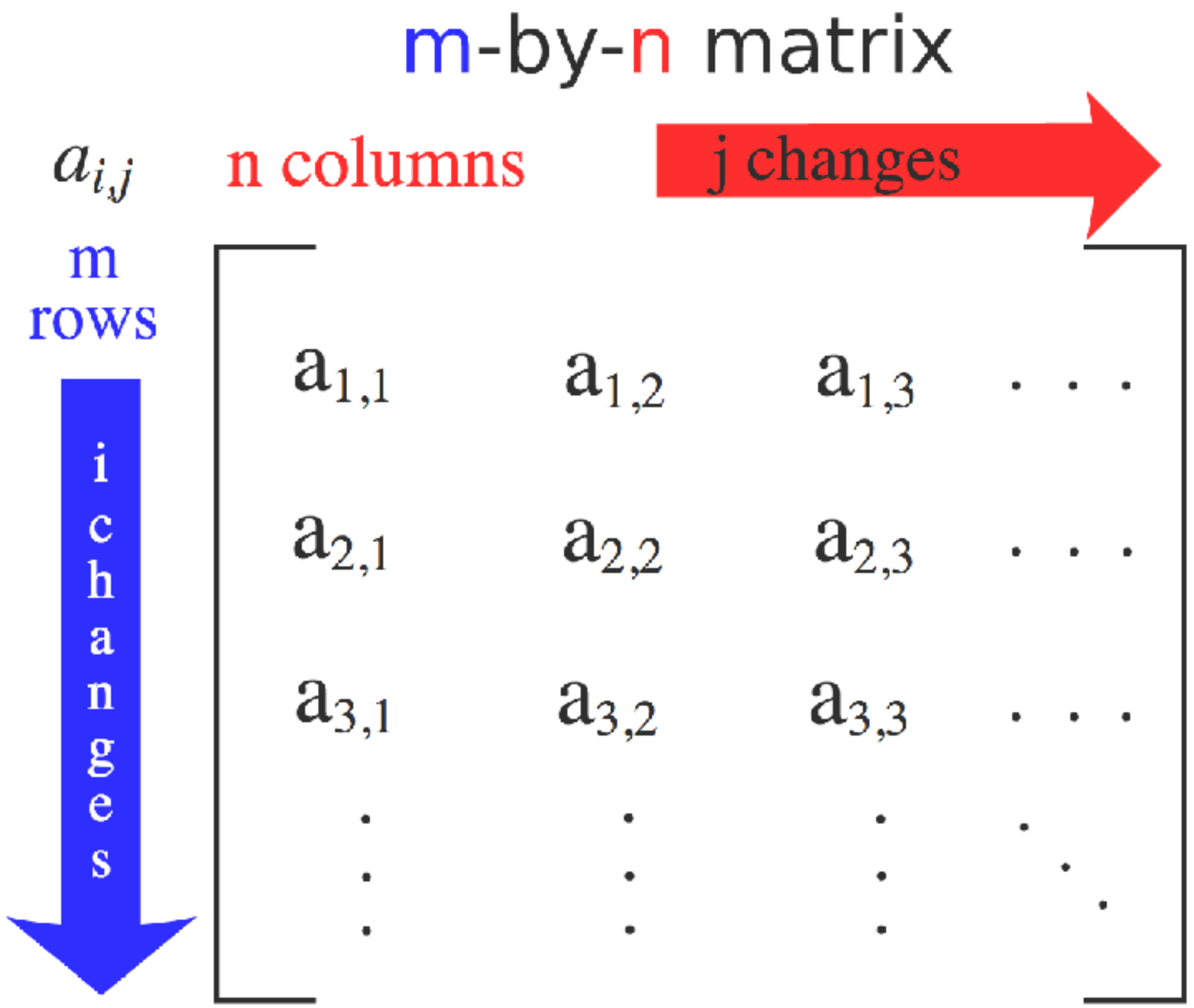
Matrix transformations.

Part 1



Matrix math.

A matrix.


m-by-n matrix
$a_{i,j}$
n columns
j changes
m rows
i changes
$a_{1,1}$ $a_{1,2}$ $a_{1,3}$ . . .
$a_{2,1}$ $a_{2,2}$ $a_{2,3}$ . . .
$a_{3,1}$ $a_{3,2}$ $a_{3,3}$ . . .

A 2x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

A 3x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

Matrix operations.

Matrix addition.

To add two matrices, add their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} + \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A+J & B+K & C+L \\ D+M & E+N & F+O \\ G+P & H+Q & I+R \end{bmatrix}$$

Matrix subtraction.

To subtract two matrices, subtract their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} - \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A-J & B-K & C-L \\ D-M & E-N & F-O \\ G-P & H-Q & I-R \end{bmatrix}$$

Matrix addition and subtraction can only happen
with matrices that are the same size!

Transpose of a matrix.

Transpose of a matrix is a matrix whose columns are the rows of the original matrix (and its rows are the columns).

$$M = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \qquad M^T = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

Matrix/scalar multiplication.

Multiply each entry of the matrix by the scalar.

$$S \times \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} S \times A & S \times B & S \times C \\ S \times D & S \times E & S \times F \\ S \times G & S \times H & S \times I \end{bmatrix}$$

Matrix/matrix multiplication.

You can only multiply two matrices
if the number of columns of the first matrix
equals the number of rows of the second.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \quad ?$$

It results in a matrix that is number of rows of first matrix by number of columns of second matrix.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For each row, find dot product with each column.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & \\ & \end{bmatrix}$$

For each row, find dot product with each column.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A\text{x}J + B\text{x}M + C\text{x}P & A\text{x}K + B\text{x}N + C\text{x}Q \\ & \end{bmatrix}$$

For each row, find dot product with each column.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

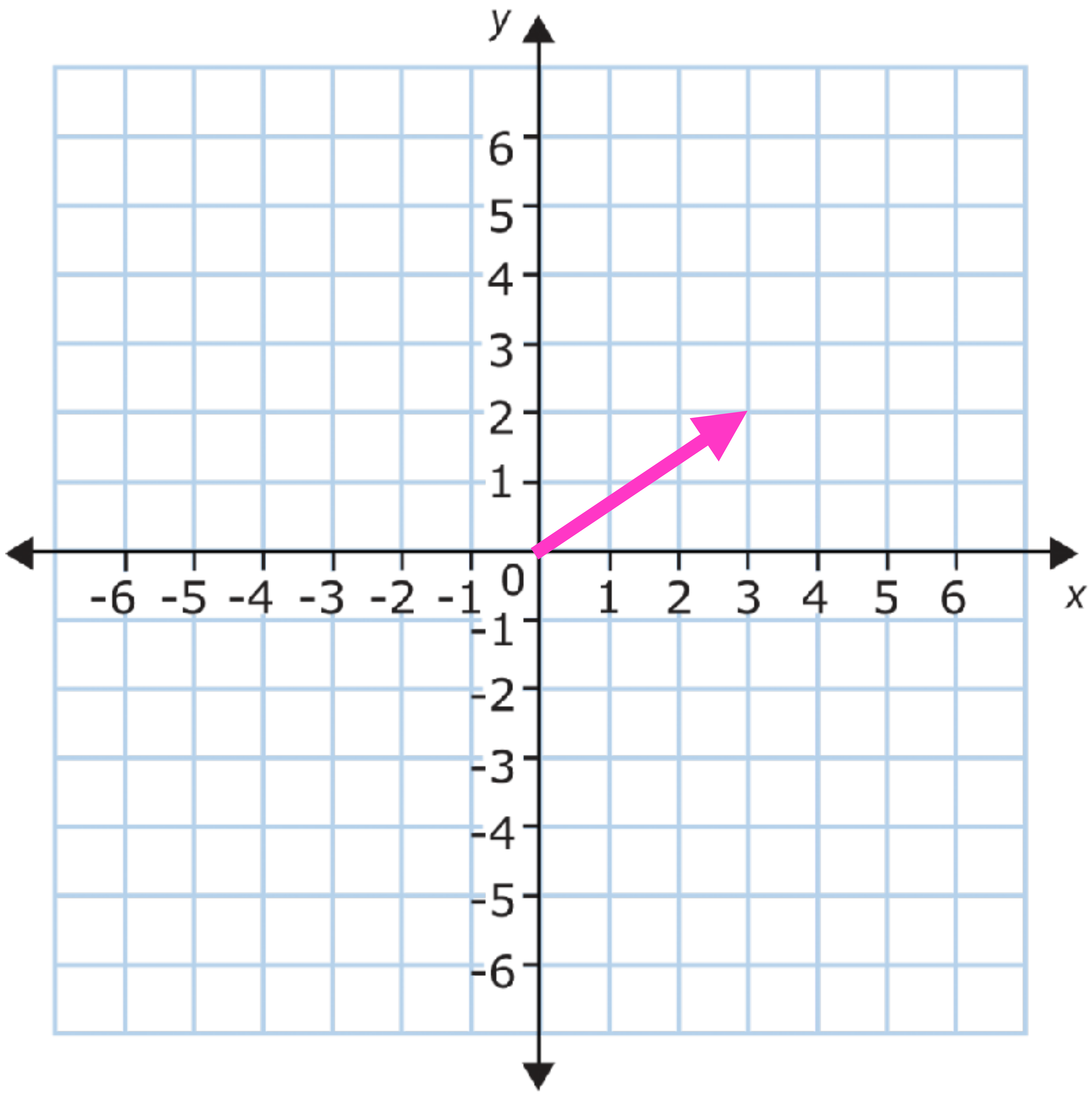
$$\begin{bmatrix} AxJ + BxM + CxP & AxK + BxN + CxQ \\ DxJ + ExM + FxP & \end{bmatrix}$$

For each row, find dot product with each column.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} AxJ + BxM + CxP & AxK + BxN + CxQ \\ DxJ + ExM + FxP & DxK + ExN + FxQ \end{bmatrix}$$

Vectors.



y
x
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6 -5 -4 -3 -2 -1
1 2 3 4 5 6

A 2 dimensional vector can be represented as a 2x1 matrix.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A 3 dimensional vector can be represented as a 3x1 matrix.

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Matrix vector multiplication.

Multiplying a matrix and a vector is just multiplying two matrices.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Row by row, multiply each column value with the each row of the vector and add them together.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ \\ \\ \end{bmatrix}$$

Row by row, multiply each column value with the each row of the vector and add them together.

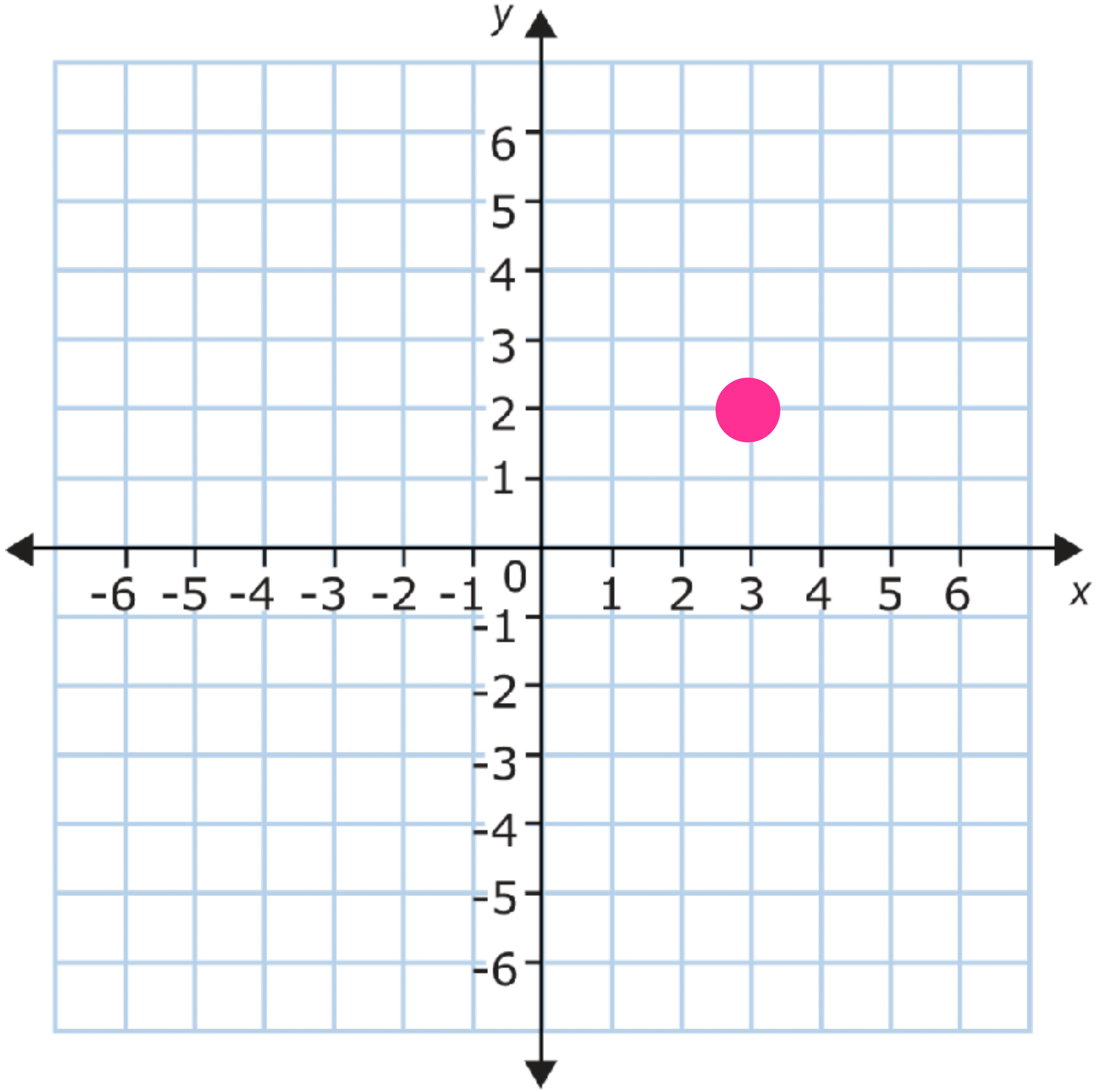
$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ \end{bmatrix}$$

Row by row, multiply each column value with the each row of the vector and add them together.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + HY + IZ \end{bmatrix}$$

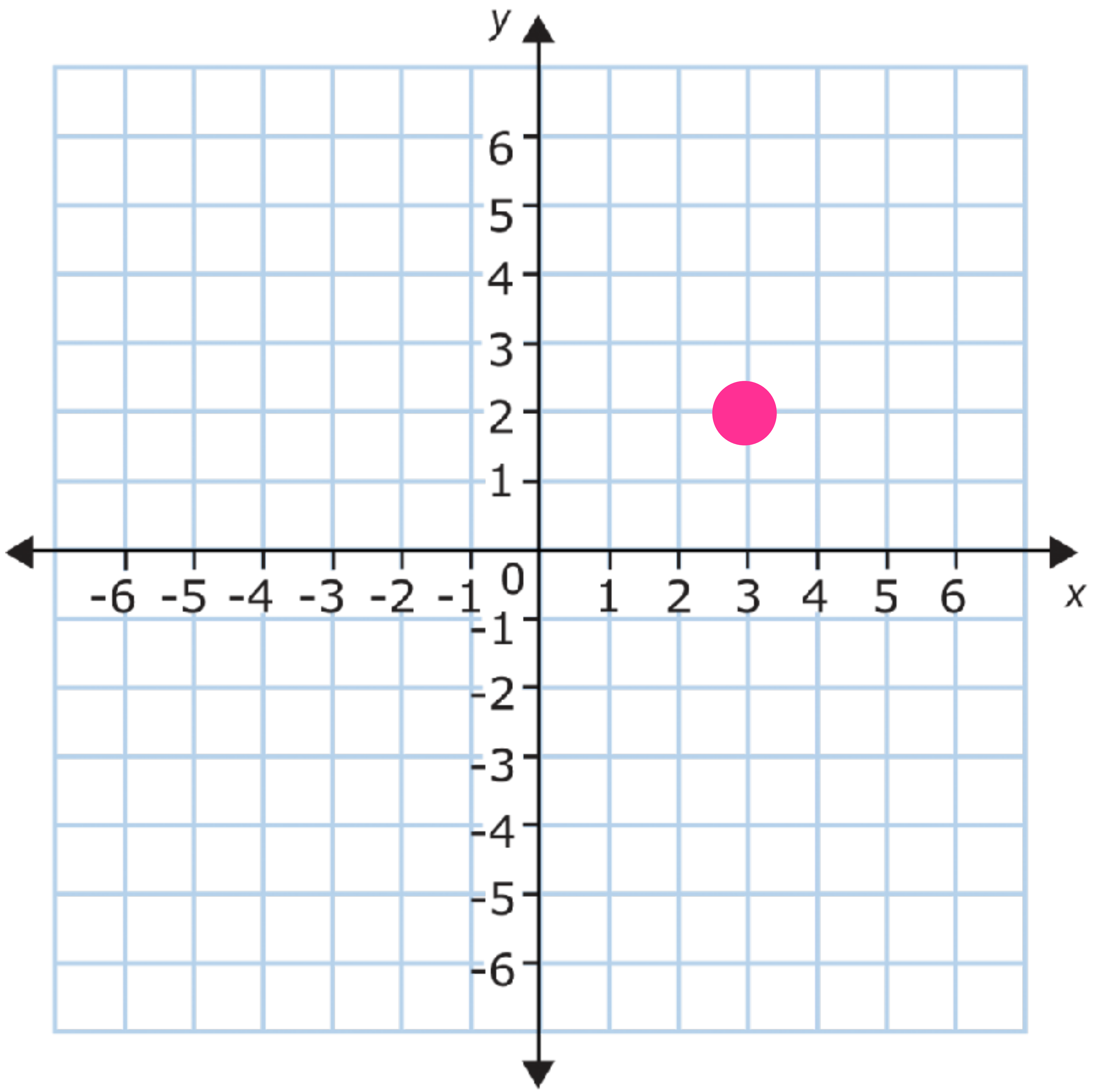
Transformation matrices.

Linear transformations stored as matrices.



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Example: scale



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Scale

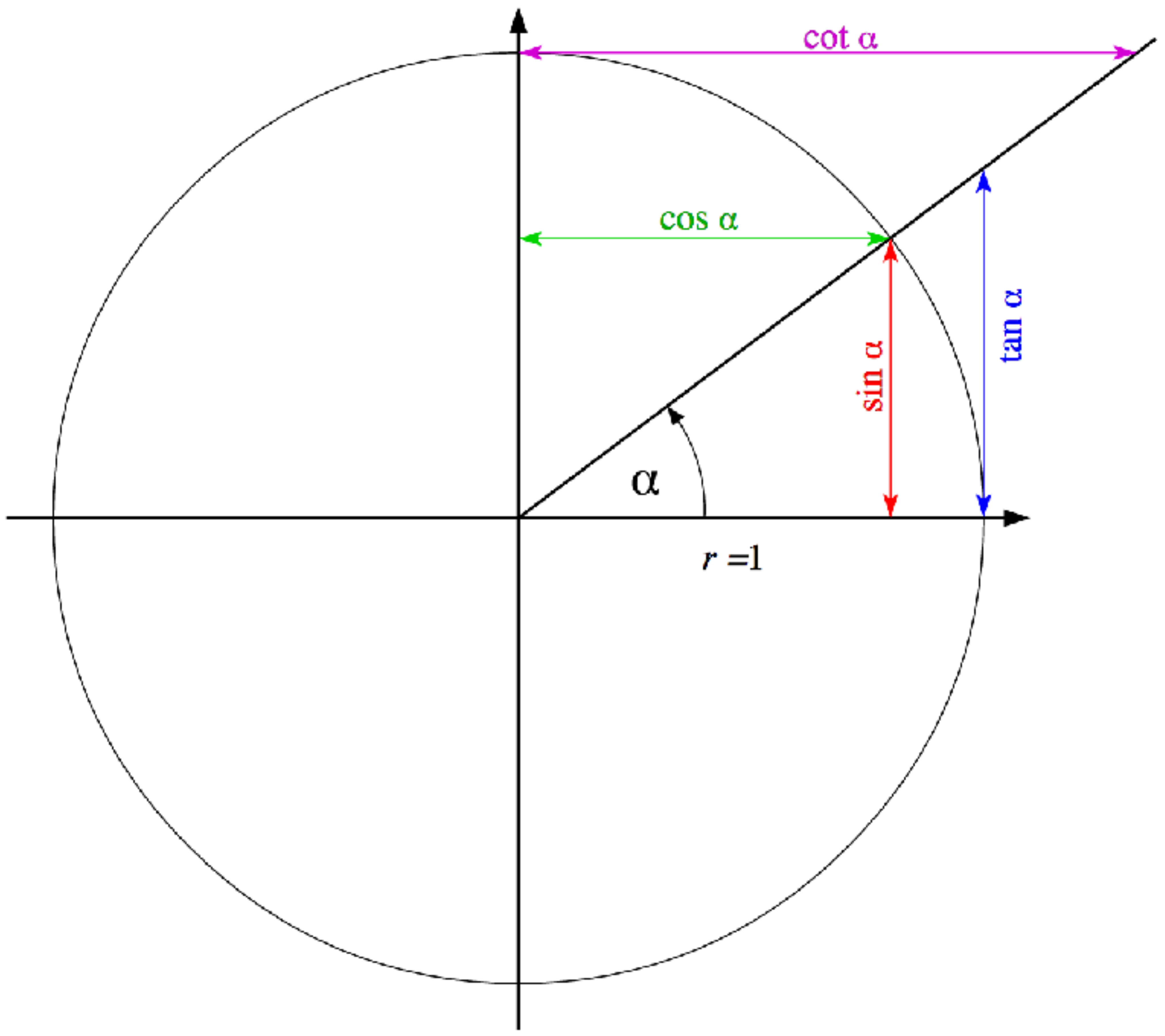
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX + BY \\ CX + DY \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} sxX + 0Y \\ 0X + syY \end{bmatrix}$$

Rotation

Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta X + -\sin\theta Y \\ \sin\theta X + \cos\theta Y \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta X + -\sin\theta Y \\ \sin\theta X + \cos\theta Y \end{bmatrix}$$

Affine transformations.

Homogeneous coordinates.

ABCD is the linear part of the affine transformation matrix.

$$\begin{bmatrix} A & B & ? \\ C & D & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Translate

Translate

$$\begin{bmatrix} 1 & 0 & Tx \\ 0 & 1 & Ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1Tx \\ 0X + 1Y + 1Ty \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Identity

Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} 1X + 0Y + 1x0 \\ 0X + 1Y + 1x0 \\ 0X + 0Y + 1x1 \end{bmatrix}$$

Multiplying affine transformation matrices.

```
matrix.identity();
matrix.Translate(5.0, 4.0);
matrix.Scale(2.0, 4.0);

// draw vertex at 3,2
```

```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity();  matrix.Translate(5.0, 4.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

```
matrix.identity();     matrix.Translate(5.0, 4.0);       matrix.Scale(2.0, 4.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 1 \end{bmatrix}$$

```
matrix.identity();
matrix.Scale(2.0, 4.0);
matrix.Translate(5.0, 4.0);

// draw vertex at 3,2
```

```
matrix.identity();
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

```
matrix.identity();    matrix.Scale(2.0, 4.0);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \end{bmatrix}$$

`matrix.identity();` `matrix.Scale(2.0f, 4.0f);` `matrix.Translate(5.0f, 4.0f);`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & 4 & 16 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \\ 1 \end{bmatrix}$$

Moving into 3D

3D identity matrix and 3D position in homogeneous coordinates.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

All transformations in 3D

X-Rotation in 3D

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Z-Rotation in 3D

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale in 3D

$$\begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Y-Rotation in 3D

$$\begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation in 3D

$$\begin{bmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection matrices are the same.

matrix.**setOrthoProjection**(l, r, b, t, n, f);

$$\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{(r+l)}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{(t+b)}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{(f+n)}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$