Prob. & Stats. - Week 5 Outline

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1 Random Variables

1.1 Discrete Random Variables

1.2 Continuous Random Variables

- Definition:
  X is a continuous random variable if there exists a positive function f_X such that
  $$P(X \in [a,b]) = \int_a^b f_X(x)dx$$
  $$P_X(A) = P(x \in A) = \int_A f_X(x)dx$$
- $f_X(x)$, the probability density function of X, which must satisfy:
  $$\int_{-\infty}^{\infty} f_X(x)dx = 1$$
  - f can be > 1.
  - f is not necessarily continuous.
- Relations between cumulative distribution function and probability density function:
  - $F_X(x) = \int_{-\infty}^{x} f_X(y)dy = P(X \leq x)$
  - $f_X(x) = \frac{d}{dx}F_x(x)$
- Observations:
  - F_X has no jumps.
  - $P(X = a) = 0$ for $\forall a$
  - X, Y are two continuous random variables that follow the same law $\Leftrightarrow F_X = F_Y$; $f_X = f_Y$
- Catalogue of Laws

– Uniform Distribution: $X \sim \text{Uni}(\alpha, \beta)$

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & otherwise \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & x \in [\alpha, \beta] \\ 1 & x \geq \beta \end{cases}$$

– Exponential Law: $X \sim \text{Exp}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \leq \alpha \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$

Important properties:

* Survival probability: $P(X > t) = e^{-\lambda t}$
* Memoryless property:

$$P(X > s + t \mid X > s) = P(X > t)$$

– Pareto Distribution: $X \sim \text{Par}(\alpha)$

$$f_X(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^\alpha} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

Generalization: $X \sim \text{Par}(x_{min}, \alpha)$

$$f_X(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}}(x_{min})^\alpha & x \geq x_{min} \\ 0 & x < x_{min} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - (\frac{x_{min}}{x})^\alpha & x \geq x_{min} \\ 0 & x < x_{min} \end{cases}$$

Note that $\text{Par}(1, \alpha) = \text{Par}(\alpha)$

Important properties of Par(α):

* Survival probability: $P(X > x) = 1/x^\alpha$

* Almost memoryless property:

$$P(X > x | X > x_0) = \left(\frac{x_0}{x}\right)^{\alpha}$$

That is, a Par(α) random variable conditioned to being larger than a certain $x_0 \geq 1$ is distributed as Par(x_0, α)

* Relation with Exponential Law:

$$X \sim \text{Par}(\alpha) \iff Y = \ln(X) \sim \text{Exp}(\alpha)$$

– Normal Distribution: $X \sim \text{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

– Standard Normal Distribution: $Z \sim$ N(0,1)

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\phi_Z(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{y^2}{2}} dy$$

Important properties:

* "Standardization":

$$X \sim N(\mu, \sigma^2) \iff Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

* Normalization integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

* Three-σ Principle:

$$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$$
$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$$
$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.99$$