Prob. & Stats. - Week 7 Outline

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1 Expectation (continue)

1.1 Expectation for some special distributions (continue)

- $X \sim \text{Par}(\alpha)$
$$E(X) = \frac{\alpha}{\alpha - 1}$$
- There are distributions for which $E(X)$ may not exist. An example is the Cauchy Distribution: $f_X = 1/[\pi(1 + x^2)]$.

1.2 Expectation of functions

If X is a random variable, $g, h : \mathbb{R} \to \mathbb{R}$ are functions, a, b are constants then:

- Discrete Case
$$E[g(X)] = \sum_x g(x)p_X(x)$$
- Continuous Case
$$E[g(X)] = \int g(x)f_X(x)dx$$
- Properties

(P1) Linearity
$$E[a\, g(X) + b\, h(X)] = aE[g(X)] + bE[h(X)]$$

(P2) Change of Units:
$$E(aX + b) = aE(X) + b$$

(P3) If X is a random variable with $E(X) = \mu$, then the random variable
$$Y = X - \mu$$
is **centered**, that is,
$$E(Y) = 0$$

1.3 Useful Expectations

- The *n-th moment* of X: $E(X^n)$
- The *exponential moment* of X: $E(e^X)$

2 Variance

2.1 Definition

The variance of X is:

$$Var(X) = E[(X - E(X))^2]$$

2.2 Properties

(P1) Change of Units:

$$Var(aX + b) = a^2 Var(X)$$

(P2) Alternative expression

$$Var(X) \;=\; E(X^2) - E(X)^2$$

(P3) If X is a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$, then the random variable

$$Z \;=\; \frac{X - \mu}{\sigma}$$

is **normalized**, that is, it satisfies:

$$E(Z) = 0 \quad , \quad Var(Z) = 1$$

3 Means and Variances of important Laws

Law	**E(X)**	**Var(X)**
$\mathrm{Ber}(p)$	p	$p(1-p)$
$\mathrm{Bi}(n,p)$	np	$np(1-p)$
$\mathrm{Geo}(r)$	$\frac{1}{r}$	$\frac{1-r}{r^2}$
$\mathrm{Poi}(\lambda)$	λ	λ
$\mathrm{Exp}(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$\mathrm{Par}(\alpha)\ \alpha > 2$	$\frac{\alpha}{\alpha-1}$	$\frac{\alpha}{(\alpha-1)^2(\alpha-2)}$
$\mathrm{Par}(\alpha)\ 1 < \alpha \leq 2$	$\frac{\alpha}{\alpha-1}$	∞
$\mathrm{N}(\mu,\sigma^2)$	μ	σ^2

4 Joint Distributions

4.1 Definitions

- Joint Probability Mass & Density Function
  - Discrete case
    X, Y are two discrete random variables on the same Ω.
    The **joint probability mass function of** X **and** Y is a function:
    $$p_{X,Y}(\ ,\ ) : A_X * B_Y \to [0,1]$$
    such that
    $$P(X \in A, Y \in B) = \sum_{x \in A} \sum_{y \in B} p_{X,Y}(x,y)$$
    The joint mass is determined through the identities:
    $$p_{X,Y}(a,b)\ =\ P(X = a, Y = b)$$
    for any a, b in the respective alphabets.
  - Continuous case
    X, Y are two continuous random variables on the same Ω.
    The **joint probability density function of** X **and** Y is a function
    $$f_{X,Y}(\ ,\ ) : \mathbb{R}^2 \to \mathbb{R}$$

such that

$$P(X \in A, Y \in B) = \int_B \int_A f_{X,Y}(x,y)dxdy$$

The joint density is determined by the identities

$$\int_{a_1}^{b_1} dx \int_{a_2}^{b_2} dy \;=\; P(a_1 \le X \le b_1, a_2 \le Y \le b_2)$$

for any real numbers $a_{\le}b_1$ and $a_2 \le b_2$.

- Marginal probability mass and density functions
  - Discrete case:

$$\begin{aligned} p_X(a) &= \sum_b p_{X,Y}(a,b) \\ p_Y(b) &= \sum_a p_{X,Y}(a,b) \end{aligned}$$

  - Continuous case:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

- Joint Cumulative Function
  - Discrete Case
    X, Y are two discrete random variables on the same Ω.
    The **joint cumulative function of X and Y** is:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \sum_{a \le x} \sum_{b \le y} p_{X,Y}(a,b)$$

  - Continuous Case
    X, Y are two continuous random variables on the same Ω.
    The **joint cumulative function of X and Y** is:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{\infty}^{y} f_{X,Y}(a,b)db\ da$$

- Generalization
  - Discrete Case
    $X_1, X_2, ..., X_n$ are discrete random variables on the same Ω.

$$p_{X_1,X_2,...,X_n}(x_1, x_2, ..., x_n) = P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n)$$

- Continuous Case
  $X_1, X_2, ..., X_n$ are continuous random variables on the same Ω.

$$P(X_1 \in A_1, X_2 \in A_2, ..., X_n \in A_n) = \int_{A_n} ... \int_{A_1} f_{X_1,...,X_n}(x_1, ..., x_n) dx_1...dx_n$$

4.2 Independence

- Definition
  Two random variables X and Y are **independent** if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B), \ \forall A, B$$

- Corollary
  X and Y are independent iff one of the following equivalent conditions hold

  - In general,
  $$F_{X,Y}(x, y) = F_X(x)\, F_Y(y)$$
  - For discrete case,
  $$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$$
  - For continuous case,
  $$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$$

4.3 Join expectations

If X, Y are random variables, g, h are joint functions, a, b are constants, then:

- Discrete Case
$$E[g(X, Y)] = \sum_x \sum_y g(x, y)\, p_{X,Y}(x, y)$$
- Continuous Case
$$E[g(X, Y)] = \int \int g(x, y)\, f_{X,Y}(x, y)\, dx\, dy$$
- Properties

(P1) Linearity

$$E[a\, g(X, Y) + b\, h(X, Y)] = a\, E[g(X, Y)] + b\, E[h(X, Y)]$$

(P2) Reduction to marginals

$$E[g(X)] = \int g(x) f_X(x) dx$$

General conclusion: Need to use the density (or mass) involving only the variables that are present.

(P3) Relations to Independence
X and Y are independent, iff

$$E[g(X)\, h(Y)] = E[g(X)]\, E[h(Y)]$$

for all functions g and h.
Remark: In particular $E(XY) = E(X)\, E(Y)$ is **NOT** enough to conclude independence.

4.4 Covariance

- Definition
  The **covariance of** X **and** Y is defined as:

$$Cov(X, Y) = E\{[X - E(X)][Y - E(Y)]\}$$

  We say that X and Y are **uncorrelated** if:

$$Cov(X, Y) = 0$$

- Properties

(P0) In all cases

$$Var(X + Y) = Var(X) + Var(Y) + 2\, Cov(X, Y)$$

(P1) If X and Y are uncorrelated, then:

$$Var(X + Y) = Var(X) + Var(Y)$$

(P2) If X and Y are independence, then X and Y are uncorrelated.
Remark:
This is only a **one-way** implication. The reverse is **NOT** correct.

(P3) Alternative Expression

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

(P4) Change of units:

$$Cov\big(a_1 X + b_1\, ,\, a_2 Y + b_2\big) \;=\; a_1\, a_2\, Cov(X, Y)$$

(P5) Relation with variance:

$$Cov(X, X) \;=\; Var(X)$$