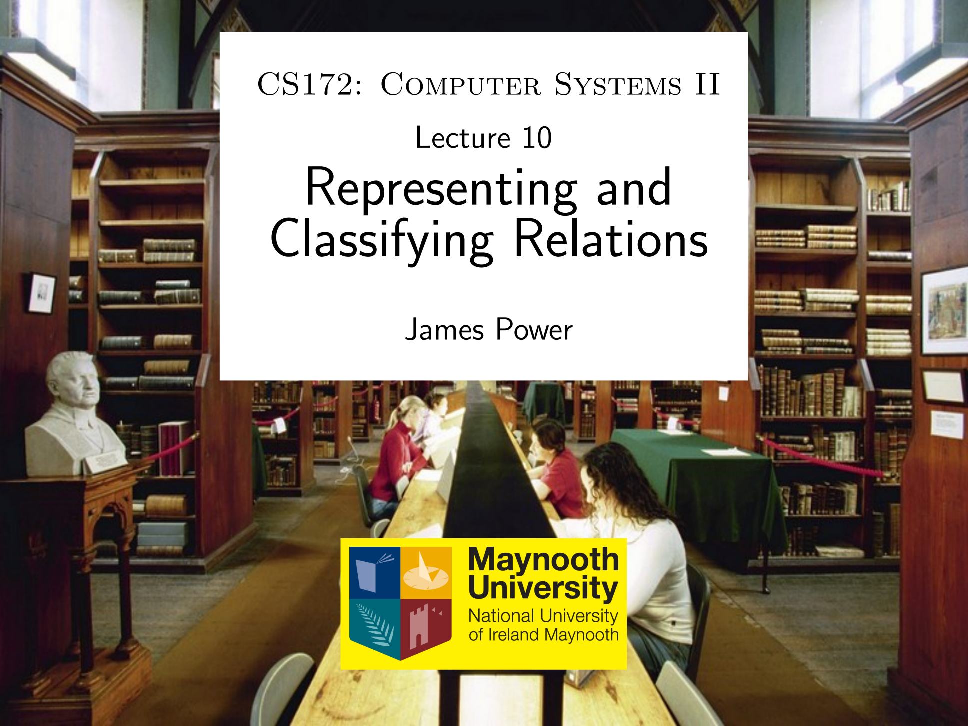

CS172: COMPUTER SYSTEMS II

Lecture 10

Representing and Classifying Relations

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Relations: summary so far

- Given two sets A and B, a **relation** from A to B is a subset of $A \times B$.

- Relations can be defined by *enumeration* or by *comprehension*.
- The main relational operations are
  - composition ($R \fatsemi S$)
  - inverse (R^{-1})
- Since a relation is just a (special kind of) set, the usual set-theory operations (union, intersection, difference) also work with relations.

Some notation

- Given some n-ary relation R, we denote *membership* of that relation by saying something like $(x_1, x_2, \ldots, x_n) \in R$
- Sometimes it is more convenient to use prefix notation: if we want to express that $x_1, x_2, \ldots, x_n$ are related by the relation R we might just write $R(x_1, x_2, \ldots, x_n)$
- When we are dealing with a binary relation, we might choose to write the relation symbol infix, as in xRy.

  The usual integer relations ($<$, $>$, etc.) are examples of binary relations that are typically written using infix notation.

Part I:

Representing relations using an adjacency matrix

Adjacency Matrix

- One way of representing a (finite) binary relation is as an adjacency matrix.
- Each entry in the adjacency matrix is a boolean value.
- Given any finite binary relation $R \subseteq A \times B$, we can construct its adjacency matrix M_R as follows:
  - We index the rows and columns of the matrix by the elements of A and B respectively (assumed to be in some fixed order).
  - For any $(a, b) \in (A \times B)$ we set the entry M_R for row a and column b to true iff $(a, b) \in R$

Adjacency Matrix: example 1

- Tudors = {Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I }
- Relation: is-a-parent-of $\subseteq$ Tudors $\times$ Tudors

	Henry VII	Henry VIII	Edward VI	Mary I	Elizabeth I
Henry VII	*F*	*T*	*F*	*F*	*F*
Henry VIII	*F*	*F*	*T*	*T*	*T*
Edward VI	*F*	*F*	*F*	*F*	*F*
Mary I	*F*	*F*	*F*	*F*	*F*
Elizabeth I	*F*	*F*	*F*	*F*	*F*

Adjacency Matrix: example 2

- Tudors = {Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I }
- Relation: lived-longer-than $\subseteq$ Tudors $\times$ Tudors

	Henry VII	Henry VIII	Edward VI	Mary I	Elizabeth I
Henry VII	*F*	*F*	*T*	*T*	*F*
Henry VIII	*T*	*F*	*T*	*T*	*F*
Edward VI	*F*	*F*	*F*	*F*	*F*
Mary I	*F*	*F*	*T*	*F*	*F*
Elizabeth I	*T*	*T*	*T*	*T*	*F*

Questions:

- How many elements in the relation?
- Who did Henry VII live longer than?
- Who lived longer then Mary I?

Adjacency Matrix: relational operations

The relational operations map quite nicely to matrix operations:

- relational application: matrix multiplication
- relational composition: matrix multiplication
- relational inverse: transpose of a matrix
- relational union: (point-wise) disjunction of the matrices
- relational intersection: (point-wise) conjunction of the matrices

For matrix multiplication:

- addition is disjunction, multiplication is conjunction.
- if you prefer, use 1 and 0 with the rule that $1 + 1 = 1$

Adjacency Matrix: examples of operators

Who was the parent of Elizabeth I?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

is-a-parent-of $\{Elizabeth\ I\}$

Adjacency Matrix: examples of operators

Who was the parent of Elizabeth I?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is-a-parent-of $\{Elizabeth\ I\}$ $\{Henry\ VIII\}$

Adjacency Matrix: examples of operators

Who were parents of either Elizabeth I *or* Henry VIII?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

is-a-parent-of $\quad \left\{ \begin{array}{c} \text{Elizabeth I,} \\ \text{Henry VIII} \end{array} \right\}$

Adjacency Matrix: examples of operators

Who were parents of either Elizabeth I *or* Henry VIII?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{is-a-parent-of} \qquad \left\{ \begin{array}{c} \text{Elizabeth I,} \\ \text{Henry VIII} \end{array} \right\} \qquad \left\{ \begin{array}{c} \text{Henry VII,} \\ \text{Henry VIII} \end{array} \right\}$$

Adjacency Matrix: examples of operators (inverse)

Who are the *children* of Henry VIII?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$$

is-a-child-of
$=$ is-a-parent-of^T

$\{HenryVIII\}$

The relation is-a-child-of is the *relational inverse* of is-a-parent-of.

Adjacency Matrix: examples of operators (inverse)

Who are the *children* of Henry VIII?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{array}{c} \text{is-a-child-of} \\ = \text{is-a-parent-of}^T \end{array} \qquad \{HenryVIII\} \qquad \left\{ \begin{array}{c} \text{Edward VI,} \\ \text{Mary I,} \\ \text{Elizabeth I} \end{array} \right\}$$

The relation is-a-child-of is the *relational inverse* of is-a-parent-of.

Adjacency Matrix: examples of operators

Who are the children of either Henry VII or Henry VIII?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$$

is-a-child-of $\quad \left\{ \begin{array}{c} \text{Henry VII,} \\ \text{Henry VIII} \end{array} \right\}$

Adjacency Matrix: examples of operators

Who are the children of either Henry VII or Henry VIII?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{is-a-child-of} \qquad \left\{ \begin{array}{c} \text{Henry VII,} \\ \text{Henry VIII} \end{array} \right\} \qquad \left\{ \begin{array}{c} \text{Henry VIII,} \\ \text{Edward VI,} \\ \text{Mary I,} \\ \text{Elizabeth I,} \end{array} \right\}$$

Adjacency Matrix: examples of operators (composition)

The relation is-a-grandparent-of $=$ is-a-parent-of $\fatsemi$ is-a-parent-of:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is-a-parent-of after is-a-parent-of $=$ is-a-grandparent-of

Adjacency Matrix: examples (inverse and composition)

What happens if I compose a relation with its inverse?
For example: is-a-parent-of ⨾ is-a-child-of:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \times \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad =$$

is-a-child-of after is-a-parent-of =

Adjacency Matrix: examples (inverse and composition)

What happens if I compose a relation with its inverse?
For example: is-a-parent-of ⨾ is-a-child-of:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

is-a-child-of after is-a-parent-of = is-a-sibling-of

- Maps a person to their *siblings* (including themselves).

Part II:

Classifying Relations

- reflexive
- symmetric
- transitive
- partial/total order
- equivalence relation

Kinds of relations: reflexive

- If A is a set, then the relation $I = \{a \in A \bullet (a, a)\}$ is called the identity relation on A.

  That is, the identity relation just maps every object to itself.

- Any binary relation R that has the property that $I \subseteq R$ called a reflexive relation.

  That is, a reflexive relation *at least* maps every object to itself, and maybe to some other objects too.

- Any binary relation R that has the property that $I \cap R = \emptyset$ called an irreflexive relation.

  That is, an *ir*reflexive relation *never* maps any object to itself.

Kinds of relations: symmetric

- Any binary relation R that has the property that $R^{-1} \subseteq R$ is called symmetric.

  i.e. whenever it contains the pair (a, b) it also contains the pair (b, a).

- An antisymmetric relation is a binary relation with the property that if it ever contains both (a, b) and (b, a) then we must have $a = b$.

  i.e. an antisymmetric relation can be reflexive, but is not otherwise symmetric.

- Aside: A relation that is both antisymmetric and also irreflexive is called *asymmetric*.

Examples: reflexivity and symmetry over $(\mathbb{N} \times \mathbb{N})$

reflexive: for all x, $R(x, x)$ irreflexive: for all x, $\neg R(x, x)$

symmetric: for all x, y, $R(x, y) \to R(y, x)$

antisymmetric: for all x, y, $(R(x, y) \wedge R(y, x)) \to x = y$

asymmetric: for all x, y, $R(x, y) \to \neg R(y, x)$

Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$					
$>$					
$==$					
$!=$					
$<=$					
$>=$					

Examples: reflexivity and symmetry over $(\mathbb{N} \times \mathbb{N})$

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asymmetric: for all x, y, $R(x, y) \rightarrow \neg R(y, x)$

Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$	✗				
$>$	✗				
$==$	✓				
$!=$	✗				
$<=$	✓				
$>=$	✓				

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Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$	✗	✓			
$>$	✗	✓			
$==$	✓	✗			
$!=$	✗	✓			
$<=$	✓	✗			
$>=$	✓	✗			

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Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$	✗	✓	✗		
$>$	✗	✓	✗		
$==$	✓	✗	✓		
$!=$	✗	✓	✓		
$<=$	✓	✗	✗		
$>=$	✓	✗	✗		

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Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$	✗	✓	✗	✓	
$>$	✗	✓	✗	✓	
$==$	✓	✗	✓	✓	
$!=$	✗	✓	✓	✗	
$<=$	✓	✗	✗	✓	
$>=$	✓	✗	✗	✓	

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reflexive: for all x, $R(x,x)$ irreflexive: for all x, $\neg R(x,x)$
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Relation	reflexive	irreflexive	symmetric	antisymmetric	asymmetric
$<$	✗	✓	✗	✓	✓
$>$	✗	✓	✗	✓	✓
$==$	✓	✗	✓	✓	✗
$!=$	✗	✓	✓	✗	✗
$<=$	✓	✗	✗	✓	✗
$>=$	✓	✗	✗	✓	✗

Transitive Relations

- Any binary relation R that has the property that whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ is called transitive.
- Any binary relation that is reflexive, transitive and also *antisymmetric* is called a partial order. *(no cycles)*

  If, in addition, for every two objects x, y we have either $(x, y) \in R$ or $(y, x) \in R$, then we call it a total order. *(a chain)*
- Any binary relation that is reflexive, transitive and also *symmetric* is called an equivalence relation.

Examples: Tudors

Relation	reflexive	symmetric	transitive
is-a-parent-of			
was-succeeded-by*			
lived-longer-than			

* assuming was-*immediately*-succeeded-by

Relation	reflexive	symmetric	transitive
sibling-of			
is-an-ancestor-of			
lived-in-reign-of			
same-gender-as			
different-gender-to			

Examples: Numeric relations

Let $x, y \in \mathbb{N}$, and suppose the following relations are all defined as sets of tuples $(x, y) \in \mathbb{N} \times \mathbb{N}$:

Relation	reflexive	symmetric	transitive
$x < y$			
$x \leq y$			
$x = y$			
$y = \sqrt{x}$			
$y = x\%2$			
$y\%2 = x\%2$			

Equivalence relations and partitions

- Given any set S, a partition of S is a set of subsets that are *collectively exhaustive* and *mutually exclusive*.

  That is, sets $S_1, \ldots, S_n$ are a partition of S if:
  - $(S_1 \cup \ldots \cup S_n) = S$ [collectively exhaustive]
  - for any sets S_i and S_j, if $i \neq j$ then $S_i \cap S_j = \emptyset$ [mutually exclusive]

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  - for any sets S_i and S_j, if $i \neq j$ then $S_i \cap S_j = \emptyset$ [mutually exclusive]

- Any *equivalence relation* over some set S automatically partitions S into a set of equivalence classes.

  (- sometimes called the *quotient set* of S)

- Example: using *equality* as the equivalence relation we partition a set into subsets each with exactly one element.

Equivalence relations: modulo example

Suppose we take the set of natural numbers, $\mathbb{N}$.

Than we can form a partition of $\mathbb{N}$ using e.g. modulo 3, getting three equivalence classes:

- $S_0 = \{n \in \mathbb{N} \mid n \,\%\, 3 = 0\}$
  $= \{0, 3, 6, 9, 12, 15, 18, \ldots\}$
- $S_1 = \{n \in \mathbb{N} \mid n \,\%\, 3 = 1\}$
  $= \{1, 4, 7, 10, 13, 16, 19, \ldots\}$
- $S_2 = \{n \in \mathbb{N} \mid n \,\%\, 3 = 2\}$
  $= \{2, 5, 8, 11, 14, 17, 20, \ldots\}$

Thus $\{S_0, S_1, S_2\}$ is a partition of $\mathbb{N}$