CS172: Computer Systems II

Lecture 4

Propositional Logic

Valid Inferences

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Inferences: previous examples

LiA §2.6

Are these valid inferences?

- John is a student
  If John is a student, then he is a human

  John is a human

- If she is guilty, she will go to jail
  She will not go to jail

  She is not guilty

What's the *algorithm* to decide this?

Definition of valid inference

- The inference from a finite set of premises $\phi_1, \ldots, \phi_k$ to a conclusion ψ is a valid inference if each valuation V with $V(\phi_1) = \ldots = V(\phi_k) = 1$ also has $V(\psi) = 1$.
- We write:

$$\phi_1, \ldots, \phi_k \models \psi$$

- Algorithm:
  1. write out a truth table for $\phi_1, \ldots, \phi_k, \psi$
  2. identify *all* the rows that make *all* of $\phi_1, \ldots, \phi_k$ true
  3. make sure ψ is also true on all these rows

The textbook (Definition 2.13) calls this "valid consequence"

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Four well-known valid inference forms

Many forms of valid inference have been known since antiquity, and have quaint Latin names:

modus ponens "the way that affirms" (something)	
modus *ponendo* ponens	modus *tollendo* ponens
"the way that affirms *by affirming*"	"the way that affirms *by denying*"
$(p \rightarrow q),\ p\ \models\ q$	$(p \vee q),\ \neg p\ \models\ q$

modus tollens "the way that denies" (something)	
modus *ponendo* tollens	modus *tollendo* tollens
"the way that denies *by affirming*"	"the way that denies *by denying*"
$\neg(p \wedge q),\ p\ \models\ \neg q$	$(q \rightarrow p),\ \neg p\ \models\ \neg q$

Where do logical laws come from?

Example: some commonly-used logical "laws"

Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
Double Negative	$\neg(\neg p) \equiv p$	
Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound	$p \vee T \equiv T$	$p \wedge F \equiv F$
De Morgan's	$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional	$(p \rightarrow q) \equiv (\neg p \vee q)$	$\neg(p \rightarrow q) \equiv (p \wedge \neg q)$

From http://integral-table.com/downloads/logic.pdf

Logical equivalence

- Two formulas ϕ and ψ are logically equivalent if $\phi \models \psi$ and $\psi \models \phi$.
- Two logically equivalent formula should agree on *every row* of their truth table.

  i.e. for any valuation V, we would have $V(\phi) = V(\psi)$

Classifying formulas

A formula is:

- unsatisfiable (or "a contradiction")
  if it is *false on every row* of its truth table.
- a tautology (or "logically valid")
  if it is *true on every row* of its truth table.
- satisfiable (or "logically contingent")
  if it is *true on at least one row* of its truth table.

Note: any inference $\phi_1, \ldots, \phi_k \models \psi$ is valid precisely when

- the formula $(\phi_1 \wedge \ldots \wedge \phi_k) \rightarrow \psi$ is a tautology
- the formula $(\phi_1 \wedge \ldots \wedge \phi_k) \wedge \neg\psi$ is unsatisfiable.

Information update

LiA §2.8

- The information content of a formula ϕ is the set of its models, that is, the valuations that assign the formula ϕ the truth-value 1.
- An update with new information ψ reduces the current set of models to the overlap or intersection of the existing models and the models of ψ.

  Any existing valuations that assign the value 0 to ψ are eliminated.

$$\text{More information} \implies \text{fewer models}$$

Note: we're skipping this sections for the moment: LiA §2.7

Information update: example

- Example: The party problem (Example 2.25)
  - John comes if Mary or Ann comes.
  - Ann comes if Mary does not come.
  - If Ann comes, John does not.
- When you have eliminated all which is impossible,
  then whatever remains, however improbable,
  must be the truth.

  - Sherlock Holmes in *The Adventure of the Blanched Soldier*

Example

Mrs White is guilty.	w
Miss Scarlet is guilty.	s
Colonel Mustard is guilty.	m

- At least one of them is guilty.
- Not all of them are guilty.
- If Mrs White is guilty, then Colonel Mustard helped her.
- If Miss Scarlet is innocent then so is Colonel Mustard.