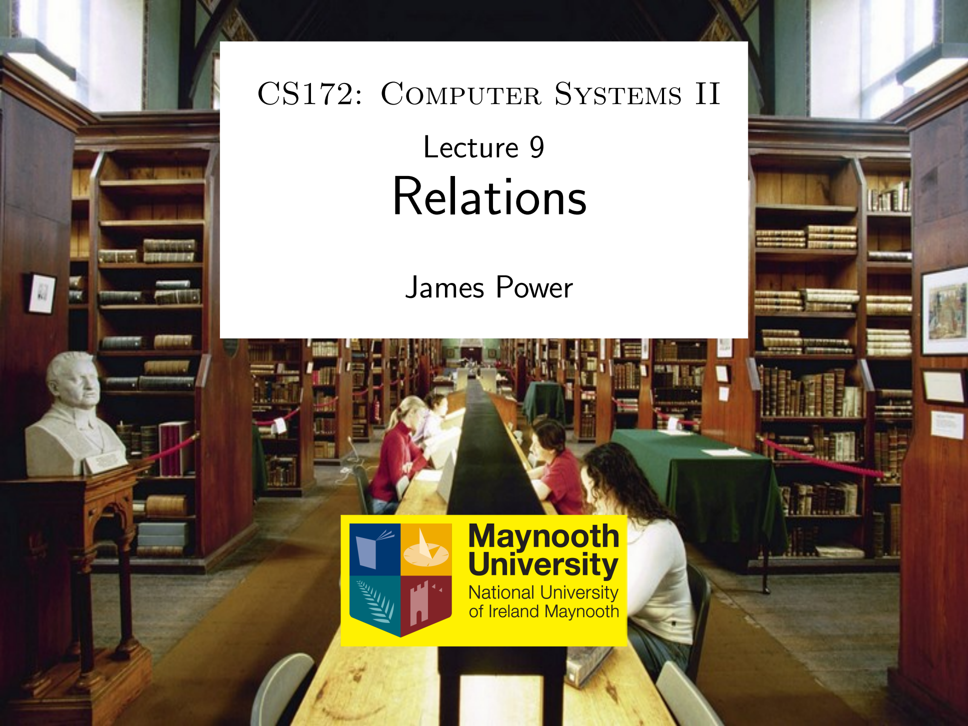
CS172: COMPUTER SYSTEMS II

Lecture 9

Relations

James Power



Maynooth University
National University of Ireland Maynooth

Relations: motivation, definition

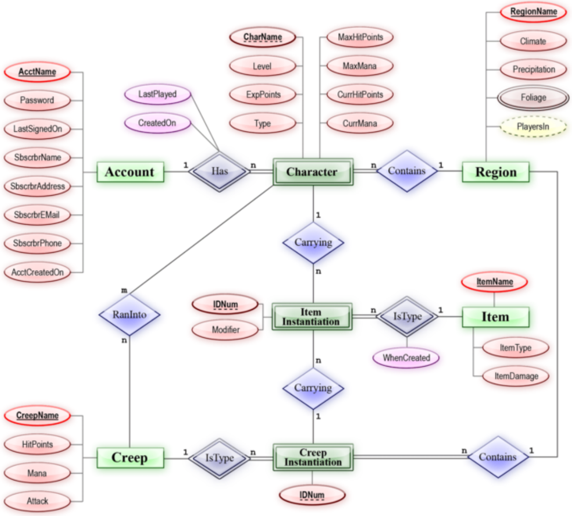
As well as describing things and their properties, it is also useful to be able to describe *relationships* between things.

- By a relation we mean a meaningful link between people, things, objects, whatever.

Textbook, §A.2

For our purposes, we will be defining relations in terms of sets.

Entity-relationship diagram



Source: Wikipedia

Running example: the Tudor monarchs

Henry VII
1457-1509

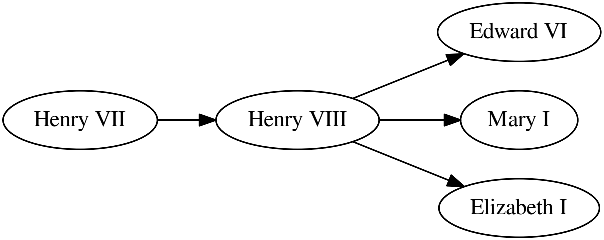
Henry VIII
1491-1547

Edward VI
1537-1553

Mary I
1516-1558

Elizabeth I
1533-1603

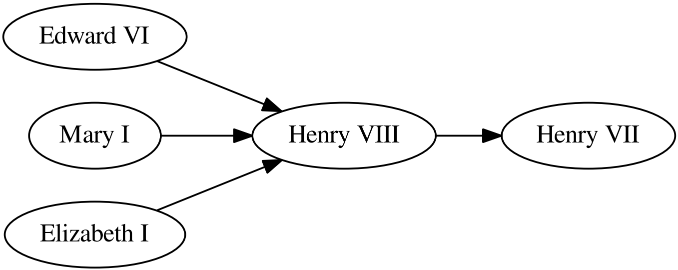
The relation: is-a-parent-of



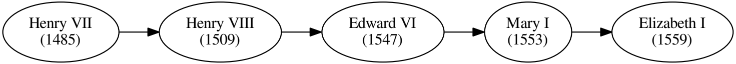
Here, the relation is represented as a directed graph

- The **nodes** represent the elements of the set
- An **edge** from A to B means that A *is a parent of* B.

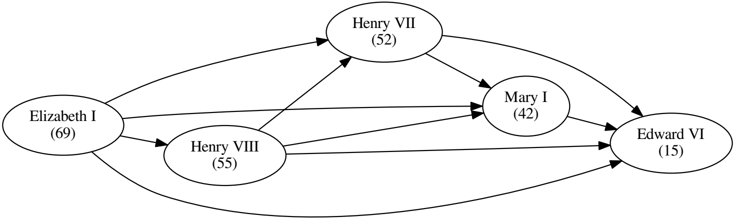
The relation: is-a-child-of



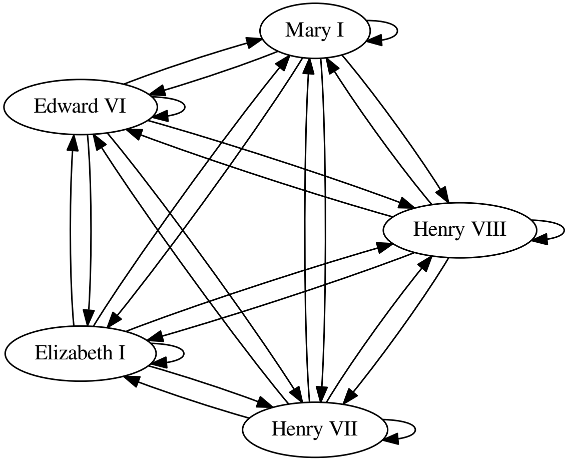
The relation: was-succeeded-by



The relation: lived-longer-than

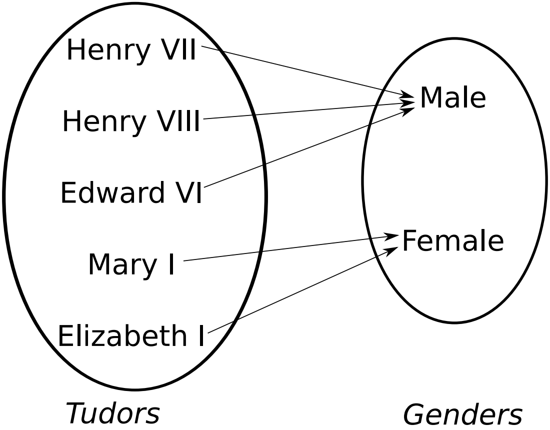


The relation: in-same-family-as



A relation over two sets: gender-of

When two sets are involved, we can picture the relation like this:



Part I:

Using sets to define relations

Ordered Pairs

As part of representing relations in set theory, we define:

- An ordered pair is an object of the form (a, b) where a and b are some already-existing objects.
- Equality:

  We define $(a, b) = (c, d)$ if any only if $a = c \ \wedge \ b = d$.

Ordered Pairs

As part of representing relations in set theory, we define:

- An ordered pair is an object of the form (a, b) where a and b are some already-existing objects.
- Equality:

  We define $(a, b) = (c, d)$ if any only if $a = c \ \wedge \ b = d$.

Related Terminology:

- For a given pair (a, b) we refer to a and b respectively as the *first* and *second* elements of the pair.
- Ordering: Note that, in general, (a, b) is different from (b, a).

  In the special case where $(a, b) = (b, a)$ we can deduce from the definition of equality that $a = b \ \wedge \ b = a$.

(Ordered) Tuples

- We can *extend* the definition of a pair to define an (ordered) triple as being an object of the form (a, b, c) ...

- We can *generalise* this by defining an n-tuple as being a sequence of n objects of the form $(x_1, x_2, \ldots, x_n)$
- We get at an element of an n-tuple by using a projection function

$$\pi_i(x_1, x_2, \ldots, x_n) \triangleq x_i \qquad \text{for } i \in \{1, \ldots, n\}$$

- In particular, for any pair (a, b)

$$\pi_1(a, b) = a$$
$$\pi_2(a, b) = b$$

Cartesian Product

We define

- The Cartesian Product of two sets A and B is the set of *all* pairs of the form (a, b), for every $a \in A$ and $b \in B$.
- Notation: the Cartesian product of A and B is the set $A \times B$

$$A \times B \triangleq \{a \in A, b \in B \bullet (a, b)\}$$

Cartesian Product

We define

- The Cartesian Product of two sets A and B is the set of *all* pairs of the form (a, b), for every $a \in A$ and $b \in B$.
- Notation: the Cartesian product of A and B is the set $A \times B$

$$A \times B \triangleq \{a \in A, b \in B \bullet (a, b)\}$$

- Examples:
  - $\{1, 2\} \times \{3, 4\} =$
  - $\{1\} \times \{3, 4\} =$
  - $\{1, 2\} \times \{blue, red, orange\} =$

Cartesian Product: Cluedo™ example

- Suppose we define three sets:
  - $Guests = \left\{ \begin{array}{l} \text{Mrs Peacock, Miss Scarlett, Reverend Green,} \\ \text{Mrs White, Colonel Mustard, Professor Plum} \end{array} \right\}$
  - $Rooms = \left\{ \begin{array}{l} \text{Library, Study, Lounge, Hall, Kitchen, Billiard} \\ \text{Room, Ballroom, Conservatory, Dining Room} \end{array} \right\}$
  - $Weapons = \left\{ \begin{array}{l} \text{Rope, Dagger, Revolver,} \\ \text{Candlestick, Lead Pipe, Spanner} \end{array} \right\}$
- then the set of *possible solutions* is given by the Cartesian product:
  $$Guests \times Rooms \times Weapons$$
- and a typical guess would be: (Colonel Mustard, Library, Revolver)

Cartesian Product: Cluedo™ example

- Suppose we define three sets:
  - $Guests = \left\{ \begin{array}{l} \text{Mrs Peacock, Miss Scarlett, Reverend Green,} \\ \text{Mrs White, Colonel Mustard, Professor Plum} \end{array} \right\}$
  - $Rooms = \left\{ \begin{array}{l} \text{Library, Study, Lounge, Hall, Kitchen, Billiard} \\ \text{Room, Ballroom, Conservatory, Dining Room} \end{array} \right\}$
  - $Weapons = \left\{ \begin{array}{l} \text{Rope, Dagger, Revolver,} \\ \text{Candlestick, Lead Pipe, Spanner} \end{array} \right\}$
- then the set of *possible solutions* is given by the Cartesian product:
  $$Guests \times Rooms \times Weapons$$
- and a typical guess would be: (Colonel Mustard, Library, Revolver)

Q: How many elements are in the set $Guests \times Rooms \times Weapons$?

Cartesian Product: special cases

- We can take the Cartesian product of a set *with itself*
- Example: $\{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- For any set A,
  - we often refer to $A \times A$ as A^2,
  - and refer to $A \times A \times A$ as A^3,
  - ... and so on
- We can take the Cartesian product of *infinite* sets too.
- Example: $\mathbb{N} \times \mathbb{N}$ is the set of all pairs of the form (a, b) for all $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

Defining a relation *in terms of sets*

We can now give a definition of relations in terms of set theory...

- Given two sets A and B we define:
  a binary relation between A and B is a subset of $A \times B$.

- Similarly, given three sets A, B and C,
  a ternary relation between them is a subset of $A \times B \times C$

- ... given n sets $S_1, S_2, \ldots, S_n$,
  an n-ary relation between them is a subset of $S_1 \times S_2 \times \ldots \times S_n$

- That is: an n-ary relation is a set of n-tuples.

Examples of relations: Tudors again

If we define the set:

Tudors $\triangleq$ {Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I }

Then we can define the relations (all are subsets of $Tudors \times Tudors$) by enumeration:

- **is-a-parent-of** $\triangleq$ { (Henry VII, Henry VIII), (Henry VIII, Edward VI), (Henry VIII, Mary I), (Henry VIII, Elizabeth I) }
- **was-succeeded-by** $\triangleq$ { (Henry VII, Henry VIII), (Henry VIII, Edward VI), (Edward VI, Mary I), (Mary I, Elizabeth I) }
- **lived-longer-than** $\triangleq$ { (Elizabeth I, Henry VII), (Elizabeth I, Henry VIII), (Elizabeth I, Mary I) (Elizabeth I, Edward VI), (Henry VIII, Henry VII), (Henry VIII, Mary I), (Henry VIII, Edward VI), (Henry VII, Mary I), (Henry VII, Edward VI), (Mary I, Edward VI) }

Relations: comprehension notation

Note we can use our *set comprehension* notation to define relations (particularly over infinite sets).

- For example, we can write:

$$L \triangleq \{ \, a \in \mathbb{N}, b \in \mathbb{N} \mid a < b^2 \bullet (a, b) \, \}$$

- Read this as:
  - For all elements a and b where $a \in \mathbb{N}$ and $b \in \mathbb{N}$,
  - pick out those where $a < b^2$,
  - and collect these in tuples of the form (a, b).

Part II:

Relational Operators

- Composition
- Inverse

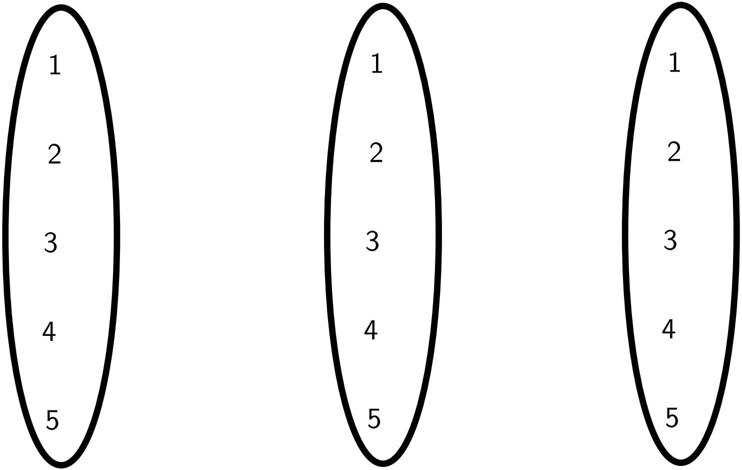
Composing two relations

Given three sets A, B and C, and two binary relations $R \subseteq (A \times B)$ and $S \subseteq (B \times C)$ we define the composition of R and S as the relation

$$R \mathbin{;} S \triangleq \{a \in A, b \in B, c \in C \mid (a, b) \in R, (b, c) \in S \bullet (a, c)\}$$

- Note that $(R \mathbin{;} S) \subseteq (A \times C)$
- Notation:
  - We use the notation as $R \mathbin{;} S$, "R then S"
  - Also written as $S \circ R$, "S after R" (not standard! opposite in textbook)
- Example: suppose I am given two relations over the set $\{1, 2, 3, 4, 5\}$
  - $\{(1, 2), (2, 3)\} \mathbin{;} \{(2, 4), (2, 5)\} \quad =$

Composition: example



$$\{(1,2),(2,3)\} \mathbin{;} \{(2,4),(2,5)\} \quad =$$

Composition: examples using comprehension

Suppose we define

- $R = \{m, n \in \mathbb{N} \mid n = m + 1 \bullet (m, n)\}$
- $S = \{m, n \in \mathbb{N} \mid n = m + 2 \bullet (m, n)\}$
- $T = \{m, n \in \mathbb{N} \mid n = 2m \bullet (m, n)\}$

Then:

- $R \mathbin{;} R =$
- $R \mathbin{;} S =$
- $T \mathbin{;} R =$
- $R \mathbin{;} T =$

The inverse of a relation

Given two sets A and B, and a binary relation $R \subseteq (A \times B)$ we define the inverse of R as the relation

$$R^{-1} \triangleq \{a \in A, b \in B \mid (a, b) \in R \bullet (b, a)\}$$

- Note that $R^{-1} \subseteq (B \times A)$
- Notation:
  - This is usually written as R^{-1}
  - Also written as $R^{\sim}$ or $R^{\smile}$ (unusual: textbook)
- Examples:
  - The inverse of $\{(1, 2), (2, 3)\}$ is $\{(2, 1), (3, 2)\}$
  - Numbers: the inverse of the *less-than* relation is just the *greater-than* relation
  - Tudors: the inverse of *is-a-parent-of* is just *is-a-child-of*

Other operations on relations

Since a relation is just a special kind of set, all the usual set operations can be applied to relations as well.

Example: suppose we define two relations over $\mathbb{N} \times \mathbb{N}$:

- $LT \triangleq \{a \in \mathbb{N}, b \in \mathbb{N} \mid a < b \bullet (a, b)\}$
- $EQ \triangleq \{a \in \mathbb{N}, b \in \mathbb{N} \mid a = b \bullet (a, b)\}$

Then

- $LT \cup EQ =$
- $LT^{-1} =$
- $EQ^{-1} = EQ =$
- $(\mathbb{N} \times \mathbb{N}) \setminus LT =$