# CS172: Computer Systems II

## Lecture 15

# Natural Deduction

## *- overview, rules for implies*

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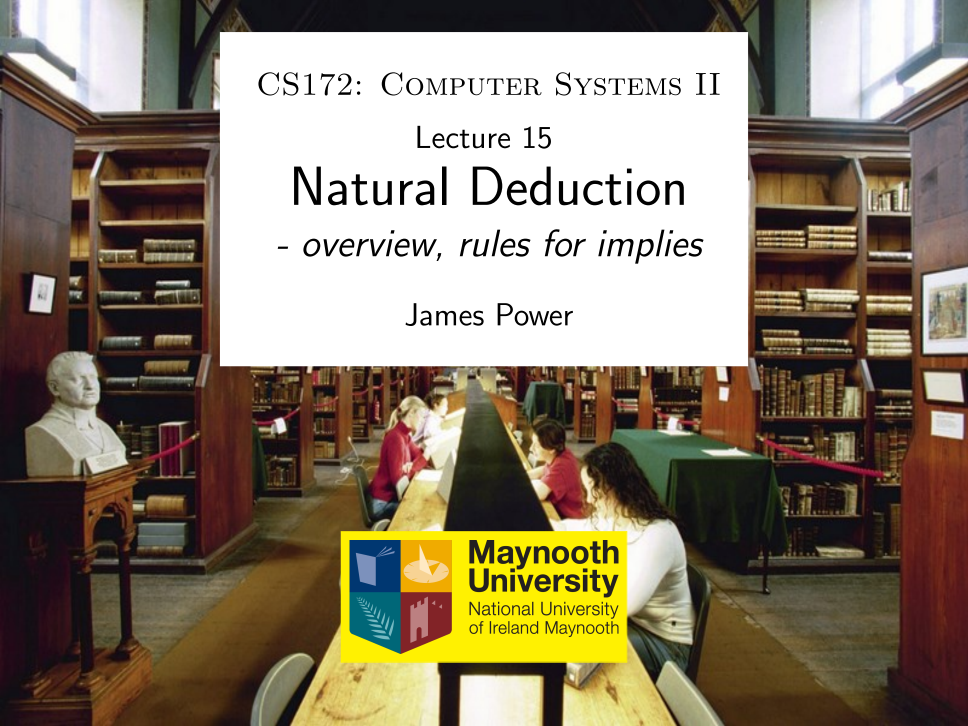

# Deciding if an inference is valid

- We already have a process for deciding if an inference using *propositional logic* is valid:
    1. Translate the premises and conclusion into propositional logic
    2. Write out the truth table for the premises and the conclusion
    3. Make sure that in every line where all the premises are true then the conclusion is also true.

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- We have a similar process for *syllogistic reasoning*.
- **Q:** How do we decide if an inference using *predicate logic* is valid?

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$$\begin{array}{c} P \to Q \\ Q \to R \\ R \to S \\ \hline P \to S \end{array}$$

Truth table must have $2^4$ rows.

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$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ R \rightarrow S \\ S \rightarrow T \\ \hline P \rightarrow T \end{array}$$

Truth table must have $2^5$ rows.

# Problems with truth tables

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$$\frac{\begin{array}{c} P_1 \rightarrow P_2 \\ P_2 \rightarrow P_3 \\ P_3 \rightarrow P_4 \\ \ldots \\ P_{n-1} \rightarrow P_n \end{array}}{P_1 \rightarrow Pn}$$

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- Doesn't mimic the "natural" way of reasoning.

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- Doesn't mimic the "natural" way of reasoning.
- Won't work for predicate logic.

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# Gentzen's "natural" deduction

Gerhard Gentzen
(1909-1945)

- Investigations into Logical Inference, *Mathematische Zeitschrift*, 1934, 1935.

  *"... the formulation of logical deduction, as it has been developed in particular by Frege, Russell and Hilbert, distances itself rather far from from the sort of deduction that mathematical proofs actually use...*

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  "*... the formulation of logical deduction, as it has been developed in particular by Frege, Russell and Hilbert, distances itself rather far from from the sort of deduction that mathematical proofs actually use...*

  *My goal was first of all to set up a formalism that came as close to actual deduction as possible*"

# Natural deduction

We'll study the natural deduction proof system for predicate logic.
Gerhard Gentzen, 1934, Dag Prawitz, 1960s (Textbook, Ch 9)

There are others:

- The Axiomatic system (Textbook §2.7, §4.8)
  Developed by David Hilbert and his group in the 1920s.
- Tableau systems (Textbook Ch 8)
  Developed by Evert Willem Beth in the 1950s.
- Resolution / SAT solvers (Textbook Ch 10)
  Developed by Davis, Putnam, Logemann and Loveland (1960)

# What does a proof look like?

**Theorem**

*Suppose $m$ and $n$ are integers.*
*If $mn$ is even, then either $m$ is even or $n$ is even.*

**Proof.**

Suppose $mn$ is even.
Then we can choose an integer $k$ such that $mn = 2k$.
If $m$ is even then there is nothing more to prove, so suppose $m$ is odd.
Then $m = 2j + 1$ for some integer $j$.
Substituting this into the equation $mn = 2k$, we get $(2j + 1)n = 2k$,
so $2jn + n = 2k$, and therefore $n = 2k - 2jn = 2(k - jn)$.
Since $k - jn$ is an integer, it follows that $n$ is even. □

# Natural deduction: the basic idea

The structure of a 'typical' proof:

- We are given
  - some *premises* that we're allowed to assume
  - a *conclusion* that we want to reach
- and we must provide
  - a set of formal steps to *transform* the premises into the conclusion

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We can build **software** to help us with this:

- Proof Checker: takes as input a formal proof and checks that we've applied the rules correctly.
- Proof Assistant: guides us (what rules to use) as we build the proof
- Theorem Prover: given the premises and conclusion, works out the proof.

Examples: Coq, Isabelle, NuPRL, Mizar

## General structure of a natural deduction proof

Gentzen's idea:

- Each of the five connectives and two quantifiers has an introduction rule and an elimination rule.
- Use the elimination rule

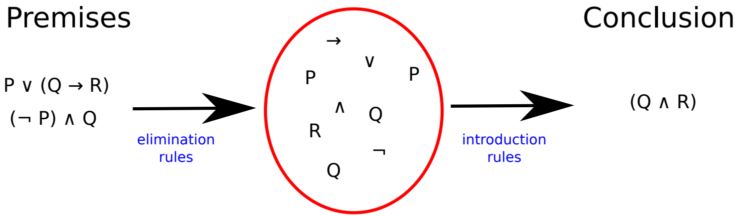
  if the connective/quantifier appears in your *premises*
- Use the introduction rule

  if the connective/quantifier appears in your *conclusion*

# General structure of a natural deduction proof

Gentzen's idea:

- Each of the five connectives and two quantifiers has an introduction rule and an elimination rule.
- Use the elimination rule to break down the *premises*
  i.e. if the connective/quantifier appears in your *premises*
- Use the introduction rule to build up the *conclusion*
  i.e. if the connective/quantifier appears in your *conclusion*



# Examining that theorem again

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Premises: $m$ and $n$ are integers. $Im \wedge In$

Conclusion: If $mn$ is even, then either $m$ is even or $n$ is even. $E(m * n) \rightarrow (Em \vee En)$

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# Introducing ‘implies’

We deduce that:

- If you want to prove a conclusion of the form $P \to Q$ then
  - Assume that $P$ is true
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We deduce that:

- If you want to prove a conclusion of the form $P \rightarrow Q$ then
  - Assume that $P$ is true
  - ... and go on to prove that $Q$ is true.

- Here $P$ is called a local assumption.

  It is only assumed temporarily, for part of the proof.

# Examining that proof one more time

**Proof.**

...

... so suppose $m$ is odd. Then $m = 2j + 1$ for some integer $j$.

$\ldots$

$\square$

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**Proof.**

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... so suppose $m$ is odd. Then $m = 2j + 1$ for some integer $j$.

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There must be a rule (in maths) that said:

- if any integer $x$ is odd, then there exists some integer $y$ such that $x = 2y + 1$.

... and we used it here.

# Eliminating ‘implies’

We deduce that:

- If you have a premise of the form $P \rightarrow Q$ then
  - if you also know that $P$ is true
  - ... you can infer that $Q$ is true.

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- The “rule in maths” that we used on the previous slide was an *implicit* premise of our theorem.

# Formal proof rules for 'implies'

$$\dfrac{\boxed{\begin{array}{c}\text{Suppose } A: \\ \vdots \\ B\end{array}}}{A \to B} \to_{\mathcal{I}} \qquad\qquad \dfrac{A \qquad A \to B}{B} \to_{\mathcal{E}}$$

In the introduction rule,

- the assumption $A$ is *local* to the sub-proof
- its *scope* is indicated by the box
- we say the assumption is *discharged* by the use of the rule.