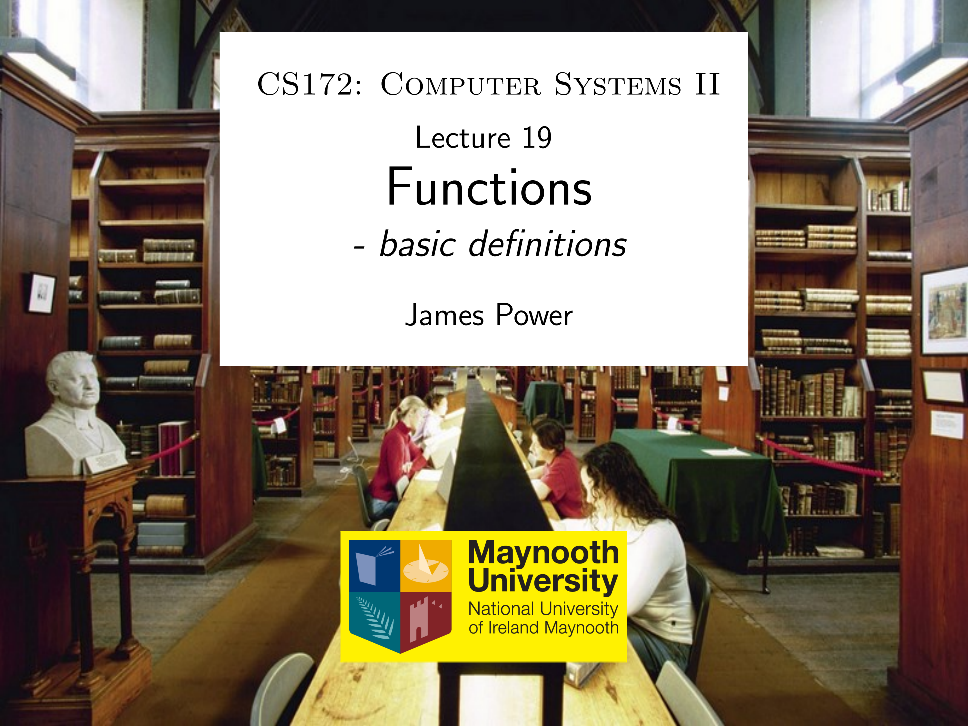

CS172: Computer Systems II

Lecture 19

Functions

- *basic definitions*

James Power

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Functions

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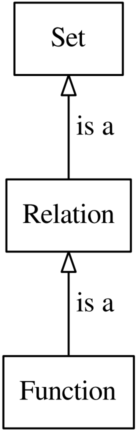
- **Notation:** Since a function has this special property, we often use a special notation for being an element of a function.

  For example, if f is a binary relation that is a function, then instead of writing $(a, b) \in f$ we usually write $f(a) = b$

Textbook, §A.5

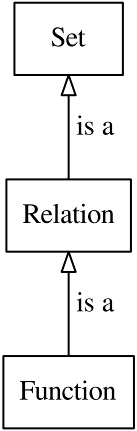
Functions, Relations and Sets (class diagram)

- A relation is a special kind of set.
- A function is a special kind of relation.



Functions, Relations and Sets (class diagram)

- A relation is a special kind of set.
- A function is a special kind of relation.
- Thus all of the *relational operators* (inverse, composition) can work with functions.
- ... and all of the *set-theory operators* (union, intersection etc.) can also work with functions.



Relations vs. functions: numerical examples

Example: consider the following two relations which are both subsets of $\mathbb{Z} \times \mathbb{Z}$:

- $S \triangleq \{a \in \mathbb{Z}, b \in \mathbb{Z} \mid b = a + 1 \bullet (a, b)\}$
- $T \triangleq \{a \in \mathbb{Z}, b \in \mathbb{Z} \mid b = \sqrt{a} \bullet (a, b)\}$

Q1: Is the relation S a function?

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i.e. $\forall\, a \cdot \forall\, b \cdot \forall\, c \cdot ((a, b) \in S \;\wedge\; (a, c) \in S)) \;\rightarrow\; (b = c)$

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Relations vs. functions: Tudor examples

If we define the set:

$$\text{Tudors} \triangleq \{\text{Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I}\}$$

Then we can define some relations (all are subsets of $\mathit{Tudors} \times \mathit{Tudors}$):

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- is-a-parent-of = { (Henry VII, Henry VIII), (Henry VIII, Edward VI), (Henry VIII, Mary I), (Henry VIII, Elizabeth I) }

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✗ This *is not* a function since Henry VIII is a parent of three different people.

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$\checkmark$ This *is* a function since in this set everyone has a unique parent.

Relations vs. functions: Tudor examples

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- was-succeeded-by = { (Henry VII, Henry VIII), (Henry VIII, Edward VI), (Edward VI, Mary I), (Mary I, Elizabeth I) }

✓ This *is* a function since each Tudor has a unique (immediate) successor.

Relations vs. functions: Tudor examples

If we define the set:

Tudors $\triangleq$ {Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I }

Then we can define some relations (all are subsets of *Tudors* $\times$ *Tudors*):

- lived-longer-than = { (Elizabeth I, Henry VII), (Elizabeth I, Henry VIII), (Elizabeth I, Mary I) (Elizabeth I, Edward VI), (Henry VIII, Henry VII), (Henry VIII, Mary I), (Henry VIII, Edward VI), (Henry VII, Mary I), (Henry VII, Edward VI), (Mary I, Edward VI) }

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This *is not* a function: for example, Elizabeth I lived longer than the other four Tudors.

Relations vs. functions: Tudor examples

If we define the set:

Tudors $\triangleq$ {Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I }

Then we can define some relations

- lived-longer-than = { (Elizabeth I, Henry VII), (Elizabeth I, Henry VIII), (Elizabeth I, Mary I) (Elizabeth I, Edward VI), (Henry VIII, Henry VII), (Henry VIII, Mary I), (Henry VIII, Edward VI), (Henry VII, Mary I), (Henry VII, Edward VI), (Mary I, Edward VI) }

This *is not* a function: for example, Elizabeth I lived longer than the other four Tudors.

Hint: since the set *Tudors* only has 5 elements, any function over this set can have *at most* five elements.

Relations vs. functions: Tudor examples

If we define the set:

$$\text{Tudors} \triangleq \{\text{Henry VII, Henry VIII, Edward VI, Mary I, Elizabeth I}\}$$

Then we can define some relations

- gender-of = { (Henry VII, male), (Henry VIII, male), (Edward VI, male), (Mary I, female) (Elizabeth I, female) }

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- gender-of $= \{$ (Henry VII, male), (Henry VIII, male), (Edward VI, male), (Mary I, female) (Elizabeth I, female) $\}$

✓ This *is* a function since each Tudor has a unique gender.

Defining functions: lambda notation

- To **define a function** we can use enumeration (for finite functions) or comprehension, since every function is a set.
- For example, we defined the function S by comprehension as:

$$S \triangleq \{a \in \mathbb{Z}, b \in \mathbb{Z} \mid b = a + 1 \bullet (a, b)\}$$

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- For example, we defined the function S by comprehension as:

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- However, since a function is a special kind of relation, we also have a special notation for defining functions.
- We can define a function much more succinctly using the lambda notation:

$$T \triangleq (\lambda\, a \in \mathbb{Z} \cdot a + 1)$$

"T is the function that takes some a, an integer, and returns $a + 1$".

Lambda notation: examples

- doubler $\triangleq (\lambda\, a \in \mathbb{N} \cdot 2 * a)$

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- f $\triangleq (\lambda\, x \in \mathbb{R} \cdot x^2 + 3x + 1)$

  f is a function taking some real number x and returning $x^2 + 3x + 1$

  f $= \{(0.5, 2.75),(-0.7, 1.61),(2, 11),\ldots\}$

Lambda notation: brief history

The concept was initially developed by Gottlob Frege in a series of papers including *Function and Concept* (1891), *What is a Function?* (1904).

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... and the basis for to anonymous functions and *lambda expressions* in Java 8 (JSR 335)

Lambda notation in Java 1.8

```
import java.util.function.*;

public class ..... {

  public static void main(String [] args) {

    IntUnaryOperator doubler = a -> 2*a;

    IntBinaryOperator pyth = (a,b) -> (int)Math.sqrt(a*a+b*b);

    DoubleUnaryOperator f = x -> x*x + 3 *x + 1;

  }
}
```