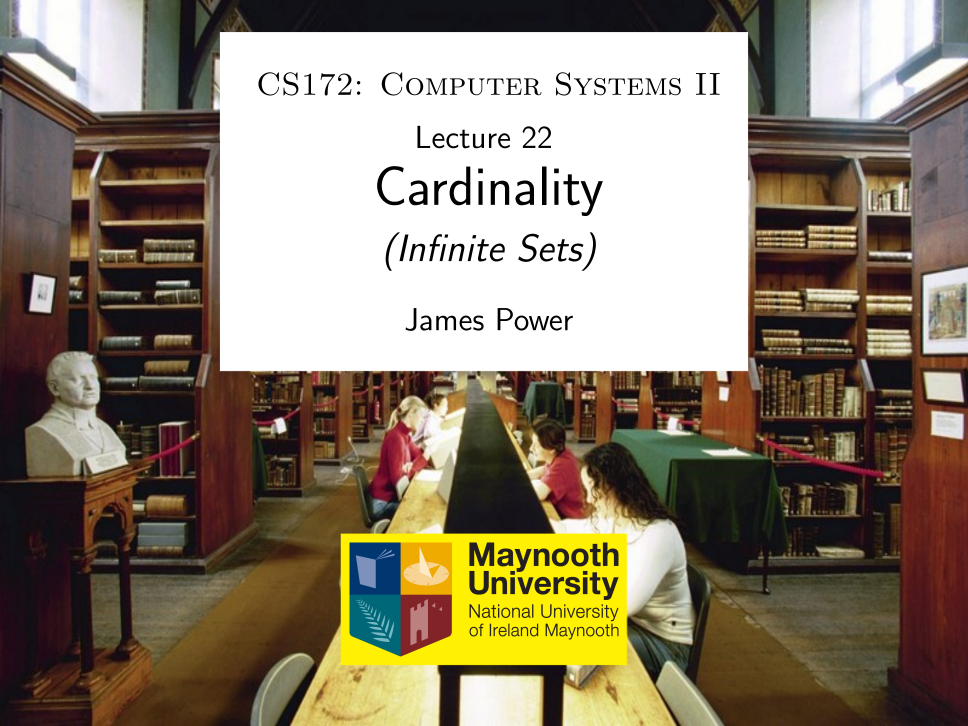

CS172: Computer Systems II

Lecture 22

Cardinality

(Infinite Sets)

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Maynooth University
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Galileo's paradox

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- For every number there is *exactly one* perfect square, and for every perfect square there is *exactly one* number that is its (positive) square root.
- The function forms a **bijection** between the natural numbers, $\{0, 1, 2, 3, \ldots.\}$, and the perfect squares, $\{0, 1, 4, 9, \ldots.\}$.

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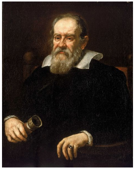

"we cannot speak of infinite quantities as being the one greater or less than or equal to another"

Galileo Galilei
Dialogues concerning two new sciences, 1638

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- we still haven't defined "infinite" yet.

Defining “Infinite sets”

Richard
Dedekind
(1831-1916)

- A set is infinite if there exists a bijection between it and one of its own proper subsets.

Was sind und was sollen die Zahlen?, 1888
(“What are numbers and what should they be?”)

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Thus the following *subsets* of $\mathbb{N}$ are all countable sets:

- $\{n \in \mathbb{N} \mid n+1\}$
- $\{n \in \mathbb{N} \mid n+k\}$, for any fixed $k \in \mathbb{N}$
- $\{n \in \mathbb{N} \mid 2 * x\}$
- $\{n \in \mathbb{N} \mid (2 * x) + 1)\}$
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Thus the following operations do not change the cardinality of $\mathbb{N}$

- Deleting an element of $\mathbb{N}$
- Deleting k elements of $\mathbb{N}$, for any fixed $k \in \mathbb{N}$
- Deleting every second element in $\mathbb{N}$
- Deleting every k^{th} element in $\mathbb{N}$, for any fixed $k > 0$

Cardinality of the integers

Suppose we consider a *superset* of $\mathbb{N}$...

What is the cardinality of the integers?

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

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That is:

- map the negative integers to the even natural numbers
- and the positive integers to the odd natural numbers

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This gives the mapping:

$$\{\ldots(-3, 6), (-2, 4), (-1, 2), (0, 0), (1, 1), (2, 3), (3, 5), \ldots\}$$

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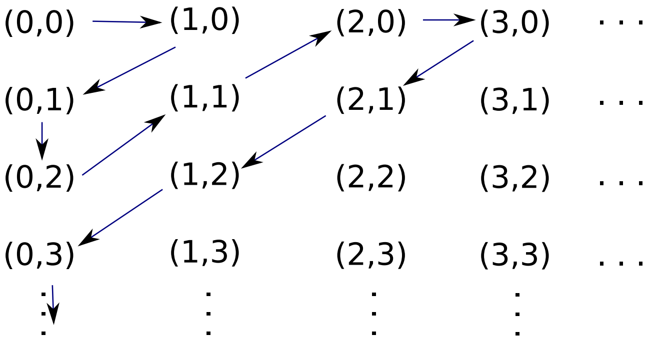

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$$= \{(0,0), (1,0), (0,1), (0,2), (1,1), (2,0), (3,0), (2,1), (1,2), (0,3), \ldots\}$$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Note each diagonal is just the (finite) set $\{i, j \in \mathbb{N} \mid i + j = k \bullet (i, j)\}$ for some fixed value of $k \in \mathbb{N}$

Defining the mapping

We can define this more formally...

- Define an *injective* function f from $(\mathbb{N} \times \mathbb{N})$ to $\mathbb{N}$ as:

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  The cardinality of $(\mathbb{N} \times \mathbb{N})$ is $\aleph_0$

Extending this to any size tuple

- The same trick - based on prime factorisation - will work for tuples of any (finite) size.
- Example: suppose we have 6-tuples of numbers, that is, elements of the set $(\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ ...
- Then we can map any 6-tuple of the form $(a_1, a_2, a_3, a_4, a_5, a_6)$ to the number
$$2^{a_1+1} * 3^{a_2+1} * 5^{a_3+1} * 7^{a_4+1} * 11^{a_5+1} * 13^{a_6+1}$$
- Example: $(3, 4, 2, 0, 2, 1)$ maps to
$$2^4 * 3^5 * 5^3 * 7^1 * 11^3 * 13^2 \quad = \quad 765,242,478,000$$
- We know that this will work for k-tuples for any given $k \in \mathbb{N}$, since there will always be enough prime numbers.

Other countable sets

The previous work implies that some other infinite sets are also countable:

- The rational numbers, $\mathbb{Q}$, since any fraction of the form $\frac{a}{b}$ can be mapped to the tuple (a, b) and then into the natural numbers.

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- Any formal language (e.g. defined by a regular expression or a context-free grammar), since a finite-length string over any given alphabet is really just a tuple of characters.
- The set of all valid Java programs (or C$\#$, or ...), since these can be arranged in order: shortest first, and then alphabetically for programs of the same length.

Hilbert's Hotel: a *Gedankenexperiment*

Imagine a hotel with infinitely many rooms, numbered $\{0, 1, 2, 3, 4, \ldots\}$, and suppose the hotel is currently full.

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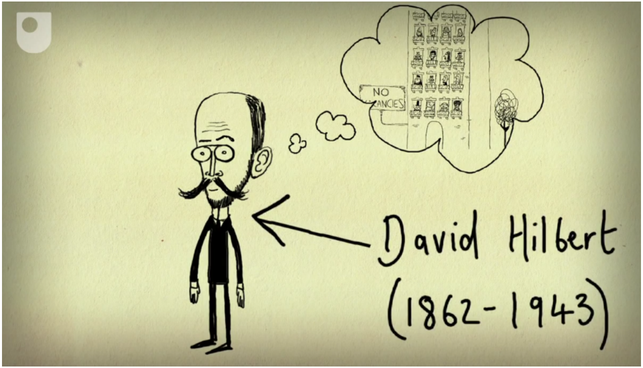

60-second video from the **Open University**:
http://www.youtube.com/watch?v=faQBrAQ87l4

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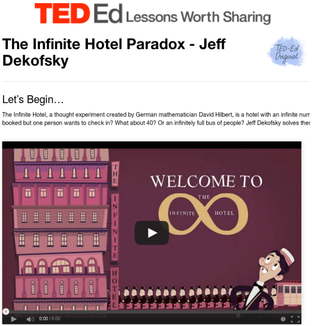

6-minute video by Jeff Dekofsky at **TedEd**:
http://ed.ted.com/lessons/the-infinite-hotel-paradox-jeff-dekofsky/

Hilbert's Hotel: a *Gedankenexperiment*

Imagine a hotel with infinitely many rooms, numbered $\{0, 1, 2, 3, 4, \ldots\}$, and suppose the hotel is currently full.

How would Hilbert accommodate:

- one new guest
- k new guests, for some given $k \in \mathbb{N}$
- a bus containing infinitely many new guests
- k buses, with each bus containing infinitely many new guests, for some given $k \in \mathbb{N}$
- infinitely many buses, with each bus containing infinitely many new guests
- k aircraft carriers, each of which contains infinitely many buses, with each bus containing infinitely many new guests, for some given $k \in \mathbb{N}$

Welcome to...

Hilbert's Hotel

- never any vacancy -

- always room for more -