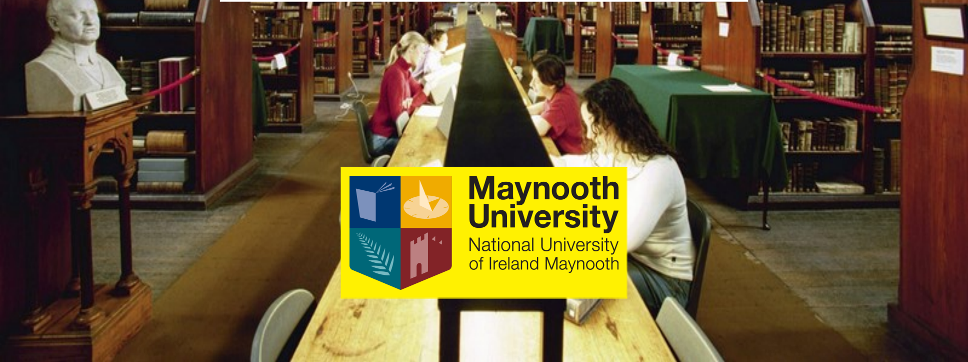
CS172: Computer Systems II

Lecture 20

Functions

- classifying

James Power



Domain, codomain and image: definition

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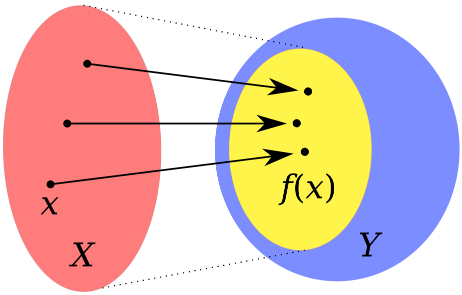
Suppose we are given some sets X and Y and a function $f \subseteq X \times Y$.



$$f : X \to Y$$

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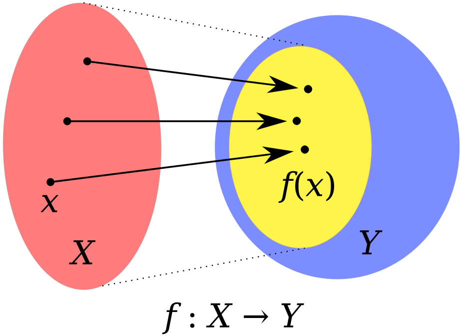

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Image from Wikipedia

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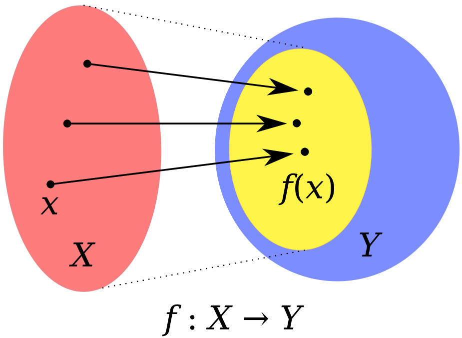

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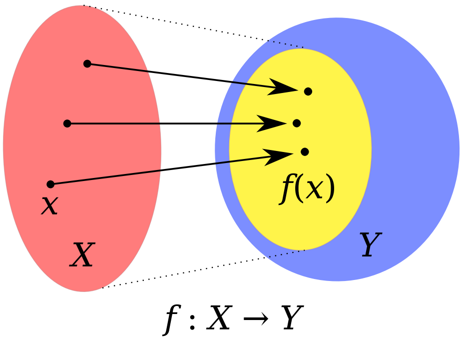

- The set X is called the domain of the function f
- The set Y is called the codomain (or range) of the function f
- The subset of the codomain Y containing elements mapped-to by f is called the image of the function f

Image from Wikipedia

Domain, codomain and image: example

For example, consider the two functions $sq, db \subseteq \mathbb{Z} \times \mathbb{Z}$ defined as

- $sq \triangleq (\lambda\, x \in \mathbb{Z} \cdot x^2)$
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In each case the domain and codomain is the set $\mathbb{Z}$, but

- The image of sq is the set $$\begin{aligned} & \{n \in \mathbb{N} \mid n^2\} \\ = \; & \{0, 1, 4, 9, 16, \ldots\} \end{aligned}$$
- The image of db is the set $$\begin{aligned} & \{n \in \mathbb{Z} \mid 2 * n\} \\ = \; & \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \end{aligned}$$

Injective and Surjective Functions

Injective and Surjective Functions

For any sets sets X and Y and a function $f \subseteq X \times Y$, we say that the function f is:

- injective (one-to-one) if each element in the image is mapped to by *just one* element in the domain; that is, if

$$(\forall x, y \cdot (x \in X \land y \in X \land f(x) = f(y)) \rightarrow x = y)$$

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Thus the *inverse* of an injective function is always a function (from the image to the domain).

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For a surjective function the image is the *whole* codomain.

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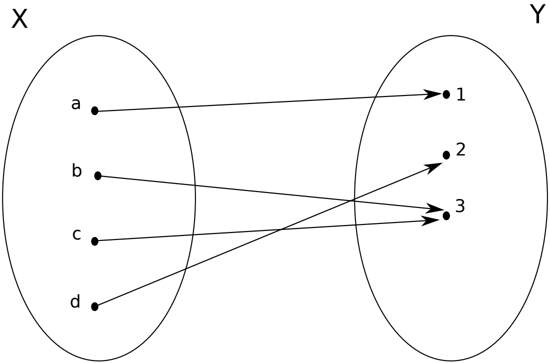
Thus if there is a bijection between two sets, then those sets have the same number of elements.

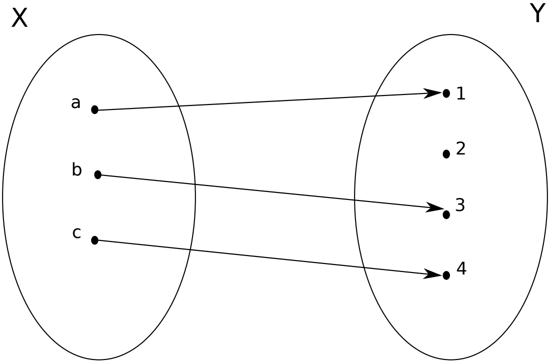
Example: Surjective but not injective



Injective? ✗

Surjective? ✓

Example: Injective but not surjective



Injective? ✓
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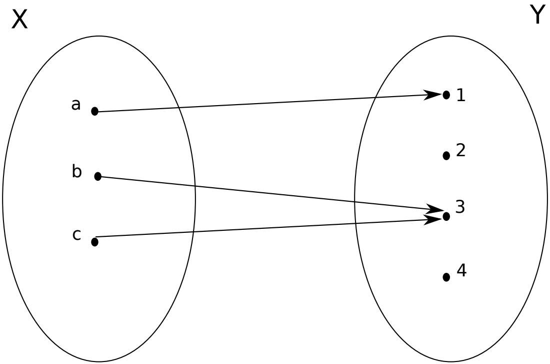
Example: Neither injective nor surjective



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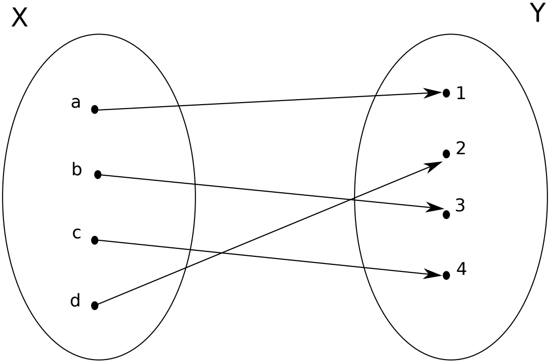
Example: Both injective and surjective (so, bijective)



Injective? ✓

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Injective and Surjective: Java example 1

Consider the function from the variable names {a,b,c} to the objects on the heap created by the following Java code:

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Person a = new Person("Tom");
Person b = new Person("Dick");
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- There is a bijective function from the variable names {a,b,c} to the three objects on the heap.

Injective and Surjective: Java example 2

Consider the function from the variable names {a,b,c} to the objects on the heap created by the following Java code:

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- There is now a surjective (but not injective) function from the variable names {a,b,c} to the *two* objects on the heap.

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Consider the function from the variable names {a,b,c} to the objects on the heap created by the following Java code:

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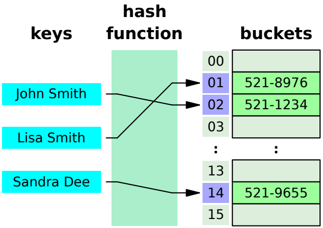
- Calling the *garbage collector* would make this function surjective again.

Injective: Hash Tables Example

- Hash tables are a data structure used to map keys to values.
- Like an array, except the index (key) can be of any type.

  Sometimes called an *associative array* (or, if the keys are strings, a *dictionary*).
- To implement this, you need a hash function that maps the key type to an integer.



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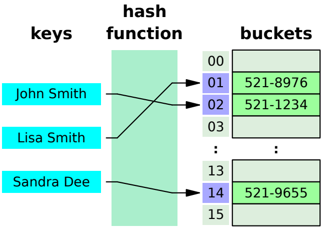

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- Ideally, the hash function would be *injective*.
- When it's not injective, you get collisions.

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$$\mathit{reciprocal} \triangleq (\lambda\, a \in \mathbb{Z} \mid a \neq 0 \cdot 1/a)$$

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- assuming we have some way of deciding which elements of the domain it won't work for...

Partial functions in CS

- *Partial* functions are interesting in Computer Science where we might have to allow for a method to be non-halting (e.g. because of an infinite loop).

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int fact(int n) {
  if (n==0) return 1;
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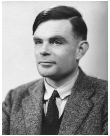

Theorem: It is not possible to write a code-analyser that will detect in advance whether *any* function is total or partial (i.e. whether it will halt or not).

- *Halting Problem*, Alan Turing, 1936

Example: classifying functions

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Example: classifying functions (continued)

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Sets, bags, and sequences

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- A bag is like a set, but we also remember *how many times* an object occurs.
- A sequence is like a set, but we also remember *where* an object occurs.

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  For example: $[\![a, a, b, b, b, c]\!]$ is the bag where a occurs twice, b occurs thrice and c occurs once.
- Notes:
  - The notation $[\![\cdots]\!]$ is not very standard.
  - The *order* of elements in a bag doesn't matter.

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we can say

$$myShopping = \{(apple, 2), (orange, 0), (pear, 3)\}$$

Operations on Bags (inherited)

- Since bags are a special kind of function/relation/set, we can use all the function/relation/set operations with them.
- But note: the union (or intersection) of two functions is a relation, but not necessarily a function.

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= \;& \{(a, 2), (b, 3), (c, 1)\} \;\cup\; \{(a, 1), (b, 2), (c, 2)\}
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Operations on Bags (inherited)

- Since bags are a special kind of function/relation/set, we can use all the function/relation/set operations with them.
- But note: the union (or intersection) of two functions is a relation, but not necessarily a function.
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- Notes:
  - Since any bag is a function, $B(x)$ is just the multiplicity of element x in bag B.
  - Can you define bag versions of intersection and difference?
  - Can you define the relation *is-a-subbag-of*?

Sequence: basic definition

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- Notes:
  - The notation $\langle \cdots \rangle$ is not very standard.
  - We usually start counting position at 1 (unlike arrays)

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we can say
$myEating = \{(1, apple), (2, pear), (3, apple), (4, pear), (5, apple)\}$

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- Since sequences are a special kind of function/relation/set, we can use all the function/relation/set operations with them.
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Operations on Sequences (inherited)

- One useful inherited operation is *cardinality*:
  - The cardinality of a finite set is the number of elements it contains.
  - **Notation:** If S is a finite set, we write $\#S$ to denote the cardinality of S.

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  Example:

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- The **empty set** $\emptyset$ is also a sequence; when we want to refer to it as a sequence we usually write $\langle \rangle$

  Naturally, $\#\langle \rangle = 0$

Operations on Sequences (new)

- One of the most important sequence-specific operations is concatenation.
- This just appends two sequences together, in order:

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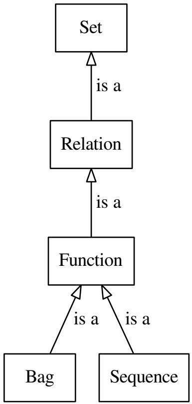
- Notes:
  - Since any sequence is a function, $s(n)$ is just the element at position n in sequence s.
  - Notation: sometimes we write $s.t$ for the concatenation of s and t.

Sets: the full “class hierarchy”



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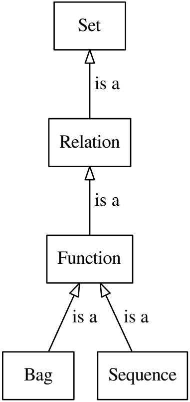

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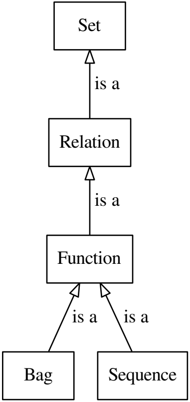

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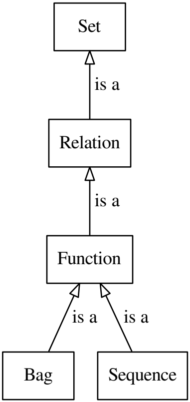

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- Database theory takes *relations* as the basic concept (Codd's relational algebra).
- The theory of computation takes *functions* as the basic concept (Church's λ-calculus).
- Language theory takes *sequences* as the basic concept (Kleene's regular expressions, Chomsky's grammars etc.).