CS172: Computer Systems II

Lecture 5

Propositional Logic

Expressiveness

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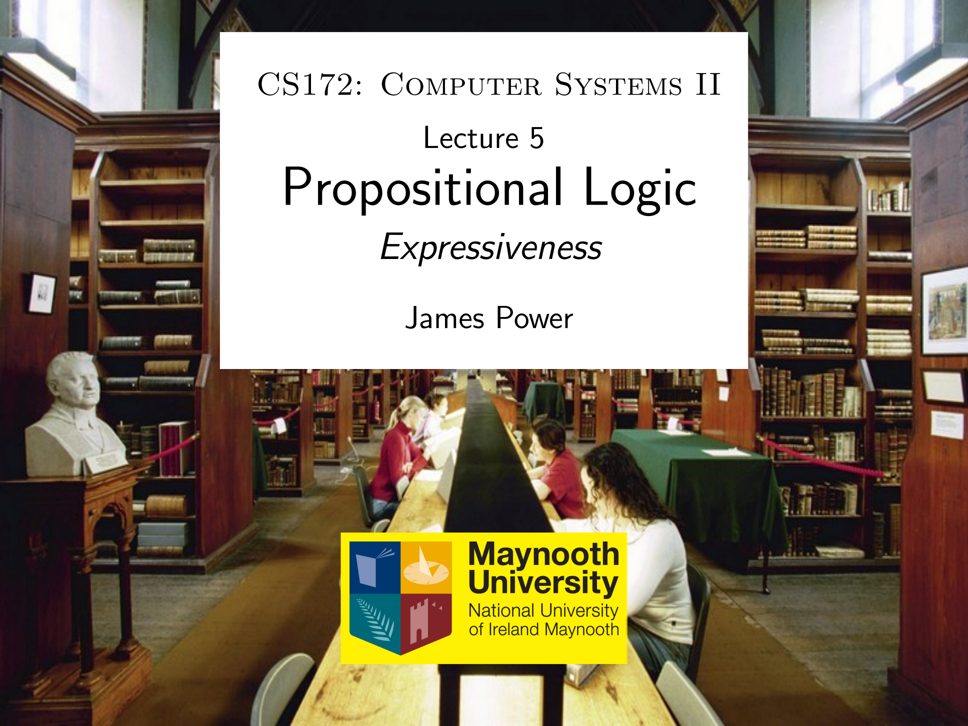

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Expressiveness: "The" five operators?

LiA §2.9

Looking at the five propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ we can ask

- *Q1: Do we need all we have?*
  Do we need all five connectives? Could we "make do" with less?

- *Q2: Do we have all we need?*
  Are the five connectives we've defined the *only* connectives?

  Could we get extra expressive power by using ternary connectives (3 operands) or more?

Q1: Do we need all we have?

- Note the following equivalences:
  1. $\phi \leftrightarrow \psi$ and $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.
  2. $\phi \rightarrow \psi$ and $\neg\phi \vee \psi$.

  3a. $\phi \wedge \psi$ and $\neg(\phi \vee \neg\psi)$.

  3b. $\phi \vee \psi$ and $\neg(\phi \wedge \neg\psi)$.

Q1: Do we need all we have?

- Note the following equivalences:
  1. $\phi \leftrightarrow \psi$ and $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.
  2. $\phi \rightarrow \psi$ and $\neg\phi \vee \psi$.

  3a. $\phi \wedge \psi$ and $\neg(\phi \vee \neg\psi)$.

  3b. $\phi \vee \psi$ and $\neg(\phi \wedge \neg\psi)$.

- Alternative:
  - Rewrite and/or/iff in terms of implies and not
  - Let $\perp$ represent the formula that is always false
  - Rewrite $\neg\phi$ to $\phi \rightarrow \perp$

A minimal set of operators

- In fact we can get away with a single operator that represents a combination of *not* with *and* (sometimes called NAND)
- This was discovered multiple times, most notably by Henry Sheffer in 1913

  In mathematical logic it is usually called the Sheffer stroke:

p	q	$p \mid q$
F	F	T
F	T	T
T	F	T
T	T	F

- Actually, *not* with *or* (NOR) will do equally well

Q2: Do we have all that we need?

What other unary operators could there be?

p	$f_1(p)$	$f_2(p)$	$f_3(p)$	$f_4(p)$
0				
1				

Q2: Do we have all that we need?

What other unary operators could there be?

p	$f_1(p)$	$f_2(p)$	$f_3(p)$	$f_4(p)$
0	0	0	1	1
1	0	1	0	1

Q2: Do we have all that we need?

What other binary operators could there be?

p	q	
0	0	
0	1	
1	0	
1	1	

Q2: Do we have all that we need?

What other binary operators could there be?

p	q	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
1	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

Q2: Do we have all that we need?

What ternary operators could there be?

p	q	r	$\cdots$	f_i	$\cdots$
0	0	0	$\cdots$	0	$\cdots$
0	0	1	$\cdots$	0	$\cdots$
0	1	0	$\cdots$	0	$\cdots$
0	1	1	$\cdots$	0	$\cdots$
1	0	0	$\cdots$	1	$\cdots$
1	0	1	$\cdots$	1	$\cdots$
1	1	0	$\cdots$	0	$\cdots$
1	1	1	$\cdots$	1	$\cdots$

- Can we define f_i as a function of p, q, r, using only our existing connectives?