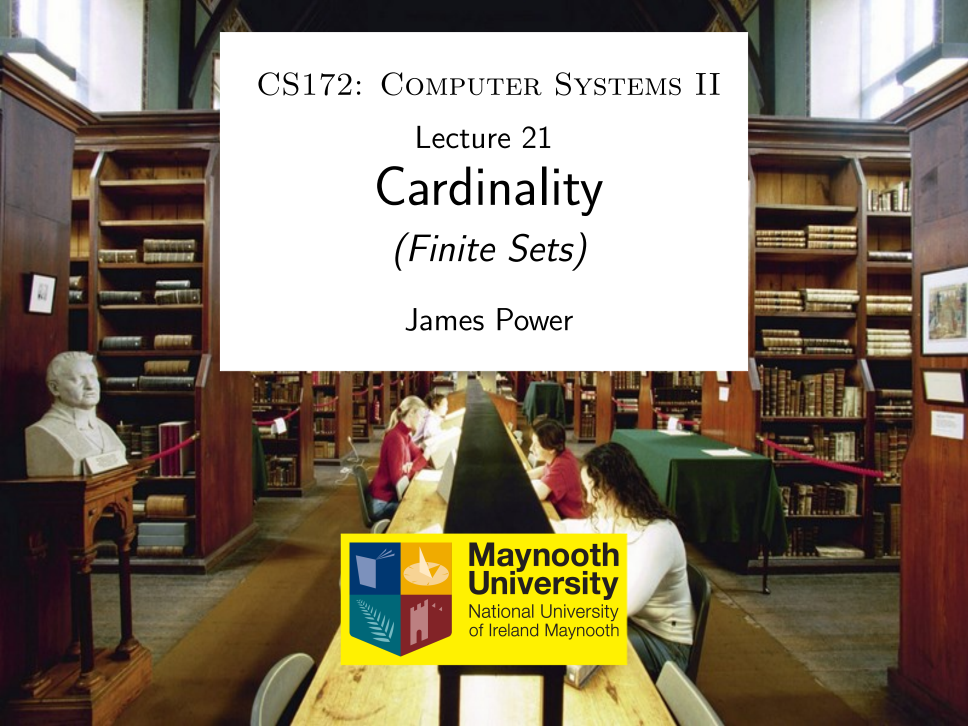

CS172: Computer Systems II

Lecture 21

Cardinality

(Finite Sets)

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The cardinality of a finite set

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Alternative terminology: we might also call this the "size" of the set S, and it is sometimes written $|S|$.

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Examples:

- $\#\{1, 2, 3\} = 3$
- $\#\{2, 4, 6, 8, 10\} = 5$
- $\#\{\} = 0$
  The empty set is the *only* set with cardinality 0.
- $\#\{(1, 1), (2, 4), (3, 9)\} = 3$
  The cardinality of a finite relation or function is just the number of tuples it contains.

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Let A and B be any **finite** sets.

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- $\#(A \cup B) = \#A + \#B - \#(A \cap B)$
- $\#(A \setminus B) = \#A - \#(A \cap B)$
- $\#(A \times B) = \#A * \#B$

Functions and cardinality

Suppose f is a **finite function** with domain A and co-domain B.

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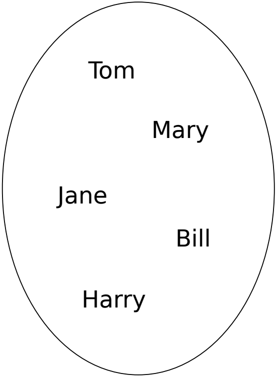
- Corollaries:

- if f is a total injective function then $\#A \leq \#B$
- if f is a total bijective function then $\#A = \#B$

Aside: counting



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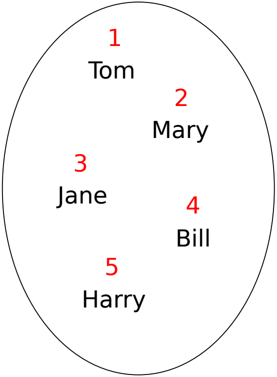
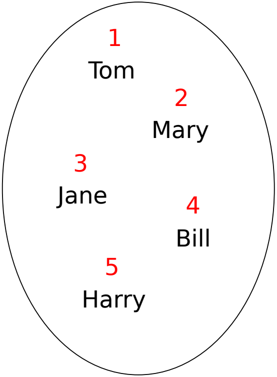


- Counting the elements in a set ...

Aside: counting = constructing a bijection



- Counting the elements in a set involves **constructing a bijection** between the elements and an initial segment of the non-zero natural numbers.

- Example:

$$\left\{ \begin{array}{c} (Tom, 1), (Mary, 2), (Jane, 3), \\ (Bill, 4), (Harry, 5) \end{array} \right\}$$

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- **Notation:** sometimes *the set* of all functions from A to B is written as B^A

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- Note that the power set is always a *set of sets*

  - i.e. a set whose elements are sets.

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$$\mathbb{P}\, S = \left\{ \begin{array}{l} \{\}, \\ \{a\}, \{b\}, \{c\}, \{d\}, \\ \{a, b\}, \{a, c\}, \{a, d\}, \\ \{b, c\}, \{b, d\}, \{c, d\}, \\ \{a, b, c\}, \{a, b, d\}, \\ \{a, c, d\}, \{b, c, d\}, \\ \{a, b, c, d\} \end{array} \right\}$$

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- Reasoning:

  for each if the n elements $x \in S$,
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- Compare this with a truth table for n propositional variables:
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- For any set S, a subset of S is basically the same as a total function from S to the set $\{0, 1\}$.

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- Let's try an easier question first...

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Counting the number of sequences (repetitions allowed)

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$$= \#S^{\#\{1..m\}}$$

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$$= n * (n-1) * (n-2) * \cdots * n - (m-1)$$

$$= \frac{n*(n-1)*(n-2)*\cdots*n-(m-1)*(n-m)*(n-m)-1*(n-m)-2*\cdots*1}{(n-m)*(n-m)-1*(n-m)-2*\cdots*1}$$

$$= \frac{n!}{(n-m)!}$$

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- So $\frac{n!}{(n-m)!}$ sequences corresponds to $\frac{n!}{m!(n-m)!}$ sets

- **Notation:** also called "n choose m", and written:

$$\frac{n!}{m!(n-m)!} = \binom{n}{m}$$

see also: the *binomial coefficient*

Number of subsets of given size: Pascal's triangle

Blaise Pascal
(1623-1662)

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1

Traité du triangle arithmétique, Blaise Pascal, 1653.

See also: *binomial coefficients*

Example: power sets

If $S = \{a, b, c, d\}$ then

$$\mathbb{P}\,S = \left\{\begin{array}{l} \{\}, \\[1ex] \{a\}, \{b\}, \{c\}, \{d\}, \\[1ex] \{a, b\}, \{a, c\}, \{a, d\}, \\ \{b, c\}, \{b, d\}, \{c, d\}, \\[1ex] \{a, b, c\}, \{a, b, d\}, \\ \{a, c, d\}, \{b, c, d\}, \\[1ex] \{a, b, c, d\} \end{array}\right.$$

Example: power sets

If $S = \{a, b, c, d\}$ then

$$\mathbb{P}\, S = \left\{ \begin{array}{lllll}
\{\}, & yyyy & & & \\[4pt]
\{a\}, \{b\}, \{c\}, \{d\}, & xyyy & yxyy & yyxy & yyyx \\[4pt]
\{a, b\}, \{a, c\}, \{a, d\}, & xxyy & xyxy & xyyx & \\
\{b, c\}, \{b, d\}, \{c, d\}, & yxxy & yxyx & yyxx & \\[4pt]
\{a, b, c\}, \{a, b, d\}, & xxxy & xxyx & & \\
\{a, c, d\}, \{b, c, d\}, & xyxx & yxxx & & \\[4pt]
\{a, b, c, d\} & xxxx & & &
\end{array} \right.$$

... writing x= "is a member", y= "not a member"

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If $S = \{a, b, c, d\}$ then

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... remember from truth tables: multiplication=and, addition=or

Example: power sets

If $S = \{a, b, c, d\}$ then

$$\mathbb{P}\,S = \left\{ \begin{array}{lll} \{\}, & yyyy & y^4 \\ \{a\}, \{b\}, \{c\}, \{d\}, & + xyyy + yxyy + yyxy + yyyx & 4xy^3 \\ \{a,b\}, \{a,c\}, \{a,d\}, & + xxyy + xyxy + xyyx & \\ \{b,c\}, \{b,d\}, \{c,d\}, & + yxxy + yxyx + yyxx & 6x^2y^2 \\ \{a,b,c\}, \{a,b,d\}, & + xxxy + xxyx & \\ \{a,c,d\}, \{b,c,d\}, & + xyxx + yxxx & 4x^3y \\ \{a,b,c,d\} & + xxxx & x^4 \end{array} \right.$$

... just adding up the terms

Example: power sets

If $S = \{a, b, c, d\}$ then

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$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example: power sets to binomial coefficients

If $S = \{a, b, c, d\}$ then

$$
\mathbb{P}\, S = \begin{cases}
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\{a\}, \{b\}, \{c\}, \{d\}, & + xyyy + yxyy + yyxy + yyyx & 4xy^3 \\
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\{a, c, d\}, \{b, c, d\}, & + xyxx + yxxx & 4x^3y \\
\{a, b, c, d\} & + xxxx & x^4
\end{cases}
$$

$$
\begin{aligned}
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
&= (x + y)^4
\end{aligned}
$$