CS172: Computer Systems II

Lecture 13

Predicate Logic: Models and Validity

James Power

Valid inference

Example of a valid inference (syllogism):

$$\frac{\begin{array}{l}\text{All politicians are rich.}\\ \text{Some students are politicians.}\end{array}}{\text{Some students are rich.}}$$

Valid inference

Example of a valid inference:

All politicians are rich.	$(\forall x \cdot Px \rightarrow Rx)$
Some students are politicians.	$(\exists y \cdot Sy \wedge Py)$
Some students are rich.	$(\exists z \cdot Sz \wedge Rz)$

where:

Px means "x is a politician"
Sx means "x is a student"
Rx means "x is rich"

Valid inference

When is an inference valid?

Reminder - Valid inference in cluedo

A1 At least one of them is guilty.

A2 Not all of them are guilty.

A3 If Mrs White is guilty, then Colonel Mustard helped her (he is guilty too).

A4 If Miss Scarlet is innocent then so is Colonel Mustard.

~~innocent~~	~~innocent~~	~~innocent~~
~~innocent~~	~~innocent~~	~~guilty~~
innocent	guilty	innocent
innocent	guilty	guilty
~~guilty~~	~~innocent~~	~~innocent~~
~~guilty~~	~~innocent~~	~~guilty~~
~~guilty~~	~~guilty~~	~~innocent~~
~~guilty~~	~~guilty~~	~~guilty~~

Reminder - Valid inference in propositional logic

$$\text{Inference:} \quad \frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

Valid inference. An inference is **valid** if and only if **every** time (**every** situation) in which **all** premises $\varphi_1, \ldots, \varphi_n$ are true, ψ is also true.

Reminder - Valid inference in syllogisms

1. **Draw the Skeleton.** Draw the domain of discourse with the three predicates.
2. **Universal step: crossing out** Apply the universal statements from the premises ("**All ... are ...**" and "**No ... is ...**") by crossing out the forbidden regions.
3. **Existential step: filling up** Apply the existential statements from the premises ("**Some ... are ...**" and "**Some ... are not ...**"), trying to put a • in an appropriate region. (This could produce several diagrams.)
4. **Check conclusion** Verify that at least one of the conclusion's representation is in all the diagrams.

Valid inference: predicate logic

An inference *in predicate logic* of the form

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

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So: what's a *model*?

Classification of formulas according to their behaviour

A formula φ in predicate logic is:

- a tautology if, for every model M, we have $M \models \varphi$.
  In this case we often simply write "$\models \varphi$".
- a contradiction if there is no model M for which $M \models \varphi$.
  In this case we often simply write "$\not\models \varphi$".
- satisfiable if there is at least one model M for which $M \models \varphi$.

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Models in other logics

So: what's a *model*?

- *in Cluedo* a model was just an assignment of innocence or guilt to each character.
- *in propositional logic* a model was a single line in the truth table (we called it a "valuation").
- *in syllogistic logic* a model is a Venn diagram with some information filled in (crossing-out or adding a '•').

Models in predicate logic: definition

How do we build a model of some predicate logic formula φ?

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1. Define some non-empty set D called the domain.
2. Provide an interpretation I that shows how each constant/predicate symbol used in φ is mapped to an object/relation in the domain.
3. Check that the formula φ is actually *true* for this domain and interpretation.

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3. Check that the formula φ is actually *true* for this domain and interpretation.

NB: Note that we can only check for truth *after* we have picked the domain and interpretation.

Models in predicate logic: example

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$x := Tom$	$T \rightarrow T$	✓

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$x := Dick$	$F \rightarrow T$	✓

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This <u>is</u> a model of the formula.

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$x := Harry$	$F \wedge F$	✗

$(\exists x \cdot)$ ✓

This is a model of the formula.

Models in predicate logic: Notation

- Reminder: A model consists of two things:
  - a domain D (sets and relations)
  - some interpretation I mapping symbols into D.
- Only **after** we have been given (or have chosen) a model can we can evaluate a formula to either "true" or "false".

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- Only **after** we have been given (or have chosen) a model can we can evaluate a formula to either "true" or "false".

- Notation:

  $\langle D, I \rangle \models \phi$ means the formula ϕ is true in the model $\langle D, I \rangle$

  $\langle D, I \rangle \not\models \phi$ means the formula ϕ is false in the model $\langle D, I \rangle$

Example: colours

1. Suppose we choose a domain D with four objects $\{o_1, o_2, o_3, o_4\}$ ... and some properties of these objects:

Property	*Objects*
is-blue	$\{o_1\}$
is-red	$\{o_4\}$
is-purple	$\{o_2, o_3\}$
is-green	$\{\ \}$
is-square	$\{o_2, o_4\}$
is-circle	$\{o_1, o_3\}$

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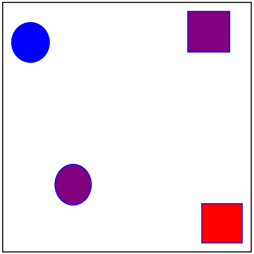

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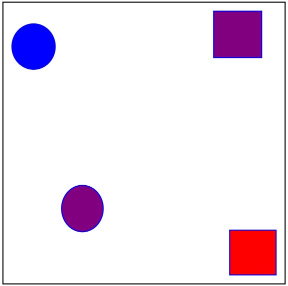
1. Suppose we choose a domain D with four objects $\{o_1, o_2, o_3, o_4\}$ ... and some properties of these objects:

Predicate	Property	*Objects*
Bx	is-blue	$\{o_1\}$
Rx	is-red	$\{o_4\}$
Px	is-purple	$\{o_2, o_3\}$
Gx	is-green	$\{\ \}$
Sx	is-square	$\{o_2, o_4\}$
Cx	is-circle	$\{o_1, o_3\}$

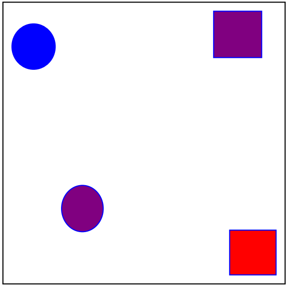
2. Then suppose we define an interpretation I for six predicate symbols (of arity 1) that map to these properties, as above.

Example: colours (continued)

D = 

$$I = \begin{array}{rcl} \text{Predicate} & \mapsto & \text{Property} \\ Bx & \mapsto & \text{is-blue} \\ Rx & \mapsto & \text{is-red} \\ Px & \mapsto & \text{is-purple} \\ Gx & \mapsto & \text{is-green} \\ Sx & \mapsto & \text{is-square} \\ Cx & \mapsto & \text{is-circle} \end{array}$$

Example: colours (continued)

D = 

I =

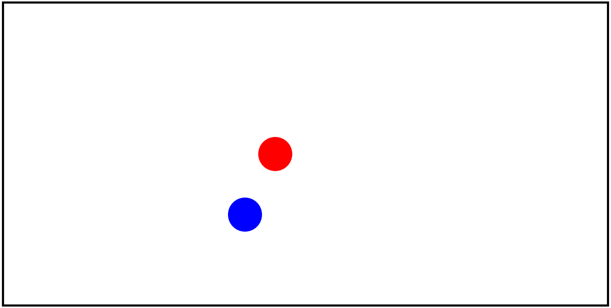
Predicate	$\mapsto$	Property
Bx	$\mapsto$	is-blue
Rx	$\mapsto$	is-red
Px	$\mapsto$	is-purple
Gx	$\mapsto$	is-green
Sx	$\mapsto$	is-square
Cx	$\mapsto$	is-circle

Q: For which of the following formula ϕ can we say that $\langle D, I \rangle \models \phi$?

1. $(\exists\, x \cdot Rx \wedge Cx)$
2. $(\forall\, x \cdot Cx \vee Sx)$
3. $(\exists\, x \cdot Gx) \vee (\exists\, x \cdot Cx)$
4. $(\exists\, x \cdot Rx) \wedge (\exists\, x \cdot Cx)$
5. $(\forall\, x \cdot Cx) \vee (\forall\, x \cdot Sx)$
6. $(\exists\, x \cdot Gx \vee Cx)$

Evaluating predicate logic formulas (1)

Colors (***R***ed, ***G***reen, ***B***lue, ***P***urple) and shapes (***S***quare, ***C***ircle).



- $\exists x Rx$
- $\forall x(Rx \to Cx)$
- $\exists x(Gx \wedge Cx)$
- $\neg\forall x \neg Rx$
- $\forall x(Rx \wedge Cx)$
- $\exists x(Gx \to Cx)$

Evaluating predicate logic formulas (1)

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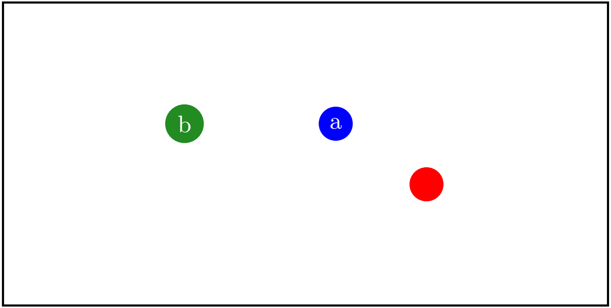
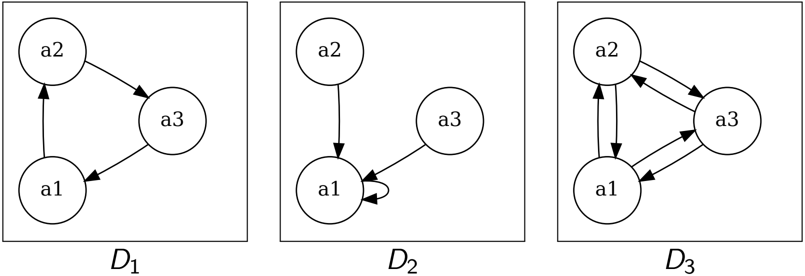


- Ba
- $\exists x Sx \vee Cb$
- $Ra \rightarrow Sb$
- $Ba \wedge Gb$
- $\neg Sa$
- $Ra \rightarrow \exists x Sx$

Another example: "sees" relation



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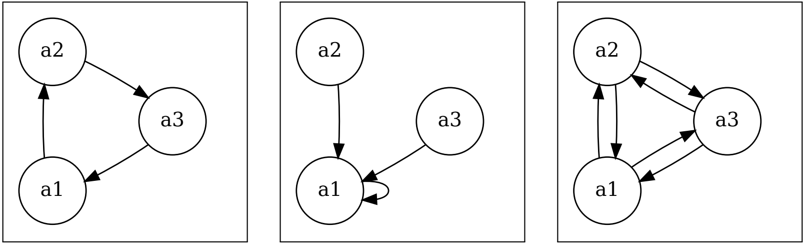

D_1 D_2 D_3

Suppose we are given three **domains**, as above.

... but just one **interpretation** in each case:

- *I* maps *Sxy* into the "sees" relation, represented by the arrows above.

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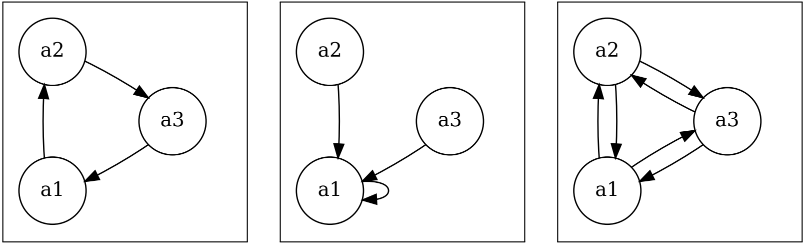

D_1 D_2 D_3

Sentence	Formula	$\langle D_1, I\rangle$	$\langle D_2, I\rangle$	$\langle D_3, I\rangle$
Everyone sees someone	$(\forall\, x \cdot (\exists\, y \cdot Sxy))$			

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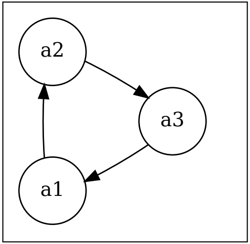
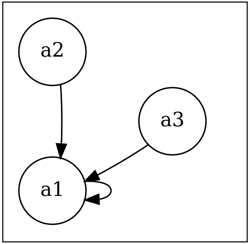
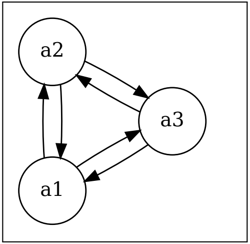


D_1



D_2



D_3

Sentence	Formula	$\langle D_1, I\rangle$	$\langle D_2, I\rangle$	$\langle D_3, I\rangle$
Everyone sees someone	$(\forall\, x \cdot (\exists\, y \cdot Sxy))$	T	T	T

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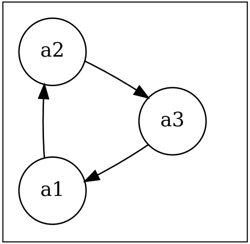

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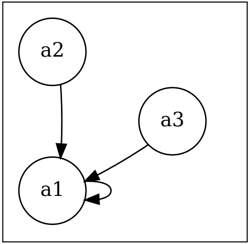
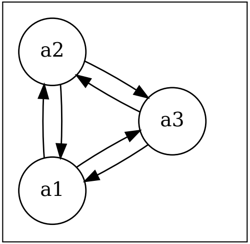


D_2



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Sentence	Formula	$\langle D_1, I\rangle$	$\langle D_2, I\rangle$	$\langle D_3, I\rangle$
Everyone sees someone	$(\forall x \cdot (\exists y \cdot Sxy))$	T	T	T
Someone sees everyone	$(\exists x \cdot (\forall y \cdot Sxy))$			

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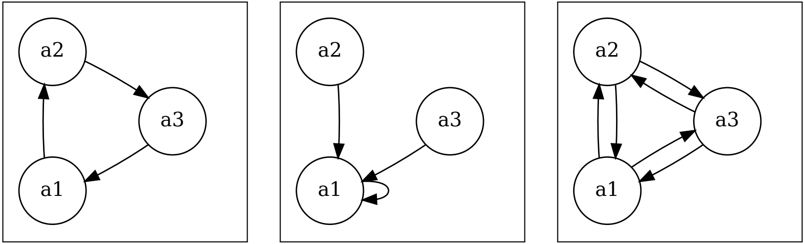

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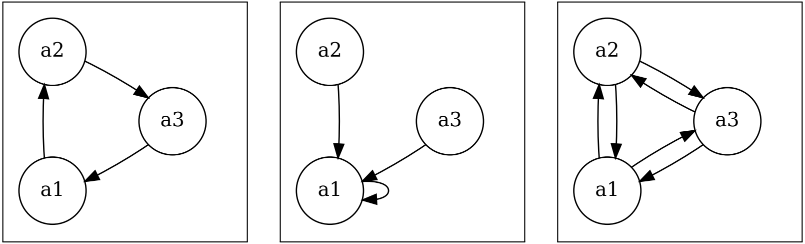

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Everyone sees someone	$(\forall x \cdot (\exists y \cdot Sxy))$	T	T	T
Someone sees everyone	$(\exists x \cdot (\forall y \cdot Sxy))$	F	F	F
Everyone is seen by someone	$(\forall x \cdot (\exists y \cdot Syx))$			

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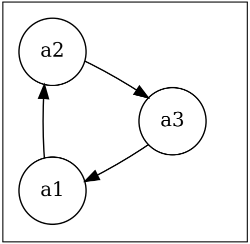

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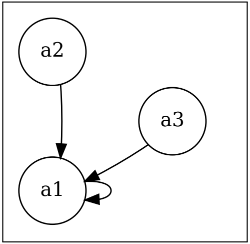

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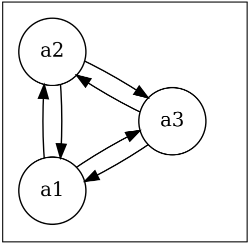

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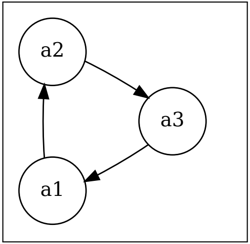
Someone sees everyone	$(\exists\, x \cdot (\forall\, y \cdot Sxy))$	F	F	F
Everyone is seen by someone	$(\forall\, x \cdot (\exists\, y \cdot Syx))$	T	F	T

Another example: "sees" relation



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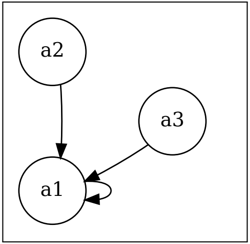
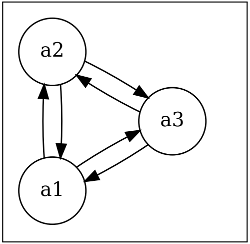


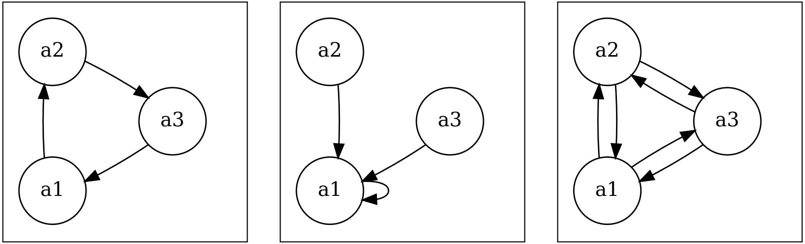
D_2



D_3

Sentence	Formula	$\langle D_1, I \rangle$	$\langle D_2, I \rangle$	$\langle D_3, I \rangle$
Everyone sees someone	$(\forall x \cdot (\exists y \cdot Sxy))$	T	T	T
Someone sees everyone	$(\exists x \cdot (\forall y \cdot Sxy))$	F	F	F
Everyone is seen by someone	$(\forall x \cdot (\exists y \cdot Syx))$	T	F	T
Someone is seen by everyone	$(\exists x \cdot (\forall y \cdot Syx))$			

Another example: "sees" relation

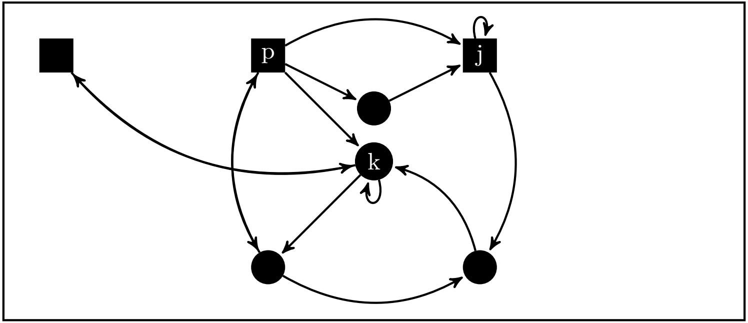


D_1 D_2 D_3

Sentence	Formula	$\langle D_1, I \rangle$	$\langle D_2, I \rangle$	$\langle D_3, I \rangle$
Everyone sees someone	$(\forall x \cdot (\exists y \cdot Sxy))$	T	T	T
Someone sees everyone	$(\exists x \cdot (\forall y \cdot Sxy))$	F	F	F
Everyone is seen by someone	$(\forall x \cdot (\exists y \cdot Syx))$	T	F	T
Someone is seen by everyone	$(\exists x \cdot (\forall y \cdot Syx))$	F	T	F

Evaluating predicate logic formulas (2)

■: **boy** •: **girl** • → ■: • **loves** ■



- $Ljk \to Lkj$
- $\neg(Ljk \wedge Lkj)$
- $\forall x(Bx \to Lxk)$
- $\forall x((Bx \vee Gx) \to \neg Lxp)$
- $Ljk \wedge Lkj$
- $(Ljk \wedge Lpk) \to (\neg Lpj \wedge \neg Lkj)$
- $\neg\forall x(Gx \to Lxx)$
- $\exists x(Gx \wedge Lpx \wedge Lxj)$