Exercise 3.2-2 Prove Equation 3.16

$$a^{\log_b c} = c^{\log_b a}$$

Take $\log_b$ both sides

$$\log_b\left(a^{\log_b c}\right) = \log_b\left(c^{\log_b a}\right)$$

$$\log_b c \, \log_b a = \log_b a \, \log_b c \qquad \text{QED.}$$

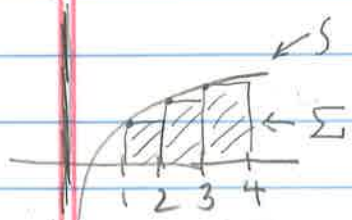
Exercise 3.2-3 Prove Equation 3.19

$$\lg(n!) = \Theta(n \lg n)$$

$$\lg(n!) = \lg(1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)(n))$$

$$= \sum_{i=1}^{n} \lg(i) \le \sum_{i=1}^{n} \lg(n) = n \lg n$$

(To show an upper bound ($i < n$))



$$\int_1^n \lg x \, dx \le \sum_{i=1}^{n} \lg(i) = \lg(n!)$$

but $\int_1^n \lg(x)\,dx = n \lg\left(\frac{n}{e}\right) + 1$

Hence $\lg(n!) = \Theta(n \lg n)$

<u>Problem 3-1(a-c)</u> $p(n) = \sum_{i=0}^{d} a_i n^i$

a) If $k \geq d$ $p(n) = O(n^k)$

By def'n of O, we need to show

$\exists c$ and $n_0 \ni$ for all $n \geq n_0$

$p(n) \leq c n^k$

worst case is all $a_i \geq 0$ and $k = d$

let $c = \sum_{i=0}^{n} a_i$

Then we have

$a_0 + a_1 n + \dots a_k n^k \leq a_0 n^k + a_1 n^k + \dots a_k n^k$

For each i $a_i n^i \leq a_i n^k$

so we have the desired result.

b) If $k \leq d$, then $p(n) = \Omega(n^k)$

Worst case when $a_0, a_1, \dots, a_{k-1}$ are all negative ($a_k > 0$ must be the case)

We must find $c > 0 \ni n > n_0 \Rightarrow$

$$0 \le cn^k \le p(n)$$

w.l.o.g. $p(n) = a_k n^k - \sum_{i=0}^{k-1} |a_i| n^i \le a_k n^k - \sum_{i=0}^{k-1} |a_i| n^{k-1}$

let $b = \max(|a_0|, |a_1|, \dots, |a_{k-1}|)$

$$p(n) \ge a_k n^k - \sum_{i=0}^{k-1} b n^i \ge a_k n^k - kbn^{k-1}$$

choose $n_0 = \frac{2kb}{a_k}$ $\qquad kb > \frac{n a_k}{2}$

for $n > n_0$, $p(n) \ge \cancel{a_k}\left(\frac{kb}{\cancel{a_k}}\right) n^{k-1} - kb n^{k-1} = kbn^{k-1}$

$$= \frac{n^k a_k}{2}$$

choose $c = \frac{a_k}{2}$

$cn^k = \frac{a_k}{2} n^k \le p(n)$ for $n > \frac{2kb}{a_k}$

c) for $d = k$ $f(n) = \Theta(n^k)$

follows from (a) + (b) + Theorem 3.1

Exercise 4.2-2 Pseudocode for Strassen

Assume matrix $A[1..n][1..n]$ and $B[1..n][1..n]$
assume $n = 2^k$ for some k

STRASSEN(A, B)
    n = A.rows
    let C be a new n×n matrix
    if n == 1
        $c_{11} = a_{11} \cdot b_{11}$
    else $A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}$ are
        $n/2 \times n/2$ as in eqn 4.9
        let $S_1, \ldots, S_{10}$ be $n/2 \times n/2$ matrices
            $S_1 = B_{12} - B_{22}$
            $S_2 = A_{11} + A_{12}$
            $S_3 = A_{21} + A_{22}$
            $S_4 = B_{21} - B_{11}$
            $S_5 = A_{11} + A_{22}$
            $S_6 = B_{11} + B_{22}$
            $S_7 = A_{12} - A_{22}$
            $S_8 = B_{21} + B_{22}$
            $S_9 = A_{11} - A_{21}$

$S_{10} = B_{11} + B_{12}$

let $P_1 \ldots P_7$ be new $n/2 \times n/2$ matrices

$P_1 = \text{STRASSEN}(A_{11}, S_1)$

$P_2 = \text{STRASSEN}(S_2, B_{22})$

$P_3 = \text{STRASSEN}(S_3, B_{11})$

$P_4 = \text{STRASSEN}(A_{22}, S_4)$

$P_5 = \text{STRASSEN}(S_5, S_6)$

$P_6 = \text{STRASSEN}(S_7, S_8)$

$P_7 = \text{STRASSEN}(S_9, S_{10})$

let C be new $n \times n$ → $C_{11} = P_5 + P_4 - P_2 + P_6$

$C_{12} = P_1 + P_2$

$C_{21} = P_3 + P_4$

$C_{22} = P_5 + P_1 - P_3 - P_7$

return C

Exercise 4.2-4

Largest $k \ni$ if you can multiply 3×3 matrices using k multiplications then you can multiply $n \times n$ in $o(n^{\lg 7})$

$$T(n) = kT(n/3) + \Theta(n^2)$$

$\lg 7 = 2.81$

To use case 1 of master method, need

$$\log_3 k \le \log_2 7$$

$\log_3 k = \frac{\ln k}{\ln 3}$ $\quad\quad$ $\log_2 7 = \frac{\ln 7}{\ln 2}$

need $\frac{\ln k}{\ln 3} < \frac{\ln 7}{\ln 2}$

$$\ln k \le \frac{\ln 7 \cdot \ln 3}{\ln 2}$$

$$k \le e^{\frac{\ln 7 \ln 3}{\ln 2}}$$

$$k \le 21.85$$

$$k = 21$$

Problem 4-1(a-f)

(a) $T(n) = 2T(n/2) + n^4$

$a = 2 \quad b = 2 \qquad n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = \Omega(n^{\log_a^b + \varepsilon})$ for $\varepsilon = 3$

$a(b(\frac{n}{2})) = 2(\frac{n}{2})^4 = \frac{n^4}{8}$ set $c = 1/8$

$\Rightarrow$ case 3 of Master Theorem applies $\boxed{T(n) = \Theta(n^4)}$

b) $T(n) = T\left(\frac{7}{10}n\right) + n$

$a = 1,\ b = \frac{10}{7}$ $\qquad \log_{10/7} 1 = 0 \qquad n^{\log_b a} = n^0 = 1$

$af(n/b) = \frac{7}{10}n$ let $c = 7/10$

apply case 3 of Master Theorem

$$\boxed{T(n) = \Theta(n)}$$

c) $T(n) = 16T\left(\frac{n}{4}\right) + n^2$

$a = 16 \quad b = 4 \quad n^{\log_b a} = n^4$

$f(n) = n^2 = O(n^{\log_b a - \epsilon})$ for $\epsilon = 2$

apply case 1: $\boxed{T(n) = \Theta(n^4)}$

d) $T(n) = 7T(n/3) + n^2$

$a = 7,\ b = 3 \qquad n^{\log_3 7} = n^{\frac{\ln 7}{\ln 3}} = n^{1.77}$

$f(n) = n^2 = \Omega(n^{\log_b a + 0.2})$

apply case 3 $\qquad 7\left(\frac{n}{3}\right)^2 = \frac{7n^2}{9}$ let $c = 7/9$

case 3 applies $\boxed{T(n) = \Theta(n^2)}$

e) $T(n) = 7T(n/2) + n^2$

$a=7$ $b=2$ $n^{\log_2 7} = n^{2.81}$

case 1 applies $f(n) = n^2 = O(n^{\log_2 7 - \varepsilon})$ for $\varepsilon = 0.8$

$$\boxed{T(n) = \Theta(n^{\lg_2 7})}$$

f) $T(n) = 2T(n/4) + \sqrt{n}$

$a=2$ $b=4$ $\log_4 2 = 0.5$

$$f(n) = n^{0.5} = \Theta(n^{\log_b a})$$

case 2 $\boxed{T(n) = \Theta(n^{1/2} \lg n)}$