THEORY OF ALGORITHMS
SOLUTIONS TO THE PROBLEMS

BAOJIAN HUA

MAY 11,2004

3.1-1 Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Proof. We only need to show that there exist constants c_1, c_2 such that for sufficiently large n, the relation

$$c_1(f(n) + g(n)) \leq \max(f(n), g(n)) \leq c_2(f(n) + g(n))$$

holds. It's easy to show that it's always hold for $c_1 = 1/2$ and $c_2 = 1$. □

3.1-2 Show that for any real constants a and b, where $b > 0$,

$$(n + a)^b = \Theta(n^b)$$

Proof. We only need to show that there exist constants c_1, c_2 such that for sufficiently large n, the relation

$$c_1 n^b \leq (n + a)^b \leq c_2 n^b$$

holds, which is

$$\sqrt[b]{c_1} n - a \leq n \leq \sqrt[b]{c_2} n - a$$

It's easy to show that it holds for sufficiently large n and $c_1 = (1/2)^b, c_2 = 2^b$. □

3.1-4 Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?
Solution. Yes. No.

□

3.1-6 Prove that the running time of an algorithm in $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

Proof. Straightforward from the definitions. □

3.2-2 Prove the equation

$$a^{\log_b c} = c^{\log_b a}$$

Proof. It's sufficient to prove $\log_b c \cdot \log_b a = \log_b a \cdot \log_b c$ □

3.1 ***Asymptotic behavior of polynomials***
Let

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

where $a_d > 0$, be a degree-d polynomial in n, and let k be a constant. Use the definition of asymptotic notations to prove the following properties.
a. If $k \geq d$, then $p(n) = O(n^k)$
b. If $k \leq d$, then $p(n) = \Omega(n^k)$

c. If $k = d$, then $p(n) = \Theta(n^k)$
d. If $k > d$, then $p(n) = o(n^k)$
e. If $k < d$, then $p(n) = \omega(n^k)$

Proof. First, we prove a more general lemma

Lemma 0.1. *Suppose both $P(n) = p_n x^n + \ldots + p_0$ and $Q(n) = q_n x^n + \ldots + q_0$ are polynomials in n (where $p_n > 0$ and $q_n > 0$), let*

$$l = \lim_{n\to\infty} \frac{P(n)}{Q(n)}$$

it is easy to show that for sufficiently large n and some constant $c, c_1 > 0$, we have

(1) *If $l = 0$, then $0 < P(n) < cQ(n)$.*
(2) *If $l = \infty$, then $0 < cQ(n) < P(n)$.*
(3) *If $l = b$ for some $b \in \mathbb{R}^+$, then $c_1 Q(n) < P(n) < cQ(n)$.*

Proof. Straightforward. □

It is easy to prove the proposition using the above lemma. □

3.2 ***Relative asymptotic growths***
Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1, \epsilon > 0$, and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box.
Solution.

	A	B	O	o	Ω	ω	Θ
a	$\lg^k n$	n^ϵ	no	no	yes	yes	no
b	n^k	c^n	no	no	yes	yes	no
c	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d	2^n	$2^{n/2}$	yes	yes	no	no	no
e	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

□