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Z-transform:

If $U_n = f(n)$ define for $n = 0, 1, 2, 3, \dots$
and $U_n = 0$, when $n < 0$

The Z-transform is denoted by $Z(U_n)$ or $Z_T(U_n)$.

$$\boxed{Z(U_n) = \sum_{n=0}^{\infty} U_n Z^{-n} = \bar{U}(z)}$$

$$\boxed{\therefore Z^{-1}[Z(U_n)] = Z^{-1}[\bar{u}(z)] = U_n}$$

Z-transform of some standard function:

1) ~~Zo~~ $Z(k^n)$

Proof: $Z(U_n) = \sum_{n=0}^{\infty} U_n Z^{-n} = \sum_{n=0}^{\infty} k^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{k}{z}\right)^n$

$$= \left(\frac{k}{z}\right)^0 + \left(\frac{k}{z}\right)^1 + \left(\frac{k}{z}\right)^2 + \dots$$

$$= 1 + r^1 + r^2 + r^3 + \dots \quad \left[\text{where } r \text{ is } \frac{k}{z}\right]$$

This converges to geometric series $\frac{a}{1-r}$

$$\boxed{Z(k^n) = \frac{1}{1-r} = \frac{1}{1-\frac{k}{z}} = \frac{z}{z-k}} \Rightarrow \bar{u}(z)$$

Z-transform of $\boxed{Z(k^{-n}) = \frac{z}{z+k}}$

Z-transfor of $Z(1^n) = \frac{z}{z-1} = Z(1)$

~~Zda~~ $Z(2^n) \neq Z(2)$

2) $Z(e^{an})$

By definition

$$z(u_n) = \sum_{n=0}^{\infty} e^{an} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^a}{z}\right)^n$$

$$= 1 + \left(\frac{e^a}{z}\right)^1 + \left(\frac{e^a}{z}\right)^2 + \left(\frac{e^a}{z}\right)^3 + \cdots$$

comparing with G.P we get :

$$= \frac{1}{1-r} = \frac{1}{1-\left(\frac{e^a}{z}\right)} = \frac{1}{\frac{z-e^a}{z}} = \boxed{\frac{z}{z-e^a} = Z(e^{an})}$$

$$\boxed{z(e^{-an}) = \frac{z}{z-e^{-a}}} \quad \therefore \quad \boxed{z[e^{2n}] = \frac{z}{z-e^2}}$$

3) $z(\sinh n\theta)$

$$\sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2}$$

$$z(\sinh n\theta) = \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{\cancel{z}-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{\cancel{z} - e^{-\theta} - \cancel{z} + e^{\theta}}{z^2 - ze^{-\theta} - ze^{\theta} + 1}\right]$$

Multiple and divide by 2.

$$= \frac{z}{\cancel{2}}\left[\frac{\cancel{2}(e^{\theta} - e^{-\theta})/2}{z^2 - 2z\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) + 1}\right]$$

$$= \boxed{z\left[\frac{\sinh\theta}{z^2 - 2z\cosh\theta + 1}\right] = Z(\sinh n\theta)}$$

4) $Z(\cosh an)$

$$\cosh an = \frac{e^{na} + e^{-na}}{2}$$

~~Z{sinh~~

$$Z(\cosh an) = \frac{1}{2}\left[\frac{z}{z-e^{a}} + \frac{z}{z-e^{-a}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-a}+z-e^{a}}{z^2 - ze^{-a} - ze^{a} + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cosh a}{z^2 - 2z\cosh a + 1}\right]$$

$$\boxed{Z(\cosh an) = z\left[\frac{z - \cosh a}{z^2 - 2z\cosh a + 1}\right]}$$

5) $Z(\cos n\theta)$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

$$Z(\cos n\theta) = \frac{z}{2}\left[\frac{1}{z-e^{i\theta}} + \frac{1}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-i\theta}+z-e^{i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$\boxed{Z(\cos n\theta) = z\left[\frac{z - \cos\theta}{z^2 - 2z\cos\theta + 1}\right]}$$

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Properties:

1) **Linearity property**: If u_n and V_n are 2 ~~district~~ discrete value functions then

$$Z_T(c_1u_n + c_2V_n) = c_1 Z(u_n) + c_2 Z(V_n)$$

Proof: $\underbrace{Z_T(c_1u_n + c_2V_n)}_{u_n} = \sum_{n=0}^{\infty}(c_1u_n + c_2V_n)z^{-n}$

$$= c_1\sum_{n=0}^{\infty} u_n z^{-n} + c_2\sum_{n=0}^{\infty} V_n z^{-n}$$

$$= c_1 Z(u_n) + c_2 Z(V_n)$$

$$\therefore \boxed{Z_T(c_1u_n + c_2V_n) = c_1 Z(u_n) + c_2 Z(V_n)}$$

2) **Damping rule:**

If $Z(u_n) = \bar{u}(z)$ then i) $Z_T(k^n u_n) = \bar{u}(z/k)$

ii) $Z_T(k^{-n}u_n) = \bar{u}(kz)$.

Proof: consider LHS:

i) $Z_T(k^n u_n) = \sum_{n=0}^{\infty}(k^n u_n)z^{-n}$

$$= \sum_{n=0}^{\infty} u_n (z|k)^{-n}$$

$$= \sum_{n=0}^{\infty} \bar{u}(z|k)$$

$$= \bar{u}(z|k)$$

$$\therefore \boxed{Z_T(k^n u_n) = \bar{u}(z|k)}$$

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ii) $z(k^{-n}u_n) = \sum_{n=0}^{\infty} (k^{-n}u_n)z^{-n}$

$$= \sum_{n=0}^{\infty} u_n (kz)^{-n}$$

$$= \bar{u}(kz)$$

$$\boxed{\therefore z(k^{-n}u_n) = \bar{u}(kz)}$$

3) <u>Right shifting rule:</u>

If $z(u_n) = \bar{U}(z)$ then

$z_T(u_{n-k}) = z^{-k}(\bar{u}(z))$, where $k>0$.

<u>Proof:</u>

$$z_T(u_{n-k}) = \sum_{n-k=0}^{\infty} u_{n-k} z^{-n}$$

$$= u_0 z^{-k} + u_1 z^{-k+1} + u_2 z^{-k+2}$$

$$= z^{-k}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots\right]$$

$$= z^{-k} \cdot z(u_n)$$

$$\boxed{\therefore z_T(u_{n-k}) = z^{-k}(\bar{u}(z))}$$

<u>NOTE:</u> $z(u_{n-k}) = z^{-k}(\bar{u}(z))$

when $k=1$,

$$\boxed{z(u_{n-1}) = \frac{\bar{u}(z)}{z}}$$

$k=2$,

$$\boxed{z(u_{n-2}) = \frac{\bar{u}(z)}{z^2}}$$

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4) **Left shifting rule:**

If $z(u_n) = \bar{U}(z)$ then

$$\boxed{Z_T(u_{n+k}) = z^k\left[\bar{U}(z) - \sum_{n=0}^{k-1} u_n z^{-n}\right]}$$, where $k>0$

NOTE: when $k=1$,

$$\boxed{z(u_{n+1}) = z^1\left[\bar{u}(z) - u_0\right]}$$

when $k=2$,

$$z(u_{n+2}) = z^2\left[\bar{u}(z) - \sum_{n=0}^{1} u_n z^{-n}\right]$$

$$\boxed{z(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]}$$

when $k=3$,

$$\boxed{z(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]}$$

Ex: Repeated question

1) S.T: ~~[illegible]~~ $Z_T(n^k) = -z\frac{d}{dz}Z_T(n^{k-1})$

$\Rightarrow$ Proof: consider: $-z\frac{d}{dz}Z_T(n^{k-1})$

$$= -z\frac{d}{dz}\sum_{n=0}^{\infty} n^{k-1} z^{-n}$$

$$= -z\sum_{n=0}^{\infty} \frac{n^k}{\cancel{n}}(-\cancel{n})z^{-n-1}$$

$$= \cancel{z}\sum_{n=0}^{\infty} n^k \frac{z^{-n}}{\cancel{z}} = \sum_{n=0}^{\infty} n^k z^{-n}$$

$$= Z_T(n^k)$$

Hence proved:

$$Z_T(n^k) = -z\frac{d}{dz}Z_T(n^{k-1})$$

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6) $Z(\sin n\theta)$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$$Z(\sin n\theta) = \frac{1}{2i}\left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2i}\left[\frac{1}{z-e^{i\theta}} - \frac{1}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2i}\left[\frac{\cancel{z} - e^{-i\theta} - \cancel{z} + e^{i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}\right]$$

$$= \frac{z}{2i}\left[\frac{2i\sin\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$\therefore \boxed{Z(\sin n\theta) = z\left[\frac{\sin\theta}{z^2 - 2z\cos\theta + 1}\right]}.$$

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NOTE: ~~to~~ $Z(n^k) = -z\frac{d}{dz} Z_T(n^{k-1})$

when $k=1$

$$Z(n^1) = -z\frac{d}{dz} Z_T(1)$$

$$= -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{\cancel{z} - 1 - \cancel{z}}{(z-1)^2}\right]$$

$$\boxed{Z(n^1) = \frac{z}{(z-1)^2}}$$

when $k=2$

$$Z(n^2) = -z \frac{d}{dz} Z_T(n^1)$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2(1) - 2z(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{\cancel{(z-1)}(z-1-2z)}{(z-1)^{\cancel{4}}}\right]$$

$$= -z\left[-z-1\right]$$

$$\boxed{Z(n^2) = \frac{z^2+z}{(z-1)^3}}$$

simillarly:

$$\boxed{Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}}$$

Ex:

1) find $Z_T\left(\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)$

$\Rightarrow$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$u_n = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$$

$$u_n = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$u_n = \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$Z(u_n) = \frac{1}{\sqrt{2}} Z_T\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

2) S.T; $Z_T\left(\frac{1}{n!}\right) = e^{1/z}$ then find $Z_T\left(\frac{1}{n+1!}\right)$ and $Z_T\left(\frac{1}{n+2!}\right)$

$\Rightarrow$ By definition of Z_T:

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \frac{1}{0!} z^{-0} + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} \cdots\cdots$$

$$= 1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} \cdots\cdots$$

Let $x = \frac{1}{z}$

$$Z\left(\frac{1}{n!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\cdots$$

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots = e^x$$

$$\therefore\ Z\left(\frac{1}{n!}\right) = e^{1/z} \rightarrow \bar{u}(z)$$

find $Z\left(\frac{1}{n+1!}\right)$

By left shifting rule ; $Z(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$

$u_n = \frac{1}{n!}$

$u_0 = \frac{1}{0!} = 1$

$u_n = \frac{1}{n!}$

$u_{n+1} = \frac{1}{n+1!}$

$$\boxed{Z\left(\frac{1}{n+1!}\right) = z\left[e^{1/z} - 1\right]}$$

$$Z(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$u_{n+2} = \frac{1}{n+2!}$

$$\boxed{Z\left(\frac{1}{n+2!}\right) = z^2\left[e^{1/z} - 1 - \frac{1}{z}\right]}.$$

3) Find: $Z_T(\sin(3n+5))$ 6M

$\Rightarrow \quad u_n = \sin(3n+5)$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$u_n = \sin 3n \cos 5 + \cos 3n \sin 5$

Applying Z transform:

$$Z(u_n) = \frac{(z \sin 3)\cos 5}{z^2 - 2z\cos 3 + 1} + \frac{z(z-\cos 3)\cdot \sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\cos 5 \sin 3 + z^2 \sin 5 - z\cos 3 \sin 5}{z^2 - 2z\cos 3 + 1}$$

$\sin a \cos b - \cos a \sin b = \sin(a-b)$

$$= \frac{z(\cos 5 \sin 3 - \cos 3 \sin 5) + z^2 \sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(\sin(3-5)) + z^2 \sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(\sin(-2)) + z^2 \sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(-\sin 2) + z^2 \sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(z \sin 5 - \sin 2)}{z^2 - 2z\cos 3 + 1}$$

(4) $Z_T\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]$

$$\Rightarrow u_n = \cosh\left(\frac{n\pi}{2}+\theta\right)$$

$$= \frac{e^{\left(\frac{n\pi}{2}+\theta\right)} + e^{-\left(\frac{n\pi}{2}+\theta\right)}}{2}$$

$$\cancel{= \frac{e^{\frac{n\pi}{2}} \cdot e^{\theta} + e^{-\frac{n\pi}{2}} \cdot e^{-\theta}}{2}}$$

$$= \frac{e^{\frac{n\pi}{2}} \cdot e^{\theta} + e^{-\frac{n\pi}{2}} \cdot e^{-\theta}}{2}$$

$$z[u_n] = \frac{1}{2}\left[e^{\theta}\frac{z}{z-e^{\pi/2}} + e^{-\theta}\frac{z}{z-e^{-\pi/2}}\right]$$

$$= \frac{1}{2}\left[\frac{ze^{\theta}(z-e^{-\pi/2}) + ze^{-\theta}(z-e^{\pi/2})}{(z-e^{\pi/2})(z-e^{-\pi/2})}\right]$$

$$= \frac{z}{2}\left[\frac{z(e^{\theta}+e^{-\theta}) - \left[e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}\right]}{z^2 - z(e^{\pi/2}+e^{-\pi/2}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z\cosh\theta - 2\cosh(\pi/2-\theta)}{z^2 - 2z\cosh(\pi/2) + 1}\right]$$

$$\therefore z[u_n] = \frac{z^2\cosh\theta - z\cosh(\pi/2-\theta)}{z^2 - 2z\cosh(\pi/2) + 1}$$

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Initial value theorem :

If $Z(u_n) = \bar{u}(z)$ then ;

$$\lim_{z\to\infty} \bar{u}(z) = u_0$$

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} = \bar{U}(z)$$

$$= u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\cdots = \bar{U}(z)$$

$$\boxed{\lim_{z\to\infty} \bar{U}(z) = u_0}$$

Applying limits OBS

$$\lim_{z\to\infty} \bar{U}(z) = u_0 + \frac{u_1}{\infty} + \frac{u_2}{\infty}$$

NOTE:
Find u_1, u_2, u_3 by using initial value theorem.

$$\Rightarrow \boxed{u_1 = z \lim_{z\to\infty} \left[\bar{u}(z) - u_0\right]}$$

$$\lim_{z\to\infty} \bar{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} \cdots \cdots$$

$$z \lim_{z\to\infty} \bar{u}(z) - u_0 = \frac{u_1}{1} + \frac{u_2}{z} + \frac{u}{z^2} \cdots$$

$$\boxed{u_2 = z^2 \lim_{z\to\infty} \left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]}$$

$$\boxed{u_3 = z^3 \lim_{z\to\infty} \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]}$$

Ex:

1) Find u_0, u_1, u_2, u_3:

If $\bar{u}(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$

$$\Rightarrow u_0 = \lim_{z\to\infty} \frac{2z^2 + 3z + 12}{(z-1)^4} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{z\to\infty} \frac{z^{\cancel{2}}\left(2+\frac{3}{z}+\frac{12}{z^2}\right)}{z^{\cancel{4}}\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} \frac{2+\frac{3}{z}+\frac{12}{z^2}}{z^2\left(1-\frac{1}{z}\right)^4}$$

$$\boxed{\therefore u_0 = 0}$$

$$u_1 = z \lim_{z\to\infty} \left[\bar{u}(z) - u_0\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{2z^2+3z+12}{(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \frac{z^{\cancel{3}}\left[2+\frac{3}{z}+\frac{12}{z^2}\right]}{z^{\cancel{4}}\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} \frac{2+\frac{3}{z}+\frac{12}{z^2}}{z\left(1-\frac{1}{z}\right)^4}$$

$$\boxed{\therefore u_1 = 0}$$

$$u_2 = z^2 \lim_{z\to\infty} \left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right]$$

$$= \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+12}{(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z^4}\left[2+\frac{3}{z}+\frac{12}{z^2}\right]}{\cancel{z^4}\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} \frac{2+\frac{3}{z}+\frac{12}{z^2}}{\left(1-\frac{1}{z}\right)^4}$$

$$\therefore \boxed{u_2 = 2}$$

$$u_3 = z^3 \lim_{z\to\infty}\left[\bar{U}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+12}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2z^4\left(1-\frac{1}{z}\right)^4}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \lim_{z\to\infty} z^7\left[\frac{2+\frac{3}{z}+\frac{12}{z^2} - 2\left(1-\frac{1}{z}\right)^4}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \lim_{z\to\infty} \frac{2z+3+\frac{12}{z} - 2z\left(1-\frac{1}{z}\right)^4}{\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2\left[(z-1)^2\right]^2}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{\cancel{2z^4}+3z^3+12z^2-\cancel{2z^4}+4z^3-2z^2+4z^3-8z^2+4z-2z^2+4z-2}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{11z^3+8z-2}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \lim_{z\to\infty} \cancel{z^6}\left[\frac{11+\frac{8}{z^2}-\frac{2}{z^3}}{\cancel{z^6}\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \lim_{z\to\infty} \left[\frac{11+\frac{8}{z^2}-\frac{2}{z^3}}{\left(1-\frac{1}{z}\right)^4}\right]$$

$$\boxed{\therefore u_3 = 11}$$

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Inverse Z-transform;

If $Z(u_n) = \bar{U}(z)$ then $Z_T^{-1}[\bar{U}(z)] = u_n$

- $Z_T^{-1}\left[\frac{z}{z-1}\right] = 1^n$
- $Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$
- $Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] = n^1$
- $Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$
- $Z_T^{-1}\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$
- $Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^n n^1$

NOTE: 1) Inverse Z transform of algebric functions by partial ~~functions~~ fraction;

- Step 1: given $\bar{U}(z) = \frac{f(z)}{g(z)}$, we need to express, $g(z)$ in terms of non repeated linear factors.
- Step 2: We need to consider $\frac{\bar{U}(z)}{z}$ in the form of proper fraction and resolve into partial fraction
- Step 3: We multiple by z to have $\bar{U}(z)$ involving varies forms of the form
- Step 4: Finally we have to compute inverse Z transform to get u_n

1) Find the inverse Z_T of $\dfrac{z}{(z-1)(z-2)}$

$\Rightarrow$ $\dfrac{z}{(z-1)(z-2)} = \bar{u}(z)$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

multiple by $(z-1)(z-2)$

$1 = A(z-2) + B(z-1)$

when $z=1$

$A = -1$

when $z=2$

$B = 1$

$$\therefore \frac{\bar{u}(z)}{z} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$\bar{u}(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$\boxed{Z_T^{-1}[\bar{u}(z)] = -1^n + 2^n}$$

2) compute Z_T^{-1} of $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$:

$\Rightarrow$ Given is non repeated.

$$\frac{3z^2+2z}{(5z-1)(5z+2)} = \bar{u}(z)$$

$$\frac{\bar{u}(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)}$$

$$= \frac{A}{(5z-1)} + \frac{B}{(5z+2)}$$

Multiply by $(5z-1)(5z+2)$

$3z+2 = A(5z+2) + B(5z-1)$

when $z = \frac{1}{5}$

$3 \times \frac{1}{5} + 2 = A\left(5 \times \frac{1}{5} + 2\right) + B\left(5 \times \frac{1}{5} - 1\right)$

$A = \frac{13}{15}$

when $z = -\frac{2}{5}$

$3 \times -\frac{2}{5} + 2 = A\left(5 \times -\frac{2}{5} + 2\right) + B\left(5 \times -\frac{2}{5} - 1\right)$

$B = \frac{-4}{15}$

$$\therefore \frac{\bar{u}(z)}{z} = \frac{13}{15} \frac{1}{5(z-\frac{1}{5})} - \frac{4}{15} \frac{1}{5(z+\frac{2}{5})}$$

$$\bar{u}(z) = \frac{13}{75} \frac{z}{(z-\frac{1}{5})} - \frac{4}{75} \frac{z}{(z+\frac{2}{5})}$$

$$z_T^{-1}\left[\bar{u}(z)\right] = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(\frac{-2}{5}\right)^n$$

$$\boxed{\therefore u_n = \frac{1}{75}\left[13\left(\frac{1}{5}\right)^n - 4\left(\frac{-2}{5}\right)^n\right]}$$

3) Obtain z_T^{-1} of $\frac{2z^2+3z}{(z+2)(z-4)}$

$$\Rightarrow \quad \bar{u}(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

$$\frac{\bar{u}(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

$$= \frac{A}{z+2} + \frac{B}{z-4}$$

Multiply by $(z+2)(z-4)$

$$2z+3 = A(z-4) + B(z+2)$$

when $z=4$

$$2(4)+3 = A(4-4) + B(4+2)$$

$$B = \frac{11}{6}$$

when $z=-2$

$$2(-2)+3 = A(-2-4) + B(-2+2)$$

$$A = \frac{1}{6}$$

$$\therefore \frac{\bar{u}(z)}{z} = \frac{1}{6} \cdot \frac{1}{(z+2)} + \frac{11}{6} \cdot \frac{1}{(z-4)}$$

$$\bar{u}(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

$$\therefore Z_T^{-1}(\bar{u}(z)) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

$$\boxed{\therefore u_n = \frac{1}{6}\left[(-2)^n + 11(4)^n\right]}$$

6/1/22

If $g(z)$ contains of the form $(z-k)$, $(z-k)^2$, $(z-k)^3$, $(z-k)^4$ . . . .

$(z-k) \downarrow z$

$(z-k)^2 \downarrow kz$

$(z-k)^3 \downarrow kz^2+k^2z$

$(z-k)^4 \downarrow kz^3+4k^2z^2+k^3z$

multiple numerator by ↑ these.

NOTE: $z^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^n \cdot n^1$

$$z^{-1}\left[\frac{4z^2+16z}{(z-4)^3}\right] = 4^n \cdot n^2$$

1) Find z_T^{-1} of $\left[\frac{8z-z^3}{(4-z)^3}\right]$:

$\Rightarrow$ $\bar{U}(z) = \frac{8z-z^3}{(4-z)^3}$

Rewrite as:

$$\bar{U}(z) = \frac{z^3-8z}{(z-4)^3}$$

$$= \frac{A}{(z-4)} + \frac{B}{(z-4)^2} + \frac{C}{(z-4)^3}$$

$$= A \cdot \frac{z}{z-4} + B\frac{4z}{(z-4)^2} + C\frac{4z^2+16z}{(z-4)^3}$$

Multiple by $(z-4)^3$

$z^3-8z = A(z-4)^2 z + B\,4z(z-4) + C[4z^2+16z]$

divide by z

$z^2-8 = A(z-4)^2 + B4(z-4) + C[4z+16]$

when $z=4$

$8 = C(32)$

$C = \frac{8}{32} = \frac{1}{4}$

when $z=-4$

$8 = 64A - 32B$

$2A - B = \frac{1}{4}$

Eq. z^2 on both side

$1 = A$.

when $z=0$.

$-8 = 16A - 16B + 16C$

$$-\frac{1}{2} = 1 - B + \frac{1}{4}$$

$$-2 = 4 - 4B + 1$$

$$-2 - 5 = -4B$$

$$B = \frac{7}{4}$$

$$\therefore \bar{U}(z) = 1 \cdot \frac{z}{z-4} + \frac{7}{4} \frac{4z}{(z-4)^2} + \frac{1}{4} \frac{4z^2+16z}{(z-4)^3}$$

$$z_T^{-1}\left[\bar{U}(z)\right] = 4^n + \frac{7}{4}\left[4^n \cdot n\right] + \frac{1}{4}\left[4^n n^2\right]$$

$$\boxed{\therefore \; U_n = \frac{4^n}{4}\left[4 + 7n + n^2\right]}$$

2) Find z_T^{-1} of $\dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

$$\Rightarrow \quad \bar{U}(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$$

$$= \frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{(z-4)}$$

$$= A \cdot \frac{z}{z-2} + B \frac{2z}{(z-2)^2} + C \frac{2z^2+4z}{(z-2)^3} + D \frac{z}{z-4}$$

Multiple by $(z-2)^3(z-4)$

$$z^3 - 20z = Az(z-2)^2(z-4) + B2z(z-2)(z-4) + C(2z^2+4z)(z-4) + Dz(z-2)^3$$

$\div z$

$$z^2 - 20 = A(z-2)^2(z-4) + 2B(z-2)(z-4) + C(2z+4)(z-4) + D(z-2)^3$$

when $z = 4$,

$$-4 = D(2)^3$$

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$$D = \frac{-1}{2}$$

when $z = 2$

$$-16 = C(-16)$$

$$C = 1$$

$$0 = A + D$$

$$A = -D$$

$$\therefore A = \frac{1}{2}$$

when $z = 0$

$$-20 = -16A + 16B - 16C - 8D$$

$$\frac{20}{8} = 2\cdot\frac{1}{2} - 2B + 2(1) - \frac{1}{2}$$

$$\frac{40}{8} = 2 - 4B + 4 - 1$$

$$5 - 5 = 4B$$

$$\therefore B = 0.$$

$$\therefore \bar{U}(z) = \frac{1}{2}\frac{z}{z-2} + \frac{2z^2+4z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

$$z_T^{-1}\left[\bar{U}(z)\right] = \frac{1}{2}\left[2^n\right] + 2^n n^2 - \frac{1}{2}4^n$$

$$\therefore U_n = 2^{-1}\cdot 2^n + 2^n n^2 - 2^{-1}2^{2n}$$

$$\boxed{U_n = 2^{n-1} + 2^n n^2 - 2^{2n-1}}$$

3) Find z_T^{-1} of $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$

$$\Rightarrow \bar{U}(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4}$$

$z^3 - 5z^2 + 8z - 4 = 0$ $\qquad$ $z_1 + z_2 + z_3 = 5$

or

$z_1 = 1, \; z_2 = 2, \; z_3 = 2$ $\qquad$ $z_1 z_2 z_3 = 4$

$\therefore$ roots are

$(z-1)(z-2)(z-2)$

$= (z-1)(z-2)^2$

$$\therefore \bar{U}(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

~~Multiple by~~ $\cancel{(z-1)(z-2)^2}$

$\cancel{4z^2 - 2z = Az}$

$$= A \cdot \frac{z}{(z-1)} + B \cdot \frac{z}{(z-2)} + \frac{C\,2z}{(z-2)^2}$$

Multiple by $(z-1)(z-2)^2$

$$4z^2 - 2z = Az(z-2)^2 + Bz(z-1)(z-2) + C2z(z-1)$$

$\div z$

$$4z - 2 = A(z-2)^2 + B(z-1)(z-2) + 2C(z-1)$$

when $z = 2$

$6 = 2C(2-1)$

$C = 3$

when $z = 1$

$A = 2.$

when $z = 0$

$-2 = 4A + 2B - 2C$

$1 = -2(2) - B + 3$

$1 = -1 - B$

$B = -2.$

$\therefore \bar{U}(z) = 2\dfrac{z}{(z-1)} - 2\dfrac{z}{(z-2)} + 3\dfrac{2z}{(z-2)^2}$

$Z_T^{-1}[\bar{U}(z)] = 2.1^n - 2.2^n + 3.2^n n.$

$$\boxed{\therefore U_n = 2 - 2^{n+1} + 3n2^n}$$

7/1/22

→ <u>solution of difference equation using z-transform:</u>

$Z[U_{n+2}] = z^2\left[\bar{U}(z) - U_0 - \dfrac{U_1}{z}\right]$

$Z[U_{n+1}] = z\left[\bar{U}(z) - U_0\right]$

$Z^{-1}[\bar{U}(z)] = U_n.$

1) solve the difference equation $U_{n+2} + U_n = 0$:

$\Rightarrow Z[U_{n+2}] + Z[U_n] = Z(0) \quad \because U_n = 0$

$z^2\left[\bar{U}(z) - U_0 - \dfrac{U_1}{z}\right] + \bar{U}(z) = 0$

$\bar{U}(z)[z^2+1] = z^2 U_0 + zU_1$

$\therefore \bar{U}(z) = U_0\dfrac{z^2}{z^2+1} + U_1\dfrac{z}{z^2+1}$

$Z_T^{-1}[\bar{U}(z)] = U_0 . \cos(n\pi/2) + U_1 \sin n\pi/2$

$$\boxed{\therefore U_n = U_0 \cos n\frac{\pi}{2} + U_1 \sin n\frac{\pi}{2}}$$

Repeated question.

2) solve the ~~Rep~~ difference equation:

$u_{n+2} + 5u_{n+1} + 6u_n = 0$

$\Rightarrow$ Applying z transform OBS;

$z[u_{n+2}] + 5z[u_{n+1}] + 6z[u_n] = 0$

$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 5z\left[\bar{u}(z) - u_0\right] + 6\bar{u}(z) = 0$

$\bar{u}(z)\left[z^2 + 5z + 6\right] = z^2u_0 + zu_1 + 5zu_0$

$\bar{u}(z)\left[(z+2)(z+3)\right] = u_0(z^2+5z) + u_1 z$

$\bar{u}(z) = u_0\frac{(z^2+5z)}{(z+2)(z+3)} + u_1\frac{z}{(z+2)(z+3)}$

$= u_0 p(z) + u_1 q(z)$ — (1)

$p(z) = \frac{z^2+5z}{(z+2)(z+3)}$

$\frac{p(z)}{z} = \frac{z+5}{(z+2)(z+3)}$

$= \frac{A}{z+2} + \frac{B}{z+3}$

Multiply by $(z+2)(z+3)$

$z+5 = A(z+3) + B(z+2)$

when $z = -3$

$B = -2$

when $z = -2$

$A = 3$

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$$\therefore \frac{P(z)}{z} = \frac{3}{z+2} + \frac{-2}{z+3}$$

$$P(z) = 3\frac{z}{z+2} - 2\frac{z}{z+3}$$

$$Z_T^{-1}[P(z)] = 3.(-2)^n - 2.(-3)^n$$

$$q(z) = \frac{z}{(z+2)(z+3)}$$

$$\frac{q(z)}{z} = \frac{1}{(z+2)(z+3)}$$

$$= \frac{A}{z+2} + \frac{B}{z+3}$$

multiple by $(z+2)(z+3)$

$$1 = (z+3)A + B(z+2)$$

when $z = -3$

$$B = -1$$

when $z = -2$

$$1 = (-2+3)A + B(-2+2)$$

$$A = 1$$

$$\therefore \frac{q(z)}{z} = \frac{1}{z+2} + \frac{-1}{z+3}$$

$$q(z) = \frac{-z}{z+3} + \frac{z}{z+2}$$

$$Z_T^{-1}[q(z)] = (-2)^n - (-3)^n$$

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$\bar{U}(z) = U_0 P(z) + U_1 q(z)$

$$= U_0\left[3\frac{z}{z+2} - 2\frac{z}{z+3}\right] + U_1\left[\frac{z}{z+2} - \frac{z}{z+3}\right]$$

$$\boxed{\text{NOTE: } z^{-1}\left[\frac{z}{z-2}\right] = 2^n \qquad z^{-1}\left[\frac{z}{z+2}\right] = (-2)^n}$$

$$\therefore z_T^{-1}\left[\bar{U}(z)\right] = U_0\left[3(-2)^n - 2(-3)^n\right] + U_1\left[(-2)^n - (-3)^n\right]$$

$$U_n = 3U_0(-2)^n - 2U_0(-3)^n + U_1(-2)^n - u_1(-3)^n$$

$$\therefore u_n = (-2)^n\left[3u_0 + u_1\right] - (-3)^n\left[2u_0 + u_1\right]$$

3) solve the difference equation:

$y_{n+2} - 4y_n = 0$; given that $y_0 = 0$ and $y_1 = 2$.

$\Rightarrow$ Apply z transforms OBS:

$$z\left[y_{n+2}\right] - 4z\left[y_n\right] = 0$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 4\bar{y}(z) = 0$$

$$\bar{y}(z)\left[z^2 - 4\right] = z^2 y_0 + z y_1$$

$$\bar{y}(z)\left[(z-2)(z+2)\right] = 2z$$

$$\bar{y}(z) = \frac{2z}{(z-2)(z+2)}$$

$$\frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)}$$

$$= \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

Multiple by $(z-2)(z+2)$

$$2 = A(z+2) + B(z-2)$$

when $z = 2$

$$A = \frac{1}{2}$$

when $z = -2$

$$B = \frac{-1}{2}.$$

$$\therefore \frac{\bar{y}(z)}{z} = \frac{1}{2}\frac{1}{(z-2)} - \frac{1}{2}\frac{1}{(z+2)}$$

$$\bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$$

$$Z_T^{-1}(\bar{y}(z)) = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

$$\boxed{\therefore y_n = \frac{1}{2}\left[2^n - (-2)^n\right]}$$

11/1/22..

1) $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$

$\Rightarrow$ Apply z transfrom OBS:

$$z[y_{n+2}] - 6z[y_{n+1}] + 9z[y_n] = z[3^n]$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 6z\left[\bar{y}(z) - y_0\right] + 9\bar{y}(z) = \frac{z}{z-3}$$

$$\bar{y}(z)[z^2-6z+9]-z^2y_0+6zy_0-zy_1=\frac{z}{z-3}$$

$$\bar{y}(z)[(z-3)^2]-y_0(z^2-6z)-zy_1=\frac{z}{z-3}$$

$$\bar{y}(z)[(z-3)^2]=y_0(z^2-6z)+zy_1+\frac{z}{z-3}$$

$$\bar{y}(z)=\frac{y_0(z^2-6z)}{(z-3)^2}+y_1\frac{z}{(z-3)^2}+\frac{z}{(z-3)^3}$$

$$\bar{y}(z)=y_0\,p(z)+y_1\,q(z)+r(z) \quad \text{—(1)}$$

consider:

$$p(z)=\frac{z^2-6z}{(z-3)^2}=\frac{A(z)}{z-3}+\frac{B(3z)}{(z-3)^2}$$

$$z^2-6z=Az(z-3)+3Bz$$

$$z-6=A(z-3)+3B$$

when $z=3$

$$B=-1$$

To find A Keep $z=0$

$$-6=-3A+3B$$

$$2+B=A$$

$$2-1=A$$

$$A=1$$

$$p(z)=\frac{z}{z-3}-\frac{3z}{(z-3)^2} \quad \text{—(a)}$$

$$q(z)=\frac{z}{(z-3)^2}=\frac{Az}{z-3}+\frac{B3z}{(z-3)^2}$$

$$z=Az(z-3)+3Bz$$

$$1=A(z-3)+3B$$

when $z=3$

$$B=1/3$$

when $z=0$

$$1 = -3A + 3B$$

$$-\frac{1}{3} + B = A$$

$$\cancel{-\frac{1}{3} - 3 = A}$$

$$\cancel{A = \frac{4}{3}}$$

$$-\frac{1}{3} + \frac{1}{3} = A$$

$$A = 0.$$

$$q(z) = \frac{1}{3} \cdot \frac{3z}{(z-3)^2} \quad \text{—(b)}$$

$$r(z) = \frac{z}{(z-3)^3} \; \cancel{= \frac{A}{z-3} + \frac{B}{(z-3)^2} + \frac{C}{(z-3)^3}}$$

$$= \frac{Az}{z-3} + \frac{B\,3z}{(z-3)^2} + \frac{C(3z^2+9z)}{(z-3)^3}$$

$$z = Az(z-3)^2 + 3Bz(z-3) + C(3z^2+9z)$$

$$1 = A(z-3)^2 + 3B(z-3) + C(3z+9)$$

when $z=3$

$$1 = 0 + 0 + C(18)$$

$$C = \frac{1}{18}$$

coefficients of z^2

$$A = 0$$

Keep $z=0$.

$$1 = -9B + 9C$$

$$-\frac{1}{9} + C = B$$

$$B = -\frac{1}{18}.$$

$$r(z) = -\frac{1}{18} \frac{3z}{(z-3)^2} + \frac{1}{18} \frac{3z^2+9z}{(z-3)^3} \quad \text{—(c)}$$

From eq(n) (1), (a), (b), (c)

$$\bar{y}(z) = y_0\left[\frac{z}{z-3} - \frac{3z}{(z-3)^2}\right] + y_1\left[\frac{1}{3} \cdot \frac{3z}{(z-3)^2}\right] + \left[-\frac{1}{18} \frac{3z}{(z-3)^2} + \frac{1}{18} \frac{3z^2+9z}{(z-3)^3}\right]$$

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Applying Z inverse OBS:

$$y_n = y_1\left[3^n - 3^n(n)\right] + y_1\left[\frac{1}{3}(3^n n^1)\right] - \frac{1}{18}(3^n n^1) + \frac{1}{18}(3^n n^2)$$

12/1/22

1) $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = 0, y_1 = 0$.

$\Rightarrow$ Apply Z transfrom OBS:

$$z(y_{n+2}) + 6z(y_{n+1}) + 9z(y_n) = z(2^n)$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 6z\left[\bar{y}(z) - y_0\right] + 9\bar{y}(z) = \frac{z}{z-2}$$

Given that $y_0, y_1 = 0$

$$\therefore z^2[\bar{y}(z)] + 6z[\bar{y}(z)] + 9\bar{y}(z) = \frac{z}{z-2}$$

$$\bar{y}(z)\left[z^2 + 6z + 9\right] = \frac{z}{z-2}$$

$$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$= \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$$

$$= A\frac{z}{z-2} + B\frac{(-z)}{z+3} + C\frac{(-3z)}{(z+3)^2}$$

Multiple by $(z-2)(z+3)^2$

$$z = Az(z+3)^2 - Bz(z-2)(z+3) - 3Cz(z-2)$$

$$1 = A(z+3)^2 - B(z-2)(z+3) - 3C(z-2)$$

when $z = -3$

$$C = \frac{1}{15}$$

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when $Z=2$

$A = \frac{1}{25}$

when $Z=0$

$1 = 9A + 6B + 6C$

$6B = 1 - \frac{9}{25} - \frac{6}{15}$

$= 1 - \frac{9}{25} - \frac{2}{5}$

$= \frac{25-9-10}{25}$

$6B = \frac{6}{25}$

$\therefore B = \frac{1}{25}$

$\therefore \bar{y}(z) = \frac{1}{25} \frac{z}{z-2} + \frac{1}{25} \frac{(-z)}{(z-(-3))} + \frac{1}{15} \frac{(-3z)}{(z-(-3))^2}$

$Z_T^{-1}(\bar{y}(z)) = \frac{1}{25} 2^n + \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n . n'$

$\therefore y_n = \frac{1}{5}\left[\frac{1}{5} 2^n + \frac{1}{5}(-3)^n + \frac{1}{3}(-3)^n . n'\right]$

2) $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$

$\Rightarrow$ Apply Z transfrom OBS;

$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 2z\left[\bar{u}(z) - u_0\right] + \bar{u}(z) = 3\frac{z}{(z-1)^2} + 5\frac{z}{(z-1)}$

$\bar{u}(z)\left[z^2 - 2z + 1\right] - u_0\left[z^2 - 2z\right] - zu_1 = 3\frac{z}{(z-1)^2} + 5\frac{z}{(z-1)}$

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$$\bar{u}(z)[z-1]^2 = \frac{3z}{(z-1)^2} + \frac{5z}{(z-1)} + u_0(z^2-2z) + zu_1$$

$$\bar{u}(z) = \frac{3z}{(z-1)^4} + \frac{5z}{(z-1)^3} + \frac{u_0(z^2-2z)}{(z-1)^2} + \frac{zu_1}{(z-1)^2}$$

Let us assume:

$$\bar{u}(z) = p(z) + q(z) + u_0 r(z) + u_1 s(z) \quad -(1)$$

$$p(z) = \frac{3z}{(z-1)^4}$$

$$= A\frac{z}{z-1} + B\frac{z}{(z-1)^2} + \frac{C(z^2+z)}{(z-1)^3} + \frac{D(z^3+4z^2+z)}{(z-1)^4}$$

$$3z = Az(z-1)^3 + Bz(z-1)^2 + Cz(z+1)(z-1) + Dz(z^2+4z+1)$$

$$3 = A(z-1)^3 + B(z-1)^2 + C(z+1)(z-1) + D(z^2+4z+1)$$

when $z=1$ $\quad D = \frac{1}{2}$ | equating coff. of z^3 $\quad 0 = A$

when $z=-1$

$3 = 4B - 2\times\frac{1}{2}$

$4 = 4B$

$B = 1$

when $z=0$

$3 = B - C + D$

$3 = 1 - C + \frac{1}{2}$

$6 = 3 - 2C$

$2C = -3$

$C = -3/2$

$$p(z) = \frac{z}{(z-1)^2} - \frac{3}{2}\frac{(z^2+z)}{(z-1)^3} + \frac{1}{2}\frac{(z^3+4z^2+z)}{(z-1)^4} \quad \rightarrow (a)$$

$$q(z) = \frac{5z}{(z-1)^3}$$

$$= A\frac{z}{z-1} + B\frac{z}{(z-1)^2} + C\frac{(z^2+z)}{(z-1)^3}$$

$$5z = Az(z-1)^2 + Bz(z-1) + C(z^2+z)$$

$$5 = A(z-1)^2 + B(z-1) + C(z+1)$$

when $z=1$ $C = \frac{5}{2}$ | Eq. coff. of z^2 $A=0$ | when $z=0$ $5 = A - B + C$ $B = \frac{5}{2} - 5 = \frac{-5}{2}$

$$q(z) = -\frac{5}{2}\frac{z}{(z-1)^2} + \frac{5}{2}\frac{z^2+z}{(z-1)^3} \quad \text{—(b)}$$

$$r(z) = \frac{z^2-2z}{(z-1)^2}$$

$$= A\frac{z}{(z-1)} + \frac{Bz}{(z-1)^2}$$

$$(z-2)z = zA(z-1) + Bz$$

$$z-2 = A(z-1) + B$$

when $z=1$ $B=-1$ | when $z=0$ $-2 = -A-1$ $A=1$

$$r(z) = \frac{z}{z-1} - \frac{z}{(z-1)^2} \quad \text{—(c)}$$

$$s(z) = \frac{z}{(z-1)^2}$$

$$= A\frac{z}{(z-1)} + \frac{Bz}{(z-1)^2}$$

$$z = Az(z-1) + Bz$$

$$1 = A(z-1) + B$$

when $z=1$ $B=1$ | when $z=0$ $1 = -A + B$ $1 = -A + 1$ $A = 0$

$$S(z) = \frac{z}{(z-1)^2} \quad \text{— (d)}$$

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From eq(n) (1), (a), (b), (c), (d)

$$\bar{u}(z) = \frac{z}{(z-1)^2} - \frac{3}{2}\frac{(z^2+z)}{(z-1)^3} + \frac{1}{2}\frac{(z^3+4z^2+z)}{(z-1)^4}$$

$$- \frac{5}{2}\frac{z}{(z-1)^2} + \frac{5}{2}\frac{z^2+z}{(z-1)^3} + \frac{zu_0}{z-1} - \frac{zu_0}{(z-1)^2}$$

$$+ \frac{zu_1}{(z-1)^2}$$

$$\cancel{z_T^{-1}[\bar{u}(z)] = \frac{z}{z-1} - \frac{3}{2}\frac{z}{(z-1)^2} + \frac{z^2+z}{(z-1)^3} + \frac{1}{2}\frac{(z^3+4z^2+z)}{(z-1)^4}}$$

$$\cancel{u_n = 1^n - \frac{3}{2}}$$

$$z_T^{-1}[\bar{u}(z)] = 1^n(n)' - \frac{3}{2}1^n n^2 + \frac{1}{2}1^n n^3 - \frac{5}{2}1^n(n)'$$

$$+ \frac{5}{2}1^n n^2 + u_0 1^n - u_0 1^n n' + u_1 1^n n'$$

$$u_n = n - \frac{3}{2}n^2 + \frac{1}{2}n^3 - \frac{5}{2}n + \frac{5}{2}n^2 + u_0 - u_0 n + u_1 n$$

$$= \frac{1}{2}n^3 + n^2 - n\left[\frac{3}{2} + u_0 - u_1\right] + u_0$$

$$\boxed{\therefore\ u_n = \frac{1}{2}n^3 + n^2 - n\left[\frac{3}{4} + u_0 - u_1\right] + u_0}$$

3) Find the equation of the system:
$y_{n+2} - 4y_{n+1} + 3y_n = u_n$ with $y_0 = 0, y_1 = 1$
$u_n = 1$, for $n = 0, 1, 2, 3,$ by z-transform:

$\Rightarrow\ y_{n+2} - 4y_{n+1} + 3y_n = 1$

Apply z transform OBS.

$$z^2\left[\bar{y}(z)-y_0-\frac{y_1}{z}\right]-4z\left[\bar{y}(z)-y_0\right]+3\bar{y}(z)=1\cdot\frac{z}{z-1}$$

$$\bar{y}(z)\left[z^2-4z+3\right]-z=\frac{z}{z-1}$$

$$\bar{y}(z)\left[(z-3)(z-1)\right]-z=\frac{z}{z-1}$$

$$\bar{y}(z)\left[(z-3)(z-1)\right]=\frac{z}{z-1}+z$$

$$\bar{y}(z)=\frac{z}{(z-1)^2(z-3)}+\frac{z}{(z-1)(z-3)}$$

$$\bar{y}(z)=p(z)+q(z) \quad \text{—(1)}$$

$$p(z)=\frac{z}{(z-3)(z-1)^2}$$

$$=\frac{Az}{z-3}+\frac{Bz}{z-1}+\frac{Cz}{(z-1)^2}$$

$$z=Az(z-1)^2+Bz(z-3)(z-1)+Cz(z-3)$$

$$1=A(z-1)^2+B(z-3)(z-1)+C(z-3)$$

when $z=1$

$C=-1/2$

when $z=3$

$A=1/4$

when $z=0$

$1=A+3B-3C$

$3B=1-\frac{1}{4}-\frac{3}{2}$

$B=-1/4$

$$p(z)=\frac{1}{4}\frac{z}{z-3}-\frac{1}{4}\frac{z}{z-1}-\frac{1}{2}\frac{z}{(z-1)^2} \quad \text{—(a)}$$

$$q(z) = \frac{z}{(z-1)(z-3)}$$

$$= A\frac{z}{z-1} + B\frac{z}{z-3}$$

$$1 = A(z-3) + B(z-1)$$

when $z=3$

$B = 1/2$

when $z = 1$

$A = -1/2$

$$q(z) = -\frac{1}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z}{z-3} \quad \text{--- (b)}$$

From eq. (1), (a), (b):

$$\bar{y}(z) = \frac{1}{4}\frac{z}{z-3} - \frac{1}{4}\frac{z}{z-1} - \frac{1}{2}\frac{z}{(z-1)^2} - \frac{1}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z}{z-3}$$

$$Z_T^{-1}[\bar{y}(z)] = \frac{3}{4}\frac{z}{z-3} - \frac{3}{4}\frac{z}{z-1} - \frac{1}{2}\frac{z}{(z-1)^2}$$

$$y_n = \frac{3}{4}\cdot 3^n - \frac{3}{4}1^n - \frac{1}{2}1^n \cdot n^1$$

$$\boxed{\therefore y_n = \frac{3}{4}\cdot 3^n - \frac{3}{4} - \frac{1}{2}n}$$

13/1/22

Numerical solution of ordinary diff. eq(n).
of first order, first degree.

consider 1^{st} order, first degree of the form
$\frac{dy}{dx} = f(xy)$ with an initial condition
$y(x_0) = y_0$ that $y = y_0$ at $x = x_0$, this
problem of finding y is called initial
value problem.

→ Tylor series method;

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) \dots$$

The solution $y(x)$ is approximated in fourier series in $(x - x_0)$ using taylor series we can find the value of y for various values of x in the neighbourhood of x_0.

We know that $\frac{dy}{dx} = y'$.

1) Use taylor series method to find y at $x = 0.1, 0.2, 0.3$. considering upto 3^{rd} degree, given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$.

$\Rightarrow$ $\frac{dy}{dx} = y' = x^2 + y^2$ $\qquad$ $y(0) = y(x_0) = 1$

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0).$$

$$y' = x^2 + y^2$$

$$y'(x_0) = x_0^2 + y_0^2 = 0 + 1$$

$$\boxed{\therefore y'_0 = 1}$$

$$y'' = 2x + 2yy'$$

$$y''(x_0) = 2x_0 + 2y_0y'_0$$

$$= 2(0) + 2(1)(1)$$

$$\boxed{\therefore y''_0 = 2}$$

$$y''' = 2[1 + yy'' + (y')^2]$$

$$y'''(x_0) = 2[1 + y_0y''_0 + (y'_0)^2]$$

$$= 2[1 + 1(2) + (1)^2]$$

$$\boxed{\therefore y'''_0 = 8}$$

$$y(x) = 1 + (x-0)1 + \frac{(x-0)^2}{2\times1}2 + \frac{(x-0)^3}{3\times2\times1}8$$

$$y(0.1) = 1 + (0.1-0)1 + (0.1-0)^2 + \frac{(0.1-0)^3}{3}\times4$$

$$= 1 + 0.1 + 0.01 + 0.00133$$

$$\boxed{\therefore y(0.1) = 1.11133}$$

$$y(0.2) = 1 + (0.2-0)1 + (0.2-0)^2 + (0.2-0)^3 \cdot \frac{4}{3}$$

$$= 1 + 0.2 + 0.04 + 0.010666$$

$$\boxed{\therefore y(0.2) = 1.25066}$$

$$y(0.3) = 1 + (0.3-0)1 + (0.3-0)^2 + (0.3-0)^3 \cdot \frac{4}{3}$$

$$= 1 + 0.3 + 0.09 + 0.036$$

$$\boxed{\therefore y(0.3) = 1.426}$$

2) Find y at $x = 1.02$, correct to five decimal places, given $dy = (xy-1)\,dx$ and $y = 2$ at $x = 1$.

$\Rightarrow$ Apply Taylor series method:

$$y_0 = 2, \quad x_0 = 1, \quad \frac{dy}{dx} = xy - 1$$

$$y' = xy - 1$$

$$y'(x_0) = x_0 y_0 - 1$$

$$= 1.2 - 1$$

$$\boxed{\therefore y'_0 = 1}$$

$$y'' = xy' + y(1)$$

$$y''(x_0) = x_0 y'_0 + y_0$$

$$= 1.1 + 2$$

$$\boxed{\therefore y''_0 = 3}$$

$$y''' = xy'' + y'(1) + y'$$

$$y'''(x_0) = x_0 y''_0 + y'_0 + y'_0$$

$$= 1.3 + 1 + 1$$

$$\boxed{\therefore y'''(x_0) = 5}$$

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$$y(1.02) = 2 + (1.02-1)1 + \frac{(1.02-1)^2}{2\times1}.3 + \frac{(1.02-1)^3}{3\times2\times1}.5$$

$$= 2 + 0.02 + 0.0006 + 0.00000666$$

$$\boxed{\therefore y(1.02) = 2.02060666}$$