HW 2

1. $\cos\theta = \dfrac{a \cdot b}{\|a\|\,\|b\|}$ $\qquad \overrightarrow{OC} = C - O = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\overrightarrow{OD} = D - O = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$$\Rightarrow \cos\theta = \frac{(-1 \cdot -1) + (1 \cdot -1) + (1 \cdot -1)}{\sqrt{(-1)^2 + 1^2 + 1^2} \cdot \sqrt{(-1)^2 + (-1)^2 + (-1)^2}} =$$

$$= \frac{-1}{\sqrt{3} \cdot \sqrt{3}} = \frac{-1}{3}$$

$\theta = \arccos(-1/3) = 109.47^\circ$ is angle b/w $\overrightarrow{OC}$ & $\overrightarrow{OD}$

$\overrightarrow{AB} = B - A = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$$\Rightarrow \cos\theta = \frac{(0 \cdot -1) + (2 \cdot 1) + (-2 \cdot 1)}{\sqrt{0^2 + 2^2 + (-2)^2} \cdot \sqrt{(-1)^2 + 1^2 + 1^2}} = \frac{0}{\sqrt{8} \cdot \sqrt{3}} = 0$$

$\theta = \arccos(0) = 90^\circ$ is the angle b/w $\overrightarrow{AB}$ & $\overrightarrow{OC}$

2. $\tilde{u} = u - \dfrac{u^T v}{v^T v} v$

$$\frac{u^T v}{v^T v} v = \frac{(1 \cdot 1) + (2 \cdot -1) + (1 \cdot 2)}{(1 \cdot 1) + (-1 \cdot -1) + (2 \cdot 2)} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix}$$

$$\tilde{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 13/6 \\ 2/3 \end{bmatrix}$$

3. Convert Q into identity matrix for 3x3

$$\left[\begin{array}{ccc|ccc} 1/3 & 2/3 & 2/3 & 1 & 0 & 0 \\ 2/3 & 1/3 & -2/3 & 0 & 1 & 0 \\ -2/3 & 2/3 & -1/3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2+R_1 \\ R_2+R_3 \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ -2/3 & 2/3 & -1/3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 2/3R_1+R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 4/3 & -1/3 & 2/3 & 2/3 & 1 \end{array}\right] \quad \begin{array}{l} R_1-R_2 \\ R_3-4/3R_2 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2/3 & -2/3 & -1/3 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ R_2+R_3 \\ \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 2/3 & -2/3 \\ 0 & 1 & 0 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & 2/3 & -2/3 & -1/3 \end{array}\right]$$

$$Q^{-1} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$

- Q is orthogonal when det of Q is ± 1
- det of $Q = 1/3 \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix} - 2/3 \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix} + 2/3 \begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}$

$$= 1/3(-1/9 + 4/9) - 2/3(-2/9 - 4/9) + 2/3(4/9 + 2/9)$$

$$= 1/3(1/3) - 2/3(-2/3) + 2/3(2/3)$$

$$= 1/9 + 4/9 + 4/9 = 1$$

4. $Tr(A) = A_{11} + A_{22} + A_{33} + A_{44} = 1 + 2 + 6 + 3 = 12$

5. $\det\begin{pmatrix} -1 & 5 & 3 \\ 11 & 4 & 2 \\ 6 & 12 & -5 \end{pmatrix} = -1 \cdot \det\begin{pmatrix} 4 & 2 \\ 12 & -5 \end{pmatrix} - 5\det\begin{pmatrix} 11 & 2 \\ 6 & -5 \end{pmatrix} + 3\det\begin{pmatrix} 11 & 4 \\ 6 & 12 \end{pmatrix}$

$= -1 \cdot (-20 - 24) - 5(-55 - 12) + 3(132 - 24)$

$= 44 + 335 + 324 = 703$

$\det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} = A_{11} \cdot A_{22} \cdot A_{33} \cdot A_{44} = 1 \cdot 2 \cdot 6 \cdot 3 = 36$

6. $D_x = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 7 & 7 \\ 7 & 5 & 2 \end{bmatrix} \Rightarrow \det(D_x) = 2(14-35) - 3(-2-49) + 2(-5-49)$

$= -42 + 153 - 108 = 3$

$D_y = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 7 \\ 2 & 7 & 2 \end{bmatrix} \Rightarrow \det(D_y) = 1(-2-49) - 2(4-14) + 2(14+2)$

$= -51 + 20 + 32 = 1$

$D_z = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & -1 \\ 2 & 5 & 7 \end{bmatrix} \Rightarrow \det(D_z) = 1(49+5) - 3(14+2) + 2(10-14)$

$= 54 - 48 - 8 = -2$

$D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 7 \\ 2 & 5 & 2 \end{bmatrix} \Rightarrow \det(D) = 1(14-35) - 3(4-14) + 2(10-14)$

$= -21 + 30 - 8 = 1$

$x = \frac{D_x}{D} = \frac{3}{1} = 3$; $y = \frac{D_y}{D} = \frac{1}{1} = 1$; $z = \frac{D_z}{D} = \frac{-2}{1} = -2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

7. $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 5 \\ 2 & 6 & 8 \end{bmatrix} \underset{\Rightarrow}{R_3 - 2R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ Rank is 2

$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 5R_1 \\ R_4 - 4R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -5 & 1 \\ 0 & 13 & -4 & -2 \\ 0 & 6 & -3 & -1 \end{bmatrix} \begin{matrix} R_3 - {}^{13}/_5 R_2 \\ R_4 - {}^{6}/_5 R_2 \\ \\ \Rightarrow \end{matrix}$

$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -5 & 1 \\ 0 & 0 & 9 & -23/5 \\ 0 & 0 & 3 & -11/5 \end{bmatrix} \underset{\Rightarrow}{R_4 - {}^{1}/_3 R_3} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -5 & 1 \\ 0 & 0 & 9 & -23/5 \\ 0 & 0 & 0 & -2/3 \end{bmatrix}$ Rank is 4