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HW 1 - Vector & Matrices

HW 1

1. $\cos\theta = \dfrac{x^T y}{\|x\|\|y\|}$

Find norm of x & y

$\|x\| = \sqrt{(-2)^2+1^2+0^2+3^2} = \sqrt{14}$

$\|y\| = \sqrt{1^2+3^2+(-2)^2+0^2} = \sqrt{14}$

Use equation to find cos θ

$\cos\theta = [-2\ 1\ 0\ 3]\begin{bmatrix}1\\3\\-2\\0\end{bmatrix} / (\sqrt{14})^2$

$= [(-2)(1)+(1)(3)+(0)(-2)+(3)(0)]/14 = \boxed{1/14}$

2. $X \times Y = \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} i - \begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} j + \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} k$

Find determinant of X×Y

$= (-2-0)i - (4-0)j + (-6-1)k$

$= -2i - 4j - 7k$

$\|X \times Y\| = \sqrt{2^2+(-4)^2(-7)^2} = \boxed{\sqrt{69}}$

Another Approach

$\|X \times Y\| = \|X\|\|Y\| \sin\theta$

$\|X\| = \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$

$\|Y\| = \sqrt{1^2+3^2+(-2)^2} = \sqrt{14}$

Use cos θ to find θ

$\theta = \cos^{-1}\left(\dfrac{x^T y}{\|x\|\|y\|}\right) = \cos^{-1}\left(\dfrac{(-2)(1)+(1)(3)+(0)(-2)}{\sqrt{5}\cdot\sqrt{14}}\right)$

$= \cos^{-1}\left(\dfrac{1}{\sqrt{70}}\right) \approx 1.4059$

$\|X \times Y\| = \sqrt{5}\cdot\sqrt{14}\cdot \sin\left(\cos^{-1}\left(\dfrac{1}{\sqrt{70}}\right)\right) = \sqrt{70}\cdot\sqrt{1-(1/\sqrt{70})^2}$

$= \sqrt{70}\cdot\sqrt{69/70} = \underline{\sqrt{69}}$

Derivation of Orthonormal vectors

3. $u_1 = v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\quad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2^2+2^2+1^2}} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$u_2 = v_2 - \text{proj}_{u_1}(v_2) = v_2 - \frac{v_2^T u_1}{u_1^T u_1} u_1$

$= \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{(2)(1)+(2)(1)+(1)(5)}{(2)^2+(2)^2+(1)^2} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

$e_2 = \frac{1}{\sqrt{(-1)^2+(-1)^2+4^2}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \frac{\sqrt{2}}{6} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix}$

$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) = v_3 - \frac{v_3^T u_1}{u_1^T u_1} u_1 - \frac{v_3^T u_2}{u_2^T u_2} u_2$

$= \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \frac{(-3)(2)+2(2)+1(1)}{2^2+2^2+1^2} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{(-3)(-1)+(2)(-1)+1(4)}{(-1)^2+(-1)^2+4^2} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \frac{(-1)}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{18} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2/9 \\ -2/9 \\ -1/9 \end{bmatrix} - \begin{bmatrix} -5/18 \\ -5/18 \\ 10/9 \end{bmatrix}$

$= \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix}$

$e_3 = \frac{1}{\sqrt{(-5/2)^2+(5/2)^2+0^2}} \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{5} \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$

$\{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \right\}$

Verify orthonormal

$$e_1^T e_2 = (2/3)(-\sqrt{2}/6) + (2/3)(-\sqrt{2}/6) + 1/3(2\sqrt{2}/3)$$

$$= (-\sqrt{2}/9) + (-\sqrt{2}/9) + 2\sqrt{2}/9 = 0$$

$$e_1^T e_3 = (2/3)(-\sqrt{2}/2) + (2/3)(\sqrt{2}/2) + (1/3)(0)$$

$$= (-\sqrt{2}/3) + \sqrt{2}/3 + 0 = 0$$

$$e_2^T e_3 = (-\sqrt{2}/6)(-\sqrt{2}/2) + (-\sqrt{2}/6)(\sqrt{2}/2) + (\sqrt{2}/2/3)(0)$$

$$= 1/6 - 1/6 + 0 = 0$$

$$\|e_1\| = (2/3)^2 + (2/3)^2 + (1/3)^2 = 1$$

$$\|e_2\| = (-\sqrt{2}/6)^2 + (-\sqrt{2}/6)^2 + (2\sqrt{2}/3)^2 = 1$$

$$\|e_3\| = (-\sqrt{2}/2)^2 + (\sqrt{2}/2)^2 + (0)^2 = 1$$

4. $AB = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 8 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & -2 \\ -1 & 4 & 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (1)(2)+(-1)(-1) & (1)(1)+(-1)(4) & (1)(6)+(-1)(3) & (1)(-2)+(-1)(1) \\ (3)(2)+(2)(-1) & (3)(1)+(2)(4) & (3)(6)+(2)(3) & (3)(-2)+(2)(1) \\ (5)(2)+(8)(-1) & (5)(1)+(8)(4) & (5)(6)+(8)(3) & (5)(-2)+(8)(1) \\ (4)(2)+(2)(-1) & (4)(1)+(2)(4) & (4)(6)+(2)(3) & (4)(-2)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 3 & -3 \\ 4 & 11 & 24 & -4 \\ 2 & 37 & 54 & -2 \\ 6 & 12 & 30 & -6 \end{bmatrix}$$

$BA = \begin{bmatrix} 2 & 1 & 6 & -2 \\ -1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 8 \\ 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (2)(1)+(1)(3)+(6)(5)+(-2)(4) & (2)(-1)+(1)(2)+(6)(8)+(-2)(2) \\ (-1)(1)+(4)(3)+(3)(5)+(1)(4) & (-1)(-1)+(4)(2)+(3)(8)+(1)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 44 \\ 30 & 35 \end{bmatrix}$$

5. $AB = \begin{bmatrix}1\\3\\5\\4\end{bmatrix}\begin{bmatrix}2 & 1 & 6 & -2\end{bmatrix} + \begin{bmatrix}-1\\2\\8\\2\end{bmatrix}\begin{bmatrix}-1 & 4 & 3 & 1\end{bmatrix}$

$$= \begin{bmatrix}(1)(2) & (1)(1) & (1)(6) & (1)(-2)\\(3)(2) & (3)(1) & (3)(6) & (3)(-2)\\(5)(2) & (5)(1) & (5)(6) & (5)(-2)\\(4)(2) & (4)(1) & (4)(6) & (4)(-2)\end{bmatrix} + \begin{bmatrix}(-1)(-1) & (-1)(4) & (-1)(3) & (-1)(1)\\(2)(-1) & (2)(4) & (2)(3) & (2)(1)\\(8)(-1) & (8)(4) & (8)(3) & (8)(1)\\(2)(-1) & (2)(4) & (2)(3) & (2)(1)\end{bmatrix}$$

$$= \begin{bmatrix}2 & 1 & 6 & -2\\6 & 3 & 18 & -6\\10 & 5 & 30 & -10\\8 & 4 & 24 & -8\end{bmatrix} + \begin{bmatrix}1 & -4 & -3 & -1\\-2 & 8 & 6 & 2\\-8 & 32 & 24 & 8\\-2 & 8 & 6 & 2\end{bmatrix} = \begin{bmatrix}3 & -3 & 3 & -3\\4 & 11 & 24 & -4\\2 & 37 & 54 & -2\\6 & 12 & 30 & -6\end{bmatrix}$$

$BA = \begin{bmatrix}2\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}3 & 2\end{bmatrix} + \begin{bmatrix}6\\3\end{bmatrix}\begin{bmatrix}5 & 8\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}4 & 2\end{bmatrix}$

$$= \begin{bmatrix}2(1) & 2(-1)\\-1(1) & -1(-1)\end{bmatrix} + \begin{bmatrix}1(3) & 1(2)\\4(3) & 4(2)\end{bmatrix} + \begin{bmatrix}6(5) & 6(8)\\3(5) & 3(8)\end{bmatrix} + \begin{bmatrix}-2(4) & -2(2)\\1(4) & 1(2)\end{bmatrix}$$

$$= \begin{bmatrix}27 & 44\\30 & 35\end{bmatrix}$$

6. $|A| = (1)(3) - (2)(-1) = 3 - (-2) = 5 \neq 0$ which means A^{-1} exist

A and A^{-1} are inverse if $AA^{-1} =$ Identity matrix

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix}$$

Verify

$A \cdot A^{-1}$ must $= I = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix} = \begin{bmatrix} (1)(3/5) + (2)(1/5) & (1)(-2/5) + (2)(1/5) \\ (-1)(3/5) + 3(1/5) & (-1)(-2/5) + (3)(1/5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7.

$$\begin{cases} 1x_1 - 1x_2 + 2x_3 + 1x_4 = 1 \\ 3x_1 + 2x_2 + 1x_3 + 4x_4 = 1 \\ 5x_1 + 8x_2 + 6x_3 + 3x_4 = 1 \\ 4x_1 + 2x_2 + 5x_3 + 3x_4 = {}^-1 \end{cases}$$

Forward Elimination of x_1

$$\Rightarrow \begin{cases} 1x_1 - 1x_2 + 2x_3 + 1x_4 = 1 \\ 5x_2 - 5x_3 + 1x_4 = {}^-2 \\ 13x_2 - 4x_3 - 2x_4 = {}^-4 \\ 6x_2 - 3x_3 - 1x_4 = {}^-5 \end{cases}$$

Forward elimination of x_2

$$\Rightarrow \begin{cases} 1x_1 - 1x_2 + 2x_3 + 1x_4 = 1 \\ 5x_2 - 5x_3 + 1x_4 = {}^-2 \\ 9x_3 - 23/5\, x_4 = 6/5 \\ 3x_3 - 11/5\, x_4 = -13/5 \end{cases}$$

Forward elimination of x_3

$$\Rightarrow \begin{cases} 1x_1 - 1x_2 + 2x_3 + 1x_4 = 1 \\ 5x_2 - 5x_3 + 1x_4 = {}^-2 \\ 9x_3 - 23/5\, x_4 = 6/5 \\ -2/3\, x_4 = {}^-3 \end{cases}$$

$\Rightarrow x_4 = {}^-3 \cdot -3/2 = 9/2$

$x_3 = (6/5 - (9/2)(-23/5))/9 = 73/30$

$x_2 = ({}^-2 - (9/2) + (5 \cdot 73/30))/5 = 17/15$

$x_1 = (1 - 9/2 - 2(73/30) - 17/15) = -19/2$

8. $$\begin{cases} 5x_1 - 1x_2 = 1 \\ -1x_1 + 5x_2 - 1x_3 = 2 \\ -1x_2 + 5x_3 - 1x_4 = 3 \\ -1x_3 + 5x_4 - 1x_5 = 4 \\ -1x_4 + 5x_5 = 5 \end{cases}$$

Forward elimination

$$\begin{cases} 5x_1 - 1x_2 = 1 \\ 24/5\, x_2 - 1x_3 = 11/5 \\ 115/24\, x_3 - 1x_4 = 83/24 \\ 551/115\, x_4 - 1x_5 = 543/115 \\ \frac{2640}{551} x_5 = \frac{3298}{551} \end{cases}$$

Solve

$\Rightarrow$
$x_5 = 3298/551 \cdot 551/2640 = 1649/1320$
$x_4 = (543/115 + 1649/1320) \cdot 115/551 = 329/264$
$x_3 = (83/24 + 329/264) \cdot 24/115 = 54/55$
$x_2 = (11/5 + 54/55) \cdot 5/24 = 175/264$
$x_1 = (1 + 175/264) / 5 = 439/1320$