HW3

1a. Derive U from A and build L based on coefficient of deriv

$$\begin{bmatrix} -2 & 1 & -2 \\ -4 & 3 & -3 \\ 2 & 2 & 4 \end{bmatrix} \begin{matrix} R2 \boxed{-2} R1 \\ R3 \boxed{+1} R1 \end{matrix} \Rightarrow \begin{bmatrix} -2 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} R3 \boxed{-3} R2$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$ (coefficient opposite is the corresponding rows in matrix L

b. Solve $LY = B$ for Y

$\Rightarrow y_1 = 1$

$\Rightarrow 2y_1 + y_2 = 4 \Rightarrow 2(1) + y_2 = 4 \Rightarrow y_2 = 4 - 2 = 2$

$\Rightarrow -y_1 + 3y_2 + y_3 = 4 \Rightarrow -(1) + 3(2) + y_3 = 4 \Rightarrow y_3 = 4 + 1 - 6 = -1$

Solve $UX = Y$

$\Rightarrow -x_3 = -1 \Rightarrow x_3 = 1$

$\Rightarrow x_2 + x_3 = 2 \Rightarrow x_2 + 1 = 2 \Rightarrow x_2 = 2 - 1 = 1$

$\Rightarrow -2x_1 + x_2 - 2x_3 = 1 \Rightarrow -2x_1 + (1) - 2(1) = 1$

$x_1 = 2/-2 = -1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & -4 & 3 \\ -1 & -4 & -1 & 3 \\ 1 & 3 & 3 & 0 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 1 & 4 & -1 \end{bmatrix} \begin{matrix} R_3 - 2R_2 \\ R_4 + R_2 \end{matrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} R_4 - R_3 \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ (Diagonal of U)

$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{L^T}$

3. $A = \begin{bmatrix} 4 & 6 & 10 \\ 6 & 25 & 19 \\ 10 & 19 & 62 \end{bmatrix} \begin{matrix} R_2 - 3/2 R_1 \\ R_3 - 5/2 R_1 \end{matrix} \Rightarrow \begin{bmatrix} 4 & 6 & 10 \\ 0 & 16 & 4 \\ 0 & 4 & 37 \end{bmatrix} R_3 - 1/4 R_2$

$\Rightarrow \begin{bmatrix} 4 & 6 & 10 \\ 0 & 16 & 4 \\ 0 & 0 & 36 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 5/2 & 1/4 & 1 \end{bmatrix}$

$A = \tilde{L}(\tilde{L}^T)$ and $\tilde{L} = L D^{1/2}$

$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 36 \end{bmatrix}$ so $D^{1/2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 5/2 & 1/4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 1 & 6 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

4. $U = (u_1 \;\; u_2 \;\; u_3)$, $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\qquad e_1 = \dfrac{1}{\sqrt{2^2+1^2+2^2}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

$u_2 = v_2 - \text{proj}_{u_1}(v_2) = v_2 - \dfrac{v_2^T u_1}{u_1^T u_1} u_1$

$u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \dfrac{0}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$e_2 = \dfrac{1}{\sqrt{1^2+(-1)^2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix}$

$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3)$

$u_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \dfrac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - (-2) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 2/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} -1/3 \\ 4/3 \\ -1/3 \end{bmatrix}$

$e_3 = \dfrac{1}{\sqrt{(-1/3)^2+(4/3)^2+(-1/3)^2}} \begin{bmatrix} -1/3 \\ 4/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/6 \\ 2\sqrt{2}/3 \\ -\sqrt{2}/6 \end{bmatrix}$

$Q = \begin{bmatrix} 2/3 & \sqrt{2}/2 & -\sqrt{2}/6 \\ 1/3 & 0 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/2 & -\sqrt{2}/6 \end{bmatrix}$

$A = QR \rightarrow Q^T A = R$

$$R = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ -\sqrt{2}/6 & 2\sqrt{2}/3 & -\sqrt{2}/6 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 2 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 2/3 & \sqrt{2}/2 & -\sqrt{2}/6 \\ 1/3 & 0 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/2 & -\sqrt{2}/6 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 0 & 2 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}}_{R}$$

5a. $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2+\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 2-\lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix}$

$= (2-\lambda)(2-\lambda)(1-\lambda) - (1)(1-\lambda)$

$= (1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

Solve for eigen values

$1-\lambda = 0 \Rightarrow \lambda = 1$

$(2-\lambda)^2 - 1 = 0 \Rightarrow \sqrt{(2-\lambda)^2} = \sqrt{1} \Rightarrow (2-\lambda) = \pm 1$

$\lambda = 2 - (\pm 1)$ so $\lambda = 1$ and 3

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For $\lambda = 1$:

$(A - \lambda I)\bar{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 + (0)x_3 = 0$ and let $x_1 = 1$ so:

$1 + x_2 = 0 \Rightarrow x_2 = -1$ & $x_3 = 0$ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Since x_3 doesn't depend on x_1 or x_2,

we can form another eigenvector with $x_3 = 1$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$:

$$(A - \lambda I)\bar{x} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 - x_2 + (0)x_3 = 0$ and let $x_1 = 1$ so:

$1 - x_2 = 0 \Rightarrow x_2 = 1$ & $x_3 = 0$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Verify:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

b. $$A = QDQ^{-1} = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix}$$

c. Only D is changing

$$A^{100} = QD^{100}Q^{-1} = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix}$$

b.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 2-\lambda \begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$= (2-\lambda)\left[(3-\lambda)(2-\lambda) - ((-1)\cdot(-1))\right] + \left[(-1)(2-\lambda)\right]$

$= (2-\lambda)[6 - 5\lambda + \lambda^2 - 1] - (2-\lambda)$

$= (2-\lambda)(\lambda^2 - 5\lambda + 5) - (2-\lambda)$

$= (2-\lambda)(\lambda^2 - 5\lambda + 5 - 1) = (2-\lambda)(\lambda^2 - 5\lambda + 4)$

$= (2-\lambda)(\lambda - 4)(\lambda - 1)$

$\Rightarrow \lambda = 1, 2, 4$

For $\lambda = 4$:

$$(A - \lambda I)\bar{x} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 - \tfrac{1}{2}R_1 \Rightarrow \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \quad R_3 - 2R_2 \Rightarrow \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-1/2\, x_2 - x_3 = 0 \Rightarrow -1/2\, x_2 = x_3$

Let $x_3 = 1$ so $x_2 = 1 \cdot -2 = -2$

$-2x_1 - x_2 = 0 \Rightarrow -2(x_1) - (-2) = 0 \Rightarrow -2x_1 = -2$

$$\Rightarrow x_1 = 1 \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2.25 \\ -4.5 \\ 2.25 \end{bmatrix} \text{ to get } |\bar{x}| = 3$$

For $\lambda = 1$:

$$(A - \lambda I)\bar{x} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_2 - x_3 = 0 \Rightarrow x_2 = x_3$. Let $x_3 = 1$ so $x_2 = 1$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ so $x_1 = 1$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$

a. Vector $x = \begin{bmatrix} 2.25 \\ -4.5 \\ 2.25 \end{bmatrix}$ would give max $f(x)$ value of $4 =$ max λ and corresponding vector.

b. Vector $x = \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$ would give min $f(x)$ value of $1 =$ min λ and corresponding vector

$A = U D V^T$

7a. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} R_3 - R_1 \\ R_4 - R_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ Matrix rank 2

b. $A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0$

$= (2-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 2$

For $\lambda = 2$:

$(A - \lambda I)\bar{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

x_1 and x_2 are independent from each other

so $v_1 = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $v_2 = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ for d_{11} & $d_{22} = \sqrt{2}$

$U_1 = A e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$U_2 = A e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$\hat{u}_1 = \frac{u_1}{\|u_1\|} = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\\0\end{bmatrix}$$

$$\hat{u}_2 = \frac{u_2}{\|u_2\|} = \frac{\sqrt{2}}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}$$

$$U = [\hat{u}_1 \; \hat{u}_2] = \begin{bmatrix}\sqrt{2}/2 & 0\\0 & \sqrt{2}/2\\\sqrt{2}/2 & 0\\0 & \sqrt{2}/2\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\\1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2 & 0\\0 & \sqrt{2}/2\\\sqrt{2}/2 & 0\\0 & \sqrt{2}/2\end{bmatrix}\begin{bmatrix}\boxed{\sqrt{2}} & 0\\0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}\boxed{1 \quad 0}\\0 \quad 1\end{bmatrix}$$

$$A = U \quad D \quad V^T$$

c. $A = U\Sigma V^T$ where rank$(A) = 1$

$$\begin{bmatrix}1 & 0\\0 & 1\\1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\\0\end{bmatrix}\begin{bmatrix}\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$$

Approximation (circle in part c)

d. $u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} -$

$\left(\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \right) \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} -$

$\left(\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \right) \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\hat{u}_3 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \qquad \hat{u}_4 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

$$A = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$