Exercises for Algorithms and Data Structures Lecture 4

Question 1. We consider a new variation on the two-player game Nim. The board consists of a set of stacks with n (> 1) chips (or matches or any other entity). Two players, Sven and Gerard, take turns making a move. The player on turn, may either (i) split a stack of chips into two stacks with a different number of chips or (ii) remove 1 or 2 chips from the same stack. The player to take the last chip wins the game. For example: Consider the game state with 2 stacks, one with 4 chips and one with 5 chips. In that case, there are seven possible moves, of which three involve splitting a stack (make sure to check whether you understand this!).

a. What are states and what are actions?

b. Construct the state space for the case of $n = 5$. Assume that Sven starts. You do not have to elaborate on states that you already elaborated on elsewhere, rather, write down who wins. Is the game winning for the starting player (with $n = 5$)? ('Winning' means that the player can always win, regardless of the moves of the opponent.)

Question 2. On a small mountain road, with on one side a steep canyon and on the other side a steep mountainside, meet two groups of goats. One group consists of n black goats and one of n white goats. The black goats come from the left and the white goats from the right. When they approach each other, there is exactly one goat-length of empty space in the middle. This situation can be represented as: `zzzz wwww`.
A goat that is in front of the empty space can move into that space. However, there is not enough to cross sideways. Instead, the goats can jump over goats of another color into an empty space. Goats will never jump over a goat of the same color.

The goats can not turn around or walk/jump backward, so the black goats only move from left to right and the white goats only move from left to right. The goal is for all goats to move to the other side. For example, the final situation for $n = 5$ would be `wwwww zzzzz`. There is only one empty spot, meaning that this procedure happens on a space of $2n + 1$.
NB. Goats of the same color can make several moves after each other, the goats do not have to move in a round-robin fashion. The following would be allowed:
`zzzz wwww` $\longrightarrow$ `zzz zwwww` $\longrightarrow$ `zz zzwwww` $\longrightarrow$ `z zzzwwww` , etc.

a. For this problem, what are states and what are actions?

b. Give an upper bound for the number of 'moves' (walking into the empty space or jumping) that must be done to come from the start position to the final position.

c. A possible upper bound for the number of states could be $\binom{2n}{n} * (2n + 1)$. Please explain this.

Question 3. We consider the following one-player game (simulation), played with n white and n black pawns. In the start position, all pawns are lined up starting (from left to right) with a white pawn followed by a black pawn (continuing this pattern). At the end of the line, there are two empty spots.

Begin position for $n = 4$: `W Z W Z W Z W Z - -`

The goal is to use a series of moves involving moving two pieces, resulting in all white pieces on the left side and all black pieces on the right side.

Final position for $n = 4$: `W W W W - - Z Z Z Z`

Specifically, a move involves 2 adjacent pawns to the empty spaces in such a way that their individual order remains the same. As an example we show three moves:

```
W Z W Z W Z W Z - -   --->   W Z W Z - - W Z W Z   --->
        W - - Z Z W W Z W Z   --->   W W W Z Z - - Z W Z
```

a. What are in this game the states and actions (for general n)?

b. Give a reasonable upper bound for the number of states as defined in **a.** (3^{2n+2} is not tight enough) and explain why.

c. **(i)** Consider the case of $n = 3$. The start position, `W Z W Z W Z - -`, has 5 possible moves, of which we will elaborate on one: `W Z - - W Z W Z`.
Draw the part of the state-space for $n = 3$ that considers all states (and the connecting actions) that are reachable within at most 2 moves from `W Z - - W Z W Z`. You can ignore the start position.
(ii) Give a sequence of moves that leads to the final position for $n = 3$.

Question 4. (former exam question)

We consider the following two-player game. It is played with $m * n$ checkers pieces, that are positioned in an m times n rectangle (m rows and n columns). There are two players, V (Vertical) and H (Horizontal), that take turns making a move. V starts.

A move for V constitutes removing a column, and a move for H constitutes removing a row. Removing a row or column from the rectangle generally splits it into two smaller rectangles (unless the row or column was removed from the edge). At the start of the game, there is only one square, but this number grows. A player in turn removes one row (H) or column (V) from exactly one of the rectangles. The player removing the last piece wins.

Consider the following sequence of moves for $m = 3$ and $n = 4$ (here, the pieces are represented by `X`-, and different rectangles are separated by commas):

```
X X X X    V     X  X X    H     X               V     X            H    X
X X X X   --->   X, X X   --->   X, X X, X X   --->    X, X X, X  --->   X, X
X X X X          X  X X          X                     X                 X
  (*)                                                                   (**)
```

V is on turn in state `(**)`. If he removes the column with 3 pieces, he will lose. Suppose he removes the column with one piece. That leaves a winning position for H. (She removes the piece in the middle row, leaving two once-piece rectangles, resulting in a win the next turn). As such, state `(**)` is losing for V.

a. What are in this game states and actions (for general m and n)?

b. Contruct the state-space for the cases $m = 2$ and $n = 4$, assumuing that V starts. Note for every state whether it is winning for V or H. Determine whether the game is winning for either V or H, and show how. You do not need to draw states that are winnable in one move. Furthermore, you do not need to elaborate on states that are equivalent to states that you already elaborated on (instead, mention who will win).

c. We consider the special case $m = 1$, consisting of a single row with n peices. We assume that V starts.
Prove that: for odd n, player V can always win. Describe a winning strategy for V. *Hint*: Show that V will win on the situation of 1 times 3, and extend this to general odd n.

d. Prove that for even n, V will always lose (H can always win). Describe a winning strategy for H.

Question 5. We consider the game tic-tac-toe, played by cross (X) and circle (O). Suppose that Cross starts.

a. How would you define a state? Draw a part of the state-space.

b. How would you define an action in this game?