Sol of Quantum Computation and Quantum Information Chapter2

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2.1

$$\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{1}$$

2.2

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|0\right\rangle + A_{21}\left|1\right\rangle = \left|1\right\rangle \Rightarrow A_{11}=0,\ A_{21}=1\\
A\left|1\right\rangle &= A_{12}\left|0\right\rangle + A_{22}\left|1\right\rangle = \left|0\right\rangle \Rightarrow A_{12}=1,\ A_{22}=0\\
\therefore A &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}
\end{aligned} \tag{2}$$

input: $\{\left|0\right\rangle, \left|1\right\rangle\}$, output: $\{\left|1\right\rangle, \left|0\right\rangle\}$

$$\begin{aligned}
A\left|0\right\rangle &= A_{11}\left|1\right\rangle + A_{21}\left|0\right\rangle = \left|1\right\rangle \Rightarrow A_{11}=1,\ A_{21}=0\\
A\left|1\right\rangle &= A_{12}\left|1\right\rangle + A_{22}\left|0\right\rangle = \left|0\right\rangle \Rightarrow A_{12}=0,\ A_{22}=1\\
A &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}
\end{aligned} \tag{3}$$

2.3

From eq (2.12)

$$\begin{aligned}
A\left|v_i\right\rangle &= \sum_j A_{ji}\left|w_j\right\rangle\\
B\left|w_j\right\rangle &= \sum_k B_{kj}\left|x_k\right\rangle
\end{aligned} \tag{4}$$

Thus

$$\begin{aligned}
BA\left|v_i\right\rangle &= B\left(\sum_j A_{ji}\left|w_j\right\rangle\right)\\
&= \sum_j A_{ji}B\left|w_j\right\rangle\\
&= \sum_{j,k} A_{ji}B_{kj}\left|x_k\right\rangle\\
&= \sum_k \left(\sum_j B_{kj}A_{ji}\right)\left|x_k\right\rangle\\
&= \sum_k (BA)_{ki}\left|x_k\right\rangle\\
\therefore (BA)_{ki} &= \sum_j B_{kj}A_{ji}
\end{aligned} \tag{5}$$

2.4

$$\begin{aligned}
I\left|v_j\right\rangle = \sum_i I_{ij}\left|v_i\right\rangle &= \left|v_j\right\rangle,\ \forall j.\\
&\Rightarrow I_{ij} = \delta_{ij}
\end{aligned} \tag{6}$$

2.5

Defined inner product on $\mathcal{C}^n$ is

$$((y_1, \cdots, y_n), (z_1, \cdots, z_n)) = \sum_i y_i^* z_i. \tag{7}$$

Verify (1) of eq (2.13).

$$\begin{aligned}
\left((y_1, \cdots, y_n), \sum_i \lambda_i (z_{i1}, \cdots, z_{in})\right) &= \sum_i y_i^* \left(\sum_j \lambda_j z_{ji}\right) \\
&= \sum_{i,j} y_i^* \lambda_j z_{ji} \\
&= \sum_j \lambda_j \left(\sum_i y_i^* z_{ji}\right) \\
&= \sum_j \lambda_j \left((y_1, \cdots, y_n), (z_{j1}, \cdots, z_{jn})\right) \\
&= \sum_i \lambda_i \left((y_1, \cdots, y_n), (z_{i1}, \cdots, z_{in})\right).
\end{aligned} \tag{8}$$

Verify (2) of eq (2.13),

$$\begin{aligned}
((y_1, \cdots, y_n), (z_1, \cdots, z_n))^* &= \left(\sum_i y_i^* z_i\right)^* \\
&= \left(\sum_i y_i z_i^*\right) \\
&= \left(\sum_i z_i^* y_i\right) \\
&= ((z_1, \cdots, z_n), (y_1, \cdots, y_n))
\end{aligned} \tag{9}$$

Verify (3) of eq (2.13),

$$\begin{aligned}
((y_1, \cdots, y_n), (y_1, \cdots, y_n)) &= \sum_i y_i^* y_i \\
&= \sum_i |y_i|^2
\end{aligned} \tag{10}$$

Since $|y_i|^2 \geq 0$ for all i. Thus $\sum_i |y_i|^2 = ((y_1, \cdots, y_n), (y_1, \cdots, y_n)) \geq 0$.
From now on, I will show the following statement,

$$((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0 \text{ iff } (y_1, \cdots, y_n) = 0. \tag{11}$$

($\Leftarrow$) This is obvious.
($\Rightarrow$) Suppose $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0$. Then $\sum_i |y_i|^2 = 0$.
Since $|y_i|^2 \geq 0$ for all i, if $\sum_i |y_i|^2 = 0$, then $|y_i|^2 = 0$ for all i. Therefore $|y_i|^2 = 0 \Leftrightarrow y_i = 0$ for all i. Thus,

$$(y_1, \cdots, y_n) = 0. \tag{12}$$

2.6

$$\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle,\ |v\rangle\right) &= \left(|v\rangle,\ \sum_i \lambda_i |w_i\rangle\right)^* \\
&= \left[\sum_i \lambda_i \left(|v\rangle,\ |w_i\rangle\right)\right]^* (\because \text{linearlity in the 2nd arg.}) \\
&= \sum_i \lambda_i^* \left(|v\rangle,\ |w_i\rangle\right)^* \\
&= \sum_i \lambda_i^* (|w_i\rangle,\ |v\rangle)
\end{aligned} \tag{13}$$

2.7

$$\begin{aligned}
\langle w|v\rangle &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1-1=0\\
\frac{|w\rangle}{\||w\rangle\|} &= \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
\frac{|v\rangle}{\||v\rangle\|} &= \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned} \tag{14}$$

2.8

If $k=1$,

$$\begin{aligned}
|v_2\rangle &= \frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|}\\
\langle v_1|v_2\rangle &= \langle v_1|\left(\frac{|w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|}\right)\\
&= \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle\langle v_1|v_1\rangle}{\||w_2\rangle - \langle v_1|w_2\rangle|v_1\rangle\|}\\
&= 0.
\end{aligned} \tag{15}$$

Suppose $\{v_1,\cdots v_n\}$ $(n\le d-1)$ is a orthonormal basis. Then

$$\begin{aligned}
\langle v_j|v_{n+1}\rangle &= \langle v_j|\left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle|v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle|v_i\rangle\|}\right) \quad (j\le n)\\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle\langle v_j|v_i\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle|v_i\rangle\|}\\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle\delta_{ij}}{\||w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle|v_i\rangle\|}\\
&= \frac{\langle v_j|w_{n+1}\rangle - \langle v_j|w_{n+1}\rangle}{\||w_{n+1}\rangle - \sum_{i=1}^n\langle v_i|w_{n+1}\rangle|v_i\rangle\|}\\
&= 0.
\end{aligned} \tag{16}$$

Thus Gram-Schmidt procedure produces an orthonormal basis.

2.9

$$\begin{aligned}
\sigma_0 &= I = |0\rangle\langle 0| + |1\rangle\langle 1|\\
\sigma_1 &= X = |0\rangle\langle 1| + |1\rangle\langle 0|\\
\sigma_2 &= Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|\\
\sigma_3 &= Z = |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned} \tag{17}$$

2.10

$$\begin{aligned}
|v_j\rangle\langle v_k| &= I_V|v_j\rangle\langle v_k|I_V\\
&= \left(\sum_p |v_p\rangle\langle v_p|\right)|v_j\rangle\langle v_k|\left(\sum_q |v_q\rangle\langle v_q|\right)\\
&= \sum_{p,q}|v_p\rangle\langle v_p|v_j\rangle\langle v_k|v_q\rangle\langle v_q|\\
&= \sum_{p,q}\delta_{pj}\delta_{kq}|v_p\rangle\langle v_q|
\end{aligned} \tag{18}$$

Thus

$$(|v_j\rangle\langle v_k|)_{pq} = \delta_{pj}\delta_{kq} \tag{19}$$

2.11

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(X - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda = \pm 1 \tag{20}$$

If $\lambda = -1$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

Thus

$$|\lambda = -1\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{22}$$

If $\lambda = 1$

$$|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

$$X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ w.r.t. } \{|\lambda = -1\rangle, \ |\lambda = 1\rangle\} \tag{24}$$

2.12

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = (1 - \lambda)^2 = 0 \Rightarrow \lambda = 1 \tag{25}$$

Therefore the eigenvector associated with eigenvalue $\lambda = 1$ is

$$|\lambda = 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

Because $|\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \neq c\, |\lambda = 1\rangle \langle \lambda = 1| = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \tag{27}$$

2.13

Suppose $|\psi\rangle, \ |\phi\rangle$ are arbitrary vectors in V.

$$\begin{aligned} (|\psi\rangle, \ (|w\rangle \langle v|) |\phi\rangle)^* &= \left((|w\rangle \langle v|)^\dagger |\psi\rangle, \ |\phi\rangle\right)^* \\ &= \left(|\phi\rangle, \ (|w\rangle \langle v|)^\dagger |\psi\rangle\right) \\ &= \langle \phi| (|w\rangle \langle v|)^\dagger |\psi\rangle . \end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned} (|\psi\rangle, \ (|w\rangle \langle v|) |\phi\rangle)^* &= (\langle \psi|w\rangle \langle v|\phi\rangle)^* \\ &= \langle \phi|v\rangle \langle w|\psi\rangle . \end{aligned} \tag{29}$$

Thus

$$\begin{aligned} \langle \phi| (|w\rangle \langle v|)^\dagger |\psi\rangle = \langle \phi|v\rangle \langle w|\psi\rangle \ \text{ for arbitrary vectors } \ |\psi\rangle, \ |\phi\rangle \\ \therefore (|w\rangle \langle v|)^\dagger = |v\rangle \langle w| \end{aligned} \tag{30}$$

2.14

$$\begin{aligned} ((a_i A_i)^\dagger |\phi\rangle, \ |\psi\rangle) &= (|\phi\rangle, \ a_i A_i |\psi\rangle) \\ &= a_i (|\phi\rangle, \ A_i |\psi\rangle) \\ &= a_i (A_i^\dagger |\phi\rangle, \ |\psi\rangle) \\ &= (a_i^* A_i^\dagger |\phi\rangle, \ |\psi\rangle) \\ \therefore (a_i A_i)^\dagger &= a_i^* A_i^\dagger \end{aligned} \tag{31}$$

2.15

$$\begin{aligned}
((A^\dagger)^\dagger \left|\psi\right\rangle,\ \left|\phi\right\rangle) &= (\left|\psi\right\rangle,\ A^\dagger \left|\phi\right\rangle) \\
&= (A^\dagger \left|\phi\right\rangle,\ \left|\psi\right\rangle)^* \\
&= (\left|\phi\right\rangle,\ A\left|\psi\right\rangle)^* \\
&= (A\left|\psi\right\rangle,\ \left|\phi\right\rangle) \\
\therefore (A^\dagger)^\dagger &= A
\end{aligned} \tag{32}$$

2.16

$$\begin{aligned}
P &= \sum_i \left|i\right\rangle \left\langle i\right|. \\
P^2 &= \left(\sum_i \left|i\right\rangle \left\langle i\right|\right)\left(\sum_j \left|j\right\rangle \left\langle j\right|\right) \\
&= \sum_{i,j} \left|i\right\rangle \left\langle i|j\right\rangle \left\langle j\right| \\
&= \sum_{i,j} \left|i\right\rangle \left\langle j\right| \delta_{ij} \\
&= \sum_i \left|i\right\rangle \left\langle i\right| \\
&= P
\end{aligned} \tag{33}$$

2.17

Proof. (⇒) Suppose A is Hermitian. Then $A = A^\dagger$. Let $\left|\lambda\right\rangle$ be eigenvectors of A with eigenvalues λ, that is,

$$A\left|\right\rangle = \lambda\left|\lambda\right\rangle. \tag{34}$$

Therefore

$$\left\langle\lambda|A|\lambda\right\rangle = \lambda\left\langle\lambda|\lambda\right\rangle = \lambda. \tag{35}$$

On the other hand,

$$\lambda^* = \left\langle\lambda|A|\lambda\right\rangle^* = \left\langle\lambda|A^\dagger|\lambda\right\rangle = \left\langle\lambda|A|\lambda\right\rangle = \lambda\left\langle\lambda|\lambda\right\rangle = \lambda. \tag{36}$$

Hence eigenvalues of Hermitian matrix are real.

(⇐) Suppose eigenvalues of A are real. From spectral theorem, normal matrix A can be written by

$$A = \sum_i \lambda_i \left|\lambda_i\right\rangle\!\left\langle\lambda_i\right| \tag{37}$$

where λ_i are real eigenvalues with eigenvectors $\left|\lambda_i\right\rangle$. By taking adjoint, we get

$$\begin{aligned}
A^\dagger &= \sum_i \lambda_i^* \left|\lambda_i\right\rangle\!\left\langle\lambda_i\right| \\
&= \sum_i \lambda_i \left|\lambda_i\right\rangle\!\left\langle\lambda_i\right| \quad (\because \lambda_i \text{ are real}) \\
&= A
\end{aligned} \tag{38}$$

Thus A is Hermitian. □

2.18

Suppose $|v\rangle$ is a eigenvector with corresponding eigenvalue λ.

$$\begin{aligned} U|v\rangle &= \lambda|v\rangle . \\ 1 &= \langle v|v\rangle \\ &= \langle v| I |v\rangle \\ &= \langle v| U^\dagger U |v\rangle \\ &= \lambda\lambda^* \langle v|v\rangle \\ &= \|\lambda\|^2 \\ \therefore \lambda &= e^{i\theta} \end{aligned} \tag{39}$$

2.19

$$X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \tag{40}$$

2.20

$$\begin{aligned} U &\equiv \sum_i |w_i\rangle \langle v_i| \\ A'_{ij} &= \langle v_i|A|v_j\rangle \\ &= \langle v_i|UU^\dagger AUU^\dagger|v_j\rangle \\ &= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\ &= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle \\ &= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr} \end{aligned} \tag{41}$$

2.21

Suppose M be Hermitian. Then $M = M^\dagger$.

$$\begin{aligned} M &= IMI \\ &= (P+Q)M(P+Q) \\ &= PMP + QMP + PMQ + QMQ \end{aligned} \tag{42}$$

Now $PMP = \lambda P$, $QMP = 0$, $PMQ = PM^\dagger Q = (QMP)^* = 0$. Thus $M = PMP + QMQ$. Next prove QMQ is normal.

$$\begin{aligned} QMQ(QMQ)^\dagger &= QMQQM^\dagger Q \\ &= QM^\dagger QQMQ \quad (M = M^\dagger) \\ &= (QM^\dagger Q)QMQ \end{aligned} \tag{43}$$

Therefore QMQ is normal. By induction, QMQ is diagonal ... (following is same as Box 2.2)

2.22

Suppose A is a Hermitian operator and $|v_i\rangle$ are eigenvectors of A with eigenvalues λ_i. Then

$$\langle v_i|A|v_j\rangle = \lambda_j \langle v_i|v_j\rangle . \tag{44}$$

On the other hand,

$$\langle v_i|A|v_j\rangle = \langle v_i|A^\dagger|v_j\rangle = \langle v_j|A|v_i\rangle^* = \lambda_i^* \langle v_j|v_i\rangle^* = \lambda_i^* \langle v_i|v_j\rangle = \lambda_i \langle v_i|v_j\rangle \tag{45}$$

Thus

$$(\lambda_i - \lambda_j) \langle v_i|v_j\rangle = 0. \tag{46}$$

If $\lambda_i \neq \lambda_j$, then $\langle v_i|v_j\rangle = 0$.

2.23

Suppose P is projector and $|\lambda\rangle$ are eigenvectors of P with eigenvalues λ. Then $P^2 = P$.

$$P|\lambda\rangle = \lambda|\lambda\rangle \text{ and } P|\lambda\rangle = P^2|\lambda\rangle = \lambda P|\lambda\rangle = \lambda^2|\lambda\rangle. \tag{47}$$

Therefore

$$\begin{aligned} \lambda &= \lambda^2 \\ \lambda(\lambda-1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned} \tag{48}$$

2.24

Def of positive $\langle v|A|v\rangle \geq 0$ for all $|v\rangle$.

Suppose A is a positive operator. A can be decomposed as follows.

$$\begin{aligned} A &= \frac{A+A^\dagger}{2} + i\frac{A-A^\dagger}{2i} \\ &= B + iC \quad \text{where } B = \frac{A+A^\dagger}{2}, \quad C = \frac{A-A^\dagger}{2i}. \end{aligned} \tag{49}$$

Now operators B and C are Hermitian.

$$\begin{aligned} \langle v|A|v\rangle &= \langle v|B+iC|v\rangle \\ &= \langle v|B|v\rangle + i\langle v|C|v\rangle \\ &= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle, \ \beta = \langle v|C|v\rangle. \end{aligned} \tag{50}$$

Since B and C are Hermitian, α, $\beta \in \mathbb{R}$. From def of positive operator, β should be vanished because $\langle v|A|v\rangle$ is real. Hence $\beta = \langle v|C|v\rangle = 0$ for all $|v\rangle$, i.e. $C = 0$.

Therefore $A = A^\dagger$.

Reference: MIT 8.05 Lecture note by Prof. Barton Zwiebach.
`https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8_05F13_Chap_03.pdf`

Proposition. 0.0.1. *Let T be a linear operator in a complex vector space V.*
If $(u, Tv) = 0$ for all $u, v \in V$, then $T = 0$.

Proof. Suppose $u = Tv$. Then $(Tv, Tv) = 0$ for all v implies that $Tv = 0$ for all v. Therefore $T = 0$. ☐

Theorem. 0.0.1. *If $(v, Av) = 0$ for all $v \in V$, then $A = 0$.*

Proof. First, we show that $(u, Tv) = 0$ if $(v, Av) = 0$. Then apply proposition 0.0.1
Suppose $u, v \in V$. Then (u, Tv) is decomposed as

$$\begin{aligned} (u, Tv) = \frac{1}{4}\Big[&(u+v, T(u+v)) - (u-v, T(u-v)) + \frac{1}{i}(u+iv, T(u+iv)) \\ &- \frac{1}{i}(u-iv, T(u-iv))\Big]. \end{aligned} \tag{51}$$

If $(v, Tv) = 0$ for all $v \in V$, the right hand side of above eqn vanishes. Thus $(u, Tv) = 0$ for all $u, v \in V$. Then $T = 0$. ☐

2.25

$$\langle\psi|A^\dagger A|\psi\rangle = \| A|\psi\rangle \|^2 \geq 0 \text{ for all } |\psi\rangle. \tag{52}$$

Thus $A^\dagger A$ is positive.

2.26

$$
\begin{aligned}
|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
|\psi\rangle^{\otimes 3} &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \\
&= \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \\
&= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned} \tag{54}
$$

2.27

$$
\begin{aligned}
X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{57}
$$

In general, tensor product is not commutable.

2.28

$$
\begin{aligned}
(A \otimes B)^* &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* \\
&= \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} \\
&= A^* \otimes B^*.
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
(A \otimes B)^T &= \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}B^T \\ \vdots & \ddots & \vdots \\ A_{n1}B^T & \cdots & A_{nm}B^T \end{bmatrix} \\
&= A^T \otimes B^T.
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
(A \otimes B)^\dagger &= ((A \otimes B)^*)^T \\
&= (A^* \otimes B^*)^T \\
&= (A^*)^T \otimes (B^*)^T \\
&= A^\dagger \otimes B^\dagger.
\end{aligned} \tag{60}
$$

2.29

Suppose U_1 and U_2 are unitary operators. Then

$$
\begin{aligned}
(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger &= U_1 U_1^\dagger \otimes U_2 U_2^\dagger \\
&= I \otimes I.
\end{aligned} \tag{61}
$$

Similarly,

$$
(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = I \otimes I. \tag{62}
$$

2.30

Suppose A and B are Hermitian operators. Then

$$
(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B. \tag{63}
$$

Thus $A \otimes B$ is Hermitian.

2.31

Suppose A and B are positive operators. Then

$$
\langle\psi| \otimes \langle\phi| \, (A \otimes B) \, |\psi\rangle \otimes |\phi\rangle = \langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \, . \tag{64}
$$

Since A and B are positive operators, $\langle\psi|A|\psi\rangle \geq 0$ and $\langle\phi|B|\phi\rangle \geq 0$ for all $|\psi\rangle, |\phi\rangle$. Then $\langle\psi|A|\psi\rangle \, \langle\phi|B|\phi\rangle \geq 0$. Thus $A \otimes B$ is positive if A and B are positive.

2.32

Suppose P_1 and P_2 are projectors. Then

$$\begin{aligned}(P_1 \otimes P_2)^2 &= P_1^2 \otimes P_2^2 \\ &= P_1 \otimes P_2.\end{aligned} \tag{65}$$

Thus $P_1 \otimes P_2$. is also projector.

2.33

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{66}$$

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{67}$$

2.34

Suppose $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

$$\begin{aligned}\det(A - \lambda I) &= (4-\lambda)^2 - 3^2 \\ &= \lambda^2 - 8\lambda + 7 \\ &= (\lambda - 1)(\lambda - 7)\end{aligned} \tag{68}$$

Eigenvalues of A are $\lambda = 1,\ 7$. Corresponding eigenvectors are $|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $|\lambda = 7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus

$$A = |\lambda = 1\rangle\langle\lambda = 1| + 7\,|\lambda = 7\rangle\langle\lambda = 7|\,. \tag{69}$$

$$\begin{aligned}\sqrt{A} &= |\lambda = 1\rangle\langle\lambda = 1| + \sqrt{7}\,|\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix}\end{aligned} \tag{70}$$

$$\begin{aligned}\log(A) &= \log(1)\,|\lambda = 1\rangle\langle\lambda = 1| + \log(7)\,|\lambda = 7\rangle\langle\lambda = 7| \\ &= \frac{\log(7)}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned} \tag{71}$$

2.35

$$\begin{aligned}\vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^{3} v_i \sigma_i \\ &= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}\end{aligned} \tag{72}$$

$$\begin{aligned}\det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\ &= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\ &= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)\end{aligned} \tag{73}$$

Eigenvalues are $\lambda = \pm 1$. Let $|\lambda_{\pm 1}\rangle$ be eigenvectors with eigenvalues ± 1.

Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $\vec{v} \cdot \vec{\sigma}$ is diagonalizable. Then

$$\vec{v} \cdot \vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}| \tag{74}$$

Thus

$$\begin{aligned}
\exp(i\theta\vec{v} \cdot \vec{\sigma}) &= e^{i\theta} |\lambda_1\rangle\langle\lambda_1| + e^{-i\theta} |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta) |\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta) |\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma}.
\end{aligned} \tag{75}$$

$\because$ Since $\vec{v} \cdot \vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthogonal. Thus

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I. \tag{76}$$

2.36

$$\begin{aligned}
\mathrm{Tr}(\sigma_1) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_2) &= \mathrm{Tr}\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = 0 \\
\mathrm{Tr}(\sigma_3) &= \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 - 1 = 0
\end{aligned} \tag{77}$$

2.37

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_i \langle i|AB|i\rangle \\
&= \sum_i \langle i|AIB|i\rangle \\
&= \sum_{i,j} \langle i|A|j\rangle \langle j|B|i\rangle \\
&= \sum_{i,j} \langle j|B|i\rangle \langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle \\
&= \mathrm{Tr}(BA)
\end{aligned} \tag{78}$$

2.38

$$\begin{aligned}
\mathrm{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle \\
&= \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) \\
&= \sum_i \langle i|A|i\rangle + \sum_i \langle i|B|i\rangle \\
&= \mathrm{Tr}(A) + \mathrm{Tr}(B).
\end{aligned} \tag{79}$$

$$\begin{aligned}
\mathrm{Tr}(zA) &= \sum_i \langle i|zA|i\rangle \\
&= \sum_i z\langle i|A|i\rangle \\
&= z\sum_i \langle i|A|i\rangle \\
&= z\,\mathrm{Tr}(A).
\end{aligned} \tag{80}$$

2.39

(1) $(A, B) \equiv \mathrm{Tr}(A^\dagger B)$.

(i)

$$\begin{aligned}\left(A, \sum_i \lambda_i B_i\right) &= \mathrm{Tr}\left[A^\dagger\left(\sum_i \lambda_i B_i\right)\right] \\ &= \mathrm{Tr}(A^\dagger \lambda_1 B_1) + \cdots + \mathrm{Tr}(A^\dagger \lambda_n B_n) \quad (\because \text{Execise 2.38}) \\ &= \lambda_1 \mathrm{Tr}(A^\dagger B_1) + \cdots + \lambda_n \mathrm{Tr}(A^\dagger B_n) \\ &= \sum_i \lambda_i \mathrm{Tr}(A^\dagger B_i)\end{aligned} \tag{81}$$

(ii)

$$\begin{aligned}(A, B)^* &= \left(\mathrm{Tr}(A^\dagger B)\right)^* \\ &= \left(\sum_{i,j} \langle i|A^\dagger|j\rangle \langle j|B|i\rangle\right)^* \\ &= \sum_{i,j} \langle i|A^\dagger|j\rangle^* \langle j|B|i\rangle^* \\ &= \sum_{i,j} \langle j|B|i\rangle^* \langle i|A^\dagger|j\rangle^* \\ &= \sum_{i,j} \langle i|B^\dagger|j\rangle \langle j|A|i\rangle \\ &= \sum_i \langle i|B^\dagger A|i\rangle \\ &= \mathrm{Tr}(B^\dagger A) \\ &= (B, A).\end{aligned} \tag{82}$$

(iii)

$$\begin{aligned}(A, A) &= \mathrm{Tr}(A^\dagger A) \\ &= \sum_i \langle i|A^\dagger A|i\rangle\end{aligned} \tag{83}$$

Since $A^\dagger A$ is positive, $\langle i|A^\dagger A|i\rangle \geq 0$ for all $|i\rangle$.

Let a_i be i-th column of A. If $\langle i|A^\dagger A|i\rangle = 0$, then

$$\langle i|A^\dagger A|i\rangle = a_i^\dagger a_i = \|a_i\|^2 = 0 \text{ iff } a_i = \mathbf{0}. \tag{84}$$

Therefore $(A, A) = 0$ iff $A = \mathbf{0}$.

(2)
(3)

2.40

$$\begin{aligned}[X, Y] &= XY - YX \\ &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \\ &= \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\0 & i\end{bmatrix} \\ &= \begin{bmatrix}2i & 0\\0 & -2i\end{bmatrix} \\ &= 2iZ\end{aligned} \tag{85}$$

$$
\begin{aligned}
[Y,Z] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= 2iX
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
[Z,X] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 2iY
\end{aligned} \tag{87}
$$

2.41

$$
\begin{aligned}
\{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= 0
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= 0
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
\{\sigma_3, \sigma_1\} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= 0
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
\sigma_0^2 &= I^2 = I \\
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = I \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I
\end{aligned} \tag{91}
$$

2.42

$$
\frac{[A,B] + \{A,B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB \tag{92}
$$

2.43

From eq (2.75) and eq (2.76), $\{\sigma_j, \sigma_k\} = 2\delta_{jk} I$. From eq (2.77),

$$
\begin{aligned}
\sigma_j\sigma_k &= \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} \\
&= \frac{2i\sum_{l=1}^3 \epsilon_{jkl}\sigma_l + 2\delta_{jk} I}{2} \\
&= \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl}\sigma_l
\end{aligned} \tag{93}
$$

2.44

By assumption, $[A,B]=0$ and $\{A,B\}=0$, then $AB=0$. Since A is invertible, multiply by A^{-1} from left, then

$$\begin{aligned} A^{-1}AB &= 0 \\ IB &= 0 \\ B &= 0. \end{aligned} \tag{94}$$

2.45

$$\begin{aligned} [A,B]^\dagger &= (AB-BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger \\ &= \left[B^\dagger, A^\dagger\right] \end{aligned} \tag{95}$$

2.46

$$\begin{aligned} [A,B] &= AB-BA \\ &= -(BA-AB) \\ &= -[B,A] \end{aligned} \tag{96}$$

2.47

$$\begin{aligned} (i\,[A,B])^\dagger &= -i\,[A,B]^\dagger \\ &= -i\left[B^\dagger, A^\dagger\right] \\ &= -i\,[B,A] \\ &= i\,[A,B] \end{aligned} \tag{97}$$

2.48

(Positive)
Since P is positive, it is diagonalizable. Then $P=\sum_i \lambda_i |i\rangle\langle i|$, $(\lambda_i \geq 0)$.

$$J=\sqrt{P^\dagger P}=\sqrt{PP}=\sqrt{P^2}=\sum_i \sqrt{\lambda_i^2}\,|i\rangle\langle i| = \sum_i \lambda_i\,|i\rangle\langle i| = P. \tag{98}$$

Therefore polar decomposition of P is $P=UP$ for all P. Thus $U=I$, then $P=P$.

(Unitary)
Suppose unitary U is decomposed by $U=WJ$ where W is unitary and J is positive, $J=\sqrt{U^\dagger U}$.

$$J=\sqrt{U^\dagger U}=\sqrt{I}=I \tag{99}$$

Since unitary operators are invertible, $W=UJ^{-1}=UI^{-1}=UI=U$. Thus polar decomposition of U is $U=U$.

(Hermitian)
Suppose $H=UJ$.

$$J=\sqrt{H^\dagger H}=\sqrt{HH}=\sqrt{H^2}. \tag{100}$$

Thus $H=U\sqrt{H^2}$.
In general, $H \neq \sqrt{H^2}$.
From spectral decomposition, $H=\sum_i \lambda_i |i\rangle\langle i|$, $\lambda_i \in \mathbb{R}$.

$$\sqrt{H^2}=\sqrt{\sum_i \lambda_i^2\,|i\rangle\langle i|}=\sum_i \sqrt{\lambda_i^2}\,|i\rangle\langle i| = \sum_i |\lambda_i|\,|i\rangle\langle i| \neq H \tag{101}$$

2.49

Normal matrix is diagonalizable, $A = \sum_i \lambda_i \left|i\right\rangle\!\left\langle i\right|$.

$$
\begin{aligned}
J &= \sqrt{A^\dagger A} = \sum_i \left|\lambda_i\right| \left|i\right\rangle\!\left\langle i\right|. \\
U &= \sum_i \left|e_i\right\rangle\!\left\langle i\right| \\
A &= UJ = \sum_i \left|\lambda_i\right| \left|e_i\right\rangle\!\left\langle i\right|.
\end{aligned}
\tag{102}
$$

2.50

Define $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $A^\dagger A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation of $A^\dagger A$ is $\det(A^\dagger A - \lambda I) = \lambda^2 - 3\lambda + 1 = 0$. Eigenvalues of $A^\dagger A$ are $\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$ and associated eigenvectors are $\left|\lambda_\pm\right\rangle = \frac{1}{\sqrt{10 \mp 2\sqrt{5}}} \begin{bmatrix} 2 \\ -1 \pm \sqrt{5} \end{bmatrix}$.

$$
A^\dagger A = \lambda_+ \left|\lambda_+\right\rangle\!\left\langle \lambda_+\right| + \lambda_- \left|\lambda_-\right\rangle\!\left\langle \lambda_-\right|. \tag{103}
$$

$$
\begin{aligned}
J = \sqrt{A^\dagger A} &= \sqrt{\lambda_+} \left|\lambda_+\right\rangle\!\left\langle \lambda_+\right| + \sqrt{\lambda_-} \left|\lambda_-\right\rangle\!\left\langle \lambda_-\right| \\
&= \sqrt{\frac{3+\sqrt{5}}{2} \cdot \frac{5-\sqrt{5}}{40}} \begin{bmatrix} 4 & 2\sqrt{5}-2 \\ 2\sqrt{5}-2 & 6-2\sqrt{5} \end{bmatrix} + \sqrt{\frac{3-\sqrt{5}}{2} \cdot \frac{5+\sqrt{5}}{40}} \begin{bmatrix} 4 & -2\sqrt{5}-2 \\ -2\sqrt{5}-2 & 6+2\sqrt{5} \end{bmatrix}
\end{aligned}
\tag{104}
$$

$$
J^{-1} = \frac{1}{\sqrt{\lambda_+}} \left|\lambda_+\right\rangle\!\left\langle \lambda_+\right| + \frac{1}{\sqrt{\lambda_-}} \left|\lambda_-\right\rangle\!\left\langle \lambda_-\right|. \tag{105}
$$

$$
U = AJ^{-1} \tag{106}
$$

I'm tired.

2.51

$$
H^\dagger H = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^\dagger \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I. \tag{107}
$$

2.52

$$
H^\dagger = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H. \tag{108}
$$

Thus

$$
H^2 = I. \tag{109}
$$

2.53

$$
\begin{aligned}
\det(H - \lambda I) &= \left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \frac{1}{2} \\
&= \lambda^2 - \frac{1}{2} - \frac{1}{2} \\
&= \lambda^2 - 1
\end{aligned}
\tag{110}
$$

Eigenvalues are $\lambda_{\pm} = \pm 1$ and associated eigenvectors are $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

2.54

Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle\langle i|$, $B = \sum_i b_i |i\rangle\langle i|$.

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_i \exp(a_i) |i\rangle\langle i|\right) \left(\sum_i \exp(b_i) |i\rangle\langle i|\right) \\
&= \sum_{i,j} \exp(a_i + b_j) |i\rangle \langle i|j\rangle \langle j| \\
&= \sum_{i,j} \exp(a_i + b_j) |i\rangle\langle j| \delta_{i,j} \\
&= \sum_i \exp(a_i + b_i) |i\rangle\langle i| \\
&= \exp(A + B)
\end{aligned} \tag{111}$$

2.55

$$H = \sum_E E |E\rangle\langle E| \tag{112}$$

$$\begin{aligned}
U(t_2 - t_1)U^\dagger(t_2 - t_1) &= \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right) \exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{iE(t_2 - t_1)}{\hbar}\right) |E\rangle\langle E|\right) \left(\exp\left(-\frac{iE'(t_2 - t_1)}{\hbar}\right) |E'\rangle\langle E'|\right) \\
&= \sum_{E,E'} \left(\exp\left(-\frac{i(E - E')(t_2 - t_1)}{\hbar}\right) |E\rangle\langle E'| \delta_{E,E'}\right) \\
&= \sum_E \exp(0) |E\rangle\langle E| \\
&= \sum_E |E\rangle\langle E| \\
&= I
\end{aligned} \tag{113}$$

Similarly, $U^\dagger(t_2 - t_1)U(t_2 - t_1) = I$.

2.56

$U = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \quad (|\lambda_i| = 1)$.

$$\begin{aligned}
\log(U) &= \sum_j \log(\lambda_j) |\lambda_j\rangle\langle\lambda_j| = \sum_j i\theta_j |\lambda_j\rangle\langle\lambda_j| \text{ where } \theta_j = \arg(\lambda_j) \\
K &= -i\log(U) = \sum_j \theta_j |\lambda_j\rangle\langle\lambda_j|.
\end{aligned} \tag{114}$$

$$K^\dagger = (-i\log U)^\dagger = \left(\sum_j \theta_j |\lambda_j\rangle\langle\lambda_j|\right)^\dagger = \sum_j \theta_j^* |\lambda_j\rangle\langle\lambda_j| = \sum_j \theta_j |\lambda_j\rangle\langle\lambda_j| = K \tag{115}$$

2.57

$$
\begin{aligned}
|\phi\rangle &\equiv \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \\
\langle\phi|M_m^\dagger M_m|\phi\rangle &= \frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle} \\
\frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} &= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} \cdot \frac{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm}|\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}
\end{aligned} \tag{116}
$$

2.58

$$
\begin{aligned}
\langle M\rangle &= \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m\langle\psi|\psi\rangle = m \\
\langle M^2\rangle &= \langle\psi|M^2|\psi\rangle = \langle\psi|m^2|\psi\rangle = m^2\langle\psi|\psi\rangle = m^2 \\
\text{deviation} &= \langle M^2\rangle - \langle M\rangle^2 = m^2 - m^2 = 0.
\end{aligned} \tag{117}
$$

2.59

$$
\begin{aligned}
\langle X\rangle &= \langle 0|X|0\rangle = \langle 0|1\rangle = 0 \\
\langle X^2\rangle &= \langle 0|X^2|0\rangle = \langle 0|X|1\rangle = \langle 0|0\rangle = 1 \\
\text{standard deviation} &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} = 1
\end{aligned} \tag{118}
$$

2.60

$$
\begin{aligned}
\vec{v}\cdot\vec{\sigma} &= \sum_{i=1}^{3} v_i\sigma_i \\
&= v_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v_2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + v_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}v_3 & v_1 - iv_2\\ v_1 + iv_2 & -v_3\end{bmatrix}
\end{aligned} \tag{119}
$$

$$
\begin{aligned}
\det(\vec{v}\cdot\vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1 \quad (\because |\vec{v}| = 1)
\end{aligned} \tag{120}
$$

Eigenvalues are $\lambda = \pm 1$.

(i) if $\lambda = 1$

$$
\begin{aligned}
\vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} - I \\
&= \begin{bmatrix}v_3 - 1 & v_1 - iv_2\\ v_1 + iv_2 & -v_3 - 1\end{bmatrix}
\end{aligned} \tag{121}
$$

Normalized eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}}\begin{bmatrix}1\\ \frac{1-v_3}{v_1-iv_2}\end{bmatrix}$.

$$
\begin{aligned}
|\lambda_1\rangle\langle\lambda_1| &= \frac{1+v_3}{2}\begin{bmatrix}1\\ \frac{1-v_3}{v_1-iv_2}\end{bmatrix}\begin{bmatrix}1 & \frac{1-v_3}{v_1+iv_2}\end{bmatrix} \\
&= \frac{1+v_3}{2}\begin{bmatrix}1 & \frac{v_1-iv_2}{1+v_3}\\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3}\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}1+v_3 & v_1 - iv_2\\ v_1 + iv_2 & 1 - v_3\end{bmatrix} \\
&= \frac{1}{2}\left(I + \begin{bmatrix}v_3 & v_1 - iv_2\\ v_1 + iv_2 & -v_3\end{bmatrix}\right) \\
&= \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})
\end{aligned} \tag{122}
$$

(ii) If $\lambda = -1$.

$$\begin{aligned}\vec{v}\cdot\vec{\sigma} - \lambda I &= \vec{v}\cdot\vec{\sigma} + I \\ &= \begin{bmatrix} v_3+1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3+1 \end{bmatrix}\end{aligned} \tag{123}$$

Normalized eigenvalue is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned}|\lambda_{-1}\rangle\langle\lambda_{-1}| &= \frac{1-v_3}{2} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1+v_3}{v_1+iv_2} \end{bmatrix} \\ &= \frac{1-v_3}{2} \begin{bmatrix} 1 & -\frac{v_1-iv_2}{1-v_3} \\ -\frac{v_1+iv_2}{1-v_3} & \frac{1+v_3}{1-v_3} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1-v_3 & -(v_1-iv_2) \\ -(v_1+iv_2) & 1+v_3 \end{bmatrix} \\ &= \frac{1}{2}\left(I - \begin{bmatrix} v_3 & v_1 - iv_2 \\ (v_1+iv_2 & -v_3 \end{bmatrix} \right) \\ &= \frac{1}{2}(I - \vec{v}\cdot\vec{\sigma}).\end{aligned} \tag{124}$$

While I review my proof, I notice that my proof has a defect. The case $(v_1, v_2, v_3) = (0, 0, 1)$, second component of eigenstate, $\frac{1-v_3}{v_1-iv_2}$, diverges. So I implicitly assume $v_1 - iv_2 \neq 0$. Hence my proof is incomplete.

Since the exercise doesn't require explicit form of projector, we should prove the problem more abstractly. In order to prove, we use the following properties of $\vec{v}\cdot\vec{\sigma}$

- $\vec{v}\cdot\vec{\sigma}$ is Hermitian
- $(\vec{v}\cdot\vec{\sigma})^2 = I$ where $\vec{v}$ is a real unit vector.

We can easily check above conditions.

$$\begin{aligned}(\vec{v}\cdot\vec{\sigma})^\dagger &= (v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)^\dagger \\ &= v_1\sigma_1^\dagger + v_2\sigma_2^\dagger + v_3\sigma_3^\dagger \\ &= v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \quad (\because \text{Pauli matrices are Hermitian.}) \\ &= \vec{v}\cdot\vec{\sigma}\end{aligned} \tag{125}$$

$$\begin{aligned}(\vec{v}\cdot\vec{\sigma})^2 &= \sum_{j,k=1}^{3} (v_j\sigma_j)(v_k\sigma_k) \\ &= \sum_{j,k=1}^{3} v_j v_k \sigma_j \sigma_k \\ &= \sum_{j,k=1}^{3} v_j v_k \left(\delta_{jk} I + i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l\right) \quad (\because \text{eqn(2.78) page78}) \\ &= \sum_{j,k=1}^{3} v_j v_k \delta_{jk} I + i \sum_{j,k,l=1}^{3} \epsilon_{jkl} v_j v_k \sigma_l \\ &= \sum_{j=1}^{3} v_j^2 I \\ &= I \quad \left(\because \sum_j v_j^2 = 1\right)\end{aligned} \tag{126}$$

Proof. Suppose $|\lambda\rangle$ is an eigenstate of $\vec{v}\cdot\vec{\sigma}$ with eigenvalue λ. Then

$$\begin{aligned}\vec{v}\cdot\vec{\sigma}\,|\lambda\rangle &= \lambda\,|\lambda\rangle \\ (\vec{v}\cdot\vec{\sigma})^2\,|\lambda\rangle &= \lambda^2\,|\lambda\rangle\end{aligned} \tag{127}$$

On the other hand $(\vec{v}\cdot\vec{\sigma})^2 = I$,

$$\begin{aligned}(\vec{v}\cdot\vec{\sigma})^2 |\lambda\rangle = I|\lambda\rangle = |\lambda\rangle \\ \therefore \lambda^2|\lambda\rangle = |\lambda\rangle .\end{aligned} \tag{128}$$

Thus $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$. Therefore $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1.

Let $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are eigenvectors with eigenvalues 1 and -1, respectively. I will prove that $P_\pm = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|$.

In order to prove above equation, all we have to do is prove following condition. (see Theorem 0.0.1)

$$\langle\psi|(P_\pm - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|)|\psi\rangle = 0 \text{ for all } |\psi\rangle \in \mathbb{C}^2. \tag{129}$$

Since $\vec{v}\cdot\vec{\sigma}$ is Hermitian, $|\lambda_1\rangle$ and $|\lambda_{-1}\rangle$ are orthonormal vector ($\because$ Exercise 2.22). Let $|\psi\rangle \in \mathbb{C}^2$ be an arbitrary state. $|\psi\rangle$ can be written as

$$|\psi\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_{\pm 1}\rangle \quad (|\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C}). \tag{130}$$

$$\begin{aligned}
\langle\psi|(P_\pm - |\lambda_\pm\rangle\langle\lambda_\pm|)|\psi\rangle &= \langle\psi|P_\pm|\psi\rangle - \langle\psi|\lambda_\pm\rangle\langle\lambda_\pm|\psi\rangle . \\
\langle\psi|P_\pm|\psi\rangle &= \langle\psi|\frac{1}{2}(I \pm \vec{v}\cdot\vec{\sigma})|\psi\rangle \\
&= \frac{1}{2} \pm \frac{1}{2}\langle\psi|\vec{v}\cdot\vec{\sigma})|\psi\rangle \\
&= \frac{1}{2} \pm \frac{1}{2}(|\alpha|^2 - |\beta|^2) \\
&= \frac{1}{2} \pm \frac{1}{2}(2|\alpha|^2 - 1) \quad (\because |\alpha|^2 + |\beta|^2 = 1) \\
\langle\psi|\lambda_1\rangle\langle\lambda_1|\psi\rangle &= |\alpha|^2 \\
\langle\psi|\lambda_{-1}\rangle\langle\lambda_{-1}|\psi\rangle &= |\beta|^2 = 1 - |\alpha|^2
\end{aligned} \tag{131}$$

Therefore $\langle\psi|(P_\pm - |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|)|\psi\rangle = 0$ for all $|\psi\rangle \in \mathbb{C}^2$. Thus $P_\pm = |\lambda_{\pm 1}\rangle\langle\lambda_{\pm 1}|$. □

2.61

$$\begin{aligned}
\langle\lambda_1|0\rangle\langle 0|\lambda_1\rangle &= \langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle \\
&= \langle 0|\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|0\rangle \\
&= \frac{1}{2}(1 + v_3)
\end{aligned} \tag{132}$$

Post-measurement state is

$$\begin{aligned}
\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\langle 0|\lambda_1\rangle\langle\lambda_1|0\rangle}} &= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \cdot \frac{1}{2}\begin{bmatrix} 1 + v_3 \\ v_1 + iv_2 \end{bmatrix} \\
&= \sqrt{\frac{1}{2}(1+v_3)}\begin{bmatrix} 1 \\ \frac{v_1 + iv_2}{1 + v_3} \end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}}\begin{bmatrix} 1 \\ \frac{1 - v_3}{v_1 - iv_2} \end{bmatrix} \\
&= |\lambda_1\rangle .
\end{aligned} \tag{133}$$

2.62

Suppose M_m is a measurement operator. From the assumption, $E_m = M_m^\dagger M_m = M_m$.

Then

$$\langle\psi|E_m|\psi\rangle = \langle\psi|M_m|\psi\rangle \geq 0. \tag{134}$$

for all $|\psi\rangle$.

Since M_m is positive operator, M_m is Hermitian. Therefore,

$$E_m = M_m^\dagger M_m = M_m M_m = M_m^2 = M_m. \tag{135}$$

Thus the measurement is a projective measurement.

2.63

$$
\begin{aligned}
M_m^\dagger M_m &= \sqrt{E_m} U_m^\dagger U_m \sqrt{E_m} \\
&= \sqrt{E_m} I \sqrt{E_m} \\
&= E_m .
\end{aligned}
\tag{136}
$$

Since E_m is POVM, for arbitrary unitary U, $M_m^\dagger M_m$ is POVM.

2.64
Read following paper:

- Lu-Ming Duan, Guang-Can Guo. Probabilistic cloning and identification of linearly independent quantum states. Phys. Rev. Lett.,80:4999-5002, 1998. arXiv:quant-ph/9804064
`https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.80.4999`
`https://arxiv.org/abs/quant-ph/9804064`

- Stephen M. Barnett, Sarah Croke, Quantum state discrimination, arXiv:0810.1970 [quant-ph]
`https://arxiv.org/abs/0810.1970`
`https://www.osapublishing.org/DirectPDFAccess/67EF4200-CBD2-8E68-1979E37886263936_176580/aop-1-2-238.pdf`

2.65

$$
\left|+\right\rangle \equiv \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}, \quad \left|-\right\rangle \equiv \frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}} \tag{137}
$$

2.66

$$
X_1 Z_2 \left(\frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} \right) = \frac{\left|10\right\rangle - \left|01\right\rangle}{\sqrt{2}} \tag{138}
$$

$$
\langle X_1 Z_2 \rangle = \left(\frac{\left\langle 00\right| + \left\langle 11\right|}{\sqrt{2}} \right) X_1 Z_2 \left(\frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} \right) = \frac{\left\langle 00\right| + \left\langle 11\right|}{\sqrt{2}} \cdot \frac{\left|10\right\rangle - \left|01\right\rangle}{\sqrt{2}} = 0 \tag{139}
$$

2.67

Suppose $W^\perp$ is the orthogonal complement of W. Then $V = W \oplus W^\perp$. Let $\left|w_i\right\rangle, \left|w_j'\right\rangle, \left|u_j'\right\rangle$ be orthonormal bases for W, $W^\perp$, $(\text{image}(U))^\perp$, respectively.

Define $U' : V \to V$ as $U' = \sum_i \left|u_i\right\rangle\left\langle w_i\right| + \sum_j \left|u_j'\right\rangle\left\langle w_j'\right|$, where $\left|u_i\right\rangle = U \left|w_i\right\rangle$.

Now

$$
\begin{aligned}
(U')^\dagger U' &= \left(\sum_{i=1}^{\dim W} \left|w_i\right\rangle\left\langle u_i\right| + \sum_{j=1}^{\dim W^\perp} \left|w_j'\right\rangle\left\langle u_j'\right| \right) \left(\sum_i \left|u_i\right\rangle\left\langle w_i\right| + \sum_j \left|u_j'\right\rangle\left\langle w_j'\right| \right) \\
&= \sum_i \left|w_i\right\rangle\left\langle w_i\right| + \sum_j \left|w_j'\right\rangle\left\langle w_j'\right| = I
\end{aligned}
\tag{140}
$$

and

$$
\begin{aligned}
U'(U')^\dagger &= \left(\sum_i \left|u_i\right\rangle\left\langle w_i\right| + \sum_j \left|u_j'\right\rangle\left\langle w_j'\right| \right) \left(\sum_i \left|w_i\right\rangle\left\langle u_i\right| + \sum_j \left|w_j'\right\rangle\left\langle u_j'\right| \right) \\
&= \sum_i \left|u_i\right\rangle\left\langle u_i\right| + \sum_j \left|u_j'\right\rangle\left\langle u_j'\right| = I .
\end{aligned}
\tag{141}
$$

Thus U' is an unitary operator. Moreover, for all $|w\rangle \in W$,

$$\begin{aligned}
U'|w\rangle &= \left(\sum_i |u_i\rangle\langle w_i| + \sum_j |u'_j\rangle\langle w'_j|\right)|w\rangle \\
&= \sum_i |u_i\rangle \langle w_i|w\rangle + \sum_j |u'_j\rangle \langle w'_j|w\rangle \\
&= \sum_i |u_i\rangle \langle w_i|w\rangle \quad (\because |w'_j\rangle \perp |w\rangle) \\
&= \sum_i U|w_i\rangle \langle w_i|w\rangle \\
&= U|w\rangle .
\end{aligned} \tag{142}$$

Therefore U' is an extension of U.

2.68

$|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.
Suppose $|a\rangle = a_0|0\rangle + a_1|1\rangle$ and $|b\rangle = b_0|0\rangle + b_1|1\rangle$.

$$|a\rangle|b\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle . \tag{143}$$

If $|\psi\rangle = |a\rangle|b\rangle$, then $a_0b_0 = 1,\ a_0b_1 = 0,\ a_1b_0 = 0,\ a_1b_1 = 1$ since $\{|ij\rangle\}$ is an orthonormal basis.
If $a_0b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$.
When $a_0 = 0$, this is contradiction to $a_0b_0 = 1$. When $b_1 = 0$, this is contradiction to $a_1b_1 = 1$.
Thus $|\psi\rangle \neq |a\rangle|b\rangle$.

2.69

Define Bell states as follows.

$$\begin{aligned}
|\psi_1\rangle &\equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} \\
|\psi_2\rangle &\equiv \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \\
|\psi_3\rangle &\equiv \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \\
|\psi_4\rangle &\equiv \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}
\end{aligned} \tag{144}$$

First, we prove $\{|\psi_i\rangle\}$ is a linearly independent basis.

$$\begin{aligned}
a_1|\psi_1\rangle + a_2|\psi_2\rangle + a_3|\psi_3\rangle + a_4|\psi_4\rangle = 0 \\
\therefore \frac{1}{\sqrt{2}}\begin{bmatrix}a_1 + a_2\\a_3 + a_4\\a_3 - a_4\\a_1 - a_2\end{bmatrix} = 0
\end{aligned} \tag{145}$$

$$\therefore \begin{cases} a_1 + a_2 = 0 \\ a_3 + a_4 = 0 \\ a_3 - a_4 = 0 \\ a_1 - a_2 = 0 \end{cases}$$

$$\therefore a_1 = a_2 = a_3 = a_4 = 0 \tag{147}$$

Thus $\{|\psi_i\rangle\}$ is a linearly independent basis.

Moreover $\| |\psi_i\rangle \| = 1$ and $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Therefore $\{|\psi_i\rangle\}$ forms an orthonormal basis.

2.70

For any Bell states we get $\langle\psi_i|E \otimes I|\psi_i\rangle = \frac{1}{2}(\langle 0|E|0\rangle + \langle 1|E|1\rangle)$.

Suppose Eve measures the qubit Alice sent by measurement operators M_m. The probability that Eve gets result m is $p_i(m) = \langle\psi_i|M_m^\dagger M_m \otimes I|\psi_i\rangle$. Since $M_m^\dagger M_m$ is positive, $p_i(m)$ are same values for all $|\psi_i\rangle$. Thus Eve can't distinguish Bell states.

2.71

From spectral decomposition,

$$\begin{aligned}
\rho &= \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \,, \quad p_i \geq 0, \quad \sum_i p_i = 1. \\
\rho^2 &= \sum_{i,j} p_i p_j \, |i\rangle \, \langle i|j\rangle \, \langle j| \\
&= \sum_{i,j} p_i p_j \, |i\rangle\langle j| \, \delta_{ij} \\
&= \sum_i p_i^2 \, |i\rangle\langle i|
\end{aligned} \tag{148}$$

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\sum_i p_i^2 \, |i\rangle\langle i|\right) = \sum_i p_i^2 \, \mathrm{Tr}(|i\rangle\langle i|) = \sum_i p_i^2 \, \langle i|i\rangle = \sum_i p_i^2 \leq \sum_i p_i = 1 \quad (\because p_i^2 \leq p_i) \tag{149}$$

Suppose $\mathrm{Tr}(\rho^2) = 1$. Then $\sum_i p_i^2 = 1$. Since $p_i^2 < p_i$ for $0 < p_i < 1$, only single p_i should be 1 and otherwise have to vanish. Therefore $\rho = |\psi_i\rangle\langle\psi_i|$. It is a pure state.

Conversely if ρ is pure, then $\rho = |\psi\rangle\langle\psi|$.

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}(|\psi\rangle \, \langle\psi|\psi\rangle \, \langle\psi|) = \mathrm{Tr}(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle = 1. \tag{150}$$

2.72

(1) Since density matrix is Hermitian, matrix representation is $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$, $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$ w.r.t. standard basis. Because ρ is density matrix, $\mathrm{Tr}(\rho) = a + d = 1$.

Define $a = (1 + r_3)/2$, $d = (1 - r_3)/2$ and $b = (r_1 - ir_2)/2$, $(r_i \in \mathbb{R})$.

In this case,

$$\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{bmatrix} = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}). \tag{151}$$

Thus for arbitrary density matrix ρ can be written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

Next, we derive the condition that ρ is positive.

If ρ is positive, all eigenvalues of ρ should be non-negative.

$$\begin{aligned}
\det(\rho - \lambda I) = (a - \lambda)(b - \lambda) - |b|^2 &= \lambda^2 - (a + d)\lambda + ad - |b^2| = 0 \\
\lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - |b|^2)}}{2} \\
&= \frac{1 \pm \sqrt{1 - 4\left(\frac{1 - r_3^2}{4} - \frac{r_1^2 + r_2^2}{4}\right)}}{2} \\
&= \frac{1 \pm \sqrt{1 - (1 - r_1^2 - r_2^2 - r_3^2)}}{2} \\
&= \frac{1 \pm \sqrt{|\vec{r}|^2}}{2} \\
&= \frac{1 \pm |\vec{r}|}{2}
\end{aligned} \tag{152}$$

Since ρ is positive, $\frac{1-|\vec{r}|}{2} \geq 0 \to |\vec{r}| \leq 1$.
Therefore an arbitrary density matrix for a mixed state qubit is written as $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

(2)
$\rho = I/2 \to \vec{r} = 0$. Thus $\rho = I/2$ corresponds to the origin of Bloch sphere.

(3)

$$\begin{aligned}
\rho^2 &= \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \ \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \\
&= \frac{1}{4}\left[I + 2\vec{r} \cdot \vec{\sigma} + \sum_{j,k} r_j r_k \left(\delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l\right)\right] \\
&= \frac{1}{4}\left(I + 2\vec{r} \cdot \vec{\sigma} + |\vec{r}|^2 I\right) \\
\mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2)
\end{aligned} \tag{153}$$

If ρ is pure, then $\mathrm{Tr}(\rho^2) = 1$.

$$\begin{aligned}
1 = \mathrm{Tr}(\rho^2) &= \frac{1}{4}(2 + 2|\vec{r}|^2) \\
&\therefore |\vec{r}| = 1.
\end{aligned} \tag{154}$$

Conversely, if $|\vec{r}| = 1$, then $\mathrm{Tr}(\rho^2) = \frac{1}{4}(2 + 2|\vec{r}|^2) = 1$. Therefore ρ is pure.

2.73

Theorem 2.6

$$\rho = \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j| = \sum_j q_j \left|\varphi_j\right\rangle\!\left\langle\varphi_j\right| \quad \Leftrightarrow \quad |\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle \tag{155}$$

where u is unitary.

The-transformation in theorem 2.6, $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$, corresponds to

$$\left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle\right] = \left[\,|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle\,\right] U^T \tag{156}$$

where $k = \mathrm{rank}(\rho)$.

$$\begin{aligned}
\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| &= \left[|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_k\rangle\right] \begin{bmatrix} \langle\tilde{\psi}_1| \\ \vdots \\ \langle\tilde{\psi}_k| \end{bmatrix} \\
&= \left[\,|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle\,\right] U^T U^* \begin{bmatrix} \langle\tilde{\varphi}_1| \\ \vdots \\ \langle\tilde{\varphi}_k| \end{bmatrix} \\
&= \left[\,|\tilde{\varphi}_1\rangle \cdots |\tilde{\varphi}_k\rangle\,\right] \begin{bmatrix} \langle\tilde{\varphi}_1| \\ \vdots \\ \langle\tilde{\varphi}_k| \end{bmatrix} \\
&= \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j| .
\end{aligned} \tag{157}$$

From spectral theorem, density matrix ρ is decomposed as $\rho = \sum_{k=1}^{d} \lambda_k \left|k\right\rangle\!\left\langle k\right|$ where $d = \dim \mathcal{H}$. Without loss of generality, we can assume $p_k > 0$ for $k = 1 \cdots, l$ where $l = \mathrm{rank}(\rho)$ and $p_k = 0$ for $k = l+1, \cdots, d$. Thus $\rho = \sum_{k=1}^{l} p_k \left|k\right\rangle\!\left\langle k\right| = \sum_{k=1}^{l} |\tilde{k}\rangle\langle\tilde{k}|$, where $|\tilde{k}\rangle = \sqrt{\lambda_k}\, |k\rangle$.

Suppose $|\psi_i\rangle$ is a state in support ρ. Then

$$|\psi_i\rangle = \sum_{k=1}^{l} c_{ik} |k\rangle, \quad \sum_k |c_{ik}|^2 = 1. \tag{158}$$

Define $p_i = \dfrac{1}{\sum_k \frac{|c_{ik}|^2}{\lambda_k}}$ and $u_{ik} = \dfrac{\sqrt{p_i} c_{ik}}{\sqrt{\lambda_k}}$.

Now

$$\sum_k |u_{ik}|^2 = \sum_k \frac{p_i |c_{ik}|^2}{\lambda_k} = p_i \sum_k \frac{|c_{ik}|^2}{\lambda_k} = 1. \tag{159}$$

Next prepare an unitary operator [1] such that ith row of U is $[u_{i1} \cdots u_{ik} \cdots u_{il}]$. Then we can define another ensemble such that

$$\left[|\tilde{\psi_1}\rangle \cdots |\tilde{\psi_i}\rangle \cdots |\tilde{\psi_l}\rangle \right] = \left[|\tilde{k}_1\rangle \cdots |\tilde{k}_l\rangle \right] U^T \tag{160}$$

where $|\tilde{\psi_i}\rangle = \sqrt{p_i} |\psi_i\rangle$. From theorem 2.6,

$$\rho = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi_k}\rangle\langle\tilde{\psi_k}|. \tag{161}$$

Therefore we can obtain a minimal ensemble for ρ that contains $|\psi_i\rangle$.

Moreover since $\rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle\langle k|$,

$$\langle\psi_i|\rho^{-1}|\psi_i\rangle = \sum_k \frac{1}{\lambda_k} \langle\psi_i|k\rangle\langle k|\psi_i\rangle = \sum_k \frac{|c_{ik}|^2}{\lambda_k} = \frac{1}{p_i}. \tag{162}$$

Hence, $\frac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle} = p_i$.

2.74

$$\begin{aligned} \rho_{AB} &= |a\rangle\langle a|_A \otimes |b\rangle\langle b|_B \\ \rho_A &= \mathrm{Tr}_B\, \rho_{AB} = |a\rangle\langle a| \,\mathrm{Tr}(|b\rangle\langle b|) = |a\rangle\langle a| \\ \mathrm{Tr}(\rho_A^2) &= 1 \end{aligned} \tag{163}$$

Thus ρ_A is pure.

2.75

Define $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

$$\begin{aligned} |\Phi_\pm\rangle\langle\Phi_\pm|_{AB} &= \frac{1}{2}(|00\rangle\langle 00| \pm |00\rangle\langle 11| \pm |11\rangle\langle 00| + |11\rangle\langle 11|) \\ \mathrm{Tr}_B(|\Phi_\pm\rangle\langle\Phi_\pm|_{AB}) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \\ |\Psi_\pm\rangle\langle\Psi_\pm| &= \frac{1}{2}(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|) \\ \mathrm{Tr}_B(|\Psi_\pm\rangle\langle\Psi_\pm|) &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \end{aligned} \tag{164}$$

2.76

Unsolved. ~~I think the polar decomposition can only apply to square matrix A, not arbitrary linear operators. Suppose A is $m \times n$ matrix. Then size of $A^\dagger A$ is $n \times n$. Thus the size of U should be $m \times n$. Maybe U is isometry, but I think it is not unitary.~~

I misunderstand linear operator.

[1] By Gram-Schmidt procedure construct an orthonormal basis $\{\boldsymbol{u}_j\}$ (row vector) with $\boldsymbol{u}_i = [u_{i1} \cdots u_{ik} \cdots u_{il}]$. Then define unitary $U = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_i \\ \vdots \\ \boldsymbol{u}_l \end{bmatrix}$.

Quoted from "Advanced Liner Algebra" by Steven Roman, ISBN 0387247661.

A linear transformation $\tau : V \to V$ is called a **linear operator** on V.[2]

Thus coordinate matrices of linear operator are square matrices. And Nielsen and Chaung say at Theorem 2.3, "Let A be a linear operator on a vector space V." Therefore A is a linear transformation such that $A : V \to V$.

2.77

$$
\begin{aligned}
\left|\psi\right\rangle &= \left|0\right\rangle \left|\Phi_+\right\rangle \\
&= \left|0\right\rangle \left[\frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)\right] \\
&= (\alpha \left|\phi_0\right\rangle + \beta \left|\phi_1\right\rangle) \left[\frac{1}{\sqrt{2}}(\left|\phi_0\phi_0\right\rangle + \left|\phi_1\phi_1\right\rangle)\right]
\end{aligned}
\tag{165}
$$

where $\left|\phi_i\right\rangle$ are arbitrary orthonormal states and $\alpha, \beta \in \mathbb{C}$. We cannot vanish cross term. Therefore $\left|\psi\right\rangle$ cannot be written as $\left|\psi\right\rangle = \sum_i \lambda_i \left|i\right\rangle_A \left|i\right\rangle_B \left|i\right\rangle_C$.

2.78

Proof. Former part.

If $\left|\psi\right\rangle$ is product, then there exist a state $\left|\phi_A\right\rangle$ for system A, and a state $\left|\phi_B\right\rangle$ for system B such that $\left|\psi\right\rangle = \left|\phi_A\right\rangle \left|\phi_B\right\rangle$. Obviously, this Schmidt number is 1.

Conversely, if Schmidt number is 1, the state is written as $\left|\psi\right\rangle = \left|\phi_A\right\rangle \left|\phi_B\right\rangle$. Hence this is a product state. □

Proof. Later part.

(⇒) Proved by exercise 2.74.

(⇐) Let a pure state be $\left|\psi\right\rangle = \sum_i \lambda_i \left|i_A\right\rangle \left|i_B\right\rangle$. Then $\rho_A = \mathrm{Tr}_B(\left|\psi\right\rangle\left\langle\psi\right|) = \sum_i \lambda_i^2 \left|i\right\rangle\left\langle i\right|$. If ρ_A is a pure state, then $\lambda_j = 1$ and otherwise 0 for some j. It follows that $\left|\psi_j\right\rangle = \left|j_A\right\rangle \left|j_B\right\rangle$. Thus $\left|\psi\right\rangle$ is a product state. □

2.79

Procedure of Schmidt decomposition.
Goal: $\left|\psi\right\rangle = \sum_i \sqrt{\lambda_i} \left|i_A\right\rangle \left|i_B\right\rangle$

- Diagonalize reduced density matrix $\rho_A = \sum_i \lambda_i \left|i_A\right\rangle\left\langle i_A\right|$.
- Derive $\left|i_B\right\rangle$, $\left|i_B\right\rangle = \dfrac{(I \otimes \left\langle i_A\right|) \left|\psi\right\rangle}{\sqrt{\lambda_i}}$
- Construct $\left|\psi\right\rangle$.

(i)

$$\frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle) \text{ This is already decomposed.} \tag{166}$$

(ii)

$$\frac{\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle}{2} = \left(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}\right) \otimes \left(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}\right) = \left|\psi\right\rangle \left|\psi\right\rangle \text{ where } \left|\psi\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}} \tag{167}$$

(iii)

$$
\begin{aligned}
\left|\psi\right\rangle_{AB} &= \frac{1}{\sqrt{3}}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle) \\
\rho_{AB} &= \left|\psi\right\rangle\left\langle\psi\right|_{AB}
\end{aligned}
\tag{168}
$$

[2]According to Roman, some authors use the term linear operator for any linear transformation from V to W.

$$\begin{aligned}
\rho_A = \mathrm{Tr}_B(\rho_{AB}) &= \frac{1}{3}\left(2\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right) \\
\det(\rho_A - \lambda I) = \left(\frac{2}{3}-\lambda\right)&\left(\frac{1}{3}-\lambda\right) - \frac{1}{9} = 0 \\
\lambda^2 - \lambda + \frac{1}{9} &= 0 \\
\lambda = \frac{1 \pm \sqrt{5}/3}{2} &= \frac{3 \pm \sqrt{5}}{6}
\end{aligned} \tag{169}$$

Eigenvector with eigenvalue $\lambda_0 \equiv \dfrac{3+\sqrt{5}}{6}$ is $|\lambda_0\rangle \equiv \dfrac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}}\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

Eigenvector with eigenvalue $\lambda_1 \equiv \dfrac{3-\sqrt{5}}{6}$ is $|\lambda_1\rangle \equiv \dfrac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

$$\rho_A = \lambda_0\,|\lambda_0\rangle\langle\lambda_0| + \lambda_1\,|\lambda_1\rangle\langle\lambda_1|\,. \tag{170}$$

$$\begin{aligned}
|a_0\rangle &\equiv \frac{(I \otimes \langle\lambda_0|)\,|\psi\rangle}{\sqrt{\lambda_0}} \\
|a_1\rangle &\equiv \frac{(I \otimes \langle\lambda_1|)\,|\psi\rangle}{\sqrt{\lambda_1}}
\end{aligned} \tag{171}$$

Then

$$|\psi\rangle = \sum_{i=0}^{1} \sqrt{\lambda_i}\,|a_i\rangle\,|\lambda_i\rangle\,. \tag{172}$$

(It's too tiresome to calculate $|a_i\rangle$)

2.80

Let $|\psi\rangle = \sum_i \lambda_i\,|\psi_i\rangle_A\,|\psi_i\rangle_B$ and $|\varphi\rangle = \sum_i \lambda_i\,|\varphi_i\rangle_A\,|\varphi_i\rangle_B$.
Define $U = \sum_i |\psi_j\rangle\langle\varphi_j|_A$ and $V = \sum_j |\psi_j\rangle\langle\varphi_j|_B$.
Then

$$\begin{aligned}
(U \otimes V)\,|\varphi\rangle &= \sum_i \lambda_i U\,|\varphi_i\rangle_A\,V\,|\varphi_i\rangle_B \\
&= \sum_i \lambda_i\,|\psi_i\rangle_A\,|\psi_i\rangle_B \\
&= |\psi\rangle\,.
\end{aligned} \tag{173}$$

2.81

Let the Schmidt decomposition of $|AR_1\rangle$ be $|AR_1\rangle = \sum_i \sqrt{p_i}\,|\psi_i^A\rangle\,|\psi_i^R\rangle$ and let $|AR_2\rangle = \sum_i \sqrt{q_i}\,|\phi_i^A\rangle\,|\phi_i^R\rangle$.
Suppose ρ^A has orthonormal decomposition $\rho^A = \sum_i p_i\,|i\rangle\langle i|$.
Since $|AR_1\rangle$ and $|AR_2\rangle$ are purifications of the ρ^A, we have

$$\begin{aligned}
|AR_1\rangle &= \sum_i \sqrt{p_i}\,|i\rangle\,|\psi_i\rangle \\
|AR_2\rangle &= \sum_i \sqrt{p_i}\,|i\rangle\,|\phi_i\rangle
\end{aligned} \tag{174}$$

where $|\psi_i\rangle$ and $|\phi_i\rangle$ are orthonormal bases on system R.

$$\begin{aligned}
\mathrm{Tr}_R(|AR_1\rangle\langle AR_1|) &= \mathrm{Tr}_R(|AR_2\rangle\langle AR_2|) = \rho^A \\
\therefore \sum_i p_i\,|\psi_i^A\rangle\langle\psi_i^A| &= \sum_i q_i\,|\phi_i^A\rangle\langle\phi_i^A| = \sum_i \lambda_i\,|i\rangle\langle i|\,.
\end{aligned} \tag{175}$$

The $|i\rangle$, $|\psi_i^A\rangle$, and $|\psi_i^A\rangle$ are orthonormal bases and they are eigenvectors of ρ^A. Hence without loss of generality, we can consider

$$\lambda_i = p_i = q_i \text{ and } |i\rangle = |\psi_i^A\rangle = |\phi_i^A\rangle\,. \tag{176}$$

Then

$$\begin{aligned}
|AR_1\rangle &= \sum_i \lambda_i |i\rangle |\psi_i^R\rangle \\
|AR_2\rangle &= \sum_i \lambda_i |i\rangle |\phi_i^R\rangle
\end{aligned} \tag{177}$$

Since $|AR_1\rangle$ and $|AR_2\rangle$ have same Schmidt numbers, there are two unitary operators U and V such that $|AR_1\rangle = (U \otimes V)|AR_2\rangle$ from exercise 2.80.

Suppose $U = I$ and $V = \sum_i |\psi_i^R\rangle\langle\phi_i^R|$. Then

$$\begin{aligned}
\left(I \otimes \sum_j |\psi_j^R\rangle\langle\phi_j^R|\right)|AR_2\rangle &= \sum_i \lambda_i |i\rangle \left(\sum_j |\psi_j^R\rangle \langle\phi_j^R|\phi_i^R\rangle\right) \\
&= \sum_i \lambda_i |i\rangle |\psi_i^R\rangle \\
&= |AR_1\rangle .
\end{aligned} \tag{178}$$

Therefore there exists a unitary transformation U_R acting on system R such that $|AR_1\rangle = (I \otimes U_R)|AR_2\rangle$.

2.82

(1)

Let $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$.

$$\begin{aligned}
\mathrm{Tr}_R(|\psi\rangle\langle\psi|) &= \sum_{i,j} \sqrt{p_i}\sqrt{p_j}\, |\psi_i\rangle\langle\psi_j|\, \mathrm{Tr}_R(|i\rangle\langle j|) \\
&= \sum_{i,j} \sqrt{p_i}\sqrt{p_j}\, |\psi_i\rangle\langle\psi_j|\, \delta_{ij} \\
&= \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho.
\end{aligned} \tag{179}$$

Thus $|\psi\rangle$ is a purification of ρ.

(2)

Define the projector P by $P = I \otimes |i\rangle\langle i|$. The probability we get the result i is

$$\mathrm{Tr}\left[P|\psi\rangle\langle\psi|\right] = \langle\psi|P|\psi\rangle = \langle\psi|(I \otimes |i\rangle\langle i|)|\psi\rangle = p_i \langle\psi_i|\psi_i\rangle = p_i. \tag{180}$$

The post-measurement state is

$$\frac{P|\psi\rangle}{\sqrt{p_i}} = \frac{(I \otimes |i\rangle\langle i|)|\psi\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i}|\psi_i\rangle|i\rangle}{\sqrt{p_i}} = |\psi_i\rangle|i\rangle . \tag{181}$$

If we only focus on the state on system A,

$$\mathrm{Tr}_R(|\psi_i\rangle|i\rangle) = |\psi_i\rangle . \tag{182}$$

(3)

($\{|\psi_i\rangle\}$ is not necessary an orthonormal basis.)

Suppose $|AR\rangle$ is a purification of ρ and its Schmidt decomposition is $|AR\rangle = \sum_i \sqrt{\lambda_i}|\phi_i^A\rangle|\phi_i^R\rangle$.

From assumption

$$\mathrm{Tr}_R\left(|AR\rangle\langle AR|\right) = \sum_i \lambda_i |\phi_i^A\rangle\langle\phi_i^A| = \sum_i p_i |\psi_i\rangle\langle\psi_i| . \tag{183}$$

By theorem 2.6, there exits an unitary matrix u_{ij} such that $\sqrt{\lambda_i}\,|\phi_i^A\rangle = \sum_j u_{ij}\sqrt{p_j}\,|\psi_j\rangle$. Then

$$\begin{aligned}
|AR\rangle &= \sum_i \left(\sum_j u_{ij}\sqrt{p_j}\,|\psi_j\rangle\right)|\phi_i^R\rangle \\
&= \sum_j \sqrt{p_j}\,|\psi_j\rangle \otimes \left(\sum_i u_{ij}\,|\phi_i^R\rangle\right) \\
&= \sum_j \sqrt{p_j}\,|\psi_j\rangle\,|j\rangle \\
&= \sum_i \sqrt{p_i}\,|\psi_i\rangle\,|i\rangle
\end{aligned} \tag{184}$$

where $|i\rangle = \sum_k u_{ki}\,|\phi_k^R\rangle$.

About $|i\rangle$,

$$\begin{aligned}
\langle k|l\rangle &= \sum_{m,n} u_{mk}^* u_{nl}\,\langle\phi_m^R|\phi_n^R\rangle \\
&= \sum_{m,n} u_{mk}^* u_{nl}\delta_{mn} \\
&= \sum_m u_{mk}^* u_{ml} \\
&= \delta_{kl}, \quad (\because u_{ij} \text{ is unitary.})
\end{aligned} \tag{185}$$

which implies $|j\rangle$ is an orthonormal basis for system R.

Therefore if we measure system R w.r.t $|j\rangle$, we obtain j with probability p_j and post-measurement state for A is $|\psi_j\rangle$ from (2). Thus for any purification $|AR\rangle$, there exists an orthonormal basis $|i\rangle$ which satisfies the assertion.

Problem 2.1

From Exercise 2.35, $\vec{n}\cdot\vec{\sigma}$ is decomposed as

$$\vec{n}\cdot\vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}| \tag{186}$$

where $|\lambda_{\pm 1}\rangle$ are eigenvector of $\vec{n}\cdot\vec{\sigma}$ with eigenvalues ± 1.

Thus

$$\begin{aligned}
f(\theta\vec{n}\cdot\vec{\sigma}) &= f(\theta)\,|\lambda_1\rangle\langle\lambda_1| + f(-\theta)\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \left(\frac{f(\theta)+f(-\theta)}{2} + \frac{f(\theta)-f(-\theta)}{2}\right)|\lambda_1\rangle\langle\lambda_1| + \left(\frac{f(\theta)+f(-\theta)}{2} - \frac{f(\theta)-f(-\theta)}{2}\right)|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \frac{f(\theta)+f(-\theta)}{2}\left(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|\right) + \frac{f(\theta)-f(-\theta)}{2}\left(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|\right) \\
&= \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\cdot\vec{\sigma}
\end{aligned} \tag{187}$$

Problem 2.2
Unsolved

Problem 2.3
Unsolved