Some Rules Of Counting That Are Useful In Probability

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September 26, 2016

Many aspects of probability will be based on counting or enumerative principles. This little note summarizes some of the more useful rules of enumeration.

Rule #1 If a particular event can occur in n different ways and a second event can happen in m different ways, then (a) the number of ways that *either* the first or the second event may occur is $m + n$; while (b) the number of ways that *both* the first and the second event may occur is mn. As an example of (a), if you are planning a trip and there are 2 different plane routes and 4 different train routes, then there are 6 different routes available for the trip. Note that the first event is choosing a plane while the second event involves choosing a train. You would choose either a train or a plane but you would not choose both a train and a plane. As an example of (b), if you have 4 different shirts and 3 different pairs of pants, then there are 12 different ways you may dress using the above shirts and pairs of pants. Note that the first event is choosing a shirt while the second event is choosing a pair of pants and you would dress wearing both a shirt and a pair of pants.

Rule #2 If n objects are selected at random, one-at-a-time, from a set of N *distinguishable* objects *with replacement* after each selection, then the number of ways this total selection can be accomplished, given that the *order of the selected objects is important* is given by N^n . This is also called the number of arrangements of N distinct objects taken n at a time with unlimited repetitions. For example, suppose there are 5 different names in a drawing and there are 3 prizes with the prizes all different and labeled as 1st, 2nd and 3rd, respectively. These are to be given away with any one person having a chance of winning from none to all of the prizes. A name is drawn, thereby announcing the winner of the first prize. This name is then returned back into the drawing. A second name is then drawn thereby announcing the winner of the second prize. It too, is returned back into the drawing. Finally another name is drawn thereby announcing the winner of the third and last prize. Note that the order in which a name is drawn is important because it determines whether that name wins the 1st, 2nd or 3rd, prize. All in all there are $5^3 = 125$ different possible outcomes which would include anything from the first person winning the 1st, 2nd and 3rd, prizes (and no one else winning any of the prizes) to the fifth person winning the 1st, 2nd and 3rd, prizes (and no one else winning any of the prizes).

Rule #3 If n objects are selected at random, one-at-a-time, from a set of N *distinguishable* objects *without replacement* after each selection, then the number of ways this total selection can be accomplished, given that *the order of the selected objects is important* is given by

$$P(N, n) = \frac{N!}{(N-n)!}.$$

This is also called the number of arrangements of N distinct objects taken n at a time without repetitions. Note that here we must have $n \leq N$ because the objects drawn are not returned back into the drawing so that it is not possible to select more than N objects. As an example of this counting rule, consider the above drawing of 5 names for 3 prizes where we now impose the restriction that any given person can win at most one prize. As a result, whenever a name is drawn, it will not be replaced. Since the prizes are still different (labeled as 1st, 2nd and 3rd), the order in which a person is chosen is important. In total, there are

$$P(5, 3) = \frac{5!}{(5-3)!} = 60$$

possible ways of announcing the winners of the 1st, 2nd and 3rd prizes ranging from the first, second and third persons winning the 1st, 2nd and 3rd prizes, respectively, to the 3rd, 4th and 5th persons winning the 3rd, 2nd and 1st prizes, respectively.

Rule #4 If n objects are selected at random, one-at-a-time, from a set of N *distinguishable* objects *without replacement* after each selection, then the number of ways this total selection can be accomplished, given that *the order of the selected objects is not important* is given by

$$\binom{N}{n} = \frac{P(N,n)}{n!} = \frac{N!}{n!(N-n)!}.$$

This is also called the number of combinations of N objects taken n at a time without repetitions. Note that here also we must have $n \leq N$ because the objects drawn are not returned back into the drawing so that it is not possible to select more than N objects. As an example, we go back to the above drawing of 5 names for 3 prizes where we now impose the restriction that any given person can win at most one prize and that all three prizes are identical. As a result, whenever a name is drawn, it will not be replaced and it does not matter whether a winner's name is drawn first, second or third. In total there are now

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$

possible ways of announcing the winners of the three prizes ranging from the first, second and third persons winning a prize, to the 3rd, 4th and 5th persons winning a prize.

Rule #5 If n objects are selected at random, one-at-a-time, from a set of N *distinguishable* objects *with replacement* being made after each selection, then the number of ways this total selection can be accomplished, given that *the order of the selected objects is not important* is given by

$$\binom{N+n-1}{n} = \frac{(N+n-1)!}{n!(N-1)!}.$$

Note here that we need not require that $n \leq N$ because we are allowing for replacement of any objects selected. As an example, we again go back to the above drawing of 5 names for 3 prizes. This time we go back to the possibility that any given person can win up to all three prizes while we still keep the restriction that all three prizes are identical. As a result, whenever a name is drawn, it will be replaced but now it does not matter whether a winner's name is drawn first, second or third since all three prizes are identical. In total there are now

$$\binom{5+3-1}{3} = \frac{(5+3-1)!}{3!(5-1)!} = 35$$

possible ways of announcing the winners of the three prizes ranging from the first person winning all of the prizes (and no one else winning any of the prizes) to the fifth person winning all of the prizes (and no one else winning any of the prizes). The above rule can be used to answer the following ball-and-urn question. If N indistinguishable balls are dispensed among n distinguishable urns, how may ways may this be accomplished? Treating the n urns as fixed, this question is equivalent to asking, how many ways can N objects be selected out of n distinguishable objects with replacement whose answer is given by the binomial coefficient

$$\binom{N+n-1}{N} = \frac{(N+n-1)!}{n!(N-1)!}.$$

Rule #6 If n objects are selected at random, with n_1 of one kind, n_2 of a second kind, n_3 of a third kind, ..., n_k of a kth kind (with $n_1 + n_2 + \cdots + n_k = n$), then the number of ways that this selection can be made given that the order of the distinguishable selected objects is important is given by the multinomial coefficient

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}.$$

For example, if there are 10 books consisting of 3 identical mathematics books, 2 identical English books, 1 history book and 4 identical physics books, the number of distinguishable ways these books may be arranged on a linear shelf is given by

$$\binom{10}{3, 2, 1, 4} = \frac{10!}{(3!)(2!)(1!)(4!)} = 12,600.$$

Rule #7 Suppose that you had d identical balls that are to be distributed among n identical urns so that no urn receives R or more balls. The number of ways this can be accomplished is given by

$$\binom{n, R}{d} = \sum_{k=0}^{[d/R]} (-1)^k \binom{n}{k} \binom{d - kR + n - 1}{n - 1}$$

where $[d/R]$ is the greatest integer less than or equal to d/R. The above quantities satisfy,

$$\sum_{d=0}^{n(R-1)} \binom{n, R}{d} x^d = \left(\frac{1 - x^R}{1 - x}\right)^n$$

and

$$\binom{n,2}{d} = \binom{n}{d} = \frac{n!}{d!(n-d)!}.$$

As an example, the number of ways to distribute 3 balls over 4 urns so that no urn gets 3 or more balls is

$$\binom{4,3}{3} = \sum_{k=0}^{[3/3]} (-1)^k \binom{4}{k} \binom{3-3k+4-1}{4-1} = \sum_{k=0}^{1} (-1)^k \binom{4}{k} \binom{6-3k}{3}$$

or

$$\binom{4,3}{3} = \binom{4}{0}\binom{6}{3} - \binom{4}{1}\binom{3}{3} = 16$$

As we proceed throughout the discussions of probability, other counting arguments will be developed. The above seven rules are intended to be used as a guide in these developments.

Permutations Results

Here we summarize the results for permutations which are ordered selections from a set S. If S is a set with n elements, then the number of permutations of size k is an ordered sequence of elements taken from the set S and these elements can be taken with repetitions allowed or with repetitions not allowed. When repetitions are allowed, an element of S is chosen, recorded and then returned to S so that it could be possibly chosen again. When repetitions are not allowed, an element of S is chosen, recorded and not returned to S so that it can not be chosen again.

Permutations With Unrestricted Repetitions

If S is a set with n elements, then the number of permutations of size k with unrestricted repetitions, taken from the set S, is

$$\left(\begin{array}{c} \text{The number of permutations of size } k \\ \text{with } \textit{unrestricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = n^k.$$

Permutations With No Repetitions

If S is a set with n elements, then the number of permutations of size k with no repetitions, taken from the set S, is

$$\left(\begin{array}{c}\text{The number of permutations of size}\\ k \text{ with no repetitions taken from}\\ \text{a set } S \text{ with } n \text{ distinct elements}\end{array}\right) = \frac{n!}{(n-k)!}.$$

Permutations With A Fixed Number Of Repetitions

Let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a set with n elements. Then the number of permutations of size k with fixed repetitions, taken from the set S, so that a_i is repeated exactly r_i times for all $i = 1, 2, 3, ..., n$, is

$$\left(\begin{array}{c}\text{The number of permutations of size } k\\ \text{(with } a_i \text{ repeated exactly } r_i \text{ times)}\\ \text{from a set } S \text{ with } n \text{ distinct elements}\end{array}\right) = \frac{k!}{r_1!r_2!r_3!\cdots r_n!}$$

with

$$r_1 + r_2 + r_3 + \cdots + r_n = k.$$

Note that this is the multinomial coefficient

$$\binom{k}{r_1, r_2, r_3, \ldots, r_n} = \frac{k!}{r_1!r_2!r_3!\cdots r_n!}.$$

Permutations with Restricted Repetitions

The number of permutations of size k taken from the set having n distinct elements, with the property that each element of S appears at least once in the permutation is

$$\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^k.$$

Combination Results

Here we summarize the results for combinations which are unordered selections. If S is a set with n elements, then the number of combinations of size k is an unordered sequence of elements taken from the set S and these elements can be taken with repetitions allowed or with repetitions not allowed. When repetitions are allowed, an element of S is chosen, recorded and then returned to S so that it could be possibly chosen again. When repetitions are not allowed, an element of S is chosen, recorded and not returned to S so that it can not be chosen again.

Combinations With No Repetitions

If S is a set with n elements, then the number of combinations of size k with no repetitions, taken from the set S, is

$$\left(\begin{array}{c}\text{The number of combinations of size} \\ k \text{ with no repetitions taken from} \\ \text{a set } S \text{ with } n \text{ distinct elements}\end{array}\right) = \frac{n!}{k!(n-k)!} = \binom{n}{k},$$

which is a binomial coefficient.

Combinations With A Fixed Number Of Repetitions

Let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a set with n elements. Then the number of combinations of size k with fixed repetitions, taken from the set S, so that a_i is repeated exactly r_i times for all $i = 1, 2, 3, ..., n$, is

$$\left(\begin{array}{c}\text{The number of combinations of size } k \\ \text{(with } a_i \text{ repeated exactly } r_i \text{ times)} \\ \text{from a set } S \text{ with } n \text{ distinct elements}\end{array}\right) = 1$$

with

$$r_1 + r_2 + r_3 + \cdots + r_n = k.$$

Combinations With Unrestricted Repetitions

If S is a set with n elements, then the number of combinations of size k with unrestricted repetitions, taken from the set S, is

$$\begin{pmatrix} \text{The number of combinations of size } k \\ \text{with } \textit{unrestricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{pmatrix} = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}.$$

Combinations With Restricted Repetitions

Let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a set with n elements. Then the number of combinations of size k, taken from S, with the property that the element a_i must be repeated at least s_i times in the combination, for all $i = 1, 2, 3, ..., n$, is

$$\binom{n+k-s_1-s_2-s_3-\cdots-s_n-1}{k-s_1-s_2-s_3-\cdots-s_n}$$

which we may also write as

$$\binom{n+k-s_1-s_2-s_3-\cdots-s_n-1}{n-1}.$$

Combinations with Restricted Repetitions

The number of combinations of size k taken from the set having n distinct elements, with the property that each element of S appears *at least once* in the combination is

$$\binom{k-1}{n-1} = \frac{(k-1)!}{(n-1)!(k-n)!}.$$

Combinations With Restricted Repetitions

Suppose that we would like to count the number of combinations of size k taken from a set $S = \{a_1, a_2, a_3, ..., a_n\}$ of size n so the number of times a_i appears between $0 \leq r_i$ and $t_i \geq r_i$ for $i = 1, 2, 3, ..., n$. The says that a_i appears at least

r_i times and at most t_i times. The basic idea is to first note that this must be the same as the number of combinations of "reduced" size

$$k' = k - (r_1 + r_2 + r_3 + \cdots + r_n)$$

taken from a same set $S = \{a_1, a_2, a_3, ..., a_n\}$ of size n so the number of times a_i appears in the reduced combination is no larger than (or at most) $t_i - r_i \geq 0$ times for $i = 1, 2, 3, ..., n$. If we let P_i be the property that a_i is repeated at least $t_i - r_i + 1$ times, for $i = 1, 2, 3, ..., n$, then we would like to count

$$|\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3 \cap \cdots \cap \bar{P}_n| = |U| - |P_1 \cup P_2 \cup P_3 \cup \cdots \cup P_n|$$

where $|U|$ equals the number of combinations of size k' taken from a set

$$S = \{a_1, a_2, a_3, ..., a_n\}$$

of size n with no restrictions and $|P_1 \cup P_2 \cup P_3 \cup \cdots \cup P_n|$ is determined using the inclusion-exclusion principle, as shown in the notes.

Combinations With Restricted Repetitions

The number of ways in which d identical balls can be distributed among n identical urns so that no urn receives R or more balls. Note that the procedure for working this out using inclusion/exclusion was already discussed. Using that procedure or Equation (20), it can be shown that, in terms of the binomial coefficients, we have

$$\binom{d, R}{n} = \sum_{j=0}^{[d/R]} (-1)^j \binom{n}{j} \binom{d - jR + n - 1}{n - 1}$$

for $n = 1, 2, 3, ...$, $d = 1, 2, 3, ...$, $R = 0, 1, 2, ..., d$, and $[d/R]$ is the greatest integer less than or equal to d/R.

The Principle of Inclusion-Exclusion

Let U is a universal set, and let A_1, A_2, A_3, ..., A_n be subsets of U. Then the number of elements in U that are in at least one of the sets A_1, A_2, A_3, ..., A_n is

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j+1} S_j$$

where S_k denotes the sum of the sizes of the intersections of all collections of k of the sets A_1, A_2, A_3, ..., A_n. We may also use this to compute

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = |U| - |A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n|.$$

For example,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

and

$$\begin{aligned} |A \cup B \cup C \cup D| \quad = \quad & |A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D|, \end{aligned}$$

and so on.

The Number of Derangements of Size n

Given an *ordered* set $S = \{a_1, a_2, a_3, ..., a_n\}$ of size n, a derangement of S is a listing of the elements of S so that no element of S is in its original position. The number D_n of derangements of size n is

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$