MEAM 520
Lecture 3: Rotations

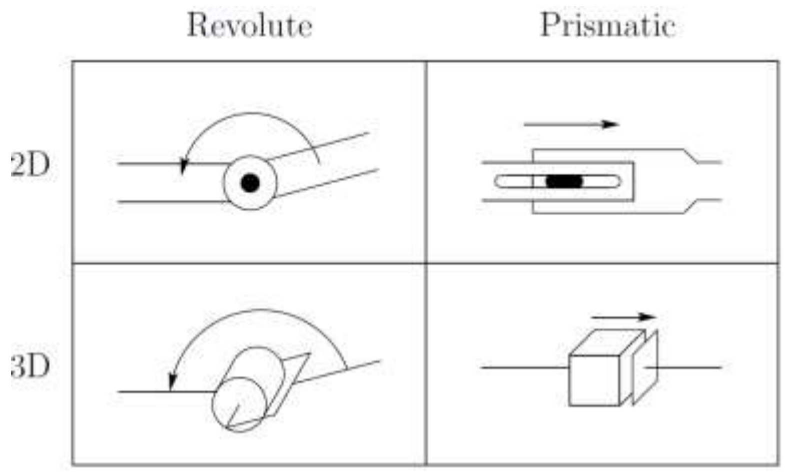
Cynthia Sung, Ph.D.

Mechanical Engineering & Applied Mechanics

University of Pennsylvania

Last Time

Manipulators are **links** connected by **joint**, which can be **R** or **P** type.



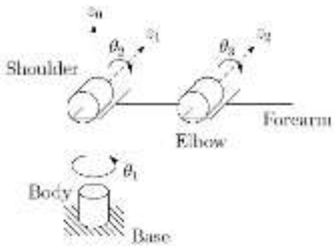
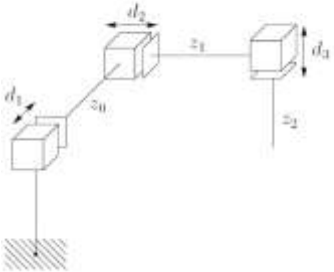
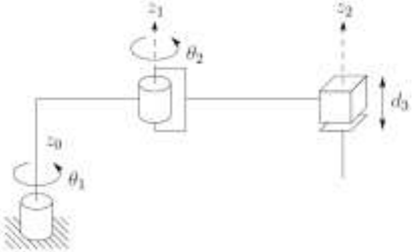
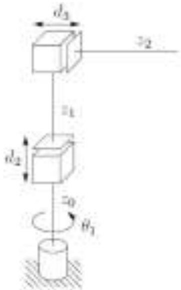
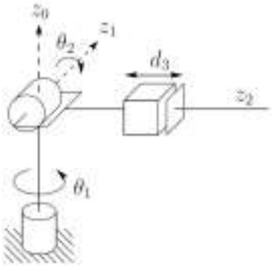
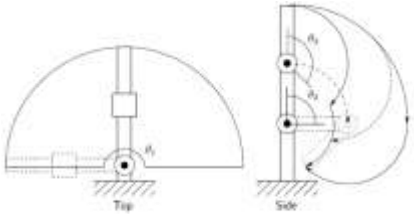
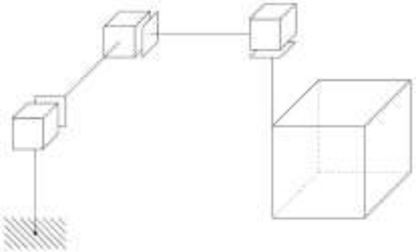
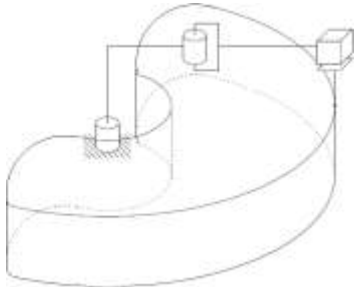
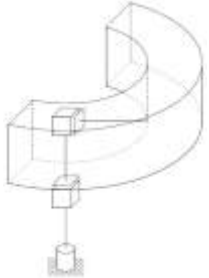
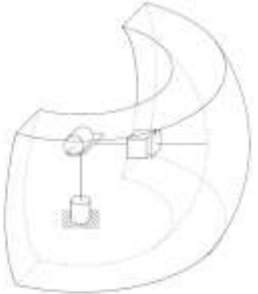
Joint variables θ and d

Zero configuration

Degrees of freedom (DOF)

Last Time

Configuration Space vs Workspace vs Task Space vs State Space

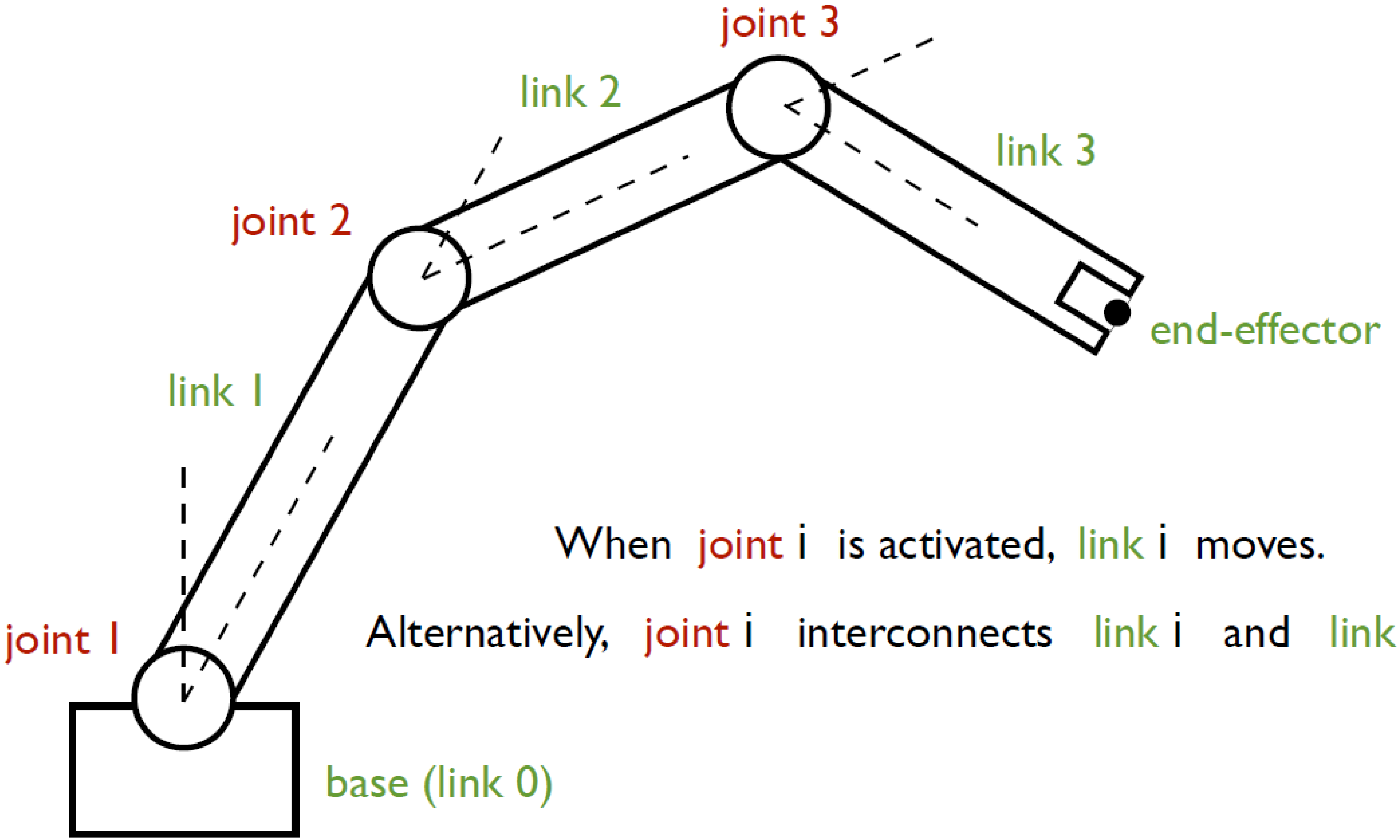
Articulated (RRR)	Cartesian (PPP)	SCARA (RRP)	Cylindrical (RPP)	Spherical (RRP)
small workspaces	gantries	speed, planar tasks	material transfer	earliest designs
				
				
				

Today: Kinematics

Kinematics is the study of motion without

reference to the causes of that motion.

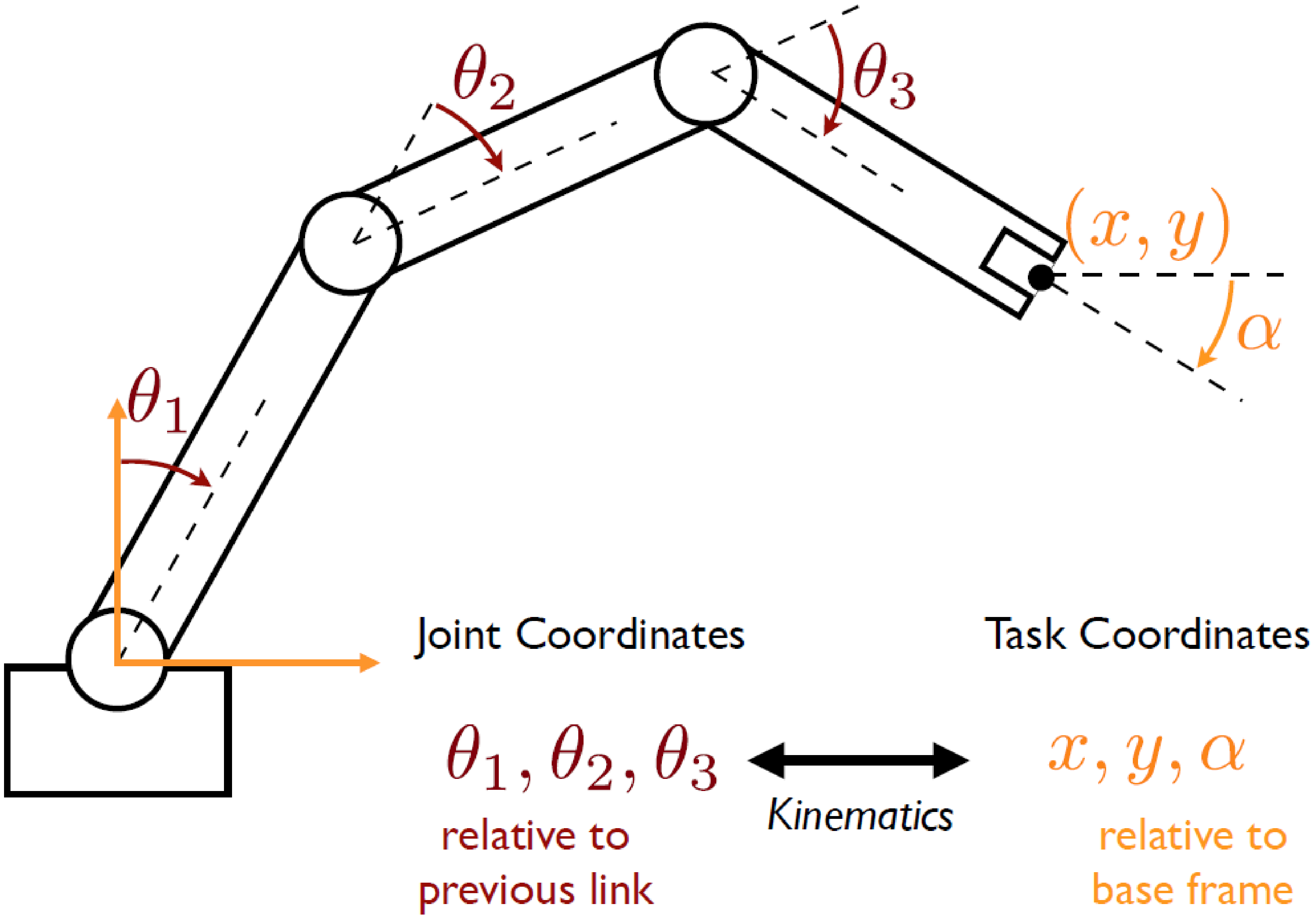
Kinematics



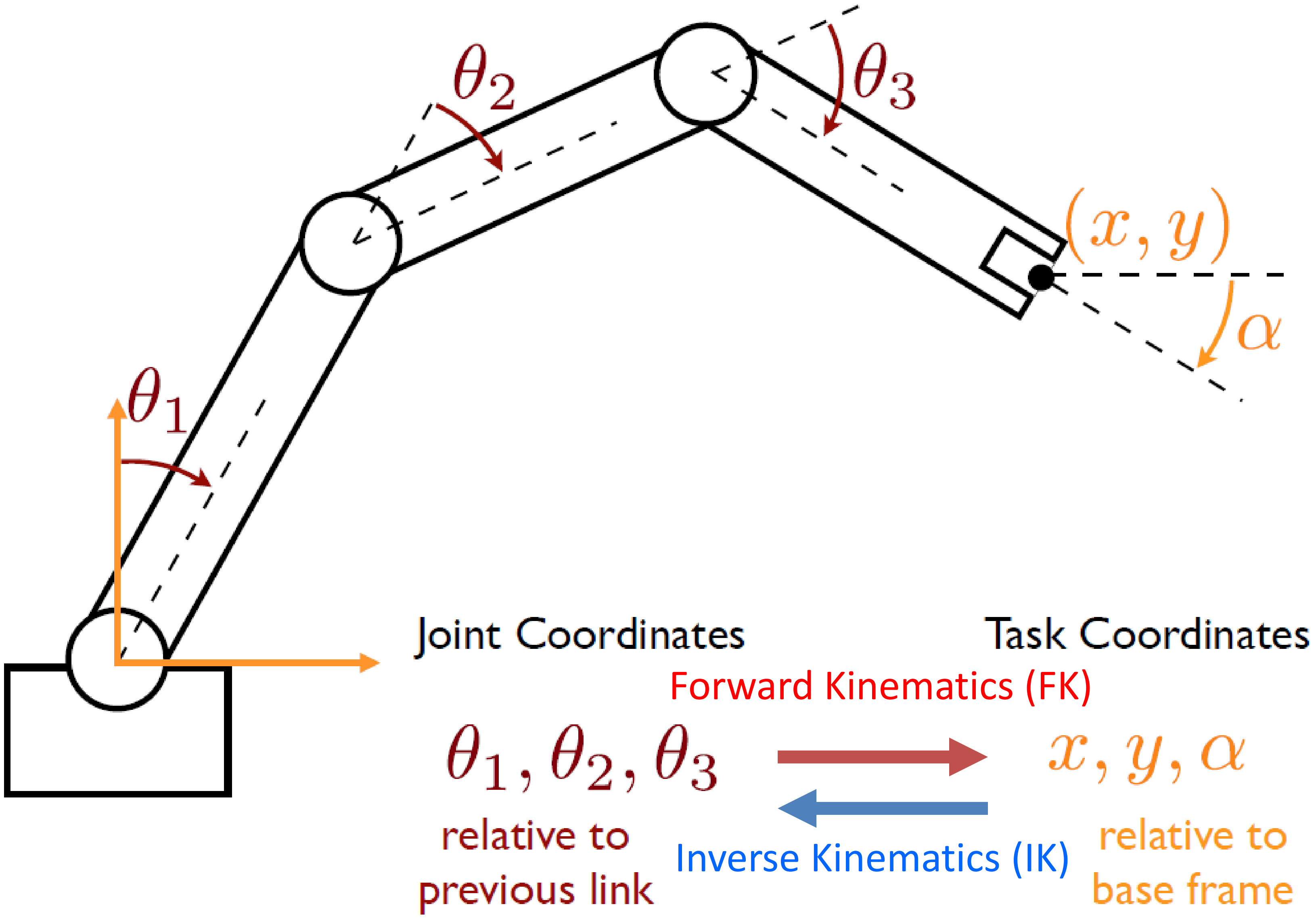
When joint i is activated, link i moves.

Alternatively, joint i interconnects link i and link i−1

Kinematics


θ_2
θ_3
(x, y)
α
θ_1
Joint Coordinates
Task Coordinates
$\theta_1, \theta_2, \theta_3$
Kinematics
x, y, α
relative to previous link
relative to base frame

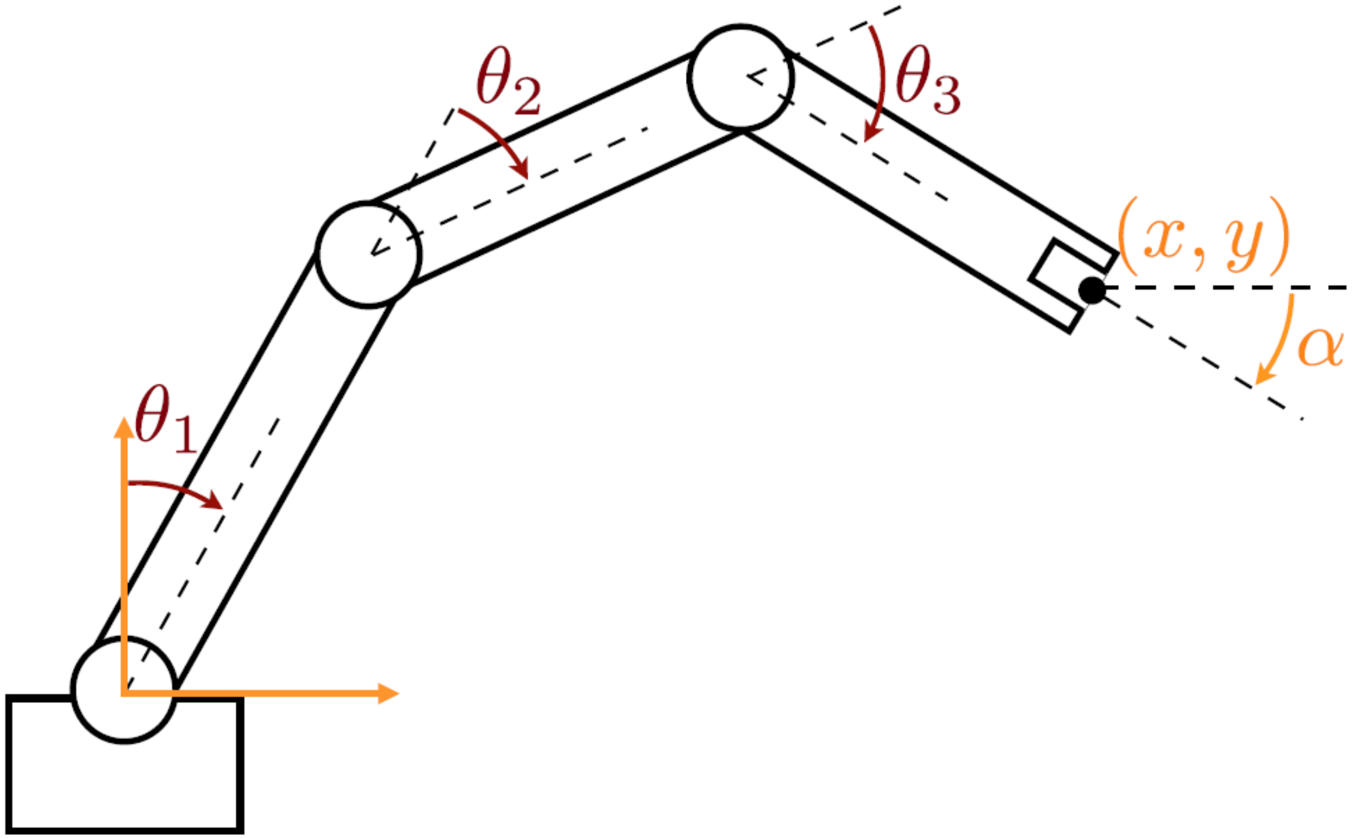
Kinematics



θ_2
θ_3
(x, y)
α
θ_1
Joint Coordinates
Task Coordinates
Forward Kinematics (FK)
$\theta_1, \theta_2, \theta_3$
x, y, α
relative to previous link
Inverse Kinematics (IK)
relative to base frame

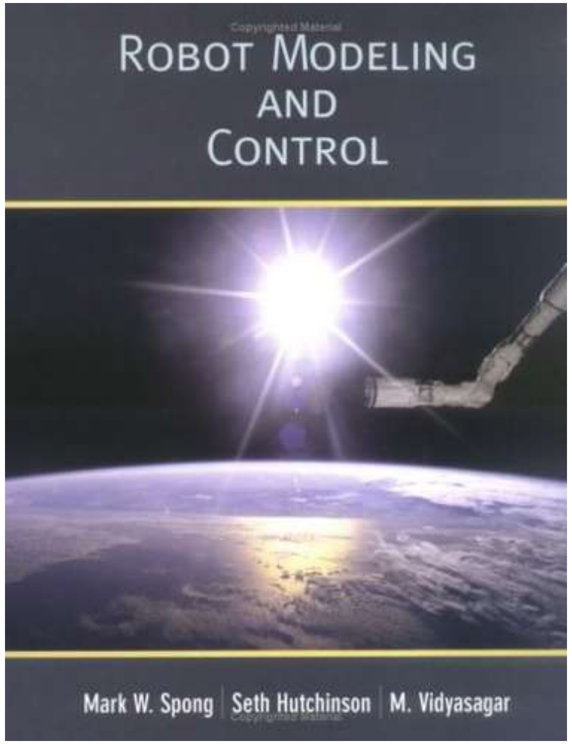
Forward Kinematics

Given the joint coordinates, what are the task coordinates?



Strategy: Break up the robot into its links

Today: Rotations in 2D and 3D



Chapter 2: Rigid Motions

- Sec. 2.intro-2.5 and B.1-B.4

Upcoming Deadlines

Lab 0 is due tomorrow by midnight!

Lab 1 posted

- Pre-lab (individual) due 9/12
- Lab due 9/19
- Sample lab report posted on Canvas
- Full grading rubric posted on Canvas

Lab 1 is posted (pre-lab due 9/12, lab due 9/19)

Lab 1: Kinematic Characterization of the Lynx

MEAM 520, University of Pennsylvania

September 5, 2018

This lab consists of two portiions, with a pre-lab due on **Wednesday, September 12, by midnight (11:59 p.m.)** and a lab report due on **Wednesday, September 19, by midnight (11:59 p.m.)**. Late submissions will be accepted until midnight on Saturday following the deadline, but they will be penalized by 25% for each partial or full day late. After the late deadline, no further assignments may be submitted; post a private message on Piazza to request an extension if you need one due to a special situation.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you submit must be your own work, not copied from any other individual or team. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. When you get stuck, post a question on Piazza or go to office hours!

Individual vs. Pair Programming

The pre-lab component of this lab must be completed and submitted individually on Canvas. For the remainder of the lab, you may work either individually or with a partner. If you do this lab with a partner, you may work with anyone you choose, but you must work with them for all parts of this assignment.

If you are in a pair, you will both turn in the same report and code (see Submission Instructions below), for which you are jointly responsible and you will both receive the same grade. Work closely with your partner throughout the lab, following these guidelines, which were adapted from "All I really needed to know about pair programming I learned in kindergarten," by Wililams and Kessler, *Communications of the ACM*, May 2000. This article is available on Canvas under Files / Supplemental Material.

- Start with a good attitude, setting aside any skepticism, and expect to jell with your partner.
- Don't start alone. Arrange a meeting with your partner as soon as you can.
- Use just one setup, and sit side by side. For a programming component, a desktop computer with a large monitor is better than a laptop. Make sure both partners can see the screen.
- At each instant, one partner should be driving (writing, using the mouse/keyboard, moving the robot) while the other is continuously reviewing the work (thinking and making suggestions).
- Change driving/reviewing roles at least every 30 minutes, *even if one partner is much more experienced than the other*. You may want to set a timer to help you remember to switch.
- If you notice an error in the equation or code that your partner is writing, wait until they finish the line to correct them.
- Stay focused and on-task the whole time you are working together.
- Take a break periodically to refresh your perspective.
- Share responsibility for your project; avoid blaming either partner for challenges you run into.
- Recognize that working in pairs usually takes more time than working alone, but it produces better work, deeper learning, and a more positive experience for the participants.

1

Forward Kinematics for Lynx robot

Pre-lab:

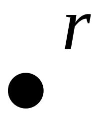
- Must be your own individual work
- Covers up to today's content

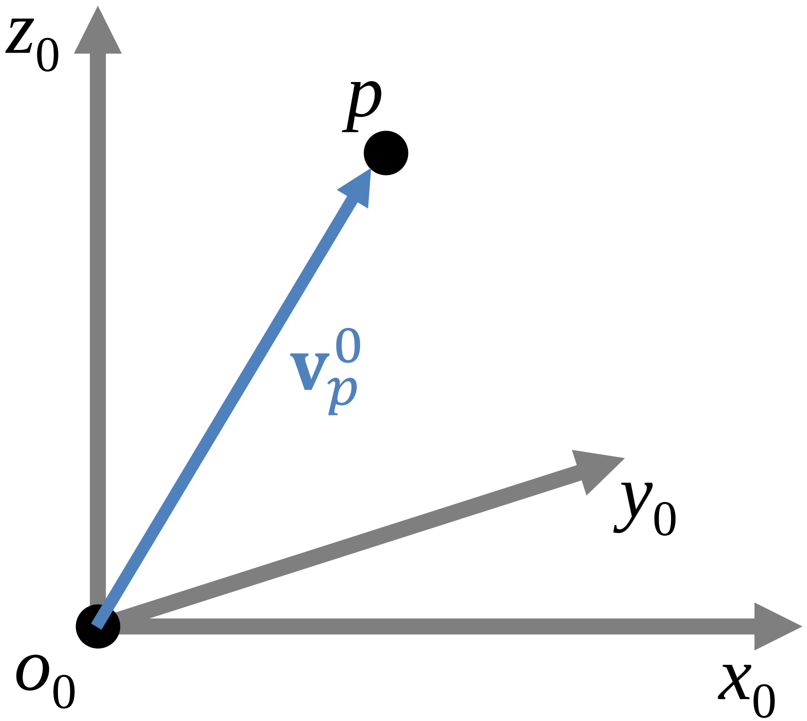
Lab:

- You are allowed to do this in pairs, but it is not required.
- Covers up to next Tuesday's content

Representing positions

A **point** exists in space as a geometric entity





Coordinate frame

- an origin (point in space)
- 2 or 3 orthogonal coordinate axes

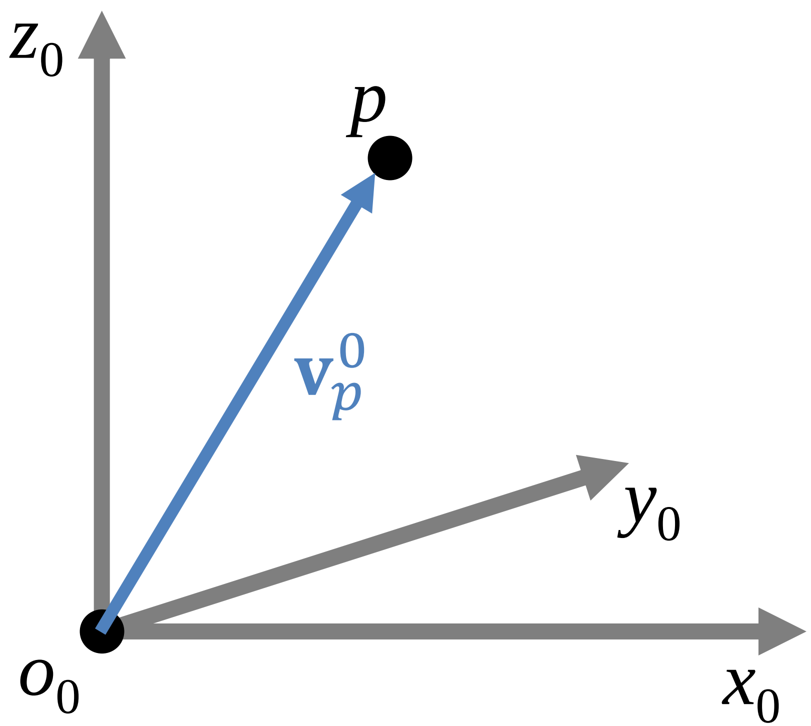
Call this frame $\boldsymbol{o_0 x_0 y_0 z_0}$ or **frame 0**

Point *p* is written as a vector

superscript = frame

$$\mathbf{v}_p^0 = p^0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x \quad y \quad z]^\top$$

Vectors



A vector has a **magnitude/length**

$$\left\|\mathbf{v}_p^0\right\| = \sqrt{x^2 + y^2 + z^2}$$

$$\left\|\mathbf{v}_p^0\right\| = \left(\left(\mathbf{v}_p^0\right)^\top \mathbf{v}_p^0\right)^{\frac{1}{2}}$$

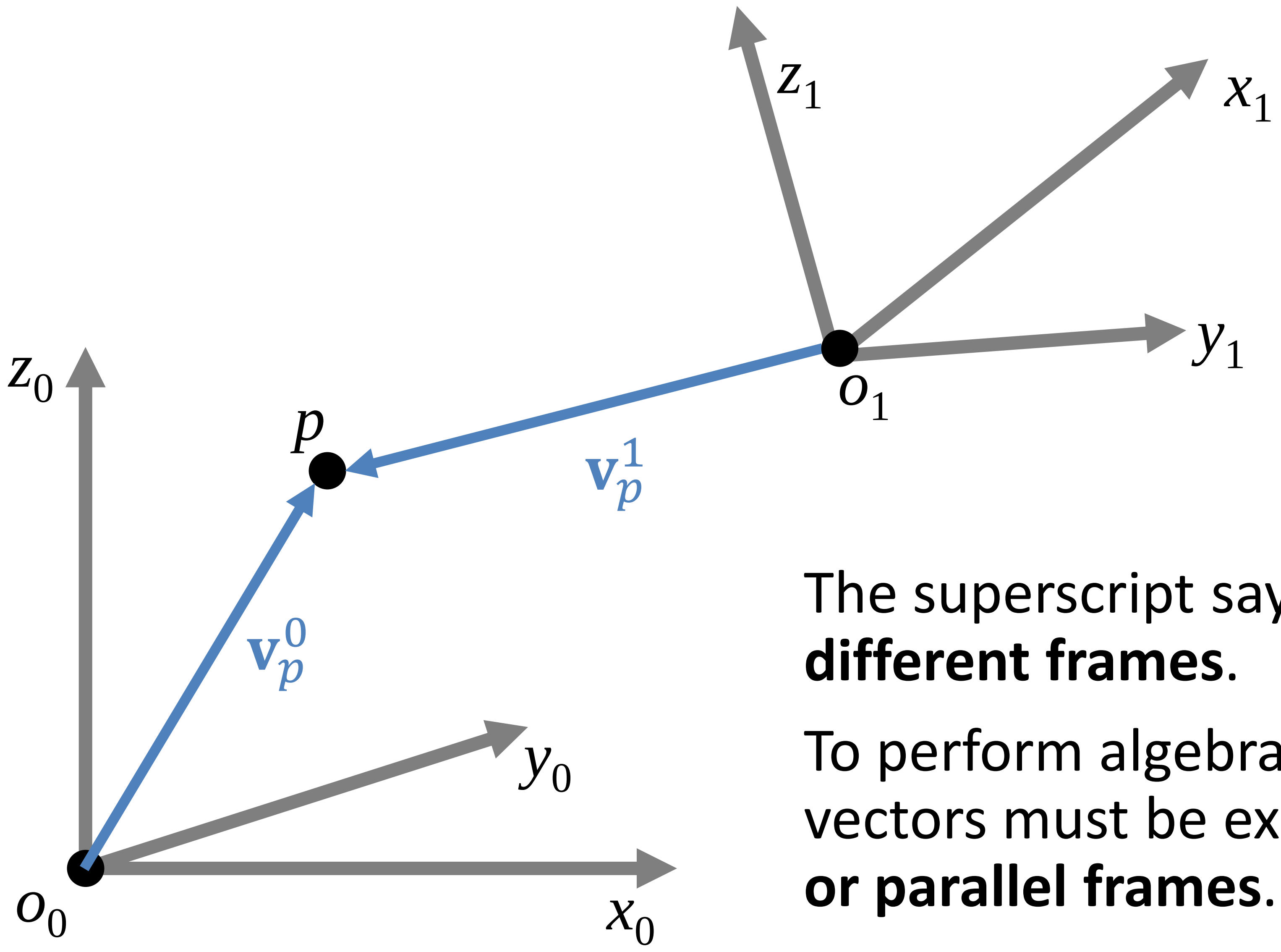
This is called the ℓ^2 **norm**.

A vector has a **direction**

$$\hat{e}_p^0 = \frac{\mathbf{v}_p^0}{\left\|\mathbf{v}_p^0\right\|}$$

This **unit vector** has length 1

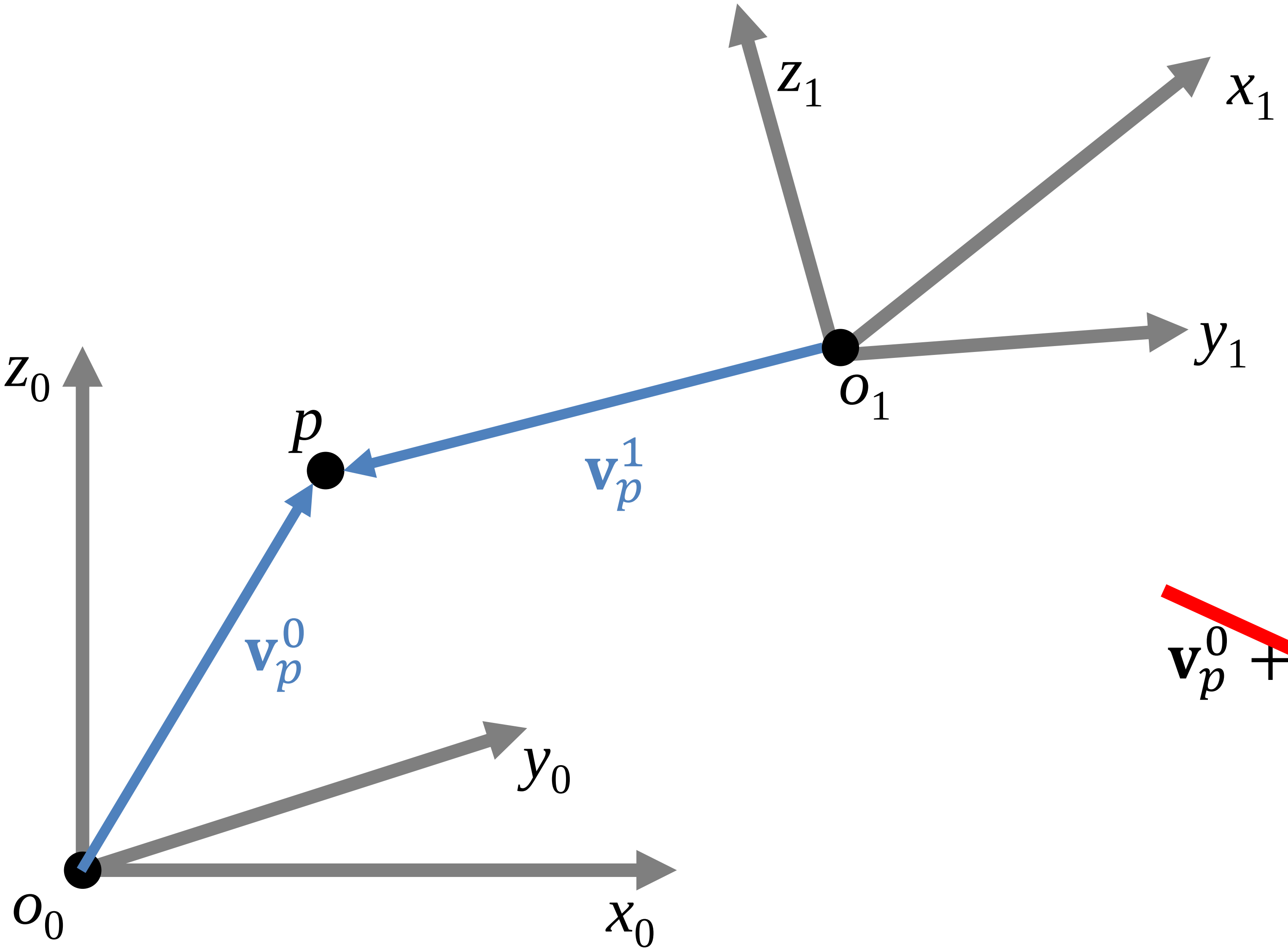
Multiple coordinate frames



The superscript says they are in **different frames**.

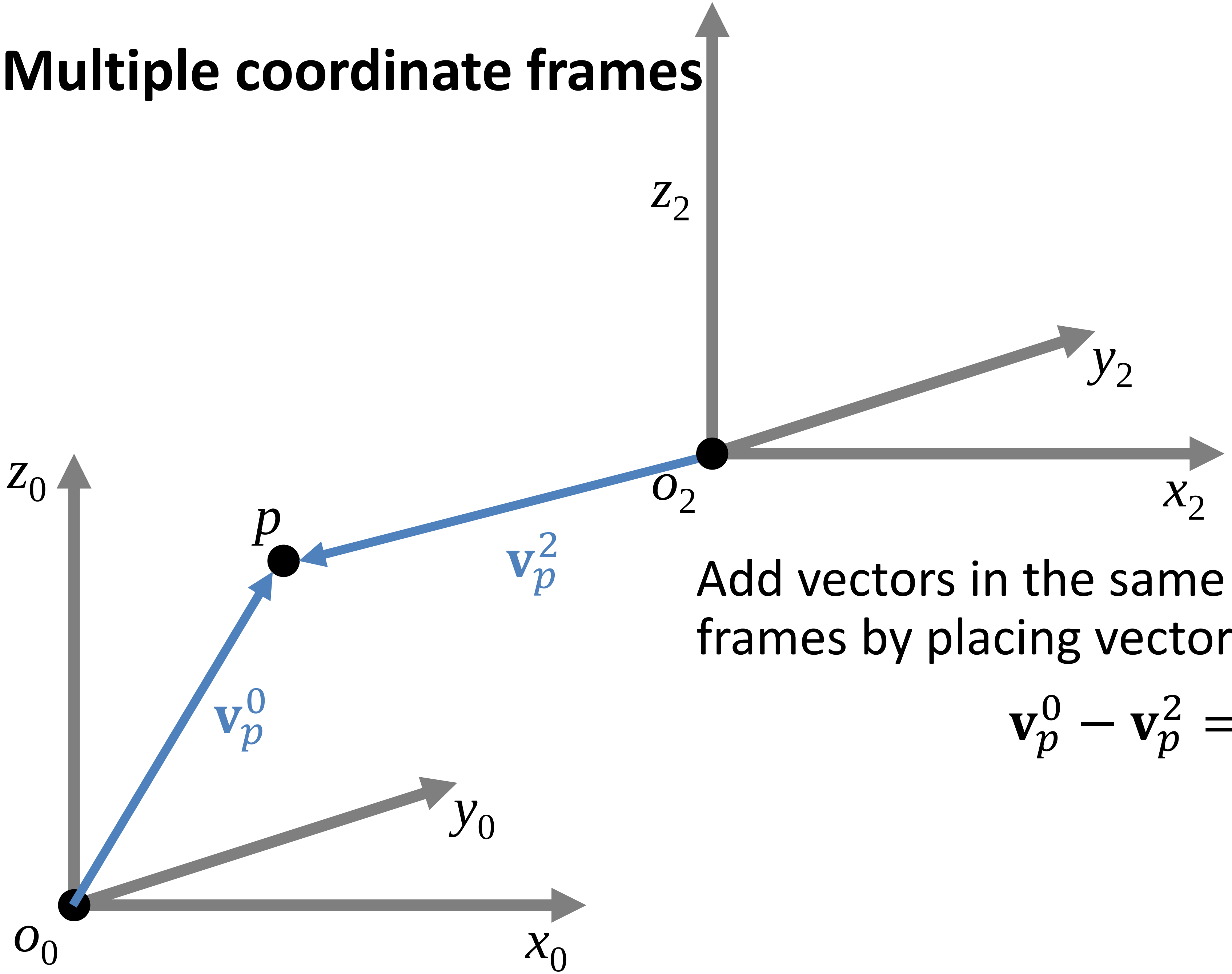
To perform algebraic manipulation, the vectors must be expressed in the **same or parallel frames**.

Multiple coordinate frames



$$\cancel{\mathbf{v}_p^0 + \mathbf{v}_p^1 = ?}$$

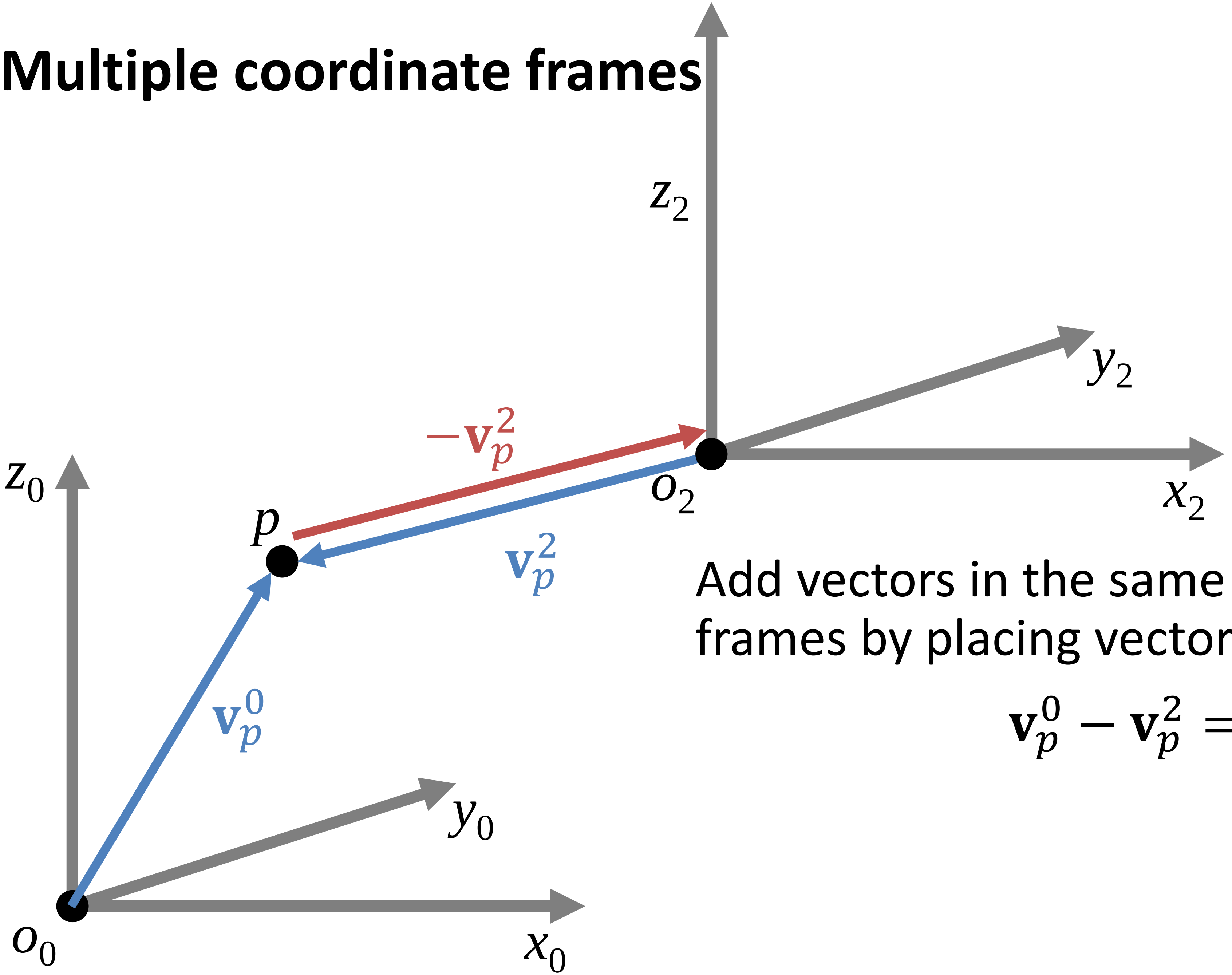
Multiple coordinate frames



Add vectors in the same or parallel frames by placing vectors tip to tail

$$\mathbf{v}_p^0 - \mathbf{v}_p^2 = ?$$

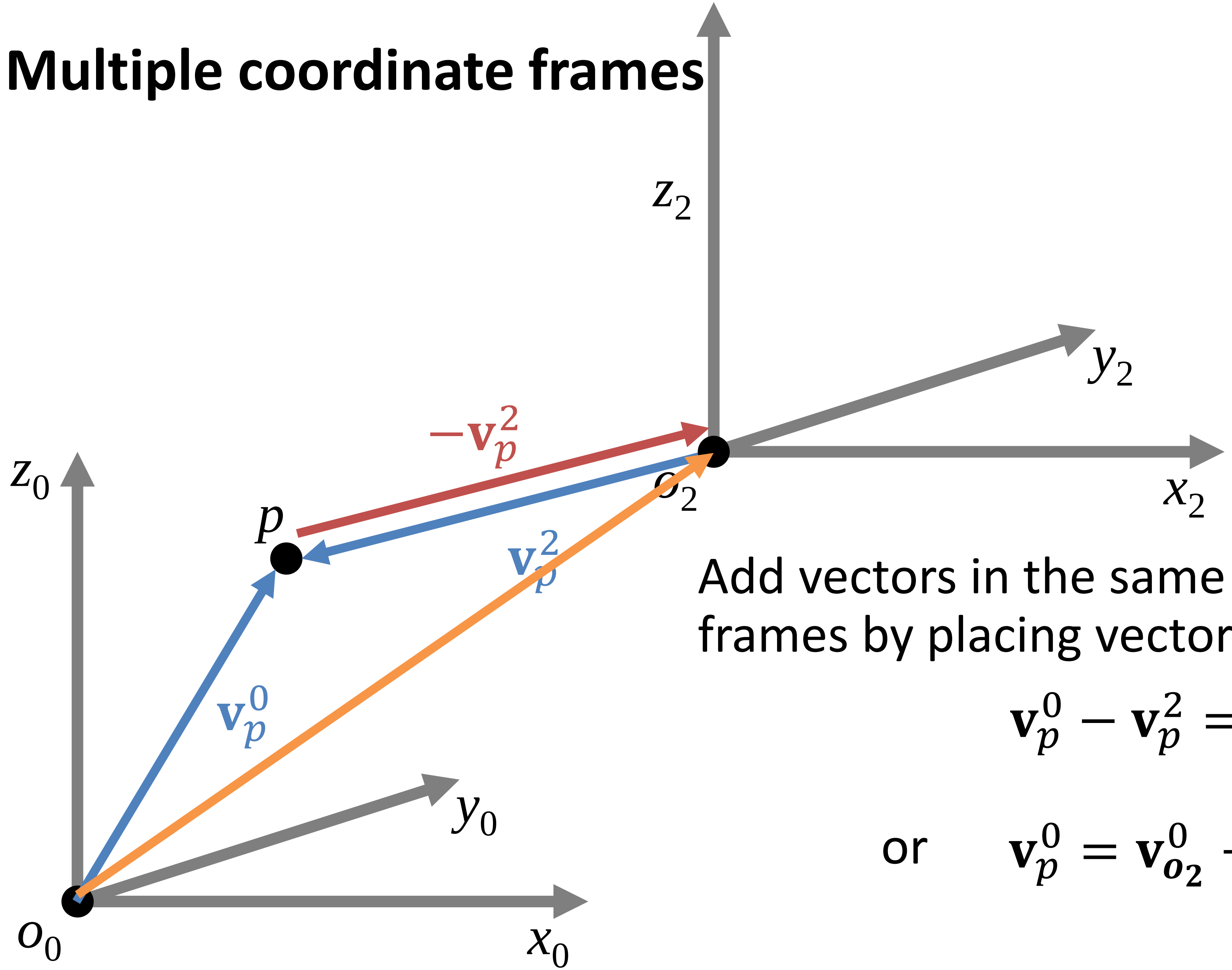
Multiple coordinate frames



Add vectors in the same or parallel frames by placing vectors tip to tail

$$\mathbf{v}_p^0 - \mathbf{v}_p^2 = ?$$

Multiple coordinate frames

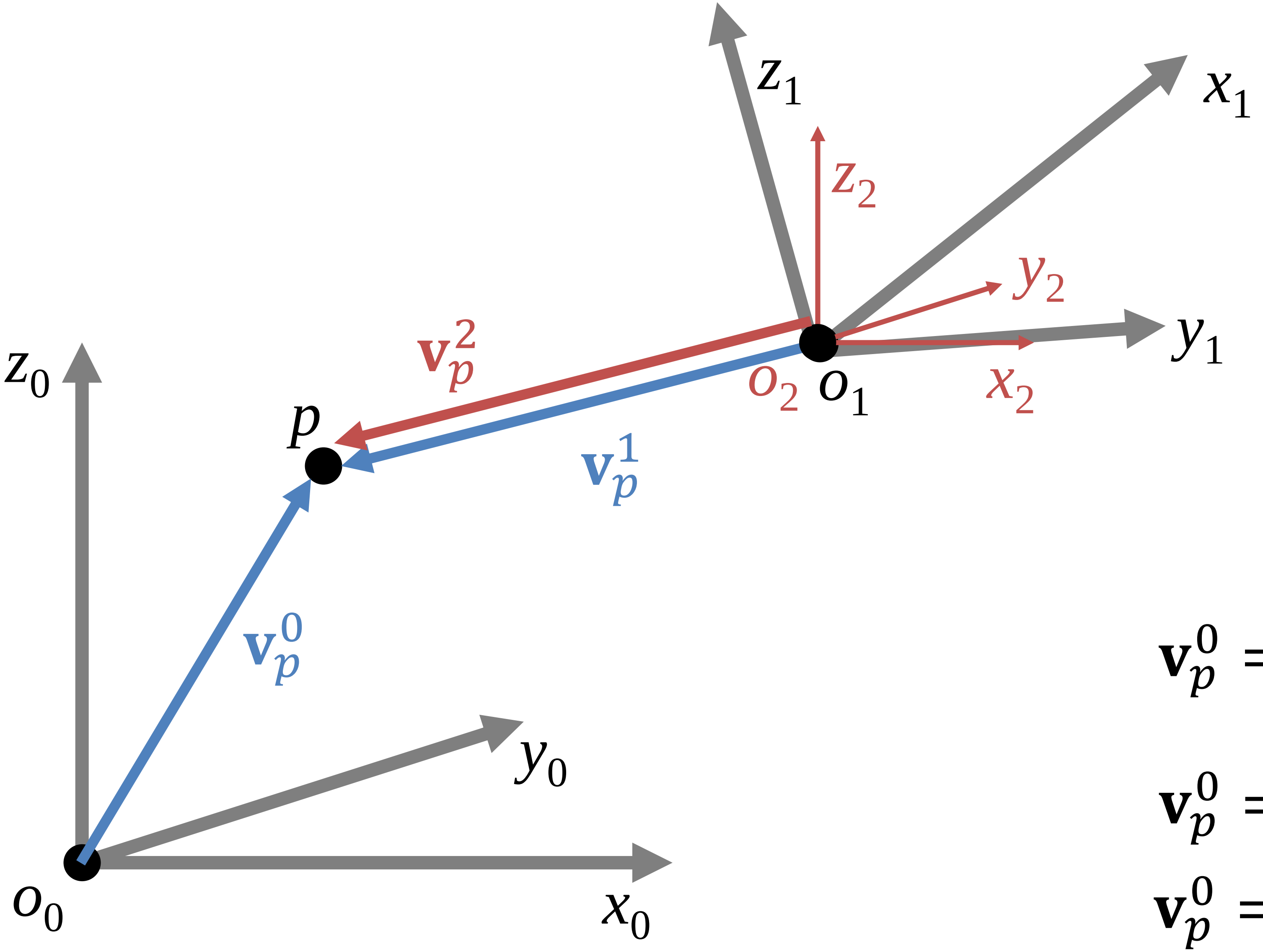


Add vectors in the same or parallel frames by placing vectors tip to tail

$$\mathbf{v}_p^0 - \mathbf{v}_p^2 = \mathbf{v}_{\boldsymbol{o_2}}^0$$

or $$\mathbf{v}_p^0 = \mathbf{v}_{\boldsymbol{o_2}}^0 + \mathbf{v}_p^2$$

How do we deal with rotated frames?

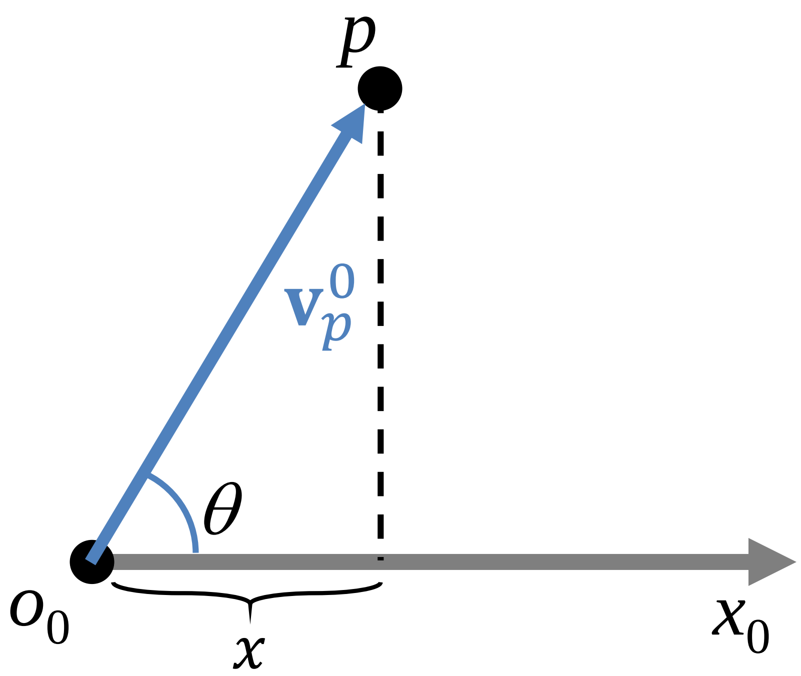


$$\mathbf{v}_p^0 = f\left(\mathbf{v}_p^1\right)$$

$$\mathbf{v}_p^0 = \mathbf{v}_{\boldsymbol{o_2}}^0 + \mathbf{v}_p^2$$

$$\mathbf{v}_p^0 = \mathbf{v}_{\boldsymbol{o_2}}^0 + \mathrm{Rotate}\left(\mathbf{v}_p^1\right)$$

Dot Products



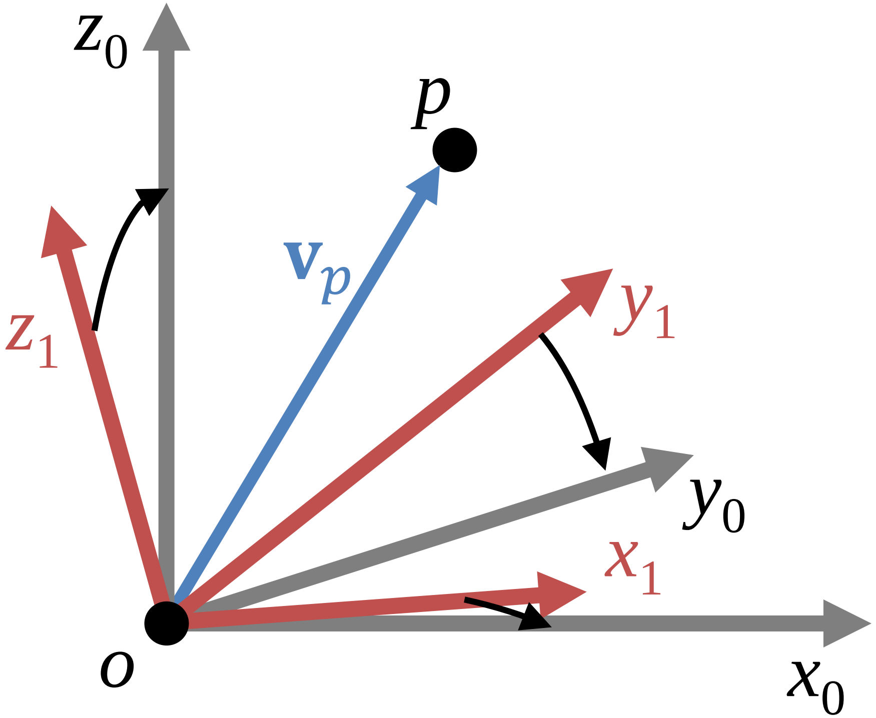
$$
\begin{aligned}
\mathbf{v}_p^0 \cdot \hat{x}_0 &= \left\|\mathbf{v}_p^0\right\| \|\hat{x}_0\| \cos\theta \\
&= \left\|\mathbf{v}_p^0\right\| (1) \cos\theta \\
&= x
\end{aligned}
$$

$$\mathbf{v}_p^0 \cdot \hat{y}_0 = y$$

$$\mathbf{v}_p^0 \cdot \hat{z}_0 = z$$

Rotation Matrices

$$\mathbf{v}_p^0 \cdot \hat{x}_0 = x \qquad \mathbf{v}_p^0 \cdot \hat{y}_0 = y \qquad \mathbf{v}_p^0 \cdot \hat{z}_0 = z$$

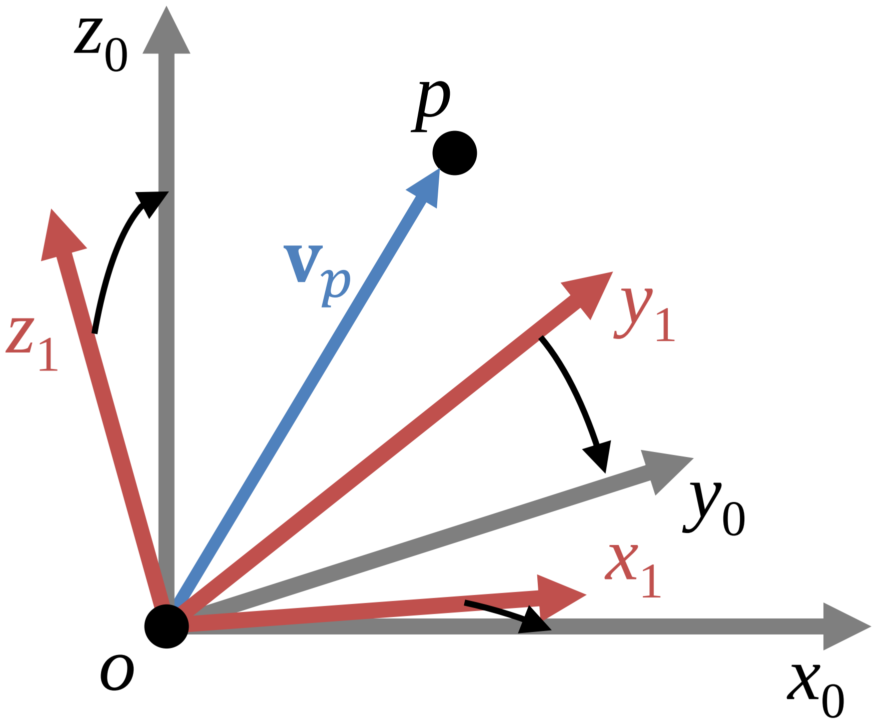


$$\mathbf{v}_p^1 = x_1\hat{x}_1 + y_1\hat{y}_1 + z_1\hat{z}_1$$

$$\mathbf{v}_p^0 = x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0) \cdot \hat{x}_0 \\ (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0) \cdot \hat{y}_0 \\ (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0) \cdot \hat{z}_0 \end{bmatrix}$$

Rotation Matrices



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0)\cdot\hat{x}_0 \\ (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0)\cdot\hat{y}_0 \\ (x_1\hat{x}_1^0 + y_1\hat{y}_1^0 + z_1\hat{z}_1^0)\cdot\hat{z}_0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{v}_p^0} = \underbrace{\begin{bmatrix} \hat{x}_1^0\cdot\hat{x}_0 & \hat{y}_1^0\cdot\hat{x}_0 & \hat{z}_1^0\cdot\hat{x}_0 \\ \hat{x}_1^0\cdot\hat{y}_0 & \hat{y}_1^0\cdot\hat{y}_0 & \hat{z}_1^0\cdot\hat{y}_0 \\ \hat{x}_1^0\cdot z_0 & \hat{y}_1^0\cdot\hat{z}_0 & \hat{z}_1^0\cdot\hat{z}_0 \end{bmatrix}}_{\mathbf{R}_1^0} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}}_{\mathbf{v}_p^1}$$

rotation from frame 1 to frame 0

subscript and superscript “cancel”

Properties of Rotation Matrices

$$\mathbf{R}_1^0 = \begin{bmatrix} \hat{x}_1^0 \cdot \hat{x}_0 & \hat{y}_1^0 \cdot \hat{x}_0 & \hat{z}_1^0 \cdot \hat{x}_0 \\ \hat{x}_1^0 \cdot \hat{y}_0 & \hat{y}_1^0 \cdot \hat{y}_0 & \hat{z}_1^0 \cdot \hat{y}_0 \\ \hat{x}_1^0 \cdot z_0 & \hat{y}_1^0 \cdot \hat{z}_0 & \hat{z}_1^0 \cdot \hat{z}_0 \end{bmatrix}$$

The rows show you the three unit vectors of the base frame expressed in the rotated frame.

$$\mathbf{R}_1^0 = \begin{bmatrix} \hat{x}_0^1 \\ \hat{y}_0^1 \\ \hat{z}_0^1 \end{bmatrix}$$

The columns show you the three unit vectors of the rotated frame expressed in the base frame.

$$\mathbf{R}_1^0 = [\hat{x}_1^0 \quad \hat{y}_1^0 \quad \hat{z}_1^0]$$

Properties of Rotation Matrices

$$\mathbf{R}_1^0 = \begin{bmatrix} \hat{x}_1^0 \cdot \hat{x}_0 & \hat{y}_1^0 \cdot \hat{x}_0 & \hat{z}_1^0 \cdot \hat{x}_0 \\ \hat{x}_1^0 \cdot \hat{y}_0 & \hat{y}_1^0 \cdot \hat{y}_0 & \hat{z}_1^0 \cdot \hat{y}_0 \\ \hat{x}_1^0 \cdot z_0 & \hat{y}_1^0 \cdot \hat{z}_0 & \hat{z}_1^0 \cdot \hat{z}_0 \end{bmatrix}$$

- Every row and column is a unit vector.
- The columns (and rows) are orthogonal.
- $(\mathbf{R}_1^0)^\top \mathbf{R}_1^0 = \mathbf{I} \implies \mathbf{R}_0^1 = (\mathbf{R}_1^0)^{-1} = (\mathbf{R}_1^0)^\top$
- If you are transforming between 2 right-handed coordinate frames, then $\det \mathbf{R}_1^0 = +1$

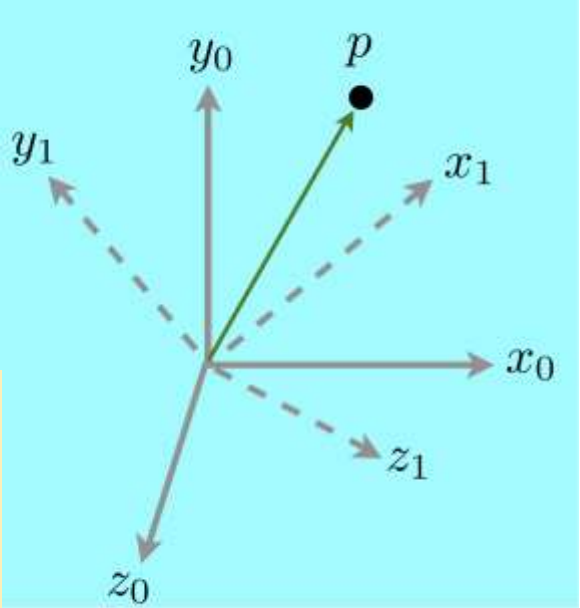
  We call these matrices SO(3) ("special orthogonal group of order 3")

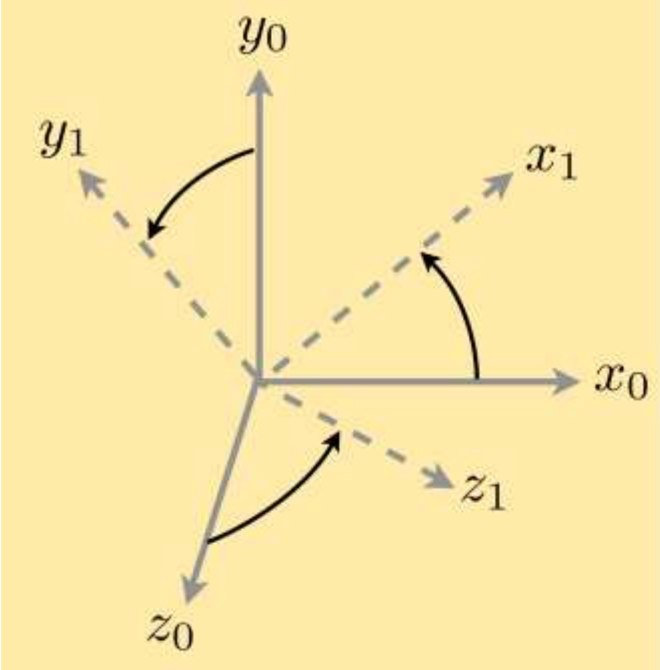
Rotation Matrices

Serve 3 purposes (p. 47 of SHV):

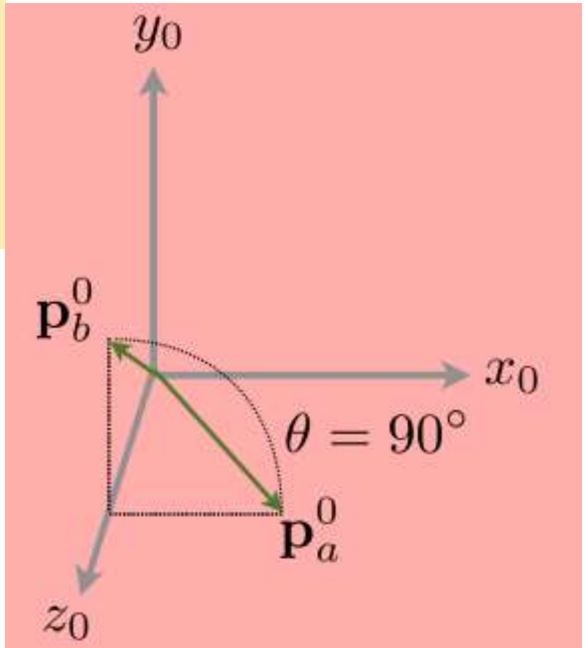
1. Coordinate transformations relating coordinates of a point p in two different frames
2. Orientation of a transformed coordinate frame with respect to a fixed frame
3. Operator taking a vector and rotating it to yield a new vector in the same coordinate frame

$$\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1$$





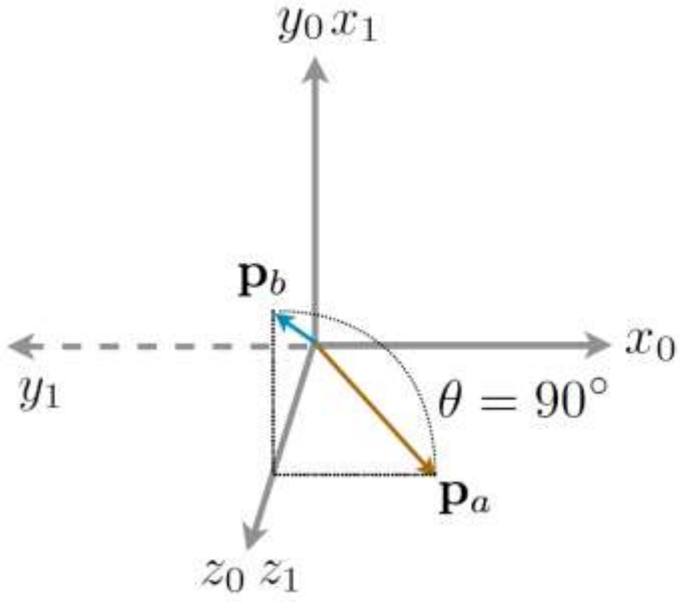
$$\mathbf{R}_1^0 = [\hat{x}_1^0 \quad \hat{y}_1^0 \quad \hat{z}_1^0]$$



$$\mathbf{p}_b^0 = \mathbf{R} \boldsymbol{p}_a^0$$

Composite Rotations

What if I want to apply multiple rotations to a vector?



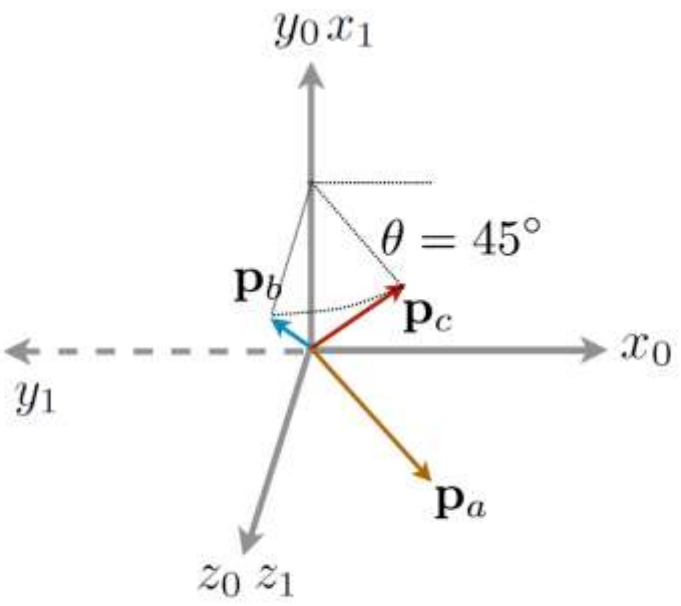
The **order** in which a sequence of rotations is performed is crucial.

Thus, the **order** in which rotation matrices are multiplied together is crucial

For example:
Rotate 45° around y_0
vs
Rotate 45° around y_1

Composite Rotations

What if I want to apply multiple rotations to a vector?



For example:

Rotate 45° around y_0

vs

Rotate 45° around y_1

$$\mathbf{p}_b = \mathbf{R}\mathbf{p}_a \qquad \mathbf{R}' = \mathbf{R}_{y,45°}$$

$$\mathbf{p}_c = ?$$

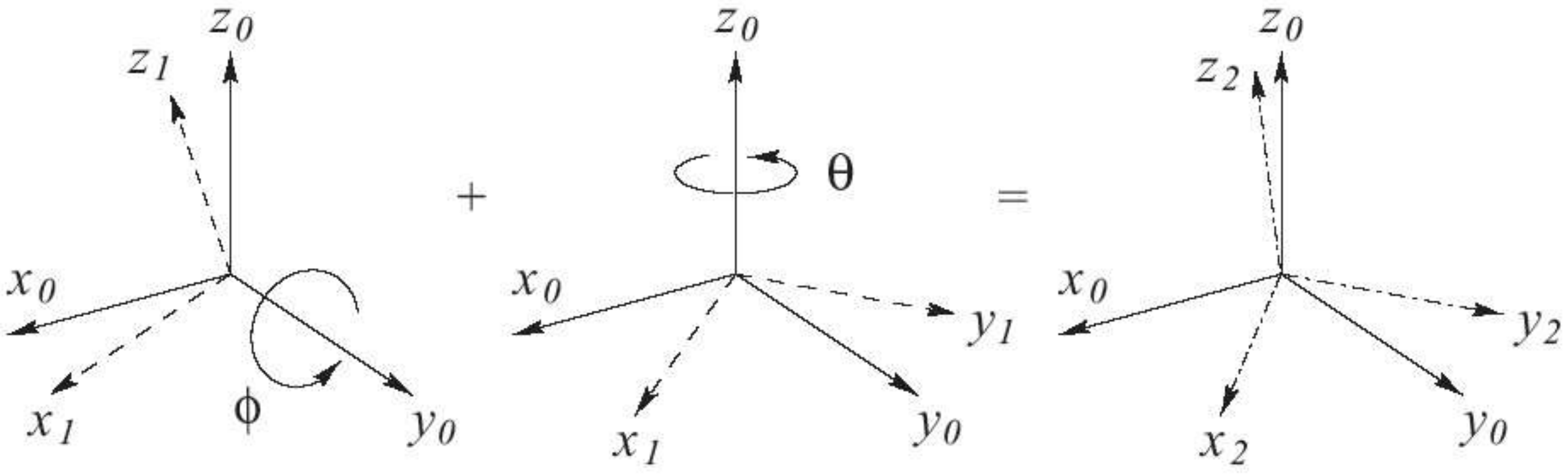
$$\mathbf{p}_c = \mathbf{R}'\mathbf{p}_b$$

$$\mathbf{p}_c = \mathbf{R}'\mathbf{R}\mathbf{p}_a$$

Compositions of Rotations with Respect to a Fixed Frame

the result of a successive rotation about a fixed frame can be found by **pre-multiplying** by the corresponding rotation matrix

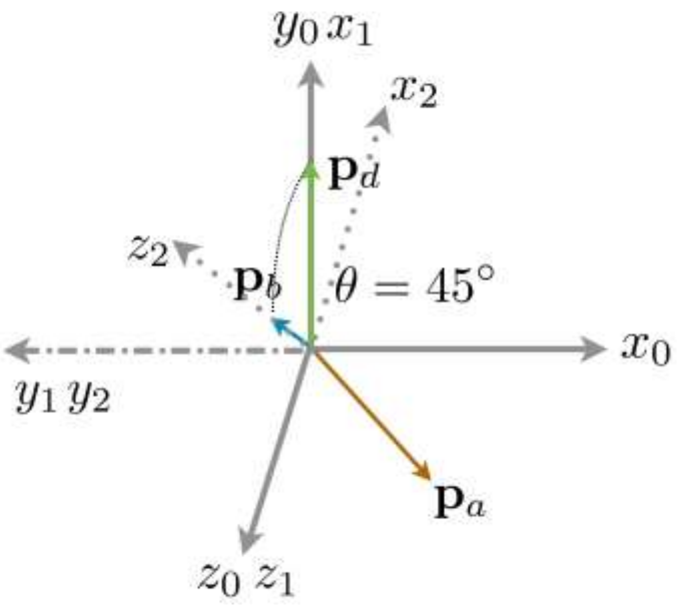
$$\mathbf{R}_2^0 = \mathbf{R}\mathbf{R}_1^0$$



Note that $\mathbf{R}$ is a rotation about the original frame

Composite Rotations

What if I want to apply multiple rotations to a vector?



For example:
Rotate 45° around y_0
vs
Rotate 45° around y_1

$$\mathbf{p}_b = \mathbf{R}\mathbf{p}_a \qquad \mathbf{R}' = \mathbf{R}_{y,45°}$$

$$\mathbf{p}_d = ?$$

$$\mathbf{p}_d^2 = \mathbf{p}_a^0$$

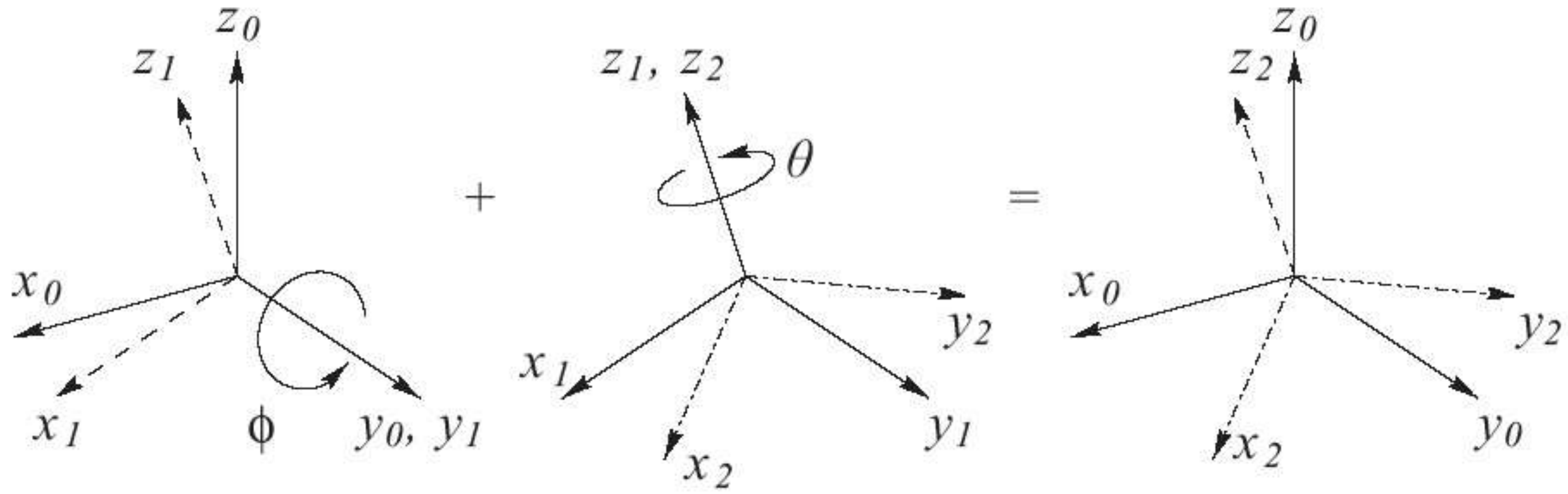
$$\mathbf{p}_d^0 = \mathbf{R}_1^0\mathbf{R}_2^1\mathbf{p}_d^2$$

$$\mathbf{p}_d = \mathbf{R}\mathbf{R}'\mathbf{p}_a$$

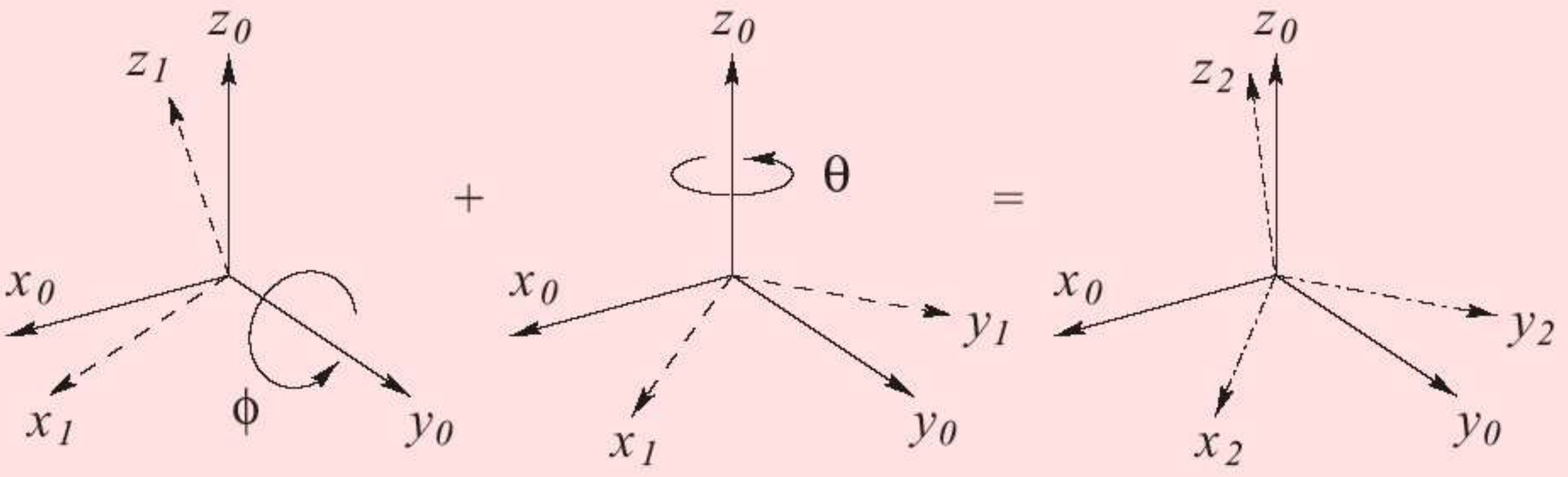
Compositions of Rotations with Respect to an Intermediate Frame

the result of a successive rotation about the current (intermediate) frame can be found by **post-multiplying** by the corresponding rotation matrix

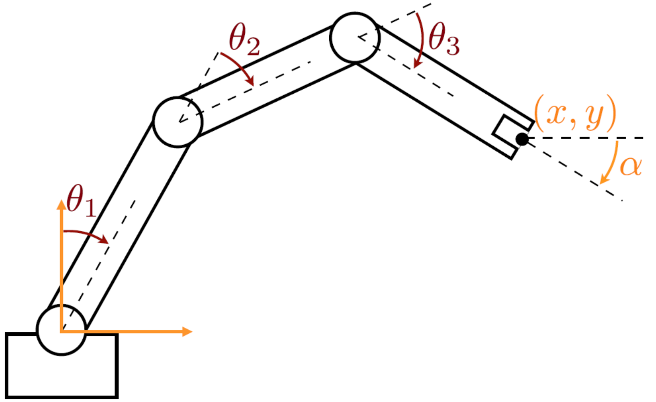
$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1$$



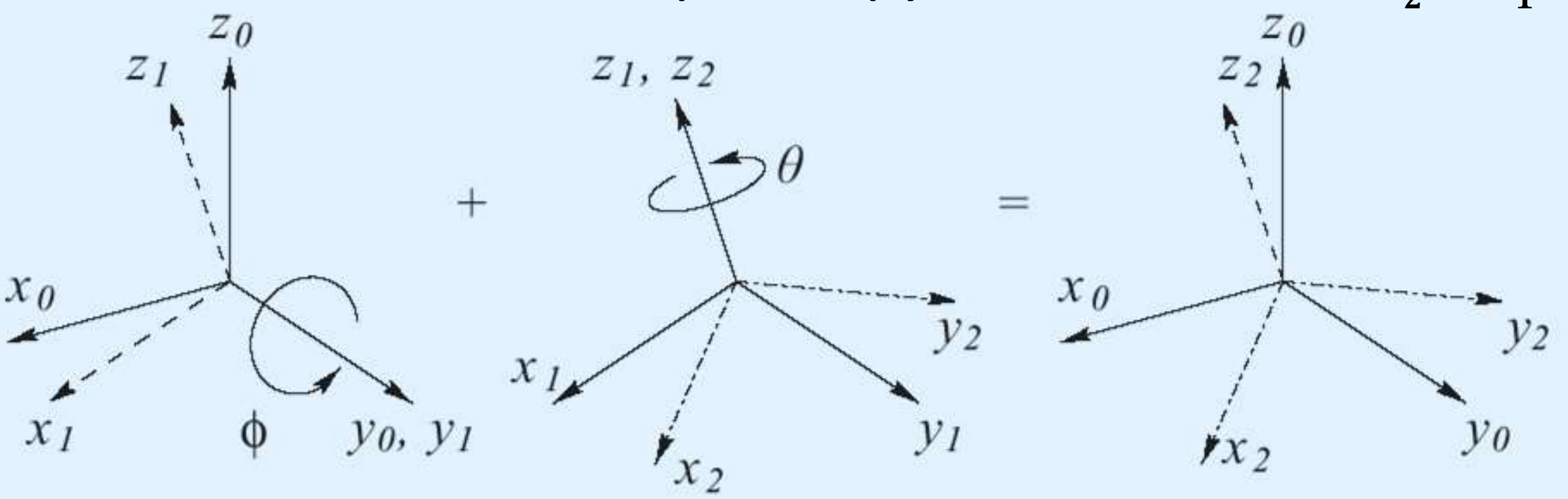
Rotation about a fixed frame? **pre-multiply** $\mathbf{R}_2^0 = \mathbf{R}\mathbf{R}_1^0$



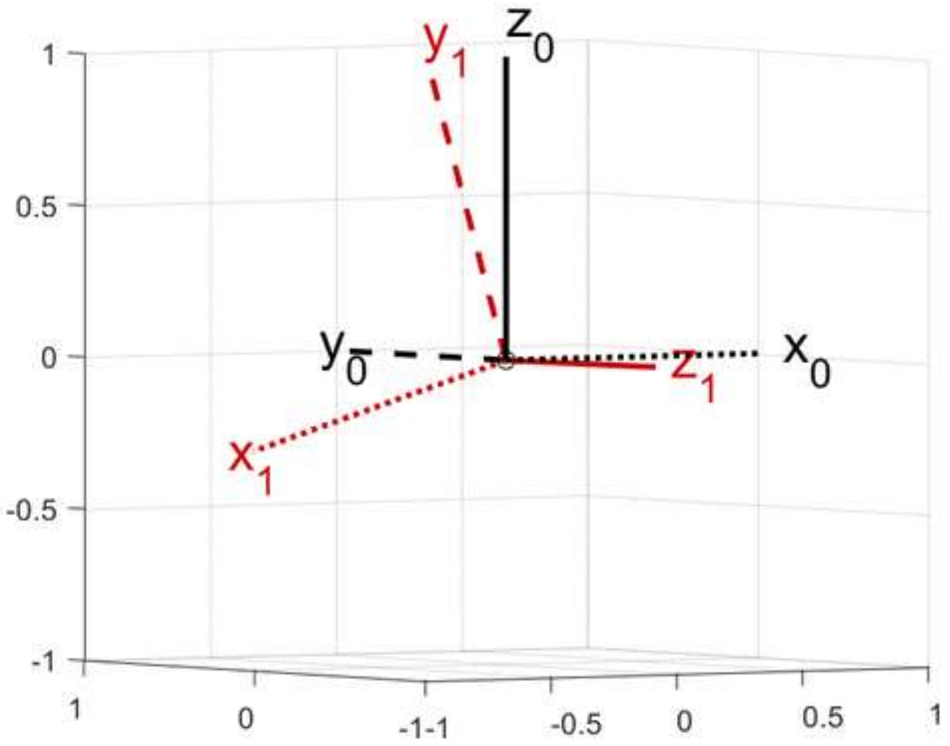
Which of these is more commonly used in robotics?



Rotation about intermediate frame? **post-multiply** $\mathbf{R}_2^0 = \mathbf{R}_1^0\mathbf{R}_2^1$



Practice: Choose the $\mathbf{R}_1^0$ matrix corresponding to the following visual representation.



$$\mathbf{R}_1^0 = \begin{bmatrix} 0.028 & 0.538 & -0.843 \\ 0.899 & 0.355 & 0.256 \\ 0.437 & -0.765 & -0.474 \end{bmatrix}$$

$$\mathbf{R}_1^0 = \begin{bmatrix} -0.434 & -0.092 & 0.895 \\ 0.842 & 0.310 & 0.443 \\ -0.318 & 0.946 & -0.057 \end{bmatrix}$$

$$\mathbf{R}_1^0 = \begin{bmatrix} -0.453 & -0.591 & -0.668 \\ 0.298 & -0.806 & 0.511 \\ -0.840 & 0.033 & 0.541 \end{bmatrix}$$

Further practice: MATLAB scripts visualizeR.m and testR.m posted on Canvas under Files > Resources

Parameterizing Rotations

$$\mathbf{R}_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot z_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$

In 3 dimensions, no more than **3 independent values** are needed to specify an arbitrary rotation.

The 9-element rotation matrix has **6 redundancies** (3 DOF)

Numerous methods have been developed to represent rotation/orientation more compactly

Euler Angles Roll/Pitch/Yaw Axis/Angle

Conventions vary, so always check definitions!

Three Special Rotation Matrices

The **basic rotation matrices** define rotations about the three coordinate axes.

The 0s and 1s are always in the row and column of the rotation axis

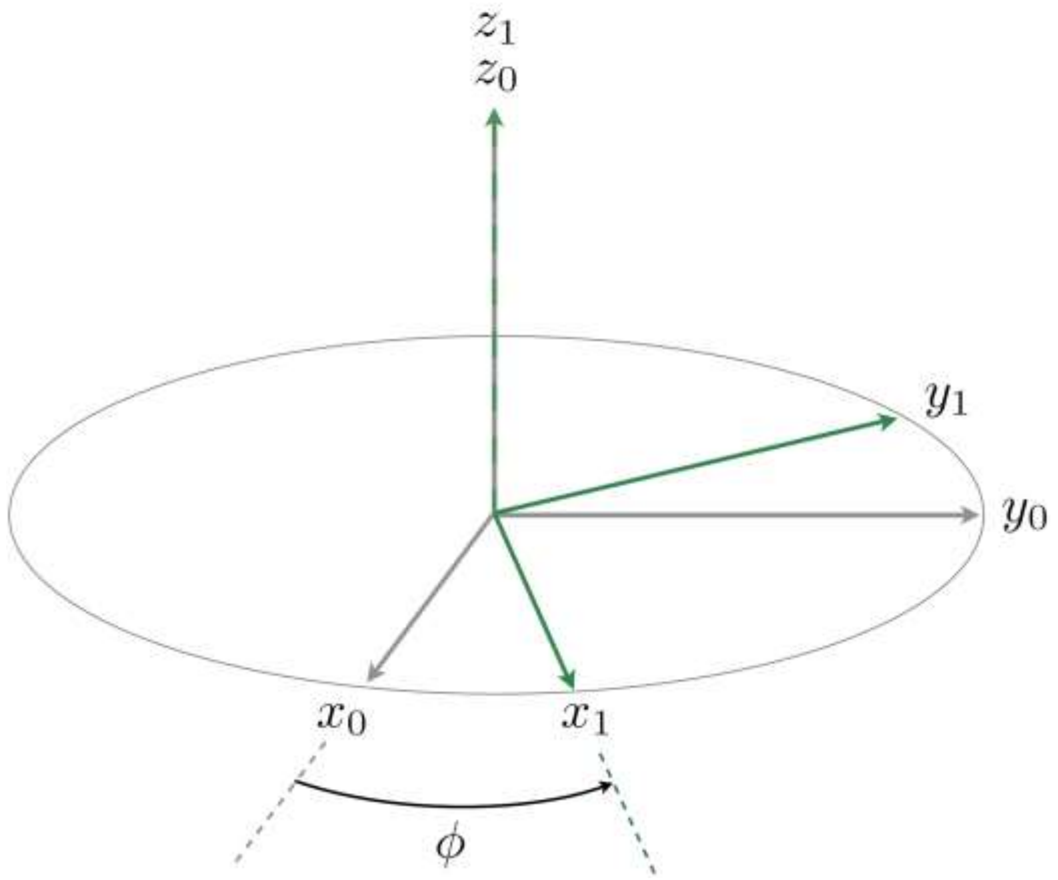
$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3 by rotating around the axes of the **current frame**.

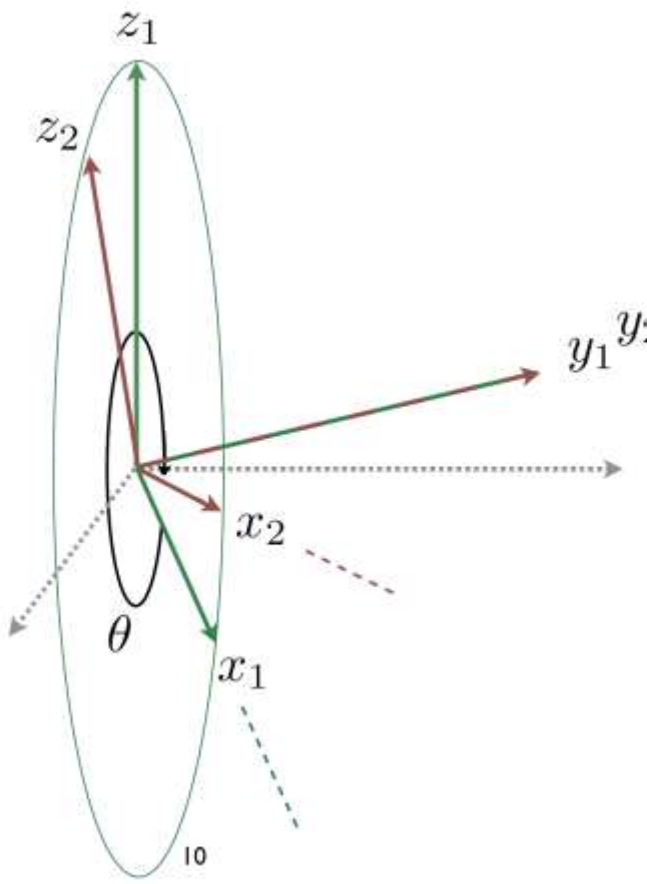


Using Z-Y-Z convention:

1. Rotate by ϕ about z_0

Euler Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3
by rotating around the axes of the **current frame**.

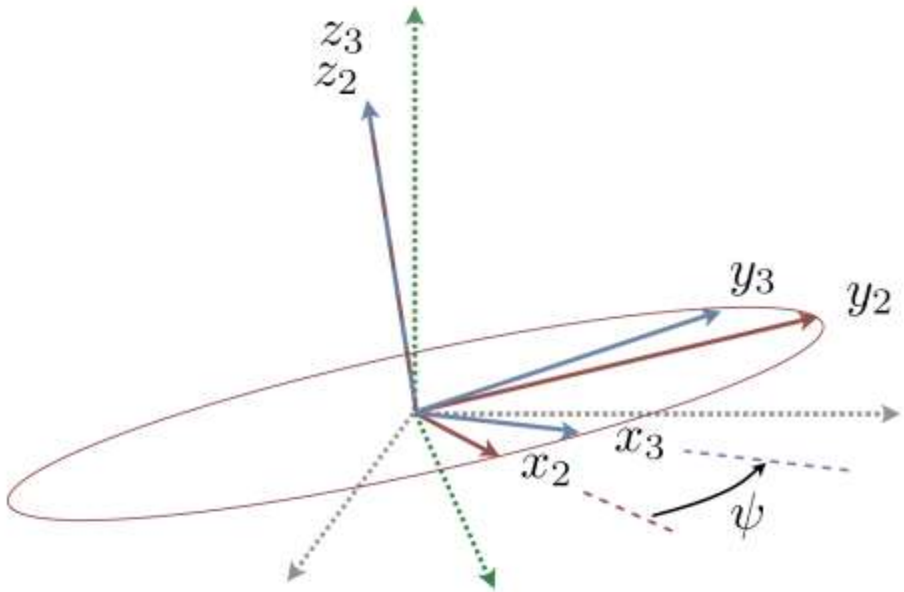


Using Z-Y-Z convention:

1. Rotate by ϕ about z_0
2. Rotate by θ about y_1

Euler Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3 by rotating around the axes of the **current frame**.



Using Z-Y-Z convention:

1. Rotate by ϕ about z_0
2. Rotate by θ about y_1
3. Rotate by ψ about z_2

Q: Should we **pre-** or **post-**multiply?

Euler Angles to Rotation Matrices

Post-multiply using the **basic rotation matrices**

$$\mathbf{R} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\,\mathbf{R}_{z,\psi}$$

$$s_\theta = sin\,\theta\,, c_\theta = \cos\theta$$

$$\mathbf{R} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}\begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Rotation Matrices to Euler Angles

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Plug in to solve for ϕ

Plug in to solve for ψ

Solve for θ

NOTE: Two solutions for θ because sign of s_θ is not known.

In general, you will end up with two sets of valid ϕ, θ, ψ values.

Example: Rotation Matrices to Euler Angles

$$\mathbf{R} = \begin{bmatrix} 0.090 & -0.785 & 0.612 \\ 0.574 & 0.544 & 0.612 \\ -0.814 & 0.296 & 0.5 \end{bmatrix}$$

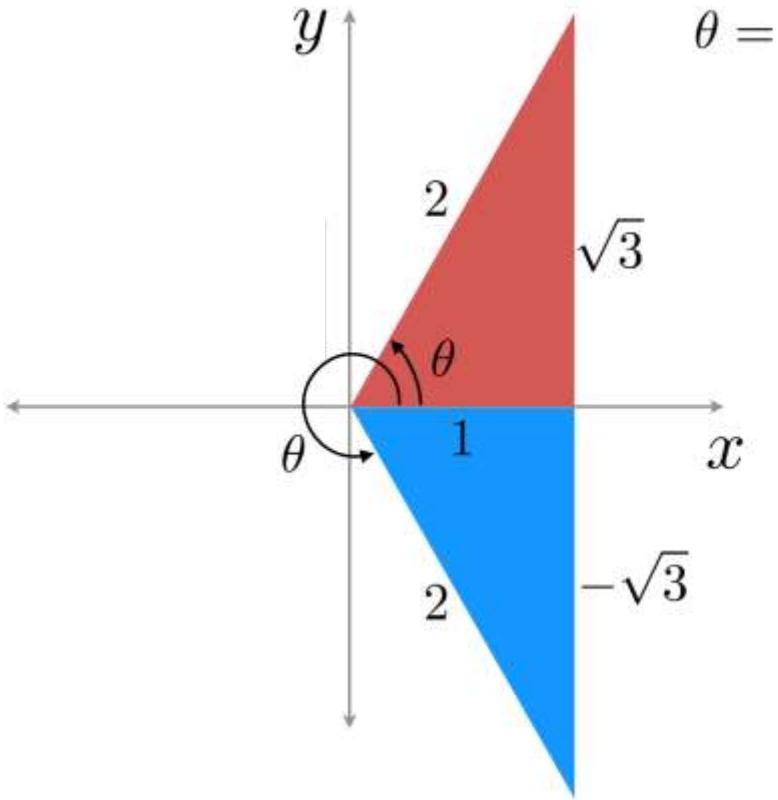
Check: Is the matrix orthonormal? ✓

Is the determinant +1? ✓

Example: Rotation Matrices to Euler Angles

$$\mathbf{R} = \begin{bmatrix} 0.090 & -0.785 & 0.612 \\ 0.574 & 0.544 & 0.612 \\ -0.814 & 0.296 & 0.5 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$



$\theta = \cos^{-1}(0.500)$

Example: Rotation Matrices to Euler Angles

$$\mathbf{R} = \begin{bmatrix} 0.090 & -0.785 & 0.612 \\ 0.574 & 0.544 & 0.612 \\ -0.814 & 0.296 & 0.5 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\theta = \frac{\pi}{3}$$

$$\sin\theta = 0.866 \Longrightarrow \begin{cases} \cos\psi = 0.940 \\ \sin\psi = 0.342 \end{cases}$$

$$\boldsymbol{\psi} = \operatorname{atan2}\left(\frac{\sin\psi}{\cos\psi}\right)$$

$$\psi = \operatorname{atan2}\left(\frac{0.342}{0.940}\right) = \frac{\pi}{9}$$

$$\theta = \frac{5\pi}{3}$$

$$\sin\theta = -0.866 \Longrightarrow \begin{cases} \cos\psi = -0.940 \\ \sin\psi = -0.342 \end{cases}$$

$$\boldsymbol{\psi} = \operatorname{atan2}\left(\frac{\sin\psi}{\cos\psi}\right)$$

$$\psi = \operatorname{atan2}\left(\frac{-0.342}{-0.940}\right) = -\frac{8\pi}{9}$$

Example: Rotation Matrices to Euler Angles

$$\mathbf{R} = \begin{bmatrix} 0.090 & -0.785 & 0.612 \\ 0.574 & 0.544 & 0.612 \\ -0.814 & 0.296 & 0.5 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\theta = \frac{\pi}{3}$$

$$\sin\theta = 0.866 \Longrightarrow \begin{cases} \cos\phi = 0.707 \\ \sin\phi = 0.707 \end{cases}$$

$$\phi = \operatorname{atan2}\left(\frac{\sin\phi}{\cos\phi}\right)$$

$$\phi = \operatorname{atan2}\left(\frac{0.707}{0.707}\right) = \frac{\pi}{4}$$

$$\theta = \frac{5\pi}{3}$$

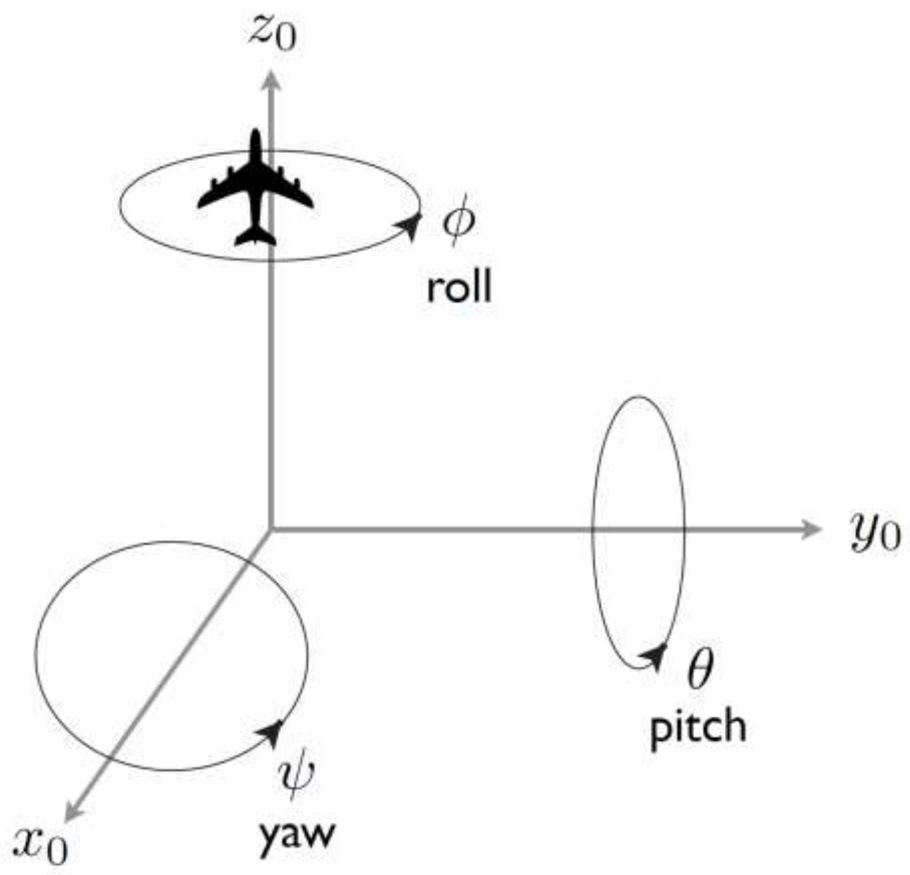
$$\sin\theta = -0.866 \Longrightarrow \begin{cases} \cos\phi = -0.707 \\ \sin\phi = -0.707 \end{cases}$$

$$\phi = \operatorname{atan2}\left(\frac{\sin\phi}{\cos\phi}\right)$$

$$\phi = \operatorname{atan2}\left(\frac{-0.707}{-0.707}\right) = -\frac{3\pi}{4}$$

Yaw, Pitch, Roll Angles

Define a set of 3 angles ϕ, θ, ψ to go from 0 → 3 by rotating around **fixed axes**.



Our book uses X-Y-Z convention.

Think about a plane flying in the z direction. Yaw is left/right, Pitch is up/down, and roll is rotating about z.

Q: Should we **pre-** or **post-**multiply?

Yaw, Pitch, Roll Angles to Rotation Matrices

Pre-multiply using the **basic rotation matrices**

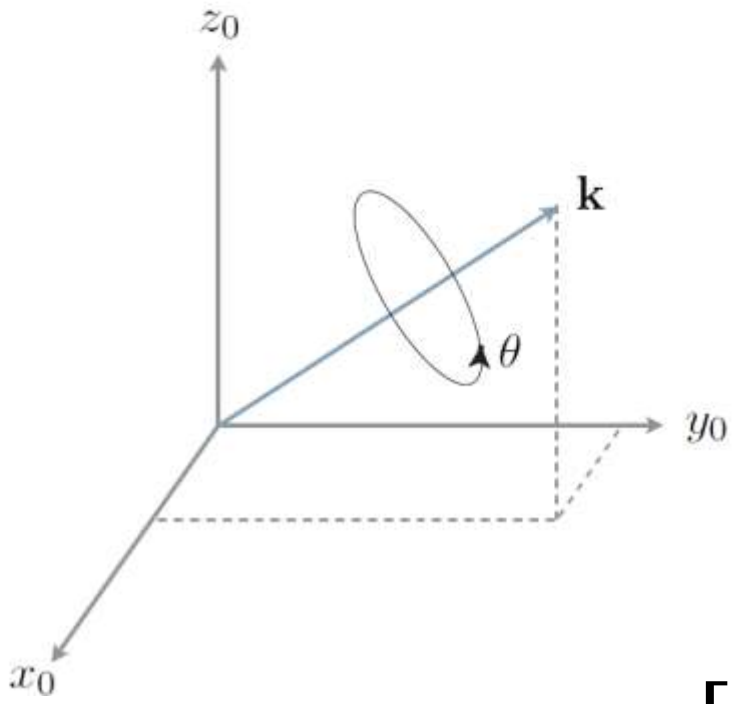
$$\mathbf{R} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\,\mathbf{R}_{x,\psi}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Angle/Axis Representation

Rotation by an angle about an axis in space



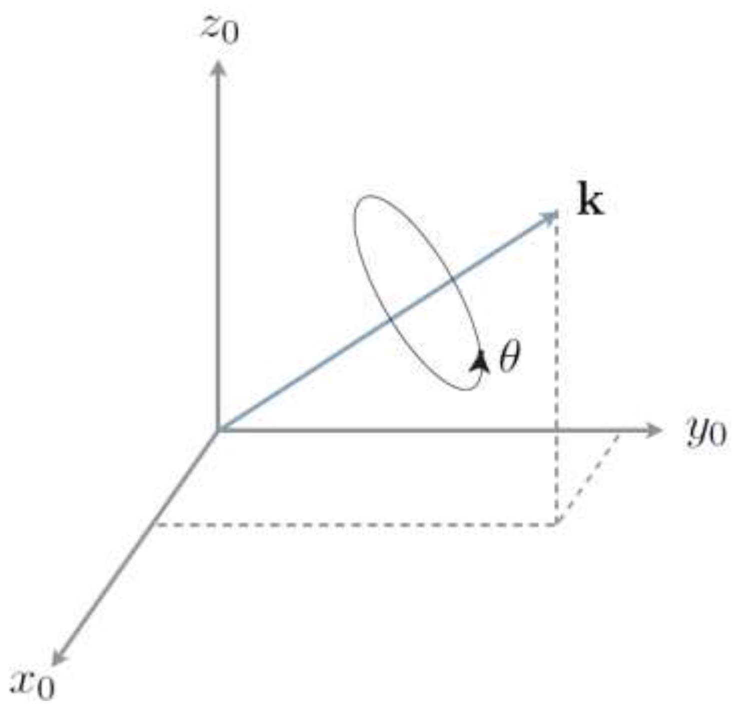
$$\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ with } \|\mathbf{k}\| = 1$$

Let $v_\theta = \operatorname{vers}\theta = 1 - c_\theta$

$$\mathbf{R}_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Angle/Axis Representation

Any rotation matrix can be represented this way!



$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

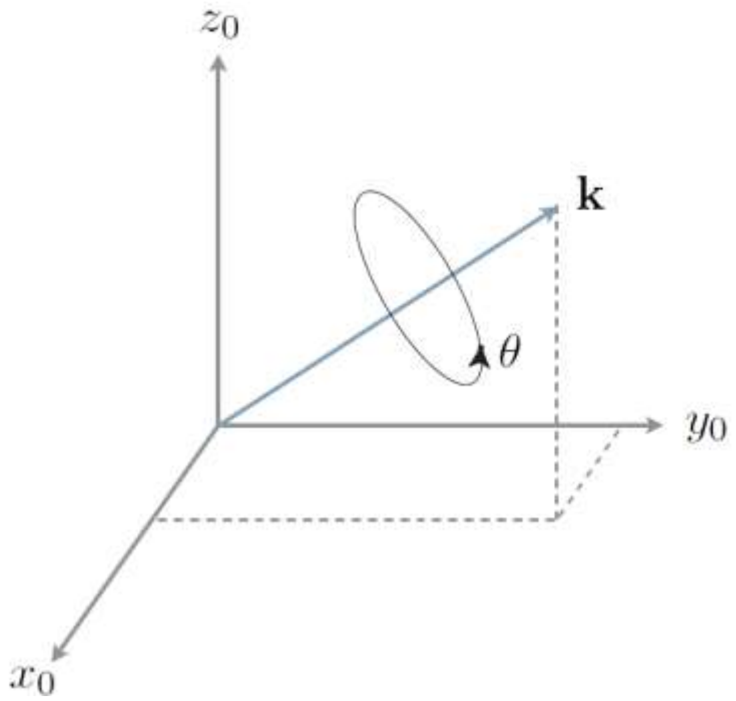
$$\mathbf{k} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Why?

Angle/Axis Representation

Any rotation matrix can be represented this way!



$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

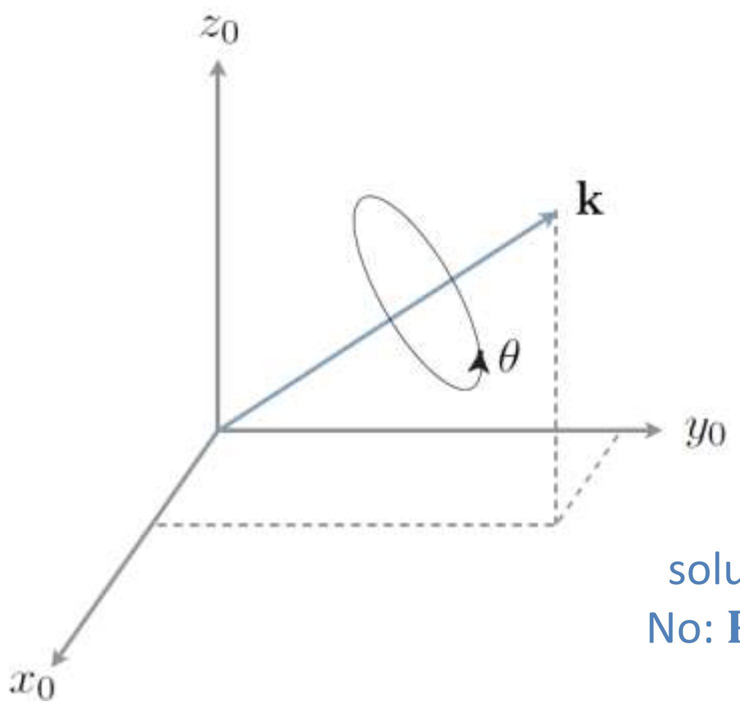
$$\mathbf{k} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Why? $\mathbf{Rk} = \mathbf{k} \Longrightarrow \mathbf{k}$ is the eigenvector of $\mathbf{R}$ corresponding to eigenvalue $\lambda = 1$

Angle/Axis Representation

Any rotation matrix can be represented this way!



Is axis/angle solution unique?
No: $\mathbf{R}_{k,\theta} = \mathbf{R}_{-k,-\theta}$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\mathbf{k} = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Why? $\mathbf{Rk} = \mathbf{k} \Longrightarrow \mathbf{k}$ is the eigenvector of $\mathbf{R}$ corresponding to eigenvalue $\lambda = 1$

Next time: Homogeneous Transformations

Chapter 2: Rigid Motions

- Read Sec. 2.6 – 2.8