MEAM 520
Lecture 22: Joint Space Dynamics

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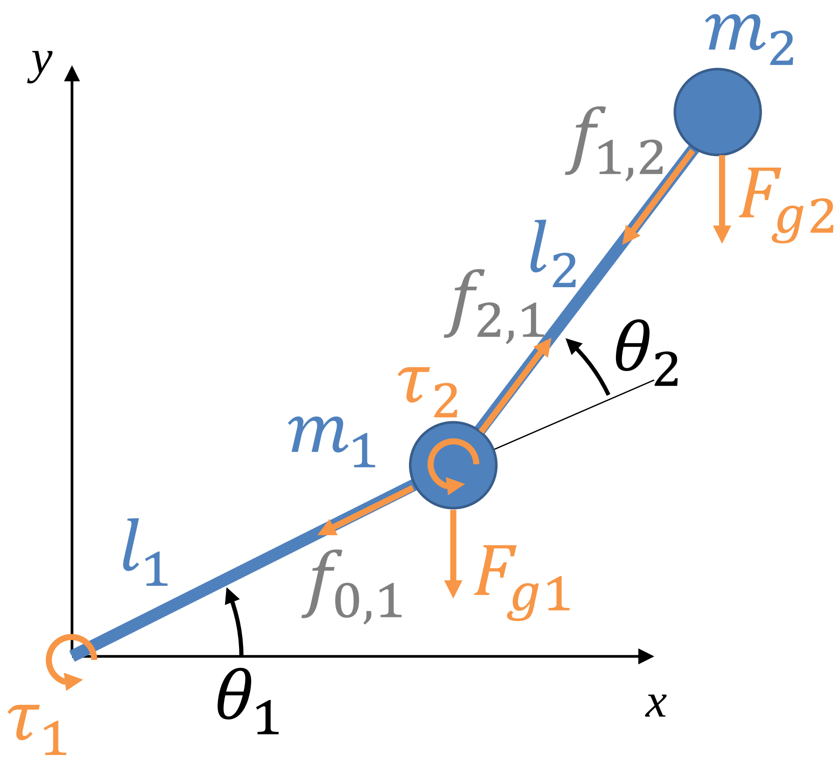
Previously: RR Manipulator w/ mass concentrated at ends of links

AKA the double pendulum



EOM:

coefficients of $\ddot{q}_i$ depend only on q

$$\boxed{[m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2l_1 l_2 c_2)]}\ddot{\theta}_1$$
$$+ \boxed{[m_2(l_2^2 + l_1 l_2 c_2)]}\ddot{\theta}_2 \boxed{- m_2 l_1 l_2 s_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)}$$
$$+ \boxed{m_1 g l_1 c_1 + m_2 g(l_1 c_1 + l_2 c_{12}}) = \tau_1$$

$$\boxed{[m_2(l_2^2 + l_1 l_2 c_2)]}\ddot{\theta}_1 + \boxed{m_2 l_2}\ddot{\theta}_2 + \boxed{m_2 l_1 l_2 s_2 \dot{\theta}_1^2}$$
$$+ \boxed{m_2 g l_2 c_{12}} = \tau_2$$

centrifugal and Coriolis terms depend on q and $\dot{q}$

gravitational terms depend only on q

Previously: Manipulator Equation

We can write this as a matrix equation

$$\tau = \underbrace{D(q)}\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

SHV uses a bit of strange notation.
Most people call this matrix *H* or *M*.

where

$D(q)$ is the *nxn* mass matrix (inertia terms)

$C(q,\dot{q})$ is the *nxn* matrix of centrifugal (square of joint velocities) and Coriolis (product of two different joint velocities) terms

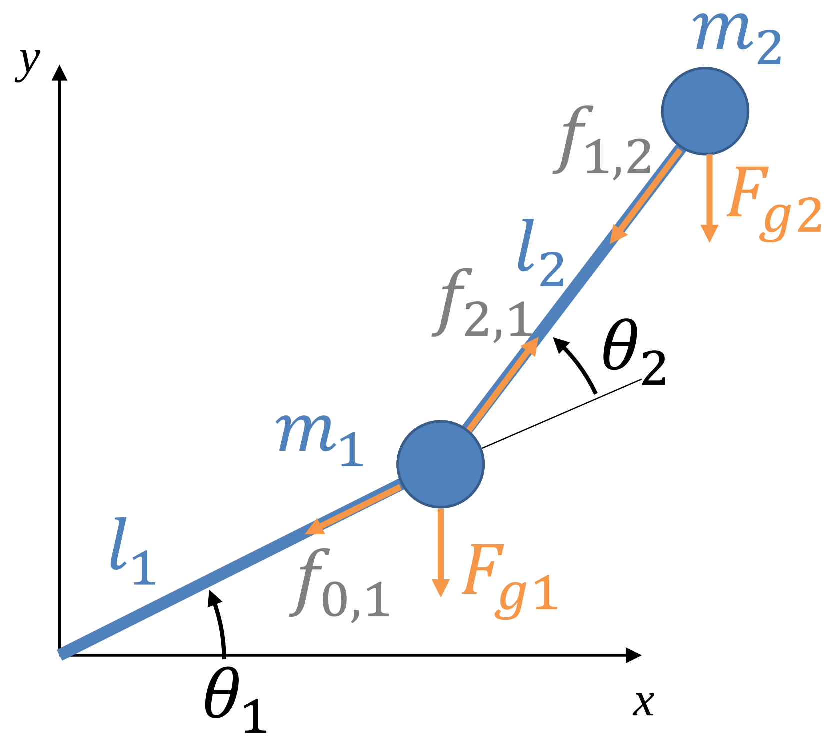
$g(q)$ is a *nx1* vector of gravitational terms

Previously: Manipulator Equation for an N(R) robot

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$D = \sum_{i=1}^{N} m_i J_{vi}^{\top} J_{vi}$$

$$g = \frac{\partial}{\partial q} \sum_{i=1}^{N} m_i \vec{g} \cdot \vec{r}_i$$

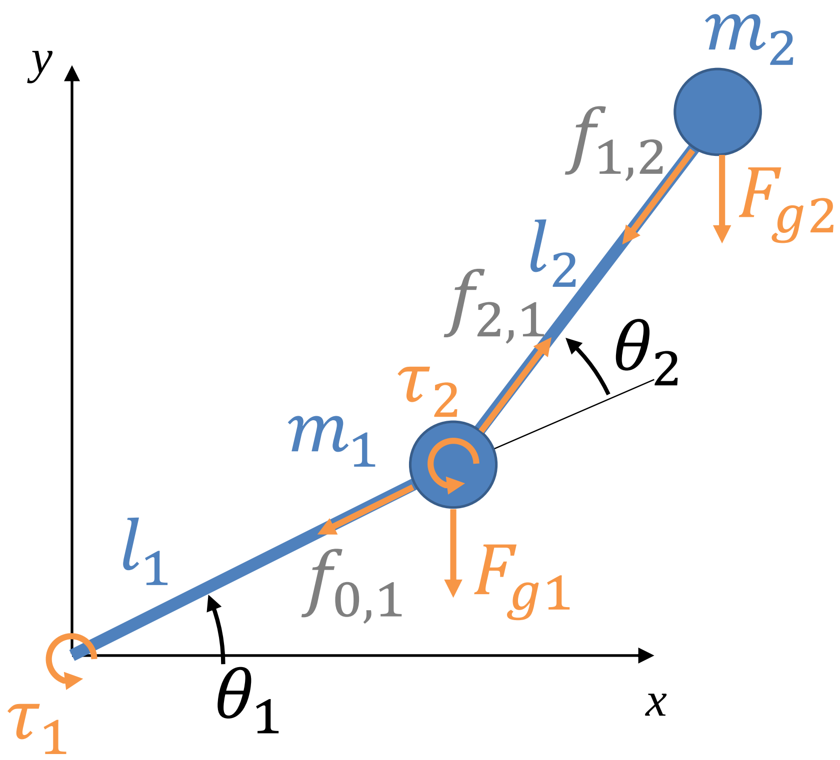


$$(C\dot{q})_k = \sum_{i,j} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j$$

or

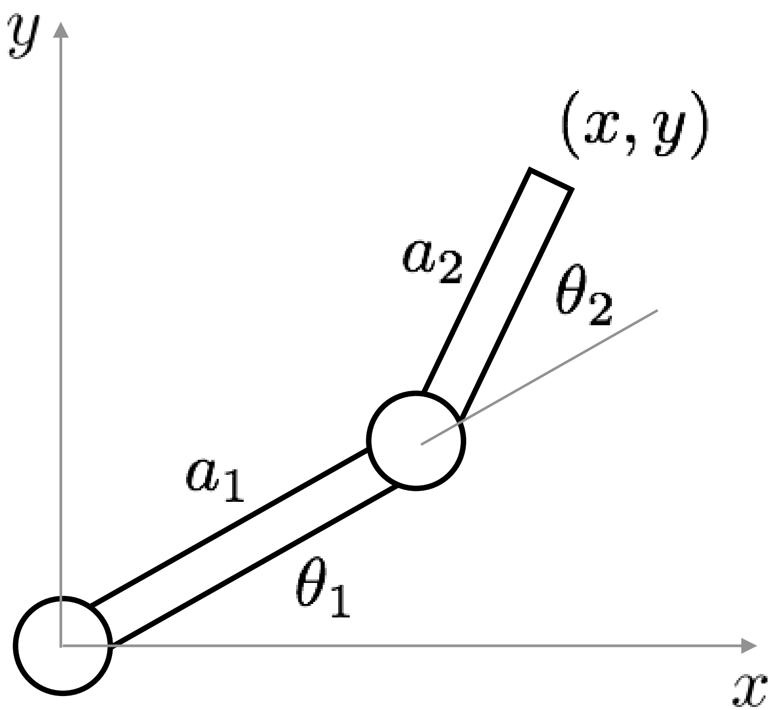
$$c_{kj} = \sum_{i} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i$$

Inertia Tensor



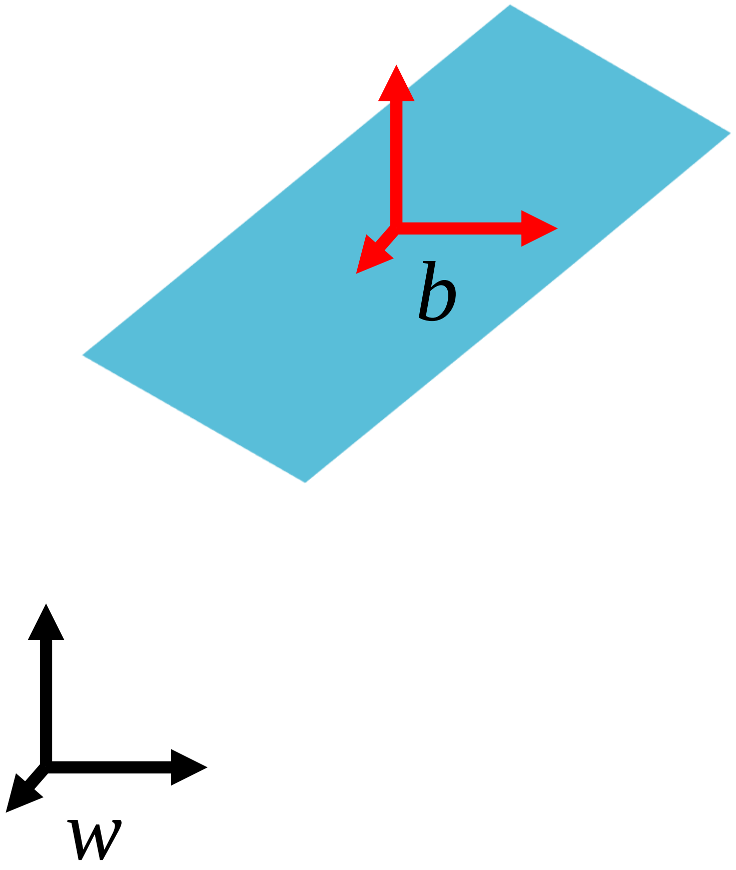
This manipulator has all the mass concentrated at the joints

Inertia Tensor



What happens when you have distributed mass?

Rigid Body Rotating in Space



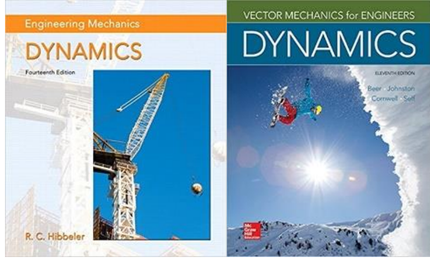
Kinetic Energy: $K = \frac{1}{2} m \vec{v}_{COM}^{\top} \vec{v}_{COM} + \frac{1}{2} \boxed{\vec{\omega}^{\top} I_{COM} \vec{\omega}}$

All of these quantities are expressed in the inertial frame (i.e., frame w)

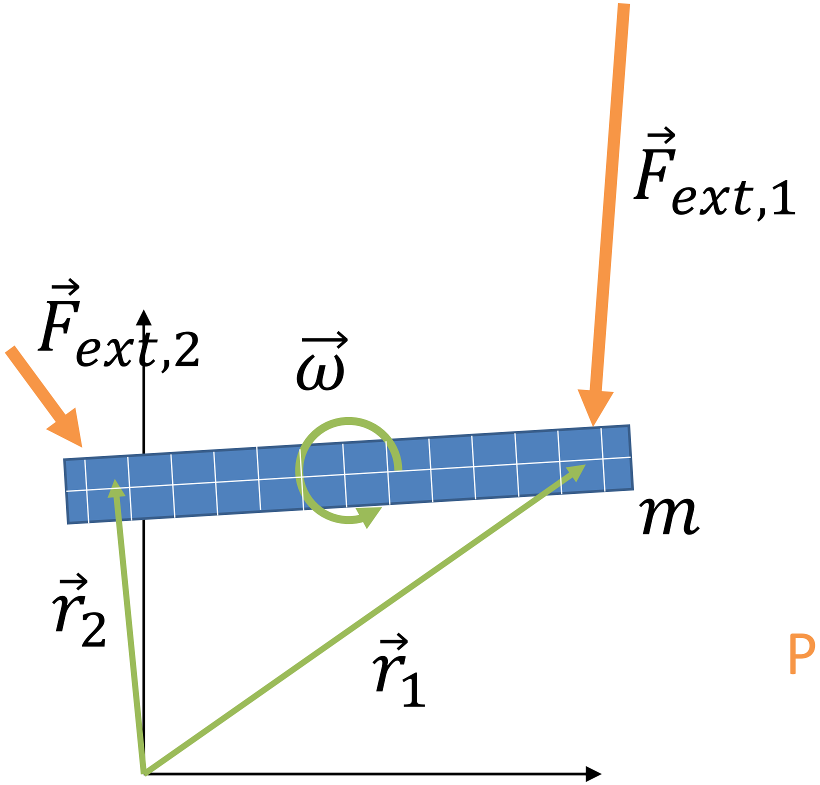
I is constant in frame b but not in frame w

What is I in frame w?

Rigid Body Dynamics



Hibbeler Ch. 21.1
Beer Ch. 18.2



$$\vec{r}_{COM} = \frac{1}{m}\int \vec{r}dm$$

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Principal Moments of Inertia

$$I_{xx} = \iiint (y^2 + z^2)dm$$

$$I_{yy} = \iiint (x^2 + z^2)dm$$

$$I_{zz} = \iiint (x^2 + y^2)dm$$

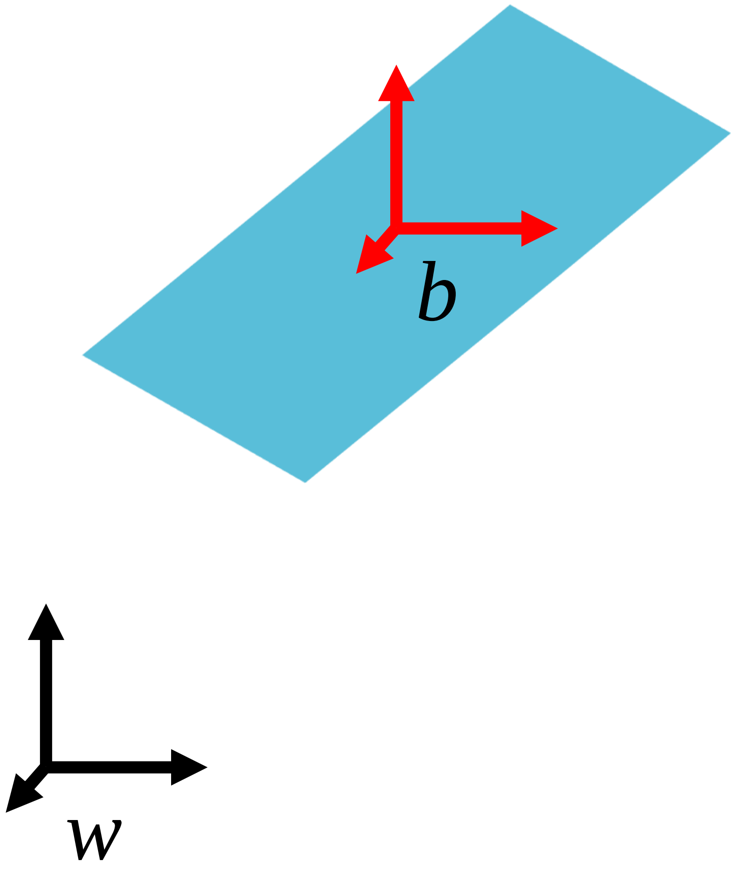
Products of Inertia

$$I_{xy} = I_{yx} = -\iiint xydm$$

$$I_{xz} = I_{zx} = -\iiint xzdm$$

$$I_{yz} = I_{zy} = -\iiint yzdm$$

Rigid Body Rotating in Space



$$\boxed{(\vec{\omega}^{w})^{\top}} I_{COM}^{w} \boxed{\vec{\omega}^{w}}$$

scalar

$$=$$

$$(\vec{\omega}^{b})^{\top} \boxed{I_{COM}^{b}} \vec{\omega}^{b}$$

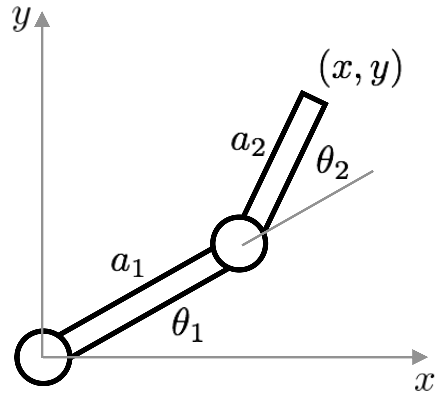
known

$$(R_{w}^{b} \vec{\omega}^{w})^{\top} I_{COM}^{b} (R_{w}^{b} \vec{\omega}^{w})$$

$$(\vec{\omega}^{w})^{\top} (R_{w}^{b})^{\top} I_{COM}^{b} R_{w}^{b} \vec{\omega}^{w}$$

$$(\vec{\omega}^{w})^{\top} \underbrace{\boxed{R_{b}^{w} I_{COM}^{b} (R_{b}^{w})^{\top}}}_{I_{COM}^{w}} \vec{\omega}^{w}$$

Manipulator Kinetic Energy



$$K_i = \frac{1}{2} m_i \vec{v}_i^\top \vec{v}_i + \frac{1}{2} \vec{\omega}_i^\top I_i \vec{\omega}_i$$

$$K_i = \frac{1}{2} \dot{q}^\top \left(m_i J_{vi}^\top J_{vi} \right) \dot{q} + \frac{1}{2} \vec{\omega}_i^\top I_i \vec{\omega}_i$$

last time

$$K_i = \frac{1}{2} \dot{q}^\top \left(m_i J_{vi}^\top J_{vi} \right) \dot{q} + \frac{1}{2} \vec{\omega}_i^\top R_i^0 I_i \left(R_i^0 \right)^\top \vec{\omega}_i$$

$$K_i = \frac{1}{2} \dot{q}^\top \left(m_i J_{vi}^\top J_{vi} \right) \dot{q} + \frac{1}{2} \left(J_{\omega i} \dot{q} \right)^\top R_i^0 I_i \left(R_i^0 \right)^\top \left(J_{\omega i} \dot{q} \right)$$

Angular velocity Jacobian: $\omega_i = J_{\omega i} \dot{q}$

$$K_i = \frac{1}{2} \dot{q}^\top \left(m_i J_{vi}^\top J_{vi} \right) \dot{q} + \frac{1}{2} \dot{q}^\top J_{\omega i}^\top R_i^0 I_i \left(R_i^0 \right)^\top J_{\omega i} \dot{q}$$

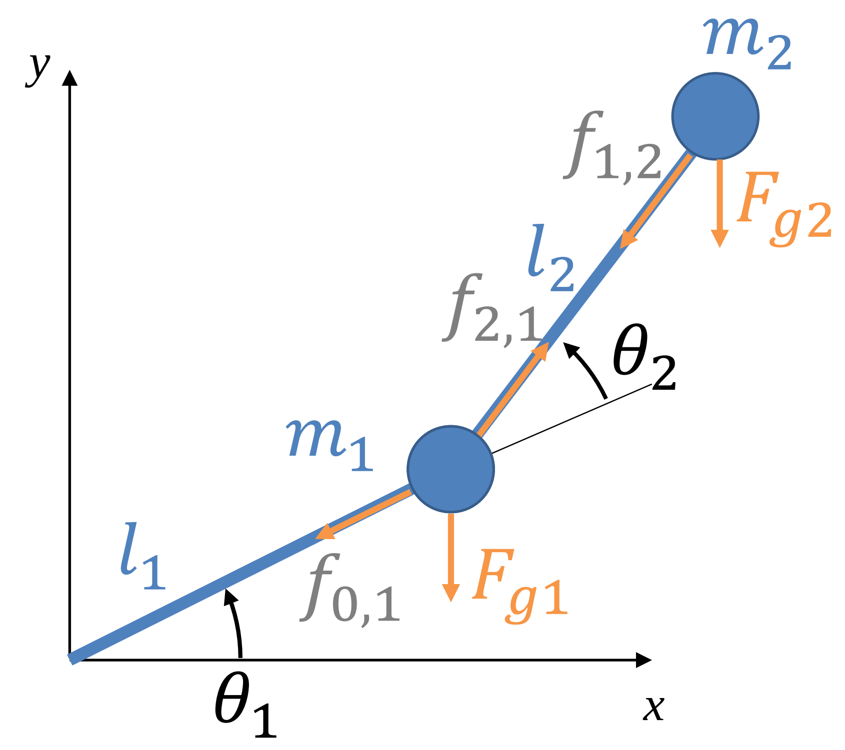
$(AB)^\top = B^\top A^\top$

$$K_i = \frac{1}{2} \dot{q}^\top \boxed{\left(m_i J_{vi}^\top J_{vi} + J_{\omega i}^\top R_i^0 I_i \left(R_i^0 \right)^\top J_{\omega i} \right)} \dot{q}$$

Inertia Matrix D

Manipulator Equation for an N(R) robot

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$



$$D = \sum_{i=1}^{N} m_i J_{vi}^{\top} J_{vi}$$

$$g = \frac{\partial}{\partial q} \sum_{i=1}^{N} m_i \vec{g} \cdot \vec{r}_i$$

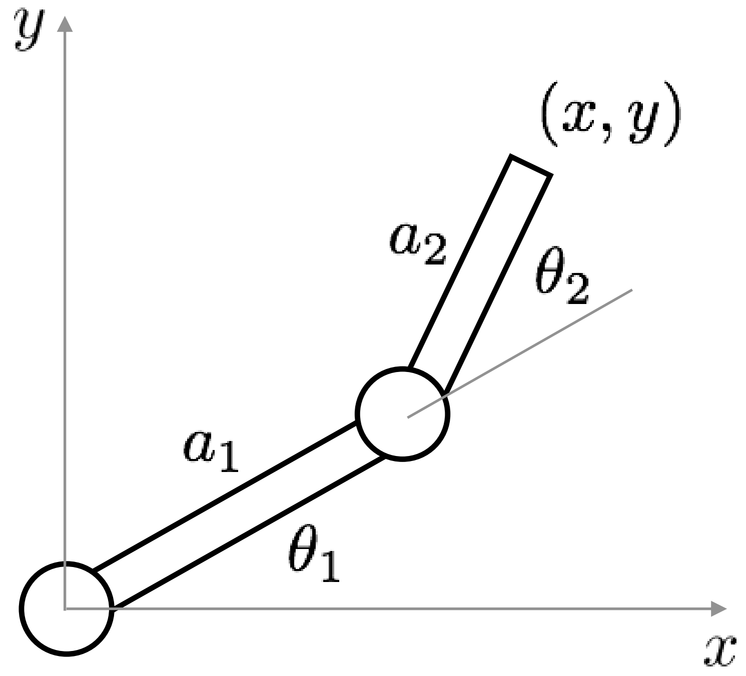
$$(C\dot{q})_k = \sum_{i,j} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j$$

or

$$c_{kj} = \sum_{i} \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i$$

Manipulator Equation for an N(R) robot

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$



$$D = \sum_{i=1}^{N}\left(m_i J_{vci}^{\top} J_{vci} + J_{\omega i}^{\top} R_i I_i R_i^{\top} J_{\omega i}\right)$$

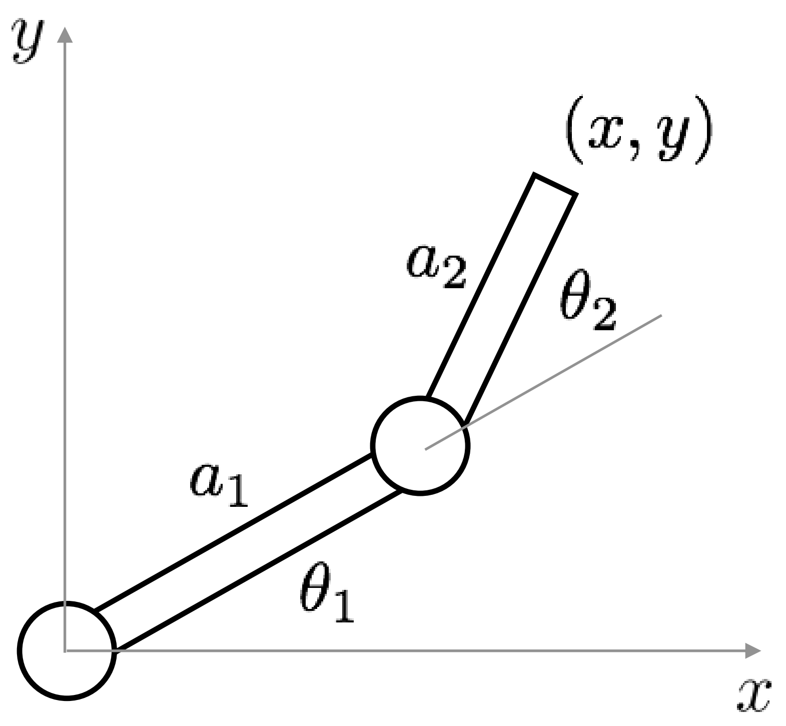
$$g = \frac{\partial}{\partial q}\sum_{i=1}^{N} m_i \vec{g} \cdot \vec{r}_i$$

$$(C\dot{q})_k = \sum_{i,j}\frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j$$

or

$$c_{kj} = \sum_{i}\frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i$$

Example: Planar RR Manipulator



Angular velocity

$$\omega_1 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}_{J_{\omega 1}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \qquad \omega_2 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}_{J_{\omega 2}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Inertia

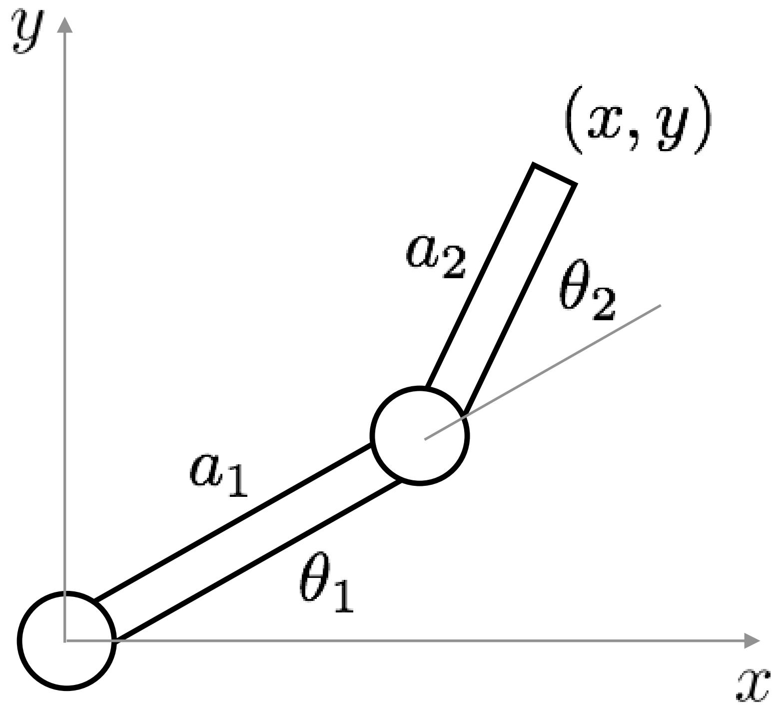
$$I_1 = \begin{bmatrix} 0 & I_{x_1 y_1} & 0 \\ I_{y_1 x_1} & 0 & 0 \\ 0 & 0 & I_{z_1 z_1} \end{bmatrix} \qquad I_2 = \begin{bmatrix} 0 & I_{x_2 y_2} & 0 \\ I_{y_2 x_2} & 0 & 0 \\ 0 & 0 & I_{z_2 z_2} \end{bmatrix}$$

Rotation Matrix

$$R_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inertia: $D_{rot} = \sum_i J_{\omega i}^{\top} R_i^0 I_i \left(R_i^0\right)^{\top} J_{\omega i} = I_{z_1 z_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_{z_2 z_2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Example: Planar RR Manipulator



EOM

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 = \tau_1$$

$$d_{21}\ddot{q}_2 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 = \tau_2$$

Inertia

$$d_{11} = m_1 a_{c,1}^2 + m_2\left(a_1^2 + a_{c,2}^2 + 2a_1 a_{c,2} c_2\right) + I_{z_1 z_1} + I_{z_2 z_2}$$

$$d_{12} = d_{21} = m_2\left(a_{c,2}^2 + a_1 a_2 c_2 + I_{z_2 z_2}\right)$$

$$d_{22} = m_2 a_{c,2}^2 + I_{z_2 z_2}$$

Christoffel Symbols

$$c_{111} = c_{122} = c_{222} = 0$$

$$c_{121} = c_{221} = -c_{112} = -m_2 a_1 a_{c,2} s_2$$

Gravity

$$g_1 = \left(m_1 a_{c,1} + m_2 a_1\right) g c_1 + m_2 a_{c,2} g c_{12}$$

$$g_2 = m_2 a_{c,2} g c_{12}$$

Newton-Euler vs. Euler-Lagrange

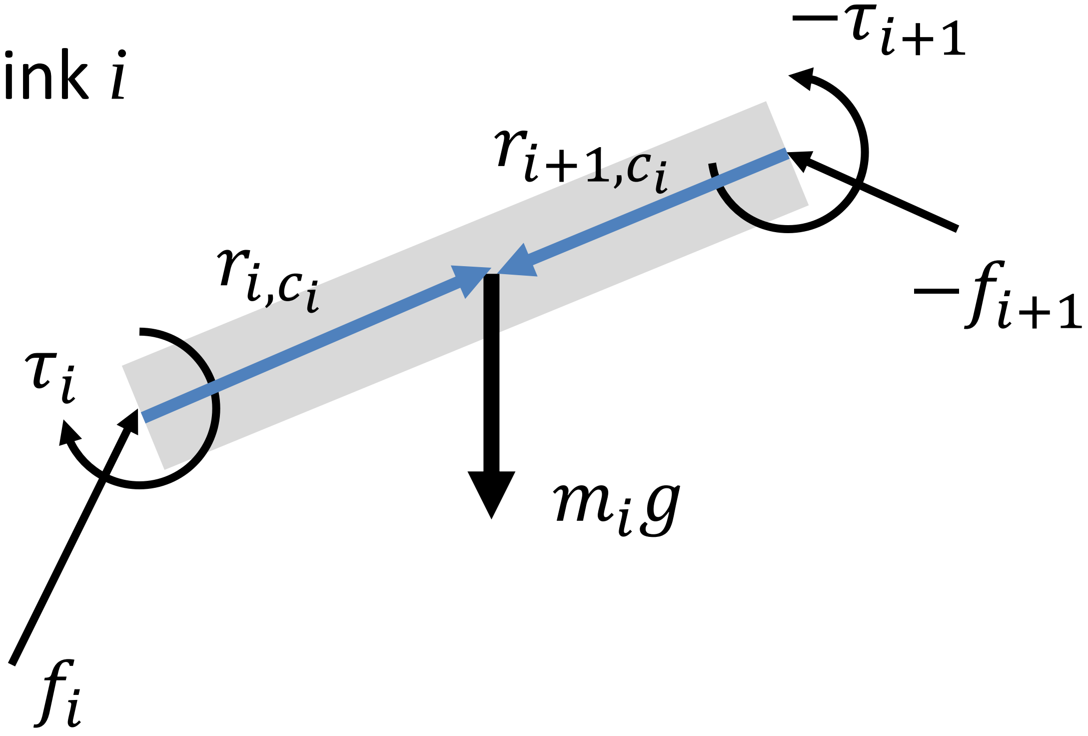
Euler-Lagrange produces a **closed-form** differential equation that describes the time evolution of the generalized coordinates

Computing these equations requires you to take partial derivatives

Newton-Euler allows you to do **recursive computation** to figure out the generalized forces for a particular time evolution

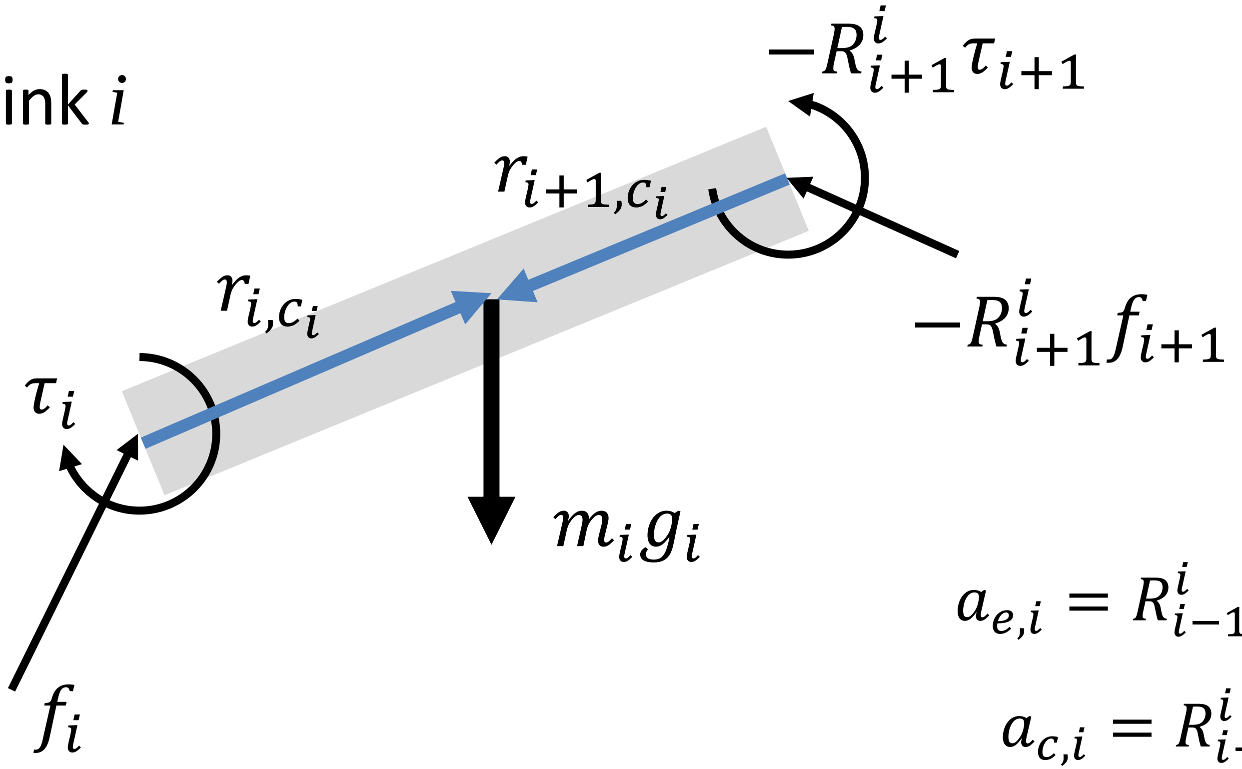
Newton-Euler

Link i



Newton-Euler (for revolute joints)

Link i



Kinematic constraints

$$\omega_i = R^i_{i-1}\omega_{i-1} + z^i_{i-1}\dot{q}_i$$

$$\alpha_i = R^i_{i-1}\alpha_{i-1} + z^i_{i-1}\ddot{q}_i + \omega_i \times z^i_{i-1}\dot{q}_i$$

$$a_{e,i} = R^i_{i-1}a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1})$$

$$a_{c,i} = R^i_{i-1}a_{e,i-1} + \dot{\omega}_i \times r_{i,c_i} + \omega_i \times (\omega_i \times r_{i,c_i})$$

Forces/Moments

$$f_i - R^i_{i+1}f_{i+1} + m_i g_i = m_i a_{c,i}$$

$$\tau_i - R^i_{i+1}\tau_{i+1} + f_i \times r_{i,c_i} - \left(R^i_{i+1}f_{i+1}\right) \times r_{i+1,c_i} = I_i\dot{\omega}_i + \omega_i \times (I_i\omega_i)$$

Newton-Euler (for revolute joints)

Start with $\omega_0 = 0, \alpha_0 = 0, a_{c,0} = 0, a_{e,0} = 0$

Solve kinematic constraints for i from 1 to n

Start with $f_{n+1} = 0, \tau_{n+1} = 0$

Solve force/moments for i from n to 1

Kinematic constraints

i terms on the left

No forces/moments!

$$\omega_i = R_{i-1}^i \omega_{i-1} + z_{i-1}^i \dot{q}_i$$

$$\alpha_i = R_{i-1}^i \alpha_{i-1} + z_{i-1}^i \ddot{q}_i + \omega_i \times z_{i-1}^i \dot{q}_i$$

$$a_{e,i} = R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1})$$

$$a_{c,i} = R_{i-1}^i a_{e,i-1} + \dot{\omega}_i \times r_{i,c_i} + \omega_i \times (\omega_i \times r_{i,c_i})$$

i-1 terms on the right

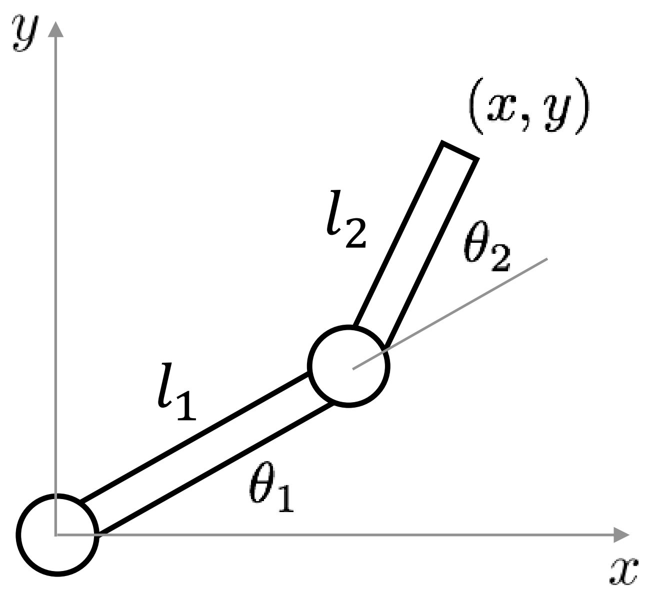
Forces/Moments

Careful! All terms written in frame i

$$f_i - R_{i+1}^i f_{i+1} + m_i g_i = m_i a_{c,i}$$

$$\tau_i - R_{i+1}^i \tau_{i+1} + f_i \times r_{i,c_i} - (R_{i+1}^i f_{i+1}) \times r_{i+1,c_i} = I_i \dot{\omega}_i + \omega_i \times (I_i \omega_i)$$

Example: Planar RR Manipulator



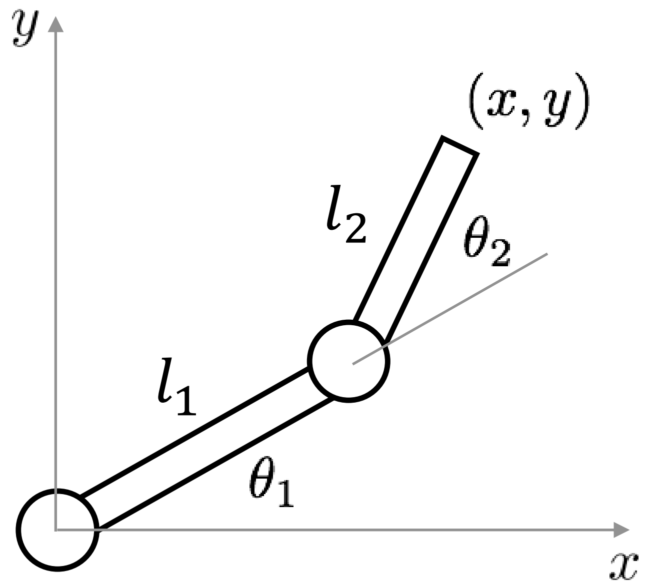
Forward Recursion

$$\omega_0 = 0, \alpha_0 = 0, a_{e,0} = 0, a_{c,0} = 0$$

$$\omega_1 = \dot{q}_1 \mathbf{z}$$
$$\alpha_1 = \ddot{q}_1 \mathbf{z}$$
$$a_{e,1} = \ddot{q}_1 \mathbf{z} \times \boxed{l_1 \mathbf{x}} + \dot{q}_1 \mathbf{z} \times (\dot{q}_1 \mathbf{z} \times \boxed{l_1 \mathbf{x}}) = [-l_1 \dot{q}_1^2 \quad l_1 \ddot{q}_1 \quad 0]^\top$$
$$a_{c,1} = [-l_{c,1} \dot{q}_1^2 \quad l_{c,1} \ddot{q}_1 \quad 0]^\top$$

$$\omega_2 = (\dot{q}_1 + \dot{q}_2) \mathbf{z}$$
$$\alpha_2 = (\ddot{q}_1 + \ddot{q}_2) \mathbf{z}$$
$$a_{e,2} = R_1^2 a_{e,1} + (\ddot{q}_1 + \ddot{q}_2) \mathbf{z} \times \boxed{l_2 \mathbf{x}} + (\dot{q}_1 + \dot{q}_2) \mathbf{z} \times \left((\dot{q}_1 + \dot{q}_2) \mathbf{z} \times \boxed{l_2 \mathbf{x}}\right)$$
$$= [-l_1 \dot{q}_1^2 c_2 + l_1 \ddot{q}_1 + s_2 - l_{c2} (\dot{q}_1 + \dot{q}_2)^2 \quad l_1 \dot{q}_1^2 s_2 + l_1 \ddot{q}_1 c_2 - l_2 (\ddot{q}_1 + \ddot{q}_2) \quad 0]^\top$$
$$a_{c,2} = [-l_1 \dot{q}_1^2 c_2 + l_1 \ddot{q}_1 + s_2 - l_{c,2} (\dot{q}_1 + \dot{q}_2)^2 \quad l_1 \dot{q}_1^2 s_2 + l_1 \ddot{q}_1 c_2 - l_{c,2} (\ddot{q}_1 + \ddot{q}_2) \quad 0]^\top$$

Example: Planar RR Manipulator



Backward Recursion

$$f_3 = 0, \tau_3 = 0$$

$$f_2 = m_2 a_{c,2} \boxed{- m_2 g_2}$$
$$\tau_2 = -f_2 \times l_{c,2}\mathbf{x} + I_2 \alpha_2 + \omega_2 \times (I_2 \omega_2)$$

$$f_1 = m_1 a_{c,1} + R_2^1 f_2 \boxed{- m_1 g_1}$$
$$\tau_1 = R_2^1 \tau_2 - f_1 \times l_{c,1}\mathbf{x} - R_2^1 f_2 \times (l_1 - l_{c,1})\mathbf{x} + I_1 \alpha_1 + \omega_1 \times (I_1 \omega_1)$$

Method Comparisons

Newton-Euler

- Complete solution for all forces and kinematic variables
- Inefficient when only a few of the system’s forces need to be solved for

Euler-Lagrange

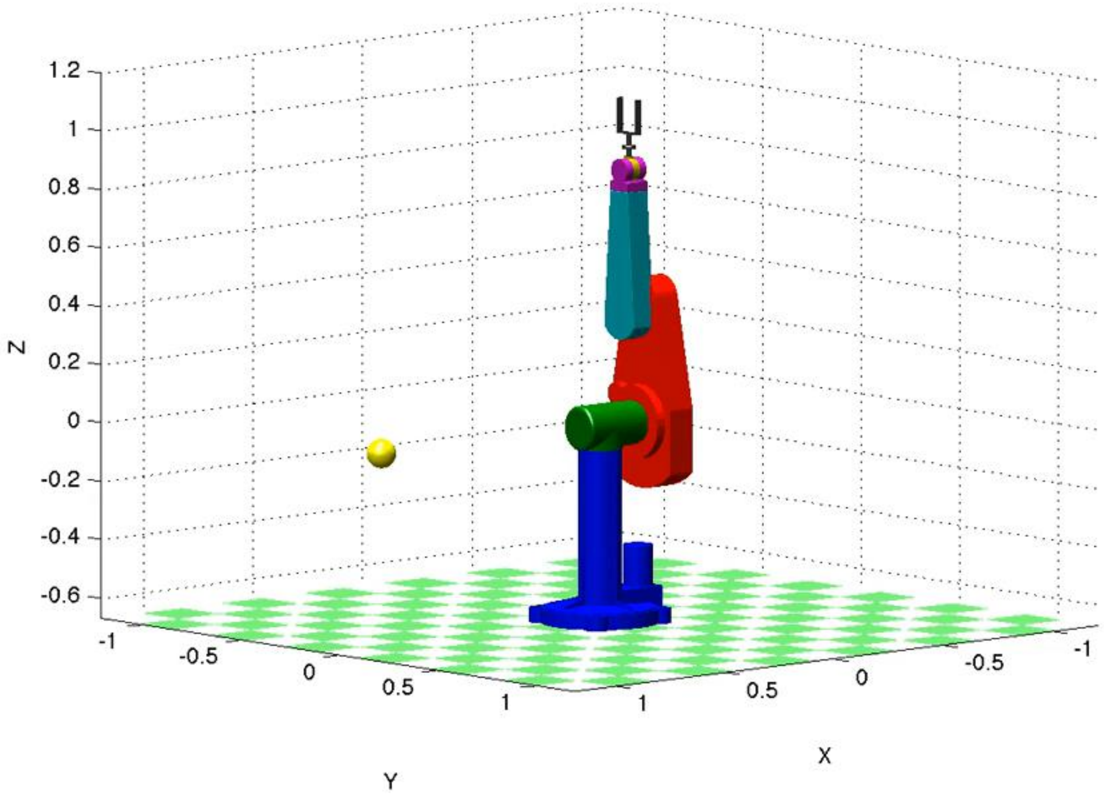
- Disregard all interactive and constraint forces that do not perform work
- Need to differentiate scalar energy functions
- Inefficient for large multi-body systems

Kane’s Method

- Generalized forces so eliminate interactive and constraint forces
- Does not employ energy functions, so no derivatives
- Lends itself to automated numerical computation

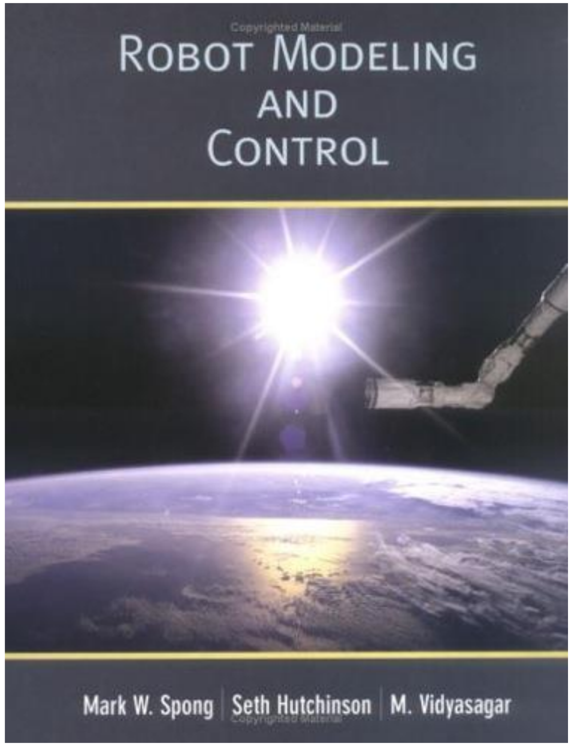
Check out the Robotics Toolbox

http://petercorke.com/wordpress/toolboxes/robotics-toolbox



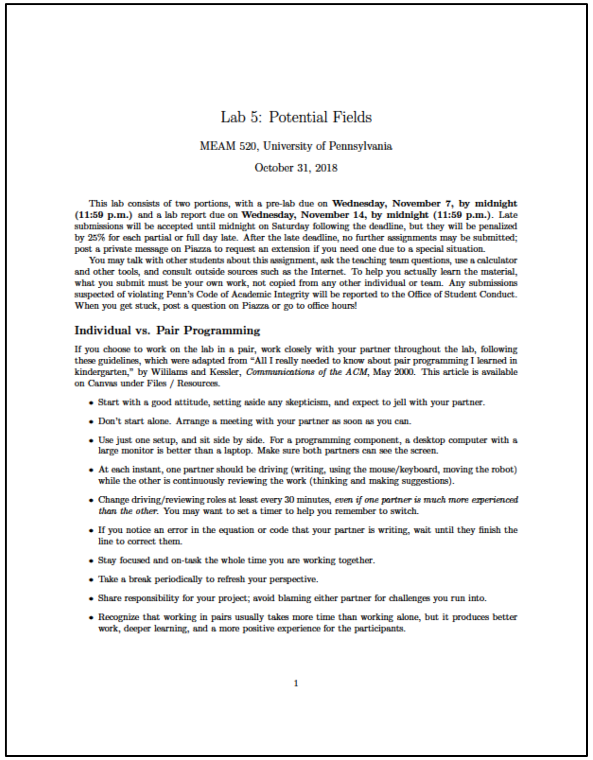
Peter Corke
Professor
Queensland University of Technology

Next time: Examples



Chapter 6: Independent Joint Control

- Read 6.3-6.4



Lab 5: Potential Fields

MEAM 520, University of Pennsylvania

October 31, 2018

This lab consists of two portions, with a pre-lab due on **Wednesday, November 7, by midnight (11:59 p.m.)** and a lab report due on **Wednesday, November 14, by midnight (11:59 p.m.)**. Late submissions will be accepted until midnight on Saturday following the deadline, but they will be penalized by 25% for each partial or full day late. After the late deadline, no further assignments may be submitted; post a private message on Piazza to request an extension if you need one due to a special situation.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you submit must be your own work, not copied from any other individual or team. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. When you get stuck, post a question on Piazza or go to office hours!

Individual vs. Pair Programming

If you choose to work on the lab in a pair, work closely with your partner throughout the lab, following these guidelines, which were adapted from "All I really needed to know about pair programming I learned in kindergarten," by Wililams and Kessler, *Communications of the ACM*, May 2000. This article is available on Canvas under Files / Resources.

- Start with a good attitude, setting aside any skepticism, and expect to jell with your partner.
- Don't start alone. Arrange a meeting with your partner as soon as you can.
- Use just one setup, and sit side by side. For a programming component, a desktop computer with a large monitor is better than a laptop. Make sure both partners can see the screen.
- At each instant, one partner should be driving (writing, using the mouse/keyboard, moving the robot) while the other is continuously reviewing the work (thinking and making suggestions).
- Change driving/reviewing roles at least every 30 minutes, *even if one partner is much more experienced than the other*. You may want to set a timer to help you remember to switch.
- If you notice an error in the equation or code that your partner is writing, wait until they finish the line to correct them.
- Stay focused and on-task the whole time you are working together.
- Take a break periodically to refresh your perspective.
- Share responsibility for your project; avoid blaming either partner for challenges you run into.
- Recognize that working in pairs usually takes more time than working alone, but it produces better work, deeper learning, and a more positive experience for the participants.

1

Lab 5: Potential Fields due 11/14