MEAM 520
Lecture 14: Velocity Kinematics

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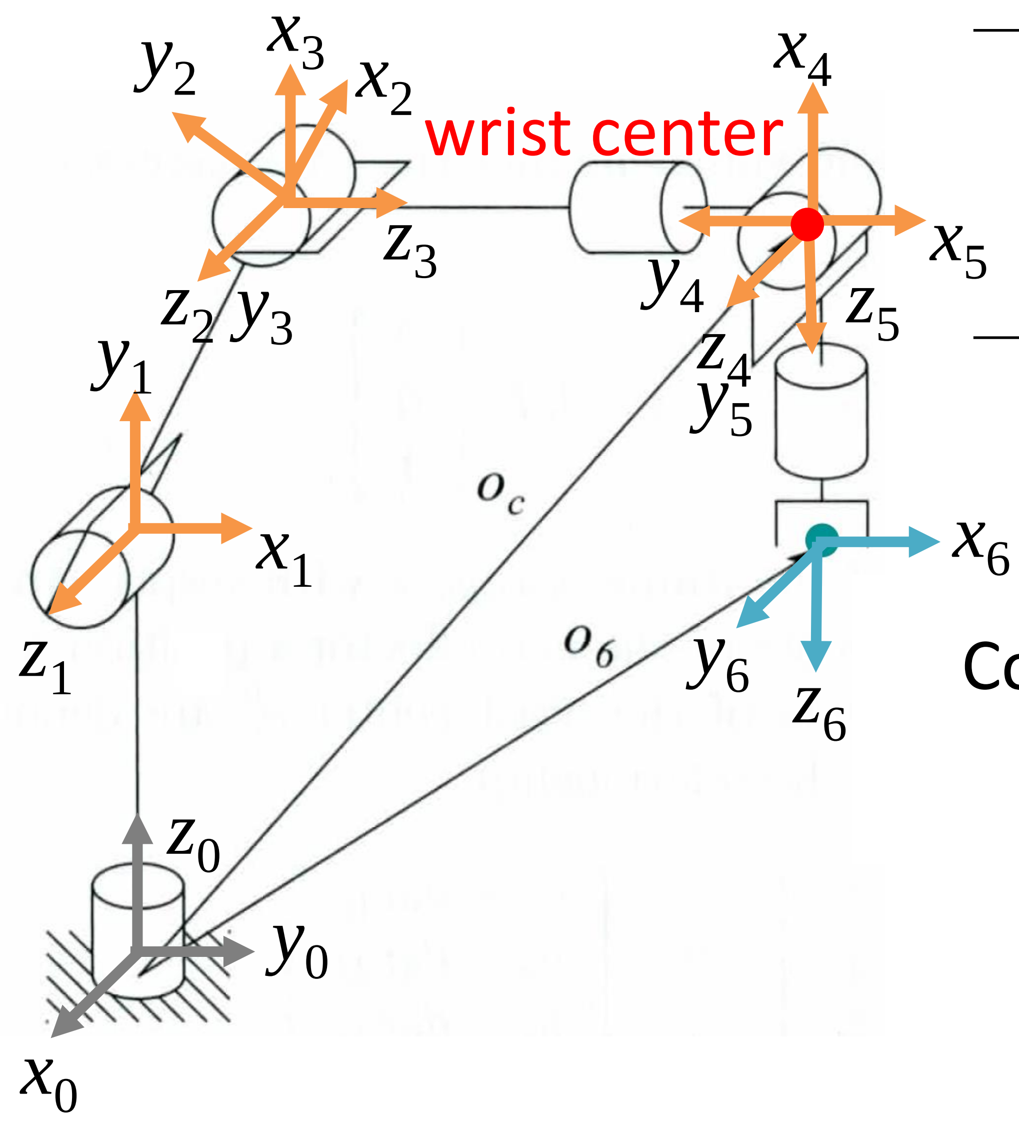
Recap of the semester so far:



Link	*x* step		*z* step	
	a_i	α_i	d_i	θ_i
1	0	$-\pi/2$	d_1	θ_1
2	a_2	0	0	θ_2
3	0	$\pi/2$	0	θ_3
4	0	$-\pi/2$	d_4	θ_4
5	0	$\pi/2$	0	θ_5
6	0	0	d_6	θ_6

DH convention

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

FK

$\boldsymbol{q}$ Configuration Space $\;\;\;\;$ $\mathbf{T}_n^0$ Task Space

IK

Kinematic Decoupling

position

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

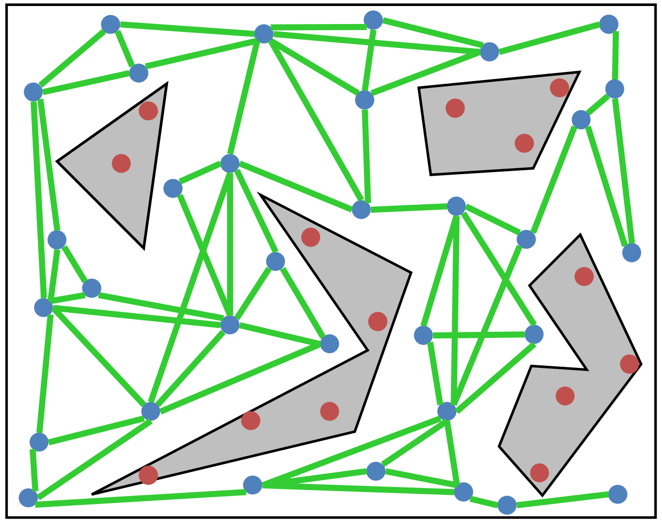
orientation

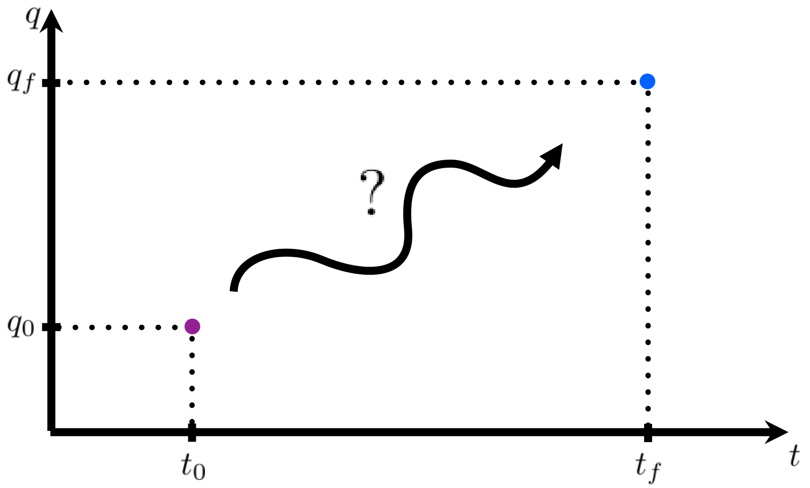
$$\mathbf{R}_6^3 = (\mathbf{R}_3^0)^{-1}\mathbf{R} = (\mathbf{R}_3^0)^{\mathrm{T}}\mathbf{R}$$

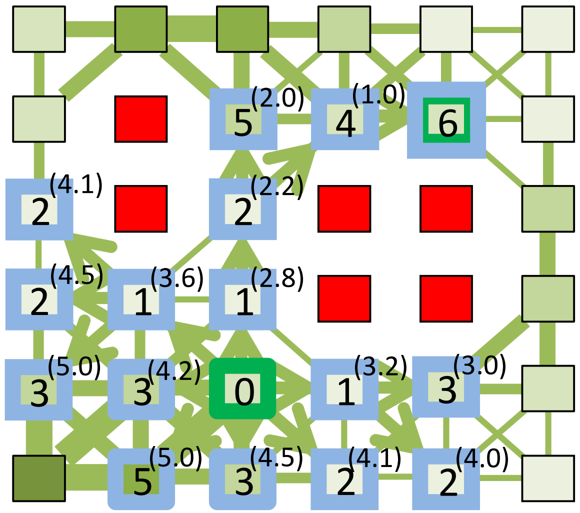
Recap of the semester so far:

Planning strategy:

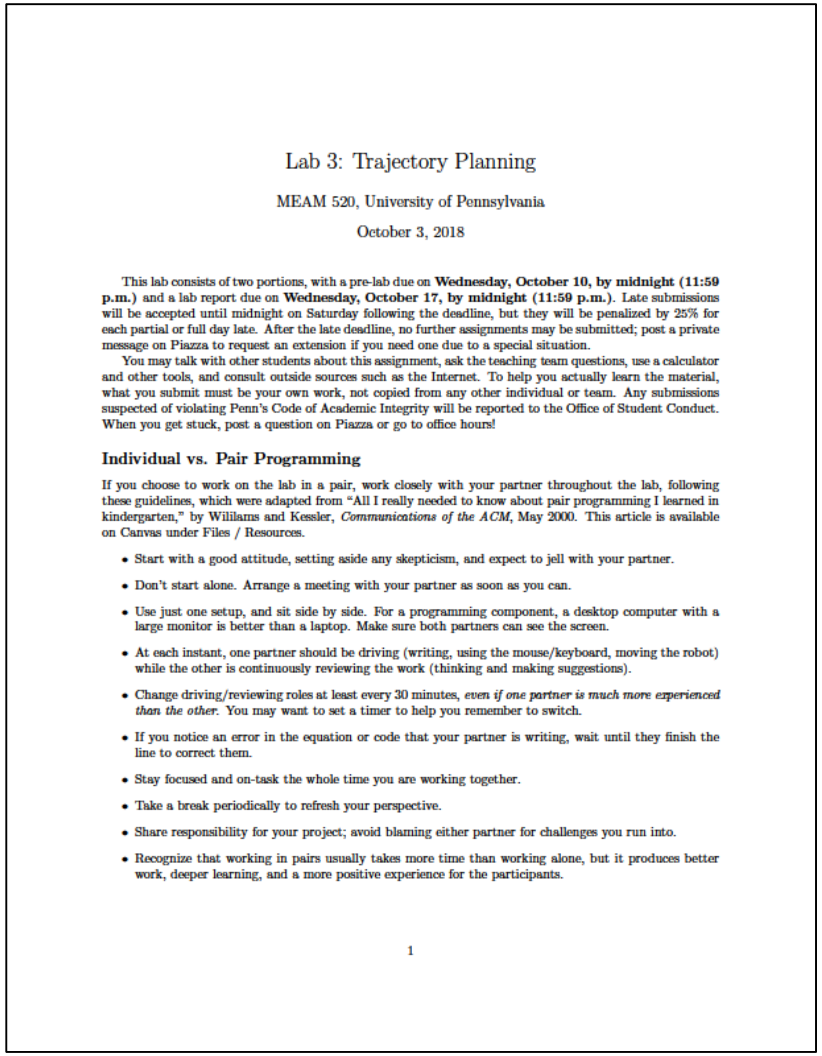
1. Convert your free C-space into a graph/roadmap
2. Find a path from q_{start} to a node q_a that is in the roadmap
3. Find a path from q_{goal} to a node q_b that is in the roadmap
4. Search the roadmap for a path from q_a to q_b







Last Minute Questions on Lab 3?



Lab 3: Trajectory Planning

MEAM 520, University of Pennsylvania

October 3, 2018

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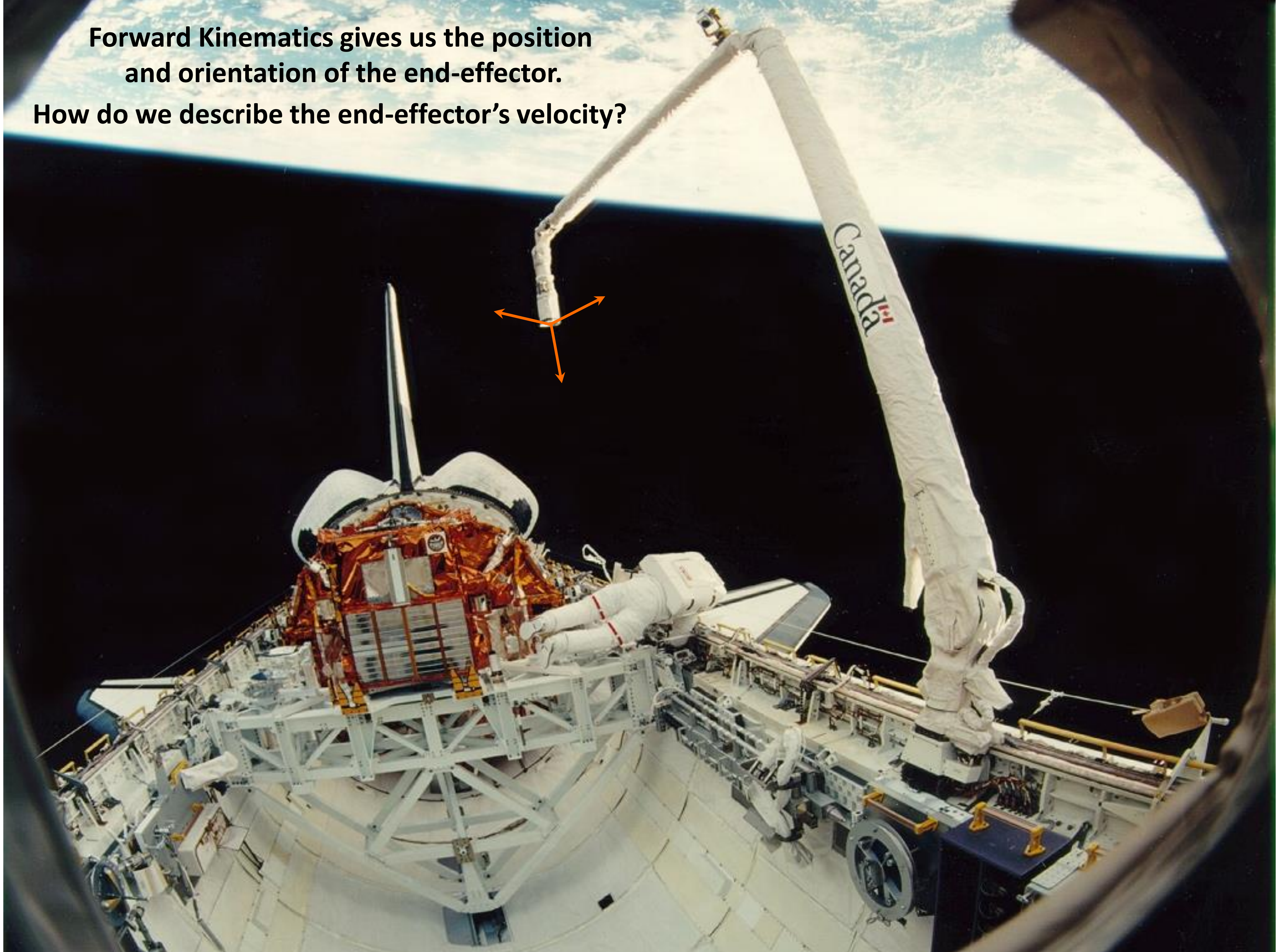
Important notes:

- All robots are points in configuration space
- Not all robots are points in the workspace/task space
- Search algorithms can be applied to graphs of arbitrary dimension
- Collision checks are often conservative

- The purpose of the lab is for you to make choices. **Explain and evaluate those choices!**

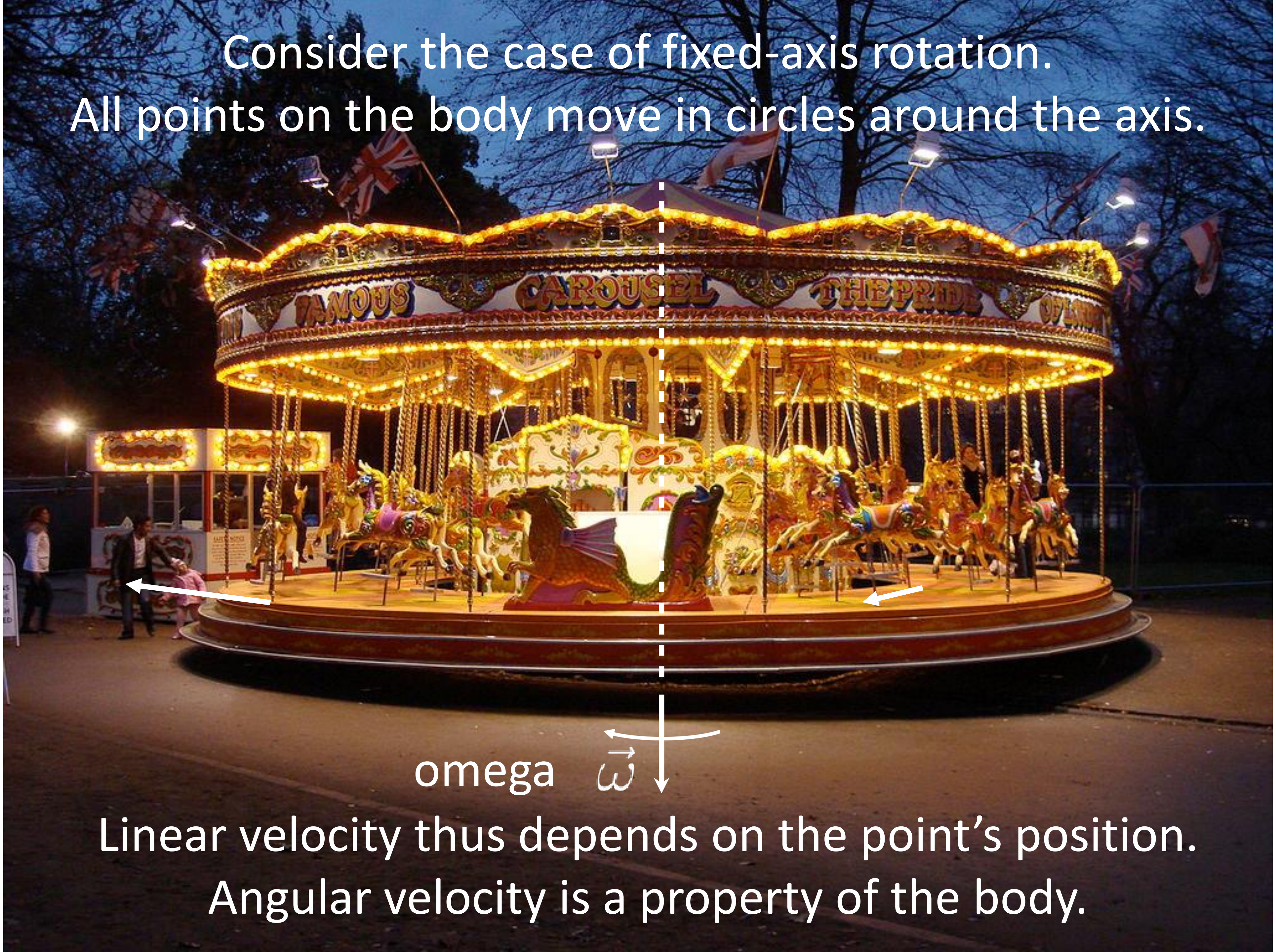
Forward Kinematics gives us the position and orientation of the end-effector.

How do we describe the end-effector’s velocity?



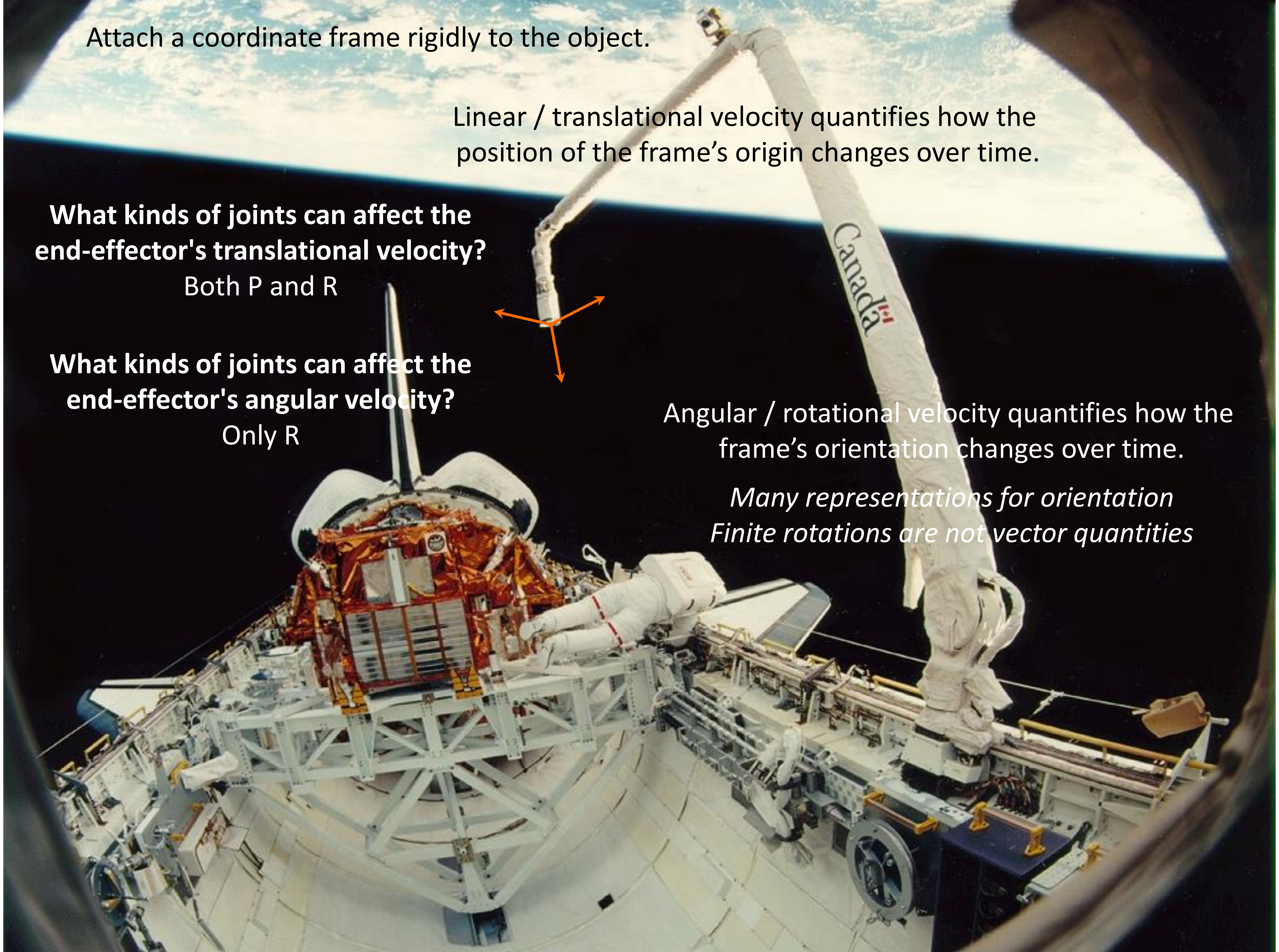
Consider the case of fixed-axis rotation.

All points on the body move in circles around the axis.



Linear velocity thus depends on the point’s position.

Angular velocity is a property of the body.



Attach a coordinate frame rigidly to the object.

Linear / translational velocity quantifies how the position of the frame’s origin changes over time.

What kinds of joints can affect the end-effector's translational velocity?
Both P and R

What kinds of joints can affect the end-effector's angular velocity?
Only R

Angular / rotational velocity quantifies how the frame’s orientation changes over time.

Many representations for orientation
Finite rotations are not vector quantities

What is the time derivative of a rotation matrix?

$$\dot{R} = \frac{dR}{dt} = ?$$

To start, consider a rotation matrix that is a function of only one variable:

$$R = R(\theta) \in SO(3)$$

$$\text{e.g., } R(\theta) = R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Angle/Axis: $\mathbf{R}_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$

$$\dot{R} = \frac{dR}{dt} = \overset{?}{\boxed{\frac{dR}{d\theta}}} \overset{\checkmark}{\boxed{\frac{d\theta}{dt}}}$$

What is the time derivative of a rotation matrix?

$$\frac{dR}{d\theta} = ?$$

What do we know about rotation matrices?

$$R\,R^T = I$$

$$\frac{d}{d\theta}\left(R\,R^T\right) = \frac{d}{d\theta}(I)$$

product rule

$$\frac{dR}{d\theta}R^T + R\frac{dR^T}{d\theta} = 0$$

Sum of two matrices equals zero.

$$\text{define } S = \frac{dR}{d\theta}R^T$$

$$S^T = \left(\frac{dR}{d\theta}R^T\right)^T = R\frac{dR^T}{d\theta}$$

$$S + S^T = 0$$

Sum of a matrix and its transpose equals zero.

Skew-Symmetric Matrices

$$S + S^T = 0$$

$$S = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

What do you know about the elements of S?

Skew-Symmetric Matrices

$$S + S^T = 0$$

$$S = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

What do you know about the elements of S?

Zeros along the diagonal.
Positive and negative values across the diagonal.

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Skew-Symmetric Matrices

$$S + S^T = 0 \qquad S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Define the operator S

$$\vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

The operator S is linear

$$S(\alpha\vec{a} + \beta\vec{b}) = \alpha S(\vec{a}) + \beta S(\vec{b})$$

But what does S do?

$$S(\vec{a})\,\vec{p} = ?$$

Talk to a partner.
What ideas do you have?

Skew-Symmetric Matrices

$$S(\vec{a})\,\vec{p} = ? = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

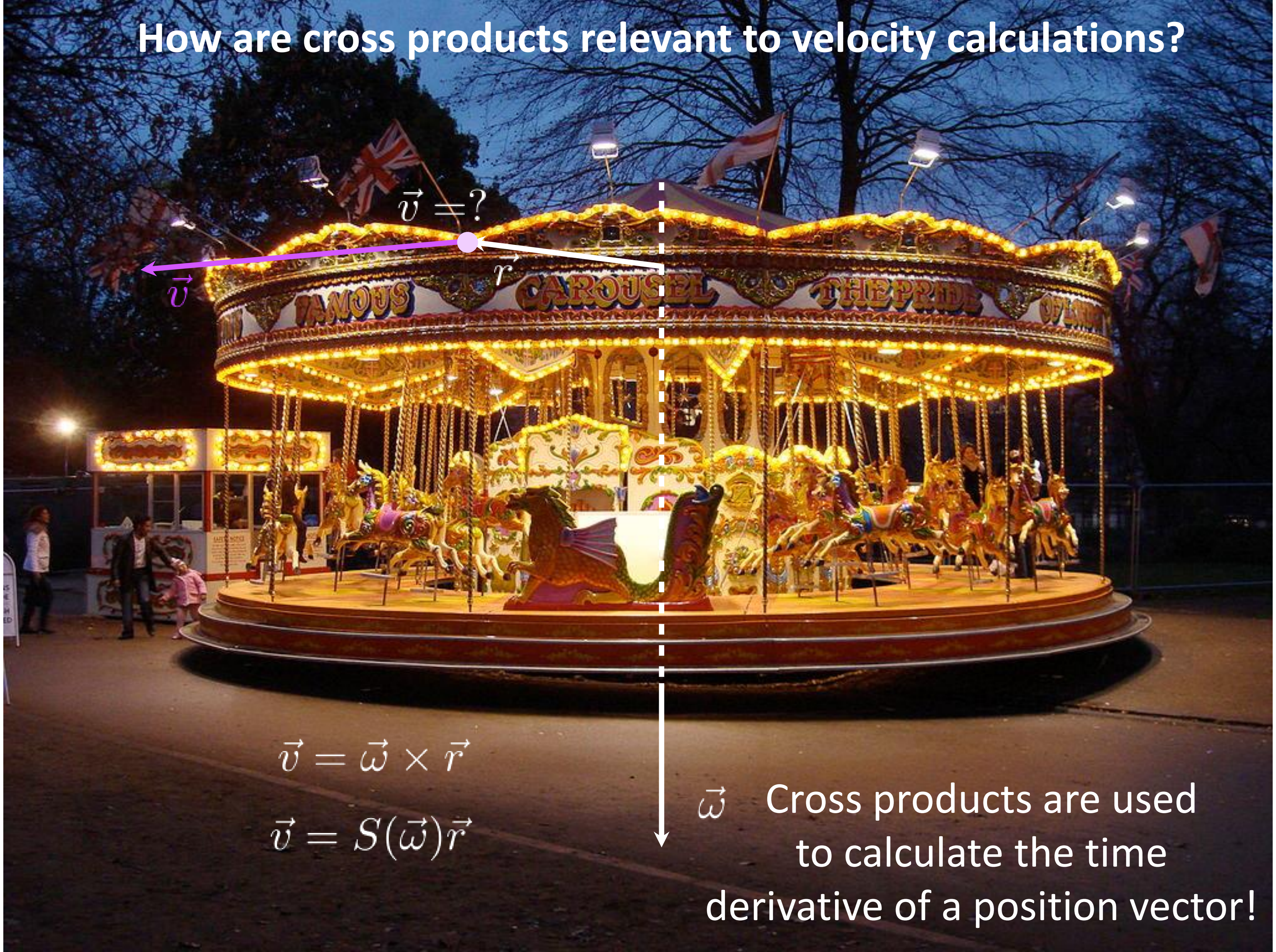
$$= \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix}$$

$$= \begin{bmatrix} a_y p_z - a_z p_y \\ a_z p_x - a_x p_z \\ a_x p_y - a_y p_x \end{bmatrix}$$

$$S(\vec{a})\vec{p} = \vec{a} \times \vec{p}$$

Skew-symmetric matrices are a matrix-based way to represent a cross-product between vectors.

How are cross products relevant to velocity calculations?



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{v} = S(\vec{\omega})\vec{r}$$

Cross products are used to calculate the time derivative of a position vector!

What is the time derivative of a rotation matrix?

$$\frac{dR}{d\theta} = ? \qquad \text{define } S = \frac{dR}{d\theta} R^T \qquad S + S^T = 0$$

This matrix is skew-symmetric.

It also contains the quantity we are seeking.

Multiply both sides on the right by R.

$$S\,R = \frac{dR}{d\theta} R^T R \qquad R^T R = I$$

$$\boxed{\frac{dR}{d\theta} = S\,R}$$

What do you get when you multiply S into R?

This crosses the vector in S into each column of R.

Computing the derivative of a rotation matrix R is equivalent to multiplying that matrix R by a skew-symmetric matrix S.

But we don't yet know how to calculate that matrix S from R!

$$\boxed{\begin{aligned} S &= \frac{dR}{d\theta} R^T \\ \frac{dR}{d\theta} &= S\,R \end{aligned}}$$

Example

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}_{x,\theta} = ?$$

Let's solve by direct calculation to discover what S must be.

$$\dot{R}_{x,\theta} = \frac{dR_{x,\theta}}{dt} = \frac{dR_{x,\theta}}{d\theta}\frac{d\theta}{dt} = SR_{x,\theta}\frac{d\theta}{dt}$$

$$S = ? = \frac{dR_{x,\theta}}{d\theta} R^T_{x,\theta}$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{i})$$

$$\vec{a} = ? = \hat{i}$$

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

S is a skew-symmetric matrix of the axis of rotation!

Example

$$\boxed{\begin{aligned} S &= \frac{dR}{d\theta}R^T \\ \frac{dR}{d\theta} &= S\,R \end{aligned}}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}_{x,\theta} = ?$$

$$\dot{R}_{x,\theta} = \frac{dR_{x,\theta}}{dt} = \frac{dR_{x,\theta}}{d\theta}\frac{d\theta}{dt} = SR_{x,\theta}\frac{d\theta}{dt}$$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{i})$$

The skew-symmetric matrix S defines the axis about which rotation is occurring.

Exactly what you would get by differentiating each element w.r.t. time.

$$\boxed{\dot{R}_{x,\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta \\ 0 & \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix}}$$

$$\dot{R}_{x,\theta} = S(\hat{i})R_{x,\theta}\dot{\theta}$$

$$\dot{R}_{x,\theta} = S(\dot{\theta}\hat{i})R_{x,\theta}$$

$$\vec{\omega} = \dot{\theta}\hat{i}$$

$$\boxed{\dot{R}_{x,\theta} = S(\vec{\omega})R_{x,\theta}}$$

Crossing omega into each column of R...

In general, you simply get S from the angular velocity vector, and you don't need to differentiate the matrix.

a skew-symmetric matrix
formed from omega

The time derivative of a rotation matrix is...

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

times the rotation matrix itself

angular velocity of rotating frame w.r.t. the fixed frame at time t

Another Example:

Frame 1 is instantaneously aligned with frame 0, and their origins are always coincident. Frame 1 has the following angular velocity vector relative to frame 0, expressed in frame 0:

$$\vec{\omega}^0_{0,1} = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$R^0_1 = ?$$

$$\dot{R}^0_1 = ?$$

Work on this individually or with a partner

Draw a diagram to understand your result

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{\omega}_{0,1}^0 = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

a skew-symmetric matrix formed from omega

$$\dot{R}_1^0 = ? = S(\vec{\omega})R_1^0$$

times the rotation matrix itself

$$S(\vec{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{\omega}_{0,1}^0 = \begin{bmatrix} 0 \text{ rad/s} \\ 2 \text{ rad/s} \\ 2 \text{ rad/s} \end{bmatrix}$$

a skew-symmetric matrix formed from omega

$$\dot{R}_1^0 = ? = S(\vec{\omega})R_1^0$$

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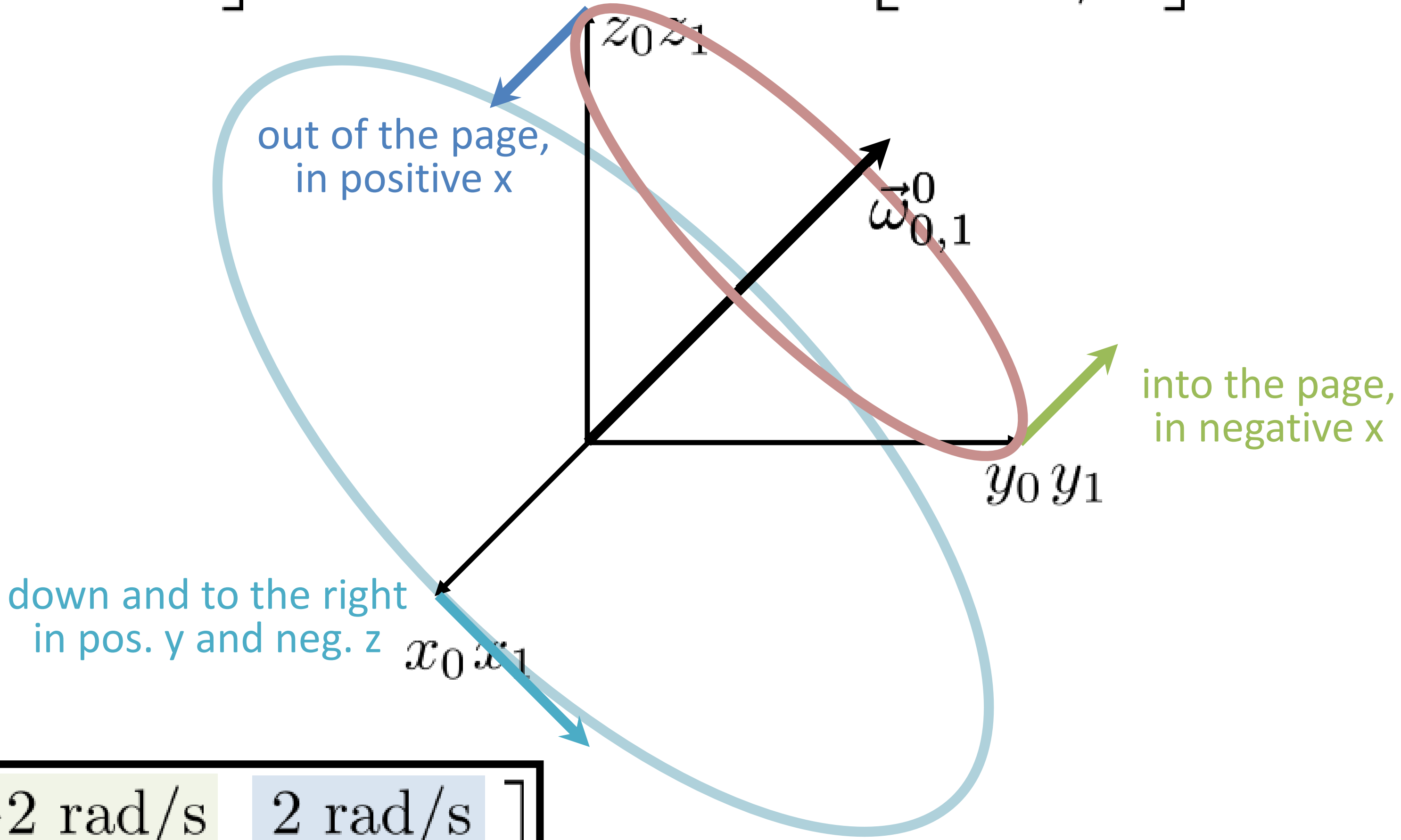


$$\boxed{\dot{R}_1^0 = \begin{bmatrix} 0 & -2 \text{ rad/s} & 2 \text{ rad/s} \\ 2 \text{ rad/s} & 0 & 0 \\ -2 \text{ rad/s} & 0 & 0 \end{bmatrix}}$$

What questions do you have?

Why is this useful?

Calculating the velocity of a point in a rotating frame.

Calculating the linear velocity of the end-effector of a robot.

Understanding how angular velocities combine on a robotic manipulator.

Calculating the velocity of a point in a rotating frame.

See SHV 4.3: Angular Velocity: The General Case

$$p^0 = R_1^0 p^1$$

A vector to a point that is fixed to frame 1, expressed in frame 0.

$$\frac{d}{dt}p^0 = ? = \dot{R}_1^0 p^1$$

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

$$= S(\vec{\omega})R_1^0 p^1$$

$$= \vec{\omega} \times R_1^0 p^1$$

$$= \vec{\omega} \times p^0$$

$$\boxed{\dot{p}^0 = S(\vec{\omega}(t))R_1^0 p^1}$$

Calculating the linear velocity of the end-effector of a robot

See SHV 4.5: Linear Velocity of a Point Attached to a Moving Frame

$$p^0 = R_1^0(t)p^1 + o_1^0(t)$$

point p is rigidly fixed in frame 1

$$\dot{p}^0 = \dot{R}_1^0 p^1 + \dot{o}_1^0$$

$$\dot{p}^0 = S(\omega^0)R_1^0 p^1 + \dot{o}_1^0$$

$$\dot{R}(t) = S(\vec{\omega}(t))R(t)$$

$$\dot{p}^0 = \omega^0 \times p^0 + \dot{o}_1^0$$

You can calculate the linear velocity of the end-effector from the angular velocity of its frame, its position relative to its frame's origin, and the linear velocity of its frame’s origin.

Understanding how angular velocities combine on a robotic manipulator

See SHV 4.4: Addition of Angular Velocities

$$R_2^0(t) = R_1^0(t) R_2^1(t)$$

Differentiate both sides with respect to time.

$$\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0$$

$$\frac{d}{dt}(R_1^0 R_2^1) = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

$$\dot{R}(t) = S(\vec{\omega}(t)) R(t)$$

$$\frac{d}{dt}(R_1^0 R_2^1) = S(\omega_{0,1}^0) R_1^0 R_2^1 + R_1^0 S(\omega_{1,2}^1) R_2^1$$

You can add angular velocity vectors!

$$\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

$$\frac{d}{dt}(R_1^0 R_2^1) = S(\omega_{0,1}^0) R_2^0 + S(R_1^0 \omega_{1,2}^1) R_2^0$$

The angular velocity of frame 2 relative to frame 0
is equal to the angular velocity of frame 1 relative to frame 0, expressed in frame 0,
plus the angular velocity of frame 2 relative to frame 1, expressed in frame 0

Uses for Skew-Symmetric Matrices

What questions do you have?

You can calculate the velocity of a point that is fixed to a rotating (but not translating) frame.

$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
\frac{d}{dt} p^0 &= ? = \dot{R}_1^0 p^1 \\
&= S(\vec{\omega}) R_1^0 p^1 \\
&= \vec{\omega} \times R_1^0 p^1 \\
&= \vec{\omega} \times p^0
\end{aligned}
$$

You can derive the fact that you can add angular velocity vectors by expressing them in the same frame.

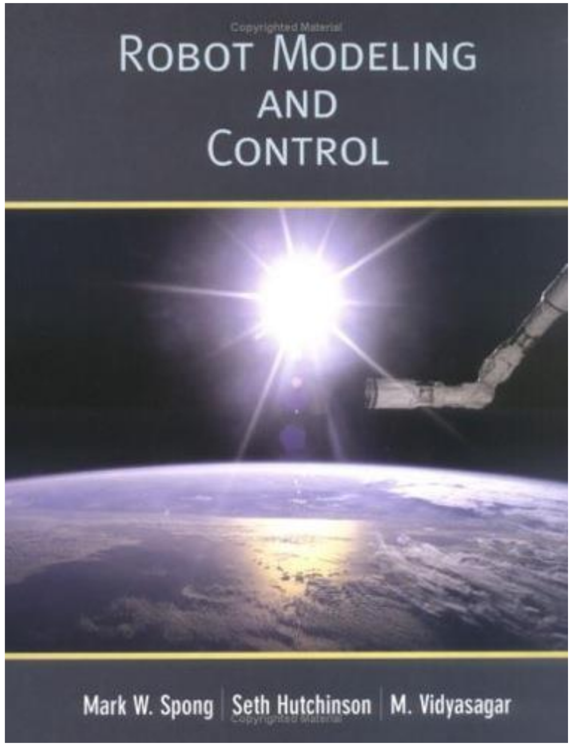
$$\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

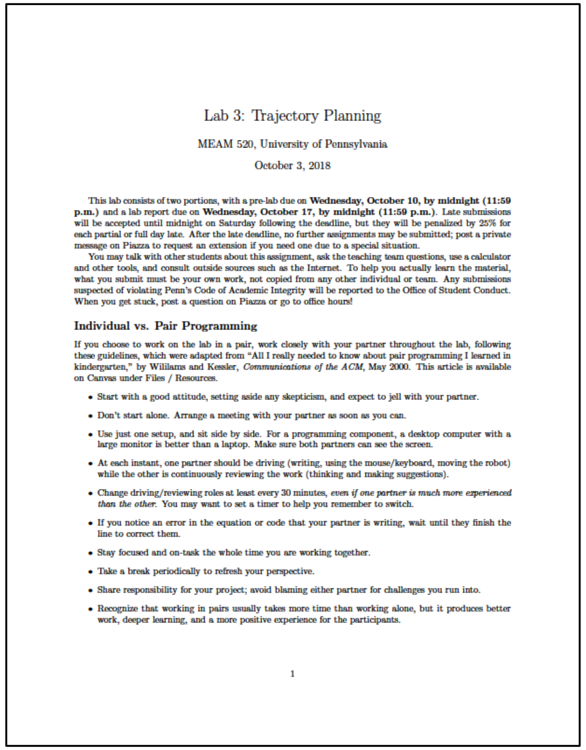
The angular velocity of frame 2 relative to frame 0 is equal to the angular velocity of frame 1 relative to frame 0, expressed in frame 0, plus the angular velocity of frame 2 relative to frame 1, expressed in frame 0

You can calculate the velocity of a point that is fixed to a rotating and translating frame.

$$
\begin{aligned}
p^0 &= R_1^0(t) p^1 + o_1^0(t) \\
\dot{p}^0 &= \dot{R}_1^0 p^1 + \dot{o}_1^0 \\
\dot{p}^0 &= S(\omega^0) R_1^0 p^1 + \dot{o}_1^0 \\
\dot{p}^0 &= \omega^0 \times p^0 + \dot{o}_1^0
\end{aligned}
$$

Next time: More Velocity Kinematics





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Chapter 4: Velocity Kinematics

- Read 4.5-4.7

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