Probability and Statistics (ENM 503)

Michael A. Carchidi

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Chapter 2 - More On Combinatorics (The Art of Counting)

The following notes are based on the textbook entitled: *An Introduction to Discrete Mathematics* by Steven Roman (2nd edition) and these notes can be viewed at

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1. Permutations with Repetitions

In much of our study of permutations and combinations in Chapter #1, we did not allow any of the objects to be used more than once. In this section, we want to consider permutations where repetitions are allowed. In the next section, we will consider combinations where repetitions are allowed. It is convenient to divide permutations with repetitions into three types, depending on whether there are any restrictions on the number of times an element can be repeated in the permutation.

Suppose that $S = \{a_1, a_2, a_3, ..., a_n\}$ is a set with n elements. (a) Let r_1, r_2, r_3, ..., r_n be nonnegative integers. An ordered arrangement of k of the elements of S, where the element a_i is repeated exactly r_i times, for all $i = 1, 2, 3, ..., n$, is called a *permutation of size* k with *fixed repetitions*, taken from the set S. The number r_i is called the repetition number of the element a_i. Of course, in order for such a permutation to exist, we must have

$$r_1 + r_2 + r_3 + \cdots + r_n = k \qquad (1)$$

(b) Let s_1, s_2, s_3, ..., s_n and t_1, t_2, t_3, ..., t_n be nonnegative integers with the property that $s_i \leq t_i$ for all $i = 1, 2, 3, ..., n$. An *ordered arrangement* of k of the elements of S, where the element a_i is repeated *at least* s_i times and *at most* t_i times, so that

$$s_i \leq r_i \leq t_i$$

for all $i = 1, 2, 3, ..., n$, is called a *permutation of size k with restricted repetitions*, taken from the set S. Note that $t_i = 0$ or $t_i = 1$ for all $i = 1, 2, 3, ..., n$ says that repetitions are really not allowed. We will also allow the possibility that any of the numbers t_i can be equal to infinity. This amounts to saying that there is no restriction on the maximum number of times the corresponding element a_i can be repeated. (c) In fact, an ordered arrangement of k of the elements of S, where any element of S can be repeated any number of times in the arrangement, is called a permutation of size k with unrestricted repetitions, taken from the set S.

Actually, the first and third types of permutations are really just special cases of the second type. A permutation with unrestricted repetitions (case c above) is just a permutation with restricted repetitions (case b above), where s_i is equal to 0 and t_i is equal to infinity, for all $i = 1, 2, 3, ..., n$. Also, a permutation with fixed repetitions (case a above) is just a permutation with restricted repetitions (case b above) in which $s_i = r_i = t_i$ for all $i = 1, 2, 3, ..., n$.

Note that $P(n, k)$ was studied in Chapter #1 and recall that this gives the number of permutations of size k, taken from a set of size n, or stated another way,

$$\left(\begin{array}{c}\text{The number of permutations of} \\ \text{size } k \text{ with } \textit{no repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements}\end{array}\right) = P(n, k) = \frac{n!}{(n-k)!},$$

These are the ones described in case (b) with $s_i = 0$ and $t_i = 1$ which says that repetitions are (in fact) not allowed.

The definitions just given can be rephrased in terms of words over an alphabet. For example, a permutation of size k, taken over an alphabet set $\sum$, with unlimited repetitions is nothing more than a word of length k over the alphabet $\sum$. The other types of permutations are just words with certain restrictions placed on the number of times each letter can be used in each word. We have phrased the definitions in terms of permutations, rather than words, because the concept of a permutation is more common in combinatorics than the concept of a word. In any case, we will leave it to you to make a complete translation of our definitions into ones using the concept of a word.

In this section, we will derive formulas for the number of permutations with unrestricted repetitions (case c above), and for the number of permutations with fixed repetitions (case a above). Unfortunately, no one has ever been able to find a simple general formula for the number of permutations with restricted repetitions (case b above). Let us first consider permutations with unrestricted repetitions by looking at a few examples.

Example #1: Permutations With Unrestricted Repetitions

Consider the set $S = \{0, 1, 2\}$. Then the permutations of size 3 with *unrestricted repetitions*, taken from S, are

000	001	002	010	011	012	020	021	022
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

and these are just the ternary words of length 3 and we see that there are $3^3 = 27$ of these. ■

Example #2: Permutations With Unrestricted Repetitions

The permutations of size 4 with unrestricted repetitions, taken from the set $S = \{0, 1\}$, are

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

and these are just the binary words of length 4 and we see that there are $2^4 = 16$ of these. ■

Permutations With Unrestricted Repetitions

It is not hard to find a formula for the number of permutations with unrestricted repetitions. In fact, if S is a set with n elements, then the number of permutations of size k with *unrestricted repetitions*, taken from the set S, is simply

$$n \times n \times n \times \cdots \times n = n^k$$

so that

$$\left(\begin{array}{c} \text{The number of permutations of size } k \\ \text{with } \textit{unrestricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = n^k.$$

Permutations With A Fixed Number Of Repetitions

Let us now turn to permutations with a fixed number of repetitions. Let

$$S = \{a_1, a_2, a_3, ..., a_n\}$$

be a set with n elements. Then the number of permutations of size k with fixed repetitions, taken from the set S, where a_i has repetition number r_i for all $i = 1, 2, 3, ..., n$, is the *multinomial coefficient*

$$\binom{k}{r_1, r_2, r_3, \ldots, r_n} = \frac{k!}{r_1! r_2! r_3! \cdots r_n!}$$

with

$$r_1 + r_2 + r_3 + \cdots + r_n = k.$$

Thus we have

$$\left(\begin{array}{c} \text{The number of permutations of size } k \\ \text{with } \textit{case a restricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = \frac{k!}{r_1! r_2! r_3! \cdots r_n!}$$

with

$$r_1 + r_2 + r_3 + \cdots + r_n = k.$$

To prove this, let us consider a set of k boxes, numbered 1 through k,

$$\begin{array}{ccccc} \square & \square & \square & \cdots & \square \\ 1 & 2 & 3 & \cdots & k \end{array}$$

Then, forming a permutation of size k, with fixed repetitions of the type specified in this analysis, amounts to filling these boxes with the elements of S in such a way that r_1 of the boxes receive the element a_1, r_2 of the boxes receive the element a_2, and so on, as illustrated below.

$$\begin{array}{ccccc} \boxed{a_1, a_1, ..., a_1} & \boxed{a_2, a_2, ..., a_2} & \boxed{a_3, a_3, ..., a_3} & \cdots & \boxed{a_k, a_k, ..., a_k} \\ 1\ (r_1) & 2\ (r_2) & 3\ (r_3) & \cdots & k\ (r_k) \end{array}$$

But, this is the same as dividing the set of k boxes into n mutually disjoint subsets. The first subset A_1 consists of those boxes that receive the element a_1, the second subset A_2 consists of those boxes that receive the element a_2, and so on, as illustrated below.

$$\begin{array}{ccccc} \boxed{\{a_1, a_1, ..., a_1\}} & \boxed{\{a_2, a_2, ..., a_2\}} & \boxed{\{a_3, a_3, ..., a_3\}} & \cdots & \boxed{\{a_k, a_k, ..., a_k\}} \\ A_1 & A_2 & A_3 & \cdots & A_k \end{array}$$

Now, since $|A_i| = r_i$ for all $i = 1, 2, 3, ..., n$, we know that there are exactly

$$\binom{k}{r_1, r_2, r_3, \ldots, r_n} = \frac{k!}{r_1!r_2!r_3! \cdots r_n!}$$

ways to divide the set of k boxes into n mutually disjoint subsets A_1, A_2, A_3, ..., A_k. Hence, this is also the number of permutations of size k, with fixed repetitions, where a_i has repetition number r_i, for all $i = 1, 2, 3, ..., n$ and this completes the proof.

Example #3: Permutations With Restricted Repetitions

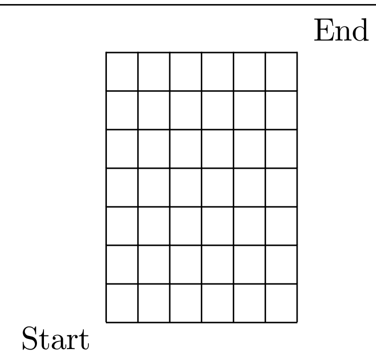
How many permutations are there of the letters in the word MISSISSIPPI? In this case, we want to count permutations of size 11 (the length of the word MISSISSIPPI), with fixed repetitions, taken from the set $S = \{\text{I, M, P, S}\}$ where I has repetition number 4, M has repetition number 1, P has repetition number 2, and S has repetition number 4. Since $k = 11$, the above result tells us that there are

$$\binom{11}{4, 1, 2, 4} = \frac{11!}{4!1!2!4!} = 34,650$$

such permutations. Comparing this with $11! = 39,916,800$, the number of permutations of 11 distinct letters shows it to be much smaller. ■

Example #4: Permutations With Restricted Repetitions

Recall that Example #3 of Chapter #1 asked the following question. There are 42 one-way streets arranged in a rectangular grid with 7 of the streets going from west to east and 6 of the streets going from south to north. A person wants to walk the 13 blocks from the most southwest point on this map to the most northeast point, always heading either east or north as illustrated in the following figure.



East is to the right and North is up

The problem is to count the number of ways that this can be done. To answer this, we use a one-to-one correspondence. We may map every walk to a sequence of E's (east) and N's (north). For example, if you walk the 13 blocks from the most southwest point on this map to the most northeast point, by walking 6 blocks east to the east most, south most part of the map and then walk 7 blocks north to the east most, north most part of the map, we have the walk

$$\text{EEEEEENNNNNNN.}$$

Another walk would be

$$\text{ENENENENENENN.}$$

We see then that each walk along the streets correspond to permutation of size $k = 13$ from the set $S = \{\text{E, N}\}$ with 6 (E's) and 7 (N's). This leads to

$$\binom{13}{6, 7} = \frac{13!}{6!7!} = 1,716$$

possible walks. ■

2. Combinations with Unrestricted Repetitions

In the previous section, we discussed permutations with repetitions. In this section, we want to discuss combinations with repetitions. We first note the *trivial case* and state that the number of combinations of size k with fixed repetitions, taken from the set S, where a_i has repetition number r_i for all $i = 1, 2, 3, ..., n$, and

$$r_1 + r_2 + r_3 + \cdots + r_n = k.$$

is just 1, *i.e.,*

$$\left(\begin{array}{c} \text{The number of combinations of size } k \\ \text{with } \textit{case a restricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = 1$$

and this is because order does not matter in a combination.

Next, we divide combinations with repetitions into only two types, the first type is one in which repetitions are now allowed and we already saw in Chapter #1 that

$$\left(\begin{array}{c} \text{The number of combinations of} \\ \text{size } k \text{ with } \textit{no repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = C(n, k) = \binom{n}{k}.$$

In this chapter, we now consider restrictions on the number of times an element can be repeated in the combination. Toward this end, let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a set with n elements. Next let s_1, s_2, s_3, ..., s_n and t_1, t_2, t_3, ..., t_n be non-negative integers with the property that $s_i \leq t_i$, for all $i = 1, 2, 3, ..., n$. (case b) An unordered selection of k of the elements of S, where the element a_i appears *at least* s_i times and *at most* t_i times, for all $i = 1, 2, 3, ..., n$, is called a combination of size k taken from the set S, with restricted repetitions. We will also allow the possibility that any of the numbers t_i can be equal to infinity. This amounts to saying that there is no restriction on the maximum number of times the corresponding element a_i can appear (case c). An unordered selection of k of the elements of S, where any element of S can appear any number of times in the selection, is called a *combination of size k with unrestricted repetitions*, taken from the set S.

Combinations with Fixed Repetitions

We have not included combinations with fixed repetitions as a separate type because there is not much to say about them since the number of such combinations must always be one, which is the trivial case mentioned earlier. For example, suppose that

$$S = \{1, 2, 3, 4, 5\}$$

and we want to look at all combinations of size 15 in which there are exactly 2 (1's), 3 (2's), 0 (3's), 4 (4's) and 6 (5's), then all we have is the single choice

$$\{1, 1, 2, 2, 2, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5\}$$

since the order of the elements in this arrangement is unimportant.

Our plan in this section is to first derive a formula for the number of combinations of size k, with unrestricted repetitions, taken from a set of size n. Let us begin with some examples of combinations with unrestricted repetitions.

Example #5: Combinations With Unrestricted Repetitions

Consider the set $S = \{0, 1, 2\}$. Then the combinations of size 2 with unrestricted repetitions, taken from S, are

$$\{0, 0\} \quad , \quad \{0, 1\} \quad , \quad \{0, 2\} \quad , \quad \{1, 1\} \quad , \quad \{1, 2\} \quad , \quad \{2, 2\}$$

and we see that there are 6 of them. The combinations of size 3 with unrestricted repetitions, taken from S, are

$$\begin{array}{ccccc} \{0,0,0\} & \{0,0,1\} & \{0,0,2\} & \{0,1,1\} & \{0,1,2\} \\ \{0,2,2\} & \{1,1,1\} & \{1,1,2\} & \{1,2,2\} & \{2,2,2\} \end{array}$$

and we see that there are 10 of these. ■

Example #6: Combinations With Unrestricted Repetitions

The combinations of size 4 with unrestricted repetitions from $S = \{0, 2\}$, are

$$\{0, 0, 0, 0\}, \quad \{0, 0, 0, 2\}, \quad \{0, 0, 2, 2\}, \quad \{0, 2, 2, 2\}, \quad \{2, 2, 2, 2\}$$

and we see that there are 5 of them. ■

As you can see from these examples, we have used set notation to denote combinations with repetitions. In fact, combinations with repetitions are very similar to ordinary sets and are frequently called *multisets*.

Combinations of Size k with Unrestricted Repetitions

Let us now turn to the problem of finding a formula for the number of combinations of size k with unrestricted repetitions, taken from a set of size n. As is usually the case, this not a hard problem to solve if we look at it in the right way. For the sake of illustration, let us compute the number of combinations of size 5 with unrestricted repetitions, taken from the set $S = \{a, b, c, d\}$ of size 4. The first step is to find a simple way to represent all of the possible combinations, so that we can count them. Our plan is to represent each combination by a sequence of 5 (x's) and 3 slashes (/'s). For example, the combination $\{a, a, b, c, d\}$ is represented by the sequence xx/x/x/x, indicating that there are 2 (a's), 1 (b), 1 (c) and 1 (d), and the combination $\{a, a, a, b, d\}$ is represented by the sequence xxx/x//x, indicating that there are 3 (a's), 1 (b), 0 (c's) and 1 (d).

In general, if we want to represent a given combination of size 5 taken from the set $S = \{a, b, c, d\}$ of size 4, with unrestricted repetitions as a sequence of 5 x's and 3 slashes, we proceed as follows. Starting from the left, we write down as many x's as there are a's in the particular combination. Then we write a slash, followed by as many x's as there are b's in the combination, then another slash followed by as many x's as there are c's in the combination, and finally another slash followed by as many x's as there are d's in the combination. Thus, each sequence has as many x' s as there are elements in the combination so that each sequence has 5 x' s. The 3 slashes serve to separate the x 's into 4 groups, one for each of the four elements of S. Notice that, in the second example above, the combination does not contain the letter c, which is why there are two slashes next to each other in the corresponding sequence of x's and slashes. Let us do a few more examples.

$$\begin{array}{lcl}
\{a, b, c, d, d\} & \Leftrightarrow & \text{x/x/x/xx} \\
\{a, b, b, b, c\} & \Leftrightarrow & \text{x/xxx/x/} \\
\{b, b, b, b, d\} & \Leftrightarrow & \text{xxxx//x} \\
\{d, d, d, d, d\} & \Leftrightarrow & \text{///xxxxx}
\end{array}$$

As you can see, by using this method each combination with unrestricted repetitions, taken from S, corresponds to exactly one sequence of 5 (x's) and 3 slashes. Also, every such sequence of x's and slashes represents exactly one combination with unrestricted repetitions, taken from S. For example, the sequence x//xx/xx corresponds to the set $\{a, c, c, d, d\}$.

In other words, we have found a *one-to-one correspondence* between the elements of the set of all combinations of size 5 taken from the set $S = \{a, b, c, d\}$ of size 4 with unrestricted repetitions and the elements of the set of all sequences consisting of 5 (x's) and 3 (/'s). Hence, the number of combinations of size 5 with unrestricted repetitions is the same as the number of sequences of 5 (x's) and 3 (/'s). But, we already know how to compute the number of such sequences. After all, a sequence of 5 (x's) and 3 (/'s) is just a permutation of size 8 with fixed repetitions, taken from the set $B = \{\text{x}, /\}$, where x has repetition number 5 and / has repetition number 3. Using the multinomial coefficients, there are

$$\binom{8}{5,3} = \frac{8!}{5!3!} = 56$$

such permutations. Hence, there are 56 combinations of size 5 with unrestricted repetitions, taken from the set S of 4 elements.

The same technique that we used here will work in general to provide us with a formula for the number of combinations of size k with unrestricted repetitions, taken from any set of size n. In fact, it should be clear that for a set of size n and a combination of size k, there will always be k (x's) and $n-1$ (/'s) and so the formula is just

$$\binom{n+k-1}{k, n-1} = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k} = C(n+k-1, k).$$

Thus we may say that

$$\left(\begin{array}{c} \text{The number of combinations of size } k \\ \text{with } \textit{unrestricted repetitions} \text{ taken} \\ \text{from a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

Recall that

$$\left(\begin{array}{c} \text{The number of combinations of size} \\ k \text{ with } \textit{no repetitions} \text{ taken from} \\ \text{a set } S \text{ with } n \text{ distinct elements} \end{array} \right) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example #7: Combinations With Unrestricted Repetitions

Consider the set $S = \{0, 1, 2\}$. Then the combinations of size 2 with unrestricted repetitions, taken from S, are

$$\{0,0\} \quad , \quad \{0,1\} \quad , \quad \{0,2\} \quad , \quad \{1,1\} \quad , \quad \{1,2\} \quad , \quad \{2,2\}$$

and we see that there are

$$\binom{3+2-1}{2} = \binom{4}{2} = \frac{4!}{2!2!} = 6$$

of them. The combinations of size 3 with unrestricted repetitions, taken from S, are

$$\begin{array}{ccccc} \{0,0,0\} & \{0,0,1\} & \{0,0,2\} & \{0,1,1\} & \{0,1,2\} \\ \{0,2,2\} & \{1,1,1\} & \{1,1,2\} & \{1,2,2\} & \{2,2,2\} \end{array}$$

and we see that there are

$$\binom{3+3-1}{3} = \binom{5}{3} = \frac{5!}{3!2!} = 10$$

of these. ■

Example #8: Combinations With Unrestricted Repetitions

The combinations of size 4 with unrestricted repetitions from $S = \{0, 2\}$, are

$$\{0,0,0,0\}, \quad \{0,0,0,2\}, \quad \{0,0,2,2\}, \quad \{0,2,2,2\}, \quad \{2,2,2,2\}$$

and we see that there are

$$\binom{2+4-1}{4} = \binom{5}{4} = \frac{5!}{4!1!} = 5$$

of them. ■

Example #9: Combinations With Unrestricted Repetitions

A certain candy store sells 6 different types of chocolates. How many ways are there to fill a box with chocolates if the box holds 10 pieces, assuming of course, that the order of the chocolates in the box is irrelevant? To solve this we note that filling a box with chocolates simply amounts to choosing a combination of size 10 with unrestricted repetitions, from a set of size 6. Hence, there are

$$\binom{6+10-1}{10} = \binom{15}{10} = \frac{15!}{10!5!} = 3003$$

different ways to fill a box with chocolates. ■

3. Combinations with Restricted Repetitions - I

Let us now turn our attention to combinations with restricted repetitions. As we pointed out earlier in this section, in general there is no simple formula for the number of combinations of size k with restricted repetitions, taken from a set of size n. Nevertheless, we can obtain a very nice formula in a special case. In particular, we want to find a formula for the number of combinations of size k, taken from a set $S = \{a_1, a_2, a_3, ..., a_n\}$, with the property that the element a_i must be repeated at least s_i times in the combination, for all $i = 1, 2, 3, ..., n$. In other words, these are combinations with restricted repetitions where the numbers t_i are all equal to infinity, and so there is no restriction on the maximum number of times an element of S may appear.

Example #10: Combinations With Restricted Repetitions

Let us start with a simple example. How many combinations of size 5 are there of the set $S = \{a, b, c\}$ of size 3, with the property that a must be repeated at least 2 times, and b must be repeated at least once in each combination? In order to answer this question, all we have to do is observe that forming such a combination simply amounts to writing down the required 2 (a's) and 1 (b) first and then appending to that, any combination of size 2 with unrestricted repetitions, taken from S. In other words, we have

$$\{a, a, b, \text{any combination of size 2 with unrestricted repetitions taken from } S\}$$

Since there are

$$\binom{3+2-1}{2} = \binom{4}{2} = 6$$

different combinations of size 2 with unrestricted repetitions, taken from S, which has 3 elements, there are also 6 combinations of size 5 with the desired restrictions. ■

We can easily generalize this line of reasoning to find the formula that we are looking for. Let us put this into a theorem. Let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a set with n elements. Then the number of combinations of size k, taken from S, with the property that the element a_i must be repeated *at least* s_i times in the combination, for all $i = 1, 2, 3, ..., n$, is

$$\binom{n + k - s_1 - s_2 - s_3 - \cdots - s_n - 1}{k - s_1 - s_2 - s_3 - \cdots - s_n} \tag{2a}$$

which we may also write as

$$\binom{n + k - s_1 - s_2 - s_3 - \cdots - s_n - 1}{n - 1}. \tag{2b}$$

Proof: We use the same reasoning as we did for the example. In order to form a combination of size k from the set S, with the required restrictions, we simply take the required s_1 (a_1's), s_2 (a_2 's), ..., s_n (a_n's) and append to this any combination of size

$$k' = k - s_1 - s_2 - s_3 \cdots - s_n$$

with unrestricted repetitions, taken from the set S of size n, and this is just

$$\binom{n + k' - 1}{k'} = \binom{n + k' - 1}{n - 1}$$

which is the desired result.

Example #11: Combinations With Restricted Repetitions

Let us consider again the candy store example above. This time, however, we want to count the number of ways to fill a box of chocolates from the 6 types, call them T_1, T_2, ..., T_6, if each box must contain at least 2 pieces of type T_1 and at least 1 piece of type T_2. In this case, filling a box is simply a matter of choosing a combination of size 10 from the set $S = \{T_1, T_2, T_3, T_4, T_5, T_6)$ in such a way that T_1 is repeated at least 2 times and T_2 is repeated at least 1 time. Hence, in this case, the numbers s_i are: $s_1 = 2$, $s_2 = 1$ and $s_i = 0$ for $i = 3, 4, 5, 6$, and so there are

$$\binom{6 + 10 - 2 - 1 - 1}{6 - 1} = \binom{12}{5} = 792.$$

different ways to fill a box with 10 pieces of chocolate, under these restrictions. This compares with 3003 ways to fill a box when there are no restrictions. ■

Example #12: Combinations With Restricted Repetitions

Suppose that a certain exam is being given on 3 consecutive days and that 25 students are required to take the exam. How many ways are there to assign each of the students to 1 of the 3 days, assuming that there must be at least 5 students assigned to each of the first 2 days? Here we are assuming that it does not matter which students are assigned to which days, only the number of students assigned to each day matters.

Assigning the 25 students to the 3 days, with these restrictions, amounts to choosing a combination of size 25, taken from the set $S = \{\text{day } 1, \text{ day } 2, \text{ day } 3\}$, where day 1 must be repeated at least $s_1 = 5$ times, day 2 must be at least repeated $s_2 = 5$ times, and day 3 need not be repeated at all ($s_3 = 0$). This leads to

$$\binom{3+25-5-5-0-1}{3-1} = \binom{17}{2} = 136.$$

ways to assign the students. ■

A Special Case

One of the most common example with restricted combinations is to determine the number of combinations of size k, taken from a set S of size n, with the property that each element of S must appear at least once in the combination. In this case,

$$s_1 = s_2 = s_3 = \cdots = s_n = 1$$

and so we have

$$\binom{n+k-1-1-\cdots-1-1}{n-1} = \binom{n+k-n-1}{n-1} = \binom{k-1}{n-1}.$$

Example #13: Combinations With Restricted Repetitions

Once again referring to candy store example, how many ways are there to fill a box of chocolates if the box must contain at least one piece of each type?

This problem is equivalent to determining the number of combinations of size 10, taken from the set S of 6 types of chocolates, with the property that each of the elements of S must be included in the combination, and this is just

$$\binom{10-1}{6-1} = \binom{9}{5}$$

or 126. ■

4. Linear Equations With Unit Coefficients

Let us consider the following problem. How many different solutions, in non-negative integers, are there of the equation

$$x + y + z = 10$$

For example,

$$\{x = 2, y = 5, z = 3\} \quad \text{and} \quad \{x = 5, y = 3, z = 2\} \quad \text{and} \quad \{x = 10, y = 0, z = 0\}$$

are all solutions to $x + y + z = 10$ in nonnegative integers, but

$$\{x = 8, y = 9, z = -3\} \quad \text{and} \quad \{x = 1/2, y = 11/2, z = 8\}$$

are not. Notice that the first two solutions above are considered to be different, even though they involve the same numbers 2, 3, and 5 since the first has $x = 2$ while the second has $x = 5$.

As it turns out, our knowledge of combinations with repetitions makes this problem very easy to solve. Consider the set $S = \{x, y, z\}$. Then any solution to the equation $x + y + z = 10$ can be represented as a combination of size 10 with unrestricted repetitions, taken from the set S. For example, the solution $x = 2$, $y = 5$, $z = 3$ can be represented by the combination

$$\{x, x, y, y, y, y, y, z, z, z\}$$

which contains 2 (x's), 5 (y's) and 3 (z's) is $x = 2, y = 5, z = 3$. On the other hand, every such combination corresponds to exactly one solution to $x+y+z = 10$. For example, the combination

$$\{x, x, x, x, y, y, y, y, z, z\}$$

which contains 4 (x's), 4 (y's) and 2 (z's) is $x = 4$, $y = 4$, $z = 2$.

In general, the combination having s_1 (x's), s_2 (y's) and s_3 (z's) corresponds to the solution $x = s_1$, $y = s_2$, $z = s_3$. In this way, we see that every solution to $x+y+z = 10$ in nonnegative integers corresponds to exactly one combination of size 10 with unrestricted repetitions, taken from S. Also, every such combination corresponds to exactly one solution to $x+y+z = 10$ in nonnegative integers. Once again, we have constructed a *one-to-one correspondence* between the nonnegative integer solutions to $x+y+z = 10$ and the combinations of size 10, with unrestricted repetitions. Therefore, the number of solutions in nonnegative integers of $x+y+z = 10$ is the same as the number of combinations of size 10 with unrestricted repetitions, taken from the set $S = \{x, y, z\}$, and this is just

$$\binom{3+10-1}{3-1} = \binom{12}{2} = 66$$

and so there are 66 different solutions, in nonnegative integers, to $x+y+z = 10$.

We can easily generalize this discussion to obtain the following result. The number of different solutions (in nonnegative integers) to the equation

$$x_1 + x_2 + x_3 + \cdots + x_n = k \tag{3a}$$

is

$$\binom{n+k-1}{k}. \tag{3b}$$

To prove this we note that every solution to Equation (3a) can be represented as a combination of size k with unrestricted repetitions, taken from the set

$$S = \{x_1, x_2, x_3, ..., x_n\}$$

of size n. In particular, the solution

$$x_1 = s_1 \quad , \quad x_2 = s_2 \quad , \quad ... \quad , \quad x_n = s_n$$

can be represented as the combination

$$\{x_1, x_1, x_1, ..., x_1, x_2, x_2, x_2, ..., ..., x_n, x_n, x_n, ..., x_n\}$$

where there are s_1 (x_1's), s_2 (x_2's), and so on up to s_n (x_n's). In this way, we see that each solution in nonnegative integers to Equation (3a) corresponds to exactly one combination of size k with unrestricted repetitions, taken from S. On the other hand, each such combination corresponds to exactly one solution to Equation (3a) in nonnegative integers. Hence, the number of solutions in *nonnegative* integers to Equation (3a) is equal to the number of combinations of size k with unrestricted repetitions, taken from S, which is equal to

$$\binom{n+k-1}{k}$$

and this completes the proof.

Example #14: Combinations With Restricted Repetitions

How many integers are there between 0 (or 000,000) and 999,999 (inclusive) with the property that the sum of their digits is equal to 5? To solve this we note that each integer between 0 and 999,999 can be written as a six-digit integer $x_1x_2x_3x_4x_5x_6$, where each digit satisfies $0 \leq x_i \leq 9$. The condition that the sum of these digits is equal to 5 is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5. \tag{4}$$

Hence, the answer to our question is the same as the number of solutions to Equation (4) that satisfy the condition $0 \leq x_i \leq 9$ for all $i = 1, 2, 3, 4, 5, 6$.

But any solution to this equation, in nonnegative integers, automatically has the property that $0 \leq x_i \leq 9$ for all $i = 1, 2, 3, 4, 5, 6$. After all, how can any of the x_i's be larger than 9 if the sum of all of the x_i's is only equal to 5, and all of the x_i's are nonnegative? Therefore, we can drop the condition $x_i \leq 9$, and so the answer to our question is the same as the number of solutions to Equation (4) in nonnegative integers and this is

$$\binom{6+5-1}{5} = \binom{10}{5} = 252$$

and so we see that there are 252 integers between 0 and 999,999 with the sum of whose digits equal to 5. ■

Solutions in Positive Integers

The result in Equation (3b) tells us how to determine the number of solutions, in *nonnegative* integers, to Equation (3a). But, sometimes it is important to know how many solutions there are, in *positive* integers, to this equation. We can use Equation (3b) to help answer this question as well. Suppose that we subtract n from both sides of Equation (3a), to get

$$x_1 + x_2 + x_3 + \cdots + x_n - n = k - n. \tag{5a}$$

This can be rewritten in the form

$$(x_1 - 1) + (x_2 - 1) + (x_3 - 1) + \cdots + (x_n - 1) = k - n.$$

and, if for convenience, we set $y_i = x_i - 1$ for all $i = 1, 2, 3, ..., n$, then we get

$$y_1 + y_2 + y_3 + \cdots + y_n = k' = k - n. \tag{5b}$$

and each solution to Equation (5a) in positive integers corresponds to exactly one solution to Equation (5b) in nonnegative integers, and vice versa. Since the number of solutions to Equation (5b) is

$$\binom{n + k' - 1}{k'} = \binom{n + k - n - 1}{k - n} = \binom{k - 1}{k - n} = \binom{k - 1}{n - 1}$$

we now see that the number of different solutions (in *positive* integers) to the equation

$$x_1 + x_2 + x_3 + \cdots + x_n = k \tag{6a}$$

is

$$\binom{k - 1}{n - 1}. \tag{6b}$$

Example #15: Combinations With Restricted Repetitions

How many ways are there to place 100 people in 3 different rooms, if each room must be occupied and if it does not matter which people go into which room? If we let x_1 be the number of people that go into room 1, x_2 be the number of people that go into room 2, and x_3 be the number of people that go into room 3, then we must have $x_1 + x_2 + x_3 = 100$ where $x_1, x_2, x_3 > 0$. Thus, the answer to our question is the same as the number of solutions to this $x_1 + x_2 + x_3 = 100$ in positive integers and this is

$$\binom{100-1}{3-1} = \binom{99}{2} = 4,851$$

and so there are 4,851 different ways to place 100 people into 3 rooms under these conditions. ■

The student should notice that a generalization to Equation (6a,b) is evident. Namely, if c_1, c_2, c_3, ..., c_n are integers (positive, negative or zero), then the number of different solutions (in integers) to the equation

$$x_1 + x_2 + x_3 + \cdots + x_n = k \tag{7a}$$

with the conditions that $x_i \geq c_i$ for all $i = 1, 2, 3, ..., n$ is

$$\binom{n + k - c_1 - c_2 - c_3 - \cdots - c_n - 1}{n-1}. \tag{7b}$$

The proof just follows by writing

$$x_1 + x_2 + x_3 + \cdots + x_n = k$$

as

$$(x_1 - c_1) + (x_2 - c_2) + (x_3 - c_3) + \cdots + (x_n - c_n) = k - c_1 - c_2 - c_3 - \cdots - c_n$$

and so

$$y_1 + y_2 + y_3 + \cdots + y_n = k'$$

with $y_i = x_i - c_i \geq 0$ for all $i = 1, 2, 3, ..., n$, and $k' = k - c_1 - c_2 - c_3 - \cdots - c_n$. Of course, if we instead require that $x_i > c_i$ for all $i = 1, 2, 3, ..., n$, then the number of different solutions (in integers) to the equation

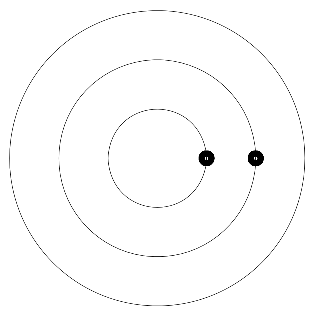
$$x_1 + x_2 + x_3 + \cdots + x_n = k$$

with the conditions that $x_i > c_i$ for all $i = 1, 2, 3, ..., n$ is

$$\binom{k - c_1 - c_2 - c_3 - \cdots - c_n - 1}{n-1}. \tag{7c}$$

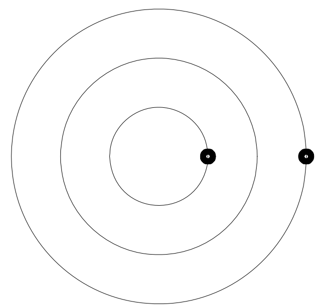
5. Distributing Balls Into Boxes (or Urns) - Optional

The same combinatorial problem frequently can be phrased in many different ways, and one of the most common ways to phrase combinatorial problems is in terms of distributing balls into boxes. For this reason, it is important to devote some time to becoming familiar with this terminology. As a simple example of an important combinatorial problem that can be phrased in terms of distributing balls into boxes, suppose that a certain atom has two electrons, and three electron shells that can contain these electrons as shown in the following figure. Suppose also that each shell is allowed to contain only one electron. The problem is to count the total number of possible electron configurations. In this case, it is not hard to see that there are exactly three possible configurations, as pictured above.



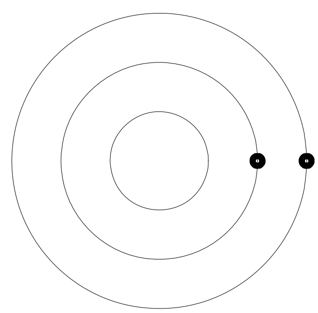
One Possible Arrangement of
Two Electrons in Three Shells

and



Another Possible Arrangement of
Two Electrons in Three Shells

and

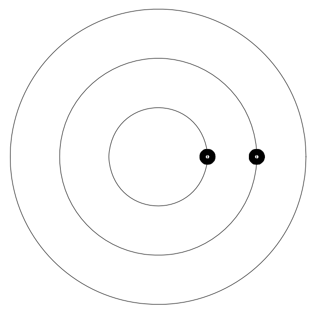


Another Possible Arrangement of
Two Electrons in Three Shells

Now, the same problem can be phrased in terms of distributing balls into boxes as follows. Suppose that we have 3 boxes and 2 balls, and we wish to place the balls into the boxes in such a way that no box receives more than 1 ball. The problem is to count the total number of ways to do this. The different possibilities are illustrated in the figure below.

$\boxed{\text{o}}\boxed{\text{o}}\boxed{-}$ and $\boxed{\text{o}}\boxed{-}\boxed{\text{o}}$ and $\boxed{-}\boxed{\text{o}}\boxed{\text{o}}$.

As you can see by comparing the earlier figures, repeated here,

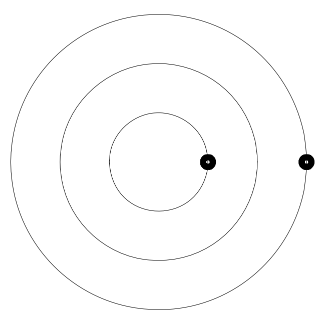


One Possible Arrangement of
Two Electrons in Three Shells

$\Leftrightarrow$

o	o	–

and

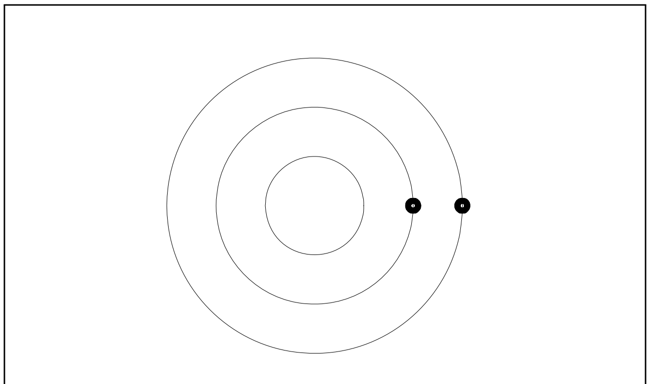


Another Possible Arrangement of
Two Electrons in Three Shells

$\Leftrightarrow$

o	–	o

and



Another Possible Arrangement of
Two Electrons in Three Shells

$\Leftrightarrow$

–	o	o

these two problems are essentially the same. The only difference is in their content.

In this section, we want to consider the problem of how to count the number of ways of distributing k balls into n boxes, under various conditions. The conditions that are generally imposed are the following.

(1) The balls can be either distinguishable or indistinguishable.

(2) The boxes can be either distinguishable or indistinguishable.

(3) The distribution can take place either with exclusion or without exclusion.

Let us discuss these terms briefly. The term “distinguishable” refers to the fact that the balls, or boxes, are marked in some way or have some feature about them that makes each one distinguishable from the others. For example, they may be numbered, each with a different number, they may each be a different color, or they may each be a different size or shape.

For the purposes of our discussion, when we speak of k distinguishable balls, we will assume that they are numbered with the consecutive integers 1 through k, and when we speak of n distinguishable boxes, we will assume that they are numbered with the consecutive integers 1 through n. The term “indistinguishable” refers to the fact that the balls, or boxes, are identical so that there is no way to tell them apart (not even by their location). In particular then, when placing indistinguishable balls into distinguishable boxes, it makes no difference which balls go into which boxes. In fact, it is not even possible to tell which balls go into which boxes! All that we are able to tell is the number of balls that end up in each box.

In our discussion, we will consider *only the case of distinguishable boxes.* As you might imagine, the case of indistinguishable boxes turns out to be much more complicated than that of distinguishable boxes and is usually studied in a course in combinatorics.

As to the third condition, the phrase "with exclusion" means that no box can contain more than one ball, and the phrase "without exclusion" means that a box may contain more than one ball.

Fortunately, we can use our knowledge of permutations and combinations to help us with the problems of distributing balls into boxes. Therefore, in each case, we will first try to rephrase the problem in terms of permutations and combinations.

Before considering the various cases, we should clear up one possible point of confusion. Namely, the order in which the balls are placed into the boxes is not important. To help keep this in mind, it is a good idea to think of the balls as being placed into the boxes at exactly the same time. Let us now turn to the four possible cases.

Case #1: Order Matters and Without Repetitions

How many ways are there to distribute k distinguishable balls into n distinguishable boxes, with exclusion? In this case, we consider putting k balls, numbered 1 through k, into n boxes, numbered 1 through n, in such a way that no box receives more than one ball. Now, we can translate this into the language of permutations and combinations as follows. Putting k distinguishable balls into n boxes, with exclusion, amounts to the same thing as making an ordered selection of k of the n boxes, where the balls do the selecting for us. The ball labeled 1 selects the first box, the ball labeled 2 selects the second box, and so on. In other words, distributing k distinguishable balls into n distinguishable boxes, with exclusion, is the same as forming a permutation of size k, taken from the set of n

boxes. This gives us the following result.

Distributing k distinguishable balls into n distinguishable boxes, with exclusion, corresponds to forming a permutation of size k, taken from a set of size n. Therefore, there are

$$P(n,k) = (n)_k = \frac{n!}{(n-k)!} \tag{8a}$$

different ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion.

Case #2: Order Matters and With Repetitions

How many ways are there to distribute k distinguishable balls into n distinguishable boxes, without exclusion? In this case, we consider putting k balls, numbered 1 through k, into n boxes, numbered 1 through n, but this time with no restriction on the number of balls that can go into each box.

Again, instead of thinking in terms of putting k balls into n boxes, we can think in terms of selecting k of the n boxes. As before, the balls do the selecting for us, but this time more than one ball may go into the same box, which means that the same box may be chosen more than once. Therefore, we are still dealing with ordered selections, or permutations, of the boxes, but now with unrestricted repetitions, and this leads to the following result.

Distributing k distinguishable balls into n distinguishable boxes, without exclusion, corresponds to forming a permutation of size k, with unrestricted repetitions, taken from a set of size n. Therefore, there are

$$n^k \tag{8b}$$

different ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion.

Case #3: Order Does Not Matters and Without Repetitions

How many ways are there to distribute k indistinguishable balls into n distinguishable boxes, with exclusion?

In this case, we have k identical balls, and we wish to place them into n distinguishable boxes in such a way that no box receives more than one ball. Once such a placement of balls has been made, then, since the balls are identical, all we can say is which boxes have received a ball and which have not. In other words, placing the balls has the same effect as simply choosing k of the n boxes. Those boxes that receive a ball are the ones that are chosen, and those that do not receive a ball are not chosen. Hence, in this case we are making unordered selections, that is, forming combinations of size k, taken from the set of n boxes, and this now gives the following result.

Distributing k indistinguishable balls into n distinguishable boxes, with exclusion, corresponds to forming a combination of size k, taken from a set of size n. Therefore, there are

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{8c}$$

different ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion.

Case #4: Order Does Not Matters and With Repetitions

How many ways are there to distribute k indistinguishable balls into n distinguishable boxes, without exclusion?

In this case, we have k identical balls, to be distributed into n distinguishable boxes, but with no restriction on the number of balls that can occupy a given box. As in the last case, since the balls are indistinguishable, we can only tell how many balls each box has received. This translates into making a choice of k of the n boxes, but with the possibility that a box may be chosen more than once. Thus, placing k balls into n boxes in this case corresponds to forming an unordered selection, or combination, of size k, taken from the set of n boxes, but with unrestricted repetitions, and this gives the following result.

Distributing k indistinguishable balls into n distinguishable boxes, without exclusion, corresponds to forming a combination of size k with unrestricted repetitions, taken from a set of size n. Therefore, there are

$$C(n+k-1,k) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!} \tag{8d}$$

different ways to distribute k indistinguishable balls into n distinguishable boxes, without exclusion.

Some More Cases - Optional

We should discuss another condition that is commonly placed on the distribution of balls into boxes, namely, the condition that no box can be empty. The next result summarizes the possibilities. We will prove part 2 of this and leave the other parts for you to prove.

(1) The number of ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion, in such a way that no box is empty, is $n!$ if $k = n$ and 0 if $k \neq n$.

(2) The number of ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion, in such a way that no box is empty, is

$$\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^k \tag{9}$$

for $k \geq n$. If $k < n$, then, of course, there is no way.

(3) The number of ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion, in such a way that no box is empty, is 1 if $k = n$ and 0 if $k \neq n$.

(4) The number of ways to distribute k indistinguishable balls into n distinguishable boxes, without exclusion, in such a way that no box is empty, is

$$\binom{k-1}{n-1}.$$

For the sake of reference, let us summarize our results in the following table.

Balls	Boxes	Exclusion	No Box Empty	# of Ways to put k balls into n boxes
Dist	Dist	With	No	$P(n,k)$
Dist	Dist	With	Yes	$n!$ if $k=n, 0$ if $k \neq n$
Dist	Dist	Without	No	n^k
Dist	Dist	Without	Yes	See Above
Indist	Dist	With	No	$C(n,k)$
Indist	Dist	With	Yes	1 if $k=n, 0$ if $k \neq n$
Indist	Dist	Without	No	$C(n+k-1,k)$
Indist	Dist	Without	Yes	$C(k-1,n-1)$

Another Way to Word Things

We may also organize some of these results using the words, drawing with replacement and drawing without replacement and whether order matters or order does not matter. In others words, suppose there are n distinct objects (labeled 1, 2, 3, ..., n) in a box, and we want to choose k of these objects one-by-one and make a list of k numbers by recording (from left to right) the number on the object drawn at the time it was drawn. We choose these objects one-by-one (and record its number) and either replace the object chosen so that it could be chosen again, or we may choose not to replace the object chosen so that it cannot be chosen more than once. In addition, in our left-to-right listing of the numbers on the objects, we could either make the order of this listing important (so that 1342 is different from 1234) or we could make it unimportant (so that 1342 is the same as 1234). We are interested in the number of different possible listings we could have if k objects are chosen one at a time (with and without replacement) from a box of n distinct objects and if the order of the objects in the list is important or not important. The table below summarizes the possible number of listings that can occur,

	Order Matters	Order Does Not Matter
With Replacement	n^k	$C(n+k-1,k)$
Without Replacement	$P(n,k)$	$C(n,k)$

where

$$P(n,k) = \frac{n!}{(n-k)!} \qquad , \qquad C(n,k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

and

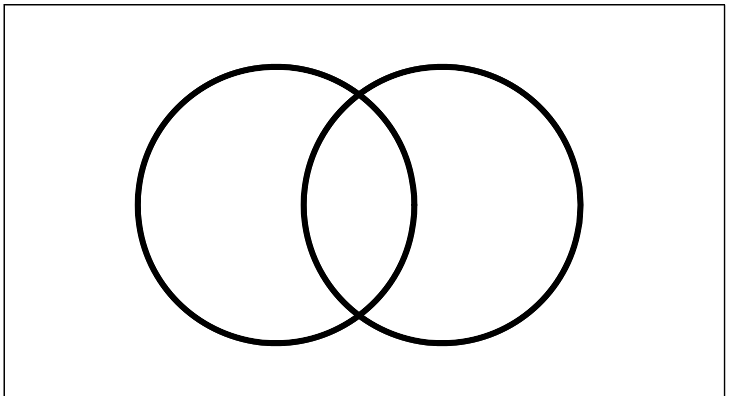
$$C(n+k-1,k) = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}.$$

6. The Principle of lnclusion-Exclusion (Two Sets)

Up to now, much of what we have done has depended, either directly or indirectly, on two important principles, namely the *multiplication* and the *addition* principles. In the next few sections, we want to study a third principle which is a generalization of the addition principle. It is called the principle of *inclusion-exclusion*. In order to describe this principle, let us begin with a simple situation. Consider a group of 60 students, and suppose that 40 of these students are taking a math class and 30 of them are taking a computer science class. Then, can we tell how many of these students are taking at least one of the two classes? The answer is no. It certainly cannot be $40 + 30 = 70$ since there are only 60 students! The reason that we cannot tell how many students are taking at least one class is that we do not know how many students are taking both classes. So let us suppose that 20 students are taking both classes. Now we can determine the number of students who are taking at least one of the classes as follows. The number $40 + 30 = 70$ is too large, but only because the students who are taking both classes are counted twice in this number. Therefore, since there are 20 students taking both classes, the total number of students taking at least one of the two classes is $40 + 30 - 20 = 50$. We can summarize this discussion in the following equation.

$$\begin{pmatrix} \text{number of students} \\ \text{taking at least one} \\ \text{of the two classes} \end{pmatrix} = \begin{pmatrix} \text{number of students} \\ \text{taking the} \\ \text{math class} \end{pmatrix} + \begin{pmatrix} \text{number of students} \\ \text{taking the} \\ \text{computer class} \end{pmatrix} - \begin{pmatrix} \text{number of students} \\ \text{taking the} \\ \text{both classes} \end{pmatrix}.$$

It might help to remember this equation if we draw a picture, as in the following figure.



Students taking math (left) & computers (right)

The large rectangular box that the two circles are in represents the entire group of 60 students. The circle on the left represents the set of 40 students who are taking the math class, and the circle on the right represents the set of 30 students who are taking the computer science class. The region of intersection of the two circles represents those 20 students who are taking both classes. These are the students that are counted twice when we take the sum of the number of students in each of the two circles, and this is precisely why we must subtract the number of students in this intersection. Since the same reasoning that we used to solve this problem will work to solve other counting problems, we should try to phrase our result in general terms, so that it will be easier to apply. This is done simply by replacing sets of students by abstract sets.

Let U be a universal set and let A and B be subsets of U, as pictured like in the above figure. We would like to know how many elements of U are in at least one of the two sets A or B. Put another way, we would like to know how many elements there are in the union $A \cup B$ of the two sets. We may use the same logic as in the example above and write

$$\begin{aligned}\begin{pmatrix}\text{number of}\\ \text{elements in } A \cup B\end{pmatrix} \quad = \quad & \begin{pmatrix}\text{number of}\\ \text{elements in } A\end{pmatrix} + \begin{pmatrix}\text{number of}\\ \text{elements in } B\end{pmatrix} \\ & - \begin{pmatrix}\text{number of}\\ \text{elements in } A \cap B\end{pmatrix}\end{aligned}$$

or

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{10}$$

The reasoning behind this equation that the sum $|A| + |B|$ counts all of the elements in $A \cup B$, but it happens to count those elements that are in the intersection $A \cap B$ twice. Therefore, in order to have them counted only once, we subtract $|A \cap B|$. Equation (10) is the simplest form of the *principle of inclusion-exclusion*.

Example #16: Inclusion-Exclusion

In a certain group of 100 kittens, 50 are male, 25 are long haired, and 10 are long-haired males. How many kittens in the group are either male or long haired, assuming that kittens come in two varieties, long haired and short haired. To answer this, let

$$U = \text{the set of all 100 kittens in the group}$$

and

$$M = \text{the set of all kittens in the group that are male}$$

and

$$L = \text{the set of all kittens in the group that are long haired.}$$

Then $M \cup L$ is the set of kittens that are either male or long haired, and so the answer to the question is $|M \cup L|$, which we can compute using the inclusion-exclusion principle. First, we observe that $|M| = 50$, $|L| = 25$, and $|M \cap L| = 10$. Therefore, according to the principle of inclusion-exclusion, we have

$$|M \cup L| = |M| + |L| - |M \cap L| = 50 + 25 - 10 = 65$$

and so there are 65 kittens that are either male or long haired. ■

The principle of inclusion-exclusion tells us how to find the number of elements in U that are in at least one of the two sets A or B. But we may also want to find the number of elements in U that are in neither of the two sets A or B, which is $|\bar{A} \cap \bar{B}|$, where $\bar{A} = U - A$ and $\bar{B} = U - B$ are the compliments of A and B in U, respectively. This problem can easily be solved by using the principle of inclusion-exclusion. First, we observe that

$$\bar{A} \cap \bar{B} = \overline{A \cup B}$$

and so the set of all elements of U that are in neither of the sets A or B is

$$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |U| - |A \cup B|, \tag{11}$$

or

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B|. \tag{12}$$

Example #17: Inclusion-Exclusion

Referring to Example #16, we ask the question, we may ask how many kittens in the group are short-haired females? Since $F = \bar{M} = U - M$ and $S = \bar{L} = U - L$, and $|F| = |U| - |M| = 100 - 50 = 50$ and $|S| = |U| - |L| = 100 - 25 = 75$, we have

$$|F \cap S| = |\bar{M} \cap \bar{L}| = |U| - |M| - |L| + |M \cap L|$$

or

$$|F \cap S| = 100 - 50 - 25 + 10 = 35$$

and so there are 35 short-haired female kittens in the group. Actually, in this case, since we had already computed $|A \cup B|$ in the previous example, it would have been a bit easier to use this information to find

$$|F \cap S| = |\bar{M} \cap \bar{L}| = |\overline{M \cup L}| = |U| - |M \cup L| = 100 - 65 = 35$$

which is the same as before. ■

Example #18: Inclusion-Exclusion

(a) How many integers between 1 and 600 have the property that they are divisible by either 3 or 5, and (b) how many integers between 1 and 600 have the property that they are divisible by neither 3 nor 5? If we let P3 be the property of being divisible by 3, and if we let P5 be the property of being divisible by 5, then these questions can be rephrased as follows. (a) How many integers are there between 1 and 600 that satisfy at least one of the properties P3 or P5? (b) How many integers are there between 1 and 600 that satisfy neither of the properties P3 or P5? Let us begin by letting

$$U = \text{the set of all integers between 1 and 600}$$

and

$$A = \text{the set of all integers in } U \text{ that satisfy property P3}$$

and

$$B = \text{the set of all integers in } U \text{ that satisfy property P5}$$

Then since $A \cup B$ is the set of all integers in U that satisfy at least one of the properties, the answer to the first question is $|A \cup B|$. In order to compute $|A \cup B|$, we need to compute $|A|$, $|B|$ and $|A \cap B|$. As to $|A|$, we must determine the number of integers between 1 and 600 that are divisible by 3. But these integers are just the multiples of 3, that is,

$$3, 6, 9, ..., 600 = 3(1), 3(2), 3(3), ..., 3(200)$$

showing that $|A| = 200$, and since

$$5, 10, 15, ..., 600 = 5(1), 5(2), 5(3), ..., 5(120)$$

we see that $|B| = 120$. Finally since $|A \cap B|$ are the set of all integers between 1 and 600 that are evenly divisible by $3 \times 5 = 15$, and

$$15, 30, 45, ..., 600 = 15(1), 15(2), 15(3), ..., 15(40)$$

we have $|A \cap B| = 40$. Using now the principle of inclusion-exclusion, we find that

$$|A \cup B| = |A| + |B| - |A \cap B| = 200 + 120 - 40 = 280$$

and so there are 280 integers between 1 and 600 that are divisible by either 3 or 5.

The answer to the second question is

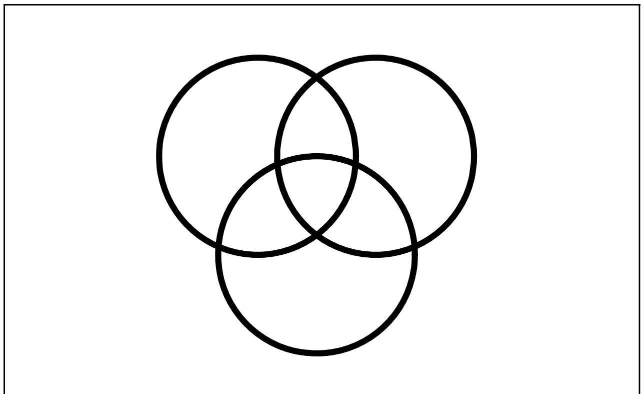
$$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |U| - |A \cup B| = 600 - 280 = 320$$

and so there are 320 integers between 1 and 600 that are divisible by neither 3 nor 5. ■

7. The Principle of lnclusion-Exclusion (Three Sets)

In this section, we want to extend the principle of inclusion-exclusion to three sets and give some important examples of its use. In the next section, we will extend the inclusion-exclusion principle to an arbitrary number of sets.

Toward this end, suppose that A, B, and C are three sets, as pictured below.



The Sets A (left circle), B (right circle), C (lower circle) and U (rectangle)

We want to find a formula for the number of elements that are in at least one of the sets; that is, we want a formula for the number of elements in the union $A \cup B \cup C$. For here we may use the result for two sets as follows.

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| \\
&\quad -|(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| \\
&\quad -(|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|)
\end{aligned}$$

or

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |B \cap C| \\
&\quad -|A \cap B| - |A \cap C| + |A \cap B \cap A \cap C|
\end{aligned}$$

or, since $A \cap B \cap A \cap C = A \cap B \cap C$, we find that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \quad (13)$$

and the proof is complete.

Example #19: Inclusion-Exclusion

Suppose that, in a group of 100 students (a) 50 students are taking math, (b) 40 students are taking computer science, (c) 35 students are taking chemistry, (d) 12 students are taking both math and computer science, (e) 10 students are taking both math and chemistry, (f) 11 students are taking both computer science and chemistry and (g) 5 students are taking all three subjects How many students are taking at least one of the subjects? To solve this, we let

$$U = \text{the set of all 100 students}$$

and

$$A = \text{the set of all students in } U \text{ who are taking math}$$

and

$$B = \text{the set of all students in } U \text{ who are taking computer science}$$

and

$$C = \text{the set of all students in } U \text{ who are taking chemistry.}$$

Then $A \cup B \cup C$ is precisely the set of all students taking at least one of the subjects, and so we want to compute $|A \cup B \cup C|$. Now $|A| = 50$, $|B| = 40$, $|C| = 35$, and since $A \cap B$ is the set of all students taking both math and computer science, we have $|A \cap B| = 12$. Similarly, we have $|A \cap B| = 10$, $|B \cap C| = 11$ and $|A \cap B \cap C| = 5$. According to Equation (13),

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

we have

$$|A \cup B \cup C| = 50 + 40 + 35 - 12 - 10 - 11 + 5 = 97.$$

In other words, exactly 97 students are taking at least one of the subjects. ■

Equation (13) can also be used to determine the number of elements in the universal set U that are in none of the sets A, B, and C. First, we write this set in the form

$$\bar{A} \cap \bar{B} \cap \bar{C} = \overline{A \cup B \cup C}$$

and then

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$$

or

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C|, \tag{14}$$

where $|A \cup B \cup C|$ is computed using Equation (13).

Example #20: Inclusion-Exclusion

In a group of 200 people, 100 like Coke, 149 like Pepsi, 83 like Seven-Up, 80 like Coke and Pepsi, 66 like Coke and Seven-Up, 45 like Pepsi and Seven-Up, and 12 like Coke, Pepsi, and Seven-Up. How many of these people like none of the soft drinks? To solve this, let us take

$$U = \text{the set of all 200 people}$$

and

$$A = \text{the set of all people in } U \text{ who like Coke}$$

and

$$B = \text{the set of all people in } U \text{ who like Pepsi}$$

and

$$C = \text{the set of all people in } U \text{ who like Seven-Up}$$

Then $\bar{A} \cap \bar{B} \cap \bar{C}$ is the set of all people in U who like none of the soft drinks. Computing first

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

we have $|A| = 100$, $|B| = 149$, $|C| = 83$, $|A \cap B| = 80$, $|A \cap C| = 66$, $|B \cap C| = 45$, and $|A \cap B \cap C| = 12$, so that

$$|A \cup B \cup C| = 100 + 149 + 83 - 80 - 66 - 45 + 12 = 153.$$

Then

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C| = 200 - 153 = 47$$

and so there are exactly 47 people in the group who do not like any of the soft drinks. ■

Incidentally, we can see again that the phrases "at least one of" and "none of' (which is similar to "neither of") are clues to the fact that the principle of inclusion-exclusion may apply.

Counting Combinations With Restricted Repetitions

One of the most important applications of the principle of inclusion-exclusion is to counting combinations with restricted repetitions. In other words, we would like to count the number of combinations of size k taken from a set $S = \{a_1, a_2, a_3, ..., a_n\}$ of size n so the number of times a_i appears stasifies $s_i \leq r_i \leq t_i$ for $i = 1, 2, 3, ..., n$. The says that a_i appears *at least* s_i times and *at most* t_i times. The basic idea is to first note that this must be the same as the number of combinations of "reduced" size

$$k' = k - (s_1 + s_2 + s_3 + \cdots + s_n)$$

taken from a same set $S = \{a_1, a_2, a_3, ..., a_n\}$ of size n so the number of times a_i appears in the reduced combination is no larger than (or *at most*) $t_i - s_i \geq 0$ times for $i = 1, 2, 3, ..., n$. If we let P_i be the property that a_i is repeated *at least* $t_i - s_i + 1$ times, for $i = 1, 2, 3, ..., n$, then we would like to count

$$|\bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3 \cap \cdots \cap \bar{P}_n| = |U| - |P_1 \cup P_2 \cup P_3 \cup \cdots \cup P_n|$$

where $|U|$ equals the number of combinations of size k' taken from a set

$$S = \{a_1, a_2, a_3, ..., a_n\}$$

of size n with no restrictions and $|P_1 \cup P_2 \cup P_3 \cup \cdots \cup P_n|$ is determined using the inclusion-exclusion principle. Let us try some examples.

Example #21: Combinations and Inclusion-Exclusion

How many combinations of size 13 are there, taken from the set $S = \{a_1, a_2, a_3\}$, with the property that a_1 can be repeated at most 4 times, a_2 can be repeated at most 5 times, and a_3 can be repeated at most 6 times? If we let P_1 be the property that a_1 is repeated at least 5 times, P_2 be the property that a_2 is repeated at least 6 times, and P_3 be the property that a_3 is repeated at least 7 times, then this question can be rephrased as follows. How many combinations of size 13 are there, taken from the set $S = \{a_1, a_2, a_3\}$, that satisfy none of the three properties P_1, P_2 , or P_3? This rewording is an indication that we should try using

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C|.$$

where U is the set of all combinations of size 13, taken from the set S, with unrestricted repetitions, A be the set of all combinations in U that satisfy property P_1, B be the set of all combinations in U that satisfy property P_2, and C be the set of all combinations in U that satisfy property P_3. Then $\bar{A} \cap \bar{B} \cap \bar{C}$ is the set of all combinations in U that satisfy none of the properties P_1, P_2 , or P_3. In order to compute this number, we first compute

$$|U| = \binom{3+13-1}{3-1} = \binom{15}{2} = 105$$

and

$$|A| = \binom{3+(13-5-0-0)-1}{3-1} = \binom{10}{2} = 45$$

and

$$|B| = \binom{3+(13-0-6-0)-1}{3-1} = \binom{9}{2} = 36$$

and

$$|C| = \binom{3+(13-0-0-7)-1}{3-1} = \binom{8}{2} = 28$$

and

$$|A \cap B| = \binom{3+(13-5-6-0)-1}{3-1} = \binom{4}{2} = 6$$

and

$$|A \cap C| = \binom{3+(13-5-0-7)-1}{3-1} = \binom{3}{2} = 3$$

and

$$|B \cap C| = \binom{3+(13-0-6-7)-1}{3-1} = \binom{2}{2} = 1$$

and

$$|A \cap B \cap C| = \binom{3+(13-5-6-7)-1}{3-1} = \binom{-3}{2} = 0.$$

Remember that

$$\binom{a}{b} = 0 \qquad \text{when} \qquad a < b.$$

This says that $A \cap B \cap C = \emptyset$ (the empty set). Thus we find that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

gives

$$|A \cup B \cup C| = 45 + 36 + 28 - 6 - 3 - 1 + 0 = 99.$$

Then

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C| = 105 - 99 = 6.$$

Thus, there are 6 combinations of the type specified in the problem. ■

Example #22: Combinations and Inclusion-Exclusion

How many combinations of size 25 are there, taken from the set $S = \{a_1, a_2, a_3\}$, with the property that a_1 must be repeated at least 2 times but at most 8 times, a_2 must be repeated at least 3 times but at most 12 times, and a_3 must be repeated at least 7 times but at most 10 times? The first step in a problem like this is to reduce it to a problem like the one in the last example. This is done simply by observing that each combination of the type described in this problem is formed by writing down the required 2 (a_1's), 3 (a_2's), and 7 (a_3's), and then appending to that any combination of "reduced" size $25-2-3-7 = 13$, taken from S, with the property that a_1 can be repeated at most $8-2 = 6$ times, a_2 can be repeated at most $12-3 = 9$ times, and a_3 can be repeated at most $10-7 = 3$ times. Thus, the number of combinations of the type described in this problem is equal to the number of combinations of size 13, taken from the set S, with the property that a_1 can be repeated at most 6 times, a_2 can be repeated at most 9 times, and a_3 can be repeated at most 3 times.

Now, this number can be determined exactly as we did in the last example, by using the principle of inclusion-exclusion. Let P_1 be the property that a_1 is repeated at least 7 times (the opposite of being repeated at most 6 times), and let P_2 be the property that a_2 is repeated at least 10 times (the opposite of being repeated at most 9 times), and let P_3 be the property that a_3 is repeated at least 4 times (the opposite of being repeated at most 3 times). Also, let U be the set of all combinations of size 13, taken from the set S, with unrestricted repetitions, let A be the set of all combinations in U that satisfy property P_1, B be the set of all combinations in U that satisfy property P_2, C be the set of all combinations in U that satisfy property P_3. Then we want to find $|\bar{A} \cap \bar{B} \cap \bar{C}|$, which we can compute using Equation (14). It should be clear that

$$|U| = \binom{3+13-1}{3-1} = \binom{15}{2} = 105$$

and

$$|A| = \binom{3+(13-7-0-0)-1}{3-1} = \binom{8}{2} = 28$$

and

$$|B| = \binom{3+(13-0-10-0)-1}{3-1} = \binom{5}{2} = 10$$

and

$$|C| = \binom{3+(13-0-0-4)-1}{3-1} = \binom{11}{2} = 55$$

and

$$|A \cap B| = \binom{3+(13-7-10-0)-1}{3-1} = \binom{-2}{2} = 0$$

and

$$|A \cap C| = \binom{3+(13-7-0-4)-1}{3-1} = \binom{4}{2} = 6$$

and

$$|B \cap C| = \binom{3+(13-0-10-4)-1}{3-1} = \binom{1}{2} = 0$$

and

$$|A \cap B \cap C| = \binom{3+(13-7-10-4)-1}{3-1} = \binom{-6}{2} = 0.$$

Thus we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

gives

$$|A \cup B \cup C| = 28 + 10 + 55 - 6 - 0 - 0 + 0 = 87.$$

Then

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C| = 105 - 87 = 18.$$

Thus, there are 18 combinations of the type specified in the problem. ■

We can also use the principle of inclusion-exclusion to help us count the number of solutions, in integers, to linear equations with unit coefficients.

Example #23: Combinations and Inclusion-Exclusion

How many solutions are there, in integers, to the equation $x + y + z = 25$, with the property that

$$2 \leq x \leq 8 \quad , \quad 3 \leq y \leq 12 \quad \text{and} \quad 7 \leq z \leq 10.$$

We may say that the number of solutions to the equation $x + y + z = 25$ that satisfy condition

$$2 \leq x \leq 8 \quad , \quad 3 \leq y \leq 12 \quad \text{and} \quad 7 \leq z \leq 10.$$

is the same as the number of combinations of size 25, taken from the set $S = \{x, y, z\}$ with the property that x is repeated at least 2 times and at most 8 times, y is repeated at least 3 times and at most 12 times, and z is repeated at least 7 times and at most 10 times. But, in the last example, we determined that this number is 18, and so there are 18 solutions to $x + y + z = 25$ that satisfy condition $2 \leq x \leq 8$, $3 \leq y \leq 12$ and $7 \leq z \leq 10$. ■

8. The Principle of lnclusion-Exclusion (Many Sets)

Our goal in this section is to extend the principle of inclusion-exclusion to an arbitrary number of sets. By writing

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

it should be clear from the pattern that

$$\begin{aligned} |A \cup B \cup C \cup D| \quad = \quad & |A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D|. \end{aligned}$$

In general, let U is a universal set, and let A_1, A_2, A_3, ..., A_n be subsets of U. Then the number of elements in U that are in at least one of the sets A_1, A_2, A_3, ..., A_n is

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j+1} S_j \tag{15}$$

where S_k denotes the sum of the sizes of the intersections of all collections of k of the sets A_1, A_2, A_3, ..., A_n. We may also use this to compute

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = |U| - |A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n|. \tag{16}$$

Derangements of n Integers - Method I

Our first application is to ordinary permutations. Let us agree to say that two permutations of the integers $1, 2, 3, ..., n$ are incompatible if they do not agree in any of their positions. For example, the permutations 3142 and 4231 are incompatible, but the permutations 3142 and 4312 are not incompatible, since they agree in their fourth positions (both have a 2). A permutation of the integers $1, 2, 3, ..., n$ is called a *derangement* of size n if it is incompatible with the permutation $123 \cdots n$. In a sense, such permutations completely "derange" the integers $1, 2, 3, ..., n$. Derangements occur in many different situations. For instance, the problem of Example #4 in Chapter #1 involves derangements of size 12.

Let us determine a beautiful formula for the number of derangements of size n. We begin by letting U be the set of all permutations of the integers $1, 2, 3, ..., n$, A_1 be the set of all permutations in U with a 1 in the first position, A_2 the set of all permutations in U with a 2 in the second position, A_3 the set of all permutations in U with a 3 in the third position, and so on up to A_n as the set of all permutations in U with an n in the nth position. Then

$$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n$$

is the set of all derangements of size n. It should be clear that $|U| = n!$,

$$|A_i| = (n-1)!$$

for each $i = 1, 2, 3, ..., n$. We also have

$$|A_i \cap A_j| = (n-2)!$$

for each $i, j = 1, 2, 3, ..., n$ and $i \neq j$, and

$$|A_i \cap A_j \cap A_k| = (n-3)!$$

for each $i, j, k = 1, 2, 3, ..., n$ and $i \neq j \neq k$, and so on. In addition, there are

$$\binom{n}{1} \quad \text{such} \quad |A_i|$$

terms, and

$$\binom{n}{2} \quad \text{such} \quad |A_i \cap A_j|$$

terms, and

$$\binom{n}{3} \quad \text{such} \quad |A_i \cap A_j \cap A_k|$$

terms, and so on. This then says that

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} (n-j)!$$

which reduces to

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{j=1}^{n} \frac{n!}{j!} (-1)^{j+1}.$$

Then

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = n! - \sum_{j=1}^{n} \frac{n!}{j!} (-1)^{j+1}$$

which reduces to

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

Thus we find that the number D_n of derangements of size n is

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}. \tag{17a}$$

Derangements of n Integers - Method II

We may also solve the derangement problem using a recurrence relation. To see how this is done, consider a derangement of n integers as pictured below.

Prohibited Value	1	2	$\cdots$	$k-1$	k	$k+1$	$\cdots$	n
Derangement	–	–	$\cdots$	–	–	–	$\cdots$	–

We begin by placing a k ($k \neq 1$) into the first place as shown below

Prohibited Value	1	2	$\cdots$	$k-1$	k	$k+1$	$\cdots$	n
Derangement	k	–	$\cdots$	–	–	–	$\cdots$	–

Then we may place either the 1 in the kth place as shown below

Prohibited Value	1	2	$\cdots$	$k-1$	k	$k+1$	$\cdots$	n
Derangement	k	–	$\cdots$	–	1	–	$\cdots$	–

which leaves us with D_{n-2} derangements of the set $\{2, 3, 4, \ldots, k-1, k+1, \ldots, n\}$, or we may choose not to place the 1 in the kth place which leaves us with D_{n-1} derangements of the set $\{2, 3, 4, \ldots, k-1, 1, k+1, \ldots, n\}$. Therefore, there are $D_{n-1} + D_{n-2}$ derangements of the integers 1 through n in which the integer k ($k \neq 1$) is moved to position 1. But since we must only require that $k \neq 1$, there are $n-1$ possible values of k (namely 2, 3, 4, ..., n), and so D_n must satisfy the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \tag{17b}$$

for $n = 3, 4, 5, ...$, with $D_1 = 0$ and $D_2 = 1$. To solve this for D_n, we begin by setting $D_n = n!R_n$, for then we find that

$$\begin{aligned} n!R_n &= (n-1)((n-1)!R_{n-1} + (n-2)!R_{n-2}) \\ &= (n-1)(n-1)!R_{n-1} + (n-1)!R_{n-2} \\ &= (n-1)!((n-1)R_{n-1} + R_{n-2}) \end{aligned}$$

or

$$nR_n = (n-1)R_{n-1} + R_{n-2} \qquad \text{or} \qquad n(R_n - R_{n-1}) = -(R_{n-1} - R_{n-2}).$$

Setting $S_n = R_n - R_{n-1}$, we then have the very simple relation

$$S_n = -\frac{S_{n-1}}{n}.$$

This says that

$$S_3 = -\frac{S_2}{3} \qquad \text{and} \qquad S_4 = -\frac{S_3}{4} = -\frac{1}{4}\left(-\frac{S_2}{3}\right) = \frac{2(-1)^2}{4!}S_2$$

and

$$S_5 = -\frac{S_4}{5} = -\frac{1}{5}\left(\frac{2(-1)^2}{4!}S_2\right) = \frac{2(-1)^3}{5!}S_2.$$

In general, we find that

$$S_n = \frac{2(-1)^{n-2}}{n!}S_2 = \frac{2(-1)^n}{n!}S_2$$

But $D_1 = 0$ and $D_2 = 1$, which says that $R_1 = 0$ and $R_2 = 1/2$. Then $S_2 = R_2 - R_1 = 1/2$. This says that

$$S_n = \frac{(-1)^n}{n!} = R_n - R_{n-1}$$

for $n = 2, 3, 4, ...$, which says that

$$\sum_{j=2}^{n}(R_j - R_{j-1}) = \sum_{j=2}^{n}\frac{(-1)^j}{j!} \qquad \text{or} \qquad R_n - R_1 = \sum_{j=2}^{n}\frac{(-1)^j}{j!}$$

which says that

$$R_n = \sum_{j=2}^{n}\frac{(-1)^j}{j!} = D_n/n! \qquad \text{or} \qquad D_n = n!\sum_{j=2}^{n}\frac{(-1)^j}{j!}.$$

Since

$$\sum_{j=0}^{1}\frac{(-1)^j}{j!} = 1 + (-1) = 0$$

we may also write this as

$$D_n = n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}$$

which now agrees with Equation (17a).

Example #24: Derangements of Size 4

The number of derangements of size 4 is

$$D_4 = 4! \sum_{j=0}^{4} \frac{(-1)^j}{j!} = 9$$

and these can be listed as follows.

2143	3142	4123
2341	3412	4312
2413	3421	4321

Because we know that there are 9 derangements of size 4, we know that this list is complete. Note that the number of derangements of size 6 is

$$D_6 = 6! \sum_{j=0}^{6} \frac{(-1)^j}{j!} = 265$$

This compares with a total of $6! = 720$ permutations of the integers $1, 2, 3, 4, 5, 6$. Therefore if a permutation of the numbers $\{1, 2, 3, 4, 5, 6\}$ is chosen at random, the probability that none of the integers is in its original place is

$$P = \frac{265}{720} = \frac{53}{144}$$

or $P = 36.8\%$.■

Example #25: Example #4 From Chapter #1

Recall that Example #4 from Chapter #1 said that your mathematics class has 12 students. On the day that your midterm exams are returned, your professor makes a total mess of things and returns the exams in such a way that every student gets someone else's exam, instead of his or her own. In how many ways can the exams be returned in such a way that every student gets someone else's exam? The answer is now simply given by

$$D_{12} = 12! \sum_{j=0}^{12} \frac{(-1)^j}{j!} = 176,214,841$$

which is probably larger than you would have guessed. If the exams are randomly handed back to the students, then the probability that no student gets their exam is

$$P_{12} = \frac{D_{12}}{12!} = \frac{176,214,841}{12!} = \frac{176,214,841}{479,001,600} = \frac{16,019,531}{43,545,600}$$

or $P_{12} \simeq 36.8\%$. It should be noted that

$$P_\infty = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \left(\frac{D_n}{n!}\right) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(-1)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} = e^{-1}$$

resulting in $P_\infty = e^{-1} \simeq 36.788\%$. ■

Example #26: Mix Up At The Dance

Six married couples come to a dance. In how many ways can the 6 men dance with the 6 women if no husband will dance with his wife? This is a problem in derangements. Assume that the wives are standing in a line, with their husbands facing them. Number the husbands, from left to right, with the integers 1 through 6. Then, if we let each permutation of the integers $1, 2, 3, 4, 5, 6$, correspond to a rearrangement of the men, it is exactly the derangements that correspond to rearrangements in which no man is paired with his wife. Hence, there are $D_6 = 265$ ways for the men to dance with the women in such a way that no man dances with his wife. The probability that no man dances with his wife is then

$$P_6 = \frac{D_6}{6!} = \frac{265}{6!} = \frac{265}{720} = \frac{53}{144}$$

or $P_6 \simeq 36.8\%$.■

Permutations with Restricted Repetitions

As a final application of the principle of inclusion-exclusion, we consider a special case of permutations with restricted repetitions. In particular, we want to find a formula for the number of permutations of size k, taken from the set $S = \{a_1, a_2, a_3, ..., a_n\}$, with the property that each element of S appears at least once in the permutation. We begin by letting U be the set of all permutations of size k with unrestricted repetitions, taken from S. Letting A_1 be the set of all permutations in U with the property that a_1 does not appear, A_2 be the set of all permutations in U with the property that a_2 does not appear, ..., A_n be the set of all permutations in U with the property that a_n does not appear.

Then $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n$ is the set of all permutations of S with the property that every element of S appears at least once in each permutation, and so again we want to use the ideas in this section to compute $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n|$.

Now, A_1 is the set of all permutations of size k with unrestricted repetitions, taken from the set S, with the additional property that a_1 does not appear. But this is the same as the set of all permutations of size k with unrestricted repetitions, taken from the set $\{a_2,\ a_3,\ a_4,\ ...,\ a_n\}$. Since there are $(n-1)^k$ of these permutations, we see that $|A_1| = (n-1)^k$. In a similar way, we have $|A_i| = (n-1)^k$ for each $i = 1, 2, 3, ..., n$ and there are

$$\binom{n}{1}$$

of these. We also have $|A_i \cap A_j| = (n-2)^k$ for each $i, j = 1, 2, 3, ..., n$ and $i \neq j$ and there are

$$\binom{n}{2}$$

of these. We also have $|A_i \cap A_j \cap A_k| = (n-3)^k$ for each $i, j, k = 1, 2, 3, ..., n$ and $i \neq j \neq k$ and there are

$$\binom{n}{3}$$

of these, and so on. Thus we find that

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (n-j)^k.$$

Then

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = |U| - \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (n-j)^k$$

or

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = n^k - \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (n-j)^k$$

which we may write as

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^k.$$

Thus we see that the number of permutations of size k, taken from the set $S = \{a_1, a_2, a_3, ..., a_n\}$, with the property that each element of S appears at least once in each permutation, is

$$\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^k.$$

This requires, of course that $k \geq n$. Note that in the special case when $k = n$, we should find that the answer is

$$n \times (n-1) \times (n-2) \times \cdots \times (2) \times (1) = n!$$

and so we should find that

$$\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^n = n!$$

which is (in fact) a true (but not obvious) statement!

We may also state this using balls and urns. In particular, the number of permutations of size k, taken from the set $S = \{a_1, a_2, a_3, ..., a_n\}$, with the property that each element of S appears at least once in each permutation is the same as the number of ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion, in such a way that no box is empty is

$$\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)^k.$$

9. Partitions of a Positive Integer - Optional

A partition of an integer n is an expression of the form

$$n = n_1 + n_2 + n_3 + \cdots + n_k$$

(with $k \leq n$) where each n_i (called a summand) is a positive integer. The number of partitions of n (denoted by P_n) equals the total number of ways in which n can be expressed as

$$n = n_1 + n_2 + n_3 + \cdots + n_k$$

(with $k \leq n$) where order is unimportant. For example, there are 15 ways to express $n = 7$ as the sum of positive integers as summarized in the following table.

Partition #	Partitions of 7	n_1	n_2	n_3	n_4	n_5	n_6	n_7
1	$7 = 7$	7	–	–	–	–	–	–
2	$7 = 6 + 1$	6	1	–	–	–	–	–
3	$7 = 5 + 2$	5	2	–	–	–	–	–
4	$7 = 5 + 1 + 1$	5	1	1	–	–	–	–
5	$7 = 4 + 3$	4	3	–	–	–	–	–
6	$7 = 4 + 2 + 1$	4	2	1	–	–	–	–
7	$7 = 4 + 1 + 1 + 1$	4	1	1	1	–	–	–
8	$7 = 3 + 3 + 1$	3	3	1	–	–	–	–
9	$7 = 3 + 2 + 2$	3	2	2	–	–	–	–
10	$7 = 3 + 2 + 1 + 1$	3	2	1	1	–	–	–
11	$7 = 3 + 1 + 1 + 1 + 1$	3	1	1	1	1	–	–
12	$7 = 2 + 2 + 2 + 1$	2	2	2	1	–	–	–
13	$7 = 2 + 2 + 1 + 1 + 1$	2	2	1	1	1	–	–
14	$7 = 2 + 1 + 1 + 1 + 1 + 1$	2	1	1	1	1	1	–
15	$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$	1	1	1	1	1	1	1

To understand one method for computing P_n, let us look at the term involving x^7 in the expansion of the product

$$(1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^7) = \prod_{k=1}^{7} (1 + x^k).$$

This gives

$$\begin{aligned}
& (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)(1+x^7) \\
= \; & (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6+x^7+x^{13}) \\
= \; & (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6+x^7+\cdots) \\
= \; & (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5+x^6+x^7+\cdots) \\
= \; & (1+x)(1+x^2)(1+x^3)(1+x^4+x^5+x^6+x^7+\cdots) \\
= \; & (1+x)(1+x^2)(1+x^3+x^4+x^5+x^6+2x^7+\cdots) \\
= \; & (1+x)(1+x^2+x^3+x^4+2x^5+2x^6+3x^7+\cdots) \\
= \; & 1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+\cdots
\end{aligned}$$

where only terms involving x^0 through x^7, inclusive, need to be included. We find then that the term involving x^7 in the expansion of the product

$$\prod_{k=1}^{7}(1+x^k)$$

has a 5 as its coefficient so that

$$\prod_{k=1}^{7}(1+x^k) = 5x^7 + (\text{other powers of } x).$$

Tracking down the five cases in which x^7 appears, we see that x^7 arises from the products

$$x^7 \quad , \quad x^1x^6 \quad , \quad x^2x^5 \quad , \quad x^3x^4 \quad \text{and} \quad x^1x^2x^4$$

and

$$7=7 \quad , \quad 7=1+6 \quad , \quad 7=2+5 \quad , \quad 7=3+4 \quad \text{and} \quad 7=1+2+4$$

are known as the partitions of the number 7 with *distinct* summands. It should be noted that the coefficient of x^6 in

$$\prod_{k=1}^{6}(1+x^k) = 4x^6 + (\text{other powers of } x),$$

which is a 4, comes from

$$x^6 \quad , \quad x^1x^5 \quad , \quad x^2x^4 \quad \text{and} \quad x^1x^2x^3$$

which, when written as

$$6 = 6 \quad , \quad 6 = 1 + 5 \quad , \quad 6 = 2 + 4 \quad \text{and} \quad 6 = 1 + 2 + 3$$

are the partitions of the number 6 with *distinct* summands. It general, it is not hard to show that the coefficient (call it D_n) of x^n in the polynomial

$$\prod_{k=1}^{n}(1 + x^k) = D_n x^n + (\text{other powers of } x),$$

gives the number of partitions of the number n with *distinct* summands. In fact

$$\prod_{k=1}^{n}(1 + x^k) = \sum_{k=0}^{n} D_k x^k + (\text{higher powers of } x),$$

gives the number of partitions of the numbers $k \le n$ with *distinct* summands.

Now can we use polynomial multiplication to get at all partitions P_n of a number n, without the "distinct summands" restriction? The answer is yes provided we choose the correct polynomial for multiplication. For example, consider the product

$$\begin{aligned}
&(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)(1 + x^2 + x^4 + x^6) \\
&\cdot(1 + x^3 + x^6)(1 + x^4)(1 + x^5)(1 + x^6)(1 + x^7).
\end{aligned}$$

The factors not of the form $1 + x^k$ are

$$\begin{aligned}
&1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 \\
&1 + x^1 + x^{1+1} + x^{1+1+1} + x^{1+1+1+1} + x^{1+1+1+1+1} + x^{1+1+1+1+1+1} + x^{1+1+1+1+1+1+1}
\end{aligned}$$

and

$$1 + x^3 + x^6 = 1 + x^3 + x^{3+3} \quad \text{and} \quad 1 + x^2 + x^4 + x^6 = 1 + x^2 + x^{2+2} + x^{2+2+2}$$

Viewing these factors in this way and not altering the factors $1+x^4$, $1+x^5$, $1+x^6$ and $1+x^7$, we see that the coefficient of x^7 in the entire product expansion can be though of as the number of ways of writing the number 7 as the sum of numbers selected from one or more of the following 7 batches, where *at most one* member may be taken from any one batch.

Batch #1	1 $1+1+1+1$ $1+1+1+1+1+1$	$1+1$ $1+1+1+1+1$ $1+1+1+1+1+1+1$	$1+1+1$ $-$ $-$
Batch #2	2	$2+2$	$2+2+2$
Batch #3	3	$3+3$	$-$
Batch #4	4	$-$	$-$
Batch #5	5	$-$	$-$
Batch #6	6	$-$	$-$
Batch #7	7	$-$	$-$

But this is just the number of $P_7 = 15$ partitions of the number 7 as summarized in the following table.

Partition #	Partition of 7\Batch #	#1	#2	#3	#4	#5	#6	#7
1	(7)	0	0	0	0	0	0	1
2	$(6)+(1)$	1	0	0	0	0	1	0
3	$(5)+(2)$	0	1	0	0	1	0	0
4	$(5)+(1+1)$	1	0	0	0	1	0	0
5	$(4)+(3)$	0	0	1	1	0	0	0
6	$(4)+(2)+(1)$	1	1	0	1	0	0	0
7	$(4)+(1+1+1)$	1	0	0	1	0	0	0
8	$(3+3)+(1)$	1	0	1	0	0	0	0
9	$(3)+(2+2)$	0	1	1	0	0	0	0
10	$(3)+(2)+(1+1)$	1	1	1	0	0	0	0
11	$(3)+(1+1+1+1)$	1	0	1	0	0	0	0
12	$(2+2+2)+(1)$	1	1	0	0	0	0	0
13	$(2+2)+(1+1+1)$	1	1	0	0	0	0	0
14	$(2)+(1+1+1+1+1)$	1	1	0	0	0	0	0
15	$(1+1+1+1+1+1+1)$	1	0	0	0	0	0	0

A look at the coefficient of x^7 in the polynomial

$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x^2+x^4+x^6)$$
$$\cdot(1+x^3+x^6)(1+x^4)(1+x^5)(1+x^6)(1+x^7)$$

shows

$$\begin{aligned}&(1+x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x^2+x^4+x^6)\\&\cdot(1+x^3+x^6)(1+x^4)(1+x^5)(1+x^6)(1+x^7)=15x^7+(\text{other powers of } x)\end{aligned}$$

so that $P_7=15$. In general, the coefficient of x^n in the polynomial

$$\left(\sum_{k=0}^{[n/1]} x^{1k}\right)\left(\sum_{k=0}^{[n/2]} x^{2k}\right)\left(\sum_{k=0}^{[n/3]} x^{3k}\right)\left(\sum_{k=0}^{[n/4]} x^{4k}\right)\cdots\left(\sum_{k=0}^{[n/n]} x^{nk}\right)$$

will give the number of partitions of n, so that

$$\left(\sum_{k=0}^{[n/1]} x^{1k}\right)\left(\sum_{k=0}^{[n/2]} x^{2k}\right)\left(\sum_{k=0}^{[n/3]} x^{3k}\right)\cdots\left(\sum_{k=0}^{[n/n]} x^{nk}\right)=P_nx^n+(\text{other powers of } x).$$

In fact,

$$\left(\sum_{k=0}^{[n/1]} x^{1k}\right)\left(\sum_{k=0}^{[n/2]} x^{2k}\right)\left(\sum_{k=0}^{[n/3]} x^{3k}\right)\cdots\left(\sum_{k=0}^{[n/n]} x^{nk}\right)=\sum_{k=0}^{n}P_kx^k+(\text{higher powers of } x)$$

give the number of partitions of all numbers $k\le n$. Note that in this expression $[x]$ is the *greatest integer less than or equal to* x. Note that for $n=7$, we have

$$\begin{aligned}&\left(\sum_{k=0}^{[7/1]} x^{1k}\right)\left(\sum_{k=0}^{[7/2]} x^{2k}\right)\left(\sum_{k=0}^{[7/3]} x^{3k}\right)\left(\sum_{k=0}^{[7/4]} x^{4k}\right)\left(\sum_{k=0}^{[7/5]} x^{5k}\right)\left(\sum_{k=0}^{[7/6]} x^{6k}\right)\left(\sum_{k=0}^{[7/7]} x^{7k}\right)\\
=&\left(\sum_{k=0}^{7} x^{1k}\right)\left(\sum_{k=0}^{3} x^{2k}\right)\left(\sum_{k=0}^{2} x^{3k}\right)\left(\sum_{k=0}^{1} x^{4k}\right)\left(\sum_{k=0}^{1} x^{5k}\right)\left(\sum_{k=0}^{1} x^{6k}\right)\left(\sum_{k=0}^{1} x^{7k}\right)\\
=&(1+x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x^2+x^4+x^6)\\
&\cdot(1+x^3+x^6)(1+x^4)(1+x^5)(1+x^6)(1+x^7)\\
=&17x^7+(\text{other powers of } x).\end{aligned}$$

More generally, it is not hard to show that if a_1, a_2, a_3, ..., a_m are *distinct* positive integers, then the coefficient of x^n in the polynomial expansion

$$\left(\sum_{k=0}^{[n/a_1]} x^{a_1k}\right)\left(\sum_{k=0}^{[n/a_2]} x^{a_2k}\right)\left(\sum_{k=0}^{[n/a_3]} x^{a_3k}\right)\left(\sum_{k=0}^{[n/a_4]} x^{a_4k}\right)\cdots\left(\sum_{k=0}^{[n/a_m]} x^{a_mk}\right)$$

gives the number of partitions of n restricted to only the positive integers: a_1, a_2, a_3, ..., a_m. In fact,

$$\left(\sum_{k=0}^{[n/a_1]} x^{a_1 k}\right)\left(\sum_{k=0}^{[n/a_2]} x^{a_2 k}\right)\left(\sum_{k=0}^{[n/a_3]} x^{a_3 k}\right)\cdots\left(\sum_{k=0}^{[n/a_m]} x^{a_m k}\right) = \sum_{k=0}^{n} R_k x^k + (\text{higher powers of } x)$$

gives the number of partitions of $k \leq n$ restricted to only the positive integers: a_1, a_2, a_3, ..., a_m (denoted by R_k). For example, if $a_1 = 2$, $a_2 = 3$ and $a_3 = 4$ and $n = 7$, then

$$\begin{aligned}\left(\sum_{k=0}^{[7/2]} x^{2k}\right)\left(\sum_{k=0}^{[7/3]} x^{3k}\right)\left(\sum_{k=0}^{[7/4]} x^{4k}\right) &= \left(\sum_{k=0}^{3} x^{2k}\right)\left(\sum_{k=0}^{2} x^{3k}\right)\left(\sum_{k=0}^{1} x^{4k}\right)\\ &= (1+x^2+x^4+x^6)(1+x^3+x^6)(1+x^4)\\ &= 2x^7 + (\text{other powers of } x)\end{aligned}$$

shows that there are 2 partitions of 7 restricted to only the positive integers: 2, 3 and 4, and these are

$$7 = 4+3 \qquad \text{and} \qquad 7 = 3+2+2.$$

As another example, if $a_1 = 1$, $a_2 = 2$ and $a_3 = 5$ and $n = 7$, then

$$\begin{aligned}\left(\sum_{k=0}^{[7/1]} x^{1k}\right)\left(\sum_{k=0}^{[7/2]} x^{2k}\right)\left(\sum_{k=0}^{[7/5]} x^{5k}\right) &= \left(\sum_{k=0}^{7} x^{1k}\right)\left(\sum_{k=0}^{3} x^{2k}\right)\left(\sum_{k=0}^{1} x^{5k}\right)\\ &= (1+x+x^2+x^3+x^4+x^5+x^6+x^7)\\ &\quad \cdot(1+x^2+x^4+x^6)(1+x^5)\\ &= 6x^7 + (\text{other powers of } x)\end{aligned}$$

shows that there are 6 partitions of 7 restricted to only the positive integers: 1, 2 and 5, and these are:

$$7 = 5+2 \quad , \quad 7 = 5+1+1 \quad , \quad 7 = 2+2+2+1$$

and

$$7 = 2+2+1+1+1 \quad , \quad 7 = 2+1+1+1+1+1$$

and

$$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

As one more example, if $a_1 = 2$ and $a_2 = 3$ and $n = 7$, then

$$\begin{aligned}\left(\sum_{k=0}^{[7/2]} x^{2k}\right)\left(\sum_{k=0}^{[7/3]} x^{3k}\right) &= \left(\sum_{k=0}^{3} x^{2k}\right)\left(\sum_{k=0}^{2} x^{3k}\right) \\ &= (1 + x^2 + x^4 + x^6)(1 + x^3 + x^6) \\ &= 1x^7 + (\text{other powers of } x)\end{aligned}$$

shows that there is only 1 partition of 7 restricted to only the positive integers: 2 and 3, and this is $7 = 3 + 2 + 2$.

Example #27: Making Change for a Dollar

It should be noted that the coefficient of x^n in the expansion

$$\left(\sum_{k=0}^{[n/a_1]} x^{a_1 k}\right)\left(\sum_{k=0}^{[n/a_2]} x^{a_2 k}\right)\left(\sum_{k=0}^{[n/a_3]} x^{a_3 k}\right)\left(\sum_{k=0}^{[n/a_4]} x^{a_4 k}\right)\cdots\left(\sum_{k=0}^{[n/a_m]} x^{a_m k}\right)$$

is also the number of solutions in *non-negative* integers to the non-unit equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_mx_m = n.$$

For example, there are 6 solutions in non-negative integers to the non-unit equation

$$x + 2y + 5z = 7$$

and these are summarized in the following table.

(x, y, z)		Partition of 7		
$(0, 1, 1)$	$\Leftrightarrow$	$7 = 2 + 5$	$=$	$1(0) + 2(1) + 5(1)$
$(2, 0, 1)$	$\Leftrightarrow$	$7 = 1 + 1 + 5$	$=$	$1(2) + 2(0) + 5(1)$
$(1, 3, 0)$	$\Leftrightarrow$	$7 = 1 + 2 + 2 + 2$	$=$	$1(1) + 2(3) + 5(0)$
$(3, 2, 0)$	$\Leftrightarrow$	$7 = 1 + 1 + 1 + 2 + 2$	$=$	$1(3) + 2(2) + 5(0)$
$(5, 1, 0)$	$\Leftrightarrow$	$7 = 1 + 1 + 1 + 1 + 1 + 2$	$=$	$1(5) + 2(1) + 5(0)$
$(6, 0, 0)$	$\Leftrightarrow$	$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$	$=$	$1(6) + 2(0) + 5(0)$

We may now answer the problem involving the number of ways there are to give change for a dollar bill using only pennies, nickels, dimes, and quarters for this is equivalent to finding the total number of solutions to the non-unit equation

$$1w + 5x + 10y + 25z = 100.$$

Buy looking at the coefficient of x^{100} in the polynomial

$$\begin{aligned} &\left(\sum_{k=0}^{[100/1]} x^{1k}\right)\left(\sum_{k=0}^{[100/5]} x^{5k}\right)\left(\sum_{k=0}^{[100/10]} x^{10k}\right)\left(\sum_{k=0}^{[100/25]} x^{25k}\right) \\ = &\left(\sum_{k=0}^{100} x^{1k}\right)\left(\sum_{k=0}^{20} x^{5k}\right)\left(\sum_{k=0}^{10} x^{10k}\right)\left(\sum_{k=0}^{4} x^{25k}\right) \end{aligned}$$

which is

$$\left(\sum_{k=0}^{100} x^{1k}\right)\left(\sum_{k=0}^{20} x^{5k}\right)\left(\sum_{k=0}^{10} x^{10k}\right)\left(\sum_{k=0}^{4} x^{25k}\right) = 242x^{100} + (\text{others powers of } x),$$

we see that there are 242 ways of giving change for a dollar bill using pennies, nickels, dimes, and quarters. ■

10. A Very Useful Result in Combinatorics - Optional

Suppose that S is a set of objects with m distinguishable properties P_1, P_2, P_3, ..., P_m, and let S_i (for $i = 1, 2, 3, ..., m$) be the set of all elements in S that have property P_i, so that

$$S_i = \{s \in S | s \text{ has property } P_i\}. \tag{18}$$

Also, let $|S_i|$ be the number of elements in S_i. In general, and object $s \in S$ may have *multiple* properties so let us define $F(t)$ as the sum of all objects in S with t ($t = 1, 2, 3, ..., m$) or more properties with *multiple counting* when more that t properties are involved. Analytically, we write

$$F(t) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m} |s_{i_1} \cap s_{i_2} \cap s_{i_3} \cap \cdots \cap s_{i_t}|, \tag{19}$$

where this sum is over all i_1, i_2, i_3, ..., i_t positive integers that satisfy the set of inequalities

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq m.$$

Finally if we let $E(t)$ be the number of elements of S that satisfy exactly t ($t = 1, 2, 3, ..., m$) of the properties, then let us prove that

$$E(t) = \sum_{j=t}^{m} (-1)^{j-t} \binom{j}{t} F(j). \tag{20}$$

Before proving this result, let us study an example so that it set us in the right frame of mind to understand the proof to follow.

Example #28

Let S be the set of all integers from 0 to 20, inclusive, so that

$$S = \{0, 1, 2, 3, ..., 19, 20\}.$$

Further, suppose we have the following 4 properties

$$\begin{aligned} P_1 &= \text{ an element is even,} \\ P_2 &= \text{ an element is greater than 13,} \\ P_3 &= \text{ an element is prime,} \\ P_4 &= \text{ an element is evenly divisible by 3.} \end{aligned}$$

Then the following table list out the elements of S along with the properties P_1, P_2, P_3 and P_4 that they possess.

$s \in S$	P_1	P_2	P_3	P_4	#	$s \in S$	P_1	P_2	P_3	P_4	#
0	✓	–	–	✓	2	11	–	–	✓	–	1
1	–	–	–	–	0	12	✓	–	–	✓	2
2	✓	–	✓	–	2	13	–	–	✓	–	1
3	–	–	✓	✓	2	14	✓	✓	–	–	2
4	✓	–	–	–	1	15	–	✓	–	✓	2
5	–	–	✓	–	1	16	✓	✓	–	–	2
6	✓	–	–	✓	2	17	–	✓	✓	–	2
7	–	–	✓	–	1	18	✓	✓	–	✓	3
8	✓	–	–	–	1	19	–	✓	✓	–	2
9	–	–	–	✓	1	20	✓	✓	–	–	2
10	✓	–	–	–	1	–	–	–	–	–	–

A check (✓) in this table indicates that the given element has the property whereas a dash ($-$)in this table indicates that the given element does not have the property. Also the column marked with the number sign (#) gives the number of properties that the given element possess. The sets S_i $(i = 1, 2, 3, 4)$ with all possible intersections and with the number of elements in each set is also provided in the next table with the element 18 underlined to help with the discussions.

Set	Elements	#
S_1	$\{0, 2, 4, 6, 8, 10, 12, 14, 16, \underline{18}, 20\}$	11
S_2	$\{14, 15, 16, 17, \underline{18}, 19, 20\}$	7
S_3	$\{2, 3, 5, 7, 11, 13, 17, 19\}$	8
S_4	$\{0, 3, 6, 9, 12, 15, \underline{18}\}$	7
$S_1 \cap S_2$	$\{14, 16, \underline{18}, 20\}$	4
$S_1 \cap S_3$	$\{2\}$	1
$S_1 \cap S_4$	$\{0, 6, 12, \underline{18}\}$	4
$S_2 \cap S_3$	$\{17, 19\}$	2
$S_2 \cap S_4$	$\{15, \underline{18}\}$	2
$S_3 \cap S_4$	$\{3\}$	1
$S_1 \cap S_2 \cap S_3$	$\emptyset$	0
$S_1 \cap S_2 \cap S_4$	$\{\underline{18}\}$	1
$S_1 \cap S_3 \cap S_4$	$\emptyset$	0
$S_2 \cap S_3 \cap S_4$	$\emptyset$	0
$S_1 \cap S_2 \cap S_3 \cap S_4$	$\emptyset$	0

If $F(t)$ is the sum of all elements in S with t $(t = 0, 1, 2, 3, 4)$ or more properties with *multiple counting* when more that t properties are involved, we see that $F(0) = |S| = 21$,

$$F(1) = |S_1| + |S_2| + |S_3| + |S_4| = 11 + 7 + 8 + 7 = 33$$

and

$$F(2) = |S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|$$

resulting in

$$F(2) = 4 + 1 + 4 + 2 + 2 + 1 = 14$$

and

$$F(3) = |S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|$$

resulting in

$$F(3) = 0 + 1 + 0 + 0 = 1$$

and finally

$$F(4) = |S_1 \cap S_2 \cap S_3 \cap S_4| = 0.$$

As a check to Equation (20), we have

$$E(0) = \sum_{j=0}^{4} (-1)^{j-0} \binom{j}{0} F(j)$$

or

$$E(0) = \binom{0}{0} F(0) - \binom{1}{0} F(1) + \binom{2}{0} F(2) - \binom{3}{0} F(3) + \binom{4}{0} F(4)$$

or

$$E(0) = (21) - (33) + (14) - (1) + (0) = 1$$

showing that there is only 1 element in S that has exactly none of the four properties: P_1, P_2, P_3, P_4, namely $\{1\}$. As a further check to Equation (20), we have

$$E(1) = \sum_{j=1}^{4} (-1)^{j-1} \binom{j}{1} F(j)$$

or

$$E(1) = \binom{1}{1}(33) - \binom{2}{1}(14) + \binom{3}{1}(1) - \binom{4}{1}(0) = 8$$

showing that there are 8 elements in S that have exactly one of the four properties: P_1, P_2, P_3, P_4, namely

$$\{4, 5, 7, 8, 9, 10, 11, 13\}.$$

As a further check to Equation (20), we have

$$E(1) = \sum_{j=2}^{4} (-1)^{j-2} \binom{j}{2} F(j)$$

or

$$E(1) = \binom{2}{2}(14) - \binom{3}{2}(1) + \binom{4}{2}(0) = 11$$

showing that there are 11 elements in S that have exactly two of the four properties: P_1, P_2, P_3, P_4, namely

$$\{0, 2, 3, 6, 12, 14, 15, 16, 17, 19, 20\}.$$

As a further check to Equation (20), we have

$$E(3) = \sum_{j=3}^{4} (-1)^{j-3} \binom{j}{3} F(j)$$

or

$$E(3) = \binom{3}{3} F(3) - \binom{4}{3} F(4) = \binom{3}{3}(1) - \binom{4}{3}(0) = 1$$

showing that there is only 1 element in S that has exactly three of the four properties, namely $\{18\}$, and finally we have

$$E(4) = \sum_{j=4}^{4} (-1)^{j-4} \binom{j}{4} F(j) = \binom{4}{4} F(4) = \binom{4}{4}(0) = 0$$

showing that there are no elements in S that have exactly all of the four properties: P_1, P_2, P_3, P_4. This example verifies Equation (20).

Let us investigate what Equation (20) is saying numerically by focusing on one element in S, namely the element 18, and ask what contribution does 18 have to $F(1)$, $F(2)$, $F(3)$, $F(4)$ and $E(1)$, as far as Equation (20) is concerned? Now the element 18 has three of the four properties, namely P_1, P_2 and P_4. Now we know that 18 should contribute nothing to $E(1)$ since $E(1)$ is counting those elements in S that have exactly one of the properties. To see how this is accomplished through the right hand side of Equation (20), let use study what 18 contributes to each of $F(1)$, $F(2)$, $F(3)$ and $F(4)$. In computing

$$F(1) = |S_1| + |S_2| + |S_3| + |S_4|,$$

the element 18 would appear in each of S_1, S_2 and S_4 and therefore contribute 3 to $F(1)$. We know that this same number

$$3 = \binom{3}{1}$$

is just the number of ways of choosing 1 property out of the three properties P_1, P_2 and P_4 that 18 possesses. In computing

$$F(2) = |S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|,$$

the element 18 would appear in the three sets $S_1 \cap S_2$, $S_1 \cap S_4$ and $S_2 \cap S_4$, and therefore contribute 3 to the value of $F(2)$, and this same number

$$3 = \binom{3}{2}$$

is just the number of ways of choosing 2 properties out of the three properties P_1, P_2 and P_4 that 18 possesses. In computing

$$F(3) = |S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|$$

the element 18 would appear in the one set $S_1 \cap S_2 \cap S_4$, and therefore contributes 1 to the value of $F(3)$, and this same number

$$1 = \binom{3}{3}$$

is just the number of ways of choosing 3 properties out of the three properties P_1, P_2 and P_4 that 18 possesses. Finally, in computing

$$F(4) = |S_1 \cap S_2 \cap S_3 \cap S_4|$$

the element 18 would not appear at all and therefore contributes 0 to the value of $F(4)$, and this same number

$$0 = \binom{3}{4}$$

is just the number of ways of choosing 4 properties out of the three properties P_1, P_2 and P_4 that 18 possesses. As a result, the contribution that the element 18 has to $F(1)$ on the right hand side of Equation (20) would be

$$\binom{1}{1}\binom{3}{1} - \binom{2}{1}\binom{3}{2} + \binom{3}{1}\binom{3}{3} - \binom{4}{1}\binom{3}{4} = 0$$

as one would expect since the element 18 does not satisfy exactly one of the four properties P_1, P_2, P_3 and P_4, since it satisfies three of these properties. If we calculate the contributions that each of the other 20 elements in S has to $E(1)$ and then sum over all the results, we should obtain the value of $E(1) = 8$. ■

A Proof of Equation (20) - Optional

Running through the above example suggest that a proof to Equation (20) would proceed as follows. What we want to do is show that the right-hand side of Equation (20) does give the result expected on the left-hand side of Equation (20). Toward this end, we may run through every element of S that should contribute anything to $E(t)$, which counts only those elements in S that have exactly t of the m properties P_1, P_2, ..., P_m. Suppose that s is some element of S that has exactly t' of the properties P_1, P_2, ..., P_m. Let use look at the three cases in which (a) $t' < t$, (b) $t' = t$ and (c) $t' > t$.

If $t' < t$, the s does not appear in the counting of $F(t)$, $F(t+1)$, etc., and so the contribution of s to the right-hand side of Equation (20), and hence to $E(t)$, would be zero, as we would require.

If $t' = t$, then s would appear only in the contribution from $F(t)$ and there it would appear only once and so s contributes 1 to the right-hand side of Equation (20) and hence to $E(t)$, again a result that we would desire.

Finally, if $t' > t$, then s would appear in contributions to

$$F(t), F(t+1), F(t+2), ..., F(t+(t'-t)) = F(t').$$

In the count for $F(t)$, $F(t+1)$, $F(t+2)$, ..., $F(t+(t'-t))$, s would contribute

$$\binom{t'}{t}, \binom{t'}{t+1}, \binom{t'}{t+2}, ..., \binom{t'}{t-(t'-t)},$$

respectively, since

$$\binom{t'}{t}$$

equals the number of ways of choosing t properties out of t' properties that s possesses, and

$$\binom{t'}{t+1}$$

equals the number of ways of choosing $t+1$ properties out of t' properties that s possesses, and

$$\binom{t'}{t+2}$$

equals the number of ways of choosing $t+2$ properties out of t' properties that s possesses, and so on, up to

$$\binom{t'}{t+(t'-t)} = \binom{t'}{t'} = 1,$$

which equals the number of ways of choosing t' properties out of t' properties that s possesses. Therefore, the total contribution of s to the right side of Equation (20) would be

$$\binom{t}{t}\binom{t'}{t} - \binom{t+1}{t}\binom{t'}{t+1} + \binom{t+2}{t}\binom{t'}{t+2} - \cdots + (-1)^{t'}\binom{t'}{t}\binom{t'}{t'}$$

or

$$\sum_{j=t}^{t'}(-1)^{j-t}\binom{j}{t}\binom{t'}{j}.$$

The binomial coefficient identity

$$\sum_{j=a}^{b}(-1)^{j-a}\binom{j}{a}\binom{b}{j} = \begin{cases} 1, & \text{when} \quad a=b \\ 0, & \text{when} \quad a<b \end{cases} \tag{21}$$

then tells us that

$$\sum_{j=t}^{t'}(-1)^{j-t}\binom{j}{t}\binom{t'}{j} = 0$$

since $t' > t$. As a result, we see that when $t' > t$, s would contribute zero to the right-hand side of Equation (20), and hence zero to $E(t)$, which is what we require.

Summarizing, we see that the element s of S contributes a non-zero amount to the right-hand side of Equation (20) only when it has exactly t of the m properties P_1, P_2, ..., P_m, in which case it contributes only 1 to the value of $E(t)$. Summing over all contributions from all the elements in s shows then that the right-hand side of Equation (20) represents the number of elements of S with exactly t of the m properties P_1, P_2, ..., P_m and the proof of Equation (20) is complete.

11. Combinations With Restricted Repetitions - II (Optional)

Using Equation (20) as a guide, we leave to the reader to prove the following result. Let us denote by the symbol

$$\binom{n, R}{d},$$

the number of ways in which d identical balls can be distributed among n identical urns so that no urn receives R or more balls. Note that the procedure for working this out using inclusion/exclusion was already discussed. Using that procedure or Equation (20), it can be shown that, in terms of the binomial coefficients, we have

$$\binom{d, R}{n} = \sum_{j=0}^{[d/R]} (-1)^j \binom{n}{j} \binom{d - jR + n - 1}{n - 1} \tag{22}$$

for $n = 1, 2, 3, \ldots$, $d = 1, 2, 3, \ldots$, $R = 0, 1, 2, \ldots, d$, and $[d/R]$ is the greatest integer less than or equal to d/R. Let us first begin with an example.

Example #29: Distributing Money

Determine the number of ways in which 10 dollars can be distributed among 6 people so that no person receives more than 3 dollars. First let's solve this using Equation (22) in which $d = 10$, $n = 6$ and $R = 3$. This leads to

$$\binom{10, 3}{6} = \sum_{j=0}^{[10/3]} (-1)^j \binom{6}{j} \binom{10 - 3j + 6 - 1}{6 - 1} = 21.$$

As a check to this, we enumerate all the these 21 possibilities in the following table.

P_1	P_2	P_3	P_4	P_5	P_6	P_1	P_2	P_3	P_4	P_5	P_6
0	2	2	2	2	2	2	1	1	2	2	2
2	0	2	2	2	2	2	1	2	1	2	2
2	2	0	2	2	2	2	1	2	2	1	2
2	2	2	0	2	2	2	1	2	2	2	1
2	2	2	2	0	2	2	2	1	1	2	2
2	2	2	2	2	0	2	2	1	2	1	2
1	1	2	2	2	2	2	2	1	2	2	1
1	2	1	2	2	2	2	2	2	1	1	2
1	2	2	1	2	2	2	2	2	1	2	1
1	2	2	2	1	2	2	2	2	2	1	1
1	2	2	2	2	1	–	–	–	–	–	–

where P_1, P_2, ..., P_6 are persons 1 through 6, inclusive. For example, the first row on the left says that person 1 gets 0 dollars and persons 2 through 6, inclusive get 2 dollars each. ■

12. An Introduction to Probability - Informal Approach

Probability Theory is a branch of mathematics that deals with methods for describing the likelihood of the outcomes of future experiments, based on certain assumptions about those experiments. As a simple example, if a coin is perfectly balanced, then we are willing to make the assumption that when the coin is tossed in the air, it is equally likely to land heads up as tails up. In a situation such as this, we would say that the probability that the coin will land with heads up (or tails up) is 1/2.

This type of statement is typical of probability theory. Of course, we can never hope to predict the outcome of a particular experiment with absolute certainty, but we can still obtain very useful information.

Probability theory was first developed, in the late seventeenth and early eighteenth centuries, in an attempt to describe the likelihood of the outcomes of various games of chance, and many of our examples will involve tossing coins, rolling dice, drawing cards, and so on. Let us begin with a few simple definitions.

The set of all possible outcomes of an experiment is called *the sample set* of the experiment. Any subset of the sample set, that is, any set of outcomes, is called an *event*. In this brief introductory section, we will deal exclusively with finite sample sets, that is, sample sets with only a finite number of elements.

Example #30: Sample Sets

(a) Consider the experiment of tossing a coin in the air and observing which side lands face up. The sample set for this experiment is the set $S = \{H, T\}$, where H represents heads and T represents tails. (b) Consider the experiment of rolling a pair of dice. The sample set for this experiment is the set S of all ordered pairs of the form (x, y), where x is the value on the first die, and y is the value on the second die. Thus,

$$S = \{(1,1),(1,2),(1,3),...,(6,4),(6,5),(6,6)\}$$

Note that the size of S is $6^2 = 36$. The event of getting a sum equal to 7 is the set

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

which has 6 elements. ■

Once the sample set for a given experiment has been determined, the next step is to assign probabilities to each of the possible outcomes. 1f the sample set has the form

$$S = \{a_1, a_2, ..., a_n\},$$

then we denote these probabilities by p_1, p_2, p_3, ..., p_n. That is,

$$\text{Probability of outcome } a_1 = P(a_1) = p_1$$

The exact method of assigning probabilities to the outcomes of an experiment depends on the assumptions made about the experiment. For instance, the assumption that a coin is fair is equivalent to the assignments

$$P(H) = \frac{1}{2} \qquad \text{and} \qquad P(T) = \frac{1}{2}.$$

The probabilities that are assigned are normalized to satisfy certain simple criteria. In particular, they must be numbers between 0 and 1,

$$0 \leq p_i \leq 1 \tag{23a}$$

for each $i = 1, 2, 3, ..., n$, and their sum must equal 1,

$$\sum_{i=1}^{n} p_i = 1. \tag{23b}$$

It should be noted that $0 \leq p_i$ for each $i = 1, 2, 3, ..., n$, along with Equation (23b) implies that $p_i \leq 1$ for each $i = 1, 2, 3, ..., n$. Once probabilities have been assigned to each outcome in a sample set, we can assign a probability to any event.

Toward this end, let E be a nonempty event in a sample set S. Then the probability of E, denoted by $P(E)$, is the sum of the probabilities of each outcome in the event. We also set $P(\emptyset) = 0$.

Example #31: Computing Probabilities

Consider the event of rolling two fair dice. The sample set for this event consists of the 36 ordered pairs described in Example #30. Since we are assuming that the dice are fair, the probabilities of each outcome must be equal, that is $p_i = 1/36$ for all $i = 1, 2, 3, ..., 36$. Thus, using the description of E in Example #30, we have

$$P(\text{getting a sum of 7}) = P(E)$$

or

$$P(E) = P((1,6)) + P((2,5)) + P((3,4)) + P((4,3)) + P((5,2)) + P((6,1))$$

or

$$P(E) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

or $P(E) = 1/6$. ■ It is not uncommon for experiments to have the property that each outcome is equally likely. In this case, if the sample set has size n, that is, if $|S| = n$, then the probability of each outcome is $1/n = 1/|S|$. Furthermore, we have

$$P(E) = \frac{|E|}{|S|}. \tag{24}$$

We also leave the proof of the following theorem as an exercise.

Theorem #1

Let S be a finite sample set. Then $P(\emptyset) = 0$, $P(S) = 1$ and $0 \leq P(E) \leq 1$ for all events $E \subseteq S$. If $\bar{E} = S - E$ is the compliment of E in S, then $P(\bar{E}) = 1 - P(E)$. If E and F are events and if $E \cap F = \emptyset$, then

$$P(E \cup F) = P(E) + P(F). \tag{25a}$$

In fact, if E_1, E_2, E_3, ..., E_n are events and if $E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n = \emptyset$, then

$$P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n). \tag{25b}$$

It is very important to keep in mind that Equations (25a,b) hold only when the events are disjoint, that is, only when $E \cap F = \emptyset$, or when $E_i \cap E_j = \emptyset$ for all i and j. In probability (which tends to have its own special vocabulary) *disjoint events* are said to be *mutually exclusive.*

Example #32: Getting Three Aces

Five cards are drawn at random (that is, each with equal probability) from a deck of 52 cards. What is the probability of getting exactly 3 aces? In this case, the sample set is the set of all possible 5-card hands, that is, the set of all combinations of size 5, taken from the set of 52 cards. This set has size

$$|S| = \binom{52}{5} = 2,598,960$$

Now we must compute the size of the event E of getting exactly 3 aces. Such a hand can be formed by first choosing 3 of the 4 aces, and this can be done in

$$\binom{4}{3} = 4$$

and then choosing 2 of the remaining 48 (non-ace) cards, which can be done in

$$\binom{48}{2} = 1,128$$

and so using the multiplication principle,

$$|E| = 4 \times 1128 = 4,512$$

and hence the probability of getting exactly 3 aces is

$$P(E) = \frac{|E|}{|S|} = \frac{4,512}{2,598,960} = \frac{94}{54,145} \simeq 0.001736$$

which is quite small. ■

Example #33: It's Sometimes Better to Compute $P(\bar{E})$ to get $P(E)$

Five digits a_1, a_2, a_3, a_4, a_5 are chosen at random. What is the probability that a_5 is the same as one of the previous 4 digits? The sample set consists of all permutations of size 5, taken from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and so $|S| = 10^5 = 100,000$. Let E be the event that a_5 is the same as one of a_1, a_2, a_3 or a_4. Determining the size of E is a bit awkward. In this case, it turns out to be easier to first determine the size of $\bar{E}$, which is the event that a_5 is different from a_1, a_2, a_3 and a_4 To determine the size of $\bar{E}$, we observe that there are 10 possibilities for a_5 , but then a_1, a_2, a_3 and a_4 must be taken from the 9 remaining digits. Thus $|E| = 10 \times 9^4 = 65,610$, and then

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{65,610}{100,000} = \frac{6561}{10000}$$

which says that

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{6561}{10000} = \frac{3439}{10000},$$

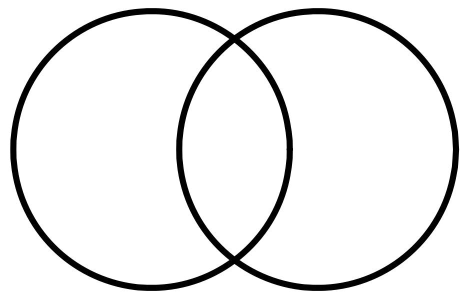
or $P(E) = 0.3439$. ■

Using Inclusion-Exclusion to Compute Probabilities

The principle of inclusion-exclusion for two sets says that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Using the following figure



Set A is on the left and set B is on the right

we may write

$$A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$$

which $A \cap \bar{B}$, $A \cap B$ and $\bar{A} \cap B$ all disjoint, we may say that

$$P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B).$$

But $A = (A \cap \bar{B}) \cup (A \cap B)$ and $(A \cap \bar{B}) \cap (A \cap B) = \emptyset$, and so

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

which says that

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

and putting this into the expression for $P(A \cup B)$, we have

$$P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(\bar{A} \cap B)$$

or

$$P(A \cup B) = P(A) + P(\bar{A} \cap B).$$

We also have $B = (\bar{A} \cap B) \cup (A \cap B)$ and $(\bar{A} \cap B) \cap (A \cap B) = \emptyset$, and so

$$P(B) = P(\bar{A} \cap B) + P(A \cap B)$$

or

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

and putting this into

$$P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

leads to

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{26}$$

Example #34: Computing $P(E \cup F)$

Five cards are chosen at random from a deck of 52 cards. What is the probability that there is at least one spade and at least one club? Let E be the probability that no spade has been drawn, and let F be the probability that no club has been drawn. We seek $P(\bar{E} \cap \bar{F})$. But

$$\bar{E} \cap \bar{F} = \overline{E \cup F}$$

and so

$$P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F}) = 1 - P(E \cup F)$$

and

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

and so

$$P(\bar{E} \cap \bar{F}) = 1 - P(E) - P(F) + P(E \cap F).$$

Now to compute $P(E)$, we must remove all 13 spades from the deck of 52 cards and write

$$P(E) = \frac{\binom{52-13}{5}}{\binom{52}{5}} = \frac{\binom{39}{5}}{\binom{52}{5}} = \frac{575757}{2598960} = \frac{2109}{9520}$$

and to compute $P(F)$, we must remove all 13 clubs from the deck of 52 cards and write

$$P(F) = \frac{\binom{52-13}{5}}{\binom{52}{5}} = \frac{\binom{39}{5}}{\binom{52}{5}} = \frac{575757}{2598960} = \frac{2109}{9520}$$

and to compute $P(E \cap F)$, we must remove all 13 spades and 13 clubs from the deck of 52 cards and write

$$P(E \cap F) = \frac{\binom{52-13-13}{5}}{\binom{52}{5}} = \frac{\binom{26}{5}}{\binom{52}{5}} = \frac{65780}{2598960} = \frac{253}{9996}.$$

Thus we find that

$$P(\bar{E}\cap\bar{F}) = 1 - \frac{2109}{9520} - \frac{2109}{9520} + \frac{253}{9996} = \frac{58201}{99960}$$

or $P(\bar{E}\cap\bar{F}) = 0.582$. ■

Independent Events

We say that two events E and F are independent if

$$P(E\cap F) = P(E)\times P(F) \tag{27}$$

Intuitively speaking, two events are independent if knowing whether one event occurs gives us no information about whether or not the other event will occur.

Example #35: Independent Events

Suppose we toss a fair coin twice. Let E be the event that the first toss results in heads, and let F be the event that the second toss results in heads. Then $P(E) = P(F) = 1/2$ and since $E\cap F$ is the event that both tosses result in heads, we have $P(E\cap F) = 1/4$. Thus we see that $P(E\cap F) = P(E)\times P(F)$ and so the events E and F are independent. ■

Example #36: Independent Events

A card is chosen at random from a deck of 52 cards. Let E be the event that the card is an ace or a deuce (2), and let F be the event that the card is an ace, king, queen or jack. Then $P(E) = 8/52 = 2/13$ and $P(F) = 16/52 = 4/13$. But since $E\cap F$ is the event that the card chosen is an ace, we have $P(E\cap F) = 4/52 = 1/13$. Now

$$P(E)\times P(F) = \frac{2}{13}\times\frac{4}{13} = \frac{8}{169} \neq P(E\cap F)$$

and so the events E and F are not independent. ■

Examples From Information Theory

Let us conclude this chapter with some examples taken from information theory. Suppose that we are sending data, in the form of 0's and l's, over a noisy communications line. Assume that, because of the noise, the probability that a bit (0 or 1) is received correctly is p, where $1/2 \leq p \leq 1$. Assume also that the event that one bit is received correctly is independent of the event that another bit is received correctly.

Example #37: Independent Events

(a) What is the probability that a message consisting of 11 bits will be received correctly? (b) What is the probability that exactly k of the n bits are received correctly? (c) What is the probability that at least k bits are received correctly?

Solutions: (a) If E_i is the event that the ith bit is received correctly, then $P(E_i) = p$ for each $i = 1, 2, 3, ..., n$, and since E_1, E_2, E_3, ..., E_n are independent, we have

$$P(\text{message received correctly}) = P(E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_n)$$

or

$$P(\text{message received correctly}) = P(E_1) \times P(E_2) \times P(E_3) \times \cdots \times P(E_n) = p^n.$$

(b) Consider the case where the first k bits are received correctly, and the rest are not. The probability of this occurring is $p^k(1-p)^{n-k}$. But this probability would be the same if any set of k bits were received correctly, and since there are

$$\binom{n}{k}$$

possibilities for the k correct bits, we have

$$P(\text{exactly } k \text{ correct bits out of } n) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, ..., n$. (c) From part (b), we have

$$P(\text{at least } k \text{ bits are received correctly}) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

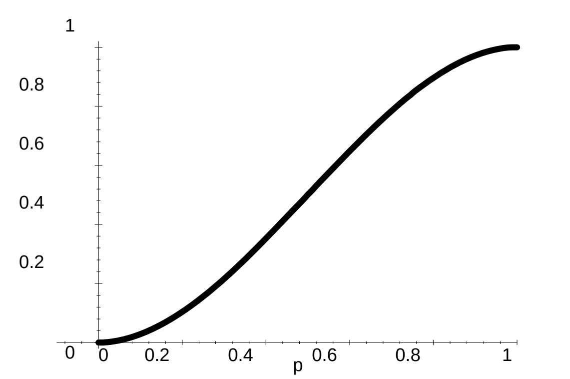
since the sets "exactly k correct bits out of n" are all disjoint. ■

Example #38: Independent Events

Suppose that, in the hope of increasing the probability of each bit being received correctly, we send it three times (in succession). Then, if exactly one error occurs in the three tries, we can (by taking the majority) still get the correct bit. What is the probability of a single bit being received (that is, interpreted) correctly? The probability that all three bits are sent correctly is p^3, and the probability of exactly one error is $3p^2(1-p)$. Hence, the probability of being able to correctly interpret that bit is

$$p^3 + 3p^2(1-p) = p^2(3-2p)$$

A plot of $p^2(3-2p)$ versus p is shown below



Plot of $p^2(3-2p)$ verses p

Here are some values for this probability

p	$p^2(3-2p)$
0.5	0.500
0.6	0.648
0.7	0.784
0.8	0.896
0.9	0.972
1.0	1.000

showing it to be pretty large. ■