Probability and Statistics (ENM 503)

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Chapter 6 - Continuous Random Variables

The following notes are based on the textbook entitled: *A First Course in Probability* by Sheldon Ross (9th edition) and these notes can be viewed at

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1. Range Spaces and Probability Density Functions

In this chapter we shall discuss some probability terminology and concepts for continuous distributions. If the range space R_X of a random variable X is a continuous interval or a union of continuous intervals, then X is called a continuous random variable. For a continuous random variable X, the probability that a value of X lies in a region R is given by

$$P(X \in R) = \int_R f(x)dx \tag{1}$$

where f is called the *probability density function* (pdf) of the random variable X, and the integration is over all points in the region R. When R is the interval $a \leq x \leq b$, then we write

$$P(a \leq x \leq b) = \int_a^b f(x)dx, \tag{2a}$$

and so

$$P(a \leq x \leq a) = P(x = a) = \int_a^a f(x)dx = 0 \tag{3a}$$

for any value of a, and also

$$P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b) = P(a < x < b) \tag{3b}$$

for all values of a and b. The fact that the probability of $X = a$ equals zero does not mean that this event can never occur. It is not likely to occur, but it can still occur.

By setting $a = x$ and $b = x + \Delta x$ in Equation (2) we see that

$$P(x \leq X \leq x + \Delta x) = \int_x^{x+\Delta x} f(x)dx = f(x)\Delta x \tag{4}$$

for small Δx and so we may view f as a *probability per length* which is why it is called a probability *density* function. Note that all pdfs must satisfy the three conditions: (i) $f(x) \geq 0$ for all x in R_X, (ii)

$$\int_{x \in R_X} f(x)dx = 1, \tag{5}$$

and (iii) $f(x) = 0$ for $x \notin R_X$.

Example #1: A Continuous Distribution

The lifetime of a laser-ray device (in years) used to inspect cracks in aircraft wings is given by the random variable X with pdf

$$f(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ \\ \frac{1}{2}e^{-x/2}, & \text{for} \quad 0 \leq x \end{cases}.$$

The probability that the lifetime of the laser-ray device is between 2 and 3 years is then given by

$$P(2 \leq X \leq 3) = \int_2^3 \frac{1}{2}e^{-x/2}dx = e^{-1} - e^{-3/2}.$$

or $P(2 \leq X \leq 3) \simeq 14.5\%$. ■

Mixed Random Variables

If the range space R_X of a random variable X is the union of discrete (D) sets and continuous (C) sets, then X is called a mixed random variable. For example

$$R_X = R_X^{\mathrm{D}} \cup R_X^{\mathrm{C}} = \{1, 2, 3\} \cup [4, 5]$$

is the set consisting of the integers 1, 2, and 3, as well as all the real numbers between 4 and 5, inclusive. Here we must have a pmf ($p(x)$) defined for all points in R_X^{D} and a pdf ($f(x)$) defined for all points in R_X^{C} so that (i) $p(x) \geq 0$ for all $x \in R_X^{\mathrm{D}}$ and $f(x) \geq 0$ for all $x \in R_X^{\mathrm{C}}$, and (ii)

$$\sum_{x \in R_X^{\mathrm{D}}} p(x) + \int_{x \in R_X^{\mathrm{C}}} f(x)dx = 1 \tag{6}$$

and (iii) $p(x) = 0$ for all $x \notin R_X^{\mathrm{D}}$ and $f(x) = 0$ for all $x \notin R_X^{\mathrm{C}}$.

Example #2: A Mixed Distribution

A box contains light bulbs that have lifetimes (in years) given by the random variable X. The probability that a given light bulb in the box is already dead is 0.1. This says that the lifetime of a light bulb chosen at random from the box satisfies $p(0) = 0.1$ and suppose that

$$f(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ Ae^{-x/2}, & \text{for} \quad 0 < x \end{cases} .$$

Then, since

$$p(0) + \int_0^\infty f(x)dx = 1 \qquad \text{we have} \qquad 0.1 + \int_0^\infty Ae^{-x/2}dx = 1$$

resulting in $A = 0.45$, and so

$$f(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ 0.1 & \text{for} \quad x = 0 \\ 0.45e^{-x/2}, & \text{for} \quad 0 < x \end{cases} .$$

The probability that the lifetime of a light bulb is between 2 and 3 years is then given by

$$P(2 \leq X \leq 3) = (0.45) \int_2^3 e^{-x/2} dx = (0.9)(e^{-1} - e^{-3/2}),$$

or $P(2 \leq X \leq 3) \simeq 13\%$. ∎

Cumulative Distribution Function

The cumulative distribution function (cdf) of a random variable X, denoted by $F(x)$, equals the probability that the random variable (discrete or continuous) X assumes a value less than or equal to x, *i.e.*, $F(x) = P(X \leq x)$ so that

$$F(x) = \sum_{x_i \leq x} p(x_i) \tag{7a}$$

when X is discrete, while

$$F(x) = \int_{t \leq x} f(t) dt = \int_{-\infty}^{x} f(t) dt \tag{7b}$$

when X is continuous.

It should be noted that if X is a continuous random variable with pdf $f(x)$, then

$$\frac{dF(x)}{dx} = f(x). \tag{7c}$$

Note that $F(x)$ must then satisfy the following three properties: (i) F is a non-decreasing function of x, so that $F(a) \leq F(b)$ whenever $a \leq b$, (ii)

$$\lim_{x \to -\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} F(x) = 1 \tag{8}$$

and (iii)

$$P(a < X \leq b) = F(b) - F(a) \tag{9}$$

for all $a < b$.

Example #3: Computing A Cumulative Distribution Function

Suppose that the pdf of a continuous distribution for a random variable X is given by

$$f(x) = Ae^{-\lambda|x|}$$

for $-\infty < \lambda < +\infty$ and for $\lambda > 0$. Let us determine the value of A (in terms of λ) and then determine the cdf of X. Toward this end, we use the fact that

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \qquad \text{we have} \qquad \int_{-\infty}^{+\infty} Ae^{-\lambda|x|}dx = 1$$

or

$$\int_{-\infty}^{0} Ae^{-\lambda|x|}dx + \int_{0}^{+\infty} Ae^{-\lambda|x|}dx = 1$$

or

$$\int_{-\infty}^{0} Ae^{-\lambda(-x)}dx + \int_{0}^{+\infty} Ae^{-\lambda x}dx = 1$$

or

$$\left.\frac{Ae^{\lambda x}}{\lambda}\right|_{-\infty}^{0} + \left.\frac{Ae^{\lambda x}}{-\lambda}\right|_{0}^{+\infty} = 1$$

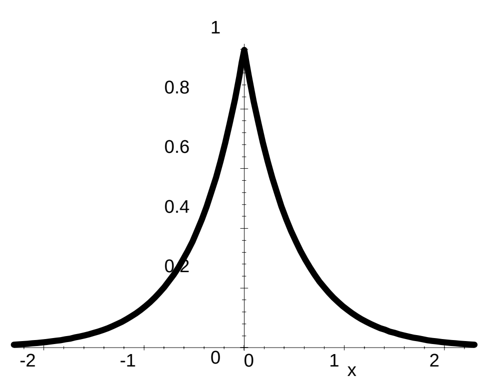
or

$$\frac{A}{\lambda} + \frac{A}{\lambda} = 1 \qquad \text{which says that} \qquad A = \frac{1}{2}\lambda.$$

Thus we find that

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$$

for $-\infty < x < +\infty$. A typical plot of f versus x is shown in the following figure.



Typical plot of f versus x

The cdf of X is given by

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{2}\lambda e^{-\lambda|t|}dt$$

For $-\infty < x < 0$, we have

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}\lambda e^{-\lambda(-t)}dt = \int_{-\infty}^{x} \frac{1}{2}\lambda e^{\lambda t}dt = \frac{1}{2}e^{\lambda t}\bigg|_{-\infty}^{x} = \frac{1}{2}e^{\lambda x}$$

and for $0 \le x < +\infty$, we have

$$F(x) = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{-\lambda(-t)}dt + \int_{0}^{x} \frac{1}{2}\lambda e^{-\lambda(t)}dt = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda t}dt + \int_{0}^{x} \frac{1}{2}\lambda e^{-\lambda t}dt$$

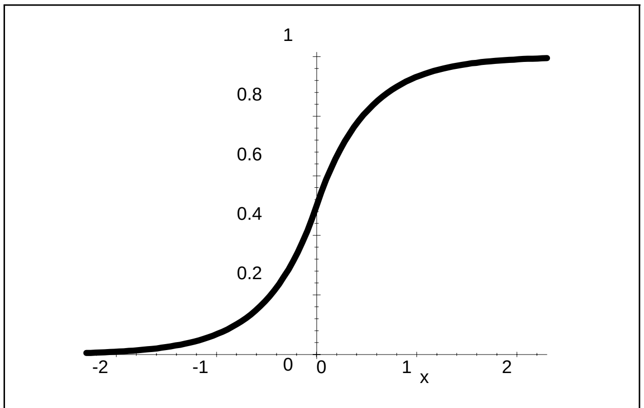
or

$$F(x) = \frac{1}{2}e^{\lambda t}\bigg|_{-\infty}^{0} - \frac{1}{2}e^{-\lambda t}\bigg|_{0}^{x} = \frac{1}{2} - \frac{1}{2}e^{-\lambda x} + \frac{1}{2} = 1 - \frac{1}{2}e^{-\lambda x}.$$

Thus we find that

$$F(x) = \frac{1}{2} \times \begin{cases} e^{\lambda x}, & \text{for } -\infty < x \le 0 \\ 2 - e^{-\lambda x}, & \text{for } 0 \le x < +\infty \end{cases}.$$

A typical plot of F versus x is shown in the following figure,



Typical plot of F versus x

and if shows that $F(-\infty) = 0$ and $F(+\infty) = 1$. ■

Example #4: A Problem in Reliability

The lifetime in hours of a certain kind of radio tube is a random variable having a pdf given by

$$f(x) = 100/x^2$$

for $x > 100$ and $f(x) = 0$ for $x \leq 100$. What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events E_i for $i = 1, 2, 3, 4, 5$ that the ith tube will have to be replaced within this time are independent. From the statement of the problem we see that

$$P(E_i) = \int_{100}^{150} \frac{100}{x^2} dx = \frac{1}{3}$$

and then the probability we seek is binomial with $n = 5$ and $p = 1/3$ as so we have

$$P = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^{5-2} = \frac{80}{243}$$

or $P \simeq 32.9\%$. ■

2. Functions of Random Variables

Suppose that Y is a random variable having known pdf $h(y)$ and known cdf $H(y)$, and suppose that X is a random variable related to Y through some known *monotonic* function g, so that

$$X = g(Y).$$

Note that a function g is monotonic if it is always increasing or always decreasing. Our goal in this section is to discuss how to compute the pdf of X from the pdf of Y. To determine the pdf $f(x)$ of the random variable X we start with its cdf $F(x)$ and write

$$F(x) = P(X \leq x) = P(g(Y) \leq x).$$

If g is always increasing, this yields

$$F(x) = P(Y \leq g^{-1}(x))$$

where g^{-1} is the inverse function of g. Therefore we see that

$$F(x) = P(Y \leq g^{-1}(x)) = H(g^{-1}(x)) = H(y)$$

with $y = g^{-1}(x)$, and so

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}H(y)$$

which via the chain rule becomes

$$f(x) = \frac{d}{dy}H(y)\frac{dy}{dx} = \frac{H'(y)}{dx/dy} = \frac{h(y)}{g'(y)}.$$

Therefore we see that

$$f(x) = \frac{h(y)}{g'(y)} \tag{10a}$$

when g is always *increasing*. On the other hand, when g is always decreasing, then

$$F(x) = P(g(Y) \leq x) = P(Y \geq g^{-1}(x)) = 1 - P(Y < g^{-1}(x))$$

or

$$F(x) = 1 - H(y)$$

with $y = g^{-1}(x)$, and so

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1 - H(y))$$

which via the chain rule becomes

$$f(x) = \frac{d}{dy}(1 - H(y))\frac{dy}{dx} = \frac{-H'(y)}{dx/dy} = \frac{-h(y)}{g'(y)}.$$

Therefore we see that

$$f(x) = \frac{h(y)}{-g'(y)} \tag{10b}$$

when g is always *decreasing*. To combine these into one expression, we note that when $g(y)$ is always increasing, $g'(y) > 0$, and when $g(y)$ is always decreasing, then $g'(y) < 0$, and so we may write both expressions for $f(x)$ as

$$f(x) = \frac{h(y)}{|g'(y)|}. \tag{10c}$$

To summarize we see that if Y is a random variable having known pdf $h(y)$ and if X is a random variable related to Y via $X = g(Y)$, for some monotonic function g, then the pdf of X is given by

$$f(x) = \frac{h(y)}{|g'(y)|} = \frac{h(g^{-1}(x))}{|g'(g^{-1}(x))|}. \tag{10d}$$

Example #5: A Uniform Distribution

As a special case of Equation (10d), suppose that Y is a random variable with pdf

$$h(y) = \begin{cases} 0, & \text{for} \quad y < 0 \\ 1, & \text{for} \quad 0 \le y < 1 \\ 0, & \text{for} \quad 1 \le y \end{cases}$$

and suppose that $X = g(Y)$. Then since $0 < Y < 1$, we have

$$R_X = \{g(0) \le X < g(1)\}$$

if g is increasing, or

$$R_X = \{g(1) \le X < g(0)\}$$

if g is decreasing, and hence

$$f(x) = \frac{h(y)}{|g'(y)|} = \frac{1}{|g'(y)|} = \frac{1}{|g'(g^{-1}(x))|}.$$

Specifically, when $X = a + (b-a)Y$ (and $a < b$), then $g(y) = a + (b-a)y$ and $g'(y) = (b-a)$, so that

$$R_X = \{x|a + (b-a)(0) \le x < a + (b-a)(1)\} = \{x|a \le x < b\}$$

and

$$f(x) = \frac{h(y)}{|g'(y)|} = \frac{1}{b-a}.$$

Example #6: A Uniform Distribution and a Cubic Function

For an example, suppose that $R_Y = \{y|0 < y < 1\}$ with $h(y) = 1$, and suppose that $X = h(Y) = Y^3$, then

$$R_X = \{x|(0)^3 \le x < (1)^3\} = \{x|0 \le x < 1\}$$

and $g(y) = y^3$, resulting in $y = g^{-1}(x) = x^{1/3}$, and $g'(y) = 3y^2$, so that

$$f(x) = \frac{h(y)}{|g'(y)|} = \frac{h(y)}{|3y^2|} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3}x^{-2/3}$$

for $0 \leq x < 1$. ■

Example #7: A Linear Function Mapping $[d, c] \rightarrow [a, b]$

Suppose that Y is a random variable with pdf $h(y)$ and defined on the interval $[d, c]$, and suppose that X is a random variable defined on a interval $[a, b]$ and suppose that

$$X = g(Y) = b + \left(\frac{b-a}{d-c}\right)(Y - c)$$

Then

$$R_X = \{x | g(c) \leq x < g(d)\} = \{x | a \leq x \leq b\}$$

and hence

$$f(x) = \frac{h(y)}{|g'(y)|} = \frac{h(y)}{\left|\frac{b-a}{d-c}\right|} = \left(\frac{d-c}{b-a}\right) h(y).$$

But from

$$x = a + \left(\frac{b-a}{d-c}\right)(y - c) \quad \text{we have} \quad y = c + \left(\frac{d-c}{b-a}\right)(x - a)$$

and so

$$f(x) = \left(\frac{d-c}{b-a}\right) h\left(c + \left(\frac{d-c}{b-a}\right)(x-a)\right).$$

Note that for the special case when $c = 0$ and $d = 1$, this becomes

$$f(x) = \left(\frac{1}{b-a}\right) h\left(\frac{x-a}{b-a}\right).$$

As an example suppose that Y is a random variable in the unit interval $[0, 1]$ with

$$h(y) = \frac{(\alpha + \beta + 1)!}{\alpha!\beta!} y^\alpha (1-y)^\beta,$$

then

$$f(x) = \left(\frac{1}{b-a}\right) \frac{(\alpha + \beta + 1)!}{\alpha!\beta!} \left(\frac{x-a}{b-a}\right)^\alpha \left(1 - \frac{x-a}{b-a}\right)^\beta$$

or simply

$$f(x) = \frac{(\alpha + \beta + 1)!}{\alpha!\beta!(b-a)^{\alpha+\beta+1}} (x-a)^\alpha (b-x)^\beta$$

for $a \leq x \leq b$. ■

3. Expectation and Other Moments

If X is a random variable with range space R_X, then the expectation value of X (if it exist) is defined by

$$E(X) = \sum_{x_i \in R_X} x_i p(x_i) \tag{11a}$$

if X is discrete,

$$E(X) = \int_{x \in R_X} x f(x) dx. \tag{11b}$$

if X is continuous, and

$$E(X) = \sum_{x_i \in R_X^{\mathrm{D}}} x_i p(x_i) + \int_{x \in R_X^{\mathrm{C}}} x f(x) dx \tag{11c}$$

if X is mixed, and it is easy to show that

$$E(\alpha X + \beta) = \alpha E(X) + \beta.$$

Example #8: A Continuous Distribution in Which $E(X)$ Does Not Exist

Note that $E(X)$ need not exist for every continuous distribution. For example, consider the pdf

$$f(x) = \frac{2/\pi}{x^2 + 1}$$

of the range space $R_X = \{x \mid 0 < x < +\infty\}$, then

$$E(X) = \int_0^{+\infty} x \frac{2/\pi}{x^2 + 1} dx = \frac{2}{\pi} \int_0^{+\infty} \frac{x}{x^2 + 1} dx$$

which also does not converge. ■ Note that when $E(X)$ does exist, then this is call the *mean* μ of the random variable.

If X is a continuous random variable with probability density function $f(x)$, then, for any real-valued function g,

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x)f(x)dx. \tag{12}$$

If n is a nonnegative integer, the quantity

$$E((X-c)^n) = \sum_{x_i \in R_X} (x_i - c)^n p(x_i) \tag{13a}$$

if X is discrete,

$$E((X-c)^n) = \int\limits_{x \in R_X} (x-c)^n f(x)dx. \tag{13b}$$

if X is continuous, and

$$E((X-c)^n) = \sum_{x_i \in R_X^{\mathrm{D}}} (x_i - c)^n p(x_i) + \int\limits_{x \in R_X^{\mathrm{C}}} (x-c)^n f(x)dx \tag{13c}$$

if X is mixed (if its exist), is called the n-th moment of X about the point c. If $c = 0$, then this is called the nth moment of X and if $c = \mu$, this is called the nth moment of X about the mean μ.

The Variance of a Random Variable X

The second moment of X about its mean μ is called the *variance* of X, and is given by

$$\begin{aligned} V(X) &= E((X-\mu)^2) = E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2, \end{aligned}$$

so that

$$V(X) = E(X^2) - (E(X))^2. \tag{14a}$$

The *standard deviation* of X is given by

$$\sigma = \sqrt{V(X)}. \tag{14b}$$

Note that the mean of X is a measure of the *central tendency* of X and the variance of X is a measure of the *spread* or variation of possible values of X around the mean. Note that μ and σ are measured in the same units.

Modes

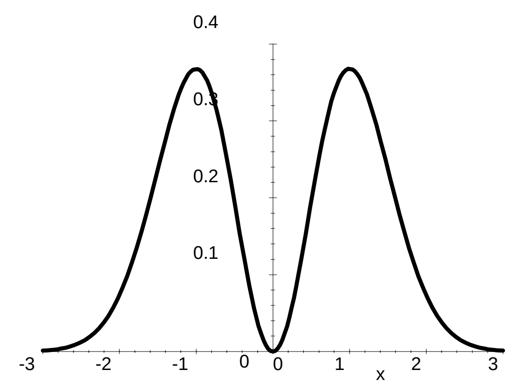
A mode of a random variable X is a value of X that occurs most frequently (if X is discrete), or it's a value of X where the pdf is a maximum (if X is continuous). Modes are not defined for mixed distributions.

Example #9: Modes Need Not Be Unique

Note that a mode need not be unique. For example, consider the bimodal distribution

$$f(x) = x^2 e^{-x^2}$$

for $-\infty < x < +\infty$. A plot of $f(x)$ versus x is shown in the figure below



Plot of $f(x) = x^2 e^{-x^2}$ versus x

and this shows that $x = -1$ and $x = +1$ are both modes for the continuous random variable X and $f'(-1) = f'(1) = 0$. ■

4. Some Important Continuous Distributions

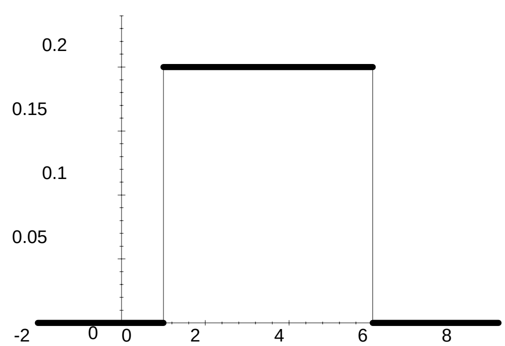
Continuous random variables can be used to describe random phenomena in which the variable of interest can take on any value in some interval or collection of intervals, *e.g.*, the time to failure for a light bulb. Ten important distributions are described in the following subsections.

4.1 The Uniform Distribution Over The Interval $[a, b)$

A random variable X is uniformly distributed over the interval $[a, b)$ if its pdf is given by

$$f(x) = \begin{cases} 1/(b-a), & \text{for} \quad a \le x < b \\ \\ 0, & \text{for} \quad \text{otherwise} \end{cases} \sim U[a, b). \tag{15a}$$

A plot of this for the interval $[1, 6)$ is shown below.

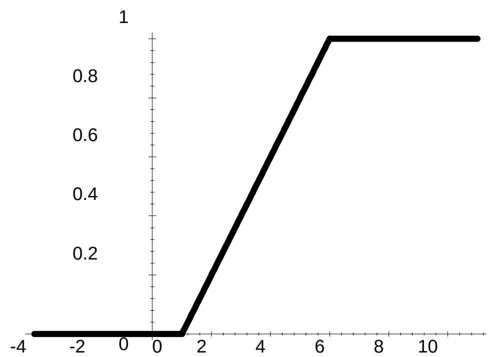


Plot of f versus x for $f(x) \sim U[1, 6)$

The cdf is given by

$$F(x) = P(X \le x) = \begin{cases} 0, & \text{for} \quad x \le a \\ \\ (x-a)/(b-a), & \text{for} \quad a \le x \le b \\ \\ 1, & \text{for} \quad b \le x \end{cases}. \tag{15b}$$

A plots of this for the interval $[1, 6)$ is shown below.



Plot of f versus x for $f(x) \sim U[1, 6)$

and it shows that $F(-\infty) = 0$ and $F(+\infty) = 0$.

Note that for the uniform distribution

$$P(x_1 < X < x_2) = F(x_2) - F(x_1) = \frac{x_2 - a}{b - a} - \frac{x_1 - a}{b - a}$$

or

$$P(x_1 < X < x_2) = \frac{x_2 - x_1}{b - a} \tag{15c}$$

is proportional to the length of the interval, for all x_1 and x_2 satisfying

$$a \leq x_1 \leq x_2 \leq b.$$

The first and second moments are given by

$$\mu \equiv E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \left(\frac{1}{b - a} \right) dx,$$

or

$$E(X) = \frac{a + b}{2}, \tag{15d}$$

(the midpoint of the interval (a, b)), and

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b x^2 \frac{1}{b - a} dx = \frac{a^2 + ab + b^2}{3}$$

so that the variance is

$$\sigma^2 \equiv V(X) = E(X^2) - (E(X))^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2,$$

or

$$V(X) = \frac{(b-a)^2}{12}. \tag{15e}$$

The uniform distribution plays a vital role in simulation. We shall see that random numbers, uniformly distributed between 0 and 1, provide the means to generate random events. We shall see how to generate random numbers and random events in a later chapter.

Example #10: Arrivals And A Uniform Distribution

A bus arrives every 20 minutes at a specified stop beginning at 6:40 AM and continuing until 8:40 AM. A certain passenger does not know the schedule, but arrives randomly between 7:00 AM and 7:30 AM every morning. Determine the probability that this passenger waits more than 5 minutes for a bus.

The passenger has to wait more than five minutes only if her arrival time is between 7:00 AM and 7:15 AM or between 7:20 AM and 7:30 AM. If X is a random variable for the number of minutes past 7:00 AM that this passenger arrives, then the desired probability is

$$P = P(0 < X < 15) + P(20 < X < 30).$$

Assuming that X is uniformly distributed on the interval $(0, 30)$, we then have

$$P = \frac{15-0}{30-0} + \frac{30-20}{30-0},$$

or $P = 5/6$. ■

Example #11: Using Monte-Carlo To Test A Derived Formula

Let R be a random variable from the uniform distribution $U[0, 1)$, and let X be the random variable computed using

$$X = \frac{\min(R, 1-R)}{\max(R, 1-R)}.$$

We want to determine the *expected value* of X, and we do this by first computing the cdf of X using

$$F(x) = P(X \leq x),$$

for $0 \leq x \leq 1$, and then use this to compute the pdf of X using $f(x) = F'(x)$. Note also that in computing $F(x)$, we will consider the cases: (a) $R \leq 1/2$ and (b) $R > 1/2$ separately. If $R \leq 1/2$, then $\min(R, 1-R) = R$, and $\max(R, 1-R) = 1-R$, and

$$X = \frac{R}{1-R}$$

and setting this less or equal to x gives

$$\frac{R}{1-R} \leq x \qquad \text{or} \qquad R \leq \frac{x}{x+1}.$$

On the other hand, if $R > 1/2$, then $\min(R, 1-R) = 1-R$, and $\max(R, 1-R) = R$, and

$$X = \frac{1-R}{R}$$

and setting this less or equal to x gives

$$\frac{1-R}{R} \leq x \qquad \text{or} \qquad R \geq \frac{1}{x+1}.$$

Therefore we have

$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= P(X \leq x \,|R \leq 1/2)P(R \leq 1/2) + P(X \leq x \,|R > 1/2)P(R > 1/2) \\
&= P\left(R \leq \frac{x}{x+1}\middle| R \leq 1/2\right)\frac{1}{2} + P\left(R \geq \frac{1}{x+1}\middle| R > 1/2\right)\frac{1}{2} \\
&= \left(\frac{2x}{x+1}\right)\frac{1}{2} + 2\left(1 - \frac{1}{x+1}\right)\frac{1}{2}
\end{aligned}$$

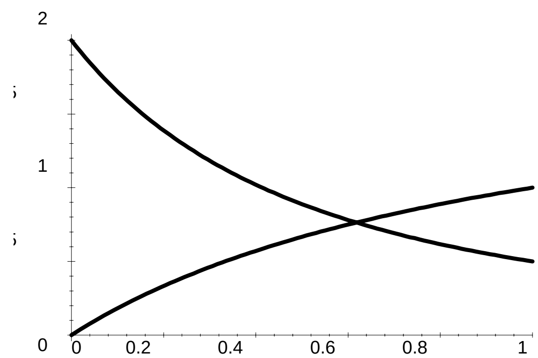
which reduces to

$$F(x) = \frac{2x}{x+1}$$

for $0 \leq x \leq 1$. Then

$$f(x) = F'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

for $0 \leq x \leq 1$ and $f(x) = 0$ elsewhere. Plots of $f(x)$ and $F(x)$ versus x for $0 \leq x \leq 1$ are shown in the figure below.



Plot of $f(x)$ and $F(x)$ versus x for $0 \leq x \leq 1$

Using the pdf $f(x)$, we then have

$$\begin{aligned} E(X) &= \int_0^1 x f(x) dx = \int_0^1 \frac{2x}{(x+1)^2} dx = 2\ln(x+1) + \frac{2}{x+1}\bigg|_0^1 \\ &= \left(2\ln(2) + \frac{2}{2}\right) - \left(2\ln(1) + \frac{2}{1}\right) \end{aligned}$$

which leads to

$$E(X) = 2\ln(2) - 1 \simeq 0.3863.$$

As a check, we may compute $E(X)$ directly as

$$E(X) = \int_0^1 X f(R) dR = \int_0^1 X dR = \int_0^1 \frac{\min(R, 1-R)}{\max(R, 1-R)} dR$$

or

$$\begin{aligned} E(X) &= \int_0^{1/2} \frac{\min(R, 1-R)}{\max(R, 1-R)} dR + \int_{1/2}^1 \frac{\min(R, 1-R)}{\max(R, 1-R)} dR \\ &= \int_0^{1/2} \left(\frac{R}{1-R}\right) dR + \int_{1/2}^1 \left(\frac{1-R}{R}\right) dR \end{aligned}$$

which reduces to

$$E(X) = \ln(2) - \frac{1}{2} + \ln(2) - \frac{1}{2} = 2\ln(2) - 1 \simeq 0.3863$$

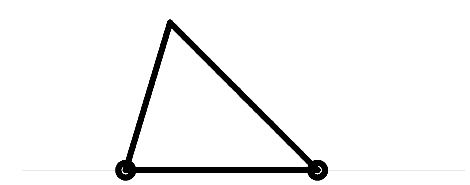
and this agrees with the result computed earlier. Let us now verify this result by running 1000 Monte-Carlo simulations on a spreadsheet as done in class. ■

Example #11: Geometry Problems - The Pipe Cleaner Problem

Consider a pipe cleaner of length 1, which is simply a bendable wire of length 1. Two points are chosen at random on this wire and then the wire is bent at these points to see if it is possible to form a triangle, as shown in the following figure.

The pipe cleaner is bent at two points

The pipe cleaner is bend at two points to form a triangle as shown below.



The pipe cleaner is bent into a triangle

Determine the probability that a triangle can be formed.

To solve this, let the two points be located at R_1 and R_2. We are given that $R_1 \sim U[0,1)$, $R_2 \sim U[0,1)$, so that their pdfs are $f_1(x) = 1$ for $0 \leq x < 1$ and $f_2(y) = 1$ for $0 \leq y < 1$. We first note that the probability P is given by

$$\begin{aligned} P &= P((\text{Triangle with } R_1 \leq R_2) \cup (\text{Triangle with } R_2 \leq R_1)) \\ &= P(\text{Triangle with } R_1 \leq R_2) + P(\text{Triangle with } R_2 \leq R_1) \end{aligned}$$

since the two events $(R_1 \leq R_2)$ and $(R_2 \leq R_1)$ are disjoint. Assuming first that $R_1 \leq R_2$, the three sides of the "possible" triangle being formed have lengths R_1, $R_2 - R_1$ and $1 - R_2$, and for these to form the sides of a triangle, we must have

$$R_1 \leq (R_2 - R_1) + (1 - R_2)$$

and

$$R_2 - R_1 \leq (R_1) + (1 - R_2)$$

and

$$1 - R_2 \leq (R_1) + (R_2 - R_1)$$

which all reduce to

$$R_1 \leq 1/2 \qquad , \qquad R_2 - R_1 \leq 1/2 \qquad , \qquad 1/2 \leq R_2.$$

Thus we want to compute

$$P(\text{Triangle with } R_1 \leq R_2) = P((R_1 \leq 1/2) \cap (R_2 - R_1 \leq 1/2) \cap (1/2 \leq R_2))$$

which is a region in the unit square that looks like a right triangle with side lengths $1/2$ and $1/2$. Thus

$$P(\text{Triangle with } R_1 \leq R_2) = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

We can also compute this probability using

$$P(\text{Triangle with } R_1 \leq R_2) = \int_0^{1/2} \int_{1/2}^{x+1/2} f_2(y) f_1(x)\, dy dx$$

or

$$P(\text{Triangle with } R_1 \leq R_2) = \int_0^{1/2} \int_{1/2}^{x+1/2} \frac{1}{1} \times \frac{1}{1} dy dx = \frac{1}{8}$$

Assuming next that $R_2 \leq R_1$, the three sides of the triangle being formed have lengths R_2, $R_1 - R_2$ and $1 - R_1$, and since R_1 and R_2 are identical distributions and independent, we may simply interchange the rolls of R_1 and R_2 and use the result of the previous calculation. Thus, the symmetry in the problem leads to

$$P(\text{Triangle with } R_2 \leq R_1) = \frac{1}{8}$$

Thus we find that

$$P = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

or $P = 25\%$. ■

Example #12: Another Pipe-Cleaner Problem

Consider the same pipe cleaner of the previous example. A point is chosen at random on this wire and then a second point is chosen at random *only to the right of* the first point. The wire is then bent at these two points to see if it is possible to form a triangle. Determine the probability that a triangle can be formed.

To solve this, let the two points be located at R_1 and R_2 with R_2 to the right of R_1. We are given that $R_1 \sim U[0,1)$, but this time we have $R_2 \sim U[R_1, 1)$, so that their pdfs are $f_1(x) = 1$ for $0 \leq x < 1$ and $f_2(y) = 1(1-x)$ for $x \leq y < 1$. The three sides of the "possible" triangle being formed have lengths R_1, $R_2 - R_1$ and $1 - R_2$, and for these to form the sides of a triangle, we must have

$$R_1 \leq (R_2 - R_1) + (1 - R_2)$$

and

$$R_2 - R_1 \leq (R_1) + (1 - R_2)$$

and

$$1 - R_2 \leq (R_1) + (R_2 - R_1)$$

which all reduce to

$$R_1 \leq 1/2 \qquad , \qquad R_2 - R_1 \leq 1/2 \qquad , \qquad 1/2 \leq R_2.$$

Thus we want to compute

$$P(\text{Triangle with } R_1 \leq R_2) = P((R_1 \leq 1/2) \cap (R_2 \leq R_1 + 1/2) \cap (1/2 \leq R_2))$$

which is a region in the unit square that looks like a right triangle with side lengths 1/2 and 1/2. Thus

$$P(\text{Triangle with } R_1 \le R_2) = \int_0^{1/2} \int_{1/2}^{x+1/2} f_2(y) f_1(x) dy dx$$

or

$$P(\text{Triangle with } R_1 \le R_2) = \int_0^{1/2} \int_{1/2}^{x+1/2} \frac{1}{1-x} \times \frac{1}{1} dy dx$$

which reduces to

$$P(\text{Triangle with } R_1 \le R_2) = \ln(2) - \frac{1}{2}$$

or $P \simeq 19.3\%$. ■

4.2 The Exponential Distribution With Parameter λ

A random variable X is said to be exponentially distributed with parameter $\lambda > 0$ if its pdf is given by

$$f(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ \\ \lambda e^{-\lambda x}, & \text{for} \quad 0 \le x \end{cases} . \tag{16a}$$

The cdf of this distribution is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t) dt$$

which reduces to

$$F(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ \\ 1 - e^{-\lambda x}, & \text{for} \quad 0 \le x \end{cases} . \tag{16b}$$

The first and second moments are given by

$$\mu \equiv E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \tag{16c}$$

and

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

so that the variance is

$$\sigma^2 \equiv V(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}. \tag{16d}$$

Note then that the mean and standard deviation are both equal to $1/\lambda$.

The Memoryless Property

The exponential distribution has been used to model interarrival times when arrivals are completely random and it has been used to model service times which are highly variable. This is because the exponential distribution is "memoryless" which says that

$$P(X > s + t \mid X > s) = P(X > t) \tag{16e}$$

for *all* $s \geq 0$ and $t \geq 0$. This follows from

$$P(X > s + t \mid X > s) = \frac{P(X > s + t \ \cap \ X > s)}{P(\ X > s)} = \frac{P(X > s + t\ )}{P(\ X > s)}$$

but

$$P(X > a) = 1 - P(X \leq a) = 1 - (1 - e^{-\lambda a}) = e^{-\lambda a}$$

and so

$$P(X > s + t \mid X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

If X denotes the lifetime of a component (like a light bulb), then the probability that the component last t time units does not depend on how long the component has already lasted. This says that a used component whose lifetime follows an exponential distribution is as good as a new component whose lifetime follows the same distribution.

We now note that the exponential distribution is the only continuous distribution which has this memoryless property. To prove this we assume that

$$P(X > s + t \mid X > s) = P(X > t)$$

for all values of s and t. This leads to

$$\frac{P(X > s + t \ \cap \ X > s)}{P(\ X > s)} = P(X > t)$$

or

$$P(X > s + t\ ) = P(X > t)P(\ X > s)$$

or

$$1 - P(X \leq s + t\ ) = (1 - P(X \leq t))(1 - P(\ X \leq s))$$

or simply

$$1 - F(t + s) = (1 - F(t))(1 - F(s))$$

which reduces to

$$F(t + s) - F(s) = F(s)(1 - F(t)).$$

If we divide by s we get

$$\frac{F(t + s) - F(s)}{s} = \frac{F(s)}{s}(1 - F(t)).$$

Now

$$\lim_{s \to 0} \left(\frac{F(t + s) - F(s)}{s} \right) = F'(t) = f(t)$$

exist while

$$\lim_{s \to 0} \left(\frac{F(s)}{s} \right) \equiv \lambda$$

exist only if $F(0) = 0$. This says that $F(t) = 0$ for all $t \leq 0$. Then we get

$$\lim_{s \to 0} \left(\frac{F(t + s) - F(s)}{s} \right) = \lim_{s \to 0} \left(\frac{F(s)}{s} \right) (1 - F(t))$$

or $F'(t) = \lambda(1 - F(t))$, which leads to

$$\frac{dF}{1 - F} = \lambda dt \qquad \text{or} \qquad \int \frac{dF}{1 - F} = \int \lambda dt$$

or

$$-\ln(1 - F) = \lambda t + C \qquad \text{or} \qquad 1 - F(t) = e^{-\lambda t - C} = e^{-\lambda t} e^{-C} = A e^{-\lambda t}$$

(with $A = e^{-C}$), resulting in $F(t) = 1 - Ae^{-\lambda t}$, and since $F(0) = 0$, we have $1 - A = 0$, or $A = 1$, and so

$$F(t) = \begin{cases} 0, & \text{for} \quad t \leq 0 \\ 1 - e^{-\lambda t}, & \text{for} \quad 0 < t \end{cases}$$

which is exponential with parameter λ and so the proof is complete.

Note that the probability that X is larger than the mean in an exponential distribution is given by

$$P(X > \mu) = 1 - P(X \leq \mu) = 1 - (1 - e^{-\lambda\mu}) = e^{-1} \simeq 0.368$$

regardless of the value of λ.

4.3 The Gamma Distribution With Parameters β and θ

The *gamma function* is a continuous extension of the factorial for non-integers and is defined by

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1}e^{-x}dx \tag{17}$$

for $0 < \beta$. Using integration by parts, we see that

$$\begin{aligned} \Gamma(\beta) &= \int_0^\infty x^{\beta-1}e^{-x}dx \\ &= x^{\beta-1}(-e^{-x})\Big|_0^\infty - \int_0^\infty (\beta-1)x^{\beta-2}(-e^{-x})dx \\ &= 0 + (\beta-1)\int_0^\infty x^{(\beta-1)-1}e^{-x}dx \\ &= (\beta-1)\Gamma(\beta-1) \end{aligned}$$

for all $\beta > 1$. Since $\Gamma(\beta) = (\beta-1)\Gamma(\beta-1)$ for all $\beta > 1$, we see that a table of values of $\Gamma(\beta)$ for $0 < \beta \leq 1$ is all that is needed to compute $\Gamma(\beta)$ for any value of $\beta > 1$. For example we may compute $\Gamma(8.3)$ using $\Gamma(\beta) = (\beta-1)\Gamma(\beta-1)$ repetitively we have

$$\begin{aligned} \Gamma(8.3) &= (7.3)\Gamma(7.3) \\ &= (7.3)(6.3)\Gamma(6.3) \\ &= (7.3)(6.3)(5.3)\Gamma(5.3) \\ &= (7.3)(6.3)(5.3)(4.3)\Gamma(4.3) \\ &= (7.3)(6.3)(5.3)(4.3)(3.3)\Gamma(3.3) \\ &= (7.3)(6.3)(5.3)(4.3)(3.3)(2.3)\Gamma(2.3) \\ &= (7.3)(6.3)(5.3)(4.3)(3.3)(2.3)(1.3)\Gamma(1.3) \\ &= (7.3)(6.3)(5.3)(4.3)(3.3)(2.3)(1.3)(0.3)\Gamma(0.3) \end{aligned}$$

resulting in $\Gamma(8.3) = (3102.5166)\Gamma(0.3)$. Then

$$\Gamma(0.3) = \int_0^\infty x^{0.3-1}e^{-x}dx = \int_0^\infty x^{-0.7}e^{-x}dx \simeq 2.991569$$

and so $\Gamma(8.3) = (3102.5166)(2.991569) \simeq 9281.39$. Using the fact that

$$\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx = 1$$

we see that when β is positive integer, then $\Gamma(\beta) = (\beta-1)!$.

The Gamma Distribution

A random variable X is gamma distributed with parameters β and θ if its pdf is given by

$$f(x) = \begin{cases} (\beta\theta)^\beta x^{\beta-1}e^{-\beta\theta x}/\Gamma(\beta), & \text{for} \quad 0 < x \\ 0, & \text{for} \quad x \leq 0 \end{cases}. \tag{18a}$$

The parameter β is called the *shape* parameter and θ is called the *scale* parameter. The first and second moments are given by

$$\begin{aligned} E(X) &= \int_0^\infty x\frac{(\beta\theta)^\beta x^{\beta-1}e^{-\beta\theta x}}{\Gamma(\beta)}dx = \frac{(\beta\theta)^\beta}{\Gamma(\beta)}\int_0^\infty x^\beta e^{-\beta\theta x}dx \\ &= \frac{(\beta\theta)^\beta}{\Gamma(\beta)}\int_0^\infty \left(\frac{u}{\beta\theta}\right)^\beta e^{-u}d\left(\frac{u}{\beta\theta}\right) = \frac{1}{\beta\theta\Gamma(\beta)}\int_0^\infty u^\beta e^{-u}du \\ &= \frac{1}{\beta\theta\Gamma(\beta)}\Gamma(\beta+1) = \frac{1}{\beta\theta\Gamma(\beta)}\beta\Gamma(\beta) \end{aligned}$$

or simply

$$E(X) = \frac{1}{\theta}, \tag{18b}$$

and

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2\frac{(\beta\theta)^\beta x^{\beta-1}e^{-\beta\theta x}}{\Gamma(\beta)}dx = \frac{(\beta\theta)^\beta}{\Gamma(\beta)}\int_0^\infty x^{\beta+1}e^{-\beta\theta x}dx \\ &= \frac{(\beta\theta)^\beta}{\Gamma(\beta)}\int_0^\infty \left(\frac{u}{\beta\theta}\right)^{\beta+1} e^{-u}d\left(\frac{u}{\beta\theta}\right) = \frac{1}{(\beta\theta)^2\Gamma(\beta)}\int_0^\infty u^{\beta+1}e^{-u}du \\ &= \frac{1}{(\beta\theta)^2\Gamma(\beta)}\Gamma(\beta+2) = \frac{1}{(\beta\theta)^2\Gamma(\beta)}(\beta+1)\beta\Gamma(\beta) \end{aligned}$$

or simply

$$E(X^2) = \frac{\beta + 1}{\beta\theta^2},$$

and hence the variance of the distribution is given by

$$\sigma^2 \equiv V(X) = E(X^2) - (E(X))^2 = \frac{\beta + 1}{\beta\theta^2} - \frac{1}{\theta^2},$$

or simply

$$V(X) = \frac{1}{\beta\theta^2} \tag{18c}$$

Note that the mode of the gamma distribution is obtained by solving

$$\frac{d}{dx}\left(\frac{(\beta\theta)^\beta x^{\beta-1}e^{-\beta\theta x}}{\Gamma(\beta)}\right) = 0$$

yielding

$$(\beta - 1 - \beta\theta x)x^{\beta-2}e^{-\beta\theta x} = 0$$

or

$$\text{mode} = \frac{\beta - 1}{\beta\theta}. \tag{18d}$$

The Sum of Exponential Random Variables

When β is an integer, the gamma distribution is related to the exponential distribution in the following way. If the random variable X is the sum of β independent, exponentially distributed random variables, each with parameter $\beta\theta$, then X has a gamma distribution with parameters β and θ. Thus if

$$X = X_1 + X_2 + X_3 + \cdots + X_\beta$$

and the pdfs of all the X_ks are the same and given by

$$f(x_k) = \begin{cases} (\beta\theta)e^{-\beta\theta x}, & \text{for } \ 0 \le x_k \\ \\ 0, & \text{for } \ x_k < 0 \end{cases},$$

and if all the X_ks are independent, then X has the pdf given by

$$f(x) = \begin{cases} (\beta\theta)^\beta x^{\beta-1}e^{-\beta\theta x}/\Gamma(\beta), & \text{for } \ 0 < x \\ \\ 0, & \text{for } \ x \le 0 \end{cases}$$

and cdf given by

$$F(x) = \frac{(\beta\theta)^\beta}{\Gamma(\beta)} \int_0^x t^{\beta-1} e^{-\beta\theta t} dt = 1 - \frac{(\beta\theta)^\beta}{\Gamma(\beta)} \int_x^\infty t^{\beta-1} e^{-\beta\theta t} dt$$

for $0 \leq x$. Note that for β equal to one, the gamma distribution reduces to the exponential distribution.

Note that if X is a gamma distribution with parameters β and θ and if Y is an exponential distribution with parameter λ. Then

$$E(X) = \frac{1}{\theta} \qquad , \qquad V(X) = \frac{1}{\beta\theta^2}$$

and

$$E(Y) = \frac{1}{\lambda} \qquad , \qquad V(Y) = \frac{1}{\lambda^2}$$

so that if $\lambda = \theta$, then $E(X) = E(Y)$, and

$$V(X) = \frac{1}{\beta\theta^2} = \frac{V(Y)}{\beta} \leq V(Y)$$

when $\beta \geq 1$, which says that the gamma distribution is less variable than an exponential distribution with the same mean.

4.4 The Erlang Distribution With Parameters k and θ

The gamma distribution with $\beta = k$ (an integer) is called the Erlang distribution of order k. If X is an Erlang distribution with parameters k and θ, then we may write X as

$$X = X_1 + X_2 + X_3 + \cdots + X_k$$

where X_i are independent and identical exponential random variables, all with parameter $\lambda = k\theta$, and so

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_k) = \frac{1}{k\theta} + \frac{1}{k\theta} + \frac{1}{k\theta} + \cdots + \frac{1}{k\theta}$$

yielding

$$\mu \equiv E(X) = \frac{k}{k\theta} = \frac{1}{\theta}.$$

In addition, we have

$$V(X) = V(X_1) + V(X_2) + \cdots + V(X_k) = \frac{1}{(k\theta)^2} + \frac{1}{(k\theta)^2} + \cdots + \frac{1}{(k\theta)^2}$$

yielding

$$\sigma^2 \equiv V(X) = \frac{k}{(k\theta)^2} = \frac{1}{k\theta^2}.$$

The cdf Of The Erlang Distribution

It is possible to obtain a very simple expression for the cdf of the Erlang distribution by using integration by parts. Toward this end, we start with

$$F(x) = \frac{(k\theta)^k}{\Gamma(k)} \int_0^x t^{k-1} e^{-k\theta t} dt = 1 - \frac{(k\theta)^k}{(k-1)!} \int_x^\infty t^{k-1} e^{-k\theta t} dt.$$

Using integration by parts we have

$$\begin{aligned} \frac{(k\theta)^k}{(k-1)!} \int_x^\infty t^{k-1} e^{-k\theta t} dt &= \frac{(k\theta)^k}{(k-1)!} \left\{ \left. \frac{t^{k-1} e^{-k\theta t}}{-k\theta} \right|_x^\infty + \frac{k-1}{k\theta} \int_x^\infty t^{k-2} e^{-k\theta t} dt \right\} \\ &= \frac{(k\theta x)^{k-1} e^{-k\theta x}}{(k-1)!} + \frac{(k\theta)^{k-1}}{(k-2)!} \int_x^\infty t^{k-2} e^{-k\theta t} dt \\ &= \frac{(k\theta x)^{k-1} e^{-k\theta x}}{(k-1)!} + \frac{(k\theta)^{k-1}}{(k-2)!} \left. \frac{t^{k-2} e^{-k\theta t}}{-k\theta} \right|_x^\infty \\ &\quad + \frac{(k\theta)^{k-1}}{(k-2)!} \frac{k-2}{k\theta} \int_x^\infty t^{k-3} e^{-k\theta t} dt \end{aligned}$$

or

$$\begin{aligned} \frac{(k\theta)^k}{(k-1)!} \int_x^\infty t^{k-1} e^{-k\theta t} dt &= \frac{(k\theta x)^{k-1} e^{-k\theta x}}{(k-1)!} + \frac{(k\theta x)^{k-2} e^{-k\theta x}}{(k-2)!} \\ &\quad + \frac{(k\theta)^{k-2}}{(k-3)!} \int_x^\infty t^{k-3} e^{-k\theta t} dt. \end{aligned}$$

Continuing in this manner we get

$$\frac{(k\theta)^k}{(k-1)!} \int_x^\infty t^{k-1} e^{-k\theta t} dt = \sum_{i=1}^{k-1} \frac{(k\theta x)^i e^{-k\theta x}}{i!} + \frac{(k\theta)^1}{0!} \int_x^\infty t^0 e^{-k\theta t} dt$$

$$= \sum_{i=1}^{k-1} \frac{(k\theta x)^i e^{-k\theta x}}{i!} + (k\theta)\left.\frac{e^{-k\theta t}}{(-k\theta)}\right|_x^\infty$$

$$= \sum_{i=1}^{k-1} \frac{(k\theta x)^i e^{-k\theta x}}{i!} + e^{-k\theta x}$$

or simply

$$\frac{(k\theta)^k}{(k-1)!}\int_x^\infty t^{k-1}e^{-k\theta t}dt = \sum_{i=0}^{k-1} \frac{(k\theta x)^i e^{-k\theta x}}{i!}$$

and so

$$F(x) = 1 - e^{-k\theta x}\sum_{i=0}^{k-1} \frac{(k\theta x)^i}{i!} \tag{19}$$

for $0 < x$, and $F(x) = 0$ for $x \leq 0$. This is just a sum of Poisson terms with mean $\alpha = k\theta x$ and so tables of cumulative Poisson distribution may be used to evaluate the cdf of Erlang distributions.

Example #13: Being In The Dark

A box contains four identical light bulbs. Each light bulb has a lifetime distribution that is exponential with a mean of 150 hours. Person A takes two of these light bulbs into a room and turns them both on at the same time and leaves them turned on. Person B takes the other two light bulbs into a different room and turns on only one of the light bulbs and then turns on the other light bulb only when and if the first light bulb burns out. (a) Compute the probability that person A will be in the dark at the end of one week (168 hours). (b) Compute the probability that person B will be in the dark at the end of one week (168 hours).

To solve part (a), let X_1, X_2, X_3, and X_4 be the random variables for the lifetime (in hours) of the four light bulbs so that

$$F_1(x) = F_2(x) = F_3(x) = F_4(x) = 1 - e^{-\lambda x}$$

with $\lambda = 1/150$. Since person A keeps both light bulbs on at the same time, this person will be in the dark after one week ($24 \times 7 = 168$ hours) only when

$$Y = \max(X_1, X_2) \leq 168.$$

It should be clear that

$$F_{\max}(y) = P(Y \leq y) = P(\max(X_1, X_2) \leq y) = P((X_1 \leq y) \cap (X_2 \leq y)).$$

Since X_1 and X_2 are independent, we have

$$F_{\max}(y) = P((X_1 \leq y) \cap (X_2 \leq y)) = P(X_1 \leq y)P(X_2 \leq y)$$

resulting in

$$F_{\max}(y) = F_1(y)F_2(y) = (1 - e^{-y/150})^2$$

and so

$$P(Y \leq 168) = (1 - e^{-168/150})^2 \simeq 0.4539.$$

To solve (b), we note that since person B keeps the light bulbs on only one at the same time, this person will be in the dark after one week ($24 \times 7 = 168$ hours) only when

$$Z = X_3 + X_4 \leq 168.$$

From our notes in class we know that Z will be an Erlang distribution with parameters $k = 2$, and $\theta = 1/300$. Then

$$F_{\text{sum}}(z) = 1 - \sum_{i=0}^{2-1} e^{-2(1/300)z} \frac{(2(1/300)z)^i}{i!}$$

which reduces to

$$F_{\text{sum}}(z) = 1 - \left(1 + \frac{z}{150}\right) e^{-z/150}.$$

Then

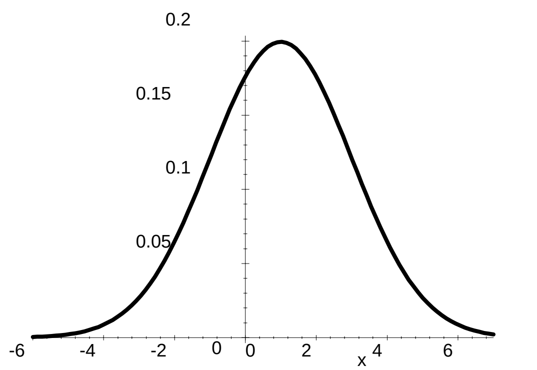
$$P(Z \leq 168) = 1 - \left(1 + \frac{168}{150}\right) e^{-168/150} \simeq 0.3083.$$

4.5 The Normal Distribution With Mean μ and Variance σ^2

A random variable with mean μ and variance $\sigma^2 > 0$ has normal distribution if it has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \sim N(\mu, \sigma^2) \tag{20a}$$

for $-\infty < x < +\infty$. A typical graph of $f(x)$ versus x is shown in the figure below for $\mu = 1$ and $\sigma = 2$.



Plot of $f(x)$ versus x for $\mu = 1$ and $\sigma = 2$

The *standard normal* distribution has $\mu = 0$ and $\sigma = 1$, is denoted by $Z \sim N(0, 1)$ and has pdf

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sim N(0,1) \tag{20b}$$

If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1), \tag{20c}$$

and if $Z \sim N(0, 1)$, then

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2). \tag{20d}$$

Some special properties of $N(\mu, \sigma^2)$ are as follows:

$$\lim_{x \to \pm\infty} f(x) = 0,$$

the pdf is symmetric about the mean μ, so that

$$f(\mu - x) = f(\mu + x), \tag{20e}$$

the mode of $N(\mu, \sigma^2)$ occurs at the mean μ. Note also that the first and second moments of the distribution are given as

$$E(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} x \exp\left\{ -\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right\} dx = \mu \tag{20f}$$

and

$$E(X^2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} x^2 \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx = \mu^2 + \sigma^2$$

so that $V(X) = \sigma^2$.

The cdf Of The Normal Distribution

The cdf of the normal distribution is given by

$$F(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\} dt,$$

and although it is not possible to evaluate this integral in closed form, various numerical methods and tables exists in order to evaluate $F(x)$. In fact one only needs to tabulate

$$\Phi(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}t^2\right) dt \tag{20g}$$

for the standard normal distribution $N(0, 1)$, having pdf

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right),$$

and then for $N(\mu, \sigma^2)$, simple substitution shows that

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right). \tag{20h}$$

Table 5.1 of the text give $\Phi(z)$ for $0 \leq z$ and for $z < 0$, we use the fact that

$$\Phi(-z) = 1 - \Phi(z). \tag{20i}$$

Example #14 - A Normal Distribution

The time to pass through a queue to begin self-service at a cafeteria has been found to be $N(15, 9)$. Determine the probability than an arriving customer waits between 14 and 17 minutes for service. This is just given by

$$P(14 < X < 17) = F(17) - F(14) = \Phi\left(\frac{17-15}{3}\right) - \Phi\left(\frac{14-15}{3}\right)$$

or

$$P(14 < X < 17) = \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) - \left\{1 - \Phi\left(\frac{1}{3}\right)\right\}$$

or simply

$$P(14 < X < 17) = \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) + \Phi\left(\frac{1}{3}\right) - 1.$$

Using a table of normal probabilities, we have

$$\Phi\left(\frac{2}{3}\right) = \Phi(0.667) = 0.7476 \qquad \& \qquad \Phi\left(\frac{1}{3}\right) = \Phi(0.333) = 0.6304$$

and so

$$P(14 < X < 17) = 0.7476 + 0.6304 - 1 = 0.378.$$

Example #15 - Another Normal Distribution

Lead-time demand, X, for an item is approximated by a normal distribution with a mean of 25 and a variance of 9. It is desired to determine a value for lead time that will be exceeded only 5% of the time. Thus the problem is to determine x_o so that $P(X > x_o) = 0.05$. This leads to

$$P(X > x_o) = 1 - P(X \leq x_o) = 0.05$$

or

$$P(X \leq x_o) = 0.95 = \Phi\left(\frac{x_o - 25}{3}\right).$$

From Table A.3 we see that $\Phi(1.645) = 0.95$, and so

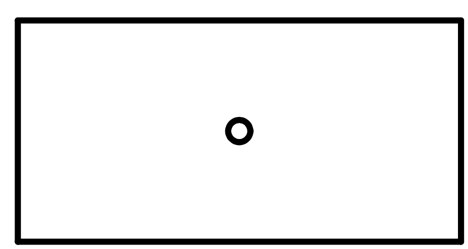
$$\frac{x_o - 25}{3} = 1.645 \qquad \text{yielding} \qquad x_o = 25 + 3(1.645) = 29.935.$$

Therefore, in only 5% of the cases will demand during lead time exceed available inventory if an order to purchase is made when the stock level reaches 30. ■

Example #16 - Killing a Target

Consider a squadron of bombers attempting to destroy an ammunition depot in the shape of some region such as of a rectangle of east-west length $L_x = 400$ meters and south-north length $L_y = 200$ meters. If the bomb hits anywhere on the depot, a hit is scored, otherwise the bomb misses its target.

The aiming point is a small circle located in the heart of the ammunition depot (which for this example is at the center point of the rectangular ammunition depot), as shown in the figure below,



The aiming point is the small circle

and the point of impact is assumed to be *normally distributed* about this aiming point with a standard deviation of $\sigma_x = 600$ meters in the east-west direction and $\sigma_y = 300$ meters in the south-north direction.

Let us compute the probability that the rectangular region is hit. Towards this end, we have $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ so that

$$\begin{aligned}
P_{\text{Hit}} &= P((-L_x/2 \leq X \leq L_x/2) \cap (-L_y/2 \leq Y \leq L_y/2)) \\
&= P(-L_x/2 \leq X \leq L_x/2) P(-L_y/2 \leq Y \leq L_y/2) \\
&= P\left(-\frac{L_x - 0}{2\sigma_X} \leq \frac{X - 0}{\sigma_X} \leq \frac{L_x - 0}{2\sigma_X}\right) P\left(-\frac{L_y - 0}{2\sigma_Y} \leq \frac{Y - 0}{\sigma_Y} \leq \frac{L_y - 0}{2\sigma_Y}\right) \\
&= \left\{\Phi\left(\frac{L_x}{2\sigma_X}\right) - \Phi\left(-\frac{L_x}{2\sigma_X}\right)\right\}\left\{\Phi\left(\frac{L_y}{2\sigma_Y}\right) - \Phi\left(-\frac{L_y}{2\sigma_Y}\right)\right\}
\end{aligned}$$

or since $\Phi(-z) = 1 - \Phi(z)$, we have

$$P_{\text{Hit}} = \left\{2\Phi\left(\frac{L_x}{2\sigma_X}\right) - 1\right\}\left\{2\Phi\left(\frac{L_y}{2\sigma_Y}\right) - 1\right\}.$$

Using the inputs $\sigma_x = 600$ meters, $\sigma_y = 300$ meters, $L_x = 400$ meters and $L_y = 200$ meters, we get

$$\begin{aligned} P_{\text{Hit}} &= \left\{2\Phi\left(\frac{400}{2(600)}\right) - 1\right\}\left\{2\Phi\left(\frac{200}{2(300)}\right) - 1\right\} \\ &= \left\{2\Phi\left(\frac{1}{3}\right) - 1\right\}\left\{2\Phi\left(\frac{1}{3}\right) - 1\right\} = \left\{2\Phi\left(\frac{1}{3}\right) - 1\right\}^2. \end{aligned}$$

Since

$$\Phi\left(\frac{1}{3}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/3} e^{-\frac{1}{2}z^2} dz \simeq 0.63056$$

we find that

$$P_{\text{Hit}} = (2(0.63056) - 1)^2$$

or $P_{\text{Hit}} \simeq 0.068$. ■

4.6 The Weibull Distribution With Parameters ν, α and β

The random variable X has a Weibull distribution with *location* parameter ν $(-\infty < \nu < +\infty)$, *scale* parameter α $(\alpha > 0)$, and *shape* parameter β $(\beta > 0)$ if its pdf has the form

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\} \tag{21a}$$

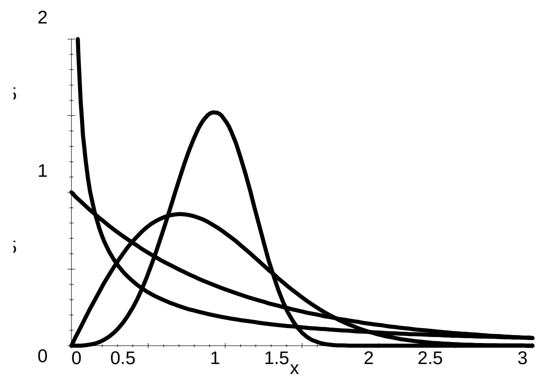
for $x \geq \nu$, and zero otherwise. When $\nu = 0$, we have

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\} \tag{21b}$$

for $x \geq 0$, and zero otherwise, and when $\nu = 0$ and $\beta = 1$, we get

$$f(x) = \frac{1}{\alpha}e^{-x/\alpha} \tag{21c}$$

which is an exponential distribution with parameter $\lambda = 1/\alpha$. Typical plots using $\nu = 0$ and $\alpha = 1$, and various values of $\beta = 1/2, 1, 2, 4$ are shown in the figure below.



Plots of the Weibull pdf for $\nu = 0$, $\alpha = 1$ and $\beta = 1/2, 1, 2, 4$

The mean and variance of the Weibull distribution are given by

$$\mu \equiv E(X) = \nu + \alpha\Gamma\left(1 + \frac{1}{\beta}\right) \tag{21d}$$

and

$$\sigma^2 \equiv V(X) = \alpha^2\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right\}, \tag{21e}$$

respectively, where $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Note that the location parameter ν has no effect on the mean and the cdf of the Weibull distribution is given by

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} \frac{\beta}{\alpha}\left(\frac{t-\nu}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{t-\nu}{\alpha}\right)^{\beta}\right\}dt \\ &= \int_{-\infty}^{\nu}(0)dt + \int_{\nu}^{x} \frac{\beta}{\alpha}\left(\frac{t-\nu}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{t-\nu}{\alpha}\right)^{\beta}\right\}dt \\ &= \exp\left\{-\left(\frac{t-\nu}{\alpha}\right)^{\beta}\right\}\Bigg|_{\nu}^{x} \end{aligned}$$

and so

$$F(x) = 1 - \exp\left\{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right\} \tag{21f}$$

for $x \geq v$ and $F(x) = 0$ for $x < \nu$. At this point the students should read through Examples 5.25 and 5.26 of the text.

4.7 The Triangular Distribution With Parameters a, b and c

A random variable X has a triangular distribution with *minimum* value a, *maximum* value c, and *mode* b if its pdf has the form

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)}, & \text{for} \quad a \leq x \leq b \\ \\ \frac{2(c-x)}{(c-b)(c-a)}, & \text{for} \quad b \leq x \leq c \end{cases} \tag{22a}$$

and zero elsewhere. The first and second moments are given by

$$\mu \equiv E(X) = \int_a^b \frac{2x(x-a)}{(b-a)(c-a)}dx + \int_b^c \frac{2x(c-x)}{(c-b)(c-a)}dx$$

or

$$E(X) = \frac{a+b+c}{3} \tag{22b}$$

and

$$E(X^2) = \int_a^b \frac{2x^2(x-a)}{(b-a)(c-a)}dx + \int_b^c \frac{2x^2(c-x)}{(c-b)(c-a)}dx$$

which reduces to

$$E(X^2) = \frac{a^2+b^2+c^2+ab+ac+bc}{6}$$

resulting in a variance of

$$V(X) = \frac{a^2+b^2+c^2+ab+ac+bc}{6} - \left(\frac{a+b+c}{3}\right)^2$$

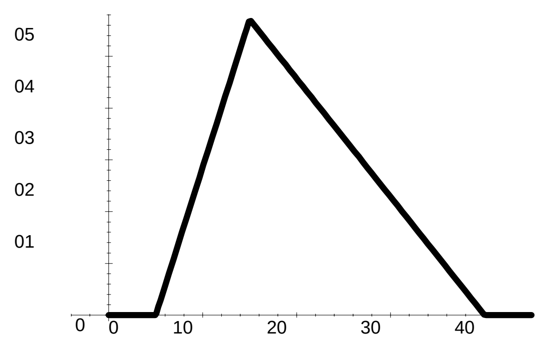
which reduces to

$$\sigma^2 \equiv V(X) = \frac{(c-a)(c-a)-(b-a)(c-b)}{18}. \tag{22c}$$

The mode is given by b, which is calculated from the mean μ using

$$b = 3\mu - (a + c). \tag{22d}$$

An example plot of the triangular distribution pdf using $a = 5$, $b = 15$ and $c = 40$ is shown in the following figure.



Plot of $f(x)$ versus x using
$a = 5$, $b = 15$ and $c = 40$

The pdf has a height given by

$$H = \frac{2}{c - a}$$

and the cdf of this distribution is give by

$$F(x) = P(X \leq x) = \begin{cases} 0, & \text{for} \quad x \leq a \\ \frac{(x-a)^2}{(b-a)(c-a)}, & \text{for} \quad a < x \leq b \\ 1 - \frac{(c-x)^2}{(c-b)(c-a)}, & \text{for} \quad b < x \leq c \\ 1, & \text{for} \quad c < x \end{cases}. \tag{22e}$$

Note for the special case when $f(x)$ is symmetric about the mode b, we have

$$c - b = b - a \qquad \text{so that} \qquad b = \frac{a + c}{2}$$

and

$$f(x) = \begin{cases} \frac{4(x-a)}{(c-a)^2}, & \text{for} \quad a \le x \le b \\ \\ \frac{4(c-x)}{(c-a)^2}, & \text{for} \quad b \le x \le c \end{cases},$$

and

$$\mu = \frac{a+c}{2} = b \qquad \text{and} \qquad \sigma^2 = \frac{(c-a)^2}{24}$$

and

$$F(x) = P(X \le x) = \begin{cases} 0, & \text{for} \quad x \le a \\ \\ \frac{2(x-a)^2}{(c-a)^2}, & \text{for} \quad a < x \le b \\ \\ 1 - \frac{2(c-x)^2}{(c-a)^2}, & \text{for} \quad b < x \le c \\ \\ 1, & \text{for} \quad c < x \end{cases}.$$

Note that the triangular distribution is used when the minimum, maximum and mean of a distribution is known. Note also that the medium of a distribution is the value of x for which $F(x) = 0.5$.

4.8 Trapezoidal Distribution With Parameters a, b, c and d:

The distribution generalizes the Uniform and Triangular Distributions. A continuous random variable X has a *trapezoidal* distribution with parameters $a \le b \le c \le d$ if its shape is a trapezoid connecting the points

$$(a, 0) \to (b, h) \to (c, h) \to (d, 0).$$

The *pdf* of this distribution is given by

$$f(x) = \begin{cases} h(x-a)/(b-a), & \text{for} \quad a \le x \le b \\ \\ h, & \text{for} \quad b \le x \le c \\ \\ h(d-x)/(d-c), & \text{for} \quad c \le x \le d \end{cases}. \tag{23a}$$

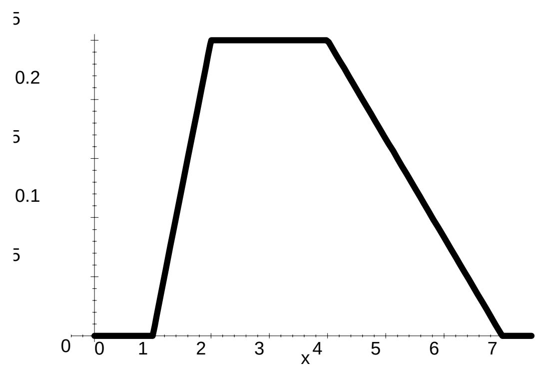
with

$$h = \frac{2}{(c+d)-(a+b)} \tag{23b}$$

and this has a range space of

$$R_X = \{x \,|\, a \le x \le d\}.$$

An example plot of the pdf using $a = 1$, $b = 2$, $c = 4$ and $d = 7$.



Plot of $f(x)$ versus x using
$a = 1$, $b = 2$, $c = 4$, $d = 7$

The parameters a, b, c, and d can take on any real values as long as $a < b < c < d$. The parameters a and d represent the minimum and maximum values of the distribution, respectively. The mean is given by

$$E(X) = \frac{1}{3}\left\{\frac{(c^2 + cd + d^2) - (a^2 + ab + b^2)}{(c + d) - (a + b)}\right\}. \tag{23c}$$

and the *cdf* is given as

$$F(x; a, b, c, d) = \begin{cases} 0, & \text{for} \quad x \le a \\ \frac{1}{2}h\frac{(x-a)^2}{b-a}, & \text{for} \quad a \le x \le b \\ \frac{1}{2}h(2x - a - b), & \text{for} \quad b \le x \le c \\ 1 - \frac{1}{2}h\frac{(d-x)^2}{d-c}, & \text{for} \quad c \le x \le d \\ 1, & \text{for} \quad d \le x \end{cases} . \tag{23d}$$

4.9 Beta Distribution With Parameters α, β, a and b

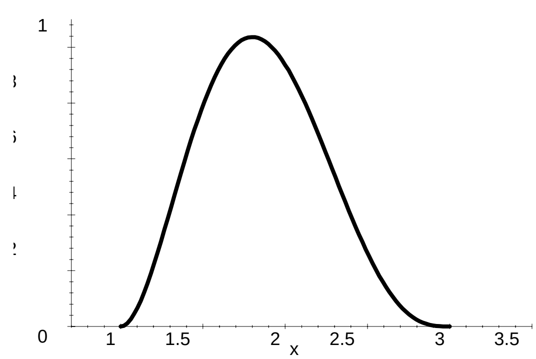
An extremely flexible distribution used to model bounded (fixed upper and lower limits) random variables is the beta distribution. The *pdf* of this distribution is given by

$$f(x) = \frac{(\alpha+\beta+1)!(x-a)^{\alpha}(b-x)^{\beta}}{\alpha!\beta!(b-a)^{\alpha+\beta+1}} \tag{24a}$$

with a range space of

$$R_X = \{x \,|\, a \le x \le b\}.$$

An example plot of the pdf using $a = 1$, $b = 3$ and, $\alpha = 2$ and $\beta = 3$ is presented below.



Plots of $f(x)$ versus x for
$a = 1$, $b = 3$, $\alpha = 2$ and $\beta = 3$

The parameters a and b can take on any real values as long as $a < b$. In addition, a is the minimum value of X while b is the maximum possible value of X. The parameters α and β can take on any positive real values. The mean and variance are given by

$$E(X) = \frac{(\beta+1)a + (\alpha+1)b}{\alpha+\beta+2} \tag{24b}$$

and

$$V(X) = \frac{(\alpha+1)(\beta+1)(b-a)^2}{(\alpha+\beta+2)^2(\alpha+\beta+3)}, \tag{24c}$$

respectively, and the *cdf* is given by

$$F(x) = \frac{(\alpha+\beta+1)!}{\alpha!\beta!}\int_a^x \frac{(t-a)^{\alpha}(b-t)^{\beta}}{(b-a)^{\alpha+\beta+1}}dt \equiv \frac{(\alpha+\beta+1)!}{\alpha!\beta!}\Psi\left(\frac{x-a}{b-a};\alpha,\beta\right)$$

where

$$\Psi(z; \alpha, \beta) = \int_0^z u^\alpha (1-u)^\beta du \tag{24d}$$

4.10 The Lognormal Distribution With Parameters μ and σ^2

A random variable X has lognormal distribution with parameters μ and σ^2 if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{1}{2}\left(\frac{\ln(x) - \mu}{\sigma}\right)^2\right\} \tag{25a}$$

for $0 < x$, and zero otherwise. The mean and variance of this distribution are given by

$$\mu_{\rm L} = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{and} \qquad \sigma_{\rm L}^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1). \tag{25b}$$

Note that if $Y \sim N(\mu, \sigma^2)$, then $X = e^Y$ has a lognormal distribution with parameters μ and σ^2 and if the mean and variance $\mu_{\rm L}$ and $\sigma_{\rm L}^2$ are known, then the parameters μ and σ^2 can be computed using

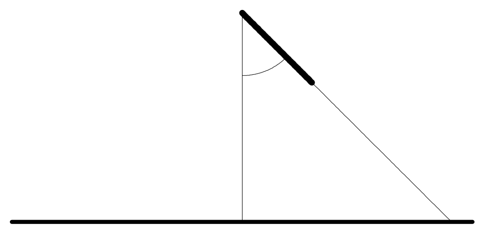
$$\mu = \ln\left(\frac{\mu_{\rm L}^2}{\sqrt{\mu_{\rm L}^2 + \sigma_{\rm L}^2}}\right) \qquad \& \qquad \sigma^2 = \ln\left(\frac{\mu_{\rm L}^2 + \sigma_{\rm L}^2}{\mu_{\rm L}^2}\right). \tag{25c}$$

4.11 The Cauchy Distribution

A random variable X is said to have a Cauchy distribution with parameters $-\infty < b < +\infty$, if its density function is given by

$$f(x) = \frac{a/\pi}{a^2 + (x-b)^2}$$

for $-\infty < x < +\infty$. An classic example of where a Cauchy distribution appears is in the problem of a laser pointer that is spun around its center, which is located a distance L from the x axis as shown in the following figure.



The laser pointing is in bold
and the angle Θ is indicated

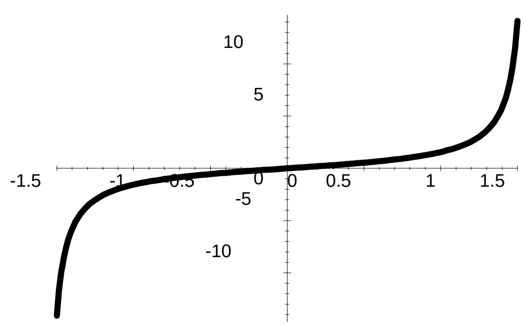
Consider the point X where the laser beam strikes the x axis when the laser pointer has stopped spinning. Assuming that the angle Θ (indicated in the figure) is a random variable that is uniform in the range $[-\pi/2, +\pi/2]$, then $\Theta \sim U[-\pi/2, +\pi/2]$ and $X = L\tan(\Theta)$. Then

$$F(x) = P(X \leq x) = P(L\tan(\Theta) \leq x) = P(\tan(\Theta) \leq x/L)$$

or

$$F(x) = P(\Theta \leq \tan^{-1}(x/L))$$

since $\tan(\theta)$, as plotted below,



Plot of $\tan(\theta)$ versus θ

is strictly increasing from $\theta = -\pi/2$ to $\theta = +\pi/2$. Thus we find that

$$F(x) = P(\Theta \le \tan^{-1}(x/L)) = \frac{\tan^{-1}(x/L) - (-\pi/2)}{\pi/2 - (-\pi/2)} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x/L).$$

Differentiating this with respect to x gives

$$f(x) = \frac{L/\pi}{L^2 + x^2}$$

which is a Cauchy distribution having $a = L$ and $b = 0$.