Document 8: Projections and Reflections in 3D

We envision a vector in 3D space. We equally have a plane we want to project the vector onto, where both the vector and the plane are at the origin. Now, lets add a line orthogonal to the plane. On the same side as the vector, on the line, is $\vec{x}^{\parallel}$, opposite it is the inverse projection of $\vec{x}$ onto the line. On the plane, we have $\vec{x}^{\perp}$, the projection on the plane. The plane, we call V.

We have four unique equations representing the relationships between all these values:

1. $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$
2. $\text{proj}_V(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$
3. $\text{ref}_L(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$
4. $\text{ref}_V(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}$

The derivation of the essential properties is as follows:

$$\text{Let } x \in \mathbb{R}^3, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{Recall } \vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}.$$

$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$$
$$\vec{x}^{\perp} = \text{proj}_V(\vec{x})$$
$$\vec{x}^{\perp} = \text{proj}_V(\vec{x}) = \vec{x} - \vec{x}^{\parallel} = \vec{x} - \text{proj}_L(\vec{x})$$
$$\vec{x}^{\perp} = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$$
$$\text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x}) = \text{proj}_L(\vec{x}) - (\vec{x} - \text{proj}_L(\vec{x})) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$$
$$\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x}) = -\text{ref}_L(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}$$

These transformations are linear transformations, and hence we may have an inverse transformation.

Definition: A function $T : X \to Y$ is invertible if $T(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. We define the inverse function of T, written T^{-1}, from Y to X, $x = T^{-1}(y)$. For matrices, consider $x \in \mathbb{R}^n \overset{A_{nxm}}{\to} y \in \mathbb{R}^m$, with $\vec{y} = A\vec{x}$. We say $A\vec{x}$ is invertible if $A\vec{x} = \vec{y}$ has a unique solution $\vec{x} \in \mathbb{R}^n$ for all $\vec{y} \in \mathbb{R}^m$. We define the inverse of A as A^{-1}.

Defining when a matrix is invertible: $A_{m\text{x}n}$ is invertible if and only if:

1. A is a square matrix, and $m = n$
2. rref$(A) = I_{m\text{x}n}$

It is useful to note that additionally for a homogeneous system of equations, $A\vec{x} = \vec{0}$, we reduce the possible cases for solutions from three, to two; There must be either infinite solutions, or just 1 solution. No solutions is not an option, and if there is one solution, then $\vec{x} = \vec{0}$.

There are a couple useful property to note when dealing with a homogeneous system. A is invertible $\Leftrightarrow$ $\vec{x} = \vec{0}$ and A is not invertible $\Leftrightarrow$ $\vec{x}$ has infinite solutions.

One last thing to note: when a matrix is invertible, $A\vec{x} = b \Rightarrow \vec{x} = A^{-1}b$.