Document 24: Diagonalization and E-Values/Vectors

Diagonalization is a really useful skill to have, and getting there involves the Eigenvalues and Eigenvectors of a matrix. The steps to get there are as follows.

1. Find characteristic polynomial
2. Find the eigenvalues and eigenvectors
3. Diagonalize A

For step one, $P_A(\lambda) = \det(A - \lambda I)$. Lets define a matrix A for this example, and compute $A - \lambda I$.

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

Then, $P_A(\lambda) = (2-\lambda) \cdot \det\left(\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 2 & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) + 2 \cdot \det\left(\begin{bmatrix} 2 & 2 \\ 2-\lambda & 2 \end{bmatrix}\right) = (2-\lambda) \cdot (\lambda^2 - 4\lambda) - 2 \cdot (-2\lambda) + 2 \cdot (2\lambda) = 6\lambda^2 - \lambda^3 = -\lambda^2(\lambda - 6)$.

Then, for step two, we identify the eigenvalues from the characteristic polynomial: we have $\lambda_1 = 0, \lambda_2 = 6$. We must also compute the eigenvectors. To compute the eigenvectors, we must solve $(A - \lambda I)\vec{v} = \vec{0}$. First, lets compute $A - \lambda_1 I$ and $A - \lambda_2 I$.

$$A - \lambda_1 I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, A - \lambda_2 I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

Then, compute $(A - \lambda_1 I)\vec{v} = \vec{0}$. Given that $\vec{v}$ has elements v_1, v_2, v_3, we have $v_1 = -v_2 - v_3, v_2 = v_2, v_3 = v_3$. Defining free parameters s, t such that $s = v_2, t = v_3$, we can rewrite the closed form for the solution.

$$\vec{v} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Lets redefine $\vec{v}$ within the scope of a new equation to compute $(A - \lambda_2 I)\vec{v} = \vec{0}$. Given that $\vec{v}$ has elements v_1, v_2, v_3, we have $v_1 = v_3, v_2 =$

$v_3, v_3 = v_3$. Lets define s, the free parameter with $s = v_3$. In closed form, we have the following.

$$\vec{v} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, we have the eigenvalues $\lambda_1 = 0, \lambda_2 = 6$. Drawing from the solutions above, define vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, the eigenvectors of the matrix as follows:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For step three, diagonalizing the matrix, lets first ensure that the matrix is diagonalizable. From the characteristic polynomial, we find the algebraic multiplicities for λ_1 and λ_2, and from the dimentions of the kernels we found earlier, we get the geometric multiplicities of λ_1 and λ_2: a.m.1= 2, g.m.1= 2, a.m.2= 1, g.m.2= 1. Since both the algebraic and geometric multiplicities equal, the matrix is diagonalizable.

The matrix A will be similar to a matrix D by a matrix S defined below. This matrix D is the diagonalization of the matrix A.

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Next, we compute the inverse of S and use properties of similarity to find D. We will do this below:

$$S^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$3S^{-1}A = \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 6 & 6 \end{bmatrix}$$

$$\therefore\ S^{-1}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

Finally, multiply by S to and use properties of similarity to compute D.

The matrix D is:

$$D = S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

These steps will always yield the matrix D, but the similarity of A and D is not part of finding D as much as it is a beneficial property of D. The diagonalized matrix is a lot more simple and easier to work with, and is similar to A, and the similarity is why it is valuable, not just a part of the process of finding it. In fact, you don't need to use similarity properties at all to find D. Consider the following:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Shortcuts to finding D are perfectly valid. Given we have $S = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$, the matrix D should be the above matrix where λ_1 is the eigenvalue that produced the eigenvector $\vec{v}_1$ and so on.