Document 2: Solving Linear Systems

Gauss-Jordan Elimination: This is a strategy for solving linear equations. One example of Guass or Gauss elimination is below.

Solve the following system: $\begin{cases} x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + x_2 - x_3 = 6 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & -3 & 5 & -4 \\ 0 & 1 & 1 & 4 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & -3 & 5 & -4 \\ 0 & 0 & \frac{8}{3} & \frac{8}{3} \end{array}\right]$$

Hence, the system is transformed into a new system.

$$\begin{gathered} 2x_1 + x_2 - x_3 = 6 \\ -3x_2 + 5x_3 = -4 \\ 8x_3 = 8 \end{gathered}$$

This system is much easier to solve, but it can be reduced further. Guass-Jordan elimination leaves only numbers along the top-left to bottom-right diagonal. When multiplying rows, we often choose a reference point we call the pivot.

Given a different set of equations, lets solve the system with Gauss-Jordan elimination. We will reduce the matrix to echelon form.

Let the system be $\begin{cases} x_1 + x_2 + x_3 + 4x_4 = 4 \\ 2x_1 + 3x_2 + 4x_3 + 9x_4 = 16 \\ -2x_1 + 3x_3 - 7x_4 = 11 \end{cases}$.

We rewrite the system in matrix form and transform it to reduced row echelon form by performing operations on each row, or adding rows to other rows.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 4 \\ 2 & 3 & 4 & 9 & 16 \\ -2 & 0 & 3 & -7 & 11 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & 4 \\ 0 & 1 & 2 & 1 & 8 \\ 0 & 2 & 5 & 1 & 19 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & -4 \\ 0 & 1 & 2 & 1 & 8 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

After reducing this matrix to echelon form, we have the following system of equations, where x_4 is considered a "free parameter".

$$\begin{cases} x_1 + 2x_4 = -1 \\ x_2 + 3x_4 = 2 \\ x_3 - x_4 = 3 \end{cases} = \begin{cases} x_1 = -1 - 2x_4 \\ x_2 = 2 - 3x_4 \\ x_3 = 3 + x_4 \\ x_4 = x_4 \end{cases}$$

In cases of a free parameter, we often use the variable t; hence, here we define $t = x_4$, and redefine our solution set using the following matrices.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

This system has infinite solutions. A system that has no solution would have a row of all zeros, with a non-zero number on the right hand side of the vertical line. Below are some useful terms to describe behaviors we have see while solving the systems.

Definition: an m x n matrix is in Row Echelon Form if

1. All zero rows appear at the bottom.
2. If a row has nonzero entries, then the first nonzero entry is 1 (This is called the leading one).
3. If a row contains a leading one, then each row above contains a leading one further to the left.

Definition: an m x n matrix is in Reduced Row Echelon Form if

1. All zero rows appear at the bottom.

2. If a row has nonzero entries, then the first nonzero entry is 1 (This is called the leading one).

3. If a row contains a leading one, then each row above contains a leading one further to the left.

4. If a column contains a leading one, then all other entries in that column are zero.