Document 13: Kernels and Images

The following equivalent properties are useful to know for kernels and images. For each property consider $T : \mathbb{R}^n \to \mathbb{R}^m, \ker(T) = \ker(A) = \{x : A\vec{x} = 0\}, A\vec{x} = y, \operatorname{im}(T) = \operatorname{im}(A) = $ span(columns space of $A$).

1. A is invertible
2. $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.
3. $\operatorname{rref}(A) = I_n$
4. $\operatorname{rank}(A) = n$ has no redundancy
5. $\ker(A) = \{\vec{0}\}$
6. $\operatorname{im}(A) = \mathbb{R}^n$

For a matrix $A$, compute the reduced row echelon form, then find the columns that have two leading ones. If these columns are $A_1, A_3$ for instance, we have found the image $\operatorname{im}(A) = \operatorname{span}\{A_1, A_3\}$. Solving $A\vec{x} = \vec{0}$ and placing it in closed form might look like $\vec{x} = sM_1 + tM_2$. Here, we have found the kernel $\operatorname{span}\{M_1, M_2\}$.

Now, lets introduce the concept of subspace.
Definition: Subspaces of $\mathbb{R}^n$. Let $w \subset \mathbb{R}^n$. W is a linear subspace of $\mathbb{R}^n$ if:

1. $\vec{0} \in \mathbb{R}^n$ is in $W$.
2. $W$ is closed under addition.
3. $W$ is closed under scalar multiplication.
4. given $T(\vec{x}) = A\vec{x}, T : \mathbb{R}^n \to \mathbb{R}^m$, $\ker(T) = \ker(A)$ is a subspace of $\mathbb{R}^n$.
5. given $T(\vec{x}) = A\vec{x}, T : \mathbb{R}^n \to \mathbb{R}^m$, $\operatorname{im}(T) = \operatorname{im}(A)$ is a subspace of $\mathbb{R}^m$.

SUMMARY TABLE: Subspace

| Dimention | $W \subset \mathbb{R}^2$ | $W \subset \mathbb{R}^3$ |
|---|---|---|
| 3 | NA | $\mathbb{R}^3$ |
| 2 | $\mathbb{R}^2$ | plane through $\vec{0}$ |
| 1 | line through $\vec{0}$ | line through $\vec{0}$ |
| 0 | $\{\vec{0}\}$ | $\{\vec{0}\}$ |

We also have the concept of redundancy and linear independence. Consider the following definition.
Definition: linear independence. Let $S = \vec{v_1}, \cdots, \vec{v_n} \in \mathbb{R}^n$.

1. $\vec{v_i} \in S$ is redundant if $\vec{v_i}$ is a linear combination of vectors in S.

2. S is a linearly independent set if none of the vectors in S is redundant.

Finally, we have the concept of bases. In 2D space, we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which can be used to create every other vector. The set that contains them would be the basis.
Definition: Let $s = \{\vec{v_1}, \cdots, \vec{v_n}\}$ such that each element $v_i$ is linearly independent of the others and $v_i \in V \subset \mathbb{R}^n$. We say $s$ forms a basis of $V$ if and only if $\forall \vec{v} \in V, \exists \vec{v_1}, \cdots, \vec{v_n}, \exists c_1, \cdots, c_n, \vec{v} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$. In other words, if each vector in $V$ can be written in coordinates using elements in vector $s$, and each element of vector $s$ is linearly independent, then $s$ is a basis.
A basis must be optimal, there can be no redundancies in a set for it to be considered a basis. Now, lets define a dimension.
Let $\{v_1, \cdots, v_p\} \in V \subset \mathbb{R}^n$, and $\{w_1, \cdots, w_q\} \in V \subset \mathbb{R}^n$. If $v_i$'s are linearly independent and $\text{span}\{w_1, \cdots, w_q\} = V$, then $p \leq q$. Given two basis sets where both sets are linearly independent and span $V$, the number of elements in each basis are less than or equal to each other (i.e. $p \leq q \wedge q \leq p$), therefore they are equal in number. Thus, all bases of $V$ must have the same number of elements.

Definition Dimension: The number of vectors in a basis of $V$ is the dimension, (i.e. Let $\mathbb{B}$ be the basis of $V$. $[[\mathbb{B}]] = \dim(V)$).