Document 7: Linear Transformations

Conceptual review of rotations:

We are given a vector $\vec{x}$ broken up into a vertical component $x_2$ and a horizontal component $x_1$, which is rotated up into quadrant 1 by an angle $\alpha$. We want to rotate this vector by an angle $\theta$ about the origin from its current position of $\alpha$ above the positive $x$-axis. First, we know $x_1 = |\vec{x}|\cos(\alpha)$ and $x_2 = |\vec{x}|\sin(\alpha)$. After the transformation, we will have $x_1\prime = |\vec{x}|\cos(\alpha+\theta)$ and $x_2\prime = |\vec{x}|\sin(\alpha+\theta)$. Therefore, $x_1\prime = |\vec{x}|\cos(\alpha)\cos(\theta) - |\vec{x}|\sin(\alpha)\sin(\theta)$ and $x_2\prime = |\vec{x}|\sin(\alpha)\cos(\theta) + |\vec{x}|\sin(\theta)\cos(\alpha)$. Hence, we have $x_1\prime = x_1\cos(\theta) - x_2\sin(\theta)$ and $x_2\prime = x_1\cos(\theta) + x_2\cos(\alpha)$.

SUMMARY TABLE: Transformations Table

| Name | Formula | Matrix Representation |
|---|---|---|
| Projection | $\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$ | $A = \begin{bmatrix} (u_1)^2 & u_1u_2 \\ u_1u_2 & (u_2)^2 \end{bmatrix}$ |
| Reflection | $\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$ | $A = \begin{bmatrix} 2(u_1)^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2(u_2)^2-1 \end{bmatrix}$ |
| Rotation | $R_\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{i\theta}\vec{z}$ | $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ |

$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $z = x_1 + x_2 i_1$, $i = \sqrt{-1}$.

Eulers Identity says: $e^{i\theta}(x_1 + x_2 i) = (\cos(\theta) + \sin(\theta)i$