Document 16: Linear Spaces

In this section, we are investigating Linear Spaces. First, it is important to define what a linear space is.

Definition: A linear space V is a set that satisfies the following:
Let $f, g, h \in V, c, k \in \mathbb{R}$ be arbitrary.

1. $(f+g)+h = f+(g+h)$
2. $f+g = g+f$
3. $\exists 0 \in V$ such that $f+0 = f$
4. $k(f+g) = kf+kg$
5. $(c+k)f = cf+kf$
6. $c(kf) = (ck)f$
7. $1 * f = f$

Revisiting a previous definition, let V be a vector space. A subset $W \subset V$ is a subspace of V if 1) $0 \in W$, 2) W is closed under addition, 3) W is closed under scalar multiplication. Lets add a few more properties.

Definition: Let V be a vector subspace $\{f_1, \cdots, f_n\} \in V$.

1. $\text{span}\{f_1, \cdots, f_n\} = V$ if $\forall f \in V, f = c_1, f_1 + \cdots + c_n f_n$.
2. $\{f_1, \cdots, f_n\} = V$ are linearly independent $\Leftrightarrow (c_1 f_1 + \cdots + c_n f_n = 0 \Leftrightarrow c_1 = \cdots = c_n = 0$.
3. $\mathbb{B} = \{f_1, \cdots, f_n\} = V$ is a basis for V if parts 1) and 2) are satisfied and the right half of the equation $f = \{f_1, \cdots, f_n\} = V$ are teh coordinates of f with respect to $\mathbb{B}$. i.e. $[f]_{\mathbb{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.
4. $T(f) = [f]_{\mathbb{B}}, T : V \to \mathbb{R}^n$ is the B-coordinate Transformation.
5. The $\mathbb{B}$ coordinate transformation is invertible and $T^{-1}(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}) = f_1 + \cdots + f_n = f$

Note, the dimension of a subspace is equal to the number of elements (the cardinality) of the basis. An infinite basis implies an infinite dimension. Now lets revisit linear transformations with this new knowledge in mind.

Definition Let V, W be arbitrary subspaces.

1. $T : V \to W$ is linear if $T(f+g) = T(f) + T(w) \wedge T(kf) = kT(f)$.
2. $\mathrm{im}(T) = \{T(f) : \forall f \in V\}$.
3. $\ker(T) = \{f \in V : T(f) = 0\}$.
4. $\mathrm{im}(T)$ is a subspace of $\mathrm{cod}(W)$ and $\ker(T)$ is a subspace of $\mathrm{dom}(V)$.
5. $\mathrm{im}(T)$ has finite dimensions $\Rightarrow \dim(\mathrm{im}(T)) \equiv \mathrm{rank}(T)$. And $\ker(T)$ has finite dimensions $\Rightarrow \dim(\ker(T)) \equiv \mathrm{nullity}(T)$.
6. If V has finite dimensions, $\Rightarrow \dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$.