Document 14: Equivalent Properties and Trace

Some additional properties which are equivalent will be noted, and Trace will be described.

Trace is a function that sums all diagonal elements. Lets prove the trace of a 3x3 matrix is a linear transformation. We define the transformation $T : \mathbb{R}^{3\times3} \to \mathbb{R}$.

Let $a_{11}, a_{12}, \cdots, a_{33}$ and $b_{11}, b_{12}, \cdots, b_{11}$ be arbitrary scalars.

$$\text{Set } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

We begin by proving $T(\vec{w} + \vec{v}) = T(\vec{w}) + T(\vec{v})$. Since A, B are arbitrary, lets apply the transformation to A, B.

$$T(A+B) = T\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}\right) = (a_{11}+b_{11})+(a_{22}+b_{22})+(a_{33}+b_{33})$$

$$\text{Now, } T(A+B) = (a_{11}+a_{22}+a_{33}) + (b_{11}+b_{22}+b_{33}) = T(A) + T(B)$$

Next we prove $T(k\vec{v}) = kT(\vec{v})$. Since A is arbitrary, lets apply the transformation to A.

$$T(kA) = T\left(\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}\right) = ka_{11} + ka_{22} + ka_{33}$$

$$\text{Now, } T(kA) = k(a_{11} + a_{22} + a_{33}) = kT(A)$$

Hence, since $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \wedge T(k\vec{v}) = kT(\vec{v})$, we have T is a linear transformation.

On the fundamental theorem of linear algebra, recall for a matrix $A_{m\times n}$, $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$. Now, $\dim(\ker(A))$ is the number of redundant column vectors, so $\dim(\ker(A)) = \mathrm{nullity}(A)$. Additionally, $\dim(\mathrm{im}(A))$ is the number of non redundant column vectors, so $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$.

Equivalent Properties (With Additions):

1. A is invertible
2. $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.
3. $\text{rref}(A) = I_n$
4. $\text{rank}(A) = n$ has no redundancy
5. $\ker(A) = \{\vec{0}\}$
6. $\text{im}(A) = \mathbb{R}^n$
7. Columns of A form a basis for $\mathbb{R}^n$
8. Columns of A span $\mathbb{R}^n$
9. Columns of A are linearly independent