Document 20: Orthogonality

Recall the following properties of orthogonality from the previous chapters. These will guide this section.

1. $\vec{v} \perp \vec{w} \Leftrightarrow \vec{v} \cdot \vec{w} = 0$
2. Length$(\vec{v}) \equiv$ norm$(\vec{v})$. $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$
3. $\vec{u}$ is a unit vector if $||\vec{u}|| = 1$

Definition of Orthogonal: $\{\vec{u}_1, \cdots, \vec{u}_m\} \in \mathbb{R}^n$ are orthogonal if $||\vec{u}_i|| = 1$ and $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Let us now generalize the projection formula. Given $\vec{x}, \{\vec{u}_1, \cdots, \vec{u}_m\}$, seeking $\vec{x}^{||} = \text{proj}_V(\vec{x})$, we get $\vec{x}^{||} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2 + \cdots + (\vec{x} \cdot \vec{u}_m)\vec{u}_m$.

There also exists some general properties of orthogonality. Let us consider the orthogonal compliment.

Definition Orthogonal Compliment: Let V be a subspace of $\mathbb{R}^n$. Then, the orthogonal compliment of V is $V^\perp = \{\vec{x} \in \mathbb{R}^n : \exists \vec{v} \in V, \vec{v} \cdot \vec{x} = 0\}$. Interestingly:

1. $V^\perp$ is also a subpsace of $\mathbb{R}^n$
2. $V^\perp = \ker(\text{proj}_V(\vec{x}))$
3. $V \cap V^\perp = \{\vec{0}\}$
4. $\dim(V) + \dim(V^\perp) = n$
5. $(V^\perp)^\perp = V$

Finding the orthogonal basis of a subspace V can be done algorithmically through what is called the Gram-Schmidt Process.

Given a basis of $V = \{\vec{v}_1, \cdots, \vec{v}_m\}$, we want the orthogonal basis $\{\vec{u}_1, \cdots, \vec{u}_m\}$. Find the vectors in the basis using the following equations:

$$\vec{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||}$$
$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{||\vec{v}_2{}^{\perp}||}, \vec{v}_2{}^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$
$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{||\vec{v}_3^{\perp}||}, \vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$
$$\vdots$$
$$\vec{u}_m = \frac{\vec{v}_m^{\perp}}{||\vec{v}_m^{\perp}||}, \vec{v}_m^{\perp} = \vec{v}_m - (\vec{u}_1 \cdot \vec{v}_m)\vec{u}_1 - \cdots - (\vec{u}_{m-1} \cdot \vec{v}_m)\vec{u}_{m-1}$$

Definition Orthogonal Transformations: Let $T : \mathbb{R}^n \to \mathbb{R}^n$. T is orthogonal if it preserves the length of vectors. $||T(\vec{x})|| = ||\vec{x}||, \forall \vec{x} \in \mathbb{R}^n$.

Definition Orthogonal Matrix: If $T(\vec{x}) = A\vec{x}$ is orthogonal, then A is an orthogonal matrix and $||A\vec{x}|| = ||\vec{x}||$

Property: Orthonormal Vectors: These vectors make up a basis for a given space. Hence, they are linearly independent and span the space.

Properties of Orthogonality & Transposes:

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal. Let $\vec{v} \perp \vec{w}$, then $T(\vec{v}) \perp T(\vec{w})$
2. T is orthogonal $\Leftrightarrow \{T(\vec{e}_1), \cdots, T(\vec{e}_n)\}$ forms an orthogonal basis of $\mathbb{R}^n$
3. $A_{n\times m}$ is orthogonal $\Leftrightarrow$ its columns form an orthogonal basis of $\mathbb{R}^n$
4. A is orthogonal $\Leftrightarrow (A^T = A^{-1} \Rightarrow A^T A = I)$
5. $A,$ and B are orthogonal $\Rightarrow AB$ is orthogonal
6. A^{-1} is orthogonal $\Leftrightarrow A$ is orthogonal
7. A is symmetric if $A^T = A$
8. A is skew symmetric if $A^T = -A$
9. $(AB)^T = B^T A^T$
10. A is invertable$\Rightarrow A^T, (A^T)^{-1}, (A^{-1})^T$ are invertable
11. $\text{rank}(A) = \text{rank}(A^T)$