Document 3: Reduced Row Echelon Form and Rank

Definition: The rank of a matrix A is the number of leading 1s in the reduced row echelon form of A.
Corollary: Consider $A_{mxn}x_{nx1} = b_m$.

1. rank$(A) \leq m \wedge$ rank$(A) \leq n$. Proof: There is at most one leading 1 in each row or colum of the reduced row echelon form.

2. Let A, b be matrices. Define an augmented matrix $[A|\, b]$. Then, rank(A)= $m \to Ax = b$ is consistent. Proof: Let c be any nonzero number. A redundancy is defined as $[0 + 0 + \cdots + 0|\, c]$. There is a leading 1 in each row, so $[0 + 0 + \cdots + 0|\, c] \notin A$

3. If rank$(A) = n$ then $Ax = b$ has at most one solution.

4. rank$(A) < n \to$ there are $n -$ rank(A) free parameters. Either $Ax = b$ is inconssitent or it has ∞-many solutions.

Definition: a matrix is considered consistent if it represents a system of linear equations that has 1 or more solutions. If the system has no solution, or equally if the matrix has a determinant of 0, then the matrix is considered inconsistent. Now, consider the following matrices.

$$A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{bmatrix}_{mxn}, B = \begin{bmatrix} b_{11} \cdots b_{1n} \\ \vdots \\ b_{m1} \cdots b_{mn} \end{bmatrix}_{mxn}, C = \begin{bmatrix} c_{11} \cdots c_{1n} \\ \vdots \\ c_{r1} \cdots c_{rn} \end{bmatrix}_{nxr}$$

Basic matrix algebra is defined as follows. Let k be a scalar, and use matrices A, B and C.

1. Equality: $A = B \leftrightarrow a_{ij} = b_{ij}$ where i and j are typical indices for rows and columns in matrices A and B.

2. Addition/Subtration: $A \pm B = [a_{ij} \pm b_{ij}]$. In other words $A \pm B$ is a matrix where every value at indices i, j is equal to $a_{ij} \pm b_{ij}$.

3. Scalar Multiplication: $kA = [ka_{ij}]$. In other words kA is a matrix where every value at indices i, j is equal to ka_{ij}.

4. Matrix Multiplication: $P_{mxr} = A_{mxn}C_{nxr}$. Where elements of P are determined using the following process. Let i represent indices for rows, drawn from the domain $1 \leq i \leq m$. Let j represent indices for columns, drawn from the domain $1 \leq j \leq r$. Elements of the matrix are defined $p_{ij} = \sum_{k=1}^{n} a_{ik}c_{kj}$.

Product of matricies are defined this way in order to preserve constancy with the composition of linear transformations.

Algebra of Vectors:

Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, A_{mxn} be a matrix with m rows and n columns.

We will use these definitions as we explore the matrix representations of vectors and performing algebra with them. Below, we have these definitions of vector algebra.

1. Scalar Multiplication: $k\vec{u} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$.
2. Dot Product (Scalar Product of Vectors): $\vec{u} \cdot \vec{v} = u_1v_1 + \cdots + u_nv_n$.
3. Outer Product (Not Cross Product): $\vec{u}\text{x}\vec{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ \vdots & & & \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{bmatrix}$
4. Matrix Multiplication: Matrices can be divided into columns and multiplied by a vector in parts. This is equivalent to simple matrix multiplication defined before. An equation demonstrating this multiplication can be found below.

$$A\vec{x} = \begin{bmatrix} A_1 | & A_2 | & \cdots | & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A_1x_1 + A_2x_2 + \cdots + A_nx_n$$

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Linear Combinations:

Finally, we say that a vector $\vec{b} \in \mathbb{R}^n$ is a linear combination if there exists scalars $c_1, \cdots, c_m$, and vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \in \mathbb{R}^n$ such that $\vec{b} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_m\vec{v_m}$. Additionally, the right hand side is considered the span of the vector.