Document 12: Subspaces of $\mathbb{R}^n$

Let X be the domain of a function f, and Y be the codomain. For arbitrary elements $x, y,\ y = f(x), x = f^{-1}(y)$. Given a transformation $T(\vec{x}) = \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$, the transformation is nonlinear, with a domain $\mathbb{R}^1$, and a codomain $\mathbb{R}^2$. The image is a unit circle. Lets try another example. Use the following definition of A for the following example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Define a transformation $T(\vec{x}) = A\vec{x}, T : \mathbb{R}^2 \to \mathbb{R}^3$. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\vec{x} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Since $\vec{x}$ is arbitrary, x_1, x_2 are arbitrary. We are summing all arbitrary combinations of these two vectors. The image this creates is a plane.

The sum of some number of terms defined by an arbitrary scalar multiplied by a specific vector is defined as the span of the specific vectors. However, there is a more formal definition.

Definition: Let $m, n \in \mathbb{R}$ be arbitrary. Let $c_1, c_2, \cdots, c_n \in \mathbb{R}$ be arbitrary. Let $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n} \in \mathbb{R}^m$ be arbitrary vector. We define the span of vectors $\vec{v_1}$ through $\vec{v_n}$ as $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$ and write $\text{span}\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$.

Lets examine some essential properties of images of linear transformations. Then, we will summarize these properties and introduce a corollary.

Properties:

1. $\vec{0}_m \in \mathbb{R}^m$ is in $\text{im}(T)$.

   Proof: $A_{m\times n}\vec{0}_n = \vec{0}_{m\times 1} \Rightarrow T(\vec{0}_n) = \vec{0}_m$

2. Let $\vec{v}_1, \vec{v}_2 \in \text{im}(T)$. Then $\vec{v}_1 + \vec{v}_2 \in \text{im}(T)$.

   Proof: Since $\vec{v}_1, \vec{v}_2 \in \text{imm}(T)$, then $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^n$ such that $T(\vec{w}_1) = \vec{v}_1, T(\vec{w}_2) = \vec{v}_2$. By the linearity of T, we have $T(\vec{w}+\vec{v}) = T(\vec{w})+T(\vec{v})$. Combining, we have $T(\vec{w}_1 + \vec{w}_2) = T(\vec{w}_1) + T(\vec{w}_2) = \vec{v}_1 + \vec{v}_2$. Since T has a codomain that is the image space, $T(\vec{w}_1 + \vec{w}_2)$ is in the image space. Since $T(\vec{w}_1 + \vec{w}_2) = \vec{v}_1 + \vec{v}_2$, the sum $\vec{v}_1 + \vec{v}_2$ is in the image space.

3. Let $\vec{v} \in \mathrm{im}(T), k \in \mathbb{R}$. Then $kv \in \mathrm{im}(T)$.

   Proof: $\vec{v} \in \mathrm{im}(T) \to \exists \vec{w}, T(\vec{w}) = \vec{v}$. Introducing our constant k, we have $kT(\vec{w}) = k\vec{v}$. By linearity, we have $kT(\vec{w}) = T(k\vec{w})$. Hence, $T(k\vec{w}) = k\vec{v}$, and $k\vec{v}$ is in the image.

Summary:

1. $\vec{0}_m \in \mathbb{R}^m$ is in $\mathrm{imm}(T)$.
2. The image is closed under addition.
3. The image is closed under scalar multiplication.

Now lets examine kernels. Kernels are subsets of the domain that satisfy the following for a given transformation $T : S \to D, T(\vec{x}) = A\vec{x}$: $A(\vec{x} \in S) = \vec{0} \in D$. So, $\ker(T)$ is the null space of A.