Document 6: Projections in Linear Space

Here are how projections work. We are given $\vec{x}$ and line l, where l may be $ax + by = c$ or $\vec{u}$. such that $||u|| = 1$. We want $\text{proj}_l(\vec{x}) = \vec{x}^{||}$. We know $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$. Parallel and perpendicular are the vector components. Lets look at what our want looks like in mathematical terms:

$$T(\vec{x}) = \vec{x}^{||}$$

$$\vec{x}^{||} = k\hat{u}$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{||} = \vec{x} - k\hat{u} \perp L(\hat{u})$$

$$\Rightarrow (\vec{x} - k\hat{u}) \cdot \hat{u} = 0$$

$$\vec{x} \cdot \vec{u} - k(\vec{u} \cdot \vec{u}) = 0$$

$$k||\hat{u}||^2 = \vec{x} \cdot \hat{u}$$

$$\Rightarrow k = \vec{x} \cdot \hat{u}$$

Our solution: $\vec{x}^{||} = (\vec{x} \cdot \hat{u})\hat{u}$, so $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \hat{u})\hat{u}$. This is how to find the projection for any matrix.

We introduce a new term, span, which lets us create planes between vectors. Consider the following.

$$\text{Let } \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, L = \text{span}\left\{\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right\} = \text{span}\{\vec{v}\}$$

$$L = c\vec{v}, ||\vec{v}|| = \sqrt{4^2 + 3^2} = 5$$

$$\text{Let } \vec{u} \text{ be a unit vector equal to } \hat{u}, \vec{u} = \frac{\vec{v}}{||\vec{v}||}$$

MATRIX REPRESENTATIONS AND FORMULAS

Projections:

$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u} = \left(\begin{bmatrix}x_1\\x_2\end{bmatrix} \times \begin{bmatrix}u_1\\u_2\end{bmatrix}\right) \times \begin{bmatrix}u_1\\u_2\end{bmatrix} = (x_1u_1 + x_2u_2)\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}(u_1)^2x_1 + u_1u_2x_2\\u_1u_2x_1 + (u_2)^2x_2\end{bmatrix} = \begin{bmatrix}(u_1)^2 & u_1u_2\\u_1u_2 & (u_2)^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

Reflections:

Begin with a line L, input vector $\vec{x}$, we want to get a reflection of the line, called $\text{ref}_L(\vec{x})$.
We begin with $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$. The reflection is $\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = \vec{x}^{\parallel} - (\vec{x} - \vec{x}^{\parallel}) = 2\vec{x}^{\parallel} - \vec{x}$.
So, $\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2\begin{bmatrix}(u_1)^2 & u_1u_2\\u_1u_2 & (u_2)^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}x_1\\x_2\end{bmatrix}$

Rotations (anticlockwise):

Visually, we have $\vec{x}$ with a certain rise and run that we want to rotate. Imagine sweeping $\vec{x}$ anticlockwise 90 degrees. The original x component would now be the y, and the original y component would now be the same magnitude of the new x component, but opposite direction. We call this new vector, orthogonal to $\vec{x}$, $\vec{y}$. If we use more formal names, lets call the x component of $\vec{x}$, x_1, and the y component of $\vec{x}$, x_2. Thus we have the following.

$$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \perp \vec{y} = \begin{bmatrix}x_2\\-x_1\end{bmatrix}$$

Both vectors are of equal magnitude. To find the $\vec{x}$ rotated anticlockwise θ degrees, we should scale each vector, $\vec{x}$ by $\cos(\theta)$, and $\vec{y}$ by $\sin(\theta)$. These two then become component vectors of our rotated vector. Thus, we have:

$$R_\theta(\vec{x}) = \cos(\theta)\vec{x} + \sin(\theta)\vec{y}$$

$$R_\theta(\vec{x}) = \cos(\theta)\begin{bmatrix}x_1\\x_2\end{bmatrix} + \sin(\theta)\begin{bmatrix}-x_2\\x_1\end{bmatrix}$$

$$R_\theta(\vec{x}) = \begin{bmatrix}\cos(\theta)x_1 - \sin(\theta)x_2\\\cos(\theta)x_2 + \sin(\theta)x_1\end{bmatrix} = \begin{bmatrix}\cos(\theta)x_1 - \sin(\theta)x_2\\\sin(\theta)x_1 + \cos(\theta)x_2\end{bmatrix}$$

$$\text{Hence, } R_\theta(\vec{x}) = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$