Document 18: Change of Bases

In this section, we are investigating changing the basis of a vector subspace. Consider the following example.

$$\text{Let } \mathbb{B}_1 = \{f_1, f_2, \cdots, f_n\}, \mathbb{B}_2 = \{g_1, g_2, \cdots, g_n\}.$$
$$f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$
$$[f]_{\mathbb{B}_1} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Then, putting the coordinates into basis 2 yields:

$$[f]_{\mathbb{B}_2} = [c_1 f_1 + c_2 f_2 + \cdots + c_n f_n]_{\mathbb{B}_2}.$$
$$[f]_{\mathbb{B}_2} = c_1[f_1]_{\mathbb{B}_2} + c_2[f_2]_{\mathbb{B}_2} + \cdots + c_n[f_n]_{\mathbb{B}_2}.$$
$$[f]_{\mathbb{B}_2} = \begin{bmatrix} [f_1]_{\mathbb{B}_2} & [f_2]_{\mathbb{B}_2} & \cdots & [f_n]_{\mathbb{B}_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$
$$[f]_{\mathbb{B}_2} = \begin{bmatrix} [f_1]_{\mathbb{B}_2} & [f_2]_{\mathbb{B}_2} & \cdots & [f_n]_{\mathbb{B}_2} \end{bmatrix} [f]_{\mathbb{B}_1}$$

We define the change of basis transformation in matrix form from $[f]_{\mathbb{B}_1}$ to $[f]_{\mathbb{B}_2}$ with $S = \begin{bmatrix} [f_1]_{\mathbb{B}_2} & [f_2]_{\mathbb{B}_2} & \cdots & [f_n]_{\mathbb{B}_2} \end{bmatrix}$.
Let D be an the standard matrix of a given a transformation $T : V \to V, T(f) = D(f) = f\prime$. Let $f \in V = \text{span}\{e^x, e^{-x}\}$. We want to find the standard matrix for the transformations in two different Bases. We know that $\mathbb{B}_2\{e^x, e^{-x}\}$ and $\mathbb{B}_1\{e^x + e^{-x}, e^x - e^{-x}\}$.

The steps to finding these matricies are always the same.

1. Apply the transformation to each of the bases elements.
2. Rewrite each output in coordinates.
3. Define column matrices in terms of the coefficients of the coordinates.
4. The solution matrix is a block matrix made up of these column vectors in order.

Lets return to our example with basis $\mathbb{B}_1$ and basis $\mathbb{B}_2$ for the transformation $T(f) = f\prime$.

For basis one:

$$\left.\begin{array}{r} T(e^x) = e^x = 1(e^x) + 0(e^{-x}) \vdash \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T(e^{-x}) = -e^{-x} = 0(e^x) + (-1)(e^{-x}) \vdash \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{array}\right\} \therefore A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For basis two:

$$\left.\begin{array}{l} T(e^x + e^{-x}) = e^x - e^{-x} = 0(e^x + e^{-x}) + 1(e^x - e^{-x}) \vdash \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ T(e^x - e^{-x}) = e^x + e^{-x} = 1(e^x + e^{-x}) + 0(e^x - e^{-x}) \vdash \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right\} \therefore B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$