$$dZ = \cancelto{0}{\frac{\partial g}{\partial t}} dt + \cancelto{Y}{\frac{\partial g}{\partial X}} dX + \cancelto{X}{\frac{\partial g}{\partial Y}} dY + \frac{1}{2} \cancelto{0}{\frac{\partial^2 g}{\partial X^2}} (dX)^2 + \frac{1}{2} \cancelto{0}{\frac{\partial^2 g}{\partial Y^2}} (dY)^2 + \cancelto{1}{\frac{\partial^2 g}{\partial X \partial Y}} dX\, dY$$

Itô product:

- $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial xy}{\partial y}\right) \xlongequal[\text{rule}]{\text{constant multiple}} \frac{\partial}{\partial x}(x) = 1$
- $\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial xy}{\partial y}\right) = \frac{\partial}{\partial y}(x) \xlongequal{x \text{ as a constant}} 0; \quad \frac{\partial^2 g}{\partial x^2} = 0$
- $\frac{\partial g}{\partial x} = \frac{\partial xy}{\partial x} = y; \quad \frac{\partial g}{\partial y} = x$
- $\frac{\partial g}{\partial t} = \frac{\partial xy}{\partial t} \xlongequal{x, y \text{ as constants}} 0$

$$= y\,dx + x\,dy + dx\,dy \Rightarrow \boxed{\frac{dZ}{Z} = \frac{dx}{x} + \frac{dy}{y} + \left(\frac{dx}{x}\right)\left(\frac{dy}{y}\right)} \quad \Uparrow$$

Itô compounding rule!

$$\boxed{\frac{dZ}{Z} = \underset{\substack{\text{random process}\\ \downarrow}}{q\,dt} + \frac{dX}{X}}$$

Itô ratio:

$$dZ = \cancelto{0}{\frac{\partial g}{\partial t}} dt + \frac{\partial g}{\partial X} dX + \frac{\partial g}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 g}{\partial X^2} (dX)^2 + \frac{1}{2} \cancelto{0}{\frac{\partial^2 g}{\partial Y^2}} (dY)^2 + \frac{\partial^2 g}{\partial X \partial Y} dX\, dY$$

- $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \frac{y}{x}}{\partial y}\right) = -\frac{1}{x^2}$
- $\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \frac{y}{x}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0; \quad \frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \frac{y}{x}}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\right) = \frac{2y}{x^3}$
- $\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2}; \quad \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}$
- $\frac{\partial g}{\partial t} = 0$

$$= \left(-\frac{y}{x^2}\right) dx + \left(\frac{1}{x}\right) dy + \left(\frac{1}{2}\right)\left(\frac{2y}{x^3}\right) dx^2 - \frac{dx\,dy}{x^2} \Rightarrow \frac{dZ}{Z} = \left(\frac{\cancel{x}}{\cancel{y}}\right)\left(-\frac{\cancel{y}}{x^{\cancel{2}}}\right) dx + \left(\frac{\cancel{x}}{y}\right)\left(\frac{1}{\cancel{x}}\right) dy + \left(\frac{\cancel{x}}{\cancel{y}}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}\cancel{y}}{x^{\cancelto{2}{3}}}\right) dx^2 - \left(\frac{\cancel{x}}{y}\right)\frac{dx\,dy}{x^{\cancel{2}}}$$

$$= \boxed{\frac{dy}{y} - \frac{dx}{x} - \left(\frac{dy}{y}\right)\left(\frac{dx}{x}\right) + \left(\frac{dx}{x}\right)^2}$$

$$\text{Itô log: } dZ = \cancelto{0}{\frac{\partial g}{\partial t}} dt + \underbrace{\frac{\partial g}{\partial X}}_{\frac{1}{X}} dX + \frac{\partial g}{\partial Y} dY + \frac{1}{2} \underbrace{\frac{\partial^2 g}{\partial X^2}}_{\frac{-1}{X^2}} (dX)^2 + \frac{1}{2} \cancelto{0}{\frac{\partial^2 g}{\partial Y^2}} (dY)^2 + \cancelto{0}{\frac{\partial^2 g}{\partial X \partial Y}} dX\, dY =$$

$$= \frac{dx}{x} + \left(\frac{1}{2}\right)\left(\frac{-1}{x^2}\right)\left(\frac{(dx)^2}{1}\right) = \boxed{\frac{dx}{x} - \frac{(dx)^2}{x^2} = \frac{dx}{x} = \left(\frac{dx}{x}\right)^2}$$

$$\text{Itô exp: } dZ = \cancelto{0}{\frac{\partial g}{\partial t}} dt + \frac{\partial g}{\partial X} dX + \cancelto{0}{\frac{\partial g}{\partial Y}} dY + \frac{1}{2} \frac{\partial^2 g}{\partial X^2} (dX)^2 + \frac{1}{2} \cancelto{0}{\frac{\partial^2 g}{\partial Y^2}} (dY)^2 + \cancelto{0}{\frac{\partial^2 g}{\partial X \partial Y}} dX\, dY \Rightarrow$$

$$\Rightarrow \frac{dZ}{Z} = \cancelto{1}{\left(\frac{1}{e^x}\right)\frac{e^x}{1}} dx + \left(\frac{1}{2}\right)\cancelto{1}{\left(\frac{1}{e^x}\right)\left(\frac{e^x}{1}\right)} dx^2 = \boxed{dx + \frac{dx^2}{2}}$$