CH3 - Black-Scholes-Merton (BSM)

Highlights:

1) Won Nobel Prize in 1997 for deriving a closed-form solution to pricing European calls & European puts — Kerry Back, ch3

2) The only **unknown** in BSM is sigma; everything else is observed — Kerry Back, ch4

3) BSM prices (C, P) can be accurately approximated w/ the Binomial Model when Δt is small: $\lim_{\Delta t \to 0} (\text{Binomial price}) = \text{BSM price}$ — Kerry Back, ch5 *

4) The BSM formulas extend to Foreign Exchange — Kerry Back, ch6-7

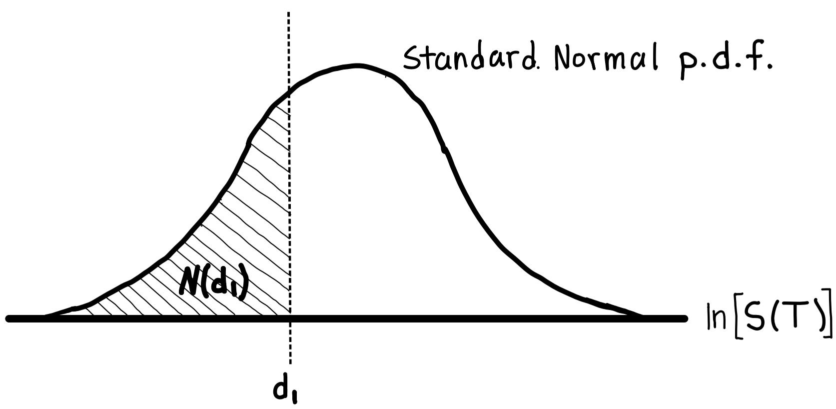
5) Stopped working after the market crash of 1987 aka "Black Monday" (risky asset returns no longer follow a normal distribution)

* for this reason, Binomial approximations (and others...) are generally accepted as the true price of exotic options, which do not have closed-form solutions — Kerry Back, ch8. Closed-form solutions for exotic options are open problems...

BSM for European stock options

$$c(t) \xlongequal{\text{Black-Scholes}} e^{-qT}S(t)N\Big(\underbrace{\frac{\log_e\left(\frac{S(t)}{K}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_1}\Big) - e^{-rT}KN\Big(\underbrace{\underbrace{\frac{\log_e\left(\frac{S(t)}{K}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_1} - \sigma\sqrt{T}}_{d_2}\Big)$$

$$p(t) \xlongequal{\text{Black-Scholes}} e^{-rT}KN\Big(\underbrace{-\frac{\log_e\left(\frac{S(t)}{K}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} + \sigma\sqrt{T}}_{-d_2}\Big) - e^{-qT}S(t)N\Big(\underbrace{-\frac{\log_e\left(\frac{S(t)}{K}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{-d_1}\Big)$$



- $N(d_1)$: the option's delta *iff* S doesn't pay dividends*
- $N(d_2)$: probability that a call (put) option will be ITM (OTM) at expiration
- $e^{-rT}S(t)N(d_1)/N(d_2)$ = expected stock price above strike price at maturity (time T)
- e^{-rT} : present value/cont. compounding
- $d_2 = d_1 - \sigma\sqrt{T}$

* $c(t) = e^{-qT}S(t)N(d_1) - e^{-rT}KN(d_2)$

$$\Longrightarrow \frac{\partial c(t)}{\partial S(t)} = \cancelto{1 \iff q=0}{e^{-qT}} N(d_1) = N(d_1)$$

$\Longrightarrow$ a call's delta $[0,1]$ since $N(d_1)$ integrates over the standard normal probability density function (total area under the curve must add to 1 or 100%).

KEY BLACK-SCHOLES ASSUMPTIONS

- no-arbitrage condition / efficient market hypothesis
- normally distributed stock price returns: $\frac{dS}{S} \sim N(\mu, \sigma)$
  where σ is treated as a constant and μ is a general random process
- continuously compounded risk-free interest rate: e^{rT}
- risk-neutrality
- cannot exercise early (European options)

Using the Fundamental Theorem if Calculus to explain N(d1)

- d_1 represents the standardizes natural log of stock price, $S(t)$, relative to the strike price, K, adjusted for volatility, σ — it is essentially a z-score:

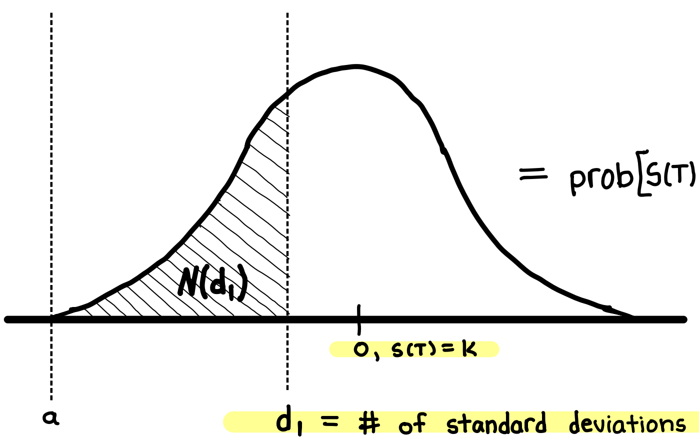
"location parameter"

$$d_1 = \frac{\log_e\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\cancelto{\sqrt{T}}{T}}{\sigma\cancelto{1}{\sqrt{T}}} \implies d_1 = \frac{\overbrace{\boxed{\log_e\left(\frac{S(t)}{K}\right)}}^{X_i} - \overbrace{\boxed{\left(q - r - \frac{1}{2}\sigma^2\right)T}}^{\bar{X}}}{\underbrace{\boxed{\sigma}}_{\text{StdDev}}}$$

"dispersion parameter"

- Which means, by the Fundamental Thm. of Calculus, $N(d_1)$ is the **net displacement** of $\ln\left(\frac{S(t)}{K}\right)$ from it's expected value, measured in standard deviations. This is because integrating d_1 w.r.t. $\ln\left(\frac{S(t)}{K}\right)$ returns the distance function.

- of course, in our application, net displacement is an accumulation of probability rather than distance & the distance function is a p.d.f.



$$= \text{prob}[S(T) < K] = \underbrace{N(d_1)}_{\text{c.d.f.}} = \int_a^{d_1} \overbrace{\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma}(x - \mu)^2\right]}^{\text{p.d.f.}} dx,$$

$$x = \ln\left(\frac{S(t)}{K}\right)$$

d_1 = # of standard deviations S(T) is from K

BSM Greeks for European options

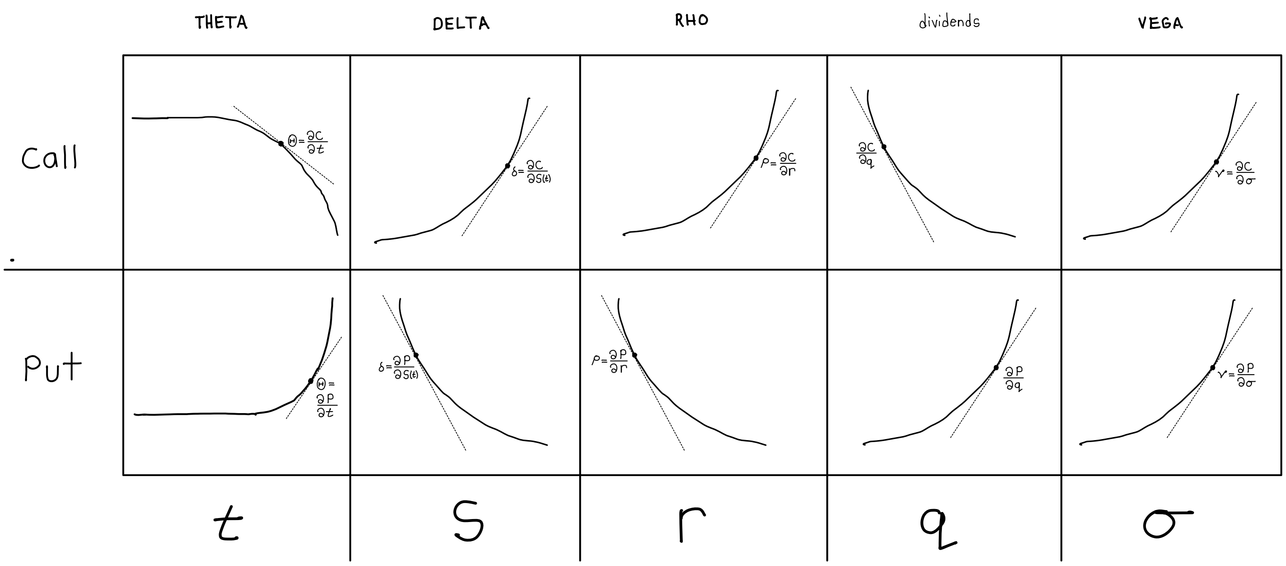
Combinations of t, S, r, q, σ which yield **large** partial derivatives ("large" in absolute value)

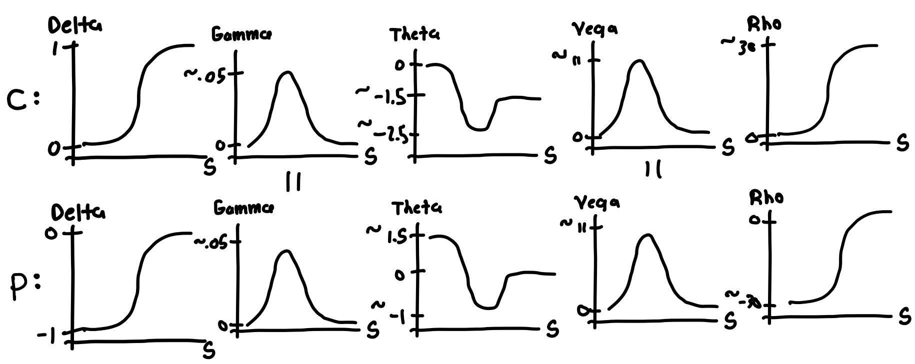
		Call				Put			
		$\frac{\partial C}{\partial S(t)}$ DELTA	$\frac{\partial C}{\partial \sigma}$ VEGA	$\frac{\partial C}{\partial t}$ THETA	$\frac{\partial C}{\partial r}$ RHO	$\frac{\partial P}{\partial S(t)}$ DELTA	$\frac{\partial P}{\partial \sigma}$ VEGA	$\frac{\partial P}{\partial t}$ THETA	$\frac{\partial P}{\partial r}$ RHO
		δ	$\mathcal{V}$	Θ	ρ	δ	$\mathcal{V}$	Θ	ρ
STOCK PRICE	S	n/a	∞	.0001	n/a	n/a	∞	∞	n/a
SIGMA (VOLATILITY)	σ	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
TIME TO MATURITY	T	.0001	∞	.0001	∞	.0001	∞	.0001	∞
RISK-FREE RATE	r	n/a	n/a	.0001	.0001	n/a	n/a	.0001	.0001
DIVIDENDS	q	.0001	.0001	∞	n/a	.0001	.0001	∞	n/a

IMPORTANT
note that Black Scholes is a closed-form solution for all but σ

"n/a" stands for "not applicable"

SHOULD NOT BE SURPRISING, GIVEN THE GREEK FORMULA DERIVATIONS





Precursor; Extensions of Black-Scholes (Chapter 7)

(1) **Margrabe's formula**: value of an option to exchange two risky assets

- since BSM makes no assumption about the currency in which S is denominated, the risky assets may be strock in different currencies
- "no real difference between a call and a put" (pg. 130)

(2) **Black's formula**: value of options on futres when interest rates are deterministic

- assumes $\frac{dF}{F} = \mu dt + \sigma dB$ i.e., assumes a constant forward-rate volatility
- assumes a "discount bond" pays $1 on T instead of assuming a constant risk-free rate (pg. 133)

(3) **Merton's formula**: sub $F(t) = \frac{e^{-qt} S(t)}{P(t,T)}$ into Black's formula

- assumes constant forward-rate volatility (inherited from Black's formula)
- the result is BSM without a constant risk-free rate