WARNING: the concept of Eigenvalues and Eigenvectors only applies to **square** matrices

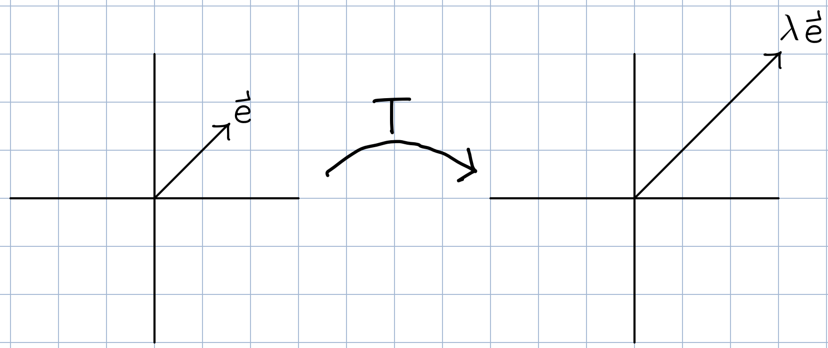
INTRODUCTION TO EIGENVALUES AND EIGENVECTORS

def: Let $A_{n \times n}$. an **eigenvector** $\vec{e}$ is a non-zero vector s.t. $A\vec{e} = \lambda \vec{e}$ for $\lambda \in \mathbb{R}$

def: a scalar $\lambda \in \mathbb{R}$ is an **eigenvalue** of A, if $\exists$ a nontrivial sol. to $A\vec{e} = \lambda\vec{e}$

$$A_{n\times n} : \mathbb{R}^n \mapsto \mathbb{R}^n$$

$$\vec{e} \mapsto A\vec{e} = \lambda\vec{e}\,,$$ λ is a scalar s.t.



i.e., **eigenvectors** are scaled by their cooresponding **eigenvalues**

THE GOAL IS TO REPLACE THE TRANSFORMATION MATRIX A WITH A CONSTANT λ

ex) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\Rightarrow A\vec{e} = \overset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}} \overset{\vec{e}}{\begin{bmatrix} -1 \\ -1 \end{bmatrix}} = -1\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \overset{\lambda}{3} \overset{\vec{e}}{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}$$

$A\vec{e} = \lambda\vec{e}$ same as $A\vec{e} = \lambda\vec{e} = \vec{0}$ same as $(A - \lambda I)\vec{e} = \vec{0}$

- λ is an eigenvalue iff $(A - \lambda I)\vec{e} = \vec{0}$ has a nontrivial sol.
- the unique solution set solves $\text{nul}(A - \lambda I)$ and is called the "eigenspace" of λ, denoted E_λ
- $A\vec{e} = \lambda\vec{e}$ iff $\boxed{\det(\lambda I_n - A) = 0}$ $\equiv$ λ is an eigenvalue of A iff $\boxed{\det(\lambda I_n - A) = 0}$

ex) $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \lambda = 2$ find $E_{\lambda=2}$

$NULL\,(A - \lambda I) = NULL\,(A - 2I)$

(1) $A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$

(2) to find the null space of a matrix is to determine which of the rows go to zero; here we solve $(A - 2I)\vec{e} = 0$ to this end.

$$(A - 2I)\,\vec{e} = 0 : \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = \frac{1}{2}x_2 - 3x_3$$

rref

optional: multiply by 0.5

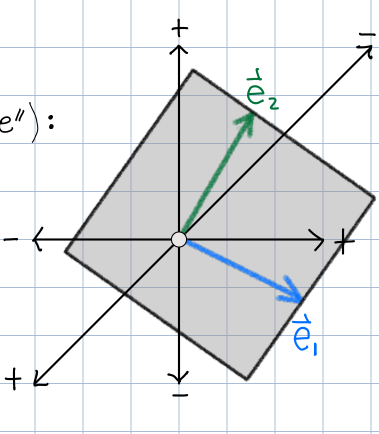
x_2 and x_3 are "free variables"

$$\Rightarrow \vec{e} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = X_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$ is the <u>full decomposition</u> of our null space,

(3) at this point, we really just want to know the **basis** for our null space (called the "eigenbasis"):

$$B_{E_{\lambda=2}} = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \overset{\vec{e}_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \overset{\vec{e}_2}{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

is a plane in $\mathbb{R}^3$ (called the "eigenspace"):



we're allowed to scale one, both, or neither vector to have a prittier representation; here I've simply multiplied the first vector by 2 in order to get rid of the fraction.

*note that the eigenbasis $B_{E_{\lambda=2}}$ is <u>not</u> **unique***