Source: ChatGPT-4; reviewed/edited by Skyler Schneekloth on 02/25/2024

COMMON RISK METRICS

1. **Sample Standard Deviation (Symmetric) aka Volatility *second moment***

   Verbal: Measures the dispersion of asset returns around the mean, capturing overall risk.

   Mathematical (daily, annualized): $\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X})^2} * \sqrt{252}$

2. **Sample Standard Deviation (Asymmetric) aka Downside Deviation**

   Verbal: Measures the dispersion of returns that fall below the mean or a target return, focusing on downside risk.

   Mathematical (daily, annualized): $\sigma_{downside} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\min(0, X_i - \overline{X}))^2} * \sqrt{252}$

3. **Sample Skewness *third moment***

   Verbal: Measures the asymmetry of the return distribution around its mean, indicating the potential for abnormal returns.

   Mathematical (approx): Skewness $= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{X_i - \overline{X}}{\sigma}\right)^3 = \text{AVERAGE}[\left(\frac{X_i - \overline{X}}{\sigma}\right)^3]$

   • For skewness and kurtosis, there's no "standard" annualization formula in the same direct manner as there is for the mean or standard deviation.

4. **Sample Kurtosis *fourth moment***

   Verbal: Measures the "tailedness" of the return distribution, indicating the risk of extreme returns. The kurtosis of a truly normal distribution is 3.

   Mathematical (approx): Kurtosis $= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{X_i - \bar{X}}{s}\right)^4 = \text{AVERAGE}[\left(\frac{X_i - \overline{X}}{\sigma}\right)^4]$

   Mathematical (approx): Excess Kurtosis $= Kurtosis - 3$

   • For skewness and kurtosis, there's no "standard" annualization formula in the same direct manner as there is for the mean or standard deviation.

5. **Jarque-Bera stat (Normality Test)**

   Verbal: A test statistic for assessing if sample data has the skewness and kurtosis matching a normal distribution.

   Mathematical: $JB = \frac{N}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right) = \frac{N}{6}\left(\text{Skewness}^2 + \frac{1}{4}(\text{Excess Kurtosis})^2\right)$

   • H_0: Data follows a normal distribution i.e., skew=0 and kurtosis=3

   • H_a: Data does not follow a normal distribution.

   • Decition Rule: reject H_0 if $JB > \chi^2_{\alpha,2}$, where $\chi^2_{\alpha,2}$ is the critical value from the chi-square distribution with 2 degrees of freedom.

6. **CAPM Beta (Systematic Risk)**

   Verbal: Measures the sensitivity of an asset's returns to the broader market (i.e., sensitivity to systematic risk), capturing market-related risk.

   Mathematical: $\beta = \frac{\text{Cov}(R_a, R_m)}{\text{Var}(R_m)} = \frac{\partial E(r)}{\partial \text{MRP}}$

7. **Non-systematic (Idiosyncratic) Risk**

   Verbal: The portion of total risk that is unique to an individual asset and can be eliminated through diversification.

   Mathematical: $\sigma^2_{\text{nonsystematic}} = \sigma^2_{\text{total}} - \sigma^2_{\text{systematic}}$ where $\sigma^2_{\text{systematic}} = \beta^2 \cdot \text{Var}(R_m)$

8. **Parametric VaR/CVaR**

   Verbal: Estimates VaR/CVaR using the normal distribution assumption and parameters (mean, variance) of returns.

   Mathematical VaR: $VaR = \mu - z_\alpha \sigma$

   Mathematical CVaR: $CVaR = \mu - \frac{\phi(z_\alpha)}{\alpha}\sigma$ where ϕ is the standard normal pdf.

   • Note: while it's standard to report CVaR in *historical* Value at Risk analysis, it is not common practice to do so for *parametric* VaR.

   • Note: Parametric VaR and CVaR can be annualized by multiplying $*\sqrt{\frac{252}{n=\text{holding period}}}$

9. **Historical VaR/CVaR**

   Verbal: Estimates VaR/CVaR based on the historical (actual) distribution of returns over a specific time period.

   Mathematical VaR: The loss L such that $P(X \leq L) = \alpha$ for a given confidence level α; $= \text{PERCENTILE}(array, \alpha)$.

   Mathematical CVaR: $CVaR = \frac{1}{\alpha}\int_0^\alpha VaR(\gamma)\,d\gamma = \text{AVERAGE}(\text{worst } \alpha \text{ of returns})$

   • Note: while it's standard to report CVaR in *historical* Value at Risk analysis, it is not common practice to do so for *parametric* VaR.

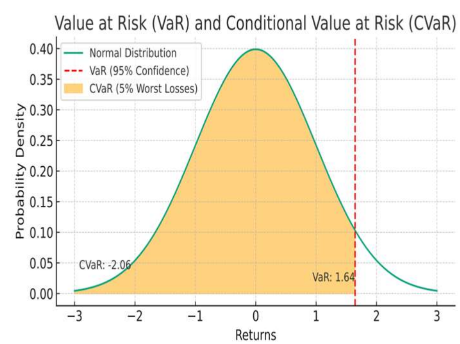
   • Note: Historical VaR and CVaR can be annualized by multiplying $*\sqrt{\frac{252}{n=\text{holding period}}}$

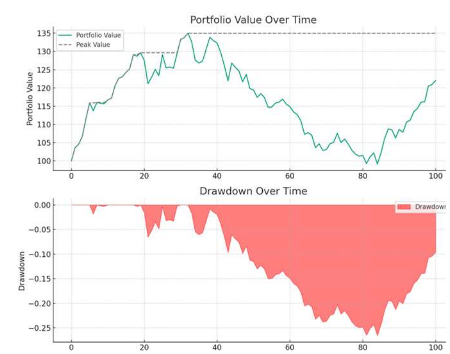
10. **Max Drawdown (an alternative to CVaR)**

    Verbal: The largest peak-to-trough decline in the value of a portfolio or asset over a specific period.

    Mathematical: $MDD = \max_{t\in[0,T]}\left(\max_{t'\in[0,t]}(V_{t'}) - V_t\right)$

CVaR versus Drawdown:





*Notice how measuring risk is equivalent to measuring the **distribution of risky asset returns** (i.e., how certain are these returns from a statistical lens?). You can test the commonly held assumption that risky asset returns are normally distributed w/ Jarque-Bera Stat—a hypothesis test—and, from there, use skewness and kurtosis to make necessary adjustments to your risk assessment.*

Portfolio-level risk calculations can be performed over a single-column vector of portfolio-level returns

Source: ChatGPT-4; reviewed/edited by Skyler Schneekloth on 02/25/2024

2-Asset Portfolio Variance Formula:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

Where:

- w_1 and w_2 are the weights of asset 1 and asset 2 in the portfolio, respectively.
- σ_1^2 and σ_2^2 are the variances of asset 1 and asset 2, respectively.
- ρ_{12} is the correlation coefficient between the returns of asset 1 and asset 2.
- σ_1 and σ_2 are the standard deviations of asset 1 and asset 2, respectively.

3-Asset Portfolio Variance Formula:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 + 2w_1w_3\rho_{13}\sigma_1\sigma_3 + 2w_2w_3\rho_{23}\sigma_2\sigma_3$$

Where:

- w_3 is the weight of asset 3 in the portfolio.
- σ_3^2 is the variance of asset 3.
- ρ_{13} and ρ_{23} are the correlation coefficients between the returns of asset 1 and asset 3, and asset 2 and asset 3, respectively.
- σ_3 is the standard deviation of asset 3.

n-Asset Portfolio Variance Formula:

$$\sigma_p^2 = \mathbf{W}^T\Sigma\mathbf{W} = \sum_{j=1}^{n}\sum_{i=1}^{n} w_iw_j\mathrm{COV}(r_i, r_j),\ \mathrm{COV}(r_i, r_j) = \rho_{ij}\sigma_i\sigma_j$$

Where:

- σ_p^2 is the variance of the portfolio.
- $\mathbf{W}$ is the column vector of the weights of the assets in the portfolio.
- Σ is the covariance matrix of the asset returns.
- $\mathbf{W}^T$ is the transpose of the weights vector.

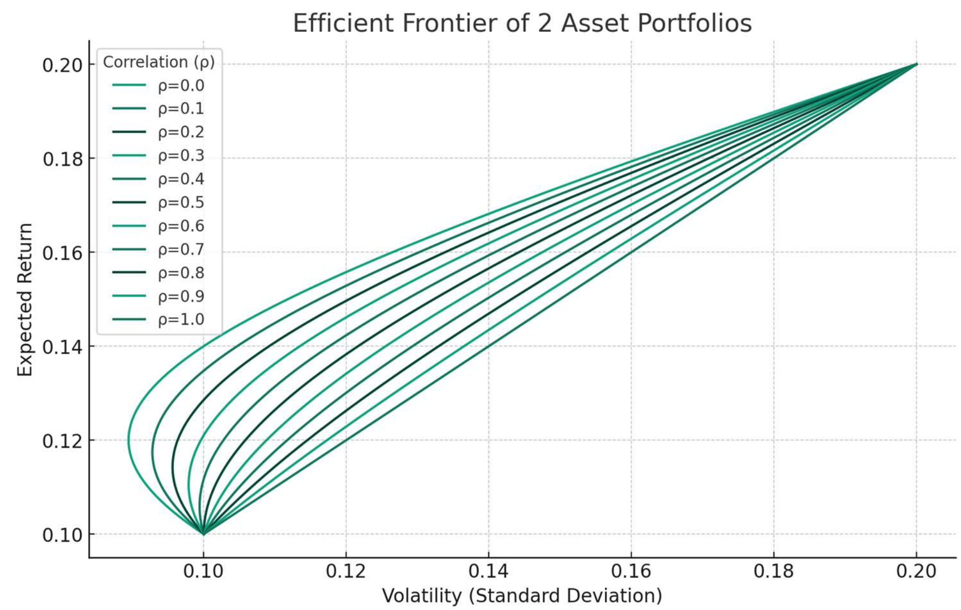
The Power of Diversification:

Suppose you build a two-asset portfolio with 40% allocated to a risky bond, B, and 60% allocated to a risky stock, S.

$$\sigma_P^2 = \boxed{(w_B\sigma_B)^2 + (w_S\sigma_S)^2} \quad \text{Asset-specific terms}$$

$$\text{Diversification terms} \quad \boxed{+2(w_B\sigma_B)\ (w_S\sigma_S)\rho_{BS}}$$

The diversification term(s) adds convexity to a two-asset combination, representing the diversification benefit:



CONCLUSION: When you add decorrelated assets to an investment portfolio, you get more return per unit of risk and the max Sharpe Ratio increases.

Source: ChatGPT-4; reviewed/edited by Skyler Schneekloth on 02/25/2024

COMMON RETURN METRICS

1. **Simple Stock Price Return**
   - **Verbal:** The percentage change in stock price over a given period.
   - **Mathematical:** $\frac{P_{\text{end}}-P_{\text{begin}}}{P_{\text{begin}}} = (\frac{P_{\text{end}}}{P_{\text{begin}}}) - 1$
2. **Simple Stock Price Return using the Midpoint Method**
   - **Verbal:** The percent change in stock price over a given period, relative to the average/midpoint of P_{begin} and P_{end}
   - **Mathematical:** $\frac{P_{\text{end}}-P_{\text{begin}}}{(P_{\text{begin}}+P_{\text{end}})/2}$
3. **Holding Period Return (HPR) aka "Total Return"**
   - **Verbal:** The total return received from holding an asset or portfolio of assets over a period of time, including dividends.
   - **Mathematical:** $\frac{P_{\text{end}}+D-P_{\text{begin}}}{P_{\text{begin}}}$
4. **Annual Percentage Return (APR)**
   - **Verbal:** APR ignores compounding and is an approximation of EAR.
   - **Mathematical:** $APR = r \cdot N \quad \Leftrightarrow \quad r = \frac{APR}{N}$
5. **Effective Annual Rate (EAR) aka Annualized Return / Discretely Compounded Return**
   - **Verbal:** EAR discretely compounds and is more precise than APR.
   - **Mathematical:** $EAR = (1+r)^N - 1 \quad \Leftrightarrow \quad r = (1+EAR)^{\frac{1}{N}} - 1$
6. **Continuous Returns aka Log Return / Continuously Compounded Return**
   - **Verbal:** The return on an investment assuming continuous compounding over time.
   - **Mathematical:** $\ln\left(\frac{P_{\text{end}}}{P_{\text{begin}}}\right)$
7. **Arithmetic Mean *first moment***
   - **Verbal:** Arithmetic means are path dependent and ignore compounding.
   - **Mathematical:** $\frac{1}{n}\sum_{i=1}^{n} r_i$
8. **Geometric Mean aka Compound Annual Growth Rate (CAGR)**
   - **Verbal:** Geometric means—often used to approximate arithmetic means—are **not** path dependent and assume exponential compounding (i.e., the mathematical formula reverse-engineers an exponentially compounded rate of return back into a period annual return with the 1/N exponent).
   - **Mathematical:** $\left(\frac{P_{\text{end}}}{P_{\text{begin}}}\right)^{\frac{1}{N}} - 1 = (\prod_{i=1}^{n}(1+R_i))^{\frac{1}{N}} - 1$
9. **Time-Weighted Rate of Return (Geometric Linking)**
   - **Verbal:** It can be difficult to determine how much money was earned on a portfolio when there are multiple deposits and withdrawals made over time. TWR eliminates the distorting effects on growth rates created by inflows and outflows of money.
   - **Mathematical:**

$$TWR = \left(\prod_{i=1}^{n}(1+HP_i)\right) - 1$$

$$HP = \frac{\text{End Value} - (\text{Initial Value} + \text{Cash Flow})}{(\text{Initial Value} + \text{Cash Flow})}$$

$$HP_i = \text{Return for sub-period } i$$

10. **P-E-R-T**
    - **Verbal:** The formula used to calculate the amount of growth or decay in a continuously compounded investment or loan over time.
    - **Mathematical:** $P = e^{rt}$
11. $(1+r)$
    - **Verbal:** 1+r format is a mathematically convenient equivalent to basic return (easier to work with in Python, R, etc.).
    - **Mathematical:** $\frac{P_{\text{end}}}{P_{\text{begin}}} = 1 + (\frac{P_{\text{end}}}{P_{\text{begin}}} - 1) = 1 + \frac{P_{\text{end}}-P_{\text{begin}}}{P_{\text{begin}}}$
12. **Jensen's Alpha**
    - **Verbal:** The intercept term on a CAPM regression line. Statistically significant alphas reflect an active manager's skill in generating excess returns over the market.
    - **Mathematical:** $\alpha = R_p - (R_f + \beta(R_m - R_f))$, want: t-stat $= \frac{\alpha}{\text{StdErr}} > c$
13. **CAPM Beta**
    - **Verbal:** Doubling as a measure of systematic/market risk, beta equals return attributed to the current market climate ("a rising tide lifts all boats"). Good passive funds have a statistically-significant beta and R-squared ≈ 1.
    - **Mathematical:** $\beta = \frac{\text{Cov}(R_i,R_m)}{\text{Var}(R_m)} = \frac{\partial E(r)}{\partial \text{MRP}}$, want: t-stat $= \frac{\beta}{\text{StdErr}} > c$
14. **Sharpe Ratio**
    - **Verbal:** Excess return per unit of total risk. Typically used to measure the risk-adjusted return of exactly <u>one</u> risky portfolio combined with the risk-free investment option.
    - **Mathematical (daily, annualized):** $SR = \frac{R_p - R_f}{\sigma_p} \times \sqrt{252}$
15. **Treynor Ratio**
    - **Verbal:** Excess return per unit of systematic risk. Typically used to measure the risk-adjusted return of multiple risky portfolios combined together.
    - **Mathematical (daily, annualized):** $TR = \frac{R_p - R_f}{\beta_p} \times 252$
16. **Information Ratio**
    - **Verbal:** Measures the excess return of a portfolio *relative to the return on a <u>benchmark</u> index*, divided by the variability of those excess returns. Quantifies an active manager's skill-level.
    - **Mathematical (daily, annualized):** $IR = \frac{R_p - R_b}{\sigma_{\text{excess}}} \times \sqrt{252}$, σ_{excess} = Tracking Error
17. **Sortino Ratio**
    - **Verbal:** A modification of the Sharpe Ratio that differentiates harmful volatility from total overall volatility by taking into account the standard deviation of negative asset returns, called downside deviation.
    - **Mathematical (daily, annualized):** Sortino Ratio $= \frac{R_p - R_f}{\sigma_{\text{downside}}} \times \sqrt{252}$
18. **Real Returns**
    - **Verbal:** Inflation-adjusted returns.
    - **Mathematical:**

$$1 + r_{\text{Real}} = \frac{1 + r_{\text{No}}}{1 + r_{\text{Inf}}}$$

    Where
    - r_{Real} = the real interest rate
    - r_{No} = the nominal interest rate
    - r_{Inf} = the inflation rate

Portfolio-level returns typically calculated as weighted averages

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Additive Property of Natural Logarithms:

- **Verbal**: The additive property of natural logarithms allows for the summing of log returns over multiple periods to obtain the total return. This is particularly useful for analyzing returns over time, as it simplifies the process of aggregating returns from different time periods.
- **Mathematical**: If you have log returns $r_1 = \ln\left(\frac{P_1}{P_0}\right)$ and $r_2 = \ln\left(\frac{P_2}{P_1}\right)$ for consecutive periods, the total log return over the two periods is simply the sum $r_1 + r_2 = \ln\left(\frac{P_2}{P_0}\right)$. This property holds because of the logarithmic identity $\ln(a) + \ln(b) = \ln(a \times b)$.

Practical Implications:

- **Verbal**: This additive feature makes log returns particularly useful for performance analysis over time, risk management, and portfolio optimization. It simplifies calculations, especially when dealing with compound returns and when comparing returns across different assets or time periods.
- **Mathematical**: For a series of prices $P_0, P_1, \ldots, P_n$, the total log return over the entire period is $\sum_{i=1}^{n} \ln\left(\frac{P_i}{P_{i-1}}\right) = \ln\left(\frac{P_n}{P_0}\right)$. This simplifies the process of calculating compounded returns over multiple periods.

Alpha vs. Beta:

*There are **three** main components of return:*

a measure of systematic risk(s),
which is *potentially rewarded* with expected return.*

Comes from passive management (unlike alpha)

Return = Cash + Beta + Alpha

r_f

a measure of superior or inferior risk-adjusted return; comes from active management (unlike beta)

Expected Returns:

Estimating expected return is all about prediction—your goal is to reasonably forecast asset-level returns. There are conventional ways to calculate expected return and some not-so-conventional ways. Here is a list of options:

- CONVENTIONAL SIMULATION-BASED MODELS
  - Monte Carlo simulation (based on a known/assumed distribution of risky asset returns); typically combined with a lattice structure such as binomial or trinomial trees.
  - Geometric Brownian Motion / Random Processes
  - Finite Difference Model(s) / Jump Diffusion
- CONVENTIONAL FACTOR MODELS
  - CAPM (single-factor): $R_i = \alpha_i + \beta(R_m - R_f) + \epsilon_i$
  - 3-factor Fama-French (1993): $R_i = \alpha_i + \beta_1(R_m - R_f) + \beta_2 SMB + \beta_3 HML + \epsilon_i$
  - Carhart's Four-Factor Model (1997): $R_i = \alpha_i + \beta_1(R_m - R_f) + \beta_2 SMB + \beta_3 HML + \beta_3 UMD + \epsilon_i$
  - 5-factor Fama-French (2015): $R_i = \alpha_i + \beta_1(R_m - R_f) + \beta_2 SMB + \beta_3 HML + \beta_3 RMW + \beta_3 CMA + \epsilon_i$
- UNCONVENTIONAL MODELS
  - Machine Learning algorithms (besides regression analysis)
  - Continuing scholarly research on expected returns. For example, [Goyal and Welch (2008)](#).

Portfolio-level returns typically calculated as weighted averages