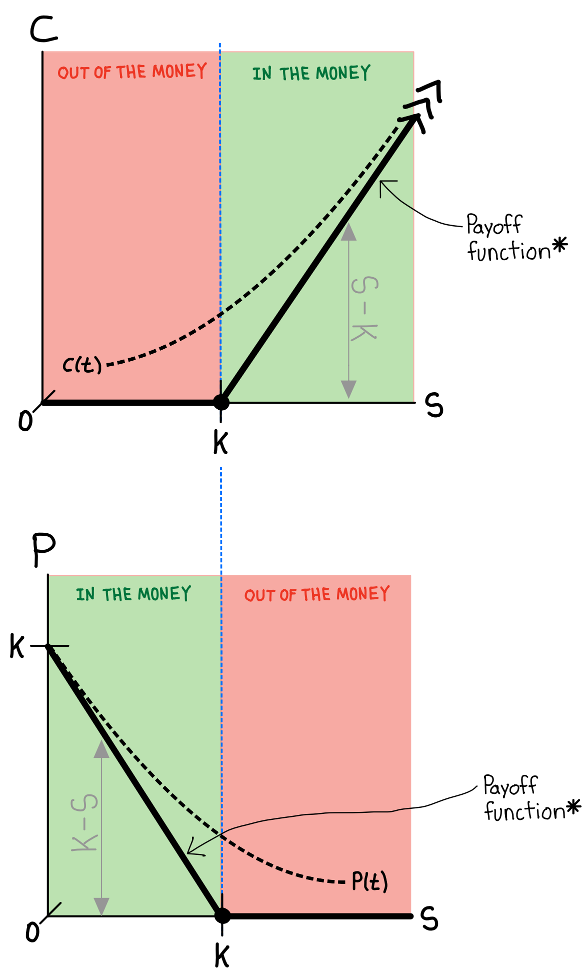
Chapter 1: Asset Pricing Basics

1.1

Call: the option to buy the underlying stock at K (strike price); the LONG position b/c betting on rising stock price, S

put: the option to sell the underlying stock at K (strike price); the SHORT position b/c betting on falling stock price, S

Activation Functions for puts and calls



*note the y-intercept at C=0, P=0 indicating that the payoff ignores acquisition costs

Standard Notation

S = stock price

K = strike price

T = strike time

S − K = profit or loss

C = Value of call option = payoff to call owner

P = Value of put option = payoff to put owner

$C = \max(0, S-K)$; $P = \max(0, K-S)$

European puts & calls vs. American puts & calls

European: cannot exercise until the options expiration date

American: can exercise up to & including the options expiration date

~ caveat #1: you pay a little extra for American puts & calls to have the embedded option to exercise early.

~ caveat #2: → p.7:p.9 while it may sometimes be profitable to exercise an American put option early (before expiration) when the underlying stock pays a large dividend far in adv. of the call's expiration,[1] more optimal choices likely exist if the call owner expects the stock price to plummet; These are:

(1) Short selling*

(2) Sell the American call option

*Short selling offers no downside protection & is therefore considered very risky

[1] there are rare conditions in which the call price falls below the calls payoff function, in which case owner can profit from exercising early

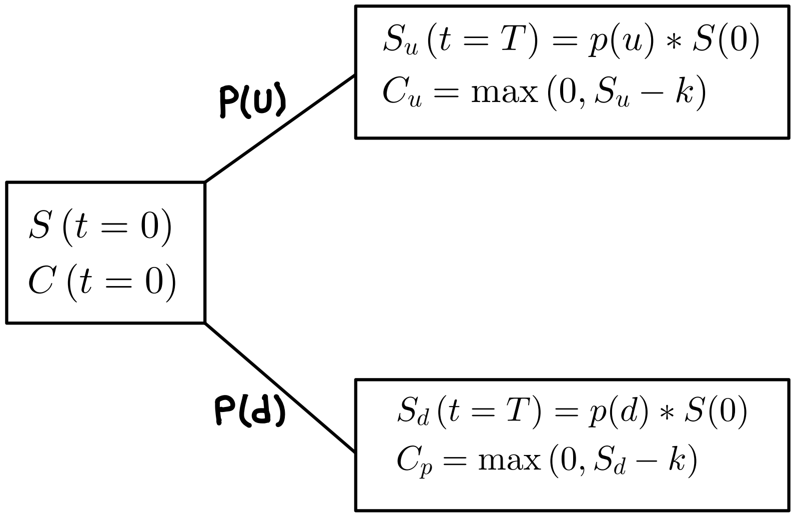
1.2

See book (page 10) for definition of continuously compounding interest

1.3

One-period Binomial Model

e.g. European call option



- no arbitrage condition: $\dfrac{S_u}{S} > e^{rT} > \dfrac{S_d}{S}$, r = risk-free rate

- $P(u) = p(\dfrac{dS(t)}{dt} > 0); P(d) = P(\dfrac{dS(t)}{dt} < 0)$

1.4, 1.5, 1.6 See supplementary notes written in LaTeX.

Covers:

- State Prices and Arrow Securities
- Discrete vs. continuous asset pricing
- “Incomplete” vs. “Complete” markets
- Introduction to estimating probabilities with numeraires

Key Takeaways:

- State Prices and Arrow Securities are fictional, theoretical concepts
- True continuous pricing models rely on Ito calculus (stochastic calculus)
- The convenience factor of numeraires in complete markets that they enable us to “calculate derivative values without having to calculate the [state] probabilities” (pg.16)

Some initial notes on Kerry Back textbook beyond CH3

- different pricing models may have different Greeks
- different pricing models may have different Numeraires (and, by extention, martingale pricing formulas)
- different pricing models may have different Assumptions
- different pricing models may have different put-call parity
- all pricing models are different in their applications, but all based on the same Fundamental Pricing Formula (pg. 19) and the Ito "machinery" in CH 2. Moreover, CH 4, 5, 6, 7 build on the Black-Scholes model whereas CH 8 changes track to focus on exotic options
- one $\frac{-B_T}{\sqrt{T}}$ for every strike date (for some reason, this term was constantly recalled in lecture!)
- Risk-neutral assumption holds througout the whole book, unless otherwise stated
- Bond Price $= Z(t,T) = e^{-r\tau} = e^{-r(T-t)} = e^{-rT}e^{rt}$
- The main novelty of Vasicek model is that it models interest rates with a stochastic process i.e., drops the assumption that interest rates are non-random.
- Vasicek is also a (second-order) ordinary differential equation!
- **CAUTION** SOFR replacing LIBOR will change some of the interest rate option pricing formulas from ch11 and ch12, or at the very least SOFR introduces some complications.

Chapter 4: Estimating and Modelling Volatility

rough notes from class

best estimate: $\hat{\sigma}_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2$

chg in vol: $\hat{\sigma}_{i+1}^2 = (1-\lambda)y_i^2 + \lambda\hat{\sigma}_i^2$, y_i is an observation at date i

$$\hat{\sigma}_{i+2}^2 = (1-\lambda)y_{i+1}^2 + \lambda\hat{\sigma}_{i+1} = (1-\lambda)y_{i+1}^2 + \lambda(1-\lambda)y_i^2 + \lambda^2\hat{\sigma}_i^2$$

$$\hat{\sigma}_{i+3}^2 = (1-\lambda)y_{i+2}^2 + \lambda(1-\lambda)y_{i+1}^2 + \lambda^2(1-\lambda)^2 y_i^2 + \lambda^3\hat{\sigma}_i^2$$

GARCH(1,1): $\sigma_{i+1}^2 = \kappa\theta + (1-\kappa)[(1-\lambda)y_i^2 + \lambda\hat{\sigma}_i^2]$

θ = unconditional variance
κ = weight placed on the unconditional variance
λ = the factor by which weights (within the conditional variance) decrease with time lags

$\Rightarrow$ low κ, high λ give you fat tails: fraction of neg. residuals <2sd from μ is less than 2.275 (z-score for $N(0,1)$)

$$y_i = \frac{\log S(t_i) - \log S(t_{i-1}) - (r - q - \frac{1}{2}\sigma_i^2)\Delta t}{\sqrt{\Delta t}}$$ w/ variance σ_i^2

Chapter 5: Introduction to Monte Carlo and Multi-Period Binomial Model

rough notes from class

- Some things about Monte Carlo...

Quadratic Variation of a Brownian Motion: $Qv = \sum(\Delta B)^2$

Monte Carlo: $Z\sqrt{\Delta t}^2, z \sim N(0,1)$

"a popular Monte Carlo trick":

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{Nt}(\pm 1) \xrightarrow{iid} B_t \;), \quad \mathbb{E}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{Nt}(\pm 1)\right| = \frac{NT}{N} = T \quad \Longrightarrow$$

(1) anytime you do an iid sample, add them up, and then divide by N: that # approaches a normal distribution

(2) generally: if you take a large enough iid sample, you get a normal distribution

Chapter 6: Foreign Exchange

$X_t S_t$ is a joint distribution

Chapter 7: Forwards, Futures, and Exchange Options

pg.129

In this chapter, we will derive three important generalizations of the Black-Scholes formula. We will derive them from the Black-Scholes formula, which shows that all of the formulas are equivalent. We will start with Margrabe's [50] formula for an option to exchange one asset for another. Standard calls and puts are special cases, involving the exchange of cash for an asset or an asset for cash. From Margrabe's formula, we will derive Black's [3] formulas for options on forward and futures contracts. Then, from Black's formulas, we will derive Merton's [51] formulas for calls and puts in the absence of a constant risk-free rate.

rough notes from class

Margrabe's

$e^{-q_1 T} S_1(0) N(d_1) - e^{-q_2 T} S_2(0) N(d_2)$

$\max(0, S_1(T) - S_2(T)) = S_2(T) \max\left(0, \frac{S_1(T)}{S_2(T)} - 1\right)$

is a valid call payoff function because BSM does not depend on currency. Payoff is struct in dollars (the "natural" currency) when asset 2 is the numeraire

$\longrightarrow$ the value of the first asset is S_1/S_2, denominated in USD

$\downarrow$ sub $S_1(0) = P(0,T')F(0)$, $S_2(0) = P(0,T')K$, $q_1 = q_2 = 0$

Black's

let $P(t,T)$ = price of discount bond when $t \leq T$

let $F(t)$ = forward rate

$C = P(0,T')F(0)N(d_1) - P(0,T')KN(d_2)$

$P = P(0,T')KN(-d_2) - P(0,T')F(0)N(-d_1)$

$C: \max(0, P(T,T')[K - F(T)])$

$C + P(0,T')K = P + P(0,T')F(0)$, K units of a discount bond is valued at $\max(F(T), K)P(T,T')$

$\downarrow$ sub $T = T'$, $F(0) = e^{-q'} S(0)/P(0,T)$

Merton's

$C = e^{-qT} S(0) N(d_1) - P(0,T) K N(d_2)$

$P = P(0,T) K N(-d_2) - e^{-qT} S(0) N(-d_1)$

$C: \max(0, F(T) - K)$

no-arbitrage: $F(t) = \dfrac{e^{-q(T-t)} S(t)}{P(t,T)}$

$\xrightarrow{\text{constant } \sigma^2 \text{ iff } P(t,T) = e^{-r(T-t)}} F(t) = e^{(r-q)(T-t)} S(t)$

Chapter 8: Exotic Options ("Exotics")

- all derivations start w/ the Fundamental Pricing Formula
- "the same Itô machinery is applied in each case to obtain a unique pricing formula(s)"

!

Still don't exactly understand how choosing the right numeraire (or any numeraire, for that matter...) enforces the martingale equality. Also, why is martingale sooooo important to the Itô stuff?

→ To get desired martingale (desired for their compatability w/ Itô calculus, which is required to derive pricing formulas):

Step 1: The martingale pricing formula is: $\frac{S(0)}{V(0)} = \mathbb{E}_{(t=0)}\left[\frac{S(T)}{V(T)}\right]$

Step 2: To obtain this, choose between numeraires:

$$V = r - q + \frac{\sigma^2}{2} \qquad\qquad Z = r - q - \frac{\sigma^2}{2}$$

- u and d are INDEPENDENT $\Rightarrow$ $u^n n^m$ follows the mn rule
- $M\left(d_1, d_1', \frac{\sqrt{T'}}{\sqrt{T''}}\right)$ where: $d_1 = -\frac{B_T}{\sqrt{T'}}$, $d_1' = -\frac{B_T}{\sqrt{T'}}$ are upper limits
- establishing lower bounds:

UNDERSTAND PARITY RULES! (used >100x in the textbook)

$$C > C - P \xlongequal{\text{Parity Rule}} \underbrace{Z(t,T)}_{e^{-q(T-t)} \text{ = price of a "discount bond" w/ FV = \$1}} S - \overbrace{e^{-r(T-K)}}^{P} K, \quad C = e^{-r(T-t)} \mathbb{E}_t^{(R)}\left[(S(T) - K)_+\right]$$

Chapter 9: Path-dependent options in the multi-period Binomial model

rough notes from class

An antithetic variant is a random variable, y, with the same mean as x and a <u>negative</u> correlation to x. It follows that the random variabl z=(x+y)/2 will have the same mean as x and a lower variance. Therefore the sam e mean of M simulations of z will be an unbiased estimate of μ but a lower stan ard deviation than the sample mean of M simulations of z" ⇒ z will converg on the closed form solution faster than x thus strengthening the numerical com tation. Besides using an antithetic, "another approach to increasing the efficiency of the M nte Carlo method is to adjust the estimated mean (option value) based on the known mean of another related/correlated variable". According to Kerry Back, the removes th need to estimate B_1 in a regression model if the relationship between x and y is 'one-for-o e' where y is a "control variate". Finally, Richardian extrapolation can be used to estimate what the option value would be w/ an infinite # of periods (=BSM price)

"The whole idea of CH 10 is to replace the partial derivatives in Black Scholes equation w/ limits of the differences" -Palle Jorgensen

- lattice:

A lattice in mathematics can be defined in several ways, depending on the context:

1. **Algebraic Lattice:** In abstract algebra, a lattice is a partially ordered set (poset) in which every pair of elements has a unique supremum (also called a least upper bound or join) and an infimum (greatest lower bound or meet). This type of lattice is studied in order theory and has applications in various areas such as algebra and computer science.

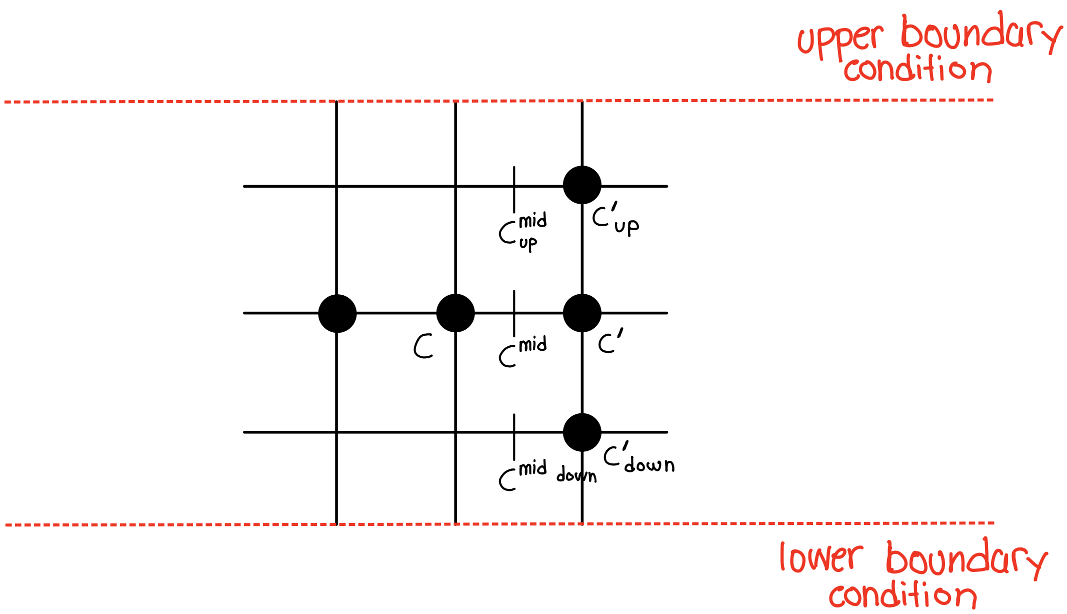
2. **Geometric Lattice:** In geometry, a lattice is a regularly spaced arrangement of points in n-dimensional space. These points often represent the positions of atoms or molecules in a crystal, or the vertices of a geometric structure. This concept is essential in solid state physics and crystallography.

3. **Lattice in Graph Theory:** In graph theory, a lattice can refer to a special kind of graph with properties related to its vertices and edges, which often exhibit a regular, repeating structure.

4. **Lattice in Data Analysis:** In data analysis, lattice can refer to a framework or structure for organizing and visualizing complex, multi-dimensional data sets.

Each of these definitions reflects a different aspect of the concept of a lattice, highlighting its versatility and importance in various fields of mathematics and science.

- Example from ch10:



- a lattice must have boundary conditions, b_i

Fundamental PDE

$$\frac{\partial^2 C}{\partial X^2} \approx \frac{C_{up} - 2C + C_{down}}{(\Delta X)^2}$$

$$\frac{\partial C}{\partial t} \approx \frac{C - C_{left}}{\Delta t} \quad \text{or} \quad \frac{C_{right} - C}{\Delta t}$$

$$rC = \frac{\partial C}{\partial t} + \nu \frac{\partial C}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial X^2}$$

$$\frac{\partial C}{\partial X} \overset{\text{CHAIN}}{=\!=\!=} \frac{\partial C}{\partial S}\frac{\partial S}{\partial X} \overset{\text{Itô Prod.}}{=\!=\!=} S\frac{\partial C}{\partial S}$$

Implicit

$$rC = \frac{C - C_{left}}{\Delta t} + \underset{\substack{\uparrow \\ r - q - \sigma^2/2}}{\nu} \left(\frac{C_{up} - C_{down}}{2\Delta X}\right) + \frac{1}{2}\sigma^2\left(\frac{C_{up} + C_{down} - 2C}{(\Delta X)^2}\right)$$

Explicit

$$rC = \frac{C_{right} - C}{\Delta t} + \nu\left(\frac{C_{up} - C_{down}}{2\Delta X}\right) + \frac{1}{2}\sigma^2\left(\frac{C_{up} + C_{down} - 2C}{(\Delta X)^2}\right)$$

crank-nicolson method

Let's modify the previous notation somewhat, writing C' for C_{right} and C'_{up} and C'_{down} for the values to the right and one step up and down, i.e., at the grid points $(t_i + \Delta t, x_j + \Delta x)$ and $(t_i + \Delta t, x_j - \Delta x)$ respectively. The obvious estimate of the call value at the midpoint $(t_i + \Delta t/2, x_j)$ is the average of C and C', so set

$$C^{\text{mid}} = \frac{C + C'}{2}\,.$$

Analogously, define

$$C^{\text{mid}}_{\text{up}} = \frac{C_{\text{up}} + C'_{\text{up}}}{2}\,, \qquad \text{and} \qquad C^{\text{mid}}_{\text{down}} = \frac{C_{\text{down}} + C'_{\text{down}}}{2}\,. \tag{10.7}$$

The formulas (10.7) give us estimates of the call value at the midpoints one space step up and one space step down from x_j—i.e., at $(t_i + \Delta t/2, x_{j+1})$ and $(t_i + \Delta t/2, x_{j-1})$. We can now estimate $\partial C/\partial X$ and $\partial^2 C/\partial X^2$ at the midpoint $(t_i + \Delta t/2, x_j)$ exactly as before:

$$\frac{\partial C}{\partial X} \approx \frac{C^{\text{mid}}_{\text{up}} - C^{\text{mid}}_{\text{down}}}{2\Delta x}\,,$$

and

$$\frac{\partial^2 C}{\partial X^2} \approx \frac{C^{\text{mid}}_{\text{up}} + C^{\text{mid}}_{\text{down}} - 2C^{\text{mid}}}{(\Delta x)^2}\,.$$

Now, (10.3) becomes

$$rC^{\text{mid}} = \frac{C' - C}{\Delta t} + \nu\left(\frac{C^{\text{mid}}_{\text{up}} - C^{\text{mid}}_{\text{down}}}{2\Delta x}\right) + \frac{1}{2}\sigma^2\left(\frac{C^{\text{mid}}_{\text{up}} + C^{\text{mid}}_{\text{down}} - 2C^{\text{mid}}}{(\Delta x)^2}\right). \tag{10.8}$$

Substituting from the formulas for C^{mid}, $C^{\text{mid}}_{\text{up}}$, and $C^{\text{mid}}_{\text{down}}$, we can re-write (10.8) as

$$\begin{aligned}\left(\frac{r}{2} + \frac{1}{\Delta t} + \frac{\sigma^2}{2(\Delta x)^2}\right)C - \left(\frac{\sigma^2}{4(\Delta x)^2} + \frac{\nu}{4\Delta x}\right)C_{\text{up}}& \\ - \left(\frac{\sigma^2}{4(\Delta x)^2} - \frac{\nu}{4\Delta x}\right)C_{\text{down}} &= \left(\frac{1}{\Delta t} - \frac{r}{2} - \frac{\sigma^2}{2(\Delta x)^2}\right)C' \\ &+ \left(\frac{\sigma^2}{4(\Delta x)^2} + \frac{\nu}{4\Delta x}\right)C'_{\text{up}} + \left(\frac{\sigma^2}{4(\Delta x)^2} - \frac{\nu}{4\Delta x}\right)C'_{\text{down}}\end{aligned} \tag{10.8'}$$

date t_{i+1}, and an integer L from which M is defined as $\boxed{M = (L-1)/2}$ (i.e., $L = 2M + 1$). The function will return the vector of values at date t_i.

We will write the boundary conditions (10.10a) and (10.10b), respectively, in the more general forms

$$C = z_1 + b_1 C_{\text{up}} , \tag{10.11a}$$

and

$$C = z_L + b_L C_{\text{down}} , \tag{10.11b}$$

where z_1, b_1, z_L and b_L are numbers to be calculated or input by the user. The equations (10.10a) and (10.10b) are the special cases in which $z_1 = \lambda_0(S - S_{\text{up}})$, $b_1 = 1$, $z_L = \lambda_\infty(S - S_{\text{down}})$, and $b_L = 1$. The additional generality in allowing b_1 and b_L to be different from one is important for many purposes, and we will see an example of it in the valuation of barrier options.

The system of equations that we want to solve is therefore

upper boundary condition → L

$$\begin{pmatrix} 1 & -b_1 & 0 & 0 & 0 \cdots 0 & 0 & 0 & 0 & 0 \\ -a_3 & a_1 & -a_2 & 0 & 0 \cdots 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & a_1 & -a_2 & 0 \cdots 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & -a_3 & a_1 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & -a_3 & a_1 & -a_2 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & 0 & -b_L & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{L-2} \\ C_{L-1} \\ C_L \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{L-2} \\ z_{L-1} \\ z_L \end{pmatrix}$$

M=(L-1)/2

lower boundary condition

where we are denoting the derivative values to be determined at date t_i across the L $(= 2M+1)$ space nodes as $C_1, \ldots, C_L$. The coefficients a_i are defined in (10.8″). The numbers $z_2, \ldots, z_{L-1}$ are the right-hand sides of (10.8″) and are determined by the coefficients a_i and the derivative values $y_1, \ldots, y_L$ at date t_{i+1}. The system of equations that must be solved to implement the implicit method is of this same form. ← only differences are the values a_i and z_i take!!!

The first equation in this array (equation (10.11a)) can be written as

$$C_1 = u_1 + b_1 C_2 , \tag{10.12a}$$

where $u_1 = z_1$. By induction, we will see that we can write, for each $j = 2, \ldots, L$,

$$C_{j-1} = u_{j-1} + b_{j-1} C_j \tag{10.12b}$$

for some coefficients u_{j-1} and b_{j-1} to be determined. The j–th equation ($j = 2, \ldots, L-1$) in the array (equation (10.8″)) is

$$-a_3 C_{j-1} + a_1 C_j - a_2 C_{j+1} = z_j .$$

Supposing (10.12b) holds and using it to substitute for C_{j-1}, we have

$$-a_3(u_{j-1} + b_{j-1} C_j) + a_1 C_j - a_2 C_{j+1} = z_j$$

$$\iff C_j = u_j + b_j C_{j+1}$$