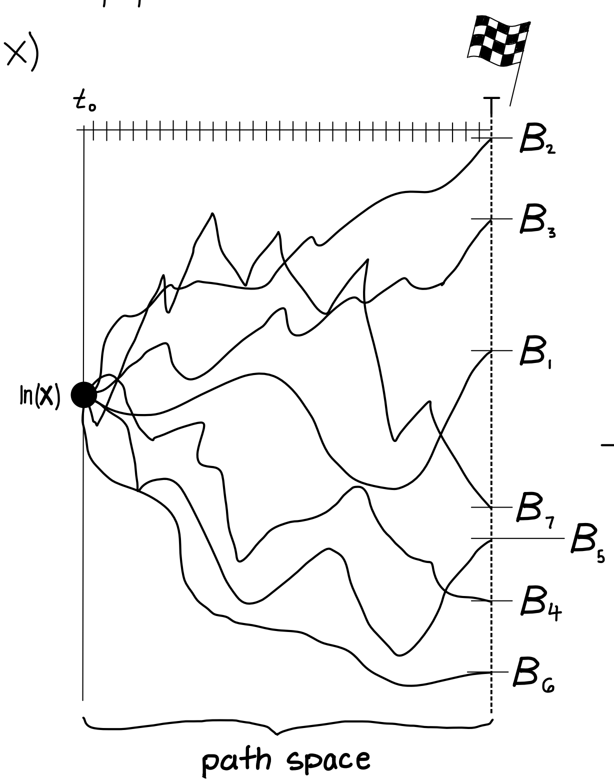
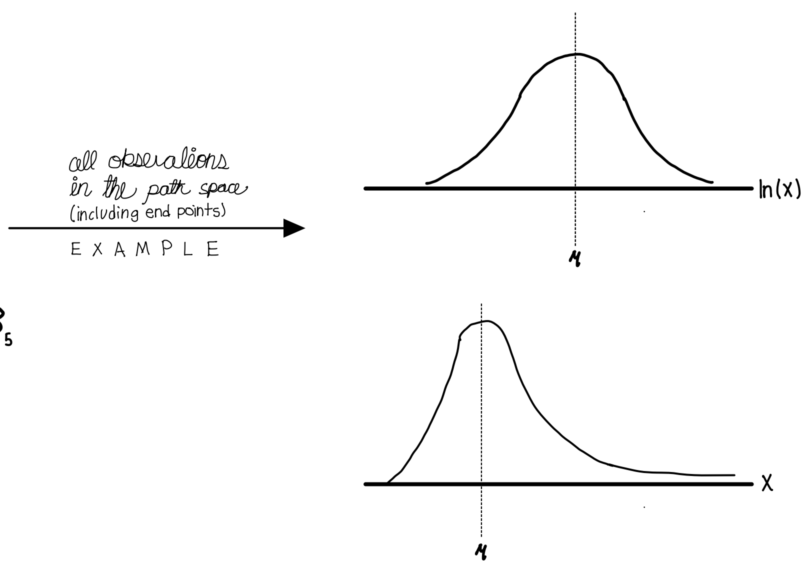
Random processes

a random process is a sequence of random values sampled from a distribution such as the normal distribution indexed by time, describing the evolution in time of a random phenomenon. The outcome is therefore probabalistic.

- a random process can be measured by its **mean**, **autocorrelation** (among the observations), **autocovariance**, **spectral density**, **average power**, **quadratic variation**, & probably several other metrics that you're unaware of.
- most popular random process: Brownian Motion, denoted B
  - ex)



where: each path is its own Brownian Motion "process"



  - our stochastic model is only as good as our assumption that ln(X) is normally distributed over time, for example. Can use certian **numeraires** to strenghen this assumption (pg. 41).

- BM was originally invented to model the erratic movement of microscopic particals in a fluid (scientific application).

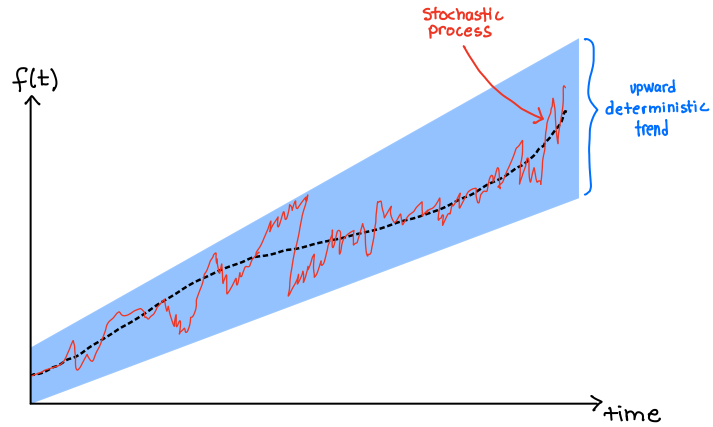
Geometric Brownian Motion (GBM)

GBM is simply Brownian Motion with (linear) drift.

Itô Process – pg. 31

$$dX(t) = \underbrace{\mu(t)dt}_{\text{drift/ deterministic}} + \underbrace{\sigma(t)dB(t)}_{\text{stochastic/ random}}$$

$dX(t)$: random variable X as a fnc. of time



- $\mu(t) = 0 \implies$ standard BM $= \sigma(t)dB(t)$
- $\mu(t)dt$ is a constant $\implies$ BM w/ constant drift *
- $\mu(t)$ is a constant, making $\mu(t)dt$ a linear fnc. of time *

* note that dt is typically held constant anyways, making these two things equivalent (if dt and $\mu(t)$ are both held constant, then so is $\mu(t)dt$).

Definition: geometric brownian motion

Given μ, $\sigma > 0$ and $x_0 > 0$. The GBM is the continuous time process, X_t, that solves the SDE

volatility/diffusion coefficient

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dB_t \\ X_0 &= x_0 \end{aligned}$$

the change in values from one element of any given BM sequence to the next element

held constant

The solution is given by

$$X_t = x_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]$$

$$(dt)^2 = 0\,,$$
$$(dt)(dB) = 0\,,$$
$$(dB)^2 = dt\,.$$

$\Downarrow$

Multiplication Table

	dt	dB
dt	0	0
dB	0	dt

<u>Probability Spaces, Martingales, and Filtration for Brownian Motion</u>

Ω denotes a probability space containing all possible "events"/sample points on which to define a BM

$\mathcal{F}$ stands for a **natural filtration** through the time index of a BM denoting the amount of information known at time t before the next step; takes on an abstract mathematical structure (an abstract algebraic form) called a "σ-algebra"

$\mathcal{F}(t)$ = amount of information at time t = $\tilde{X}$'s path up to & including X_t = the BM up to time t

$\mathcal{F}(t)$ is independently and identically distributed (iid) in time

information accumulates over time s.t. $\mathcal{F}(0) \subset \mathcal{F}(1) \subset \mathcal{F}(2) \subset \dots \subset \Omega$

"filtration" refers to <u>filtering out</u> future information from the BM at time t.

3.3.4 Martingale Property for Brownian Motion

Theorem 3.3.4. *Brownian motion is a martingale.*

PROOF: Let $0 \le s \le t$ be given. Then

$$\begin{aligned}
\mathbb{E}\big[W(t)\big|\mathcal{F}(s)\big] &= \mathbb{E}\big[(W(t)-W(s)) + W(s)\big|\mathcal{F}(s)\big] \\
&= \mathbb{E}\big[W(t)-W(s)\big|\mathcal{F}(s)\big] + \mathbb{E}\big[W(s)\big|\mathcal{F}(s)\big] \\
&= \mathbb{E}\big[W(t)-W(s)\big] + W(s) \\
&= W(s).
\end{aligned}$$

a random walk

does <u>NOT</u> hold for GBM, in which $\mu \neq 0$

Monte Carlo martingale gambling strategy:

$$c(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{P}^*}[S(T)-K]$$

$$Z = \frac{\text{underlying } X}{\text{numeraire } Y}$$

$$Z_t = \mathbb{E}\Big(Z_u \,\Big|\, \underbrace{\text{all information available up to \& including } t = \text{all the sim. events up to \& including } t}_{\mathcal{F}(t)}\Big)$$