Word Problem

The three cases of rates, interest, dividends, and yields from Chapter 11 stem from the idea that you can use a variety of methods to estimate a yield curve. Mostly, this pertains to how the analyst obtains discount factors for the present value formula.

Pricing Method #1:

Discount each cash flow by a constant value, which is the bond's Yield to Maturity/Internal Rate of Return. However, this assumes that proceeds from the investment can always be reinvested at the same rate. This is not generally practical, while methods #2 and #3 are more robust.

Pricing Method #2:

Instead of using the Yield to Maturity as a discount factor, you may treat each cash flow in the Present Value decomposition as a discount, zero-coupon bond. We can derive a spot curve from a list of discount bond prices (since the interest rate on these bonds are embedded into its price). This method avoids the requirement for assuming that bond investors are always able to find yields such that equation 11.1 holds exactly.

Pricing Method #3:

According to Kerry Back's textbook: The most popular method of estimating the yield curve from bond prices is to fit a "cubic spline" (pg.238). This is no more than a series of cubic interpolations between "knot points" along the yield curve for which the discount bond yields are unobservable. This is sometimes necessary to match the compounding frequency of a coupon-paying bond, in order to price it.

Calculate the price of a 2-year, semi-annual coupon paying bond with $1 face value and a 6% annual coupon rate using

1) the **spot curve** method

2) the **Bootstrap** method

3) the **Yield to Maturity** method

step 1 you observe the following discount bond prices on BBG Terminal (Chapter 11, Kerry Back textbook):

Assume the following discount bond prices for each of the following exercises:

* $P(0, 0.5) = 0.995$
* $P(0, 1.0) = 0.988$
* $P(0, 1.5) = 0.978$
* $P(0, 2.0) = 0.966$

$P(0, 2.5) = 0.951$
$P(0, 3.0) = 0.935$
$P(0, 3.5) = 0.916$
$P(0, 4.0) = 0.896$
$P(0, 4.5) = 0.874$
$P(0, 5.0) = 0.850$

t T

11.7. Consider a two-year coupon bond with $1 face value and a semi-annual coupon of $0.03, with the first coupon being six months away.

(a) Compute the price of the bond.
(b) Compute the yield to maturity of the bond using the Excel solver tool.

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step 2 obtain discount factors and solve for price

(1) Spot Curve method

Using spot curve:

$$e^{-rT} = p \Rightarrow \ln(p) = \ln(e^{-rT}) = -rT \Rightarrow -r = \frac{\ln(p)}{T} \Rightarrow r = \frac{-\ln(p)}{T} \quad \ldots$$

$$P = \frac{\$0.03}{(1.010025)^{0.5}} + \frac{\$0.03}{(1.012073)^{1}} + \frac{\$0.03}{(1.010025)^{1.5}} + \frac{\$1.03}{(1.012073)^{2}} \approx \boxed{\$1.0841}$$

see next page for all decimals

Using YTM:

$$P = \frac{\$0.03}{(1+YTM)^{1}} + \frac{\$0.03}{(1+YTM)^{2}} + \frac{\$0.03}{(1+YTM)^{3}} + \frac{\$1.03}{(1+YTM)^{4}} \approx \boxed{\$1.0841}$$

)

$$YTM = IRR \approx \underbrace{\frac{C + \frac{F-P}{n}}{(F+P)/2}}$$

THIS FORMULA IS AN APPROXIMATION FORMULA; BETTER PRACICE TO CALCULATE YTM AS THE BONDS INTERNAL RATE OF RETURN (IRR)

(2) Bootstrap Method

- see next page for extracting/estimating discount factors via Bootstrap (Wiley CFA textbook, Pg. 483)
- Bootstrap-based YTM method identical process to spot curve method (above), but w/ a slightly different bond price...

(1) Spot Curve method

	A	B	C	D	E
1	Maturity	Price	Cont. Compounded Bond Yield		
2	0.5	0.995	0.010025083647088600	=-LN(B2)/A2	
3	1	0.988	0.012072581234269200	=-LN(B3)/A3	
4	1.5	0.978	0.014830405964879800	=-LN(B4)/A4	
5	2	0.966	0.017295722384809500	=-LN(B5)/A5	
6					
7		*CF*	=(1+C2)^A2	=B8/C8	
8	PV1	0.03	1.005000041615470	0.029850745	
9	PV2	0.03	1.012072581234270	0.029642143	
10	PV3	0.03	1.022327884062320	0.029344793	
11	PV4	1.03	1.034890586782430	0.995274296	
12				**1.0841119765321600**	<- Price (Present Value)
13				=SUM(D8:D11)	

(2) Bootstrap method

	A	B	C	D	E
1	Maturity	Price	Cash Flow		
2	0.5	0.995	0.03		
3	1	0.988	0.03		
4	1.5	0.978	0.03		
5	2	0.966	1.03		
6					
7	Price (Present Value):		**1.08381**		
8			=SUMPRODUCT(B2:B5,C2:C5)		

(3) YTM-based present value calculations using Spot Curve method vs. Bootstrap method

	A	B	C	D
1	(1)		(2)	
2	ytm		ytm	
3	0.852209842900442%	=IRR(A5:A9)	0.859533876291962%	=IRR(C5:C9)
4				
5	-1.0841119765321600		-1.08381	
6	0.03		0.03	
7	0.03		0.03	
8	0.03		0.03	
9	1.03		1.03	
10				
11	price		price	
12	**$1.08411197652247**		**$1.08381**	
13	=-PV(A3,4,0.03,1,0)		=-PV(C3,4,0.03,1,0)	

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Implemented in:
BondPrices_SpotCurve.xlxs
on T7Sheild drive!

11.7)

(a) $$P = \frac{\$0.03}{(1.010025)^{.5}} + \frac{\$0.03}{(1.012073)^{1}} + \frac{\$0.03}{(1.01483)^{1.5}} + \frac{\$1.03}{(1.017295)^{2}} = \boxed{\$1.0841}$$

(b) $$\overset{\text{Excel}}{=\!=} IRR(-\$1.0841, \$0.03, \$0.03, \$0.03, \$1.03) = \boxed{.85\%}$$

(c)

T	DCF	T*DCF
.5	$\frac{\$0.03}{(1.010025)^{.5}}$	.0149
1	$\frac{\$0.03}{(1.012073)^{1}}$	.0296
1.5	$\frac{\$0.03}{(1.01483)^{1.5}}$	.044
2	$\frac{\$1.03}{(1.017295)^{2}}$	1.99

time-weighted Present Value

$$\longrightarrow \frac{\Sigma(T*DCF)}{\$1.0841} \overset{\text{Macaulay Durr.}}{=\!=} \underbrace{\boxed{1.92 \text{ yrs.}}}_{\text{always less than maturity!!!}}$$

Present Value

11.8)

(a) $$P = \frac{\$0.04}{(1.010025)^{.5}} + \frac{\$1.04}{(1.012073)^{1}} = \boxed{\$1.0674}$$

(b) $$\overset{\text{Excel}}{=\!=} IRR(-\$1.0674, \$0.04, \$1.04) = \boxed{.6\%}$$

(c)

T	DCF	T*DCF
.5	$\frac{\$0.04}{(1.010025)^{.5}}$	.0199
1	$\frac{\$1.04}{(1.012073)^{1}}$	1.027

time-weighted Present Value

$$\longrightarrow \frac{\Sigma(T*DCF)}{\$1.0674} \overset{\text{Macaulay Durr.}}{=\!=} \underbrace{\boxed{0.98 \text{ yrs.}}}_{\text{always less than maturity!!!}}$$

Present Value

11.9)

(a)

$$P = \frac{\$0.02}{(1+r(1))} + \frac{\$0.02}{(1+r(2))} + \frac{\$0.02}{(1+r(3))} + \frac{\$0.02}{(1+r(4))} + \frac{\$0.02}{(1+r(5))} + \frac{\$0.02}{(1+r(6))}$$

$$+ \frac{\$0.02}{(1+r(7))} + \frac{\$0.02}{(1+r(8))} + \frac{\$0.02}{(1+r(9))} + \frac{\$1.02}{(1+r(10))} = \boxed{\$1.0393}$$

(b)

$$\overset{\text{Excel}}{=\!=} \text{IRR}(-\$1.0393, \$0.02, \$0.02, \$0.02, \$0.02, \$0.02, \$0.02, \$0.02, \$0.02, \$0.02, \$1.02) = \boxed{1.6\%}$$

(c)

T	DCF	T*DCF
.5	$\frac{\$0.02}{(1+r(1))}$	.0099
1	$\frac{\$0.02}{(1+r(2))}$	.0197
1.5	$\frac{\$0.02}{(1+r(3))}$	.0293
⋮	⋮	⋮
4.5	$\frac{\$0.02}{(1+r(9))}$	.0788
5	$\frac{\$1.02}{(1+r(10))}$	4.346

time-weighted Present Value

$$\longrightarrow \frac{\Sigma(T*DCF)}{\$1.0393} \overset{\text{Macaulay Durr.}}{=\!=} \underbrace{\boxed{4.58 \text{ yrs.}}}_{\text{always less than maturity !!!}}$$

Present Value

Using Principal Components as a Hedging Tool

11.10)

pg 249 "to hedge a liability against the factor, we want this coef. (11.19) multiplied by its dollar amount to equal the corresponding coef. for the asset multiplied by its dollar value"

- Treating the \$100 short as the liability, we get:

$$\$100\left(-\underbrace{\boxed{\sum_{1}^{4}\frac{e^{-y_j\tau_j}C_j\tau_j}{P}}}_{\text{Macaulay Duration}}\beta_{j1}\right) = \$100\left(-\sum_{1}^{4}1.92\,\beta_{j1}\right) = -\$192\sum_{1}^{4}\beta_{j1} \overset{\text{pg.249}}{=\!=} x(-4.85)\sum_{1}^{10}\beta_{j1}$$

- From the data on pg. 247 we can obtain values for $\sum_{1}^{4}\beta_{j1}$ and $\sum_{1}^{10}\beta_{j1}$ s.t.

$$-\$192\sum_{1}^{4}\beta_{j1} = x(-4.85)\sum_{1}^{10}\beta_{j1} \xrightarrow{\text{pg.247}} -\$192\,(1.2708) = x(-4.85)(2.1885) \Rightarrow x = \boxed{\$24.33}$$

- K=1 because a "duration hedge" is one that protects against a <u>PARALLEL SHIFT</u> in the yield curve, but not other factors such as changes in slope or changes in curvature/convexity

Duration Metrics

The **duration** of a bond measures the sensitivity of the bond's full price (including accrued interest) to changes in the bond's yield-to-maturity or, more generally, to changes in benchmark interest rates.

There are several types of bond duration. In general, these can be divided into **yield duration** and **curve duration**.

Yield duration is the sensitivity of the bond price with respect to the changes in bond's own yield-to-maturity, or interest rate.	**Curve duration** is the sensitivity of the bond price (or more generally, the market value of a financial asset or liability) with respect to a benchmark yield curve.

* market-Value weighted for portfolio-level calculations

- ## Macaulay duration*

  - $$\frac{\partial \text{ PRESENT VALUE}}{\partial \text{ DISCOUNT RATE}} \approx D = \left\{\frac{1+r}{r} - \frac{1+r+[N\times(c-r)]}{c\times[(1+r)^N-1]+r}\right\} - \underbrace{\boxed{\left(\frac{t}{T}\right)}}_{\text{off-coupon adjustment}} \xrightarrow{\text{zero coupon}} N$$

    zero coupon → $\frac{1+r(1-N)}{r}$; zero coupon → 0

    actual/actual day-count conversio

    where **c** is the coupon rate per period. $\xrightarrow{\text{zero coupon}} 0$

    r = yield to maturity (at times denoted by "y"

    = no. years it takes to pay back the principal investment

    n

- The Macaulay duration is usually expressed in annual terms. To convert it to an annual duration, divide the Macaulay duration by the number of coupon payment periods per year.

- ## Modified duration*

  - $\approx$ Macaulay Duration $\div \left(1+\frac{r}{n}\right) \xrightarrow{\text{zero coupon}} \frac{N}{1+r}$

    = a linear price elasticity w.r.t. r

We can also **approximate modified duration (AMD)** directly:

$$\text{AMD} = \frac{(\text{PV}_-) - (\text{PV}_+)}{2\times(\Delta\text{Yield})\times(\text{PV}_0)} \implies \text{appx. Macauly durr} = \text{AMD}(1+r)$$

where $\mathbf{PV_0}$ is the price of the bond at the current yield, $\mathbf{PV_+}$ is the price of the bond if the yield increases (by ΔYield), and $\mathbf{PV_-}$ is the price of the bond if the yield decreases (by ΔYield).

- Modified duration $\overset{\text{always!}}{<}$ Macaulay duration

- PVBP aka DV01 $= PV_r - PV_{r+1bp}$
  - practically, pg.184: $PVBP = (PV_- - PV_+)/2$
  - $PVBP_{portfolio} = \sum_{i=1}^{k} w_i PVBP_i$, par-value wieghted

- The **price value of a basis point (PVBP)** is an estimate of the change in the full price given a 1 bp change in the yield-to-maturity.

- Yield Convexity $= \frac{\partial^2 \text{ PRESENT VALUE}}{\partial \text{ DISCOUNT RATE}^2} \xrightarrow{\text{zero coupon}} \frac{N(N+1)}{(1+r)^2} \approx N^2$

Like modified duration, convexity can be approximated.

- The **approximate convexity** is calculated by the following:

$$\text{AConv} = \frac{(\text{PV}_-) + (\text{PV}_+) - [2\times(\text{PV}_0)]}{(\Delta\text{Yield})^2\times(\text{PV}_0)}$$

- The **money convexity** of the bond is the annual convexity multiplied by the full price.

- ## Key Rate Duration (KRD)

Interpretation: the price sensitivity to a key rate change is the key rate duration.

$$\frac{\Delta P_i}{P} = -KRD_i * \Delta y(T_i), \quad KRD_i = -\frac{1}{P}\frac{\partial P}{\partial y T_i} \approx \frac{CF_i * \overbrace{Z(0,T_i)}^{Z(0,T_i) = \frac{P_i}{Par} = \text{the discount factor extracted via Bootstrap}}}{P}(T_i)$$

the sum of basic key rate shifts equals the approximated total shift in the yield curve:

$$\frac{\Delta P}{p} = -\sum[KRD_i * \Delta y(T_i)]$$

- ## Curve Convexity

$$\text{Conv} \approx \sum_{i=1} \frac{CF_i * Z(0,T_i)}{P}(T_i)^2$$

- The **effective convexity** of a bond is a curve convexity statistic that measures the secondary effect of a change in a benchmark yield curve.

$$\text{EffConv} = \frac{[(\text{PV}_-)+(\text{PV}_+)]-[2\times(\text{PV}_0)]}{(\Delta\text{Curve})^2\times(\text{PV}_0)}$$

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