TYPES OF SERIES WITH CONVERGENCE AND DIVERGENCE

- A <u>series</u> is a decomposed summation in which the decomposition reveals a <u>numerical pattern</u>
- A series either <u>converges</u> or <u>diverges</u> as $n \longrightarrow \infty$
- If diverges, the series does not have a finite limit
- Radius of Convergence: measures the distance from the center point (c), within which the series converges. Denoted by R.
- Interval of Convergence: The specific set of x values for which a Power Series converges; directly related to the radius of convergence <u>but includes end points (must test each end point to determine if they should be included in the set)</u>

<u>Geometric Series (infinite series)</u>

Form: $$\sum_{k=1}^{n=\infty} ar^{k-1} = (a)\,r^0 + (a)\,r^1 + (a)\,r^2 + (a)\,r^3 + \ldots + (a)\,r^{n-1} + (a)\,r^n$$

where: $a \neq 0$

The hallmark feature of a geometric series is that r gets "powered up" with each term s.t.

e.g.#1 $\quad a = \frac{4}{3},\ r = \frac{1}{3}: \quad \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \ldots$

e.g.#2 $\quad a = \frac{51}{100},\ r = \frac{1}{100}: \quad \frac{51}{100} + \frac{51}{1000} + \frac{51}{10000} + \frac{51}{100000} + \ldots$

CONVERGES: $|a| < 1 \longrightarrow$ converges to: $\frac{a}{1-r}$

DIVERGES: $|a| \geq 1$

a special convenience unique to geometric series

Proof: Let: $(-r)\; S_n = ar^0 + ar^1 + ar^2 + \ldots + ar^n\,(-r)$

$$\Rightarrow S_n - rS_n = [a + \cancel{ar^1} + \cancel{ar^2} + \ldots + \cancel{ar^n}] + [-\cancel{ar^1} - \cancel{ar^2} - \ldots - \cancel{ar^n} - ar^{n+1}]$$

$$= \frac{a - ar^{n+1}}{(1+r)} = \frac{S_n \cancel{(1+r)}}{\cancel{(1+r)}}$$

Taking the limit of the n^{th} partial sum from above, we get:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^{n+1}}{(1+r)} \; \underset{\text{otherwise, the limit D.N.E.}}{\overset{r^{n+1} \text{ goes to } 0 \iff |r|<1}{=\!=}} \; \frac{a}{1+r}$$

IMPORTANT: a geometric series is a power series with constant coefficients

Power Series

Forms: $\sum_{k=0}^{n=\infty} a_k x^k$ or $\sum_{k=0}^{n=\infty} a_k(x - c)^k$; $\sum_{k=1}^{n=\infty} a_k x^{k-1}$ or $\sum_{k=1}^{n=\infty} a_k(x - c)^{k-1}$

where: c is sometimes used to center x. *

notes:
- when we start w/ $k=1$, x^{k-1} is required to preserve the constant/intercept term in our decomposition
- Power Series is the most powerful approximation tool in all of mathematics
- 98% of all Power Series can be studied using the Absolute Ratio test

Tool for finding convergence/divergence: Absolute Ratio Test (works 98% of the time)

Q: what if the limit/direct comparison test is inconclusive?

A: use the Ratio Test to compare the series with itself; most useful for series involving $k!$, r^k, or k^k (only works for positive series)

numerator is the next term in the series

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \begin{cases} L < 1: \sum_{n=1}^{\infty} a_n \text{ Converges} \\ L = 1: \text{INCONCLUSIVE} \\ L > 1: \sum_{n=1}^{\infty} a_n \text{ Diverges} \end{cases}$$

denominator is the curr. term in the series

ex #1) $a_n = \frac{n^n}{n!}$; $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$: $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} =$

$= (n+1)^{n+1-1} = (n+1)^n$

$$= \lim_{n\to\infty} \frac{n^n / n!}{(n+1)^{n+1} / (n+1)!} = \lim_{n\to\infty} \frac{\boxed{(n+1)^{n+1}}}{\boxed{(n+1)}\cancel{n!}} \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} =$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)}{n}\right]^n = \lim_{n\to\infty} \left[1 + \frac{1}{n}\right]^n \xrightarrow[\text{power}]{\text{indeterminant}} e = L > 1$$

ex #2) determine the behavior of $\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

step 1 $a_n = \frac{2^n}{n!}$; $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$ $\boxed{= 2^{(k+1)-k} = 2^{1+k-k} = 2^1 = 2}$

step 2 $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^n / n!}{2^{n+1} / (n+1)!} = \lim_{n\to\infty} \frac{\boxed{(2^{n+1})}(n!)}{\boxed{2}(n+1)!} =$$

$$= \lim_{n\to\infty} \frac{2n!}{(n+1)!} = 2 \lim_{n\to\infty} \frac{n!}{(n+1)!} = 2 \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{n!}(n+1)} = \lim_{n\to\infty} \frac{2}{(n+1)} =$$

THIS COMMON TRICK IS THE REAL POWER/UTILITY OF THE RATIO TEST!

$(n+1)! = (n+1)\underbrace{n(n-1)(n-2)\ldots(1)}_{n!} \implies$

$\implies (n+1)! = (n+1)n!$

$$\xrightarrow[\text{Substitution}]{\text{Direct}} 0 = L < 1 \implies \text{converges}$$

* given a power series centered at c, one of three posible scenarios will occur

① will only conv rge at $x=c$

Radius of Convergence $(R) = 0$

Interval of convergence is $\{c\}$

② will converge for every value of x

Radius of Convergence $(R) = \infty$

Interval of convergence: $(-\infty, \infty)$

③ will converge for values of x that satisfy the inequality $|x-c| < R$

Radius of Convergence (R) is determined using the appropriate test. For example, a geometric series converges if $|r| < 1$, and $|r| = |x-c|$ in a standard power series, meaning $R = 1$.

Interval of Convergence:

$-R < |x-c| < R$; $-R \le |x-c| < R$; $-R < |x-c| \le R$; $-R \le |x-c| \le R$

check End Points $-R$ and R to determine if the interval of convergence is a closed or open set.

#1, #2, and #3 above constitute the convenient properties of Power Series which make them easy to work with to transform a function into a power series (so we can exploit the convenient mathematical properties of power series in order to determine the functions behavior),

1st: Try to obtain $\frac{a}{1-[f(x)]}$ algebraically, which equates to a geometric series of the form $\sum_{k=0}^{\infty} a[f(x)]^k$

2nd: If #1 fails, use the following Theorem for representing functions f(x) as a Power Series

given: $f(x)$

want: $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$

use: the Power Rules

$$\int f(x)\,dx = \sum_{k=0}^{\infty} a \frac{(x-c)}{k+1} + c$$

NOT geometric, but still Power Series

$$f'(x) = \sum_{\boxed{k=1}}^{\infty} a_n \left[K(x-c)^{k-1}\right]$$

Since the derivative of a constant =0, we must start at K=1

Differentiation & integration of a power series will NOT change the radius of convergence (R), but it MIGHT change the interval of convergence (by closing or opening the set at either end point)

Taylor Series and Maclaurin Series

- a Maclauren Series is just a taylor series centered at zero (c=0)

- a Tayor Series is used to find Power Series representations for functions $f\left(x\right)$

- Both are ordinary differential equations

$$\text{Taylor Series } T_n=\sum_{k=0}^{n=\infty}\frac{f^{[k]}(c)}{k!}\left(x-c\right)^k$$

$$\approx f\left(c\right)+f^{'}(c)\left(x-c\right)+\frac{f^{''}(c)}{2!}\left(x-c\right)^2+\ldots+\frac{f^{[\text{n'th derivative}]}\left(c\right)}{n!}\left(x-c\right)^n$$

To solve the Taylor series expantion for a given f(x), which is to convert/simlify the expression into a Power Series:

step 1 derive f(x) n number of times w.r.t. x

step 2 evaluate each derivative at x=c (gives us the numerators f'(c), f''(c), f'''(c), f''''(c), ...

step 3 calculate the coefficients for each (x-c)

note: there is a special trick when f(x) is a trig. fnc.

Telescoping Series (infinite series)

Form: $\sum_{k=1}^{n=\infty}$ fraction − fraction

Tool for finding convergence/divergence: n'th term divergence test (also works for geometric)

only reveals if a series diverges / tells us nothing about convergence.

Given $\sum_{k=1}^{n} a_k$, if $\lim_{n\to\infty} \sum_{k=1}^{n} a_k \neq 0$ then $\sum_{k=1}^{n} a_k$ diverges

Tool for finding convergence/divergence: Partial Fractions

example: Show that $\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$ converges & find it's sum.

PDF: $\frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3} \longrightarrow A=1,\ B=2$

$$\Longrightarrow \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{k+3} \xrightarrow[\text{decomp.}]{\text{series}} \left(\frac{1}{3} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \ldots + \left(\cancel{\frac{1}{n+2}} - \frac{1}{n+3}\right)$$

$$= \frac{1}{3} + \frac{1}{n+3} \Longrightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \boxed{\frac{1}{3}}$$

Harmonic Series (infinite series)

Form: $\sum_{k=1}^{n=\infty} \frac{1}{k}$

Tool for finding convergence/divergence: Integral Test

1st caveat: only works for POSITIVE infinite series

2nd caveat: the value that the improper integral converges to is NOT the same value that the series converges to; we can only say it "converges"

example: $$\sum_{k=1}^{\infty} \frac{1}{k} \xrightarrow[\text{integral}]{\text{convert into an}} \int_1^{\infty} \frac{1}{k}\,dk = \lim_{b\to\infty} \int_1^{b} \frac{1}{k}\,dk = \lim_{b\to\infty} [\ln k]_1^b = \lim_{b\to\infty} [\ln(b) - \cancel{\ln(1)}^{0}] \xrightarrow[\text{substitution}]{\text{Direct}} \infty \Longrightarrow \text{diverges}$$

Tool for finding convergence/divergence: P-Series Test

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^p}}_{\text{Harmonic Series}} \begin{cases} p > 1: & \text{converges} \\ p \leq 1: & \text{diverges} \end{cases}$$

Arithmetic Mean: A Closer Look

Definition and Formula

The arithmetic mean is the sum of all values in a dataset divided by the total number of values. The formula for arithmetic mean is:

Arithmetic Mean = (Sum of all values) / (Total number of values)

Calculating the Arithmetic Mean: Step-by-Step Process

1. Add all the values in the dataset.
2. Count the total number of values.
3. Divide the sum of values by the total number of values.

Properties of the Arithmetic Mean

- The arithmetic mean is influenced by outliers or extreme values.
- It is the most commonly used measure of central tendency due to its simplicity.

Practical Examples in Finance

Calculating Average Investment Returns

A private equity firm may calculate the arithmetic mean of its past investment returns to gauge its historical performance. For example, if the firm's annual returns were 8%, 12%, and 15% over the past three years, the arithmetic mean return would be (8 + 12 + 15) / 3 = 11.67%.

Determining Mean Salary for a Job Role

Companies often use the arithmetic mean to calculate the average salary for specific job roles, helping them remain competitive in the job market. If a company pays its analysts $50,000, $60,000, and $70,000, the arithmetic mean salary would be ($50,000 + $60,000 + $70,000) / 3 = $60,000.

Geometric Mean: A Valuable Tool for Finance Professionals

Definition and Formula

The geometric mean is the nth root of the product of all values in a dataset, where n is the total number of values. The formula for geometric mean is:

$$\text{Geometric Mean} = (\Pi(\text{value1} * \text{value2} * \ldots * \text{valuen}))\hat{}(1/n)$$

Calculating the Geometric Mean: Step-by-Step Process

1. Multiply all the values in the dataset.
2. Count the total number of values.
3. Take the nth root of the product, where n is the total number of values.

Applications in Finance

Evaluating Compounded Investment Returns

The geometric mean is useful for assessing the performance of investments that compound over time. For instance, consider an investment that generates returns of 10%, -5%, and 15% over three years. The geometric mean return would be ((1.10 * 0.95 * 1.15)^(1/3)) - 1 = 6.26%, which better reflects the compound annual growth rate (CAGR) than the arithmetic mean.

Comparing Investment Opportunities with Different Time Horizons

The geometric mean allows investors to compare investments with varying time frames by normalizing returns to a consistent annualized rate.

Asian Options

Asian options are path-dependent and their payoffs are based on he arithmetic average. However, there is no true closed-form solution for Asian options because—while log returns may be normally distributed—their arithmetic averages are not.

Geometric averages have the property that if their underlying values are normally distributed, then so is the geometric average of those values. Therefore, we can obtain a very good estimate for the closed-form solution to Asian options by using the geometric average instead of the arithmetic average in our pricing models.

Harmonic Mean: An Alternative Measure in Specific Cases

The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of a dataset. The formula for the harmonic mean is:

$$\text{Harmonic Mean} = n / (1/\text{value1} + 1/\text{value2} + \ldots + 1/\text{valuen})$$

When to Use the Harmonic Mean

The harmonic mean is most useful when dealing with data involving rates or ratios, such as speed, efficiency, or price-earnings ratios.

Applications in Finance

Calculating Average Rates of Return for Varying Investment Periods

The harmonic mean can help investors determine the average return of multiple investments with different holding periods. For example, if an investor has three investments with holding periods of 1, 2, and 3 years and annual returns of 10%, 15%, and 20%, respectively, the harmonic mean holding period would be $(3 / (1/1 + 1/2 + 1/3)) \approx 1.63$ years.

Assessing the Efficiency of Financial Ratios

The harmonic mean can be used to evaluate the average efficiency of financial ratios, such as the price-earnings ratio, by considering the reciprocal of the ratio as a rate.