INVERTIBLE LINEAR TRANSFORMATIONS

def: a linear transformation $T(A)$ is invertible if the standard matrix A is invertible

- so, we always want A to be invertible.
- a matrix is **invertible** *iff* it's determinant $\neq 0$ and "**singular**" if else (e.g., $\underset{2\times 2}{A^{-1}} = \frac{1}{\det(A)}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ is undefined when $\det(A) = 0$)

$$\left.\begin{array}{l} \bigstar \text{ Req. for invertibility } \bigstar \\ \square \text{ Full Rank} \\ \square \text{ Square Matrix} \\ \square \ \det(A) \neq 0 \end{array}\right\} \longleftrightarrow \begin{array}{l} \bigstar \text{ Req. for a unique, nontrivial solution } \bigstar \\ \square \ A \text{ in } A\vec{x}=\vec{b} \text{ is consistent} \\ \square \ A \text{ in } A\vec{x}=\vec{b} \text{ is invertible} \end{array}$$

SUBSPACES OF $\mathbb{R}^n$

ex)

dimensions	$\mathbb{R}^4$
0	$\{\vec{0}\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
1	lines $(\mathbb{R}^1)$ thru $\vec{0}$, $\vec{0}$ is the origin
2	planes $(\mathbb{R}^2)$ thru $\vec{0}$
3	$\mathbb{R}^3$ thru $\vec{0}$
4	$\mathbb{R}^4$ thru $\vec{0}$

- "domain" aka "range" aka "vector space" aka "column space" all refer to the **span**, $\mathbb{R}^n$, in which vectors exist.

- when a matrix is viewed as a function on vectors, it's outputs $(A\vec{x})$ are "copies" of $\mathbb{R}^n$

Recall:

"T is a map on vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$ and a linear transformation on $\vec{x}$"

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\underbrace{n = \#\text{ cols. in } A \text{ to start with}, \quad m = \#\text{ independent rows.}}_{\text{when we're talking about a finding a basis for } H}$

"domain" "co-domain" $\stackrel{\text{def}}{=}$ image(T)

"subspace"

$T(\vec{x}): \quad \vec{x} \xmapsto{\text{"maps"}} A\vec{x}, \quad \vec{x} \subset \mathbb{R}^n, \; A\vec{x} \subset \mathbb{R}^m$

FINDING BASES FOR A COLUMN SPACE

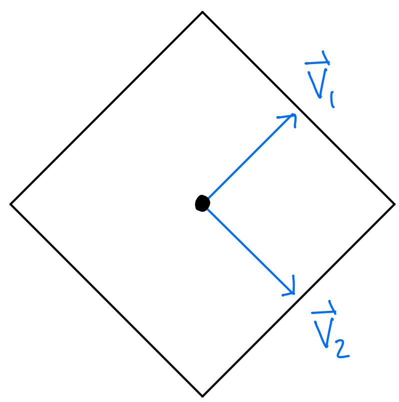
def: a **basis** B of subspace $H \subset \mathbb{R}^n$ is a linearly independent set s.t. span$\{B\} = H$

Two conditions for a set of vectors to form a basis:

(1) the vectors contained by the basis are **linearly independent**

(2) the vectors contained by the basis are a **spanning set** for H

$H = \mathbb{R}^2$:



$B = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$

✱ infinitly many vectors per subspace H ✱

✱ finitly many vectors per basis B ✱

Since the basis vectors must be linearly independent, a basis will <u>never</u> contain the 0 vector. Even though span{H} will still contain the 0 vector, like always.

The basis for a subspace $H \subset \mathbb{R}^n$ are all the vectors in A whose columns are linearly independent

In this situation:

- **A** represents the subspace **H** (aka "vector space" aka "column space" col(A))

So, to "find a basis for H" is to find the columns in **A** that are linearly independent of each other. We do this via row reduction i.e., by putting A into RREF.

- Recall: each linearly independent column will have a pivot in rref(A)

ex) find a basis for
$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}_{3\times 4} \sim \dots \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 8 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot col. pivot col.

the basis col(A) is:
$$B_{col(A)} = \left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}$$

and col(A) equals:
$$Span \left\{ \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \right\}$$

- **A** is a "map" from $\mathbb{R}^4$ to $\mathbb{R}^3$ so $A : \mathbb{R}^4 \longmapsto \mathbb{R}^3$

The basis of our null space in H is the solution to $A\vec{x} = 0$. For example:

5. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find a basis for $\mathrm{Nul}\,A$.

1 / 1 point

(◉) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Correct

Good work! The null space $\mathrm{Nul}\,A$ is the solution set of the system $A\vec{x} = \vec{0}$. The matrix A is already in reduced row echelon form, and x_2 and x_3 are free variables. The first row of A leads to the equation $x_1 + x_2 + x_3 = 0$, which implies $x_1 = -x_2 - x_3$. The general solution of $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, they form a basis of $\mathrm{Nul}\,A$.

5. Find a basis for $\mathrm{Nul}\,A$.

( ) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

(◉) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

( ) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Correct

Good work! The null space $\mathrm{Nul}\,A$ is the solution set of the system $A\vec{x} = \vec{0}$. The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable. The first row tells us that $x_1 + x_3 = 0$, so $x_1 = -x_3$. The second row tells us that $x_2 = 0$. Thus the solutions to the homogeneous equation are vectors of the form $x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. As a result,

$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the null space of A.

★ DIMENSION AND RANK

def: the **dimension** or **dimensionality** of a subspace H, denoted dim(H), equals the number of vectors in any **basis** for H

def: the **rank** of a matrix A the **dimention** of the subspace $H \subset \mathbb{R}^n$ that the A-matrix reprents. So, $rk(A) = dim(H) = \#$ of pivots in A

- Recall: to find A's pivots, you must put A into rref(A)

FOR ALL INTENTS AND PURPOSES, THESE TWO DEFINITIONS ARE IDENTICAL.

def: the **nullity** of a matrix A or subspace H equals # of free variables ≡ # linearly *dependent* vectors

def: a matrix is **full rank** when it has no nullity ≡ a matrix without any "redundancy" (i.e., a "nonsingular" matrix) ≡ a matrix w/o any free variables or linearly dependent column vectors ≡ how "big" the output is

- Recall: the inverse of a matrix A^{-1} exists *iff* A is square and **full rank**

- note that if a matrix is **full rank** it does not necessarily have to be a **square** matrix; "full rank" simply means that the matrix's rows or columns (whichever is fewer) are all linearly independent. For a **SQUARE matrix** A this logically means the # of linearly independent rows must equal # linearly independent columns (which must be the case for A^{-1} to exist).

Thm: the **Rank-Nullity Theorem** says $rk(A) + nul(A) = n$, $n = \#$ of matrix's columns

$$A_{m \times n}: \mathbb{R}^n \longmapsto \mathbb{R}^m \equiv \vec{x} \longmapsto A\vec{x}$$

The point of the Rank-Nullity theorem is, as a column vector of A $\vec{v}_i \subset col(A)$, you either get "sent to zero" during the transformation (i.e., during row reduction) or you dont; the theorem is a statement about **domain**.

Thm: the **Invertible Matrix Theorem** says, for **square, invertible** matrices only, then all of the following are 100% true or 100% false

- the columns of A form a basis for A ≡ col(A) is "onto" ||
- col(A) = $\mathbb{R}^n$ s.t. T: $\mathbb{R}^n \longmapsto \mathbb{R}^m$ implies no dimensionality reduction
- T: $\mathbb{R}^n \longmapsto \mathbb{R}^m$ is "one-to-one" (n=m) i.e., ∃ a pivot ∀ column ∈ A
- dim(col(A)) = n
- rk(A) = n
- nul(A) = $\{\vec{0}\}$
- dim(nul(A)) = 0
- det(A) ≠ 0

equivalent statements to: "**matrix A is invertible**"

def: let B be a basis for subspace H.

if $B = \{\vec{v}_1, \ldots \vec{v}_k\}$, then $\vec{x} \in H$ can be written as a linear combination of <u>ALL</u> vectors in B: $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_k$

$\Longrightarrow$ the **coordinate vector** of $\vec{x}$ is $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$

★ <u>summary of how to find the standard basis for H = col(A)</u>

step 1 rref(A) to identify A's pivot column vectors

step 2 take A's pivot colums as a **basis** for H: B_H = {pivot columns in A} ⊆ col(A)

equal iff A is already full rank; proper subset if else

★ <u>conclusion</u>: the whole idea is that we want our square matrices to be invertible b/c then we can use super efficient factorization algorithms to solve systems of linear equations; but if A is <u>not</u> full rank we can form a basis for col(A) that will help circumvent this issue... que: next up, we'll see how bases help us achieve invertibility