

PSET2, Part 2: Proof Writing

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Problem Eight: Yablo's Paradox

A *logical paradox* is a statement that isn't true and isn't false. One of the simplest paradoxes is the *Liar's paradox*, which is the following:

(P) : Statement (P) is false.

If statement (P) is true, then by its own admission statement (P) is false – a contradiction! That means that statement (P) isn't true.

On the other hand, suppose that statement (P) is false. Then we know that “statement (P) is false” is false, meaning that statement (P) is true – a contradiction! This means that statement (P) isn't false either.

Since statement (P) isn't true and isn't false, it's a paradox.

Paradoxes often arise as a result of self-reference. In the Liar's Paradox, the paradox arises because the statement directly refers to itself. However, it's not the only paradox that can arise from self-reference. This problem explores a beautiful, subtle paradox called *Yablo's paradox*.

Consider the following collection of infinitely many statements numbered S_0, S_1, S_2, \dots , where there is a statement S_n for each natural number n . These statements are ordered in a list as follows:

- (S_0) : All statements in this list after this one are false.
- (S_1) : All statements in this list after this one are false.
- (S_2) : All statements in this list after this one are false.
- ...

More generally, for each $n \in \mathbb{N}$, the statement (S_n) is

(S_n) : All statements in this list after this one are false.

Surprisingly, the interplay between these statements makes every statement in the list a paradox.

- i. Prove, by contradiction, that there does not exist a natural number n where statement (S_n) is true.

Something to ponder: how do you negate a universally-quantified statement?

- ii. Prove, by contradiction, that there does not exist a natural number n where statement (S_n) is false.

Your results from parts (i) and (ii) collectively establish that every statement in the list above is paradox, since none of them are true and none of them are false.

Now, consider the following modification to this list. Instead of infinitely many statements, suppose that there are “only” 10,000,000,000 statements. Specifically, suppose we have these statements:

- (T_0) : All statements in this list after this one are false.
- (T_1) : All statements in this list after this one are false.
- (T_2) : All statements in this list after this one are false.
- ...

Problem Eight: Yablo's Paradox (1,999,999,999): All statements in this list after this one are false.

Problem Nine: Square and Triangular Numbers

Problem Ten: Tournament Champions

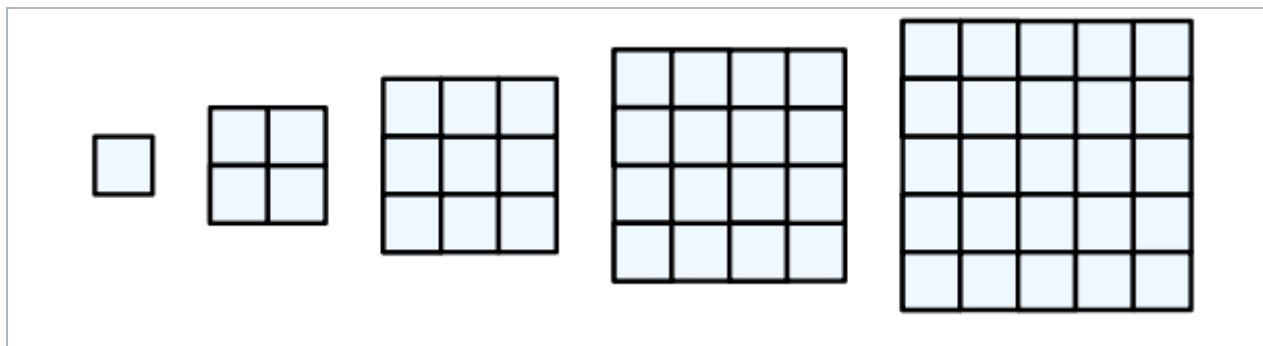
There's still a lot of statements here, but not infinitely many of them. Interestingly, these statements are all perfectly consistent with one another and do not result in any paradoxes.

- iii. For each statement in the above list, determine whether it's true or false and explain why your choices are consistent with one another.

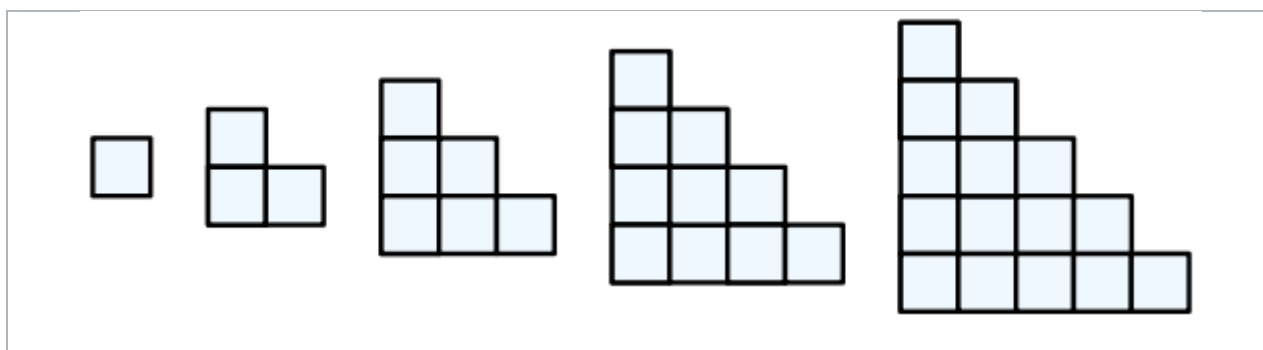
Going forward, don't worry about paradoxes in CS103. We won't talk about any more statements like these. 😊

Problem Nine: Square and Triangular Numbers

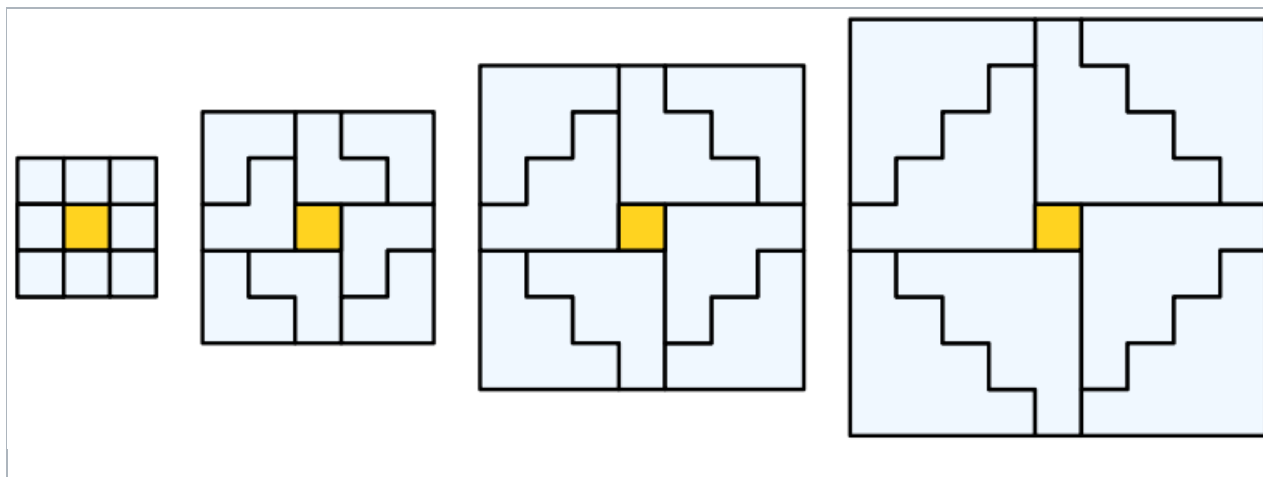
A natural number n is called a **square number** when there exists a natural number k where $n = k^2$. The first few square numbers are 0, 1, 4, 9, 16, 25, ... Intuitively, the square numbers count the number of squares in a square grid:



A natural number n is called a **triangular number** when there exists a natural number k where $n = \frac{k(k+1)}{2}$. The first few triangular numbers are 0, 1, 3, 6, 10, 15, ... Intuitively, the triangular numbers count the number of squares in these figures:



Prove for all natural numbers n that n is a triangular number if and only if $8n + 1$ is a square number. There's a lovely geometric intuition for this shown here:



Just to make sure you didn't miss it, the statement to prove is a biconditional rather than a standard implication.

You may want to make use of a result we proved in lecture.

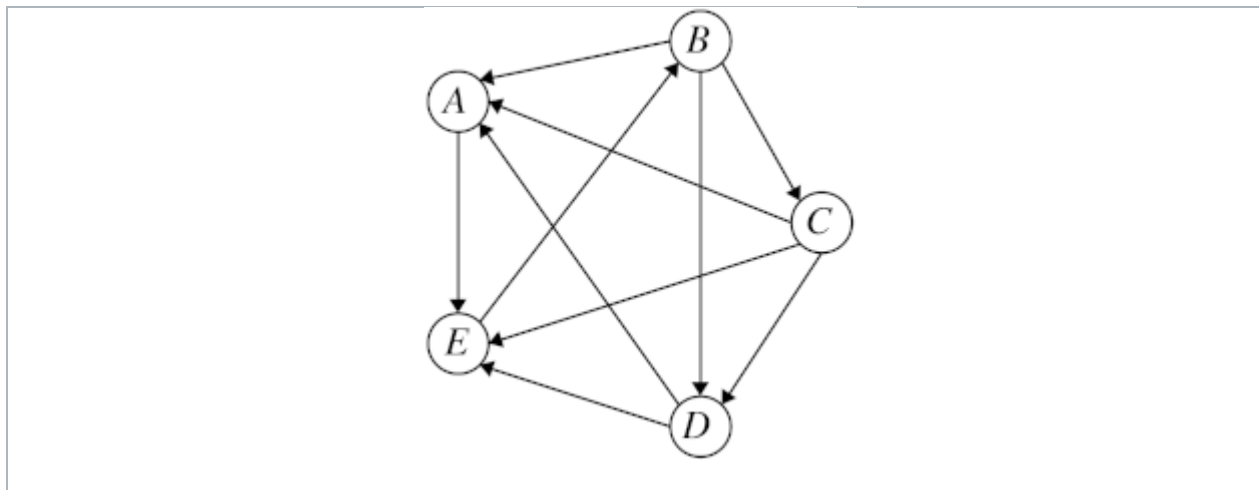
Problem Ten: Tournament Champions

A **tournament** is a contest among $n \geq 0$ players. Each player plays exactly one game against each other player, and either wins or loses the game (let's assume that there are

Problem Eight: Yablo's Paradox can visually represent a tournament by drawing a circle for each player

Problem Nine: Squares and Irregular Numbers are and drawing arrows between pairs of players to indicate who won each game. For example,

Problem Ten: Tournament Champions tournament:



In this tournament, player A beat player E , but lost to players B , C , and D .

Now, a new definition. A **tournament champion** is a player c where, for each player p that c lost to, there is a player q where c won against q and q won against p .

In the tournament shown above, player B is a champion: she won against A , C , and D , and although she lost to E , she won against D , who in turn won against E . Player C is also a champion: she won against A , D , and E , and the one player she lost to (B) is covered because C won against E , who in turn won against B . Player A , however, is not a tournament champion. To see this, A didn't win against C , nor is there any player who A won against who in turn won against C .

- i. Is player D a tournament champion? How about player E ? No justification is required.

You might have an intuition for what a tournament champion is, but to determine the answers to these questions, you'll have to look back at the definition.

Here's an amazing fact about tournaments.

- ii. Let T be an arbitrary tournament and c be a player in that tournament. Prove the following statement: if c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion in T .

Remember that proofwriting heavily involves leveraging definitions. Intuitively, it makes sense that someone who won the most games did the best in a tournament, but that's not what's being asked here. Instead, you're asked to show that if someone won the most games (or tied for winning the most games), then they specifically meet the criteria described above that characterize a tournament champion.

*A great question to ponder – what has to happen for someone to **not** be a tournament champion? Don't guess based on your intuition; negate the definition of a tournament champion and see what happens. You might want to draw a picture of what happens in this case.*

A corollary of the result from (ii) is that every tournament with at least one player has at least one tournament champion – pick someone who won the most games or is tied for winning the most games.

The result you proved in part (ii) of this problem is noteworthy for several reasons. First, this is a fundamental result about tournaments – many aspects of tournaments can be better understood by looking at its champions. Second, it's an example of a style of proof called a **proof by extreme case**. In a proof of that sort, you pick some “extreme” object, usually the biggest or smallest object of some type, then prove that it has some property. Here, you proved that the “winningest” player must be a tournament champion. Keep this idea in mind for the future – you'll likely get some more mileage out of it!

Tournaments have many applications in computational complexity theory and in the emerging field of *computational social choice theory*. A former CS103 TA, Michael Kim, studied them extensively when doing his PhD here in computer science!

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Problem Ten: Tournament Champions

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