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Hairy black hole entropy and the role of solitons in three dimensions

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Abstract

Scalar fields minimally coupled to General Relativity in three dimensions are considered. For certain families of self-interaction potentials, new exact solutions describing solitons and hairy black holes are found. It is shown that they fit within a relaxed set of asymptotically AdS boundary conditions, whose asymptotic symmetry group coincides with the one for pure gravity and its canonical realization possesses the standard central extension. Solitons are devoid of integration constants and their (negative) mass, fixed and determined by nontrivial functions of the self-interaction couplings, is shown to be bounded from below by the mass of AdS spacetime. Remarkably, assuming that a soliton corresponds to the ground state of the sector of the theory for which the scalar field is switched on, the semiclassical entropy of the corresponding hairy black hole is exactly reproduced from Cardy formula once nonvanishing lowest eigenvalues of the Virasoro operators are taking into account, being precisely given by the ones associated to the soliton.

This provides further evidence about the robustness of previous results, for which the ground state energy instead of the central charge appears to play the leading role in order to reproduce the hairy black hole entropy from a microscopic counting.

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I. INTRODUCTION

The microscopic origin of black hole entropy, a question raised right after the pioneering work of Bekenstein and Hawking during the 1970's, still remains as a challenging unsolved puzzle. Nevertheless, the most compelling current proposals appear to converge in the sense that, regardless the precise mechanisms and assumptions, an emergent conformal symmetry in two dimensions, endowed with a suitable central extension, allows to reproduce the semiclassical entropy of different classes of black holes from a microscopic counting (see e.g. [1–5], as well as [6–8] and references therein).

One of the simplest and clear examples is the one provided by Strominger [2]. This proposal relies on an observation pushed forward by Brown and Henneaux during the 1980's [9] and currently interpreted in terms of the AdS/CFT correspondence [10]. As follows from [9], since the asymptotic symmetries of General Relativity with negative cosmological constant in three dimensions correspond to two copies of the Virasoro algebra, a consistent quantum theory of gravity should then be described in terms of a conformal field theory in two dimensions, with a central charge given by

$$c = \frac{3l}{2G} \ , \tag{1}$$

where G and l stand for the Newton constant and the AdS radius, respectively. Thus, in [2] it was assumed that if the CFT fulfills some physically sensible properties, the physical states form a consistent unitary representation of the conformal algebra, so that the asymptotic growth of the number of states must be given by Cardy formula [11]. Remarkably, the result precisely agrees with semiclassical entropy of the BTZ black hole [12, 13] provided the central charge is exactly given by (1).

Nonetheless, there are known examples for which this proposal has to be refined, since for them the central charge does not play the leading role in order to reproduce the semiclassical black hole entropy from a microscopic counting. Indeed, as explained in [14] the asymptotic growth of the number of states can be expressed only in terms of the spectrum of the Virasoro operators without making any explicit reference to the central charges, so that the relevant quantities that allow to reproduce the black hole entropy turn out to be the lowest eigenvalues of the Virasoro operators. This can be seen as follows. If the spectrum of the Virasoro operators $L_0^{\pm}$, whose eigenvalues are given by $\Delta^{\pm}$, is such that their lowest

eigenvalues, denoted by $\Delta_0^\pm$, are nonvanishing (i.e., for $\Delta_0^\pm \neq 0$), Cardy formula reads (see e.g. [11, 15–17])

$$S = 2\pi\sqrt{\frac{(c^+ - 24\Delta_0^+)}{6}\left(\Delta^+ - \frac{c^+}{24}\right)} + 2\pi\sqrt{\frac{(c^- - 24\Delta_0^-)}{6}\left(\Delta^- - \frac{c^-}{24}\right)} \ , \tag{2}$$

where it is assumed that the ground state is non degenerate. As pointed out in [14], *on a cylinder*, the zero mode of the Virasoro operators gets shifted according to

$$\tilde{L}_0^\pm := L_0^\pm - \frac{c^\pm}{24} \ , \tag{3}$$

so that formula (2) can be naturally written as

$$S = 4\pi\sqrt{-\tilde{\Delta}_0^+\tilde{\Delta}^+} + 4\pi\sqrt{-\tilde{\Delta}_0^-\tilde{\Delta}^-} \ , \tag{4}$$

where $\tilde{\Delta}^\pm$ correspond to the eigenvalues of $\tilde{L}_0^\pm$, while $\tilde{\Delta}_0^\pm$ stand for the lowest ones. Therefore, from (4) it is apparent that the asymptotic growth of the number of states can be precisely obtained if one only knew the spectrum of $\tilde{L}_0^\pm$ without knowledge about the central charges.

Note that when the lowest eigenvalues of the Virasoro operator vanish, i.e. for $\Delta_0^\pm = 0$, or equivalently $\tilde{\Delta}_0^\pm = -\frac{c^\pm}{24}$, formula (4) reduces to its more familiar form, given by,

$$S = 2\pi\sqrt{\frac{c^+}{6}\tilde{\Delta}^+} + 2\pi\sqrt{\frac{c^-}{6}\tilde{\Delta}^-} \ . \tag{5}$$

In this case, as shown in [2], assuming that the eigenvalues of $\tilde{L}_0^\pm$ are given by the corresponding canonical generators according to

$$\tilde{\Delta}^\pm = \frac{1}{2}(Ml \pm J) \ , \tag{6}$$

where M and J stand for the mass and the angular momentum, respectively, then formula (5) precisely reproduces the semiclassical entropy of the BTZ black hole.

Therefore, for instances such that the lowest eigenvalues of the Virasoro operators do not vanish, i.e., for $\Delta_0^\pm \neq 0$, formula (5) does not apply. Remarkably, as explained in [14], in these cases the semiclassical black hole entropy can still be successfully reproduced by virtue of the generic formula given by (4) once the ground state configuration is suitably identified. A concrete example where this effect occurs is provided by the existence of hairy black holes found in [18] for General Relativity minimally coupled to a self-interacting scalar field in three dimensions. The action is given by

$$I[g_{\mu\nu}, \phi] = \frac{1}{\pi G}\int d^3x\sqrt{-g}\left[\frac{R}{16} - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] \ , \tag{7}$$

and the self-interaction potential, expanded around $\phi = 0$, is assumed to be of the form

$$V(\phi) = -\frac{1}{8l^2} - \frac{3}{8l^2}\phi^2 - \frac{1}{2l^2}\phi^4 + \mathcal{O}(\phi^6) \ , \tag{8}$$

so that the first term corresponds to the cosmological constant $\Lambda = -1/l^2$, while the second one is the mass term, with $m^2 = -3/(4l^2)$, fulfilling the Breitenlohner-Freedman bound [19, 20]. As shown in [18], the scalar field is able to acquire slow fall-off at infinity, so that the action (7) admits a suitable set of asymptotically AdS boundary conditions in a relaxed sense as compared with the one of Brown and Henneaux [9], which nevertheless possesses the same asymptotic symmetries, i.e., they are also left invariant under the conformal group in two dimensions, spanned by two copies of the Virasoro algebra. The asymptotic conditions are given by:

$$\phi = \frac{\chi}{r^{1/2}} + \alpha\frac{\chi^3}{r^{3/2}} + O(r^{-5/2}) \tag{9}$$

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} - \frac{4l^2\chi^2}{r^3} + O(r^{-4}) & g_{tt} &= -\frac{r^2}{l^2} + O(1) \\ g_{tr} &= O(r^{-2}) & g_{\varphi\varphi} &= r^2 + O(1) \\ g_{\varphi r} &= O(r^{-2}) & g_{t\varphi} &= O(1) \end{aligned} \tag{10}$$

where $\chi = \chi(t, \varphi)$ is an arbitrary function, and α is an arbitrary constant. One of the effects of relaxing the asymptotic conditions is reflected through the fact that the generators of the asymptotic symmetries acquire a nontrivial contribution from the scalar field. Following the Regge-Teitelboim approach [21], the canonical generators were found to be given by

$$Q(\xi) = \frac{1}{16\pi G}\int d\varphi \left\{ \frac{\xi^\perp}{lr}\left((g_{\varphi\varphi} - r^2) - 2r^2(lg^{-1/2} - 1)\right) + 2\xi^\varphi \pi^r{}_\varphi + \xi^\perp \frac{2r}{l}\left[\phi^2 - 2l\frac{\phi\partial_r\phi}{\sqrt{g_{rr}}}\right]\right\} , \tag{11}$$

where the reference background is chosen to be the massless BTZ black hole. The corresponding Poisson brackets were shown to span two copies of the Virasoro algebra with the standard central charges $c^+ = c^- = c$, where c is given by eq. (1).

An exact analytic hairy black hole solution whose asymptotic behavior fits within the boundary conditions given by eqs. (9), (10) was found in [18] for the following self-interaction potential

$$V_{1,\nu}(\phi) = -\frac{1}{8l^2}\left(\cosh^6\phi + \nu\sinh^6\phi\right) \ , \tag{12}$$

which belongs to the class defined in (8). As shown in [14], for this specific potential, the field equations corresponding to (7) also admit an exact analytic soliton solution, being

such that the metric and the scalar field are regular everywhere and fulfill the boundary conditions (9) and (10). The soliton turns out to be devoid of integration constants, and it has a precise fixed (negative) mass M_0 determined by the Newton constant and the self-interaction parameter ν. This fact naturally suggests to regard the soliton as the ground state of the "hairy sector", for which the scalar field is switched on. Remarkably, assuming that the lowest eigenvalues of the Virasoro operators are determined by the global charges of the soliton, according to eq. (6), i.e., $\tilde{\Delta}_0^{\pm} = \frac{l}{2} M_0$, the asymptotic growth of the number of states, given by eq. (4), reduces to

$$S = 4\pi l \sqrt{-M_0 M} \ , \tag{13}$$

where $M = \frac{2}{l} \tilde{\Delta}^{\pm}$ corresponds to the mass of the hairy black hole, which exactly reproduces its semiclassical entropy $S = \frac{A}{4G}$.

One may wonder whether this result is just a curiosity of the particular model specified by the potential in (12), or actually corresponds to a generic feature of hairy black holes. In this article we construct new examples that strongly support the latter possibility. This is carried out for different self-interaction potentials within the class in eq. (8), being simultaneously involved enough so as to provide non trivial lowest eigenvalues for the Virasoro operators in the hairy sector, as well as sufficiently simple in order to find exact analytic hairy black holes and their corresponding solitons.

In what follows we show the existence of new analytic hairy black hole and soliton solutions for different classes of self-interaction potentials. In the next section a one-parameter family of potentials which differs from (12) is considered, while in section III a class of potentials that depend on two parameters is discussed. Section IV is devoted to the explicit microscopic computation of the entropy of the hairy black holes mentioned above in terms of their corresponding solitons by means of formula (4) which reduces to eq. (13) in the static case. Final remarks are given in section V. Appendix A includes a description of certain analytic functions that become relevant in order to describe the properties of hairy black holes and solitons, and finally, appendix B is devoted to discuss the new exact solutions in the conformal (Jordan) frame.

II. CASE 1: HAIRY BLACK HOLES AND SOLITONS FOR A UNIPARAMETRIC FAMILY OF POTENTIALS

Let us consider the following class of self-interaction potentials,

$$V_{0,\nu}(\phi) = -\frac{1}{16l^2}\left(\nu\cosh^8\phi - \nu\cosh^4\phi + 2\cosh^6\phi\left(1-\nu\ln\cosh^2\phi\right)\right), \tag{14}$$

which, apart from the AdS radius l, depends on a single parameter ν. Around $\phi = 0$, the potential behaves as

$$V_{0,\nu}(\phi) = -\frac{1}{8l^2} - \frac{3}{8l^2}\phi^2 - \frac{1}{2l^2}\phi^4 - \frac{94+5\nu}{240l^2}\phi^6 + \mathcal{O}(\phi^8)\ , \tag{15}$$

and it then falls within the family defined in eq. (8) that is consistent with the boundary conditions given by (9) and (10). Note that $V_{0,\nu}(\phi)$ does not overlap with $V_{1,\nu}(\phi)$ in eq. (12) for any value of ν. As mentioned above, the self interaction (14) turns out to be involved enough so as to possess a ground state in the hairy sector with nontrivial lowest eigenvalues for the Virasoro operators, but nevertheless it becomes simple in order to produce exact analytic hairy black holes and solitons. This is discussed next.

A. Black hole

The field equations that correspond to the action (7) with $V = V_{0,\nu}(\phi)$ admit an exact solution, whose the line element reads

$$ds^2 = -\frac{r^2}{l^2}h(r)dt^2 + \frac{l^2r^2dr^2}{(r+a)^4h(r)} + r^2d\varphi^2\,, \tag{16}$$

with

$$h(r) = 1 + \nu\left(\frac{a}{r+a} + \ln\frac{r}{r+a}\right)\ , \tag{17}$$

and the scalar field is given by

$$\phi = \operatorname{arctanh}\sqrt{\frac{a}{r+a}}\,. \tag{18}$$

The coordinates range as $-\infty < t < \infty$, $r > 0$, $0 \leq \varphi < 2\pi$, and the solution depends on a single non-negative integration constant a. Remarkably, the scalar field is regular everywhere and the solution describes a hairy black hole provided $\nu > 0$, otherwise the singularity at the origin $r = 0$ becomes naked. There is a single horizon located at

$$r_+ = a\Phi_\nu\ , \tag{19}$$

where Φ_ν depends only on the parameter ν and it is given by

$$\Phi_\nu := \frac{-W(-e^{-1-\frac{1}{\nu}})}{1+W(-e^{-1-\frac{1}{\nu}})} \ . \tag{20}$$

Here W stands for the Lambert W function, defined as

$$W(z)e^{W(z)} = z \ , \tag{21}$$

which for $-1/e < z < 0$ has an upper branch that ranges according to $-1 < W(z) < 0$ (see e.g. [22])[40]. For latter purposes it is worth pointing out that for $0 < \nu < \infty$, the function Φ_ν is bounded as

$$0 < \Phi_\nu^2 < \frac{\nu}{2} \ . \tag{22}$$

Further details about Φ_ν are included in appendix A.

The Hawking temperature of the hairy black hole (16), (18) can be obtained demanding regularity of the Euclidean solution at the horizon and it is found to be proportional to the integration constant a, which reads

$$T = \frac{a\nu}{4\pi l^2 \Phi_\nu} \ , \tag{23}$$

while its entropy is given by

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G} = \frac{\pi \Phi_\nu}{2G} a \ . \tag{24}$$

Asymptotically, this hairy black hole behaves as

$$\phi = \frac{a^{1/2}}{r^{1/2}} - \frac{1}{6}\frac{a^{3/2}}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right) \ , \tag{25}$$

with

$$g_{tt} = -\frac{r^2}{l^2} + \frac{a^2\nu}{2l^2} + \mathcal{O}\left(\frac{1}{r}\right) \ , \tag{26}$$

$$g_{rr} = \frac{l^2}{r^2} - \frac{4l^2 a}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \ , \tag{27}$$

$$g_{\varphi\varphi} = r^2 \ , \tag{28}$$

and then falls within the set defined by eqs. (9), (10) with $\chi = \sqrt{a}$ and $\alpha = -1/6$. Therefore, the solution is asymptotically AdS in a relaxed sense as compared with the standard one [9]. The mass of the hairy black hole can be readily computed by virtue of eq. (11), yielding[41]

$$M = Q(\partial_t) = \frac{\nu a^2}{16 G l^2} \ . \tag{29}$$

Note that the scalar field cannot be switched off keeping the mass fixed. Indeed, the solution depends on a single integration constant, so that for $\phi \to 0$, the geometry approaches the one of the massless BTZ black hole.

B. Soliton

For the self-interaction potential $V_{0,\nu}(\phi)$ in eq. (14), the field equations also admit the following solution:

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2 r^2 dr^2}{(r + \frac{2l\Phi_\nu}{\nu})^4 H(r)} + r^2 H(r) d\varphi^2 \,, \tag{30}$$

with

$$H(r) = 1 + \frac{2l\Phi_\nu}{r + \frac{2l\Phi_\nu}{\nu}} + \nu \ln \frac{r}{r + \frac{2l\Phi_\nu}{\nu}} \,, \tag{31}$$

and

$$\phi(r) = \operatorname{arctanh} \sqrt{\frac{1}{1 + \frac{\nu r}{2l\Phi_\nu}}} \,, \tag{32}$$

where the coordinates range according to $-\infty < t < \infty$, $0 \leq \varphi < 2\pi$, and

$$\frac{2l\Phi_\nu^2}{\nu} \leq r < \infty \,, \tag{33}$$

with $\nu > 0$. Note that the solution is devoid of integration constants and depends only on the self-interaction parameter ν and the AdS radius l. It is simple to verify that the solution is smooth and regular everywhere. Indeed, the behaviour of the solution around the origin, located at $r = \frac{2l\Phi_\nu^2}{\nu}$ can be seen from the expansion of (31), given by

$$H\left(r\right) = \frac{\nu^2}{2l\Phi_\nu^2(1+\Phi_\nu)^2}\left(r - \frac{2l\Phi_\nu^2}{\nu}\right) + \mathcal{O}\left[\left(r - \frac{2l\Phi_\nu^2}{\nu}\right)^2\right] \,,$$

so that defining $\hat{t} = \frac{2\Phi_\nu^2}{\nu}t$, and $\rho^2 = r^2 H(r)$, the metric approaches to the one of Minkowski spacetime,

$$ds^2 \to -d\hat{t}^2 + d\rho^2 + \rho^2 d\varphi^2 \,.$$

Analogously, the form of the scalar field around $\rho = 0$ is given by

$$\phi(\rho) = \operatorname{arctanh} \sqrt{\frac{1}{1+\Phi_\nu}} - \frac{\nu\left(1+\Phi_\nu\right)^{3/2}}{8l^2\Phi_\nu^4}\rho^2 + \mathcal{O}\left(\rho^4\right) \,.$$

The asymptotic behavior of (32) and (30) is given by

$$\phi = \left(\frac{2l\Phi_\nu}{\nu}\right)^{1/2} \frac{1}{r^{1/2}} - \frac{1}{6}\left(\frac{2l\Phi_\nu}{\nu}\right)^{3/2} \frac{1}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right) \;, \tag{34}$$

with

$$\begin{aligned} g_{tt} &= -\frac{r^2}{l^2} \;, \\ g_{rr} &= \frac{l^2}{r^2} - \frac{2l\Phi_\nu}{\nu}\frac{4l^2}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \;, \\ g_{\varphi\varphi} &= r^2 + \mathcal{O}(1) \;, \end{aligned} \tag{35}$$

and then belongs to the class of relaxed asymptotically AdS conditions defined by eqs. (9) and (10) with $\chi = \left(\frac{2l\Phi_\nu}{\nu}\right)^{1/2}$ and, as for the hairy black hole discussed in II A, $\alpha = -1/6$. The mass of this solution can then be obtained by virtue of eq. (11) which yields

$$M_{\rm sol} = -\frac{\Phi_\nu^2}{4G\nu} \;. \tag{36}$$

In sum, this solution is regular everywhere, shares the same causal structure with AdS spacetime, and since it has a fixed finite mass, it describes a soliton.

Note that by virtue of (22), which holds for the allowed range of the self-interaction coupling, $\nu > 0$, the soliton mass $M_0 = M_{\rm sol}$ becomes bounded according to

$$-\frac{1}{8G} < M_0 < 0 \;. \tag{37}$$

III. CASE 2: HAIRY BLACK HOLES AND SOLITONS FOR SELF-INTERACTION POTENTIALS WITH TWO PARAMETERS.

Here we consider a wider class of self-interaction potentials being such that not only interpolates, but generalizes the ones considered above. This is given by

$$\begin{aligned} V_{\lambda,\nu}(\phi) = &-\frac{\nu}{16l^2}\sinh^2\phi\left[\lambda(\lambda+1) - 2\lambda(\lambda+2)\cosh^2\phi + (1+4\lambda+\lambda^2)\cosh^4\phi - (\lambda-1)\cosh^6\phi\right] \\ &- \frac{\cosh^6\phi - \lambda^2\sinh^6\phi}{8l^2(\lambda-1)}\left[(\lambda-1) + \nu\ln(\lambda - (\lambda-1)\cosh^2\phi)\right], \end{aligned} \tag{38}$$

which depends on two parameters ν, λ, and l stands for the AdS radius. The behavior of $V_{\lambda,\nu}$ around $\phi = 0$ reads

$$V_{\lambda,\nu}(\phi) \underset{\phi\to 0}{\longrightarrow} -\frac{1}{8l^2} - \frac{3}{8l^2}\phi^2 - \frac{1}{2l^2}\phi^4 - \frac{94 - 30\lambda^2 + 5\lambda^2\nu - 10\nu\lambda + 5\nu}{240l^2}\phi^6 + \mathcal{O}(\phi^8)\,, \tag{39}$$

so that it belongs to the class defined in eq. (8). Note that for $\lambda = 0$ the self interaction (38) reduces to $V_{0,\nu}(\phi)$ in Eq. (14), while after redefining $\nu = 6\frac{\tilde{\nu}+1}{(\lambda-1)^2}$, in the limit $\lambda \to 1$, the potential (38) acquires the form of $V_{1,\tilde{\nu}}(\phi)$ in eq. (12).

In what follows, exact hairy black holes and soliton solutions for this self interaction are explicitly found.

A. Black hole

In the case of $V = V_{\lambda,\nu}(\phi)$, the field equations possess an analytic solution. The metric is given by

$$ds^2 = \Omega^2(r)\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2\right) , \tag{40}$$

where

$$\begin{aligned} \Omega^2(r) &= \left(\frac{\lambda(r+b)-b}{\lambda(r+b)}\right)^2 , \\ f(r) &= \frac{r^2}{l^2} - \frac{\nu}{\lambda(\lambda-1)l^2}\left(\frac{(\lambda-1)^2}{2}b^2 - b(\lambda-1)r + \lambda r^2 \ln\left(1 + b\frac{\lambda-1}{\lambda r}\right)\right) , \end{aligned} \tag{41}$$

and the scalar field reads,

$$\phi = \operatorname{arctanh}\sqrt{\frac{b}{\lambda(r+b)}} . \tag{42}$$

The solution depends on a single integration constant b, and the coordinates range as $-\infty < t < \infty$, $0 \leq \varphi < 2\pi$, and $r > r_s$, where $r = r_s$ stands for the location of the curvature singularity specified below. The asymptotic conditions (9), (10) are also fulfilled with $\alpha = -\frac{1}{6}(1+3\lambda)$ and $\chi = (b/\lambda)^{1/2}$, which means that the hairy solution is well defined provided $b/\lambda > 0$. Indeed, making the shift $r \to r + \frac{b}{\lambda}$, the asymptotic behavior of the solution reads

$$\phi = \frac{(b/\lambda)^{1/2}}{r^{1/2}} - \frac{1}{6}(1+3\lambda)\frac{(b/\lambda)^{3/2}}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right) \tag{43}$$

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} - \frac{b}{\lambda}\frac{4l^2}{r^3} + O(r^{-4}) \qquad & g_{tt} &= -\frac{r^2}{l^2} + O(1) \\ g_{tr} &= O(r^{-2}) & g_{\varphi\varphi} &= r^2 + O(1) \\ g_{\varphi r} &= O(r^{-2}) & g_{t\varphi} &= O(1) \end{aligned}$$

The mass can then be computed by virtue of eq. (11), which gives

$$M = \frac{b^2}{16Gl^2}\frac{(\lambda-1)^2}{\lambda^2}\nu . \tag{44}$$

As in the case of the solution found in the previous section, this one depends on a single integration constant b, and it is such that the massless BTZ black hole in vacuum ($\phi = 0$) is recovered for $b \to 0$.

It is simple to verify that the solution describes a hairy black hole with a regular scalar field on and outside the event horizon provided $\nu > 0$, which ensures the mass (44) is positive. Thus, remarkably, although the self interaction looks somehow involved, positivity of the hairy black hole energy still goes by hand with cosmic censorship.

The event horizon is located at $r = r_+$, with

$$r_+ = \frac{b}{\lambda}(1-\lambda)\mathcal{R}_{\lambda,\nu} \ , \tag{45}$$

where the function $\mathcal{R}_{\lambda,\nu}$ is defined as the real root of

$$\mathcal{R}_{\lambda,\nu}^2 - \frac{\nu}{(\lambda-1)}\left[\frac{\lambda}{2} + \mathcal{R}_{\lambda,\nu} + \mathcal{R}_{\lambda,\nu}^2 \log\left(1 - \frac{1}{\mathcal{R}_{\lambda,\nu}}\right)\right] = 0 \ , \tag{46}$$

which only holds for $\nu > 0$.

In the case of $\lambda > 1$ there is a curvature singularity at $r_s = 0$, and since the function $\mathcal{R}_{\lambda,\nu}$ ranges as $-\infty < \mathcal{R}_{\lambda,\nu} < 0$, it is always surrounded by the event horizon.

For $\lambda < 1$ the curvature singularity is located at $r_s = b\frac{1-\lambda}{\lambda}$, and it is also always cloaked by the event horizon since in this case the function $\mathcal{R}_{\lambda,\nu}$ ranges according to $1 < \mathcal{R}_{\lambda,\nu} < \infty$.

The Hawking temperature and hairy black hole entropy are given by

$$T = \frac{\nu}{4\pi l^2}\frac{b}{\lambda}\frac{1-\lambda}{\Upsilon_{\lambda,\nu}} \ , \tag{47}$$

$$S = \frac{A}{4G} = \frac{\pi}{2G}(1-\lambda)\frac{b}{\lambda}\Upsilon_{\lambda,\nu} \ , \tag{48}$$

respectively, where the function

$$\Upsilon_{\lambda,\nu} := \frac{(1-\lambda)\mathcal{R}_{\lambda,\nu}(\mathcal{R}_{\lambda,\nu}-1)}{\lambda + (1-\lambda)\mathcal{R}_{\lambda,\nu}} \ , \tag{49}$$

fulfills $\Upsilon_{\lambda,\nu}(1-\lambda) > 0$, so that the temperature and the entropy are manifestly positive, and it is bounded as

$$\Upsilon_{\lambda,\nu}^2 < \frac{\nu}{2} \ . \tag{50}$$

Further details about the functions $\mathcal{R}_{\lambda,\nu}$ and $\Upsilon_{\lambda,\nu}$ are revisited in Appendix A.

B. Soliton

The field equations for the self-interaction potential $V_{\lambda,\nu}$ in (38), with $\nu > 0$, also admit the following soliton solution:

$$\phi(r) = \operatorname{arctanh}\sqrt{\frac{\gamma_{\lambda,\nu}}{r+(1+\lambda)\gamma_{\lambda,\nu}}}\ , \tag{51}$$

with

$$ds^2 = -\frac{(r+\gamma_{\lambda,\nu})^2(r+\lambda\gamma_{\lambda,\nu})^2}{l^2(r+(1+\lambda)\gamma_{\lambda,\nu})^2}dt^2 + \left(\frac{r+\lambda\gamma_{\lambda,\nu}}{r+(1+\lambda)\gamma_{\lambda,\nu}}\right)^2\left(\frac{dr^2}{g(r)}+l^2 g(r)d\varphi^2\right)\ , \tag{52}$$

where

$$g(r) = \frac{\nu\,\gamma_{\lambda,\nu}}{l^2}\left(r+(2+\lambda-\lambda^2)\frac{\gamma_{\lambda,\nu}}{2}\right)+\frac{1}{l^2}(r+\gamma_{\lambda,\nu})^2\left(1+\frac{\nu}{1-\lambda}\ln\left(\frac{r+\lambda\gamma_{\lambda,\nu}}{r+\gamma_{\lambda,\nu}}\right)\right)\ , \tag{53}$$

and $\gamma_{\lambda,\nu} := \frac{2l\Upsilon_{\lambda,\nu}}{(1-\lambda)\nu}$ is a two-parametric constant.

The coordinates range according to $-\infty < t < \infty$, $0 \leq \varphi < 2\pi$, and

$$\frac{2l\Upsilon^2_{\nu,\lambda}}{\nu} \leq r < \infty\ . \tag{54}$$

This solution depends only on the parameters of the potential, ν, λ and the AdS radius l. Thus, as in the case discussed in Sec. II B, the soliton has no integration constants and it is simple to verify that the solution is smooth and regular everywhere. The soliton also fulfills the asymptotic conditions in eqs. (9) and (10), with $\chi = (\gamma_{\lambda,\nu})^{1/2}$ and $\alpha = -\frac{1}{6}(1+3\lambda)$. Indeed, for $r \to \infty$ the solution behaves as

$$\phi = \frac{(\gamma_{\lambda,\nu})^{1/2}}{r^{1/2}} - \frac{1}{6}(1+3\lambda)\frac{(\gamma_{\lambda,\nu})^{3/2}}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right)$$

$$\begin{aligned}
g_{rr} &= \frac{l^2}{r^2} - \gamma_{\lambda,\nu}\frac{4l^2}{r^3} + O(r^{-4}) & g_{tt} &= -\frac{r^2}{l^2} + O(1)\\
g_{tr} &= O(r^{-2}) & g_{\varphi\varphi} &= r^2 + O(1)\\
g_{\varphi r} &= O(r^{-2}) & g_{t\varphi} &= O(1)
\end{aligned}$$

Note that the constant α coincides with the one of the hairy black hole for the same potential $V_{\lambda,\nu}$.

The soliton mass can then be readily obtained from eq. (11), which gives

$$M_{\text{sol}} = -\frac{\Upsilon^2_{\nu,\lambda}}{4G\nu}\ , \tag{55}$$

and by virtue of (50) turns out to be bounded exactly as in eq. (37), i.e.,

$$-\frac{1}{8G} < M_0 < 0\ .$$

As a concluding remark of this section, one can verify that in the limits $\lambda \to 0$ and $\lambda \to 1$, not only the potentials $V_{0,\nu}$, and $V_{1,\nu}$ in eqs. (14) and (12) are recovered from $V_{\lambda,\nu}$, respectively, but also their corresponding hairy black holes and solitons described above, as well as those in Refs. [14, 18]. In the case $\lambda \to 0$, this can be explicitly see as follows[42]:

Taking the limit $\lambda \to 0$ with $\frac{b}{\lambda} \to a$, the solution defined by the metric (40) and the scalar field (42) becomes the black hole solution of section II A given by (16) and (18), provided the radial coordinates is shifted as $r \to r + a$.

In the case of the soliton solution, the function Φ_ν is exactly recovered from the limit $\lambda \to 0$ in (46) and (49), i.e. $\Upsilon_{0,\nu} = \Phi_\nu$. In this way, the soliton that corresponds to the two parametric potential case, described by eqs. (52) and (51), reduces to the uniparametric one in (30) and (32).

In sum, the self-interaction potentials considered here were shown to be simple enough so as to obtain exact and physically sensible hairy black hole solutions, and at the same time, sufficiently involved in order to provide analytic solitons whose masses are fixed and determined by nontrivial functions of the self-interaction parameters, as it can be explicitly seen from eqs. (36), (55). The link between soliton masses and the entropy of their corresponding hairy black hole entropies is discussed next.

IV. MICROSCOPIC ENTROPY OF HAIRY BLACK HOLES: SOLITON MASS AND ITS ROLE IN THE ASYMPTOTIC GROWTH OF THE NUMBER OF STATES.

The class of hairy black holes found here was shown to have positive mass and it can be seen that they share many of the features with the one previously found in [18]. In fact, for any of the self interactions discussed above, $V_{0,\nu}$ and $V_{\lambda,\nu}$, it occurs that for some fixed value of the energy, the same theory admits the existence of at least two different static and circularly symmetric black holes. Namely, for a precise value of the mass, apart from the hairy black hole that is dressed with a nontrivial scalar field, in vacuum one may also have the static BTZ black hole. Furthermore, it is worth highlighting that, since both black

holes depend on a single integration constant, the hairy and BTZ black holes cannot be smoothly deformed into each other, due to that fact that for a fixed mass, the scalar field cannot be switched off. Following [14], this observation naturally suggests that the hairy and the vacuum black holes belong to different disconnected sectors. In the vacuum sector, the energy spectrum of the static BTZ black hole possesses a continuous part bounded from below by zero, a gap describing naked conical singularities, and a ground state that corresponds to AdS spacetime, having a negative mass given by

$$M_0 = \frac{2}{l}\tilde{\Delta}_0^{\pm} = -\frac{c^{\pm}}{12l} = -\frac{1}{8G} \ . \tag{56}$$

In the hairy sector the situation is similar, since the energy spectrum also consists of a continuous part being bounded from below by zero that describes the hairy black holes, a gap that corresponds to naked singularities, and remarkably, a ground state that turns out to be consistently identified with the solitons described above. Indeed, the solitons possess negative fixed masses, given by eqs. (36) and (55), being completely determined by the fundamental constants of the theory. The soliton solutions were also found to be smooth, regular everywhere, and devoid of integration constants. Furthermore, they naturally provide the completion of the hairy sector spectrum, since they not only fulfill the same asymptotic conditions as the hairy black holes, but they also have precisely the same boundary conditions, because the value of the constant α, in eq. (9) coincides for both kinds of configurations. Besides, unitarity of the dual theory for $c^{\pm} > 1$ (see e.g. [31]), together with the fact that the asymptotic growth of the number of states, given by (4), is well defined only for negative lowest eigenvalues of the shifted Virasoro operators, impose the following bounds on $\tilde{\Delta}_0^{\pm}$:

$$-\frac{c^{\pm}}{24} \leq \tilde{\Delta}_0^{\pm} < 0 \ . \tag{57}$$

Remarkably, full agreement is found from the bulk theory, since as expressed by eq. (37) this bound is precisely fulfilled by the soliton mass that corresponds to the ground state of the hairy sector, and according to (56) it is saturated in vacuum.

According to [14], the semiclassical entropy of the black holes under consideration can then be suitably reproduced in terms of the microscopic counting provided the ground state configuration is identified as the soliton or the AdS spacetime, for the hairy and vacuum sectors, respectively.

Therefore, in the vacuum sector, since the lowest eigenvalues of the Virasoro operators

$\tilde{\Delta}_0^{\pm}$, are given by eq. (56), as explained in introduction, the asymptotic growth of the number of states given by (4) reduces its standard form in eq. (5). Thus, by virtue of (6) the semiclassical entropy of the BTZ black hole is exactly reproduced as in Ref. [2].

The microscopic entropy of the hairy black holes discussed here can then be obtained assuming that $\tilde{\Delta}_0^{\pm}$ are determined by the global charges of their corresponding solitons, which according to eq. (6) are given by $\tilde{\Delta}_0^{\pm} = \frac{l}{2} M_0$, where M_0 stands for the soliton mass. In the case of static hairy black holes, as the ones discussed here, the asymptotic growth of the number of states, given by eq. (4), reduces to

$$S = 4\pi l \sqrt{-M_0 M} \ , \tag{58}$$

where M is the hairy black hole mass. For the uniparametric potential $V_{0,\nu}$, the semiclassical entropy of the hairy black hole, given by (24) is then successfully reproduced from a microscopic counting once the black hole and soliton masses, given by (29) and (36) are replaced into eq. (58). Explicitly, this reads

$$S = 4\pi l \sqrt{\frac{\Phi_\nu^2}{4G\nu} \times \frac{\nu a^2}{16 G l^2}} = \frac{\pi \Phi_\nu}{2G} a = \frac{A}{4G} \ .$$

Analogously, for the more generic potential $V_{\lambda,\nu}$, taking into account that the black hole and soliton masses are given by (44) and (55) respectively, formula (58) reduces to

$$S = 4\pi l \sqrt{\frac{\Upsilon_{\lambda,\nu}^2}{4G\nu} \times \frac{b^2}{16 G l^2} \frac{(\lambda-1)^2}{\lambda^2} \nu} = \frac{\pi}{2G} (1-\lambda) \frac{b}{\lambda} \Upsilon_{\lambda,\nu} = \frac{A}{4G} \ ,$$

in full agreement with (48).

V. FINAL REMARKS

New asymptotically AdS hairy black holes and solitons were shown to exist for General Relativity minimally coupled to a self-interacting scalar field in three dimensions. Different self-interaction potentials were engineered in order to obtain nontrivial analytic results, with the purpose of testing the robustness of regarding the soliton as the ground state of the hairy sector, and its key role in a microscopic counting of hairy black hole entropy. Our results then confirm that this proposal successfully goes beyond the example previously discussed in [14] and naturally point towards the fact that this mechanism should correspond to a generic feature of hairy black holes[43].

In the microcanonical ensemble, i.e., for a fixed value of the mass, since the theory admits hairy and vacuum black holes, it is natural to wonder which is the preferred configuration. By virtue of eqs. (58) and (37) (or equivalently (57)), the quotient of the entropies of the vacuum and hairy black holes fulfills

$$\frac{S_{BTZ}}{S_{\rm hbh}} = \sqrt{\frac{M_{AdS}}{M_{\rm sol}}} > 1 \ ,$$

where M_{AdS}, and $M_{\rm sol}$ stand for the mass of AdS and the soliton, respectively. Therefore, the vacuum black hole turns out to be the thermodynamically preferred configuration. This result could be readily extended for the (grand) canonical ensemble, as well as for the rotating case through applying a boost in the "$t - \varphi$" cylinder. It would then also be interesting to compare it with the one that could be obtained from the mechanical stability of the hairy solutions.

As an ending remark, an interesting feature of the hairy black holes reported here and their corresponding solitons that is worth to be highlighted, is that their Euclidean continuations turn out to be diffeomorphic provided their temperatures are related by an S-modular transformation.

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Appendix A: The functions Φ_ν and $\Upsilon_{\lambda,\nu}$

As shown in sections II and III, the functions Φ_ν and $\Upsilon_{\lambda,\nu}$ become relevant in order to analytically describe the geometric and physical properties of the hairy black holes and

solitons that correspond to the self interactions $V_{0,\nu}$ and $V_{\lambda,\nu}$, respectively. Some of their useful properties in this context are detailed in this appendix.

· The function Φ_ν is defined in terms of the Lambert W function according to eq. (20). It is a monotonically increasing function of the parameter ν, as it is depicted in Fig. 1. Its behaviour around $\nu \to 0$ is given by

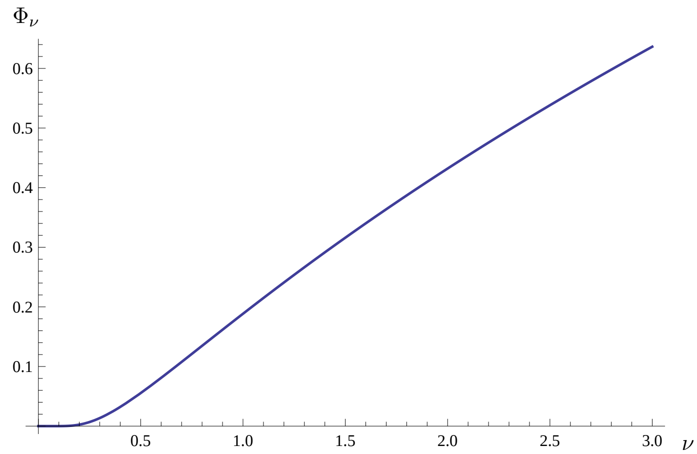

FIG. 1: Plot of the monotonically increasing function Φ_ν.

$$\Phi_\nu \xrightarrow[\nu\to 0]{} e^{-1-\frac{1}{\nu}} + 2e^{-2-\frac{2}{\nu}} + \frac{9}{2}e^{-3-\frac{3}{\nu}} + \cdots \ , \tag{A1}$$

while for $\nu \to \infty$, reads

$$\Phi_\nu \xrightarrow[\nu\to \infty]{} \sqrt{\frac{\nu}{2}} - \frac{2}{3} + \mathcal{O}\left(\nu^{-1/2}\right) \ . \tag{A2}$$

Therefore, by virtue of eqs. (A1) and (A2), it can be readily checked that the bound (22) is fulfilled.

· The function $\Upsilon_{\lambda,\nu}$ is defined in terms of the function $\mathcal{R}_{\lambda,\nu}$ through eq. (49), where $\mathcal{R}_{\lambda,\nu}$ stands for the real root of (46). Further details about $\mathcal{R}_{\lambda,\nu}$ are revisited in Fig. 2. It is simple to verify that it fulfills $\Upsilon_{\lambda,\nu}(1-\lambda) > 0$. In order to prove that the bound (50) holds, which reads

$$\Upsilon^2_{\lambda,\nu} < \frac{\nu}{2} \ , \tag{A3}$$

it is useful to recall eq. (46), so that this inequality is equivalently expressed as:

$$F(\mathcal{R}_{\lambda,\nu}) > 0 \quad \text{if} \quad \lambda > 1 \ , \tag{A4}$$

$$F(\mathcal{R}_{\lambda,\nu}) < 0 \quad \text{if} \quad \lambda < 1 \ , \tag{A5}$$

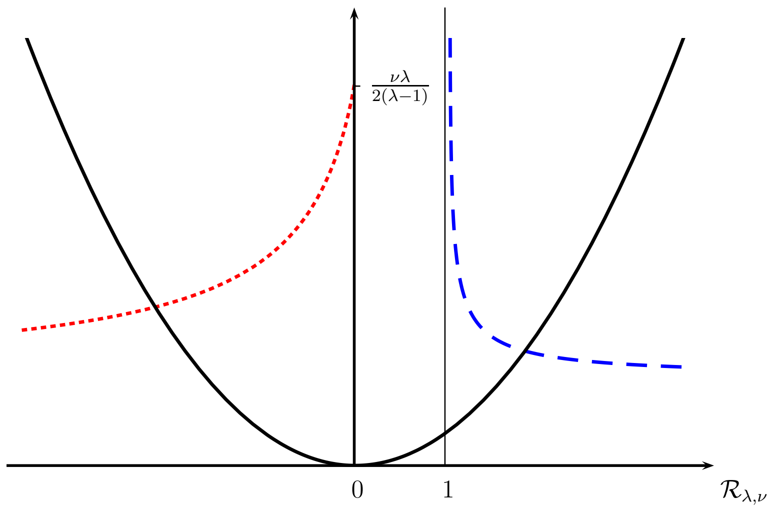

FIG. 2: The plot represents the real zeros of eq. (46), which reads $\mathcal{R}_{\lambda,\nu}^2 = \frac{\nu}{(\lambda-1)}\left[\frac{\lambda}{2} + \mathcal{R}_{\lambda,\nu} + \mathcal{R}_{\lambda,\nu}^2 \log\left(1 - \frac{1}{\mathcal{R}_{\lambda,\nu}}\right)\right]$. The solid line corresponds to $\mathcal{R}_{\lambda,\nu}^2$, while the function at the right hand side, for the case $\lambda > 1$, is depicted by the dotted line, and for $\lambda < 1$ is given by the dashed line. The intersection of these curves then shows the existence of a real root of (46), which for $\lambda > 1$ lies within the range $-\infty < \mathcal{R}_{\lambda,\nu} < 0$. This root approaches to $\sqrt{\frac{\nu\lambda}{2(\lambda-1)}}$ for $\nu \to 0$. For $\lambda < 1$, the plot shows that the real root is in the range $1 < \mathcal{R}_{\lambda,\nu} < \infty$.

where

$$F(\mathcal{R}_{\lambda,\nu}) := \frac{(\lambda + (1-\lambda)\mathcal{R}_{\lambda,\nu})^2}{2(\lambda-1)(\mathcal{R}_{\lambda,\nu}-1)^2} - \frac{\lambda}{2} - \mathcal{R}_{\lambda,\nu} - \mathcal{R}_{\lambda,\nu}^2 \log\left(1 - \frac{1}{\mathcal{R}_{\lambda,\nu}}\right) . \tag{A6}$$

In the case of $\lambda > 1$, when $\mathcal{R}_{\lambda,\nu} \to -\infty$ the function $F(\mathcal{R}_{\lambda,\nu})$ approaches to $-\frac{2}{3\mathcal{R}_{\lambda,\nu}} > 0$, while for $\mathcal{R}_{\lambda,\nu} \to 0$ tends to $F(\mathcal{R}_{\lambda,\nu}) \to \frac{\lambda}{2(\lambda-1)} > 0$. Hence, since the function $F(\mathcal{R}_{\lambda,\nu})$ is monotonically increasing in the range $-\infty < \mathcal{R}_{\lambda,\nu} < 0$, the inequality (A4) holds.

Finally, for $\lambda < 1$, when $\mathcal{R}_{\lambda,\nu} \to \infty$ the function $F(\mathcal{R}_{\lambda,\nu})$ tends to $-\frac{2}{3\mathcal{R}_{\lambda,\nu}} < 0$, while if $\mathcal{R}_{\lambda,\nu} \to 1$ it approaches to $F(\mathcal{R}_{\lambda,\nu}) \to \frac{1}{2(\lambda-1)(\mathcal{R}_{\lambda,\nu}-1)^2} < 0$. Therefore, as $F(\mathcal{R}_{\lambda,\nu})$ is a monotonically increasing function in the range $1 < \mathcal{R}_{\lambda,\nu} < \infty$, the inequality (A5) is satisfied.

It is simple to verify that in the case of $\lambda = 0$, the function $\Upsilon_{\lambda,\nu}$ reduces to Φ_ν defined above, i.e.,

$$\Upsilon_{0,\nu} = \Phi_\nu \, . \tag{A7}$$

Following the same procedure that allows to recover the potential $V_{1,\tilde{\nu}}$ in (12) from $V_{\lambda,\nu}$ in (38), that consists on redefining $\nu = 6\frac{\tilde{\nu}+1}{(\lambda-1)^2}$, and then taking the limit $\lambda \to 1$, the function

$\Upsilon_{\lambda,\nu}$ can be shown to fulfill

$$\lim_{\lambda\to 1}\,(\lambda-1)^2\Upsilon^2_{\lambda,6(1+\tilde{\nu})/(\lambda-1)^2}=\Theta^2_{\tilde{\nu}}\ , \tag{A8}$$

where

$$\Theta_{\tilde{\nu}}:=2(z\bar{z})^{\frac{2}{3}}\frac{z^{\frac{2}{3}}-\bar{z}^{\frac{2}{3}}}{z-\bar{z}},\ \text{with}\quad z=1+i\sqrt{\tilde{\nu}}. \tag{A9}$$

As shown in [18] and [14], the function $\Theta_{\tilde{\nu}}$ is the relevant one in order to obtain an analytic description of the hairy black holes and solitons that correspond to the self interaction $V_{1,\tilde{\nu}}$.

Appendix B: Solutions in the conformal (Jordan) frame

The three-dimensional hairy black hole and soliton solutions for a self-interacting scalar field minimally coupled to General Relativity discussed here, acquire an appealing form in the conformal frame. The action in eq. (7), after applying a conformal transformation, followed by a scalar field redefinition of the form

$$\hat{g}_{\mu\nu}=\left(1-\hat{\phi}^2\right)^{-2}g_{\mu\nu}\quad\text{and}\quad\hat{\phi}=\tanh\left(\phi\right)\ , \tag{B1}$$

reduces to the one for General Relativity with cosmological constant and a conformally coupled self-interacting scalar field, given by

$$I[\hat{g},\hat{\phi}]=\frac{1}{\pi G}\int d^3x\sqrt{-\hat{g}}\left(\frac{\hat{R}+2l^{-2}}{16}-\frac{1}{2}(\nabla\hat{\phi})^2-\frac{1}{16}\hat{R}\hat{\phi}^2-\hat{V}(\hat{\phi})\right)\ . \tag{B2}$$

It can be shown that the potentials $\hat{V}_{0,\nu}$, $\hat{V}_{1,\nu}$, and $\hat{V}_{\lambda,\nu}$ coming from their corresponding counterparts in the Einstein frame discussed above, given by eqs. (14), (12) and (38), respectively, are the only three branches of self interactions that are compatible with static and spherically symmetric solutions of the form $d\hat{s}^2=-f(\rho)dt^2+\frac{d\rho^2}{f(\rho)}+\rho^2d\theta^2$, $\hat{\phi}=\hat{\phi}(\rho)$. As shown below, by means of (B1), this class of solutions corresponds to the hairy black holes described in sections II A and III A as well as the ones in Refs. [39] and [18] in the conformal frame. The explicit form of the self interactions $\hat{V}_{0,\nu}$, $\hat{V}_{1,\nu}$, and $\hat{V}_{\lambda,\nu}$, including also their corresponding soliton solutions are discussed in what follows.

1. Solutions for $\hat{V}_{1,\nu}$

In the conformal frame, the self interaction (12) is mapped into

$$\hat{V}_{1,\nu}=-\frac{\nu}{8l^2}\hat{\phi}^6\ . \tag{B3}$$

This potential is the only one that turns out to be singled out by requiring the matter piece of the action (B2) to be conformally invariant; i.e., unchanged under local rescalings of the form $\hat{g}_{\mu\nu} \to \lambda^2(x)\hat{g}_{\mu\nu}$, and $\hat{\phi} \to \lambda^{-1/2}(x)\hat{\phi}$. In the case of $\nu \geq -1$, hairy black holes solutions were found in [18], which reduces to the one previously found in [39] for $\nu = 0$. Solutions describing solitons were also found for the same range of the self-interaction coupling in [14].

2. Solutions for $\hat{V}_{0,\nu}$

The self interaction (14) in the conformal frame, is mapped into

$$\hat{V}_{0,\nu} = \frac{\nu}{16l^2}\frac{\hat{\phi}^2 - 2}{1 - \hat{\phi}^2}\hat{\phi}^2 - \frac{\nu}{8l^2}\ln\left(1 - \hat{\phi}^2\right) \ , \tag{B4}$$

whose behavior around $\hat{\phi} = 0$ reads

$$\hat{V}(\hat{\phi}) \sim -\frac{\nu}{48l^2}\hat{\phi}^6 - \frac{\nu}{32l^2}\hat{\phi}^8 + \mathcal{O}(\hat{\phi}^{10}) \ . \tag{B5}$$

Hairy black hole: In this case the metric is given by

$$d\hat{s}^2 = -f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 d\theta^2 \ , \tag{B6}$$

with

$$f(\rho) = \frac{\rho^2}{l^2} + \frac{\nu}{l^2}\left(a\rho + \rho^2\ln\left(1 - \frac{a}{\rho}\right)\right) \ , \tag{B7}$$

where the coordinates range as $-\infty < t < \infty$, $a < \rho < \infty$, $0 \leq \varphi < 2\pi$. The scalar field also acquires a very simple form, that reads

$$\hat{\phi}(\rho) = \sqrt{\frac{a}{\rho}} \ . \tag{B8}$$

The hairy black hole solution, then depends on a single non-negative integration constant a, that parametrizes the location of the event horizon, for $\nu > 0$, at $\rho = \rho_+$, with

$$\rho_+ = \frac{a}{1 + W(-e^{-1-\frac{1}{\nu}})} \ , \tag{B9}$$

where W stands for the Lambert W function.

Soliton: The potential (B4) also admits an additional exact solution. The scalar field is given by

$$\hat{\phi}(\rho) = \sqrt{\frac{1}{\rho}} \ , \tag{B10}$$

and the metric reads

$$d\hat{s}^2 = -\rho^2 dt^2 + \frac{d\rho^2}{g(\rho)} + \frac{4l^4\Phi_\nu^2}{\nu^2} g(\rho) d\varphi^2 \ , \tag{B11}$$

with

$$g(\rho) = \frac{\rho^2}{l^2} + \frac{\nu}{l^2}\left(\rho + \rho^2 \ln\left(1 - \frac{1}{\rho}\right)\right) \tag{B12}$$

Note that the solution is devoid of integration constants. The coordinates range according to $1 + \Phi_\nu \leq \rho < \infty$, $-\infty < t < \infty$, and $0 \leq \varphi < 2\pi$. The metric and the scalar field are regular everywhere. Since the mass does not depend on the choice of frame, it is given by (36) and the solution then describes a soliton.

3. Solutions for $\hat{V}_{\lambda,\nu}$

In the conformal frame, the two-parametric potential (38) is mapped into

$$\begin{aligned} \hat{V}_{\lambda,\nu}(\hat{\phi}) = & \frac{\lambda^2}{8l^2}\hat{\phi}^6 - \frac{\nu}{8l^2(\lambda-1)}\left(1 - \lambda^2\phi^6\right)\ln\left(\frac{1-\lambda\phi^2}{1-\phi^2}\right) \\ & - \frac{\nu}{8l^2}\frac{1}{1-\phi^2}\left(\phi^2 + \frac{\lambda-1}{2}\phi^4 + \frac{\lambda(\lambda-1)}{2}\phi^6 - \frac{\lambda(\lambda+1)}{2}\phi^8\right) \ , \end{aligned} \tag{B13}$$

whose behavior around $\hat{\phi} = 0$ is of the form

$$V_{\lambda,\nu}(\hat{\phi}) = -\frac{\nu(\lambda-1)^2 - 6\lambda^2}{48l^2}\hat{\phi}^6 + \frac{\nu(\lambda-1)^3}{32l^2}\hat{\phi}^8 + \mathcal{O}(\hat{\phi}^{10}) \ .$$

Hairy black hole: The field equations derived from (B2) admit a static circularly symmetric solution whose metric is given by

$$d\hat{s}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2 \ , \tag{B14}$$

where $f(r)$ is expressed in eq. (41), and the scalar field reads

$$\hat{\phi}(r) = \sqrt{\frac{b}{\lambda(r+b)}} \ . \tag{B15}$$

The solution then depends on a single integration constant b, and it is well-defined provided $b/\lambda > 0$. The coordinates range as $-\infty < t < \infty$, $r_s < r < \infty$, and $0 \leq \varphi < 2\pi$. This solution describes a hairy black hole for $\nu > 0$, with an event horizon at $r = r_+$, with r_+ given by (45), which surrounds the singularity at $r = r_s$, located precisely as explained in Section III A.

Soliton: The self interaction (B13) also admits soliton solution described by the metric

$$d\hat{s}^2 = -\rho^2 dt^2 + \frac{d\rho^2}{g(\rho)} + l^2\lambda^2\gamma_{\lambda,\nu}^2 g(\rho) d\varphi^2 \,, \tag{B16}$$

with

$$g(\rho) = \frac{\rho^2}{l^2} - \frac{\nu}{\lambda(\lambda-1)l^2}\left(\frac{(\lambda-1)^2}{2} - (\lambda-1)\rho + \lambda\rho^2 \ln\left(1 + \frac{(\lambda-1)}{\lambda\rho}\right)\right) \,, \tag{B17}$$

where the scalar field is given by

$$\hat{\phi}(\rho) = \sqrt{\frac{1}{\lambda(\rho+1)}} \,. \tag{B18}$$

The soliton possesses no integration constants, and taking in account that the coordinates range as $\frac{1}{\lambda} + \frac{(1-\lambda)\Upsilon_{\lambda,\nu}}{\lambda} \leq \rho < \infty$, $-\infty < t < \infty$, $0 \leq \varphi < 2\pi$, it is simple to verify that the solution is is regular everywhere.

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[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy,* Phys. Lett. B **379**, 99 (1996) [hep-th/9601029].

[2] A. Strominger, *Black hole entropy from near-horizon microstates,* JHEP **9802**, 009 (1998) [arXiv:hep-th/9712251].

[3] D. Birmingham, I. Sachs and S. Sen, *Entropy of three-dimensional black holes in string theory,* Phys. Lett. B **424**, 275 (1998) [hep-th/9801019].

[4] S. Carlip, *Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole,* Class. Quant. Grav. **22**, R85 (2005) [gr-qc/0503022].

[5] M. Guica, T. Hartman, W. Song and A. Strominger, *The Kerr/CFT Correspondence,* Phys. Rev. D **80**, 124008 (2009) [arXiv:0809.4266 [hep-th]].

[6] S. Carlip, *Effective Conformal Descriptions of Black Hole Entropy,* arXiv:1107.2678 [gr-qc].

[7] M. Cvetič and F. Larsen, *Conformal Symmetry for General Black Holes,* arXiv:1106.3341 [hep-th].

[8] M. Cvetič and F. Larsen, *Conformal Symmetry for Black Holes in Four Dimensions,* arXiv:1112.4846 [hep-th].

[9] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,* Commun. Math. Phys. **104**, 207 (1986).

[10] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity,* Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory,* Phys. Lett. B **428**, 105 (1998) [hep-th/9802109]; E. Witten, *Anti-de Sitter space and holography,* Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[11] J. L. Cardy, *Operator Content Of Two-Dimensional Conformally Invariant Theories,* Nucl. Phys. B **270**, 186 (1986).

[12] M. Bañados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time,* Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].

[13] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole,* Phys. Rev. D **48**, 1506 (1993) [arXiv:gr-qc/9302012].

[14] F. Correa, C. Martínez and R. Troncoso, *Scalar solitons and the microscopic entropy of hairy black holes in three dimensions,* JHEP **1101**, 034 (2011) [arXiv:1010.1259 [hep-th]].

[15] S. Carlip, *Entropy from conformal field theory at Killing horizons,* Class. Quant. Grav. **16**, 3327 (1999) [arXiv:gr-qc/9906126].

[16] M. I. Park, *Fate of three-dimensional black holes coupled to a scalar field and the Bekenstein-Hawking entropy,* Phys. Lett. B **597**, 237 (2004) [arXiv:hep-th/0403089].

[17] F. Loran, M. M. Sheikh-Jabbari and M. Vincon, *Beyond Logarithmic Corrections to Cardy Formula,* JHEP **1101**, 110 (2011) [arXiv:1010.3561 [hep-th]].

[18] M. Henneaux, C. Martínez, R. Troncoso and J. Zanelli, *Black holes and asymptotics of 2+1 gravity coupled to a scalar field,* Phys. Rev. D **65**, 104007 (2002) [arXiv:hep-th/0201170].

[19] P. Breitenlohner and D. Z. Freedman, *Positive Energy In Anti-De Sitter Backgrounds And Gauged Extended Supergravity,* Phys. Lett. B **115**, 197 (1982); *Stability In Gauged Extended Supergravity,* Annals Phys. **144**, 249 (1982).

[20] L. Mezincescu and P. K. Townsend, *Stability At A Local Maximum In Higher Dimensional Anti-De Sitter Space And Applications To Supergravity,* Annals Phys. **160**, 406 (1985).

[21] T. Regge and C. Teitelboim, *Role Of Surface Integrals In The Hamiltonian Formulation Of General Relativity,* Annals Phys. **88**, 286 (1974).

[22] Weisstein, Eric W. "Lambert W-Function." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/LambertW-Function.html`. See also S. R. Valluri, D. J. Jeffrey and R. M. Corless, *"Some applications of the Lambert W function to physics,* Can. J.

Phys. **78**, 823 (2000), and references therein.

[23] G. Barnich, *Conserved charges in gravitational theories: Contribution from scalar fields,* [arXiv:gr-qc/0211031].

[24] G. Barnich, *Boundary charges in gauge theories: Using Stokes theorem in the bulk,* Class. Quant. Grav. **20**, 3685 (2003) [arXiv:hep-th/0301039].

[25] J. Gegenberg, C. Martínez and R. Troncoso, *A finite action for three dimensional gravity with a minimally coupled scalar field,* Phys. Rev. D **67**, 084007 (2003) [arXiv:hep-th/0301190].

[26] G. Clément, *Black hole mass and angular momentum in 2+1 gravity,* Phys. Rev. D **68**, 024032 (2003) [gr-qc/0301129].

[27] M. Hortaçsu, H. T. Özçelik and B. Yapışkan, *Properties of solutions in (2+1)-dimensions,* Gen. Rel. Grav. **35**, 1209 (2003) [gr-qc/0302005].

[28] M. Bañados and S. Theisen, *Scale invariant hairy black holes,* Phys. Rev. D **72**, 064019 (2005) [hep-th/0506025].

[29] Y. S. Myung, *Phase transition for black holes with scalar hair and topological black holes,* Phys. Lett. B **663**, 111 (2008) [arXiv:0801.2434 [hep-th]].

[30] N. Lashkari, *Holographic Symmetry-Breaking Phases in AdS_3/CFT_2,* JHEP **1111**, 104 (2011) [arXiv:1011.3520 [hep-th]].

[31] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Springer, New York, USA (1997).

[32] A. Pérez, D. Tempo and R. Troncoso, *Gravitational solitons, hairy black holes and phase transitions in BHT massive gravity,* JHEP **1107**, 093 (2011) [arXiv:1106.4849 [hep-th]].

[33] H. A. González, D. Tempo and R. Troncoso, *Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes,* JHEP **1111**, 066 (2011) [arXiv:1107.3647 [hep-th]].

[34] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Massive Gravity in Three Dimensions,* Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:0901.1766 [hep-th]].

[35] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, *More on Massive 3D Supergravity,* Class. Quant. Grav. **28**, 015002 (2011) [arXiv:1005.3952 [hep-th]].

[36] J. Oliva, D. Tempo and R. Troncoso, *Three-dimensional black holes, gravitational solitons, kinks and wormholes for BHT masive gravity,* JHEP **0907**, 011 (2009) [arXiv:0905.1545 [hep-th]].

[37] G. Giribet, J. Oliva, D. Tempo and R. Troncoso, *Microscopic entropy of the three-dimensional rotating black hole of BHT massive gravity,* Phys. Rev. D **80**, 124046 (2009) [arXiv:0909.2564 [hep-th]].

[38] E. Ayón-Beato, A. Garbarz, G. Giribet and M. Hassaïne, *Lifshitz Black Hole in Three Dimensions,* Phys. Rev. D **80**, 104029 (2009) [arXiv:0909.1347 [hep-th]].

[39] C. Martínez and J. Zanelli, *Conformally dressed black hole in 2+1 dimensions,* Phys. Rev. D **54**, 3830 (1996) [arXiv:gr-qc/9604021].

[40] Note that, for $-1/e < z < 0$, the lower branch of the Lambert W function ranges as $-\infty < W(z) < -1$. Hereafter the lower branch is not considered since the solution would describe a naked singularity and it would then be ruled out by cosmic censorship.

[41] It would be interesting to compare this result with the ones that could be obtained from different approaches that are adapted to deal with scalar fields and relaxed AdS asymptotics, as the ones in Refs. [23–25]. Further results along the lines of [26–30], previously found for the hairy black hole of Ref. [18], would also worth to be explored for the new solutions found here.

[42] In the limit $\lambda \to 1$, following a similar procedure, the hairy black hole and soliton solutions of [14, 18] can be recovered from those in Sec. II.

[43] It is worth pointing out that similar results have also been recently found in [32, 33] in the context of BHT massive gravity [34, 35]; namely for asymptotically AdS hairy black holes and solitons in vacuum [36, 37], as well as for black holes and solitons with Lifshitz asymptotics [33, 38].