We have the following Hamiltonian in XXX-Heisenberg model in one dimension with periodic boundary conditions.

$$\widehat{H} = -J\sum_{j=1}^{N}\left(\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \hat{S}_j^z\hat{S}_{j+1}^z\right)$$

if we compute the commutator of $\widehat{H}$ and spin vector operator $\vec{S} = \left(\hat{S}^x, \hat{S}^y, \hat{S}^z\right)$, this is

$$\left[\widehat{H}, \vec{S}\right] = \left[-J\sum_{j=1}^{N}\left(\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \hat{S}_j^z\hat{S}_{j+1}^z\right), \left(\hat{S}^x, \hat{S}^y, \hat{S}^z\right)\right] = -J\sum_{j=1}^{N}\left[\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \hat{S}_j^z\hat{S}_{j+1}^z, \left(\hat{S}^x, \hat{S}^y, \hat{S}^z\right)\right]$$

$$= -J\sum_{j=1}^{N}\left(\left[\hat{S}_j^y\hat{S}_{j+1}^y + \hat{S}_j^z\hat{S}_{j+1}^z, \hat{S}^x\right], \left[\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^z\hat{S}_{j+1}^z, \hat{S}^y\right], \left[\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y, \hat{S}^z\right]\right) = \vec{0}$$

so we can use eigenvectors of any component of $\vec{S}$.

For preference let use S_z then the we can use the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Indeed the number of $N_\uparrow$ and $N_\downarrow$ is fixed. Also the total number $N = N_\uparrow + N_\downarrow$ is fixed.

In this sense it's useful to express $\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y$ in terms of operators $\hat{S}_+$ and $\hat{S}_-$ defined by

$$\hat{S}^+ = \hat{S}^x + i\hat{S}^y$$
$$\hat{S}^- = \hat{S}^x - i\hat{S}^y$$

and the operators $\hat{S}^x$ and $\hat{S}^y$ expressed in terms of these operators

$$\hat{S}^x = \frac{1}{2}\left(\hat{S}^+ + \hat{S}^-\right)$$
$$\hat{S}^y = \frac{i}{2}\left(\hat{S}^- - \hat{S}^+\right)$$

then

$$\hat{S}_j^x\hat{S}_{j+1}^x = \left(\frac{1}{2}\left(\hat{S}_j^+ + \hat{S}_j^-\right)\right)\left(\frac{1}{2}\left(\hat{S}_{j+1}^+ + \hat{S}_{j+1}^-\right)\right) = \frac{1}{4}\left(\hat{S}_j^+\hat{S}_{j+1}^+ + \hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^-\hat{S}_{j+1}^+ + \hat{S}_j^-\hat{S}_{j+1}^-\right)$$

$$\hat{S}_j^y\hat{S}_{j+1}^y = \left(\frac{i}{2}\left(\hat{S}_j^- - \hat{S}_j^+\right)\right)\left(\frac{i}{2}\left(\hat{S}_{j+1}^- - \hat{S}_{j+1}^+\right)\right) = -\frac{1}{4}\left(\hat{S}_j^-\hat{S}_{j+1}^- - \hat{S}_j^-\hat{S}_{j+1}^+ - \hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^+\hat{S}_{j+1}^+\right)$$

so

$$\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y = \frac{1}{2}\left(\hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^-\hat{S}_{j+1}^+\right)$$

and the hamiltonian now is

$$\widehat{H} = -\frac{J}{2}\sum_{j=1}^{N}\left(\hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^-\hat{S}_{j+1}^+ + 2\hat{S}_j^z\hat{S}_{j+1}^z\right)$$

we know that

$$\hat{S}^+|\uparrow\rangle = 0 \qquad \hat{S}^+|\downarrow\rangle = |\uparrow\rangle \qquad \hat{S}^-|\uparrow\rangle = |\downarrow\rangle \qquad \hat{S}^-|\downarrow\rangle = 0 \qquad \hat{S}^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle \qquad \hat{S}^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$$

Construction of the basis

Now we will explote that $N_\uparrow$ and $N_\downarrow$ are constants. Also we explote translational invariance and periodic boundary conditions.

So we will have a set of representative states $|r_k\rangle = \hat{P}_k \frac{|r\rangle}{\sqrt{\langle r|\hat{P}_k|r\rangle}}$. And we will apply $\widehat{H}$ to these states

$$\langle r_q|\widehat{H}|r_k\rangle = \frac{\langle r_q|\widehat{H}\hat{P}_k|r\rangle}{\sqrt{\langle r|\hat{P}_k|r\rangle}} = \frac{\langle r_q|\hat{P}_k\widehat{H}|r\rangle}{\sqrt{\langle r|\hat{P}_k|r\rangle}} = \langle r_q|\hat{P}_k\left(-\frac{J}{2}\sum_{j=1}^{N}\left(\hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^-\hat{S}_{j+1}^+ + 2\hat{S}_j^z\hat{S}_{j+1}^z\right)\right)\hat{c}_{N,\sigma}^\dagger\cdots\hat{c}_{j,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger\frac{|0\rangle}{\sqrt{\langle r|\hat{P}_k|r\rangle}}$$

$$= -\frac{J}{2}\sum_{j=1}^{N}\frac{\langle r_q|\hat{P}_k\left(\hat{S}_j^+\hat{S}_{j+1}^- + \hat{S}_j^-\hat{S}_{j+1}^+ + 2\hat{S}_j^z\hat{S}_{j+1}^z\right)\hat{c}_{N,\sigma}^\dagger\cdots\hat{c}_{j,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle}{\sqrt{\langle r|\hat{P}_k|r\rangle}}$$

Since we know that

$$\left[\hat{S}_l^z, \hat{c}_{m,\sigma}^\dagger\right] = \frac{1}{2}\delta_{lm}\hat{c}_{m,\sigma}^\dagger$$

$$\left[\hat{S}_l^+, \hat{c}_{m,\sigma}^\dagger\right] = \frac{1}{2}\delta_{lm}\hat{c}_{m,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\uparrow}$$

$$\left[\hat{S}_l^-, \hat{c}_{m,\sigma}^\dagger\right] = \frac{1}{2}\delta_{lm}\hat{c}_{m,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\downarrow}$$

so for each term in the hamiltonian we will have

$$\hat{S}_j^z\hat{S}_{j+1}^z|r\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\hat{S}_j^z\hat{S}_{j+1}^z\hat{c}_{j,\sigma}^\dagger\hat{c}_{j+1,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\hat{S}_j^z\hat{c}_{j,\sigma}^\dagger\hat{S}_{j+1}^z\hat{c}_{j+1,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle$$

$$= \hat{c}_{N,\sigma}^\dagger\cdots\left(\frac{1}{2}\hat{c}_{j,\sigma}^\dagger + \hat{c}_{j,\sigma}^\dagger\hat{S}_j^z\right)\left(\frac{1}{2}\hat{c}_{j+1,\sigma}^\dagger + \hat{c}_{j+1,\sigma}^\dagger\hat{S}_{j+1}^z\right)\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\left(\frac{1}{2}\hat{c}_{j,\sigma}^\dagger + \hat{c}_{j,\sigma}^\dagger\hat{S}_j^z\right)\left(\frac{1}{2}\hat{c}_{j+1,\sigma}^\dagger\right)\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle$$

$$= \frac{1}{4}\hat{c}_{N,\sigma}^\dagger\cdots\hat{c}_{j,\sigma}^\dagger\hat{c}_{j+1,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle = \frac{1}{4}|r\rangle$$

$$\hat{S}_j^+\hat{S}_{j+1}^-|r\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\hat{S}_j^+\hat{S}_{j+1}^-\hat{c}_{j,\sigma}^\dagger\hat{c}_{j+1,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\hat{S}_j^+\hat{c}_{j,\sigma}^\dagger\hat{S}_{j+1}^-\hat{c}_{j+1,\sigma}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle$$

$$= \hat{c}_{N,\sigma}^\dagger\cdots\left(\frac{1}{2}\hat{c}_{j,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\uparrow} + \hat{c}_{j,\sigma}^\dagger\hat{S}_j^+\right)\left(\frac{1}{2}\hat{c}_{j+1,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\downarrow} + \hat{c}_{j+1,\sigma}^\dagger\hat{S}_{j+1}^-\right)\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle = \hat{c}_{N,\sigma}^\dagger\cdots\left(\frac{1}{2}\hat{c}_{j,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\uparrow} + \hat{c}_{j,\sigma}^\dagger\hat{S}_j^+\right)\left(\frac{1}{2}\hat{c}_{j+1,\bar{\sigma}}^\dagger\delta_{\bar{\sigma},\downarrow}\right)\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle$$

$$= \frac{1}{4}\delta_{\bar{\sigma}_{j+1},\downarrow}\delta_{\bar{\sigma}_j,\uparrow}\hat{c}_{N,\sigma}^\dagger\cdots\hat{c}_{j,\bar{\sigma}}^\dagger\hat{c}_{j+1,\bar{\sigma}}^\dagger\cdots\hat{c}_{1,\sigma}^\dagger|0\rangle$$