Planar Inverse Kinematics

Lecture 5

Winter 2023

ROBOTICS

Review: Forward Kinematics

- The complete FK equation is assembled from the individual homogeneous transformations

$$H = A_1(q_1)...A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

- R_n^0 represents orientation of the end effector
- o_n^0 represents location of end effector in frame 0

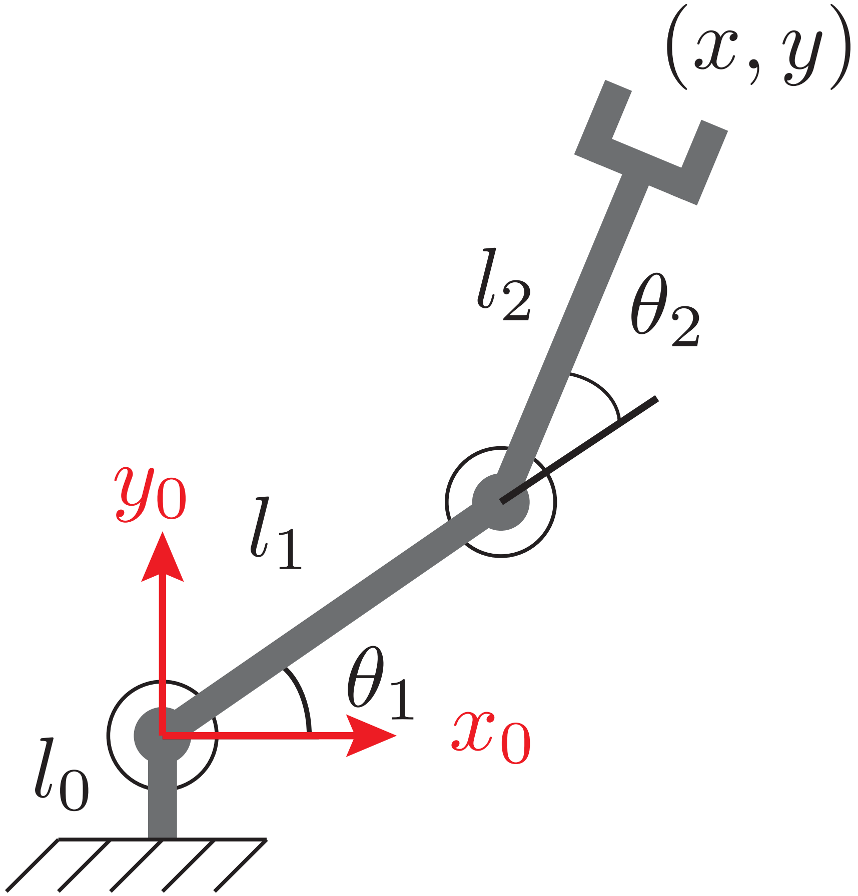
Inverse Kinematics

- Forward Kinematics:
  - Finds end-effector pose $H_n^0(q)$ as function of configuration q

$$H = A_1(q_1)...A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

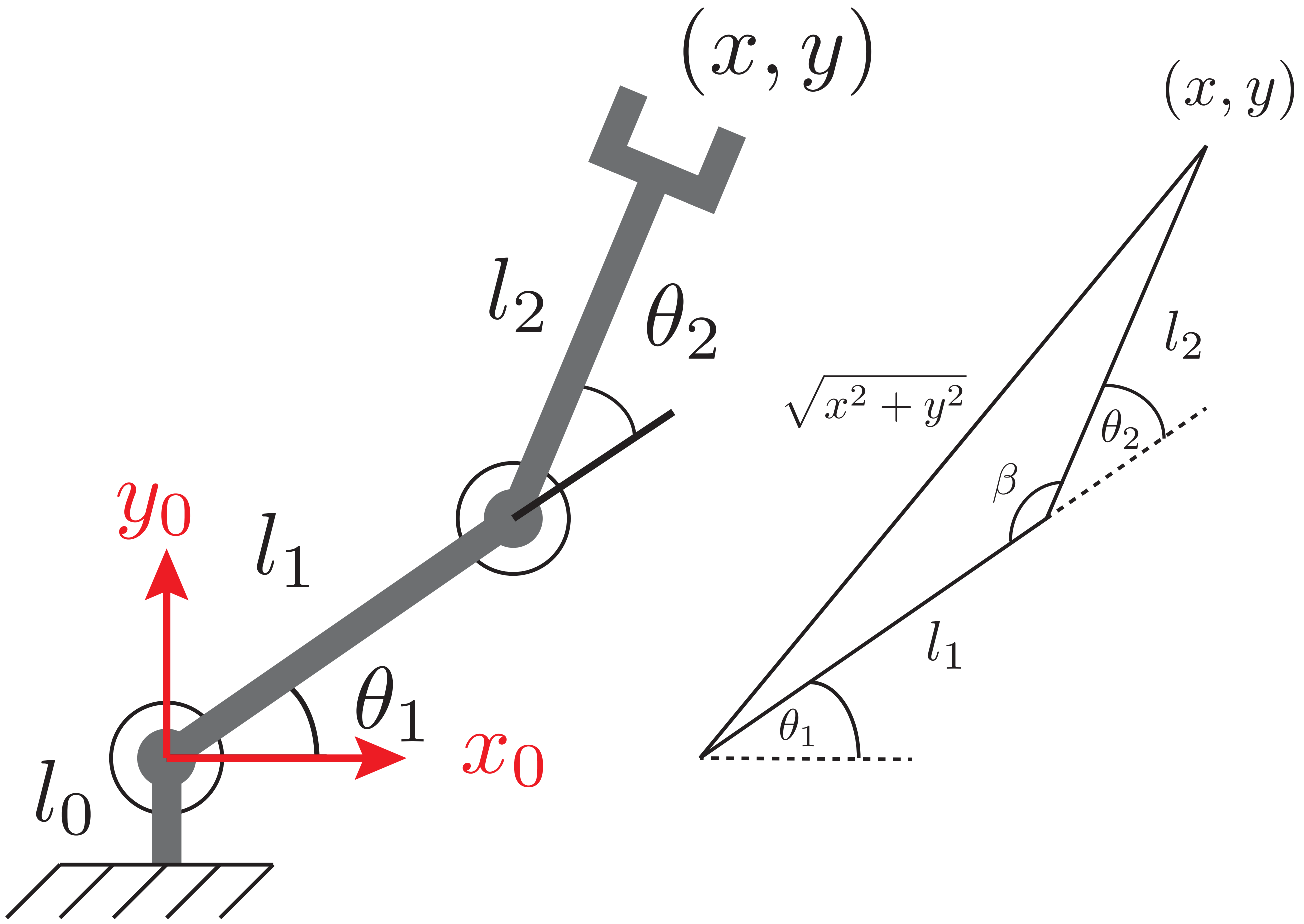
- Inverse Kinematics:
  - Given desired end effector pose H
  - Find all joint variables $q_1, \dots, q_n$
  - In 2D we have 3 independent equations with n unknowns
  - In 3D we have 6 independent equations with n unknowns

RR Arm



- 2 Joints, 2DOF
- Can control position
- No control over orientation

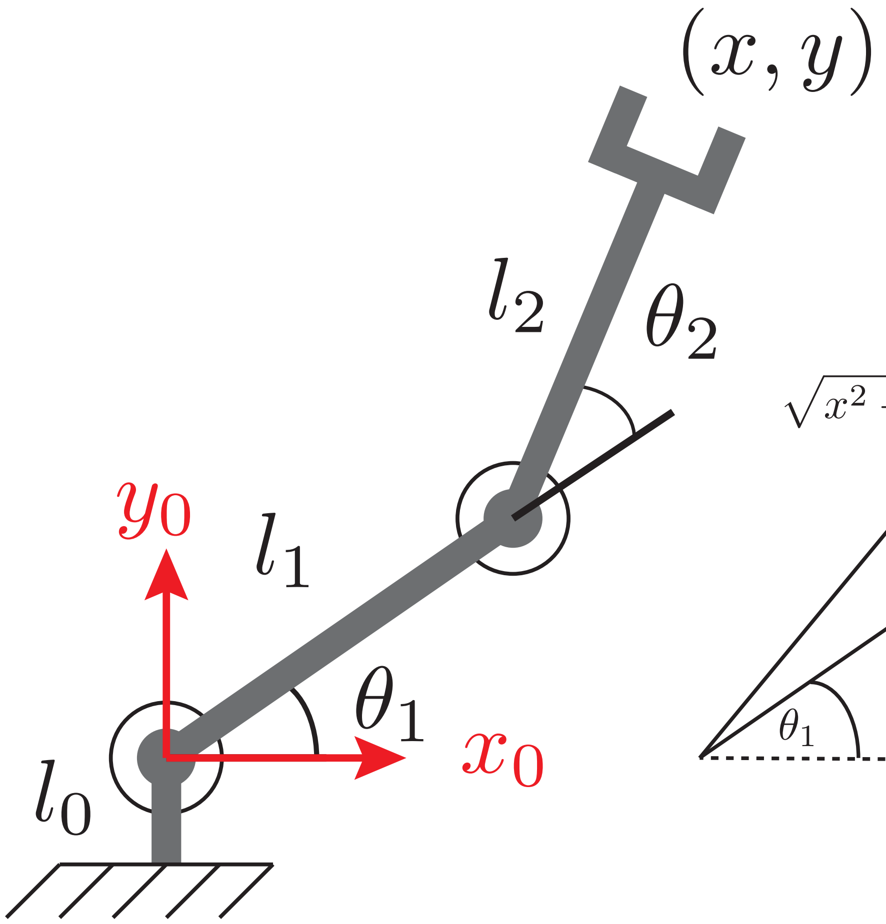
Geometrical Solution

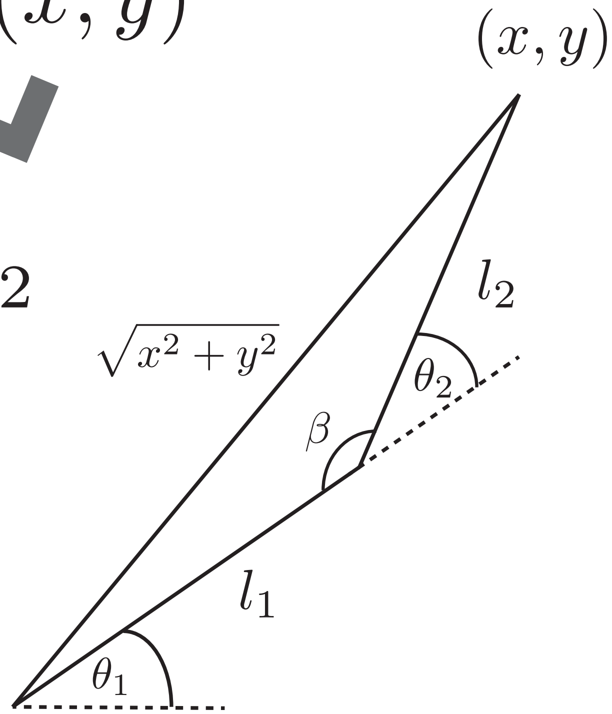


- Use Law of Cosines to find β
- $\theta_2 = \pi - \beta$

$$x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos\beta$$

Geometrical Solution





$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\beta$$

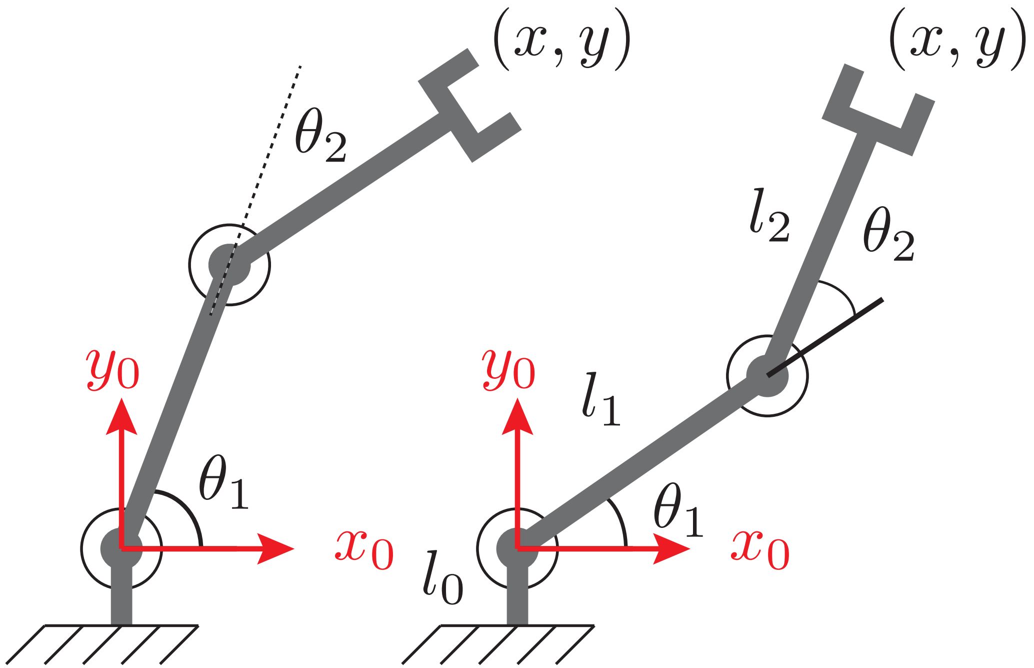
$$\cos\beta = \frac{l_1^2 + l_2^2 - (x^2 + y^2)}{2l_1 l_2}$$

$$\cos(\pi - \theta) = -\cos\theta$$

$$\cos\theta_2 = \frac{(x^2 + y^2) - l_1^2 - l_2^2}{2l_1 l_2}$$

$$\boxed{\theta_2 = \cos^{-1}\left(\frac{(x^2 + y^2) - l_1^2 - l_2^2}{2l_1 l_2}\right)}$$

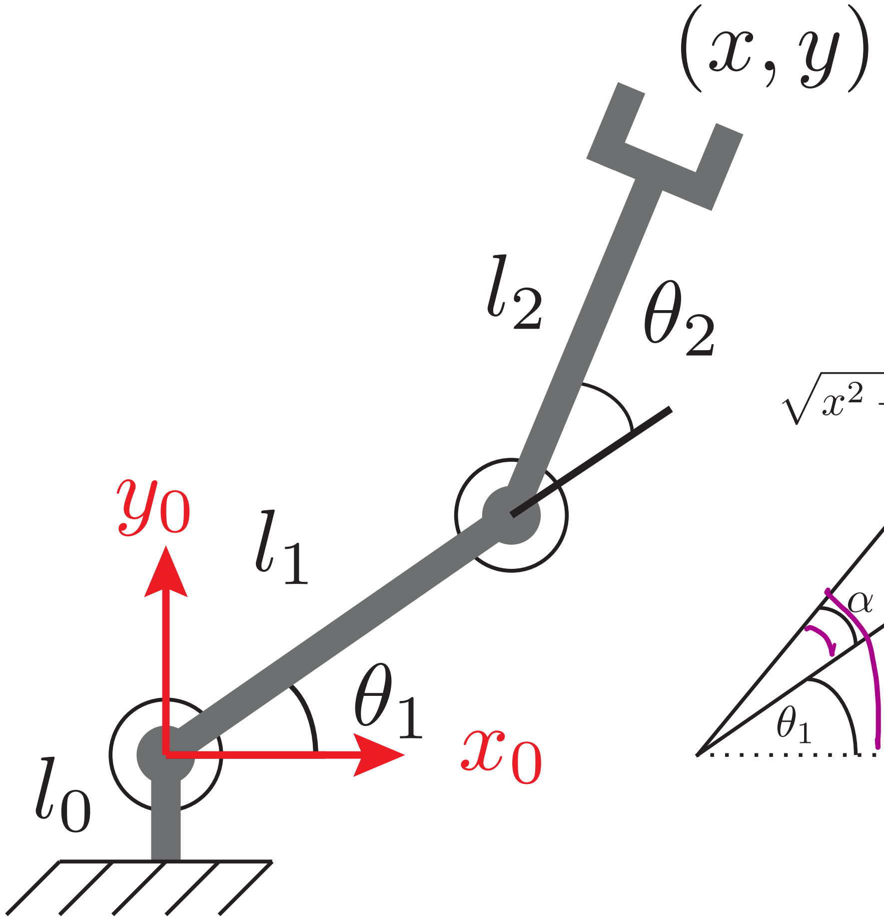
Degeneracies

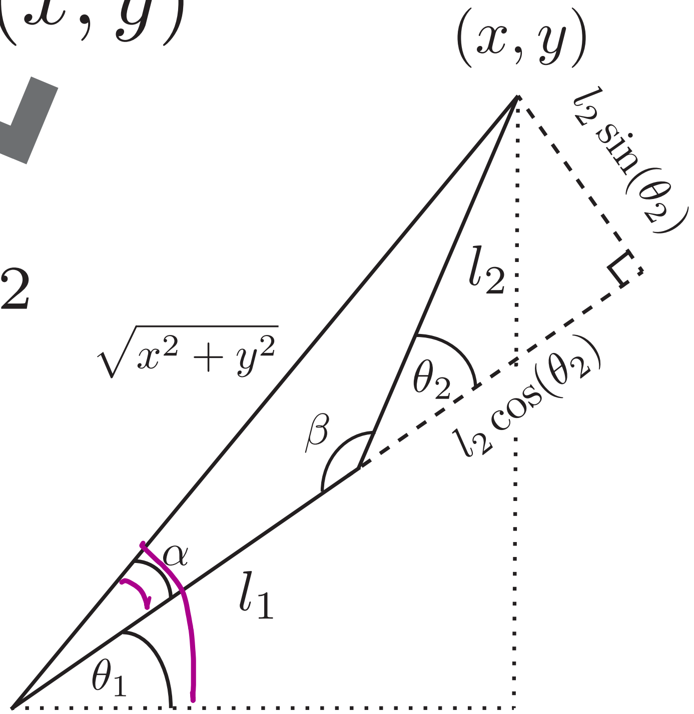


$$\cos(\theta) = \cos(-\theta)$$

- 2 solutions for $\boldsymbol{\theta_2}$
- We refer to these as “elbow up” and “elbow down” configuration
- The configuration depends on $\boldsymbol{\theta_1}$ as well

Geometrical Solution





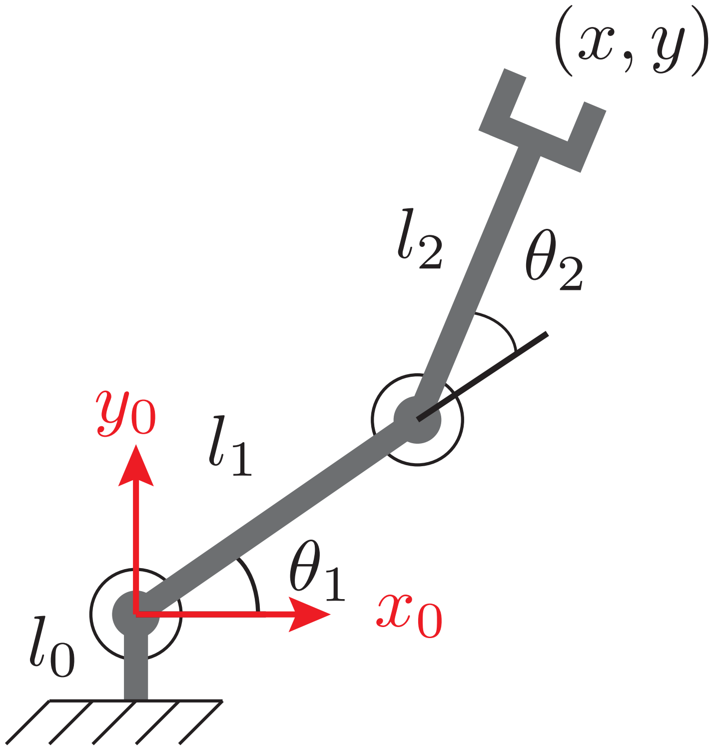
$$\tan \alpha = \frac{l_2 \sin \theta_2}{l_1 + l_2 \cos \theta_2}$$

$$\alpha = \operatorname{atan2}(l_2 \sin \theta_2, l_1 + l_2 \cos \theta_2)$$

$$\theta_1 = \operatorname{atan2}(y, x) - \alpha$$

$$\boxed{\theta_1 = \operatorname{atan2}(y, x) - \operatorname{atan2}(l_2 \sin \theta_2, l_1 + l_2 \cos \theta_2)}$$

RR IK Solution Summary



- Find 2 solutions for θ_2

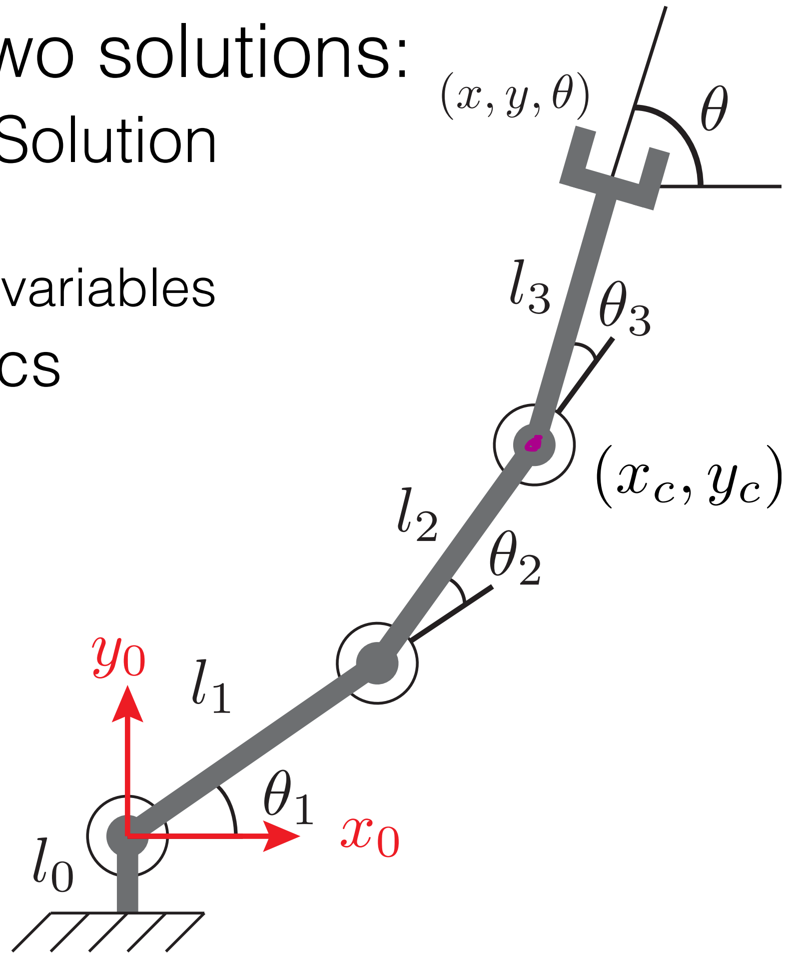
$$\boxed{\theta_2 = \cos^{-1}\left(\frac{(x^2 + y^2) - l_1^2 - l_2^2}{2 l_1 l_2}\right)}$$

- Find solution for θ_1 depending on choice of θ_2

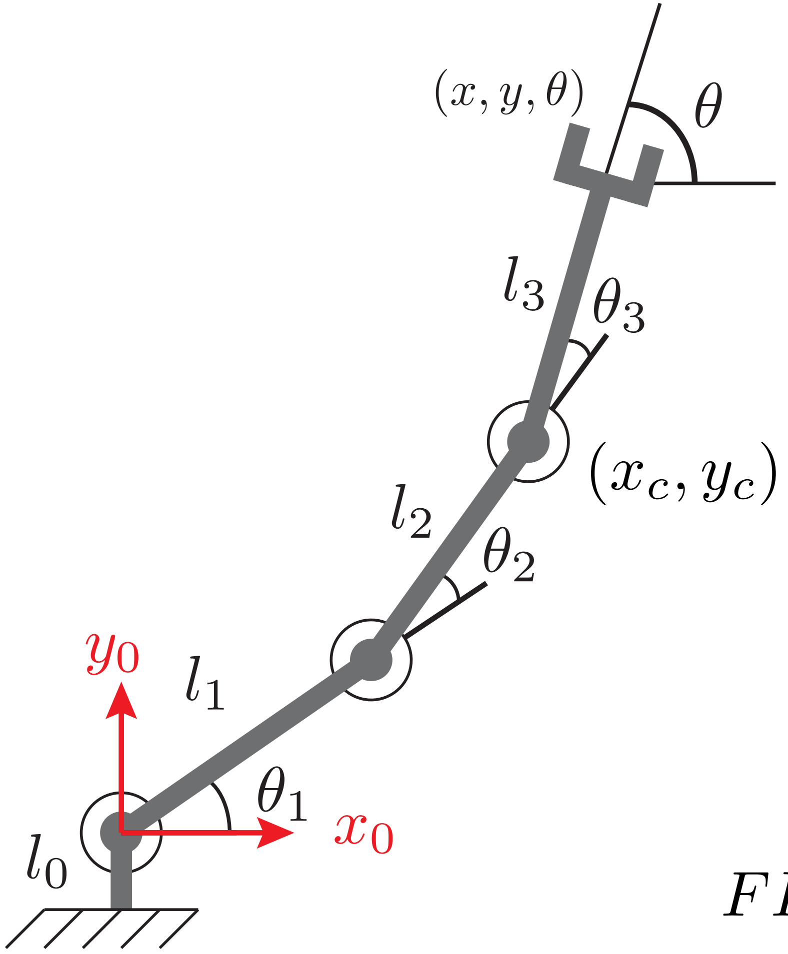
$$\boxed{\theta_1 = \operatorname{atan2}(y, x) - \operatorname{atan2}(l_2 \sin\theta_2, l_1 + l_2 \cos\theta_2)}$$

Kinematic Decoupling

- Decompose problem into two solutions:
  - Inverse Position Kinematics Solution
    - Find wrist center (x_c, y_c)
    - Find shoulder and elbow joint variables
  - Inverse Orientation Kinematics
    - find wrist joint variable



3-link RRR Arm

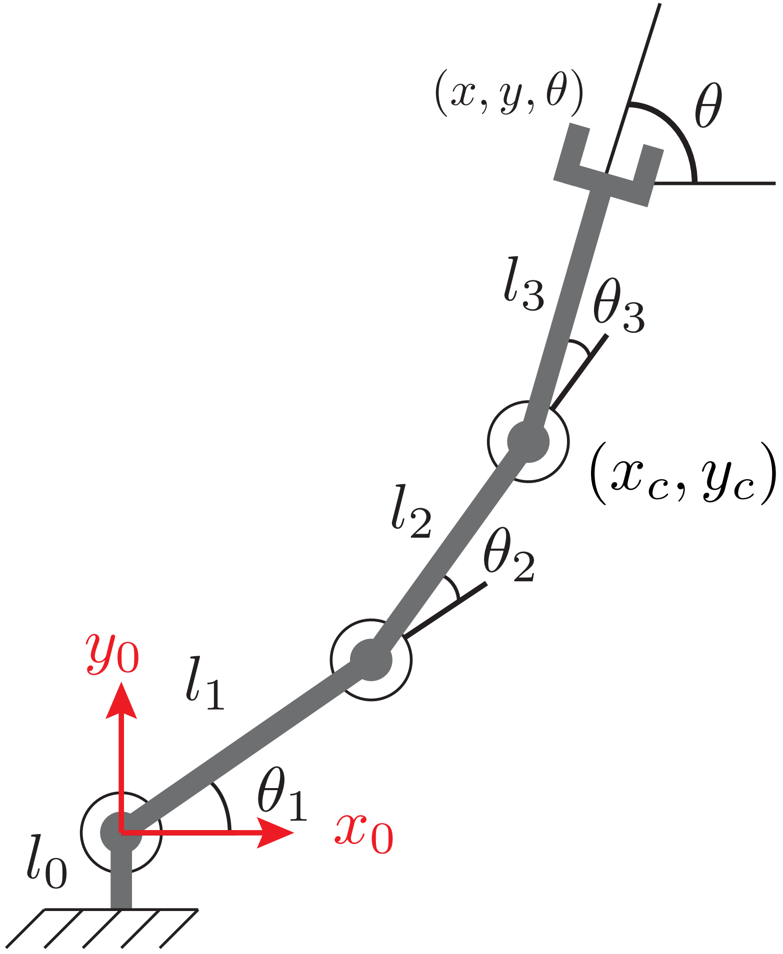


- Given (x, y, θ) can find (x_c, y_c) then use 2 link RR manipulator equations to find θ_1 and θ_2.

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - l_3 R_3^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$FK : f(q_1, q_2, q_3) = \begin{bmatrix} c_{123} & -s_{123} & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 \end{bmatrix}$$

3-link RRR Arm



- Using FK for the wrist point and desired θ find θ_3.
- By inspection $\theta_3 = \theta$ - $(\theta_1 + \theta_2)$
- Generally, we want to solve for R_3^2:

$$R = R_2^0 R_3^2$$

$$R_3^2 = (R_2^0)^{-1} R$$

$$\begin{bmatrix} c_3 & -s_3 \\ s_3 & c_3 \end{bmatrix} = \begin{bmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{bmatrix} \begin{bmatrix} c_{123} & -s_{123} \\ s_{123} & c_{123} \end{bmatrix}$$

Summary

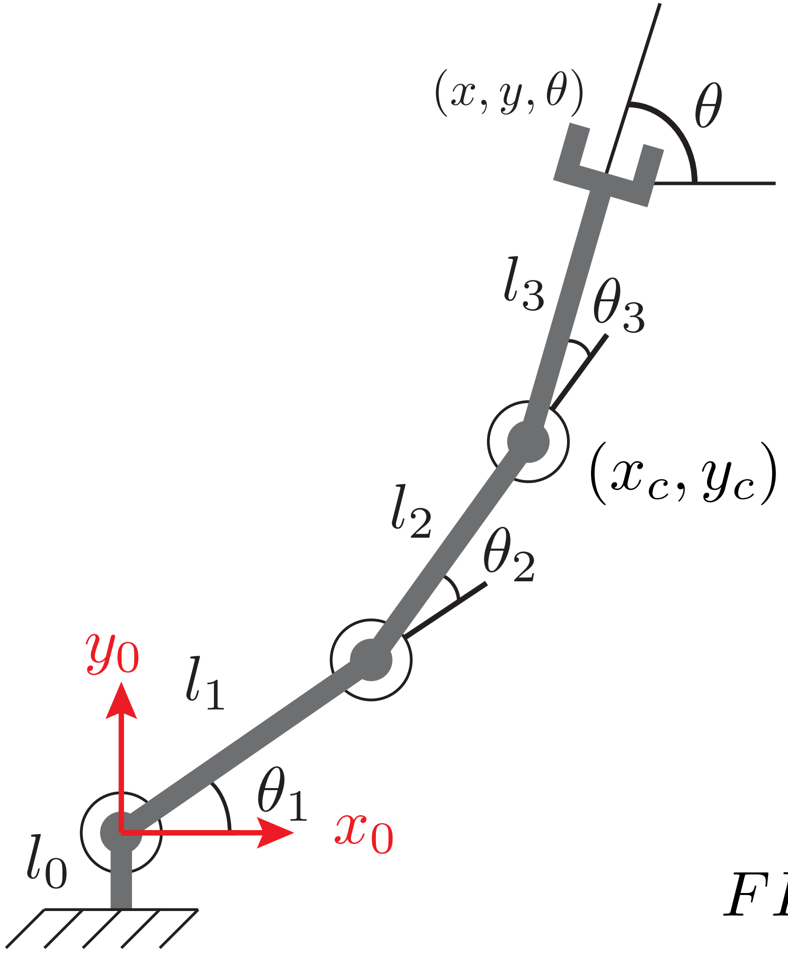
- Inverse Kinematics equations can be solved in closed form using geometry.
- Closed form solutions are convenient, fast to compute, and provide explicit knowledge of degeneracies and singularities.
- Kinematic decoupling can be used to solve for orientation as well as position so long as the kinematics of the end effector position and orientation are independent.
- We will use the planar IK solution as part of the full 3D IK solution in the next lecture.

3D Geometrical Inverse Kinematics

Winter 2023

ROBOTICS

Review: Kinematic Decoupling

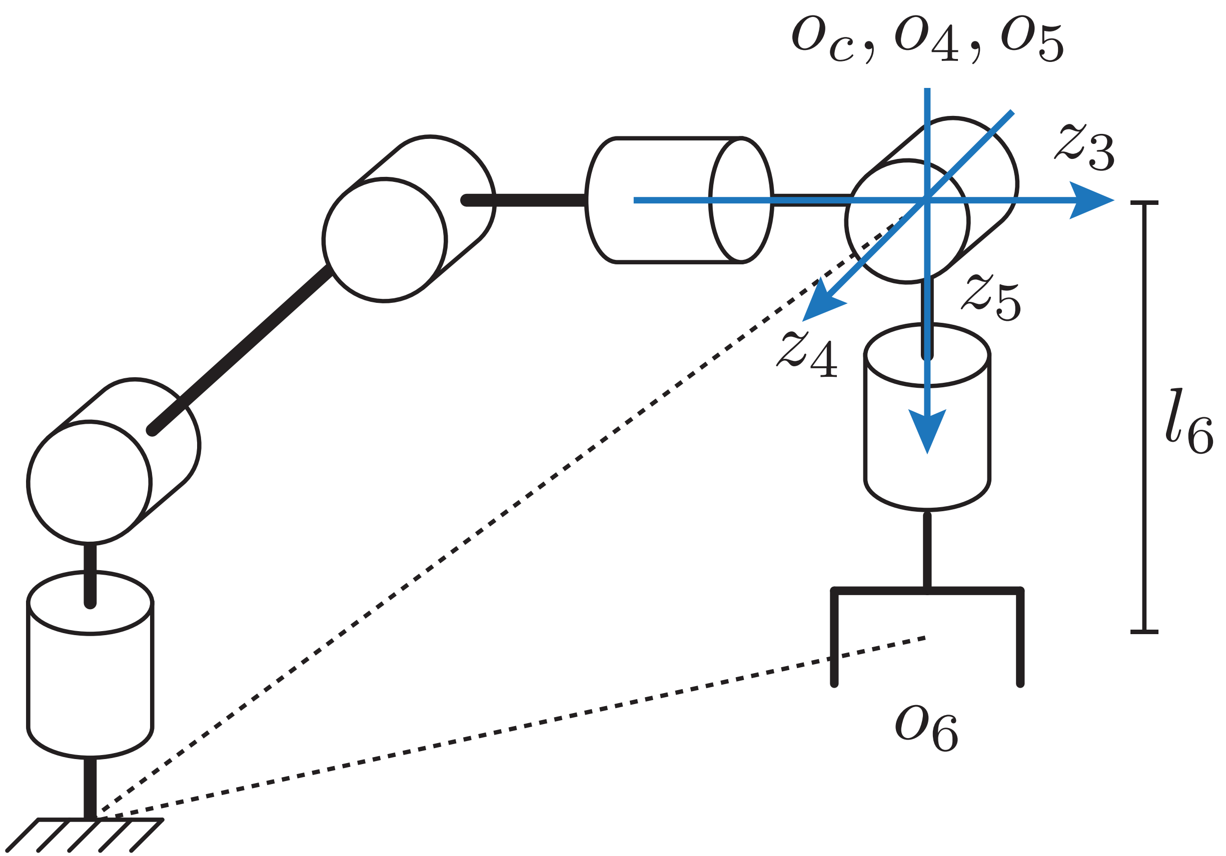


- Given (x, y, θ) can find (x_c, y_c) then use 2 link RR manipulator equations to find θ_1 and θ_2.

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - l_3 \underbrace{R_3^0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Direction of } x_3 \text{ in base frame}}$$

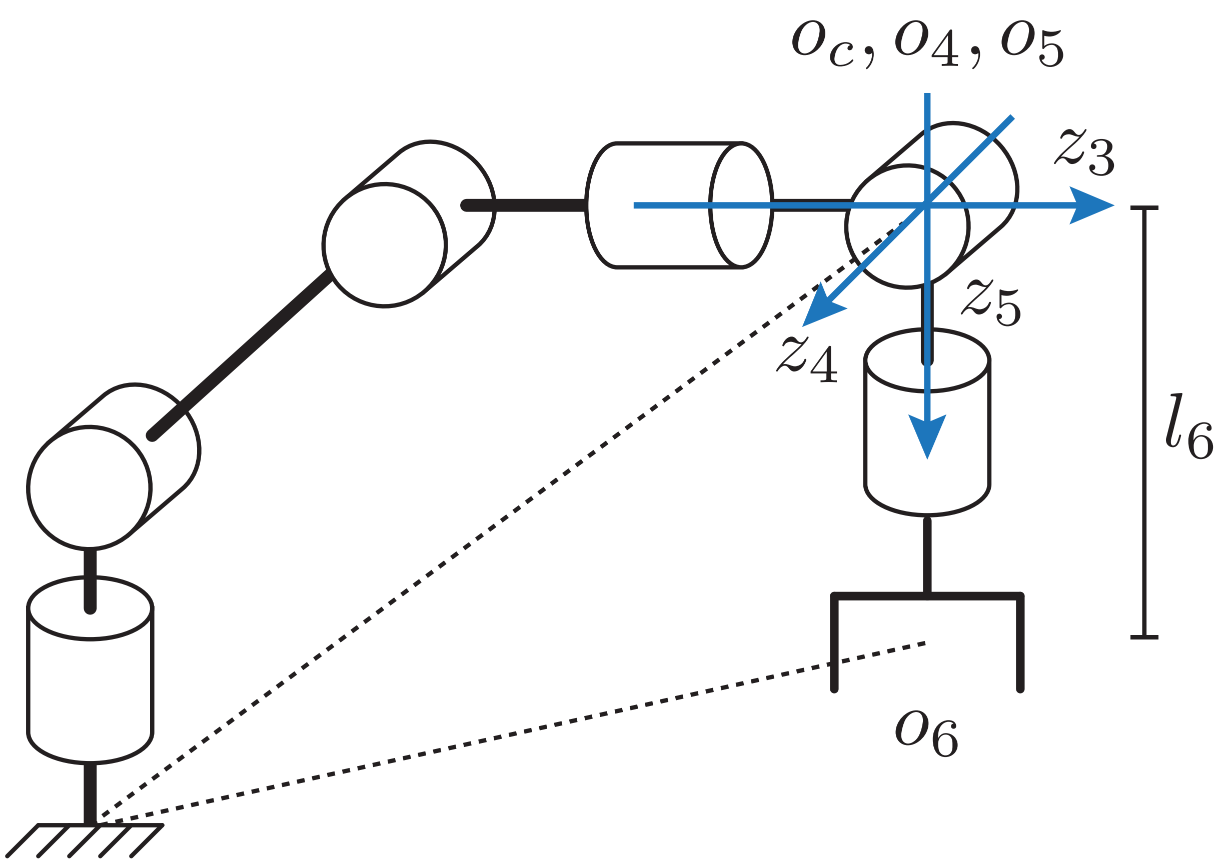
$$FK : f(q_1, q_2, q_3) = \begin{bmatrix} c_{123} & -s_{123} & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 \end{bmatrix}$$

6DOF with Spherical Wrist



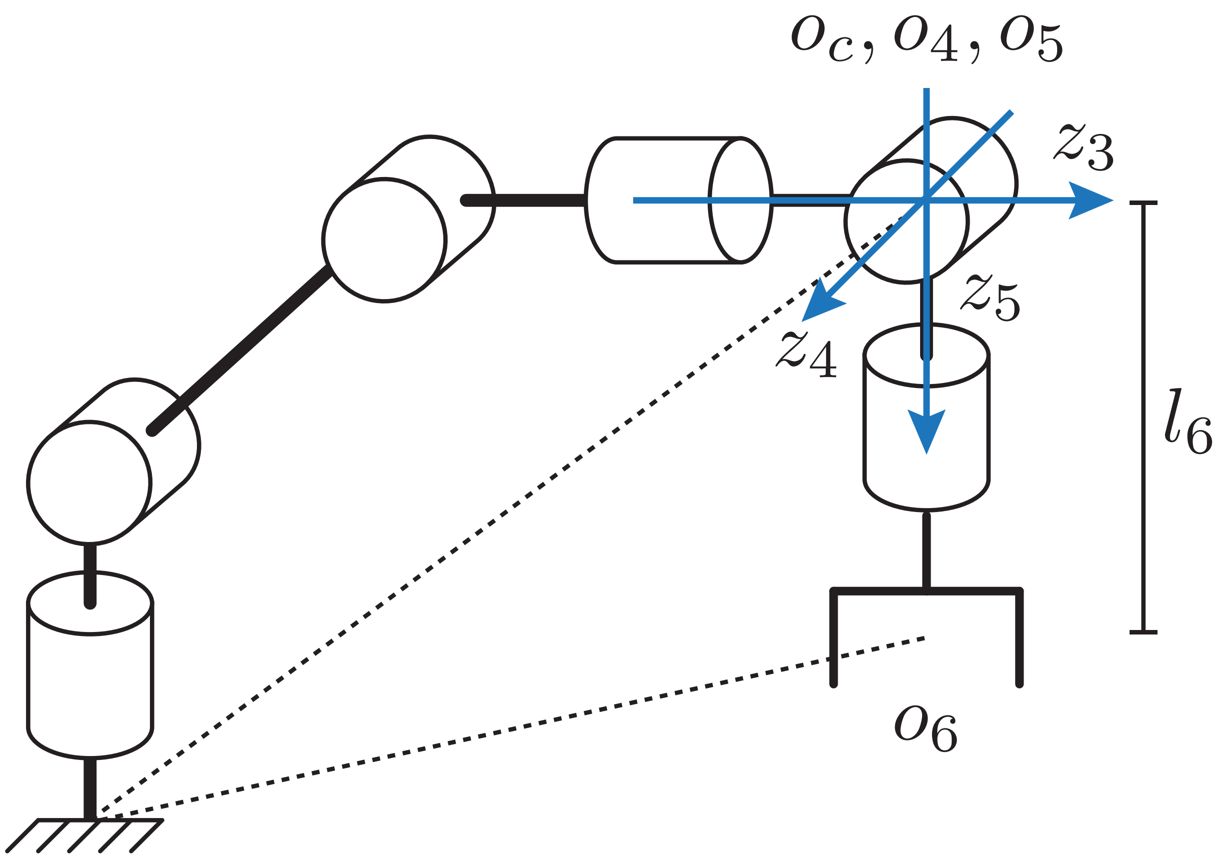
- Axes of last 3 joints intersect at point $\boldsymbol{o_c}$
- Decouple into 2 solutions:
- Inverse position at wrist center $\boldsymbol{o_c}$
- Inverse orientation of wrist center R_6^3

6DOF with Spherical Wrist



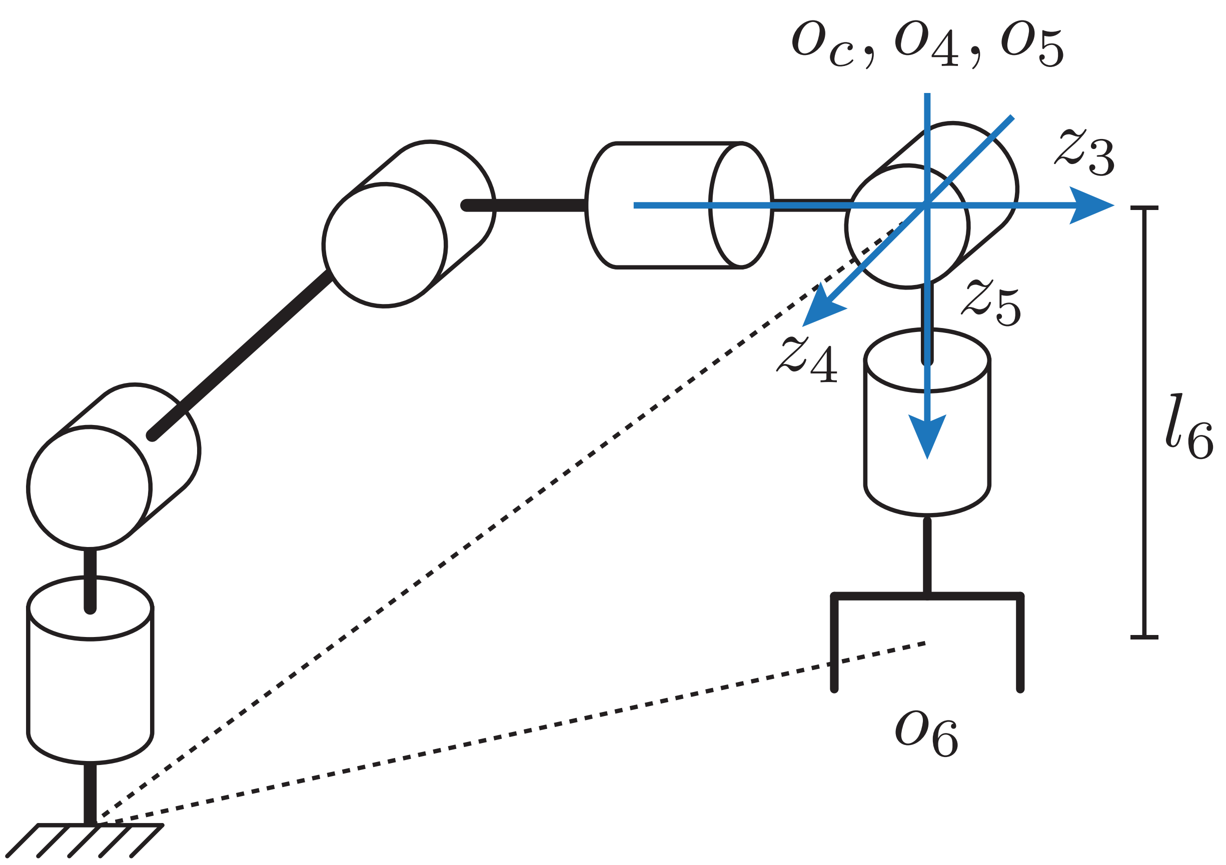
- Represent desired end effector pose as the homogeneous transform H
- Separate out R and $\boldsymbol{o}$ from H
- IK problem: given R and $\boldsymbol{o}$, find $q_1, \ldots, q_6$
- The wrist center $\boldsymbol{o}_c$ only depends on $q_1, \ldots, q_3$

Finding o_c



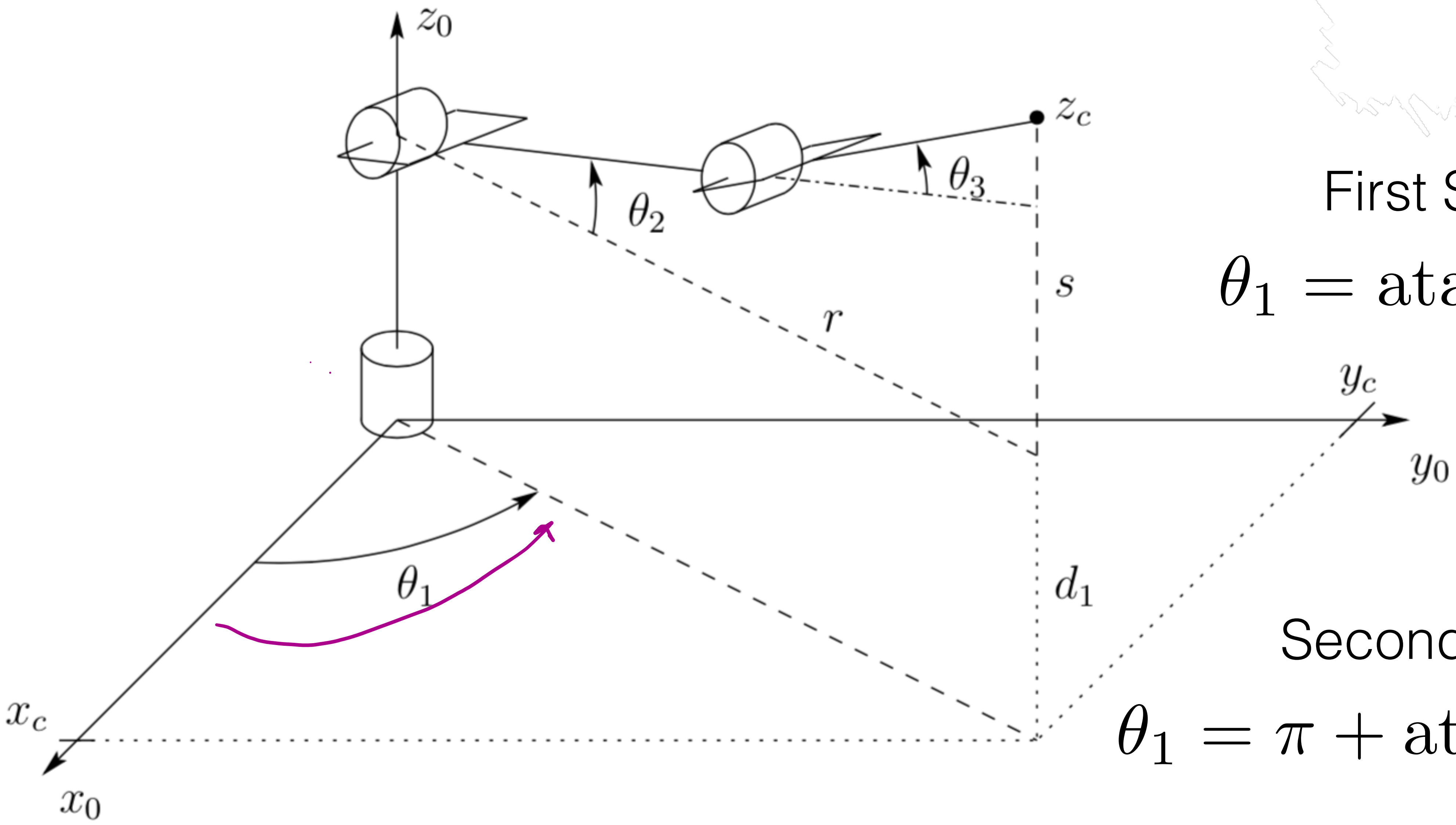
- The origin of the end effector frame is a distance l_6 along z_5 from the wrist center o_c
- $o = o_c^0 + l_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- The 3rd column of R represents the direction of z_6 in the base frame
- $o_c^0 = o - l_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Finding the other joints



- $o_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - l_6 r_{13} \\ o_y - l_6 r_{23} \\ o_z - l_6 r_{33} \end{bmatrix}$
- After finding o_c, we can use 2D planar IK solution to find other joints

Base Joint - θ_1



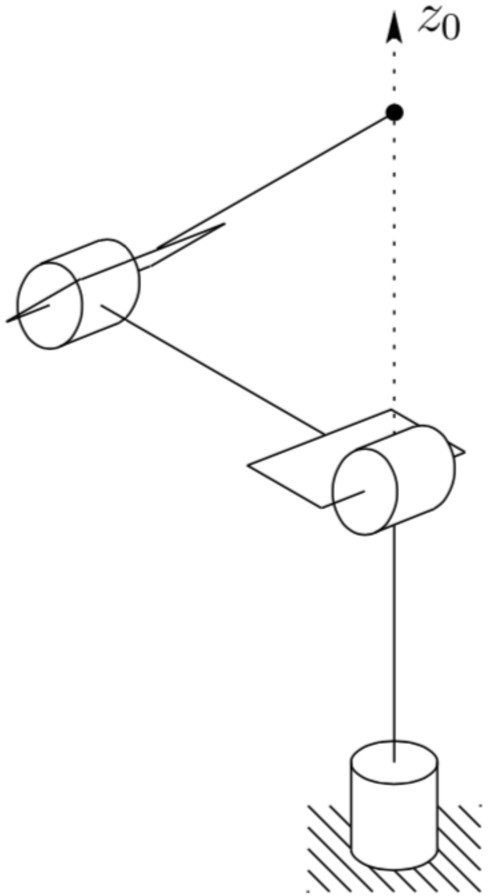
First Solution:

$$\theta_1 = \text{atan2}(y_c, x_c)$$

Second Solution:

$$\theta_1 = \pi + \text{atan2}(y_c, x_c)$$

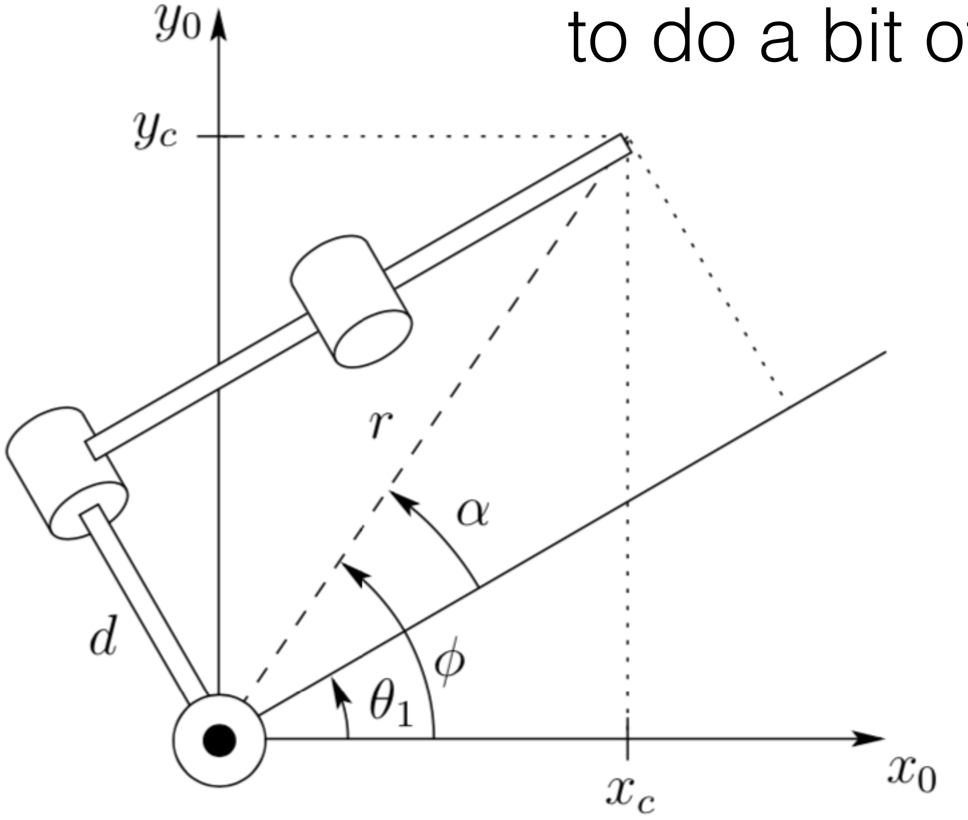
Singular Configuration

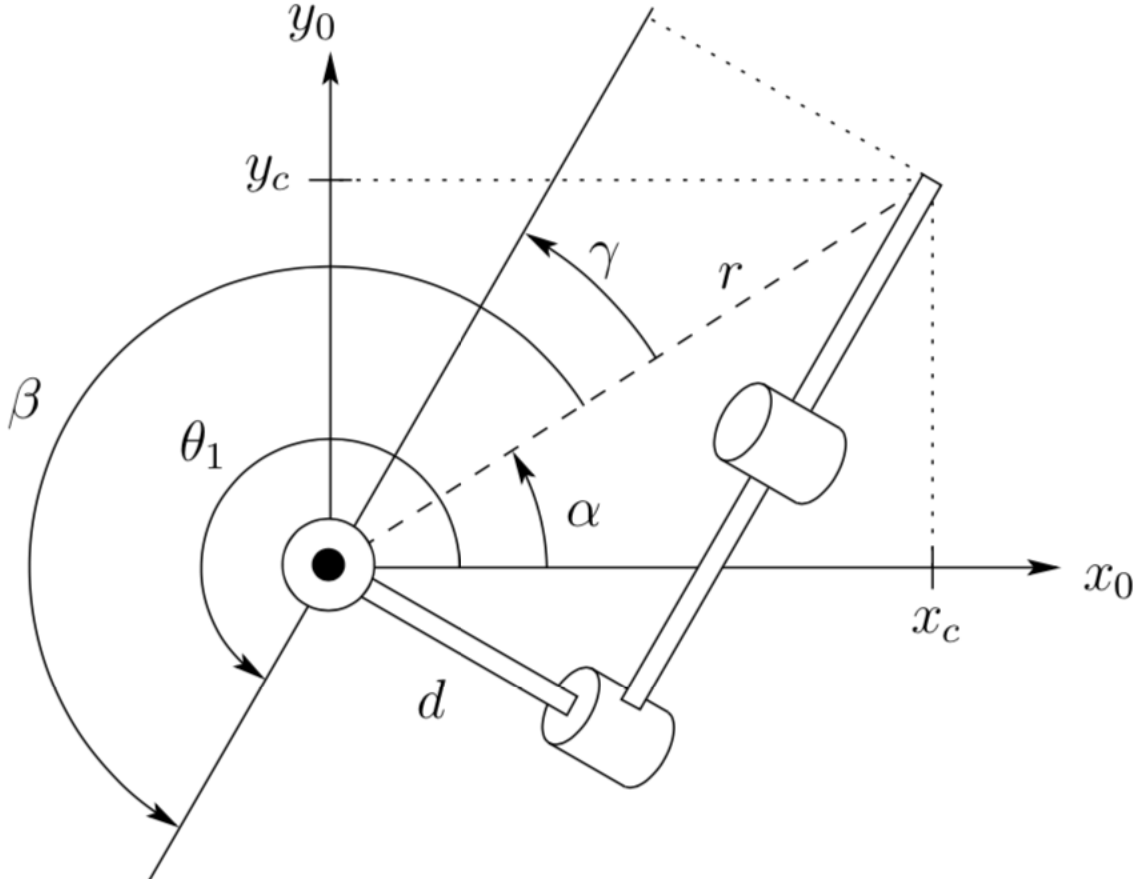


- Solutions for θ_1 are valid unless $x_c = y_c = 0$
- Infinite solutions exist…

Offset

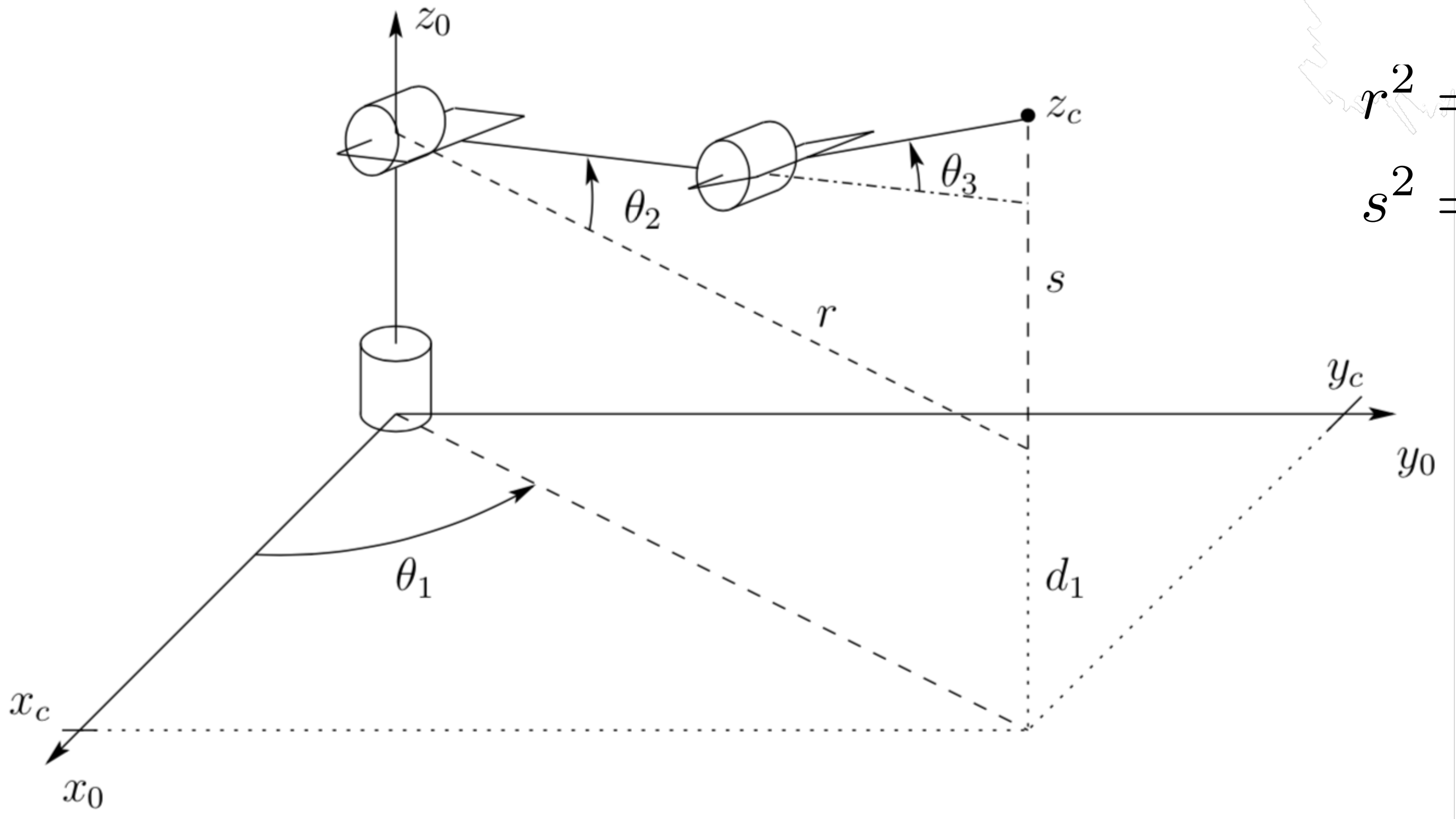
- If there is a link offset, you have to do a bit of geometry





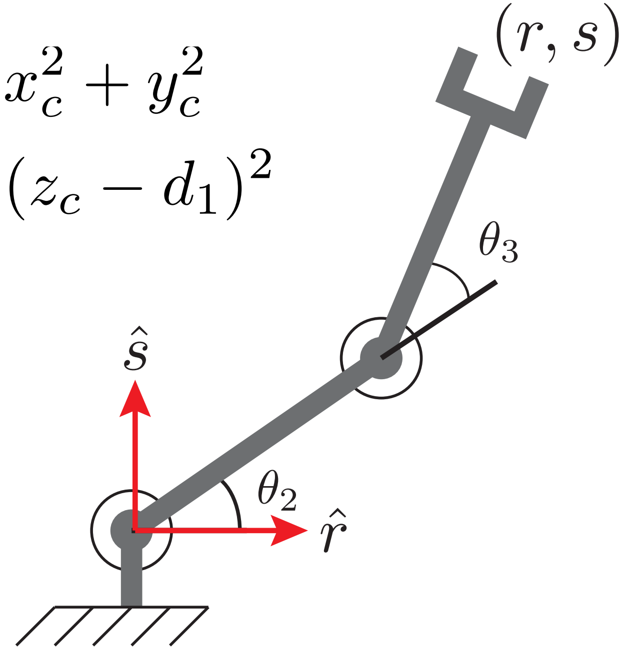
Wrist & Elbow - θ_2, θ_3

- Same result as the Planar RR manipulator we covered previously…



$$r^2 = x_c^2 + y_c^2$$

$$s^2 = (z_c - d_1)^2$$



Inverse Orientation Kinematics

- Now that we know $\theta_1, \theta_2, \theta_3$, can find orientation of the wrist R_3^0 using forward kinematics
- Then find expression for remaining joints angles

$$R_3^0 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$

$$R = R_3^0 R_6^3$$

known known unknown

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

Determining Angles from Rotation Matrix

- For ZYZ Euler Angles (ϕ, θ, ψ)

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

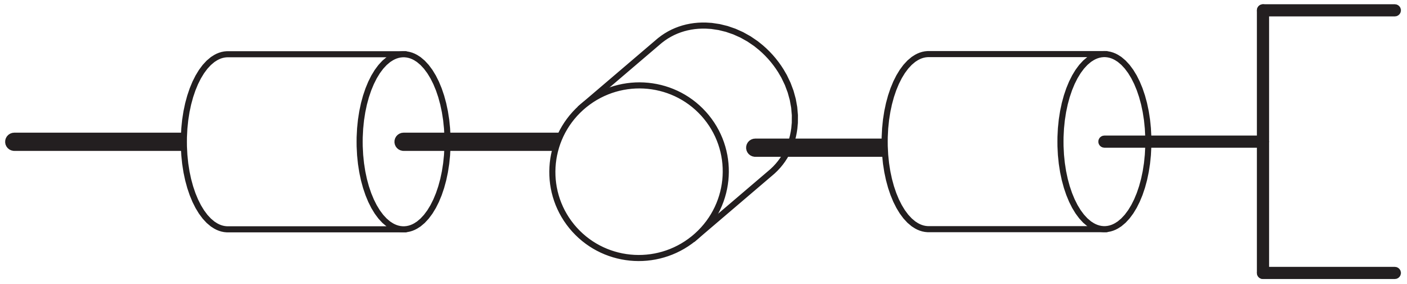
- Case 1: r_{13} and/or r_{23} is non-zero, so $r_{33} \neq \pm 1$
- $c_\theta = r_{33}$, $s_\theta = \pm\sqrt{1 - r_{33}^2}$ so $\theta =$ atan2$(\pm\sqrt{1 - r_{33}^2}, r_{33})$
- If $s_\theta > 0$
  - $\phi =$ atan2(r_{23}, r_{13})
  - $\psi =$ atan2$(r_{32}, -r_{31})$
- If $s_\theta < 0$
  - $\phi =$ atan2$(-r_{23}, -r_{13})$
  - $\psi =$ atan2$(-r_{23}, r_{31})$

Determining Angles from Rotation Matrix

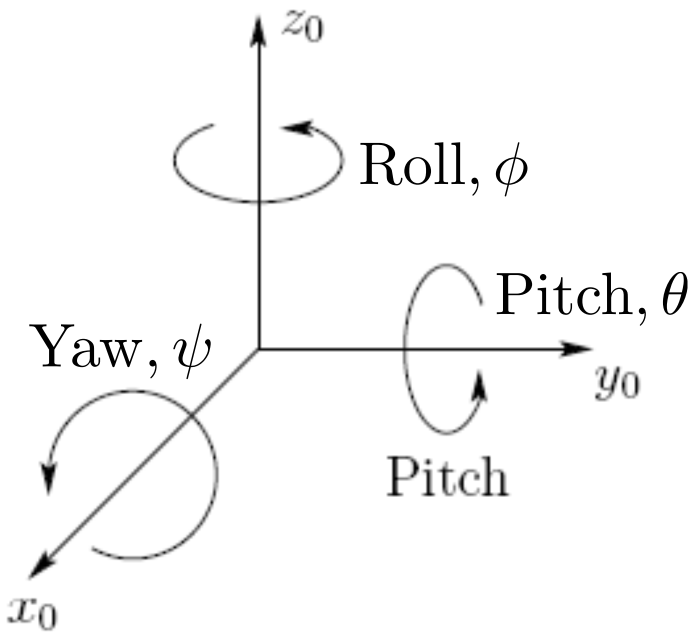
- For ZYZ Euler Angles (ϕ, θ, ψ)

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- Case 2: $r_{13} = r_{23} = 0$, $r_{33} = \pm 1$ so $s_\theta = 0$
- $\phi + \psi = \operatorname{atan2}(r_{21}, r_{11})$ therefore infinitely many solutions!

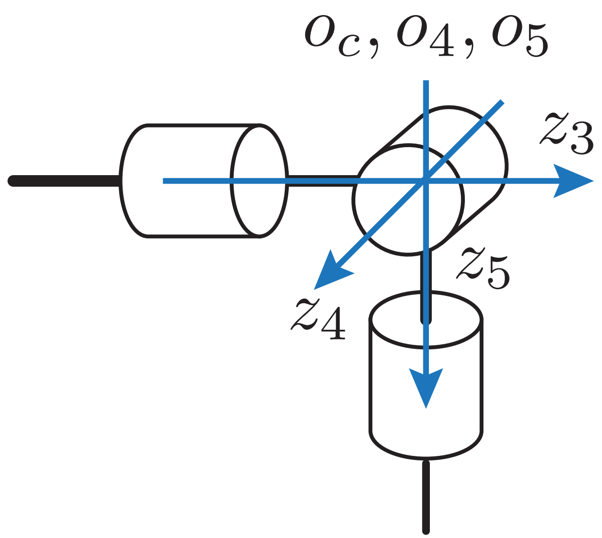


Roll, Pitch, Yaw



$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

Wrist Joints - $\theta_4, \theta_5, \theta_6$



- Kinematics are equivalent to ZYZ Euler angles
- Will be different if using other Euler angles

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

$$\theta_4 = \text{atan2}(r_{23}, r_{13})$$

$$\theta_5 = \text{atan2}(\pm\sqrt{1 - r_{33}^2}, r_{33})$$

$$\theta_6 = \text{atan2}(r_{32}, -r_{31})$$