Forward Kinematics

Lecture 4

Fall 2022

M | ROBOTICS

Forward Kinematics

If we know our state in configuration space…

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix}$$

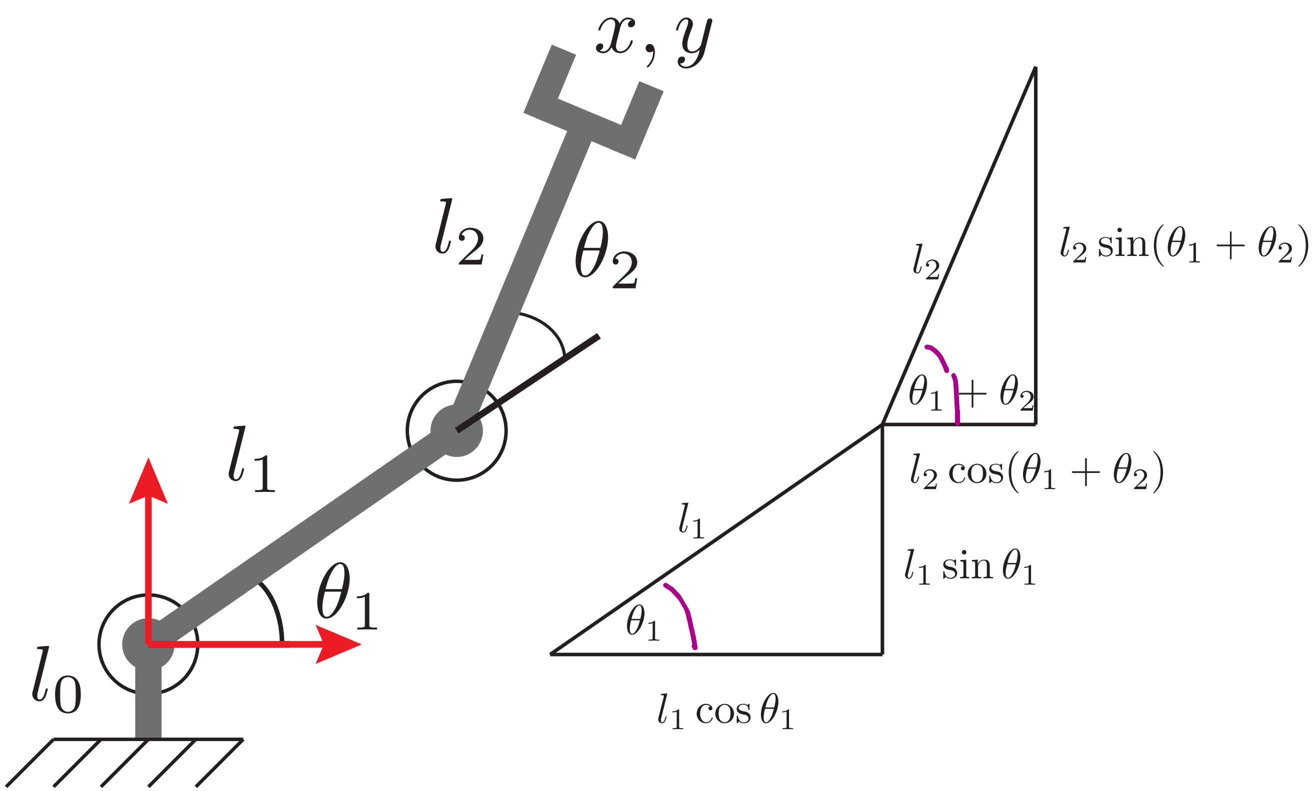
What is our end effector position in the workspace?

$$\mathbf{x} = \begin{bmatrix} x & y & z & \theta & \psi & \phi \end{bmatrix}$$

Forward kinematic map is a vector function:

$$\mathbf{x} = f(\mathbf{q})$$

Planar Arm - Geometry



$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Pose:

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix}$$

$$c_i = \cos \theta_i$$
$$c_{ij} = \cos(\theta_i + \theta_j)$$

Joint variables: $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

End effector pose: $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$

Forward Kinematics Equation

Compose kinematics from rigid body transformations

Pose of the end effector in the base frame:

$$H = A_1(q_1)...A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

Review Homogeneous Transformations

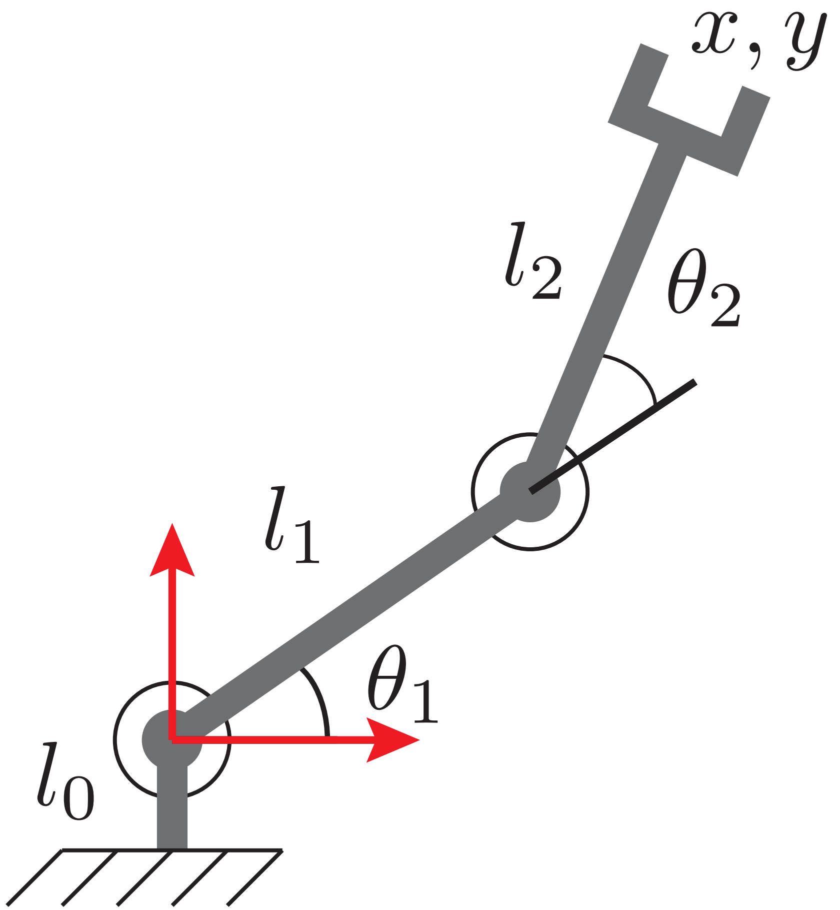
- Rigid body motion can be represented as a homogeneous transformation.
- 3DOF in 2D
- Homogeneous transformation matrix is a product of homogeneous rotation and translation matrices.

$$
\begin{array}{ccc} & \text{Rotation} & \text{Translation} \\ H = & \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \end{array}
$$

$$
H = \begin{bmatrix} R & & d \\ 0 & 0 & 1 \end{bmatrix}; R \in SO(2), d \in \mathbb{R}^2
$$

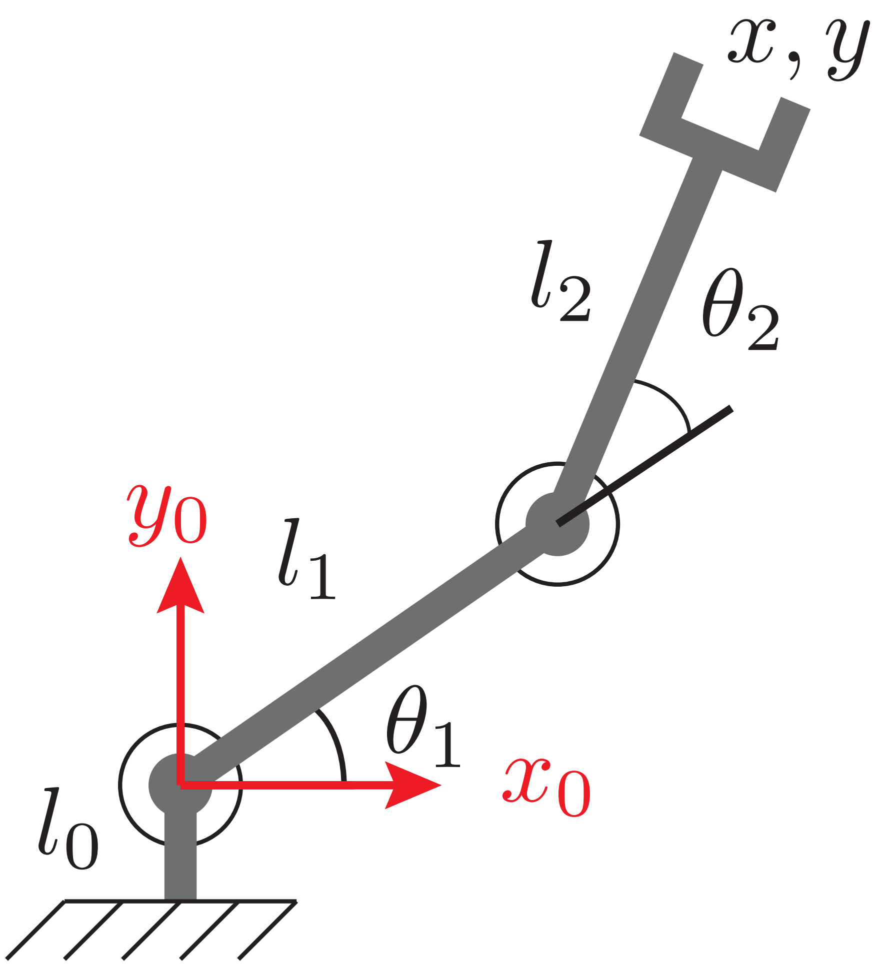
Serial Chain Conventions

- n links, $n - 1$ joints
- Each joint is single DOF
- Joints numbered 1 to n
- Links numbered 0 to n
- Joint i connects link $i - 1$ to link i
- Joint i moves link i



Assigning Frames

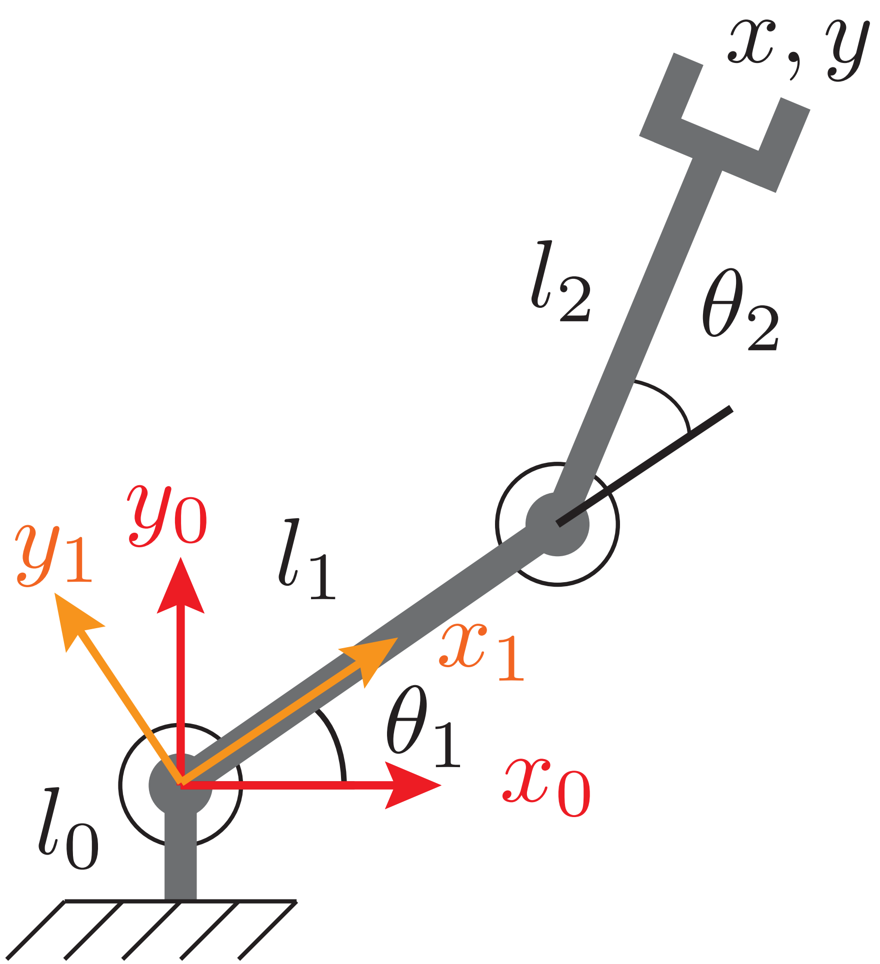
- Assign the base frame conveniently on the axis of joint 1



Assigning Frames

- Assign frame 1 attached to link 1

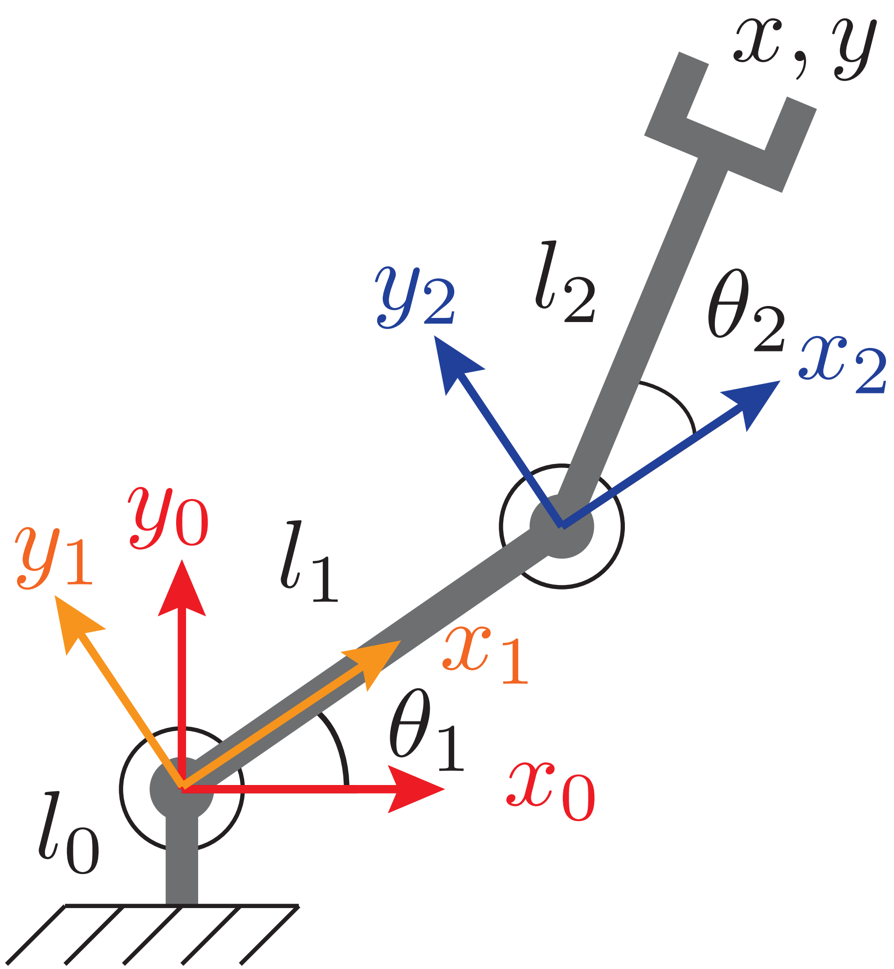
$$A_1 = H_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Assigning Frames

- Assign frame 2 at the end of link 1

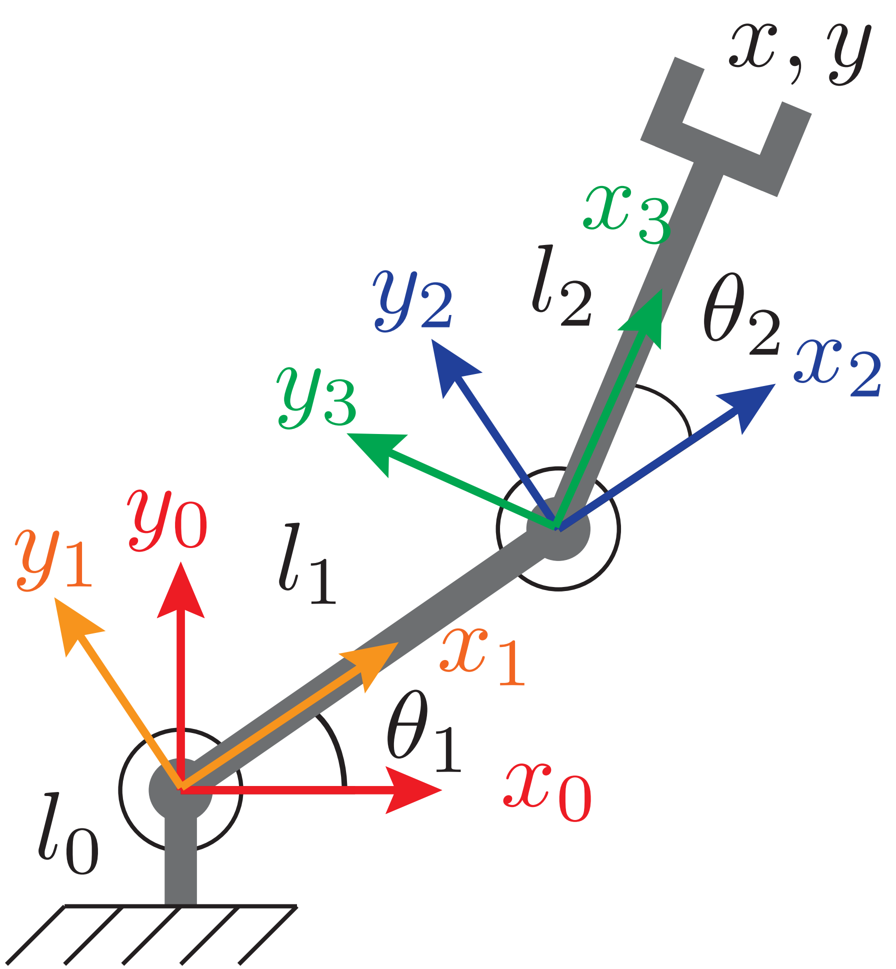
$$A_2 = H_2^1 = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Assigning Frames

- Assign frame 3 attached to link 2

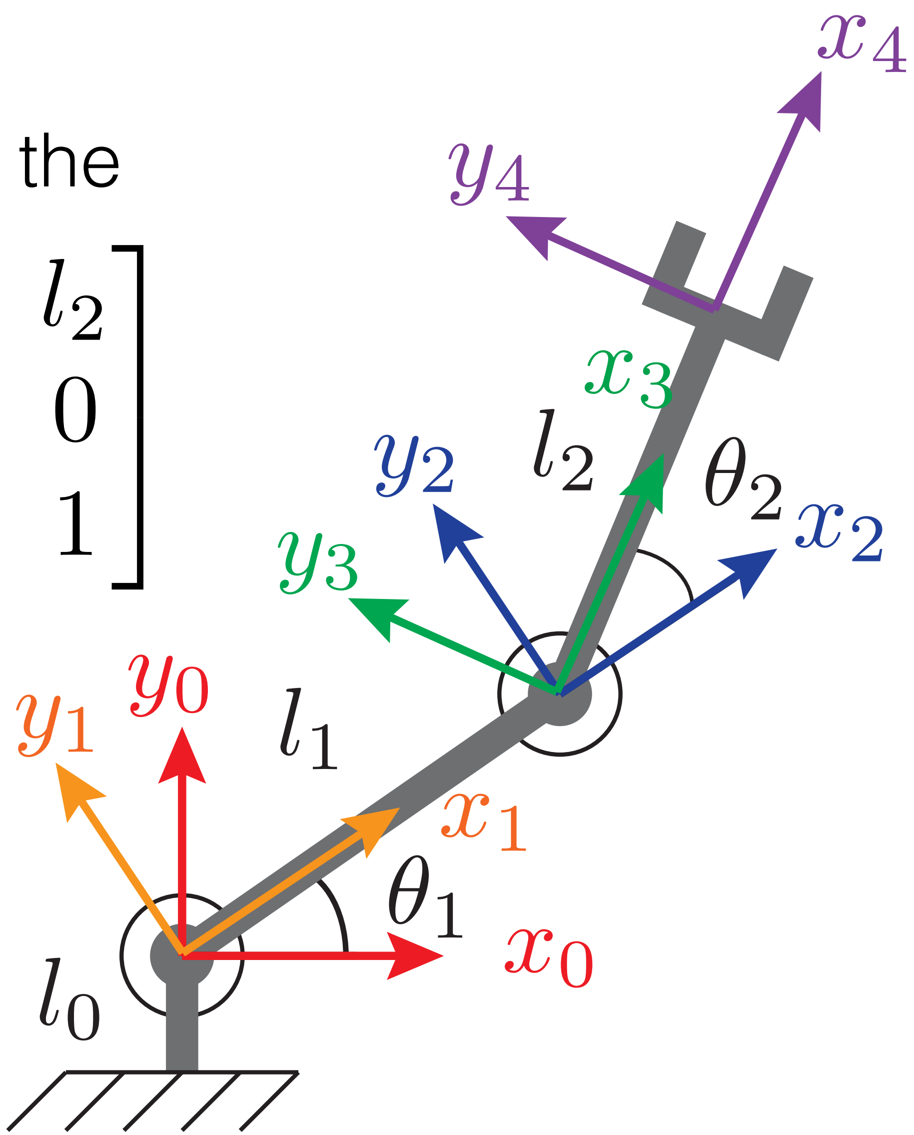
$$A_3 = H_3^2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Assigning Frames

- Finally place frame 4 at the end effector

$$A_4 = H_4^3 = \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Forward Kinematics Equations

$$H_4^0 = H_1^0 H_2^1 H_3^2 H_4^3$$

$$H_1^0 H_2^1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & l_1 c_1 \\ s_1 & c_1 & l_1 s_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 H_4^3 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & l_2 c_2 \\ s_2 & c_2 & l_2 s_2 \\ 0 & 0 & 1 \end{bmatrix}$$

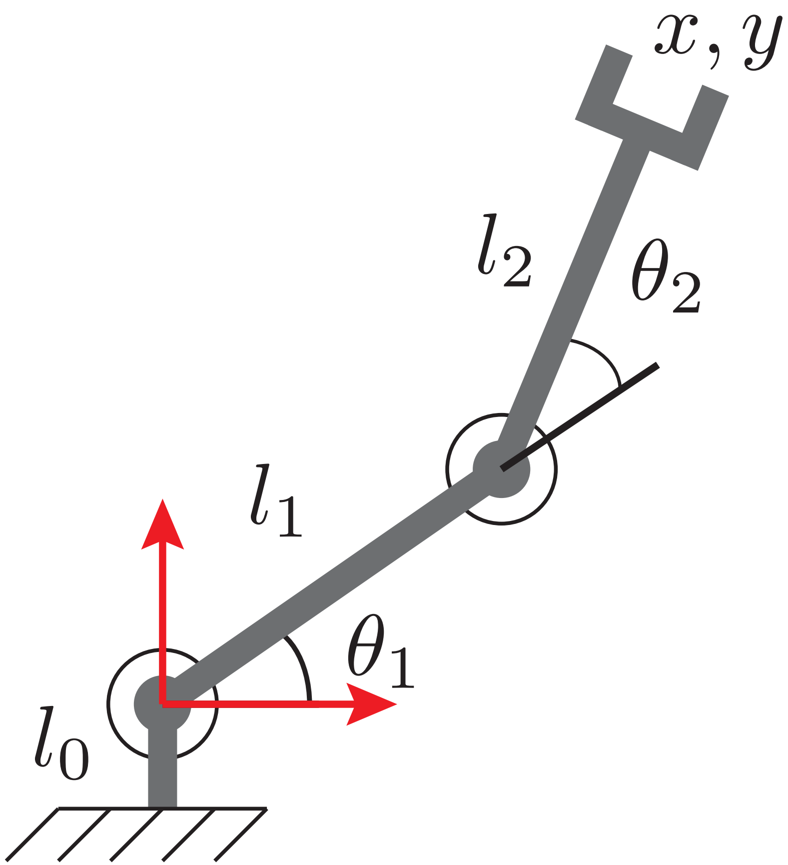
$$H_4^0 = \begin{bmatrix} c_1 & -s_1 & l_1 c_1 \\ s_1 & c_1 & l_1 s_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & l_2 c_2 \\ s_2 & c_2 & l_2 s_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{12} & -s_{12} & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & l_1 s_1 + l_2 c_{12} \\ 0 & 0 & 1 \end{bmatrix}$$

sum and difference formulas for sine and cosine

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \ \cos\beta + \cos\alpha \ \sin\beta \\ \cos(\alpha + \beta) &= \cos\alpha \ \cos\beta - \sin\alpha \ \sin\beta \\ \sin(\alpha - \beta) &= \sin\alpha \ \cos\beta - \cos\alpha \ \sin\beta \\ \cos(\alpha - \beta) &= \cos\alpha \ \cos\beta + \sin\alpha \ \sin\beta \end{aligned}}$$

Planar Arm Kinematics



Geometrical Result:

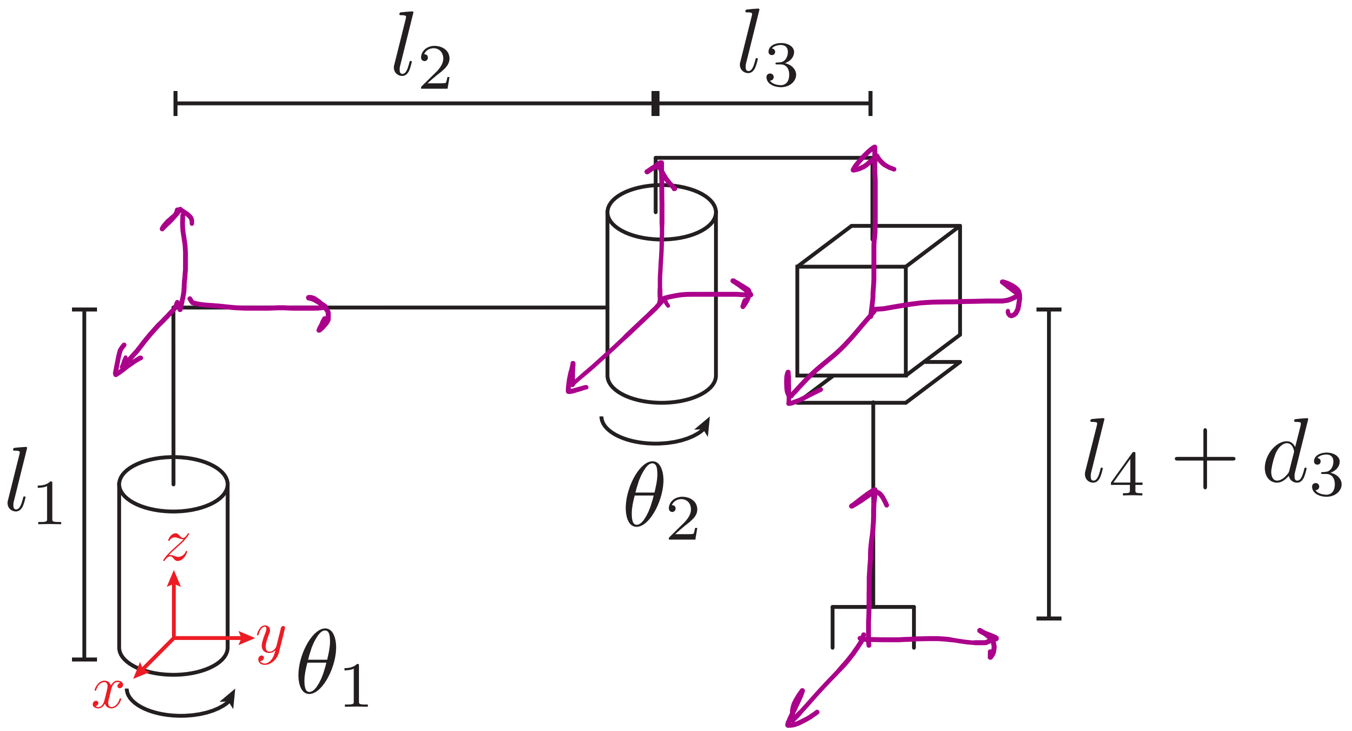
$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix}$$

Matrix Result:

$$H_4^0 = \begin{bmatrix} c_{12} & -s_{12} & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & l_1 s_1 + l_2 c_{12} \\ 0 & 0 & 1 \end{bmatrix}$$

Kinematics in 3D

$$H = R_{z,\theta_1} T_{z,l_1} T_{y,l_2} R_{z,\theta_2} T_{y,l_3} T_{z,-(l_4+d_3)}$$



Denavit-Hartenberg Convention

- Arbitrary transformations require 6 parameters
- Simplify to 4 parameters by careful choice of frames
- Standard in robotics for several decades
- More modern approaches exist

DH Parameters

$$A_i = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}$$

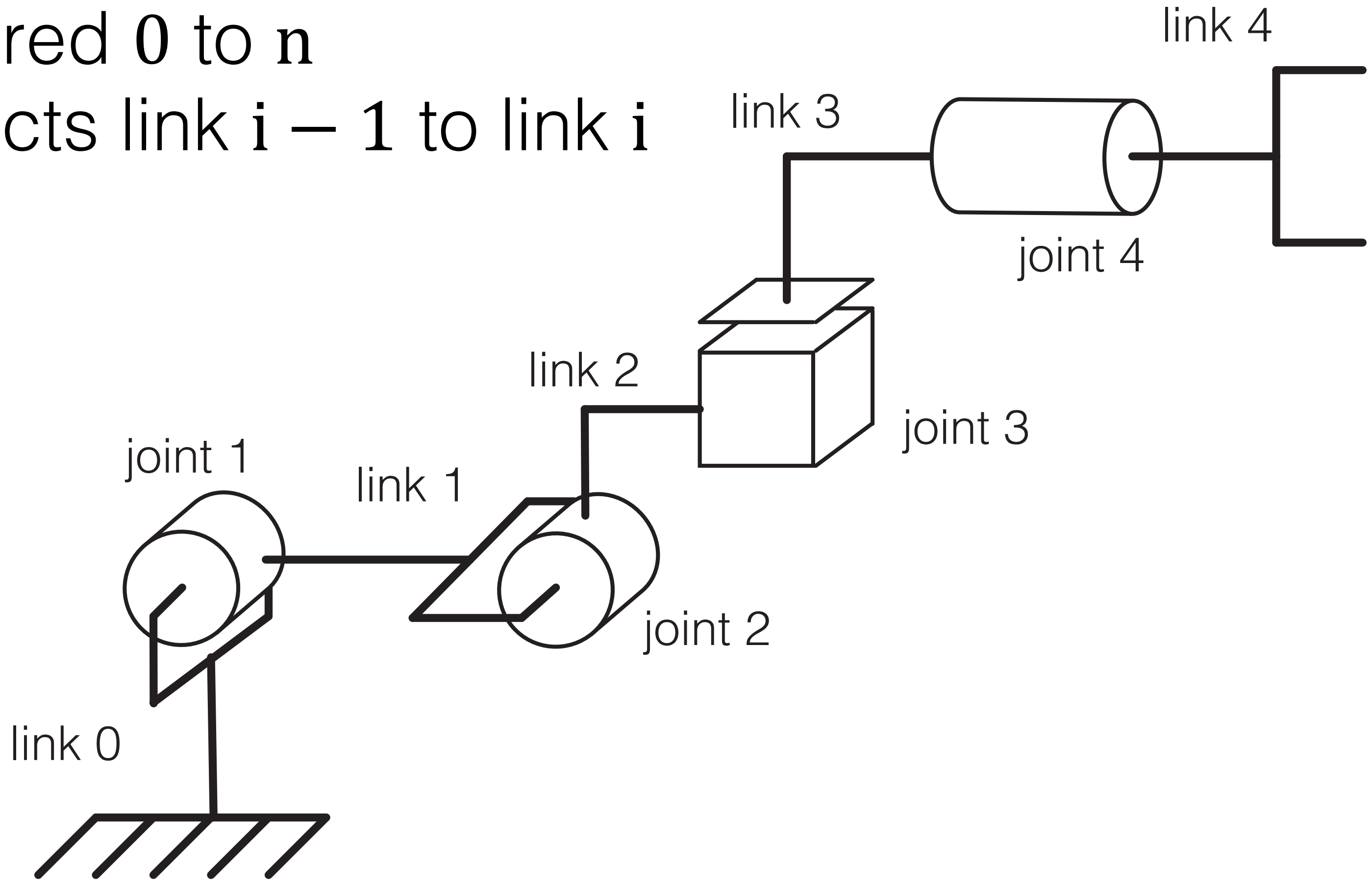
$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH Parameters: $\begin{bmatrix} \theta_i & d_i & a_i & \alpha_i \end{bmatrix}$

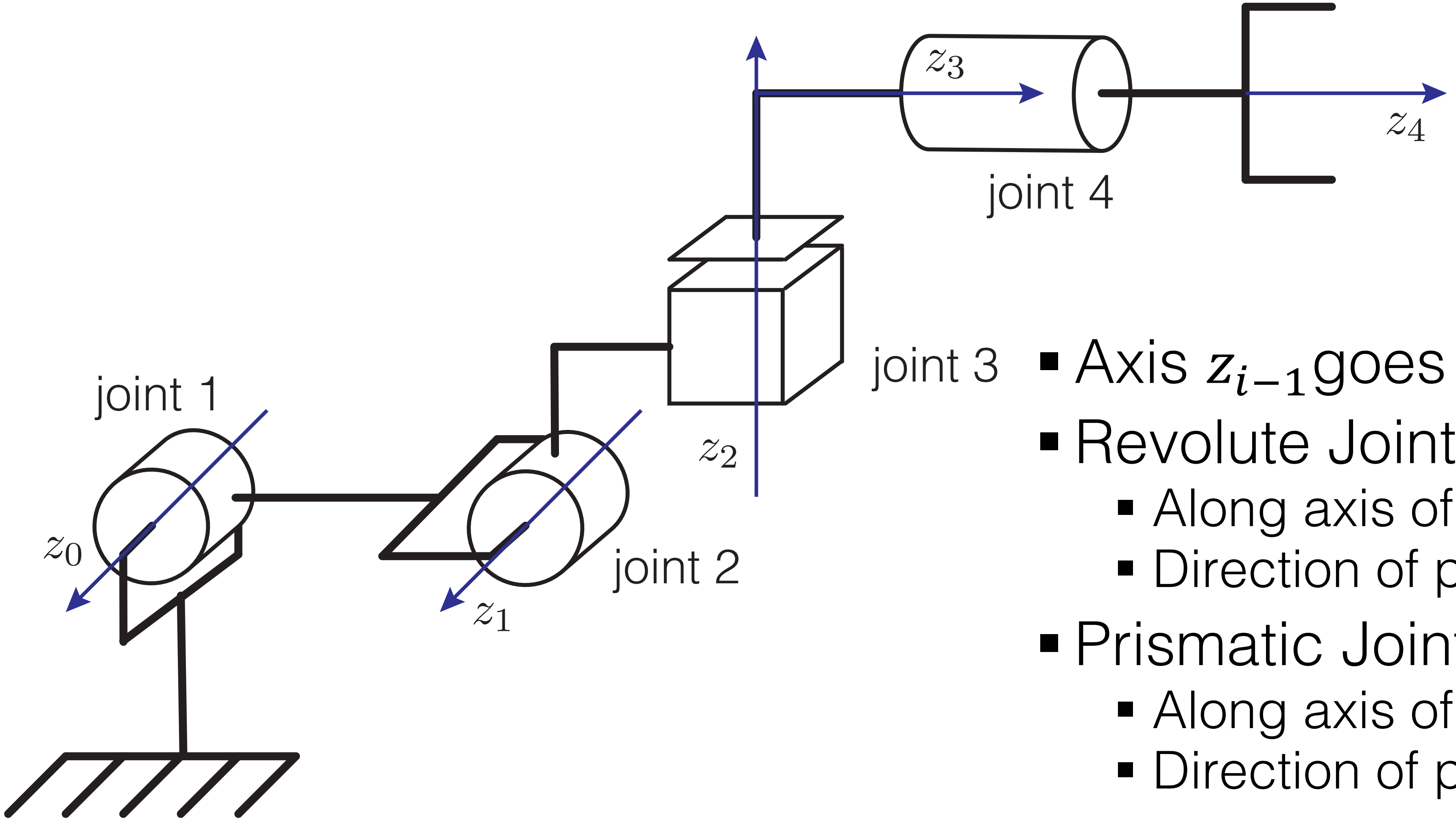
Joint Angle	Joint Offset	Link Length	Link Twist

DH – Identify Links & Joints

- Joints numbered 1 to n
- Links numbered 0 to n
- Joint i connects link $i-1$ to link i

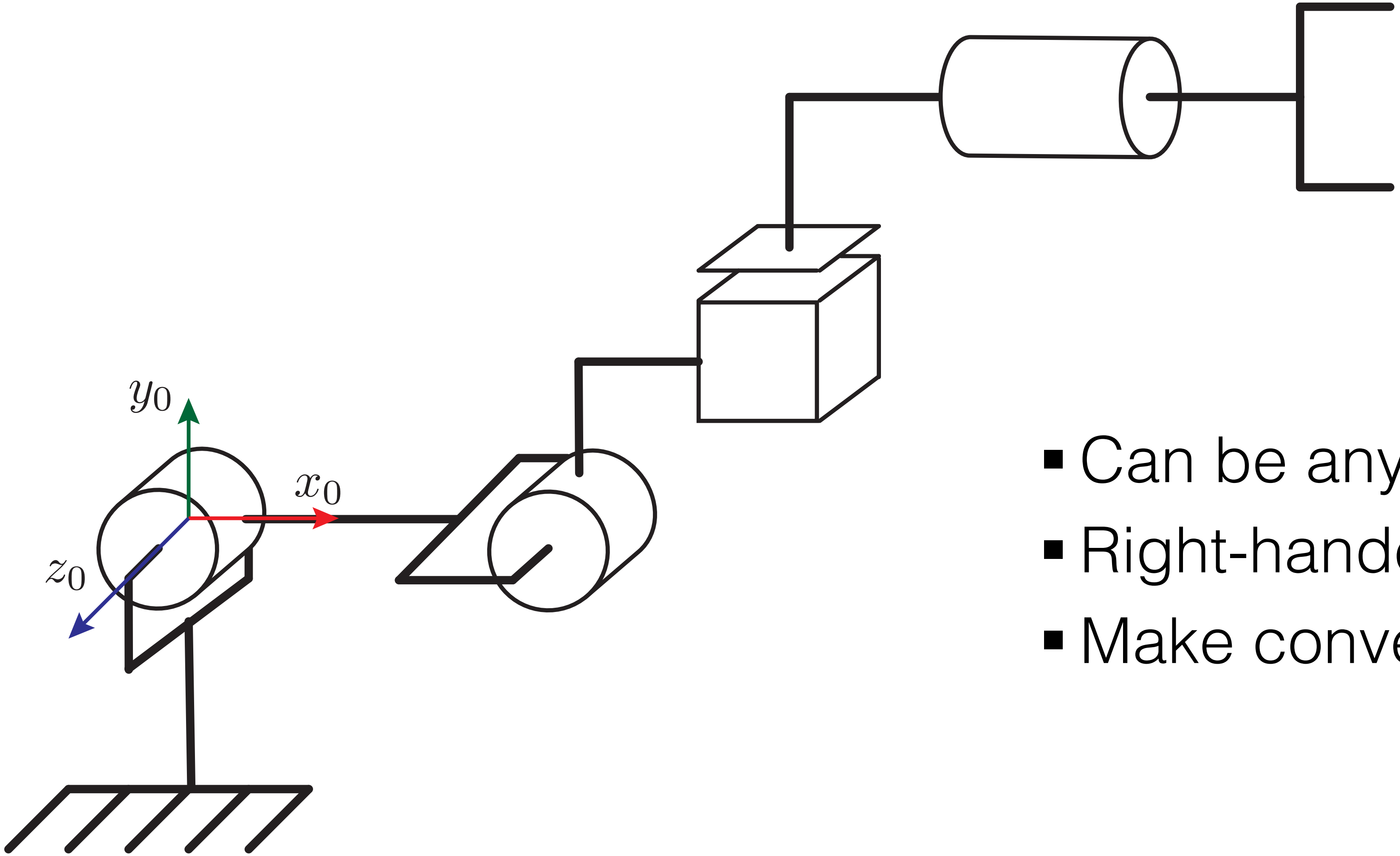


DH - Find z axes



- Axis z_{i-1} goes through joint i
- Revolute Joint
  - Along axis of rotation
  - Direction of positive rotation
- Prismatic Joint
  - Along axis of displacement
  - Direction of positive displacement

DH – Assign Base Frame



- Can be anywhere on axis z_0
- Right-handed
- Make convenient choice

DH - Assign Other Frames

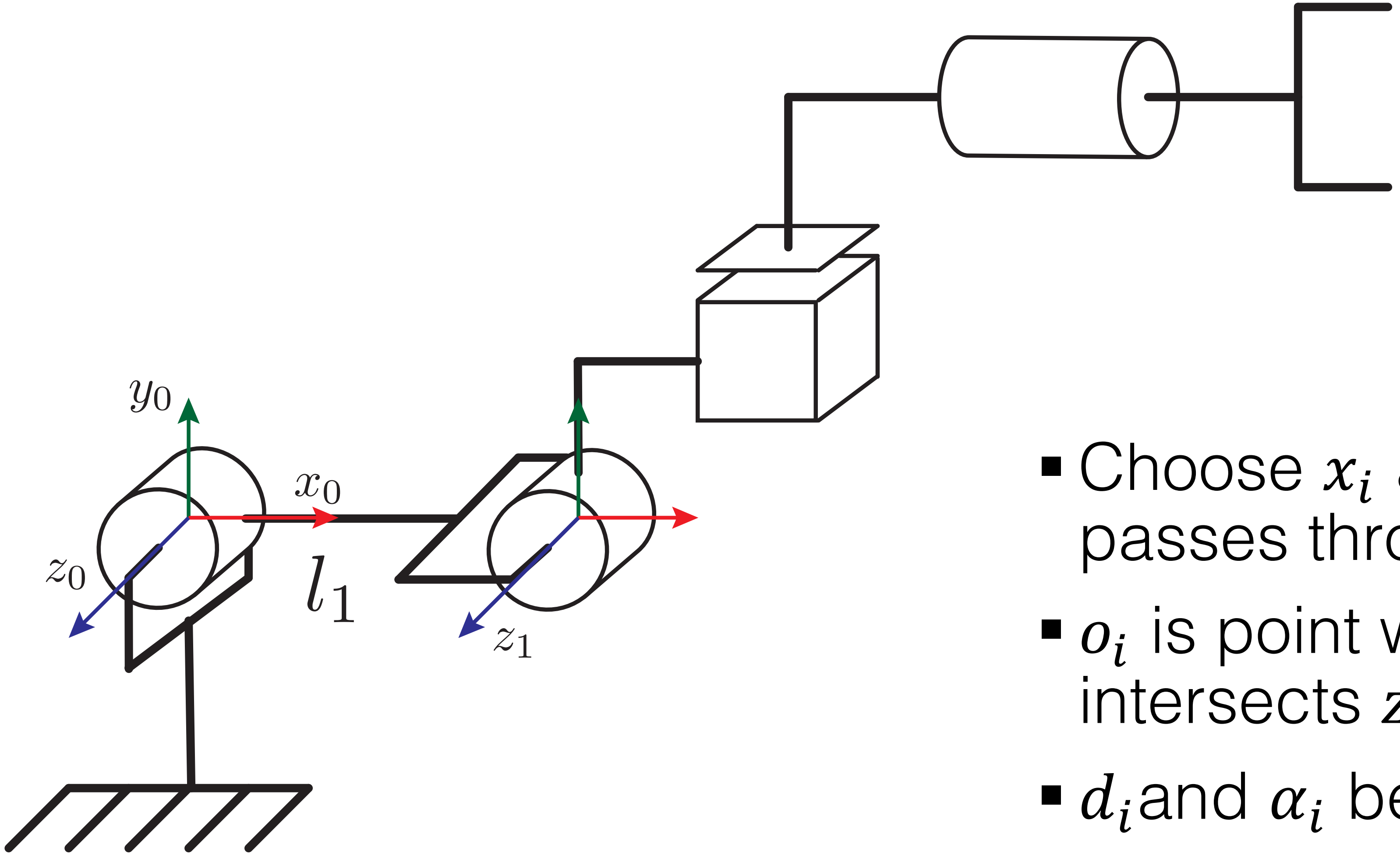
Rules:

- DH1: Axis x_i perpendicular to Axis z_{i-1}
- DH2: Axis x_i intersects Axis z_{i-1}

3 Cases:

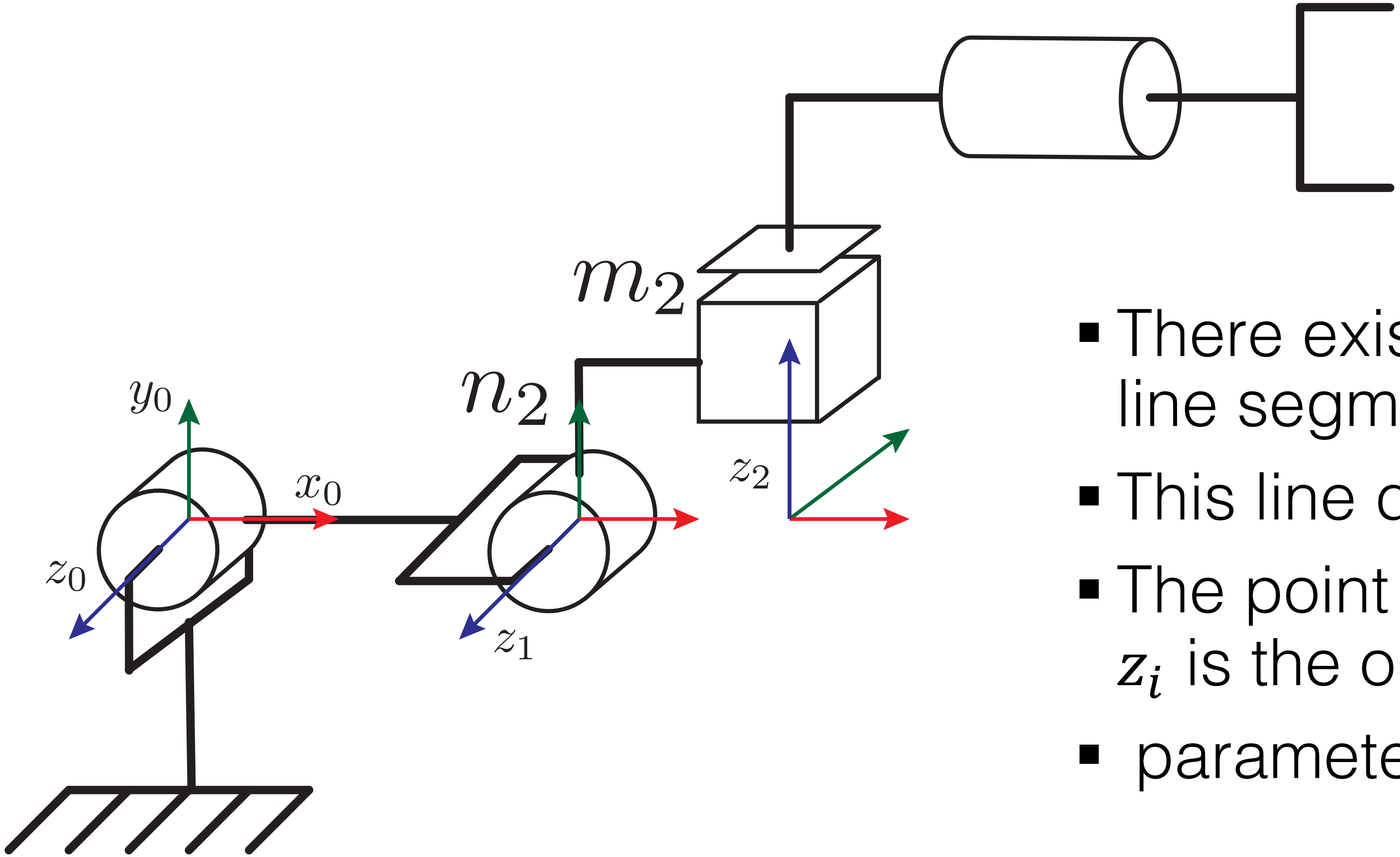
- Axis z_i and z_{i-1} parallel
- Axis z_i and z_{i-1} not co-planar
- Axis z_i and z_{i-1} intersect

Axis z_i and z_{i-1} parallel



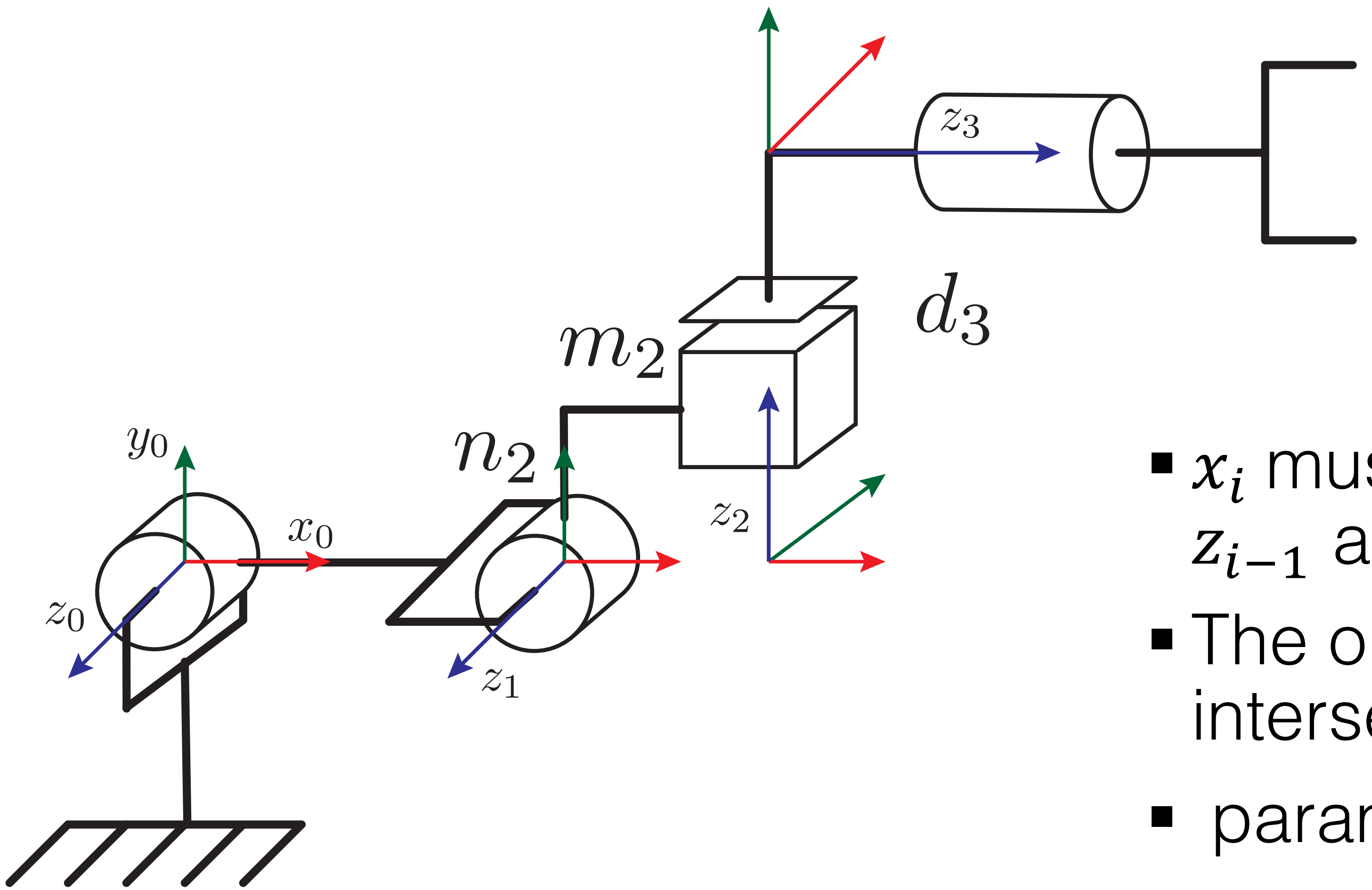
- Choose x_i along the normal that passes through o_{i-1} and z_i axis
- o_i is point where normal intersects z_i
- d_i and α_i become zero
- In this example a_i becomes l_1

Axis z_i and z_{i-1} not co-planar



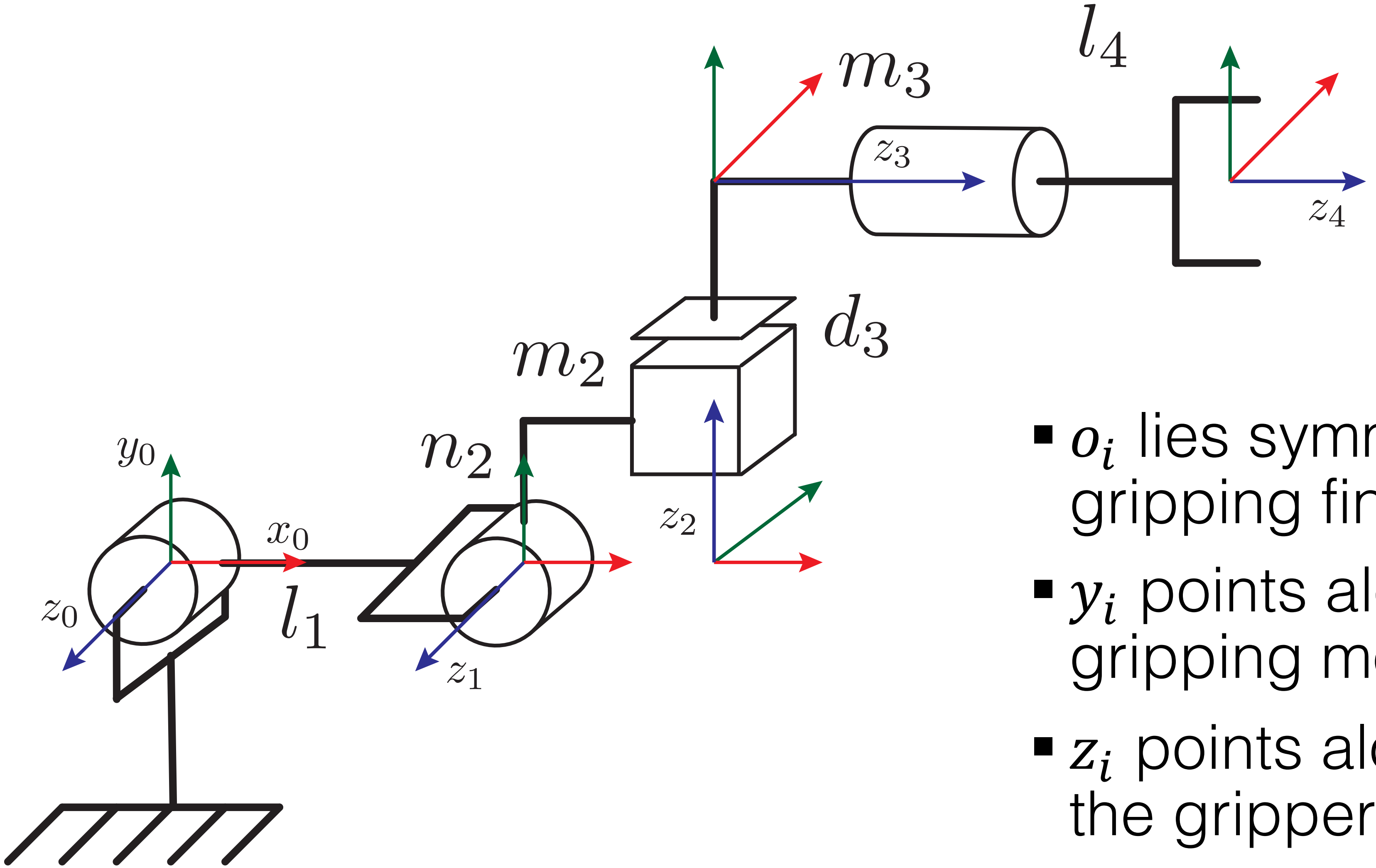
- There exists unique shortest line segment from z_{i-1} to z_i
- This line defines x_i
- The point where is intersects z_i is the origin o_i
- parameters?

Axis z_i and z_{i-1} intersect



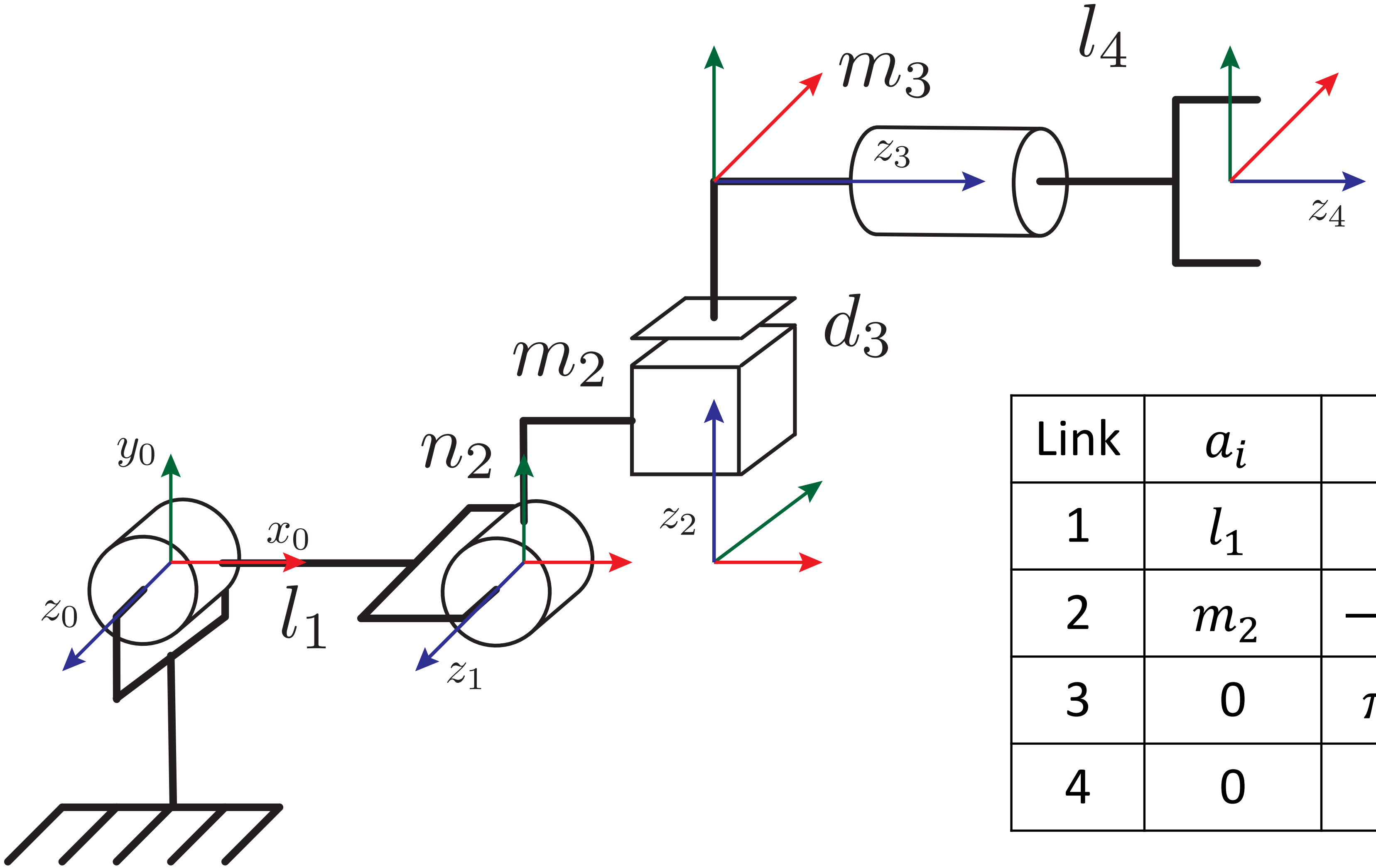
- x_i must be orthogonal to both z_{i-1} and z_i
- The origin o_i goes on the intersection of z_{i-1} and z_i
- parameters?

End Effector



- o_i lies symmetrically between gripping fingers
- y_i points along direction of gripping motion
- z_i points along the reach of the gripper

DH Table



Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1^*
2	m_2	$-\pi/2$	0	θ_3^*
3	0	$\pi/2$	$n_2 + d_3^*$	$\pi/2$
4	0	0	$m_3 + l_4$	θ_4^*

Constructing FK equation

- Each row in the DH table represents a homogeneous transformation matrix which is constructed a follows:

$$A_i = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics solution

- The complete FK equation is assembled from the individual homogeneous transformations

$$H = A_1(q_1)...A_n(q_n) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

- R_n^0 represents orientation of the end effector
- o_n^0 represents location of end effector in frame 0

Exponential Representation of Rotation

Lecture 14.1

Fall 2022

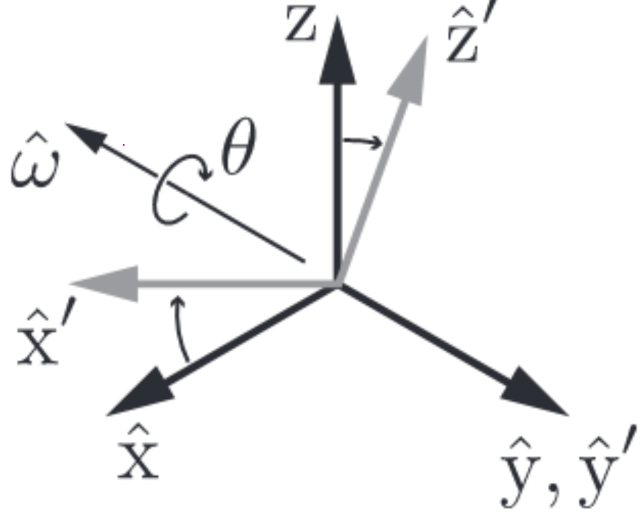
3D Rotation

Rotation around any single axis:

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



Rotation around arbitrary axis:

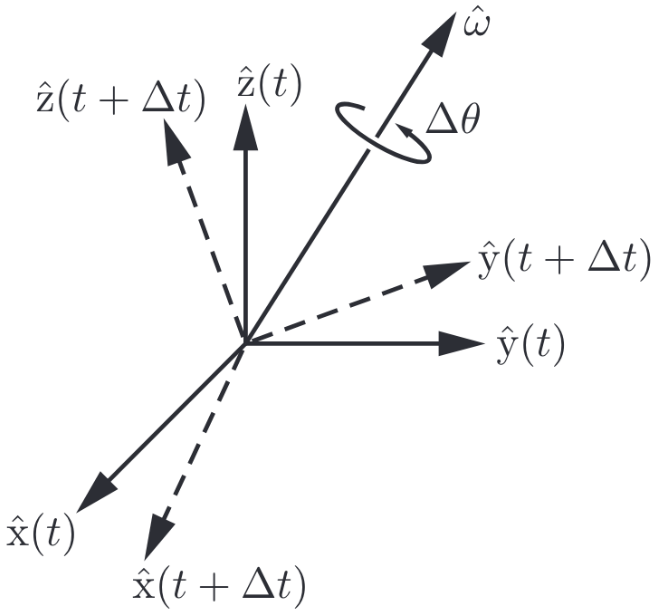
$$\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$$

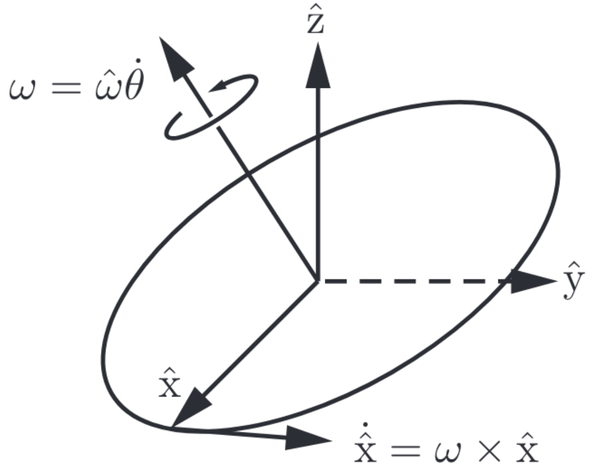
$$\mathrm{Rot}(\hat{\omega}, \theta) =$$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

Angular Velocity

Consider instantaneous angular velocity about an axis





Angular velocity:

$$\omega = \hat{\omega}\dot{\theta}$$

$$\dot{\hat{x}} = \omega \times \hat{x}$$

$$\dot{\hat{y}} = \omega \times \hat{y}$$

$$\dot{\hat{z}} = \omega \times \hat{z}$$

Angular Velocity – Rotation Matrix

- Must choose reference frame to describe ω
- Let $\{s\}$ be the fixed "world" or "space" frame and $\{b\}$ be the "body" frame
- Let $R(t)$ be the rotation matrix describing orientation of the body frame $\{b\}$ in the space frame $\{s\}$

$$R(t) = \begin{bmatrix} r_1(t) & r_2(t) & r_3(t) \end{bmatrix}$$

- Where $r_1(t), r_2(t), r_3(t)$ describe $\hat{x}$,$\hat{y}$,$\hat{z}$ in fixed-frame coordinates

Angular Velocity – Rotation Matrix

- Given the rotation R at time t:

$$R(t) = \begin{bmatrix} r_1(t) & r_2(t) & r_3(t) \end{bmatrix}$$

- The time rate of change of R is:

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R$$

Skew Symmetric Matrix

- Instead of using cross product…

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R$$

- We can define a skew symmetric matrix to produce the same result

Given a vector $x = [x_1 \; x_2 \; x_3]^{\mathrm{T}} \in \mathbb{R}^3$, define

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Matrix Exponential

Now consider the vector linear differential equation

$$\dot{x}(t) = Ax(t), \tag{3.40}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant, and the initial condition $x(0) = x_0$ is given. From the above scalar result one can conjecture a solution of the form

$$x(t) = e^{At}x_0 \tag{3.41}$$

where the **matrix exponential** e^{At} now needs to be defined in a meaningful way. Again mimicking the scalar case, we define the matrix exponential to be

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \tag{3.42}$$

Exponential Representation of Rotation

- Consider the angular velocity of a point

$$\dot{p} = [\hat{\omega}]p$$

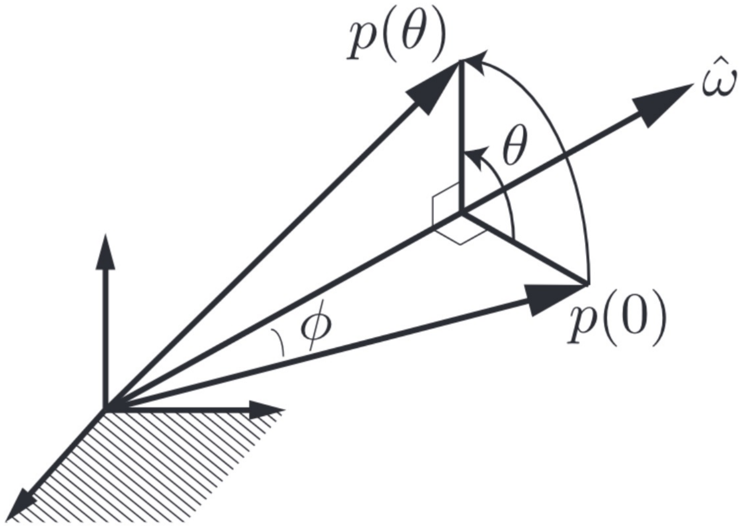
- A solution to this differential equation is

$$p(t) = e^{[\hat{\omega}]t}p(0)$$

- If we normalize velocity (e.g. 1 rad/s)

$$p(\theta) = e^{[\hat{\omega}]\theta}p(0)$$

- Interpret as rotation from $t = 0$ to $t = \theta$ at 1 rad/s



Rodrigues' Formula

$$
\begin{aligned}
e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\
&= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2
\end{aligned}
$$

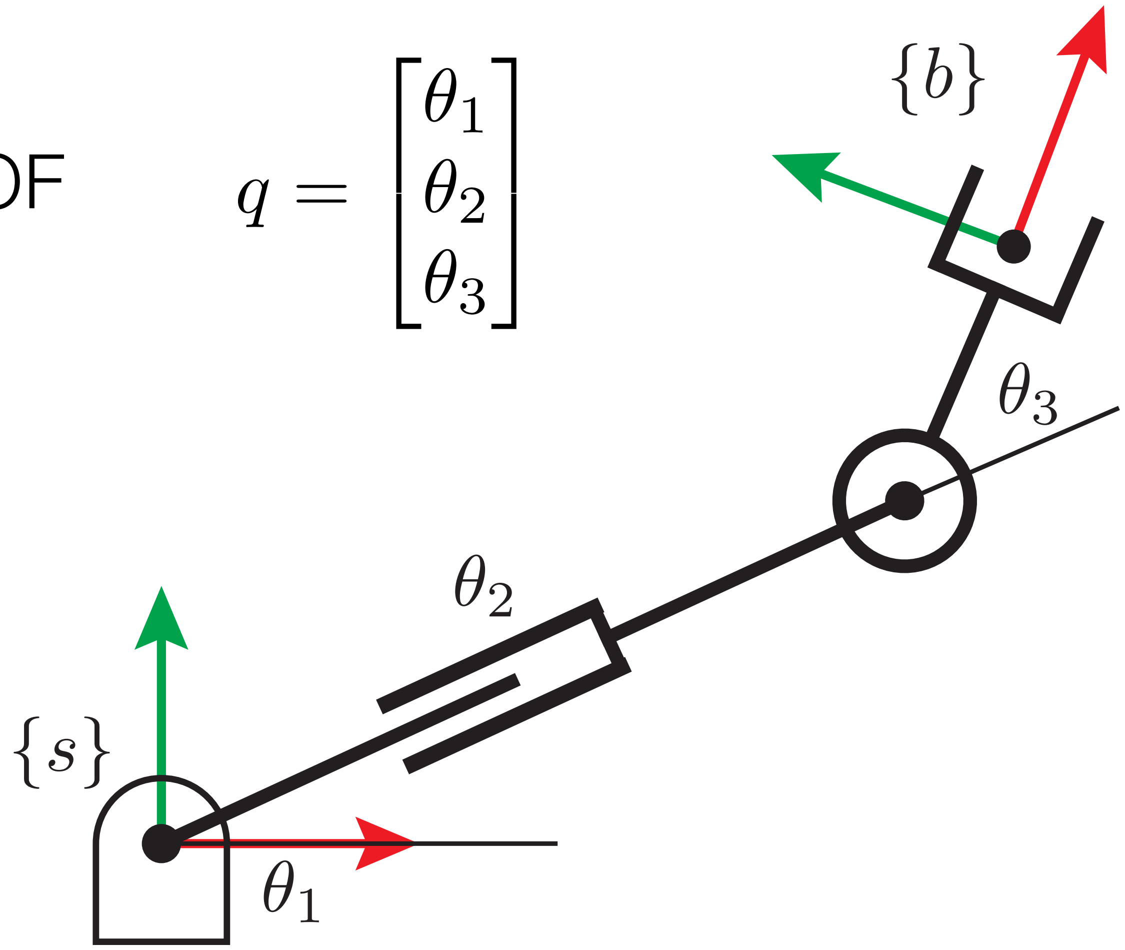
$$
\begin{aligned}
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots
\end{aligned}
$$

$$
\boxed{\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3)}
$$

Forward Kinematics with PoX

- Define home position
- Find screw vector for each DOF
- Use PoX formula to find pose

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

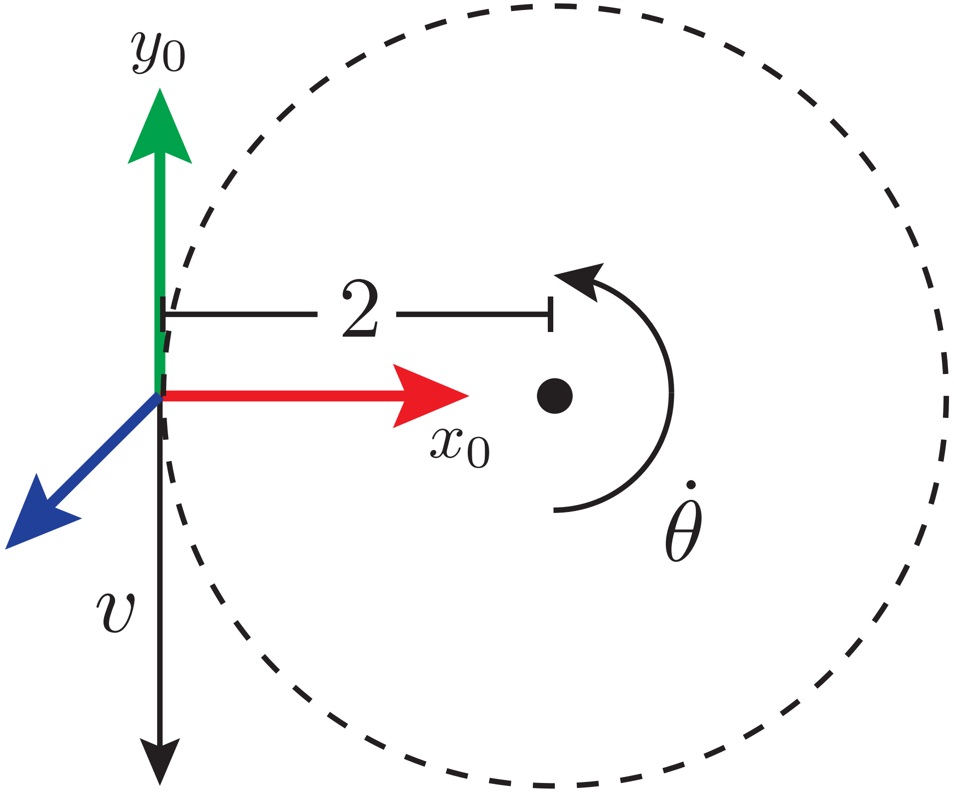


Screws

- ω is angular velocity about axis of rotation
- v is linear velocity of origin

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathcal{S}\dot{\theta}$$

- Screw is normalized twist



$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_\omega \\ \mathcal{S}_v \end{bmatrix} = \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of origin when } \dot{\theta} = 1 \end{bmatrix}$$

Constructing Rigid Body Transformations

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

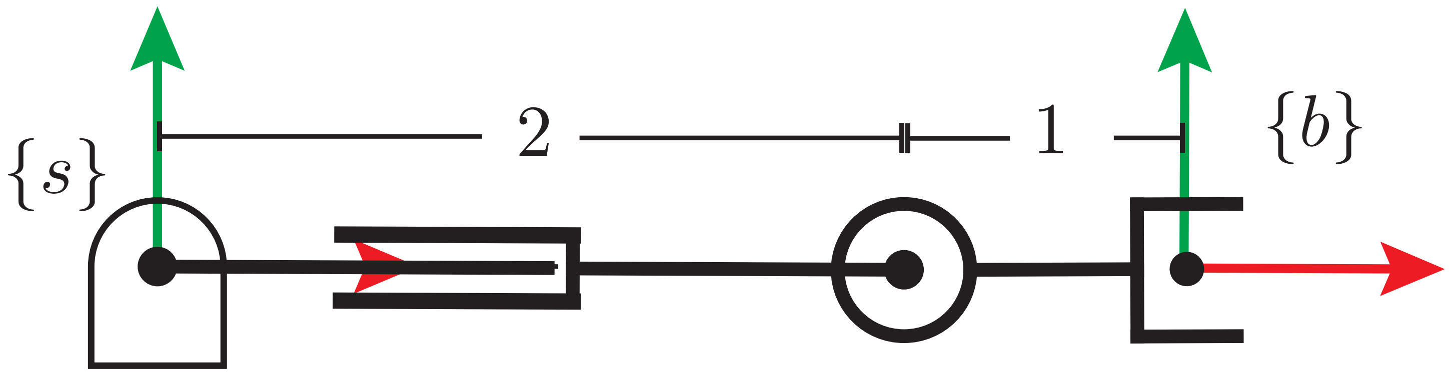
$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$$

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Home Position

- M is home position, where all joint variables are 0

$$M = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

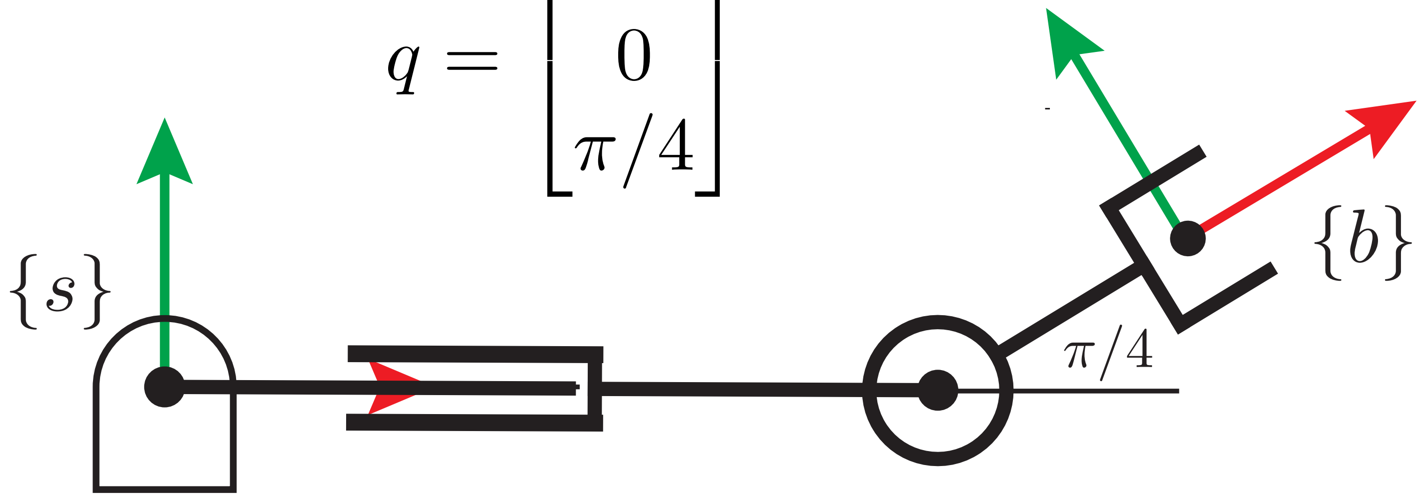


Screw Vector – Joint 3

- How does the joint motion move the $\{s\}$ frame?

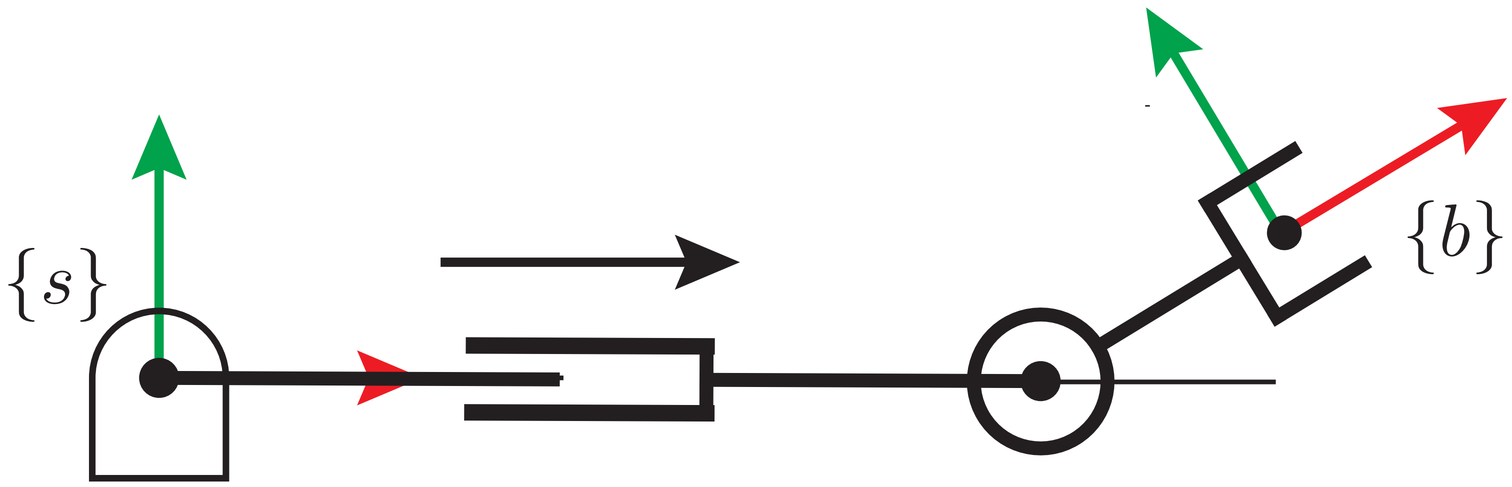
$$q = \begin{bmatrix} 0 \\ 0 \\ \pi/4 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$



Screw Vector – Joint 2

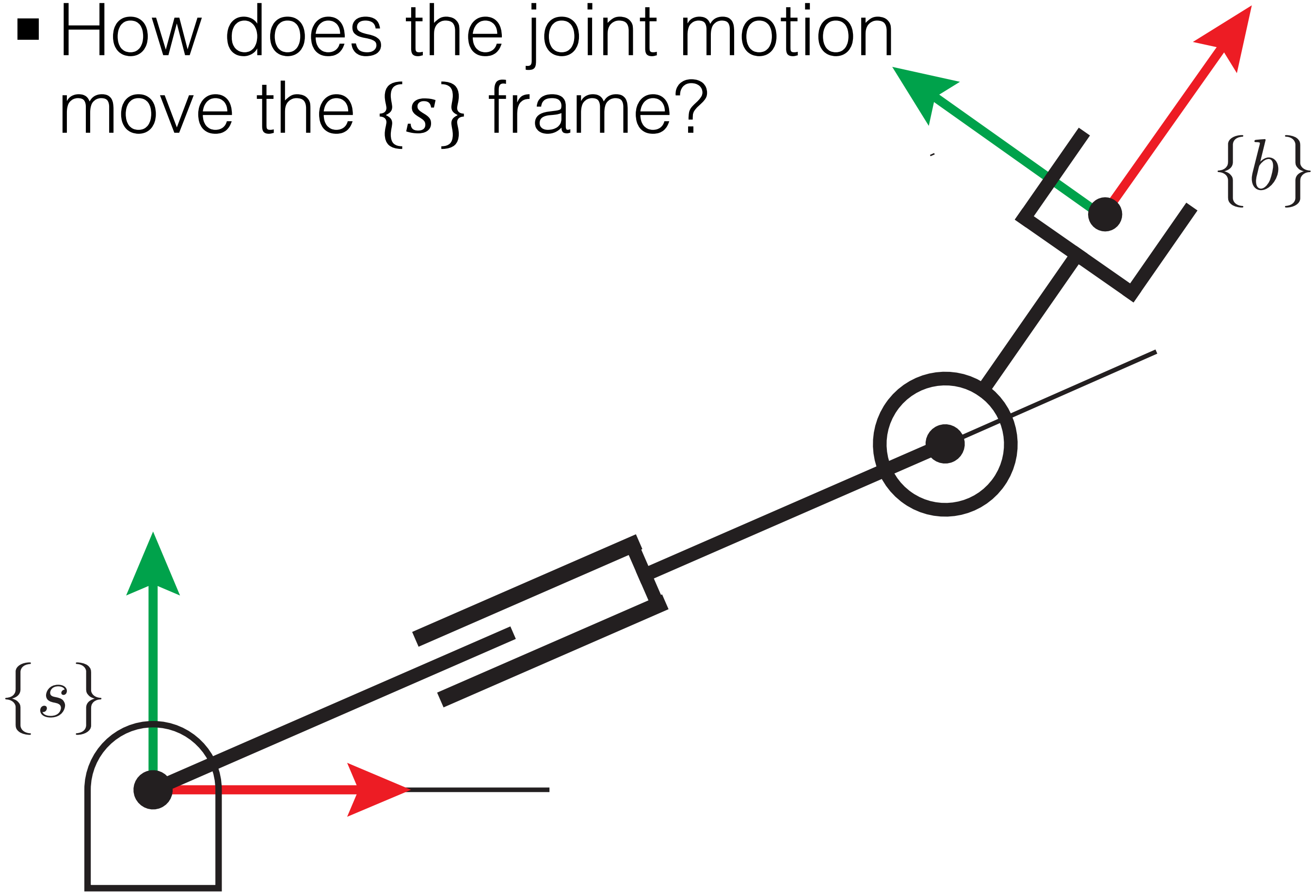
- How does the joint motion move the $\{s\}$ frame?



$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Screw Vector – Joint 1

- How does the joint motion move the $\{s\}$ frame?



$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Forward Kinematic Map

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$T(q) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$