不等式证明

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一、函数不等式证明

1.证明:$\forall x\in R,xe^{x}\geqslant e^{x}-1$

2.证明:$\forall x\in\left(0,\dfrac{\pi}{2}\right),\dfrac{2}{\pi}x<\sin x<x<\tan x$

3.证明:$\forall x>0,\dfrac{x}{1+x}<\ln(1+x)<x$

4.证明:$\forall a,b$恒有$a^{2}+b^{2}\geqslant 2ab$

二、数列不等式的证明

1.证明:$\forall n>0,\ln(n+1)<\sum_{i=1}^{n}\dfrac{1}{i}<1+\ln n$

2.设正值连续函数$f(x)$单调递减,证明:$\forall n>0$

$$\int_{1}^{n+1}f(x)dx\leqslant\sum_{i=1}^{n}f(i)\leqslant\int_{0}^{n}f(x)dx$$

3.证明:$\forall n>1,\dfrac{1}{2(n+1)}<\int_{0}^{\frac{\pi}{4}}\tan^{n}x\,dx<\dfrac{1}{2(n-1)}$

4.证明:$\forall n\geqslant 1,\dfrac{\sqrt{\pi}}{2\sqrt{n+1}}<\int_{0}^{\frac{\pi}{2}}\sin^{2n}x\,dx<\dfrac{\sqrt{\pi}}{2\sqrt{n}}$

三、常用积分不等式

1.函数$f(x)$在$[a,b]$上连续则$\left|\int_{a}^{b}f(x)dx\right|\leqslant\int_{a}^{b}|f(x)|dx$

2.柯西不等式:$f(x)$、$g(x)$在$[a,b]$上连续则

$$\left[\int_{a}^{b}f(x)g(x)dx\right]^{2}\leqslant\int_{a}^{b}f^{2}(x)dx\int_{a}^{b}g^{2}(x)dx$$

3.切比雪夫不等式:若连续函数$f(x),g(x)$在$[a,b]$上连续

若$f(x),g(x)$单调性一致,则$(b-a)\int_{a}^{b}f(x)g(x)dx\geqslant\int_{a}^{b}f(x)dx\int_{a}^{b}g(x)dx$

若$f(x),g(x)$单调性相反,则$(b-a)\int_{a}^{b}f(x)g(x)dx\leqslant\int_{a}^{b}f(x)dx\int_{a}^{b}g(x)dx$

4.泰勒中值定理涉及的不等式

若$f(x)$在$x=x_0$邻域二阶可导,则

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^{2}$$

若$f''(x)>0$,则$(x) \geqslant f(x_0)+f'(x_0)(x-x_0)$

若$f''(x)<0$,则$(x) \leqslant f(x_0)+f'(x_0)(x-x_0)$

四、历史经典例题难题

1.$f(x)$在$[0,1]$上连续,且$0<m \leqslant f(x) \leqslant M$,证明:

$$1 \leqslant \int_0^1 \frac{1}{f(x)}dx \int_0^1 f(x)dx \leqslant \frac{(m+M)^2}{4mM}$$

2.求极限: $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n}$

3. 设函数$f(x)$有连续的二阶导数，且有$f(a)=f(b)=0$

证明：

①$\int_a^b f(x)dx=\frac{1}{2}\int_a^b (x-a)(x-b)f''(x)dx$

②$\left|\int_a^b f(x)dx\right| \leqslant \frac{(b-a)^3}{12} \bullet \max\limits_{a\leqslant x\leqslant b}|f''(x)|$

③存在$\xi\in[a,b]$使,$|f'(\xi)|>\frac{4}{(b-a)^2}\int_a^b f(x)dx$

4.证明: $\frac{5}{2}\pi < \int_0^{\frac{\pi}{2}} e^{\sin x}dx < 2\pi e^{\frac{1}{4}}$ (提示:无穷级数)

5.证明: $\int_0^{+\infty} \frac{\sin x}{x}dx$条件收敛

6.设函数$f(x)$非负连续,且最大值M在(a,b)内取得,证明:

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b [f(x)]^n dx}=M$$

7.求 $\lim\limits_{n\to\infty} \sqrt[n]{\int_0^1 (e^x+x-1)^n dx}$

8.设函数$f(x)$ $[a,b]$上有连续的导函数,且$f(a)=0$,证明:

$$\int_a^b f^2(x)dx \leqslant \frac{(b-a)^2}{2} \int_a^b [f'(x)]^2 dx$$

9.设$f(x,y)$在D:$x^2+y^2\leqslant 1$上具有连续偏导数且$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=e^{-(x^2+y^2)}$

求$\iint\limits_D \left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)dxdy$

10.设$f(x,y,z)$在Ω:$x^2+y^2+z^2\leqslant 1$上具有二阶连续偏导数且

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\sqrt{x^2+y^2+z^2}\text{求}\iiint_{\Omega}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}\right)dxdydz$$

11.设$f(x)$在$[a,b]$上具有连续导数,证明:

$$\lim_{n\to\infty} n\left[\int_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)\right]=\frac{b-a}{2}[f(a)-f(b)]$$

12.计算$\lim\limits_{n\to\infty}\left(\frac{n}{2}-\frac{1}{n}-\frac{2}{n}-\cdots-\frac{n}{n}\right)$

13.函数$f(x)$在$[0,1]$上连续,证明:$\int_0^1 e^{f(x)}dx\int_0^1 e^{-f(x)}dx\geqslant 1$

14.设$f(x)$在$\left[-\frac{1}{a},a\right]$上非负可积$(a>0)$,且$\int_{-\frac{1}{a}}^{a} xf(x)dx=0$

证明:$\int_{-\frac{1}{a}}^{a} x^2 f(x)dx\leqslant\int_{-\frac{1}{a}}^{a} f(x)dx$

15.证明:$\frac{1}{2}<\int_0^1\frac{dx}{\sqrt{4-x^2+x^3}}dx<\frac{\pi}{6}$

16.设$f(x)$在$[0,1]$上为正直连续函数,证明:$\ln\int_0^1 f(x)dx\geqslant\int_0^1\ln f(x)dx$

17.证明:$\cos\left(\frac{1}{n}\sum_{k=1}^{n}\theta_k\right)>\frac{1}{n}\sum_{k=1}^{n}\cos\theta_k$,其中$\theta_i$满足:$-\frac{\pi}{2}<\theta_1<\cdots<\theta_n<\theta_n$

18.① 证明:$x\in[0,1]$,证明:

$$x-\frac{1}{6}x^3\leqslant\sin x\leqslant x$$

②计算$\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\sin\frac{1}{n+k}$

19. 函数$f(x)\in C[0,2]$,$\int_0^2 f(x)dx=0$,$\int_0^2 xf(x)dx=a>0$

证明:$\exists\xi\in[0,2]$使得$|f(\xi)|\geqslant a$

20. 设函数$f(x)$在$[0,1]$是正值连续函数且满足$f^2(t)\leqslant 1+2\int_0^t f(s)ds$

证明:$\forall t\in[0,1]$,满足$f(t)\leqslant t+1$

21.设$f(x)$在$[0,1]$上一阶导数连续,证明:

$$\int_0^1|f(x)|dx\leqslant\max\left\{\int_0^1|f'(x)|dx,\left|\int_0^1 f(x)dx\right|\right\}$$