Problem Solving 6: Reflection and Transmission

OBJECTIVES:

1. To learn the use of phase matching condition.

2. To learn the calculation of reflection and transmission from an ideal conductor.

3. To learn the calculation of reflection and transmission from a dielectric slab.

REFERENCE: Chapter 5, Waves in Media

PROBLEM SOLVING STRATEGIES

A. Write down the field distributions

For either TE or TM waves, use the Maxwell's Equations for plane wave solutions to derive the total field distributions

$$\boxed{\begin{aligned}&\vec{k}\cdot\vec{D}=0\\&\vec{k}\times\vec{E}=\omega\vec{B}\\&\vec{k}\cdot\vec{B}=0\\&\vec{k}\times\vec{H}=-\omega\vec{D}\end{aligned}}$$

These are the time-harmonic source-free Maxwell's Equations for plane wave solutions. We can see

B. Phase Matching Condition

For a plane wave propagates from region 1 to region 2, phase matching condition requires that the tangential wave vector is continuous at a plane boundary,

$$\boxed{\hat{n}\times\vec{k}_i=\hat{n}\times\vec{k}_r=\hat{n}\times\vec{k}_t} \text{ or } \boxed{\vec{k}_i^{\parallel}=\vec{k}_r^{\parallel}=\vec{k}_t^{\parallel}}$$

where $\vec{k}_i$, $\vec{k}_r$ and $\vec{k}_t$ are the wave vectors of the incident, reflected and transmitted waves, $\hat{n}$ is the surface normal. If we let θ_i, θ_i and θ_t denote the angles of incident, reflected and transmitted waves, respectively, then we have

$$\boxed{\theta_i=\theta_r} \text{ and } \boxed{k_i\sin\theta_i=k_t\sin\theta_t}$$

C. Boundary Conditions

To solve the reflection and transmission coefficients, we need to use appropriate boundary conditions.

For source-free problems, the boundary conditions for dielectrics:

$$\boxed{\begin{aligned}&\hat{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0\\&\hat{n}\times\left(\vec{H}_1-\vec{H}_2\right)=0\end{aligned}}$$

the boundary conditions for ideal conductors:

$$\boxed{\begin{aligned}\hat{n}\times\vec{E}_1 &= 0\\ \hat{n}\times\vec{H}_1 &= \vec{J}_s\end{aligned}}$$

PROBLEM 1

A plane wave in free space is incident at an angle θ on a conducting half-space. For large $\sigma/\omega\varepsilon_0$, show that the transmitted wave is almost perpendicular to the boundary.

Solution:

$k_x = k_0\sin\theta$ => $k_z = \sqrt{\omega^2\mu_0\left(\varepsilon_0 - j\,\sigma/\omega\right) - k_x^2} \approx \sqrt{-j\omega\mu_0\sigma} = \sqrt{\omega\mu_0\sigma}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$k_{zR} \gg k_x$ => $\theta_t = \tan^{-1}\left(k_x/k_{zR}\right) \approx 0$

PROBLEM 2

A plane wave is incident from free space ($z > 0$) on a perfect electric conductor ($z < 0$) with the incident electric field

$$\vec{E}_i = \left(\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{z} - j\hat{y}\right)\exp\left(-j\frac{1}{\sqrt{2}}k_0 x + j\frac{1}{\sqrt{2}}k_0 z\right)$$

What is the reflected electric field ?

Solution:

The wave vector of the incident wave is

$$\vec{k}_i = \hat{x}\frac{1}{\sqrt{2}}k_0 - \hat{z}\frac{1}{\sqrt{2}}k_0$$

According to the phase matching condition, the wave vector of the reflected wave is

$$\vec{k}_r = \hat{x}\frac{1}{\sqrt{2}}k_0 + \hat{z}\frac{1}{\sqrt{2}}k_0$$

Thus the reflected electric field can be written by

$$\vec{E}_r = \vec{E}_{r0}\exp\left(-j\frac{1}{\sqrt{2}}k_0 x - j\frac{1}{\sqrt{2}}k_0 z\right)$$

The boundary conditions requires $\hat{n}\times\vec{E}_1 = 0$, where $\vec{E}_1 = \vec{E}_i + \vec{E}_r$ is the total field in the reflected region. As a consequence, we have $\vec{E}_{r0} = \left(-\frac{1}{\sqrt{2}}\hat{x} + C\hat{z} + j\hat{y}\right)$, C is an unknown constant.

Then we apply the relation of $\vec{k}_r \cdot \vec{E}_{r0} = 0$, we get $C = \frac{1}{\sqrt{2}}$.

Thus, the reflected electric field is

$$\vec{E}_r = \left(-\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{z} + j\hat{y} \right) \exp\left(-j\frac{1}{\sqrt{2}}k_0 x - j\frac{1}{\sqrt{2}}k_0 z \right)$$

PROBLEM 3

Rainbow arc often appears when sunlight shines on water droplets after a brief shower late in the afternoon. When a sun ray is refracted as it enters the raindrop, total internally reflected from inside the drop, and refracted again as it leaves the drop and passes to the observer.

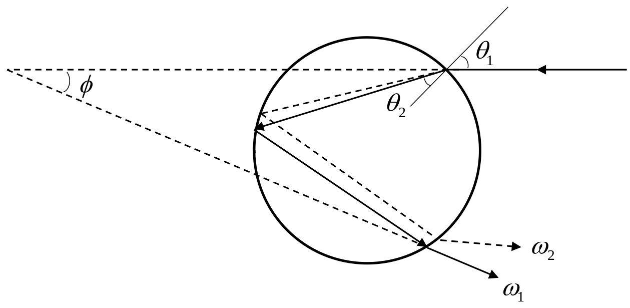

Figure S1

(a) Consider the ray path with only one internal reflection. Show that the scattering angle ϕ between the incident ray and the exit ray is $2(2\theta_2 - \theta_1)$, where θ_1 is the incident angle and θ_2 is the refracted angle.

(b) For a sphere with a radius a where $ka \gg 1$, the direction where the scattering angle is stationary ($d\phi/d\theta_1 = 0$) corresponds to the least cancellation between different rays and hence a large scattering amplitude. Show that the maximum scattering angle ($\phi_{\max}$) occurs at $\theta_1 = \sin^{-1}\sqrt{(4-n^2)/3}$ and $\phi_{\max} \approx 42°$ for n = 4/3, with the scattering angle between the incident ray and scattered ray $\theta_s = 138°$.

(c) The refractive index for a raindrop is n = 1.330 for red light (λ = 0.7μm), n = 4/3 = 1.333 for orange light, and n = 1.342 for violet light (λ = 0.4μm). Determine the scattering angles for the red and violet light rays. What are the relative positions of the different color bands in a rainbow?

Solutions:

(a) $\phi = 2\theta_2 - 2(\theta_1 - \theta_2) = 2(2\theta_2 - \theta_1)$

(b) $n_{air}\sin\theta_1 = n_{raindrop}\sin\theta_2$ => $\theta_2 = \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$

=> $\phi = 2\left(2\sin^{-1}\left(\dfrac{\sin\theta_1}{n}\right) - \theta_1\right)$

=> $\dfrac{d\phi}{d\theta_1} = 2\dfrac{d}{d\theta_1}\left(2\sin^{-1}\left(\dfrac{\sin\theta_1}{n}\right) - \theta_1\right) = 0$

=> $\dfrac{2}{n}\cos\theta_1 = \sqrt{1-\left(\dfrac{\sin\theta_1}{n}\right)^2}$ => $\sin\theta_1 = \sqrt{\dfrac{4-n^2}{3}}$

Here, we use the identity $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$.

For n = 4/3, $\sin\theta_1 = 0.86$, $\phi = 2(2\theta_2 - \theta_1) \approx 42°$

(c) For red light, θ_1 = 59.58 °, θ_2 = 40.42 °, ϕ_{max} = 42.52 °, and θ_s = 137.5 °; for violet light θ_1 = 58.89 °, θ_2 = 39.64 °, ϕ_{max} = 40.78 °, and θ_s = 139 °. The outer arc of the rainbow will be red, and violet will be on the inner arc of the rainbow.

PROBLEM 4

A plane wave is normally incident from free space ($z < 0$) on a medium slab with the thickness d, permittivity ε_1 and permeability $\mu_1 = \mu_0$. Calculate the reflection and transmission of the wave.

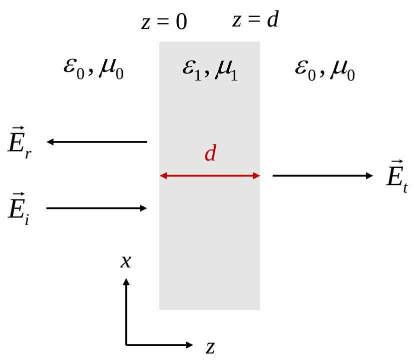

Figure S2

Solution:

For normal incidence, TE and TM waves are identical for isotropic media and we can denote it as a TEM wave. So we only consider x-polarized wave in the following discussion.

In region 1 ($z < 0$), the total fields can be written by

$$
\begin{aligned}
\vec{E}_1 &= \hat{x}\exp\left(-jk_0 z\right) + \hat{x}R\exp\left(jk_0 z\right) \\
\vec{H}_1 &= \frac{\vec{k}_i \times \vec{E}_i}{\omega\mu_0} + \frac{\vec{k}_r \times \vec{E}_r}{\omega\mu_0} \\
&= \frac{\hat{z}k_0 \times \hat{x}\exp\left(-jk_0 z\right)}{\omega\mu_0} + \frac{-\hat{z}k_0 \times \hat{x}R\exp\left(jk_0 z\right)}{\omega\mu_0} \\
&= \hat{y}\frac{k_0}{\omega\mu_0}\left[\exp\left(-jk_0 z\right) - R\exp\left(jk_0 z\right)\right]
\end{aligned}
$$

where $\vec{k}_i = \hat{z}k_0$ and $\vec{k}_r = -\hat{z}k_0$ are the incident and reflected wave vectors.

In region 2 ($0 < z < d$), the total field distributions are

$$
\begin{aligned}
\vec{E}_2 &= \hat{x}A\exp\left(-jk_1 z\right) + \hat{x}B\exp\left(jk_1 z\right) \\
\vec{H}_2 &= \frac{\hat{z}k_1 \times \hat{x}A\exp\left(-jk_1 z\right)}{\omega\mu_1} + \frac{-\hat{z}k_1 \times \hat{x}A\exp\left(jk_1 z\right)}{\omega\mu_1} \\
&= \hat{y}\frac{k_1}{\omega\mu_1}\left[A\exp\left(-jk_1 z\right) - B\exp\left(jk_1 z\right)\right]
\end{aligned}
$$

In region 3 ($z > d$), the total field distributions are

$$
\begin{aligned}
\vec{E}_3 &= \hat{x}T\exp\left(-jk_0 z\right) \\
\vec{H}_3 &= \frac{\hat{z}k_0 \times \hat{x}T\exp\left(-jk_0 z\right)}{\omega\mu_0} \\
&= \hat{y}\frac{k_0}{\omega\mu_0}T\exp\left(-jk_0 z\right)
\end{aligned}
$$

Boundary conditions require that the tangential electric and magnetic fields are continuous at $z = 0$ and $z = \mathrm{d}$, that is,

$$
\begin{aligned}
E_{1x}\left(z=0\right) &= E_{2x}\left(z=0\right) \\
H_{1y}\left(z=0\right) &= H_{2y}\left(z=0\right) \\
E_{2x}\left(z=d\right) &= E_{3x}\left(z=d\right) \\
H_{2y}\left(z=d\right) &= H_{3y}\left(z=d\right)
\end{aligned}
$$

then we obtain

$$
\begin{cases}
1 + R = A + B \\
\dfrac{k_0}{\omega\mu_0}\left(1-R\right) = \dfrac{k_1}{\omega\mu_1}\left(A-B\right) \\
A\exp\left(-jk_1 d\right) + B\exp\left(jk_1 d\right) = T\exp\left(-jk_0 z\right) \\
\dfrac{k_1}{\omega\mu_1}\left[A\exp\left(-jk_1 d\right) - B\exp\left(jk_1 d\right)\right] = \dfrac{k_0}{\omega\mu_0}T\exp\left(-jk_0 d\right)
\end{cases}
$$

Let $\omega\mu_0/k_0 = \sqrt{\mu_0/\varepsilon_0} = \eta_0$ and $\omega\mu_1/k_1 = \sqrt{\mu_1/\varepsilon_1} = \eta_1$ represent the wave impedances, we have

$$\begin{cases} 1+R=A+B \\ \eta_1(1-R)=\eta_0(A-B) \\ A\exp(-jk_1d)+B\exp(jk_1d)=T\exp(-jk_0d) \\ \eta_0\left[A\exp(-jk_1d)-B\exp(jk_1d)\right]=\eta_1 T\exp(-jk_0d) \end{cases}$$

=>

$$\begin{cases} \eta_1(2-A-B)=\eta_0(A-B) \\ \eta_0\left[A\exp(-jk_1d)-B\exp(jk_1d)\right]=\eta_1\left[A\exp(-jk_1d)+B\exp(jk_1d)\right] \end{cases}$$

=>

$$\begin{cases} A(\eta_1+\eta_0)+B(\eta_1-\eta_0)=2\eta_1 \\ A(\eta_1-\eta_0)\exp(-j2k_1d)=-B(\eta_1+\eta_0) \end{cases}$$

=>

=>

$$A=\frac{2\eta_1(\eta_1+\eta_0)}{(\eta_1+\eta_0)^2-(\eta_1-\eta_0)^2\exp(-j2k_1d)}$$
$$B=-\frac{2\eta_1(\eta_1-\eta_0)}{(\eta_1+\eta_0)^2-(\eta_1-\eta_0)^2\exp(-j2k_1d)}\exp(-j2k_1d)$$

=>

$$R=A+B-1=\frac{(\eta_1^2-\eta_0^2)\left[1-\exp(-j2k_1d)\right]}{(\eta_1+\eta_0)^2-(\eta_1-\eta_0)^2\exp(-j2k_1d)}$$
$$T\exp(-jk_0d)=\frac{4\eta_1\eta_0}{(\eta_1+\eta_0)^2-(\eta_1-\eta_0)^2\exp(-j2k_1d)}\exp(-jk_1d)$$

If we use the single-interface reflection coefficient at normal incidence

$$R_{01}=\frac{\eta_1\cos\theta_i-\eta_0\cos\theta_t}{\eta_1\cos\theta_i+\eta_0\cos\theta_t}=\frac{\eta_1-\eta_0}{\eta_1+\eta_0}$$

Then the reflection and transmission coefficients of a slab can be written by

$$R=A+B-1=\frac{R_{01}\left[1-\exp(-j2k_1d)\right]}{1-R_{01}^2\exp(-j2k_1d)}$$
$$T\exp(-jk_0d)=\frac{1-R_{01}^2}{1-R_{01}^2\exp(-j2k_1d)}\exp(-jk_1d)$$

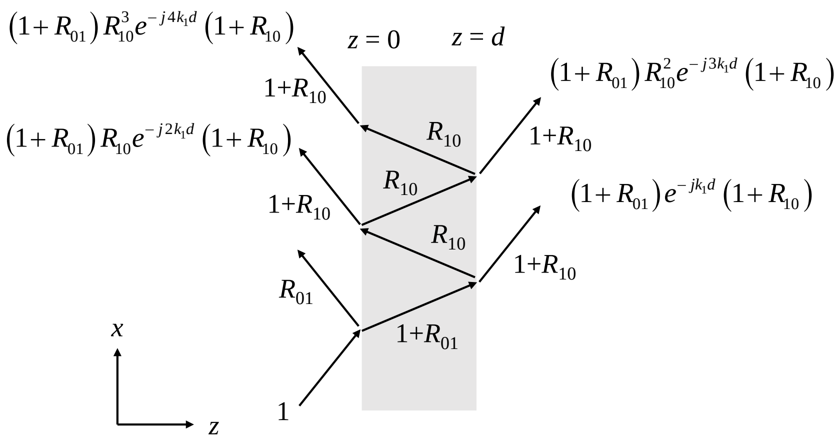

Figure S3

From Figure S3, we can consider the reflected wave as superposition of multiply reflected waves from the two-layer interfaces, that is,

$$
\begin{aligned}
R &= R_{01}+(1+R_{01})\exp(-j2k_1d)R_{10}(1+R_{10})+(1+R_{01})\exp(-j4k_1d)R_{10}^3(1+R_{10})+\cdots \\
&= R_{01}+(1+R_{01})(1+R_{10})\exp(-j2k_1d)R_{10}\left[1+\exp(-j2k_1d)R_{10}^2+\exp(-j4k_1d)R_{10}^4+\cdots\right] \\
&= R_{01}+(1+R_{01})(1+R_{10})\exp(-j2k_1d)R_{10}\frac{1}{1-\exp(-j2k_1d)R_{10}^2} \\
&= R_{01}-R_{01}\frac{(1-R_{01}^2)\exp(-j2k_1d)}{1-R_{01}^2\exp(-j2k_1d)} \\
&= \frac{R_{01}\left[1-\exp(-j2k_1d)\right]}{1-R_{01}^2\exp(-j2k_1d)}
\end{aligned}
$$

$$
\begin{aligned}
T &= (1+R_{01})\exp(-jk_1d)(1+R_{10})\left[1+R_{10}^2\exp(-j2k_1d)+R_{10}^4\exp(-j4k_1d)+\cdots\right] \\
&= (1+R_{01})\exp(-jk_1d)(1+R_{10})\frac{1}{1-R_{10}^2\exp(-j2k_1d)} \\
&= \frac{1-R_{01}^2}{1-R_{01}^2\exp(-j2k_1d)}\exp(-jk_1d)
\end{aligned}
$$

where $R_{10}=-R_{01}$.