Chapter 3: Maxwell's Equations and Electromagnetic Waves

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3.1 The Displacement Current

In Chapters 1 and 2, we learned that If a current-carrying wire possesses certain symmetry, the magnetic field can be obtained by using Ampere's law:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \tag{3.1}$$

where I_{enc} is the conduction current passing through the surface bound by the closed path. In addition, as a consequence of the Faraday's law of induction, a changing magnetic field can produce an electric field, according to

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{3.2}$$

One might then wonder whether or not the converse could be true, namely, a changing electric field produces a magnetic field. If so, then the right-handed side of Eq. (3.2) will have to be modified to reflect such symmetry between $\vec{E}$ and $\vec{B}$.

To see how magnetic fields can be generated by a time-varying electric field, consider a capacitor which is being charged. During the charging process, the electric field strength increases with time as more charge is accumulated on the plates. The conduction current that carries the charges also produces a magnetic field.

In order to apply Ampere's law to calculate this field, let us choose path P shown in Figure 3.1 to be the Amperian loop.

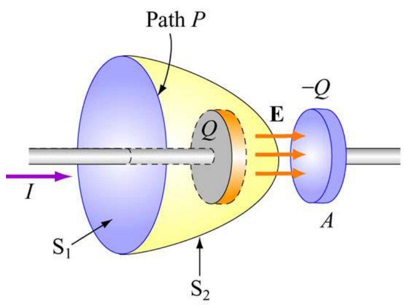

Figure 3.1 Surfaces S_1 and S_2 bound by Path P

If the surface bounded by the path is the flat surface S_1, then the enclosed current is $I_{enc} = I$. On the other hand, if we choose S_2 to be the surface bounded by the path, then $I_{enc} = 0$ since no current passes through S_2.

Maxwell showed that the ambiguity can be resolved by adding to the right-hand side of the Ampere's law an extra term

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} \tag{3.3}$$

which is called the "*displacement current*".

In Figure 3.1, the electric flux which passes through S_2 is given by

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{S} = EA = \frac{Q}{\varepsilon_0} \tag{3.4}$$

where A is the area of the capacitor plates. From Eq. (3.3), we readily see that the displacement current I_d is related to the rate of increase of charge on the plate by

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt} \tag{3.5}$$

The generalized Ampere's (or the Ampere-Maxwell) law now reads

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + I_d \right) = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \tag{3.6}$$

3.2 Gauss's Law for Magnetism

We have seen that Gauss's law for electrostatics states that the electric flux through a closed surface is proportional to the charge enclosed (Figure 3.2(a)).

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{in}}{\varepsilon_0} \tag{3.7}$$

However, despite intense search effort, no isolated magnetic monopole has ever been observed. Hence, $Q_{in} = 0$ and Gauss's law for magnetism becomes

$$\oiint_S \vec{B} \cdot d\vec{S} = 0 \tag{3.8}$$

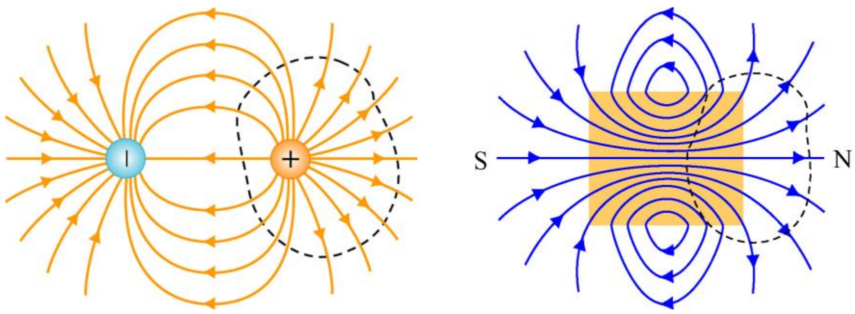

Figure 3.2 Gauss's law for (a) electrostatics, and (b) magnetism.

This implies that for a bar magnet, the field lines that emanate from the north pole to the south pole outside the magnet return within the magnet and form a closed loop, which is shown in Figure 3.2(b). That is, there is no source or sink.

3.3 Maxwell's Equations

We now have four equations which form the foundation of electromagnetic phenomena:

Law	Equation	Physical Interpretation
Gauss's law	$\oiint_S \vec{E}\cdot d\vec{S} = \dfrac{Q_{in}}{\varepsilon_0}$	Electric flux through a closed surface is proportional to the charged enclosed
Faraday's law	$\oint_C \vec{E}\cdot d\vec{l} = -\dfrac{d\Phi_B}{dt}$	Changing magnetic flux produces an electric field
Magnetic Gauss's law	$\oiint_S \vec{B}\cdot d\vec{S} = 0$	The total magnetic flux through a closed surface is zero
Ampere-Maxwell law	$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \dfrac{d\Phi_E}{dt}$	Electric current and changing electric flux produces a magnetic field

Collectively they are known as Maxwell's equations. An important consequence of Maxwell's Equations, as we will shall later in this Chapter, is the prediction of the existence of electromagnetic waves that can travel with speed of light $c = 1/\sqrt{\varepsilon_0\mu_0}$. The reason is due to the fact that a changing electric field produces a magnetic field and vice versa, and the coupling between the two fields leads to the generation of electromagnetic waves. The prediction was confirmed by H. Hertz in 1887.

3.4 Review of Vector analysis

A vector $\vec{A}$ has a magnitude and a direction, which can be represented graphically by a straight-line element of length proportional to the magnitude of $\vec{A}$ and with an arrow pointing in the direction of $\vec{A}$. In a Cartesian coordinate system, we write in terms of the three Cartesian components A_x, A_y, and A_z (Figure 3.3).

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \tag{3.9}$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are unit vectors with the scalar product $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$. Furthermore $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$.

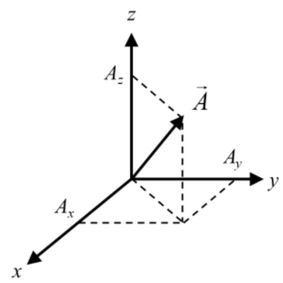

Figure 3.3 Projection of $\vec{A}$ in a Cartesian coordinate system.

Vector Addition and Subtraction

The addition of two vectors $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ and $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$ in Cartesian components follows that

$$\vec{A} \pm \vec{B} = \hat{x}\left(A_x \pm B_x\right) + \hat{y}\left(A_y \pm B_y\right) + \hat{z}\left(A_z \pm B_z\right) \tag{3.10}$$

Scalar Dot Product

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\cdot\vec{B}$, is a scalar number,

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{3.11}$$

Vector Cross Product

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\times\vec{B}$, is a vector. In terms of their Cartesian component,

$$\vec{A}\times\vec{B} = \hat{x}\left(A_y B_z - A_z B_y\right) + \hat{y}\left(A_z B_x - A_x B_z\right) + \hat{z}\left(A_x B_y - A_y B_x\right) \tag{3.12}$$

For the three orthogonal unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ it is seen that $\hat{x} = \hat{y}\times\hat{z}$, $\hat{y} = \hat{z}\times\hat{x}$, $\hat{z} = \hat{x}\times\hat{y}$.

The direction of $\vec{A}\times\vec{B}$ follows the right-hand rule, i.e., when the fingers of the right hand rotate from $\vec{A}$ to $\vec{B}$, the thumb of the right hand points in the direction of $\vec{A}\times\vec{B}$. Thus the vector $\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ and the plane containing them.

Operation of Three Vectors

For three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we have

$$\begin{aligned} &\vec{C}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) \\ &= \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} \end{aligned} \tag{3.13}$$

$$\begin{aligned}\vec{C}\times\left(\vec{A}\times\vec{B}\right) &= \hat{x}\left[C_y\left(A_xB_y - A_yB_x\right) - C_z\left(A_zB_x - A_xB_z\right)\right] \\ &\quad + \hat{y}\left[C_z\left(A_yB_z - A_zB_y\right) - C_x\left(A_xB_y - A_zB_y\right)\right] \\ &\quad + \hat{z}\left[C_x\left(A_zB_x - A_xB_z\right) - C_y\left(A_xB_y - A_yB_x\right)\right] \\ &= \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)\left(C_xB_x + C_yB_y + C_zB_z\right) \\ &\quad - \left(C_xA_x + C_yA_y + C_zA_z\right)\left(\hat{x}B_x + \hat{y}B_y + \hat{z}B_z\right) \\ &= \vec{A}\left(\vec{C}\cdot\vec{B}\right) - \left(\vec{C}\cdot\vec{A}\right)\vec{B}\end{aligned} \tag{3.14}$$

Notice that the vector $\vec{C}\times\left(\vec{A}\times\vec{B}\right)$ is perpendicular to $\vec{C}$ and lies in the plane determined by $\vec{A}$ and $\vec{B}$.

3.4.1 Del Operator

Del operator is a differential quantity, which is also called the “gradient operator.” It can be written as

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{3.15}$$

Its function is to simplify the mathematical expression of physical formulas. The following can be proved in Cartesian coordinates or in vector form:

$$\nabla\cdot\left(\vec{E}\times\vec{H}\right) = \vec{H}\cdot\left(\nabla\times\vec{E}\right) - \vec{E}\cdot\left(\nabla\times\vec{H}\right) \tag{3.16}$$

$$\nabla\cdot\left(\nabla\times\vec{A}\right) = 0 \tag{3.17}$$

$$\nabla\times\left(\nabla\Phi\right) = 0 \tag{3.18}$$

$$\nabla\times\left(\nabla\times\vec{E}\right) = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} \tag{3.19}$$

where

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3.20}$$

is the Laplacian operator in the rectangular coordinate system.

3.4.2 Gradient of a Scalar

The idea of gradient is a vector, which means that the directional derivative of a function at that point is maximized along that direction. That’s to say, the function changes fastest along that direction at that point.

The gradient of a scalar function Φ (x, y, z) is a vector quantity written as

$$\nabla\Phi = \hat{x}\frac{\partial\Phi}{\partial x} + \hat{y}\frac{\partial\Phi}{\partial y} + \hat{z}\frac{\partial\Phi}{\partial z} \tag{3.21}$$

Gradient can be denoted by the symbol *grad* , so that the gradient of a scalar becomes

$$\nabla\Phi = grad\ \Phi \tag{3.22}$$

The change of Φ by the infinitesimal amounts *dx*, *dy*, *dz* is

$$d\Phi = \left(\frac{\partial \Phi}{\partial x}\right)dx + \left(\frac{\partial \Phi}{\partial y}\right)dy + \left(\frac{\partial \Phi}{\partial z}\right)dz = \left(\hat{x}\frac{\partial \Phi}{\partial x} + \hat{y}\frac{\partial \Phi}{\partial y} + \hat{z}\frac{\partial \Phi}{\partial z}\right)\cdot\left(\hat{x}dx + \hat{y}dy + \hat{z}dz\right)$$
$$= \left(\nabla\Phi\right)\cdot\left(d\vec{l}\right) = \left|\nabla\Phi\right|\left|dl\right|\cos\theta \tag{3.23}$$

where θ is the angle between $\nabla\Phi$ and $d\vec{l}$. Now, if we fix the magnitude $\left|d\vec{l}\right|$ and search around in various directions (that is, vary θ), the maximum change in Φ evidentially occurs when $\theta = 0$. That is, for a fixed distance $\left|d\vec{l}\right|$, $d\Phi$ is greatest when $\vec{l}$ moves in the same direction of $\nabla\Phi$. Thus:

The gradient $\nabla\Phi$ points in the direction of maximum increase of the function Φ.

Moreover:

The magnitude $|\nabla\Phi|$ gives the slope (rate of increase) along this maximum direction.

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the direction of the gradient. Now measure the slope in that direction (rise over run). That is the magnitude of the gradient.

What would it mean for the gradient to vanish? If $\nabla\Phi = 0$ at (x, y, z), then $d\Phi = 0$ for small displacements about the point (x, y, z). This is, then, a stationary point of the function $\Phi(x, y, z)$. It could be a maximum (a summit), a minimum (a valley), a saddle point (a pass), or a “shoulder.” This is analogous to the situation for functions of one variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

Example 2.1. Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector)

Solution

$$\nabla r = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z}$$
$$= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$
$$= \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r}$$

Does this make sense? Well, it says that the distance from the origin increases most rapidly in the radial direction, and that its rate of increase in that direction is 1. . . just what you’d expect.

3.4.3 Divergence of a Vector

If there is a vector field $\vec{F}$ and a closed surface S in space, then the flux through the surface is

$$\Phi = \oint_S \vec{F} \cdot d\vec{S} \tag{3.24}$$

In Cartesian coordinates, Eq. (3.24) can be written as

$$\Phi = \oint_S \vec{F} \cdot d\vec{S} = \int_S F_x dydz + F_y dzdx + F_z dxdy \tag{3.25}$$

When the area of a closed surface $\vec{S}$ is infinitely contracted at a certain point, the limit of the flux through the closed surface $\vec{S}$ divides the volume surrounded by $\vec{S}$ is called the *divergence* of vector field $\vec{F}$ at this point, which is expressed by

$$\nabla \cdot \vec{F} = div\ \vec{F} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{F} \cdot d\vec{S}}{\Delta V} \tag{3.26}$$

It's obvious that $\Delta V = dxdydz$. Try to compare Eq. (3.25) and Eq. (3.26), we can draw the conclusion that

$$\nabla \cdot \vec{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot \left(\hat{x}F_x + \hat{y}F_y + \hat{z}F_z \right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{3.27}$$

Geometrical Interpretation: The name divergence is well chosen, for $\nabla \cdot \vec{F}$ is a measure of how much the vector $\vec{F}$ spreads out (diverges) from the point in question. For example, the vector function in Figure 3.4a has a large (positive) divergence (if the arrows pointed in, it would be a negative divergence), the function in Figure 3.4b has zero divergence, and the function in Figure 3.4c again has a positive divergence. (Please understand that $\vec{F}$ here is a function—there's a different vector associated with every point in space. In the diagrams, of course, I can only draw the arrows at a few representative locations.)

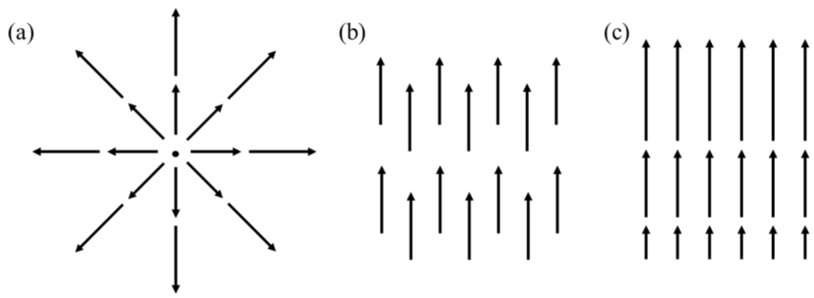

Figure 3.4

Example 2.2. Suppose the functions in Figure 3.4 are $\vec{F}_a = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{F}_b = \hat{z}$, and $\vec{F}_c = z\hat{z}$. Calculate their divergences.

Solution

$$\nabla \cdot \vec{F}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

as anticipated, this function has a positive divergence.

$$\nabla \cdot \vec{F}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0+0+0 = 0,$$

as expected.

$$\nabla \cdot \vec{F}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0+0+1 = 1.$$

3.4.4 Divergence theorem

Applying the definition of divergence to a large volume V containing an infinite number of infinitesimal differential volumes (Figure 3.5), we note that integrating the divergence over the volume surfaces shared by adjacent differential volumes will have no contribution because the surface normals point in the opposite directions and thus cancel. The result is the divergence theorem or Gauss theorem

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_V \left(\nabla \cdot \vec{F}\right) dV \tag{3.28}$$

The divergence theorem states that the volume integral of the divergence of the vector field $\vec{F}$ is equal to the total outward flux $\vec{F}$ through the surface S enclosing the volume.

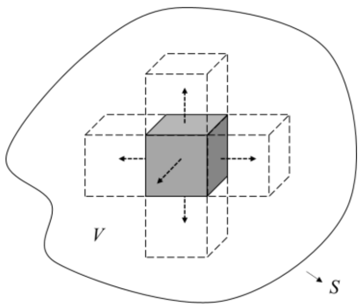

Figure 3.5 Derivation of divergence theorem

3.4.5 Curl of a Vector

We developed the concept of divergence, a local property of a vector field, by starting from the surface integral over a large closed surface. In the same spirit, let us consider the line integral of some vector field $\vec{F}(x, y, z)$, taken around a closed path, some curve C that comes back to join itself. The curve C can be visualized as the boundary of some surface S that spans it. A good name for the magnitude of such a closed path line integral is *circulation*; we shall use Γ (capital gamma) as its symbol:

$$\Gamma = \oint_C \vec{F} \cdot d\vec{l} \tag{3.29}$$

In the integrand, $d\vec{l}$ is the element of path, an infinitesimal vector locally tangent to C (Figure 3.6). The area density of the circulation is written by

$$\lim_{\Delta S \to 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{\Delta S} \tag{3.30}$$

The rule for sign is that the direction of $\hat{n}$ and the sense in which C is traversed in the line integral shall be related by a right-hand-screw rule. The limit we obtain by this procedure is a scalar quantity that is associated with the point P in the vector field $\vec{F}$, and with the direction $\hat{n}$. We could pick three directions, such as $\hat{x}$, $\hat{y}$, and $\hat{z}$, and get three different numbers. It turns out that these numbers can be considered components of a vector. We call the vector "curl $\vec{F}$". That is to say, the number we get for the limit with $\hat{n}$ in a particular direction is the component, in that direction, of the vector curl $\vec{F}$. To state this in an equation, the curl of a vector field is defined as

$$\left(\nabla \times \vec{F}\right) \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{\Delta S} \tag{3.31}$$

where $\hat{n}$ is the unit vector normal to the curve C. In Cartesian coordinates, curl can be calculated in this way.

$$\begin{aligned} \nabla \times \vec{F} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \times (\hat{x}F_x + \hat{y}F_y + \hat{z}F_z) \\ &= \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \end{aligned} \tag{3.32}$$

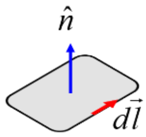

Figure 3.6

Geometrical Interpretation: The name curl is also well chosen, for $\nabla \times \vec{F}$ is a measure of how much the vector $\vec{F}$ swirls around the point in question. Thus the three functions in Fig. 3.4 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 3.7 have a substantial curl, pointing in the z direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero curl. A whirlpool would be a region of large curl.

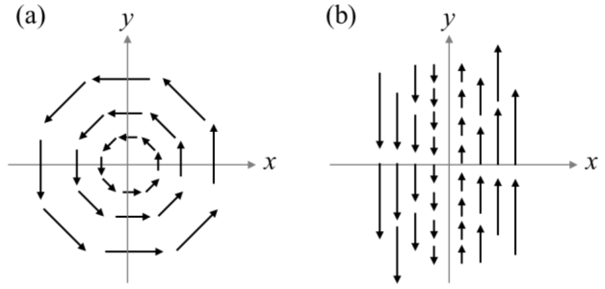

Figure 3.7

Example 2.3. Suppose the function sketched in Figure 3.7a is $\vec{F}_a = -y\hat{x} + x\hat{y}$, and that in Figure 3.7b is $\vec{F}_b = x\hat{y}$. Calculate their curls.

Solution

$$\nabla \times \vec{F}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{z}, \quad \nabla \times \vec{F}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{z}$$

As expected, these curls point in the +*z* direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out". . . it just "swirls around.")

3.4.6 Stokes' theorem

We apply the curl to an open surface *S*, subdivide into N differential areas (Figure 3.8). For a differential area ΔS_i bounded by a contour C_i and with a surface normal $\hat{n}_i$, we have $\Delta\vec{S}_i = \hat{n}_i \Delta S_i$ and

$$\Delta\vec{S}_i \cdot \left(\nabla \times \vec{F}\right) = \oint_{C_i} \vec{F} \cdot d\vec{l} \tag{3.33}$$

Adding the contributions of all *N* differential areas, we find

$$\lim_{\substack{\Delta S_i \to 0 \\ N \to \infty}} \sum_{i=1}^{N} \Delta\vec{S}_i \cdot \left(\nabla \times \vec{F}\right)_i = \oint_C \vec{F} \cdot d\vec{l} \tag{3.34}$$

Since the common part of the contours in two adjacent elements is traversed in opposite directions by the two contours, the net contribution of all the common parts in the interior sums to zero and only the contribution from the external contour *C* bounding the open surface *S* remains in the line integral on the right-hand side. The left-hand side becomes a surface integral, and the result is Stokes' theorem:

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{l} \tag{3.35}$$

Stokes' theorem states that the surface integral of the curl of the vector field $\vec{F}$ over an open surface S is equal to the closed line integral of the vector along the contour C enclosing the open surface.

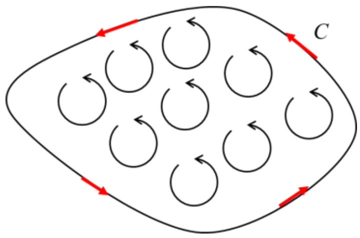

Figure 3.8 Derivation of Stokes' theorem

3.5 Differential Maxwell Equations

Applying Stokes theorem to the Ampere's law and Faraday's law and applying the divergence theorem to Gauss's and continuity laws, we find the Maxwell equations in the differential form

$$\begin{aligned} &\nabla\cdot\vec{E}=\rho/\varepsilon_0 && \text{(Gauss's Law)} \\ &\nabla\times\vec{E}=-\partial\vec{B}/\partial t && \text{(Faraday's Law)} \\ &\nabla\cdot\vec{B}=0 && \text{(Magnetic Gauss's Law)} \\ &\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\left(\partial\vec{E}/\partial t\right) && \text{(Ampere's Law)} \end{aligned} \tag{3.36}$$

where ρ and J are the free charge and the conduction current densities, respectively.

3.6 Wave Equation

First we shall investigate solutions to the Maxwell equations in regions devoid of source, namely in regions where $\vec{J}=0$ and $\rho=0$. This is of course does not mean that there is no source anywhere in all space. Sources must exist outside the regions of interest in order to produce fields in these regions. Thus in source-free regions in free space, the Maxwell equations become

$$\begin{aligned} &\nabla\cdot\vec{E}=0 \\ &\nabla\times\vec{E}=-\partial\vec{B}/\partial t \\ &\nabla\cdot\vec{B}=0 \\ &\nabla\times\vec{B}=\mu_0\varepsilon_0\left(\partial\vec{E}/\partial t\right) \end{aligned} \tag{3.37}$$

Use the vector identity of $\nabla\times\nabla\times\vec{E}=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$, we can get

$$\nabla\times\left(\nabla\times\vec{E}\right)=-\frac{\partial}{\partial t}\left(\nabla\times\vec{B}\right)=-\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E} \tag{3.38}$$

Noticing that $\nabla\left(\nabla\cdot\vec{E}\right)=0$. Substitute this formula into Eq. (3.37), we can get the Helmholtz equation, which can be written as

$$\boxed{\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} = 0} \tag{3.39}$$

Similarly, we can get an equation about $\vec{B}$.

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{B} = 0 \tag{3.40}$$

3.7 Wave Solution

Let's consider for simplicity an electromagnetic wave propagating in the +*x*-direction, with the electric field $\vec{E}$ pointing in the +*y*-direction and the magnetic field $\vec{B}$ in the +*z*-direction, as shown in Figure 3.9 below.

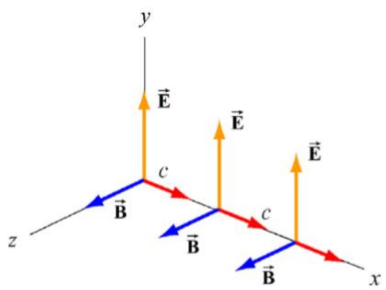

Figure 3.9 A plane electromagnetic wave

The results of Eq.(3.39) and Eq.(3.40) may be summarized as:

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) E_y(x,t) = 0 \\ \left(\frac{\partial^2}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) B_z(x,t) = 0 \end{aligned} \tag{3.41}$$

For $v = 1/\sqrt{\mu_0 \varepsilon_0}$, the general form of a one-dimensional wave equation is given by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) y(x,t) = 0 \tag{3.42}$$

d'Alembert's general solution

Let's define two variables $\xi = x - vt, \eta = x + vt$, from which we can derive the d'Alembert's general solution.

Using the Differential Rule of Compound Function to make the following deductions.

$$\begin{aligned}
&\frac{\partial y}{\partial x}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{\partial y}{\partial \xi}+\frac{\partial y}{\partial \eta}\\
&\frac{\partial^2 y}{\partial x^2}=\frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial \xi}+\frac{\partial y}{\partial \eta}\right)=\frac{\partial^2 y}{\partial \xi^2}+2\frac{\partial^2 y}{\partial \xi\partial \eta}+\frac{\partial^2 y}{\partial \eta^2}\\
&\frac{\partial y}{\partial t}=\frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial t}+\frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial t}=-v\frac{\partial y}{\partial \xi}+v\frac{\partial y}{\partial \eta}\\
&\frac{\partial^2 y}{\partial t^2}=\frac{\partial y}{\partial t}\left(\frac{\partial y}{\partial \xi}+\frac{\partial y}{\partial \eta}\right)=v^2\frac{\partial^2 y}{\partial \xi^2}-2v^2\frac{\partial^2 y}{\partial \xi\partial \eta}+v^2\frac{\partial^2 y}{\partial \eta^2}
\end{aligned} \tag{3.43}$$

From Eq.(3.42) and Eq.(3.43), we can conclude that $\partial^2 y/\partial\xi\partial\eta=0$. If the partial derivative with respect to ξ of $\partial y/\partial\eta$ is zero, then $\partial y/\partial\eta$ cannot depend on ξ. That means $\partial y/\partial\eta$ must be a function of η alone, so you can write $\partial y/\partial\eta=F(\eta)$. F represents the function of η how y changes with η. This equation can be integrated to give

$$\begin{aligned}
y&=\int F(\eta)\,d\eta+constant\\
&=f(\eta)+g(\xi)\\
&=f(x+vt)+g(x-vt)
\end{aligned} \tag{3.44}$$

This is the general solution to the classical one-dimensional wave equation, and it tells you that every wavefunction $y(x,t)$ that satisfies the wave equation can be interpreted as the sum of two waves propagating in opposite directions with the same speed. That's because $f(x+vt)$ represents a disturbance moving in the negative x-direction at speed v and $g(x-vt)$ represents a disturbance. moving in the positive x-direction also at speed v.

In some problems, initial boundary conditions are required to determine the value of wave solutions at any location and time

$$\begin{aligned}
&y(x,0)=f(x)+g(x)=I(x)\\
&\frac{\partial y(x,t)}{\partial t}\Big|_{t=0}=V(x)
\end{aligned} \tag{3.45}$$

and obtain the d'Alembert's general solution

$$y(x,t)=\frac{1}{2}I(x+vt)+\frac{1}{2}I(x-vt)+\frac{1}{2v}\int_{x-vt}^{x+vt}V(s)\,ds \tag{3.46}$$

Separation of Variables

In the case of the one-dimensional wave equation Eq. (3.42), the solution $y(x,t)$ is assumed to be the product of a function $X(x)$ that depends only on x and another function $T(t)$ that depends on t. Thus $y(x,t)$ = $X(x)T(t)$, and the classical wave equation becomes

$$\frac{\partial^2\left[X(x)T(t)\right]}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[X(x)T(t)\right] \tag{3.47}$$

But the time function $T(t)$ has no x-dependence and the spatial function $X(x)$ has no time dependence, so T comes out of the first derivative and X comes out of the second

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} \tag{3.48}$$

Notice that the left side depends only on x and the right side depends only on t. This means that both the left side and the right side of this equation must be constant. You can set that constant (called the "separation constant") equal to $-k^2$ and write

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} &= -k^2 X \\ \frac{\partial^2 T}{\partial t^2} &= -v^2 k^2 T \end{aligned} \tag{3.49}$$

Solution to this equation is

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) \\ T(t) &= C\cos(kvt) + D\sin(kvt) \end{aligned} \tag{3.50}$$

3.8 Plane Wave, Poynting Vector and Energy

So that one possible solution to the wave equations is

$$\begin{aligned} \vec{E} &= \hat{y}E_y(x,t) = \hat{y}E_0 \cos k(x - vt) = \hat{y}E_0 \cos(kx - \omega t) \\ \vec{B} &= \hat{z}B_z(x,t) = \hat{z}B_0 \cos k(x - vt) = \hat{z}B_0 \cos(kx - \omega t) \end{aligned} \tag{3.51}$$

The angular wave number k is related to the wavelength λ by

$$k = \frac{2\pi}{\lambda} \tag{3.52}$$

and the angular frequency ω (with the unit rad/s) is

$$\omega = kv = 2\pi\frac{v}{\lambda} = 2\pi f \tag{3.53}$$

where f is the linear frequency (with the unit 1/s or Hz). In empty space the wave propagates at the speed of light, $v = c = 2.9979 \times 10^8\ (\mathrm{m/s})$. The characteristic behavior of the sinusoidal electromagnetic wave is illustrated in Figure 3.10.

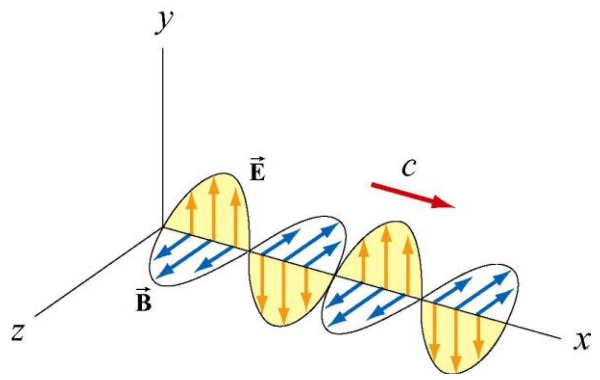

Figure 3.10 Plane electromagnetic wave propagating in the +*x* direction.

From $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$ and noting that $\vec{B}$ is in the +*z* direction, we have

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \tag{3.54}$$

Incorporating the first equation of Eq.(3.51), the solution of Eq.(3.54) is

$$kE_0 = \omega B_0 \tag{3.55}$$

or

$$\boxed{\frac{E_0}{B_0} = \frac{\omega}{k} = c} \tag{3.56}$$

We know that electric and magnetic fields store energy, the energy densities are

$$\begin{aligned} u_E &= \frac{1}{2}\varepsilon_0 E^2 \\ u_B &= \frac{1}{2\mu_0} B^2 \end{aligned} \tag{3.57}$$

Thus, energy can also be carried by the electromagnetic waves which consist of both fields. Consider a plane electromagnetic wave passing through a small volume element of area A and thickness dz, as shown in Figure 3.11.

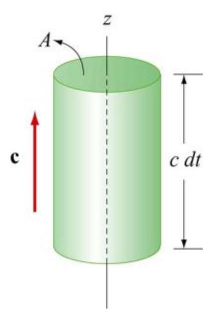

Figure 3.11 Electromagnetic wave passing through a volume element

The total energy in the volume element is given by

$$dU = \left(u_E + u_B\right) Adz = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) Acdt \tag{3.58}$$

Since the amount of time it takes for the wave to move through the volume element is $dt = dx / c$. Thus, one may obtain the rate of change of energy per unit area, denoted with the symbol *S*, as

$$S = \frac{1}{A}\frac{dU}{dt} = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \tag{3.59}$$

The SI unit of S is W / m^2 or $J / \left(m^2 s\right)$. Noting that $E = cB$ and $c = 1 / \sqrt{\mu_0 \varepsilon_0}$, the above expression may be rewritten as

$$S = \frac{c}{2}\left(\varepsilon_0 cEB + \frac{EB}{c\mu_0}\right) = \frac{EB}{2\mu_0}\left(\mu_0 \varepsilon_0 c^2 + 1\right) = \frac{EB}{\mu_0} \tag{3.60}$$

In general, the rate of the energy flow per unit area may be described by the Poynting vector $\vec{S}$ (after the British physicist John Poynting), which is defined as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \tag{3.61}$$

with $\vec{S}$ pointing in the direction of propagation.

The intensity of the wave, I , defined as the time average of S , is given by

$$I \equiv \langle S \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0} \tag{3.62}$$

Example: Solar Constant

At the upper surface of the Earth's atmosphere, the time-averaged magnitude of the Poynting vector, $\langle S \rangle = 1.35 \times 10^3$ W/m^2, is referred to as the solar constant.

(a) Assuming that the Sun's electromagnetic radiation is a plane sinusoidal wave, what are the magnitudes of the electric and magnetic fields?

(b) What is the total time-averaged power radiated by the Sun? The mean Sun-Earth distance is R $= 1.50 \times 10^{11}$ m

Solution:

The time-averaged Poynting vector is related to the amplitude of the electric field by

$$\langle S \rangle = \frac{c}{2}\varepsilon_0 E_0^2 \tag{3.63}$$

Thus, the amplitude of the electric field is

$$E_0 = \sqrt{\frac{2\langle S \rangle}{c\varepsilon_0}} = \sqrt{\frac{2\left(1.35\times10^3\,W/m^2\right)}{\left(3\times10^8\,m/s\right)\left(8.85\times10^{-12}\,C^2/N\cdot m^2\right)}} = 1.01\times10^3\,V/m \tag{3.64}$$

The corresponding amplitude of the magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1.01\times10^3\,V/m}{3\times10^8\,m/s} = 3.4\times10^{-6}T \tag{3.65}$$

Note that the associated magnetic field is less than one-tenth the Earth’s magnetic field.

(b) The total time averaged power radiated by the Sun at the distance *R* is

$$\begin{aligned}\langle P \rangle &= \langle S \rangle A = \langle S \rangle 4\pi R^2 \\ &= \left(1.35\times10^3\,W/m^2\right)4\pi\left(1.5\times10^{11}\,m\right)^2 = 3.8\times10^{26}W\end{aligned} \tag{3.66}$$

For a spherical wave, the intensity at a distance r from the source is

$$I = \langle S \rangle = \frac{\langle P \rangle}{4\pi R^2} \tag{3.67}$$

which decreases as $1/r^2$ .On the other hand, the intensity of a plane wave (Figure 3.12b) remains constant and there is no spreading in its energy.

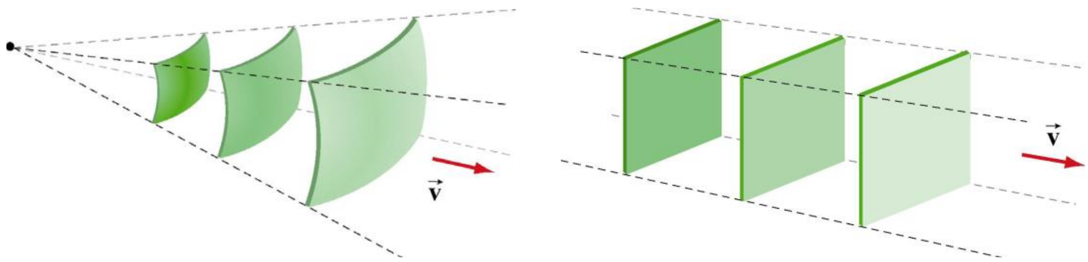

Figure 3.12 (a) a spherical wave, and (b) plane wave.

2.1 Standing wave

Let us examine the situation where there are two sinusoidal plane electromagnetic waves, one traveling in the +*x*-direction, with

$$\begin{aligned}E_{1y}(x,t) &= E_0\cos\left(k_1x-\omega_1t\right) \\ B_{1z}(x,t) &= B_0\cos\left(k_1x-\omega_1t\right)\end{aligned} \tag{3.68}$$

and the other traveling in the –*x* direction, with

$$\begin{aligned} E_{2y}(x,t) &= -E_0\cos(k_2x+\omega_2 t) \\ B_{2z}(x,t) &= B_0\cos(k_2x+\omega_2 t) \end{aligned} \tag{3.69}$$

For simplicity, we assume that these electromagnetic waves have the same wavelengths ($k_1 = k_2 = k,\ \omega_1 = \omega_2 = \omega$). Using the superposition principle, the electric field and the magnetic fields can be written as

$$E_y(x,t) = E_{1y}(x,t) + E_{2y}(x,t) = E_0\left[\cos(kx-\omega t) - \cos(kx+\omega t)\right] \tag{3.70}$$

and

$$B_z(x,t) = B_{1z}(x,t) + B_{2z}(x,t) = B_0\left[\cos(kx-\omega t) + \cos(kx+\omega t)\right] \tag{3.71}$$

Using the identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{3.72}$$

The above expressions may be rewritten as

$$E_y(x,t) = E_{1y}(x,t) + E_{2y}(x,t) = 2E_0\sin kx\sin\omega t \tag{3.73}$$

and

$$B_z(x,t) = B_{1z}(x,t) + B_{2z}(x,t) = 2B_0\cos kx\cos\omega t \tag{3.74}$$

The waves described by Eqs.(3.73) and (3.74) are *standing waves*, which do not propagate but simply oscillate in space and time.

Eq.(3.73) shows that the total electric field remains zero at all times if $\sin kx = 0$, or

$$x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2}, n = 0,1,2\cdots \tag{3.75}$$

The planes that contain these points are called the *nodal planes* of the electric field. On the other hand, when $\sin kx = \pm 1$, or

$$x = \left(n+\frac{1}{2}\right)\frac{\pi}{k} = \left(\frac{n}{2}+\frac{1}{4}\right)\lambda, n = 0,1,2\cdots \tag{3.76}$$

the amplitude of the field is at its maximum $2E_0$. The planes that contain these points are the *anti-nodal planes* of the electric field.

For the magnetic field, using the same method, we can get the *nodal planes*

$$x = \left(\frac{n}{2} + \frac{1}{4}\right)\lambda, n = 0,1,2\cdots \tag{3.77}$$

the *anti-nodal planes*

$$x = \frac{n\lambda}{2}, n = 0,1,2\cdots \tag{3.78}$$

The Poynting vector for the standing wave is

$$\begin{aligned}\vec{S} &= \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(\hat{y}2E_0 \sin kx \sin \omega t\right)\times\left(\hat{z}2B_0 \cos kx \cos \omega t\right) \\ &= \frac{E_0 B_0}{\mu_0}\sin 2kx \sin 2\omega t\end{aligned} \tag{3.79}$$

The time average of S is

$$I = \left\langle \vec{S} \right\rangle = \frac{E_0 B_0}{\mu_0}\sin 2kx\left\langle \sin 2\omega t\right\rangle = 0 \tag{3.80}$$

The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions to form the standing wave exactly cancel each other, with no net energy transfer.

3.9 Phase Velocity and Group Velocity

To learn phase velocity and group velocity, let's start with wave packet. A wave packet is the sum of a group of plane waves in a small area of space, as shown in Figure 3.11.

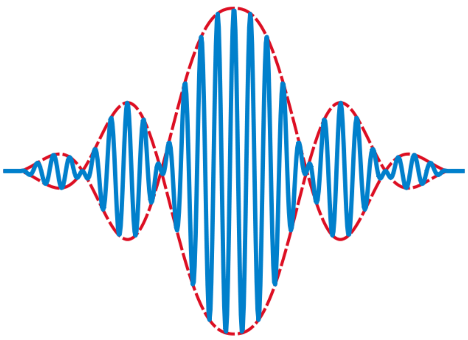

Figure 3.11 wave packet

Phase velocity is the displacement velocity of a single frequency in a wave packet. And group velocity is the moving speed of wave packet.

For example, plane waves propagating in the $+x$-direction can be expressed as

$$E = E_0 \cos\left(kx - \omega t\right) \tag{3.81}$$

the phase velocity is

$$\boxed{v_p = \frac{\omega}{k}} \tag{3.82}$$

Assuming the expressions of two plane waves are

$$\begin{aligned} E_1 &= E_0 \cos\left(k_1 x - \omega_1 t\right) \\ E_2 &= E_0 \cos\left(k_2 x - \omega_2 t\right) \end{aligned} \tag{3.83}$$

Superposition of these two plane waves to get a new one

$$E = E_1 + E_2 = 2E_0 \cos\left(\frac{\omega_1 - \omega_2}{2} t - \frac{k_1 - k_2}{2} x\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t - \frac{k_1 + k_2}{2} x\right) \tag{3.84}$$

Since $\omega_1 - \omega_2 \ll \omega_1 \, or \, \omega_2$ and $k_1 - k_2 \ll k_1 \, or \, k_2$, $\cos\left[0.5\left(\omega_1 - \omega_2\right)t - 0.5\left(k_1 - k_2\right)x\right]$ changes very slowly, it represents the wave packet.

So that the group velocity is

$$\boxed{v_g = \frac{\Delta x}{\Delta t} = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\partial \omega}{\partial k}} \tag{3.85}$$

3.10 Polarization

The polarization of a wave is conventionally defined by the time variation of the tip of the electric field vector $\vec{E}$ at a fixed point in space. If the tip moves along a straight line, the wave is linearly polarized. When the locus of the tip is circle, the wave is circularly polarized. For an elliptically polarized wave, the tip of $\vec{E}$ describes an ellipse. If the right-hand thumb points in the direction of propagation while the fingers point in the direction of the tip motion, the wave is defined as right-handed polarized. The wave is left-handed polarized when it is described by the left-hand thumb and fingers.

Consider the following wave solution

$$\vec{E}\left(z,t\right) = \hat{x}E_x + \hat{y}E_y = \hat{x}\cos\left(kz - \omega t\right) + \hat{y}A\cos\left(kz - \omega t + \varphi\right) \tag{3.86}$$

with $A > 0$. The wave propagates in the +z direction. From the temporal view point,

$$\vec{E}\left(t\right) = \hat{x}\cos\left(\omega t\right) + \hat{y}A\cos\left(\omega t - \varphi\right) \tag{3.87}$$

We now study polarization for the following special cases:

(1) $\varphi = 2m\pi$, where m = 0, 1, 2, … is an integer. We have

$$\vec{E}\left(t\right) = \hat{x}\cos\left(\omega t\right) + \hat{y}A\cos\left(\omega t\right) \tag{3.88}$$

The tip of the electric field vector moves along a line as shown in Figure 3.12a. The wave is linearly polarized.

(2) $\varphi = (2m+1)\pi$, where m = 0, 1, 2, … is an integer. We have

$$\vec{E}(t) = \hat{x}\cos(\omega t) - \hat{y}A\cos(\omega t) \tag{3.89}$$

The tip of the electric field vector moves along a line as shown in Figure 3.12b. The wave is linearly polarized.

(3) $\varphi = \pi/2$, and A = 1. We have

$$\vec{E}(t) = \hat{x}\cos(\omega t) + \hat{y}\sin(\omega t) \tag{3.90}$$

It can be seen that while the x component is at its maximum the y component is zero. As time progresses, the y component increases and the x component decreases. The tip of $\vec{E}$ rotates from the positive E_x axis to the positive E_y axis (Figure 3.12c). Besides, $E_x^2 + E_y^2 = 1$. Thus the wave is right-hand circularly polarized.

(4) $\varphi = -\pi/2$, and A = 1. We have

$$\vec{E}(t) = \hat{x}\cos(\omega t) - \hat{y}\sin(\omega t) \tag{3.91}$$

As time progresses, the y component increases and the x component decreases. The tip of $\vec{E}$ rotates from the positive E_x axis to the negative E_y axis (Figure 3.12d). Besides, $E_x^2 + E_y^2 = 1$. Thus the wave is left-hand circularly polarized.

(5) $\varphi = \pm\pi/2$. We have

$$\vec{E}(t) = \hat{x}\cos(\omega t) \pm \hat{y}A\sin(\omega t) \tag{3.92}$$

The wave is right-hand elliptically polarized for $\varphi = \pi/2$ and left-hand elliptically polarized for $\varphi = -\pi/2$, as shown in Figures 3.12e and 3.12f.

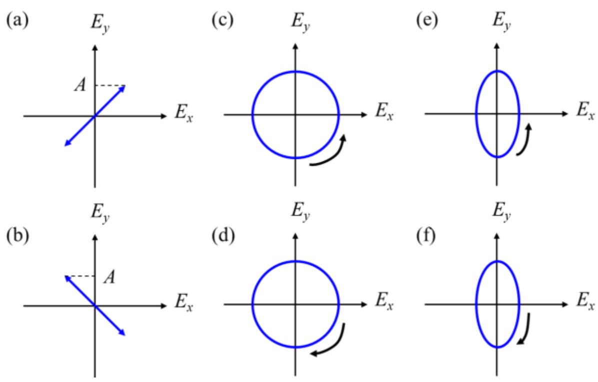

Figure 3.12 Polarizations. (a, b) Linear polarization. (c) right-hand and (d) left-hand circular polarization. (e) right-hand and (f) left-hand elliptical polarization.

3.11 Electromagnetic Spectrum

The electromagnetic spectrum covers electromagnetic waves with frequencies ranging from below 1 Hz to above 10^{25} Hz, corresponding to wavelengths from thousands of kilometers down to a fraction of the size of an atomic nucleus. This frequency range is divided into separate bands, and the electromagnetic waves within each frequency band are called by different names; beginning at the low frequency (long wavelength) end of the spectrum these are: radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays at the high-frequency (short wavelength) end. The electromagnetic waves in each of these bands have different characteristics, such as how they are produced, how they interact with matter, and their practical applications.

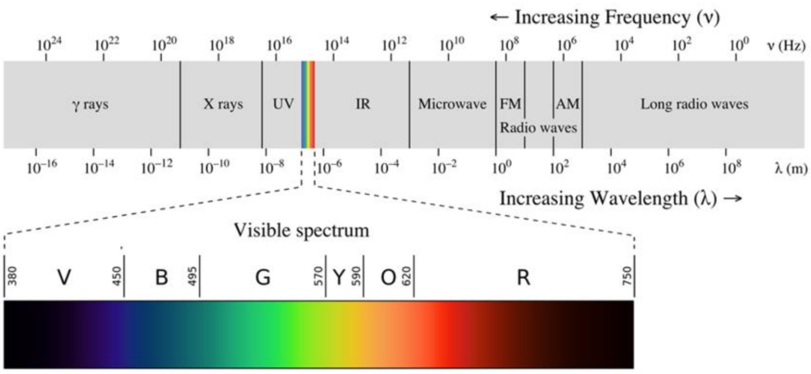

Figure 3.13 Electromagnetic Spectrum

3.12 Additional Problems

See Reference [1].

Reference:

[1] MIT Physics 8.02 : <<Electricity & Magnetism>>, by Sen-ben Liao, Peter Dourmashkin, and John W. Belcher.

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