Course Review (I)

Outline

- L1. Electrostatics
- L2. Magnetostatics and Faraday's Law
- L3. Maxwell's Equations and Electromagnetic Waves
- L4. Boundary conditions
- L5. Waves in Media

L1. Electrostatics

Electrostatics

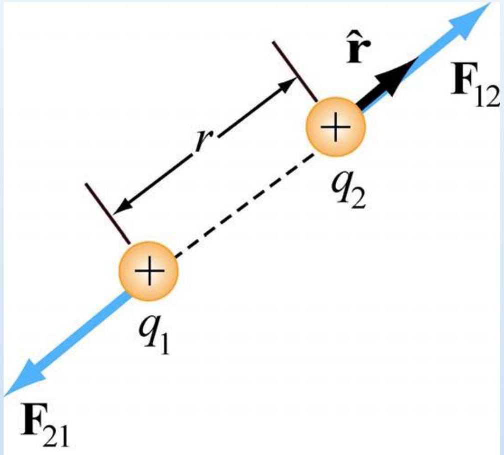
Coulomb's law:

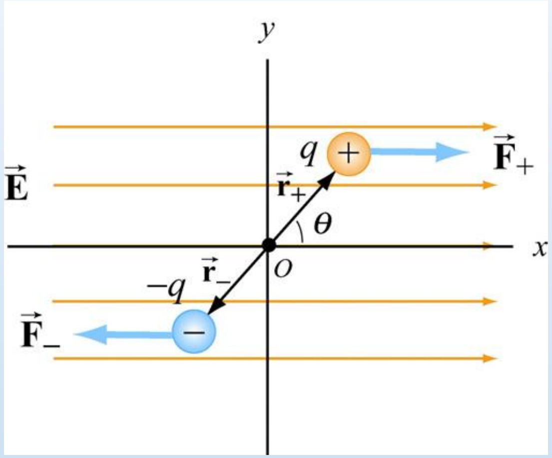
$$\overrightarrow{F}_{12} = k_e \frac{q_1 q_2}{r^2} \hat{r}$$

(Force by q_1 on q_2)

$$\vec{\tau} = \overrightarrow{p}_e \times \overrightarrow{E}$$

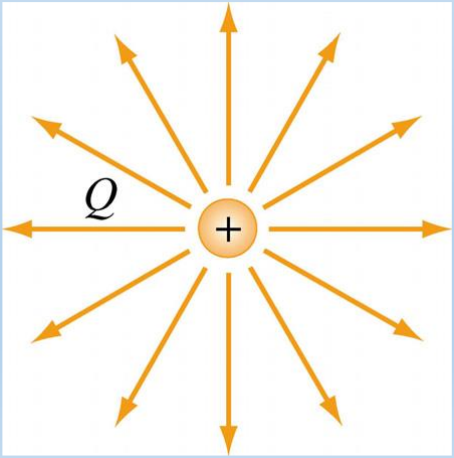
(Torque on electric dipole)



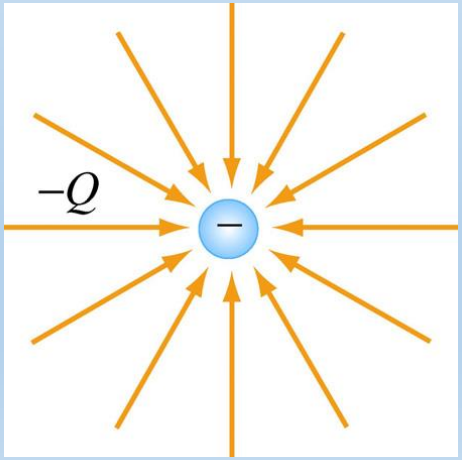


Electric Dipole Moment $\boxed{\overrightarrow{p}_e = q\vec{l}}$

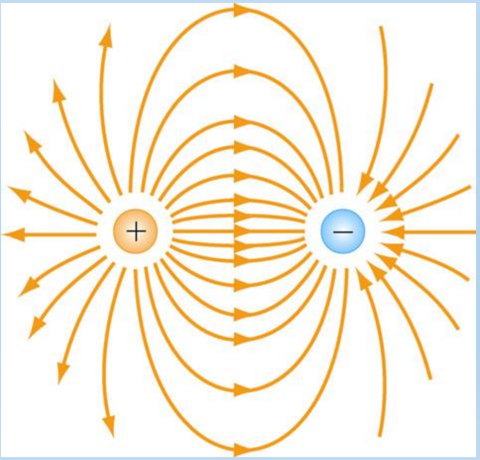
Electric Field Lines:



Positive Charge

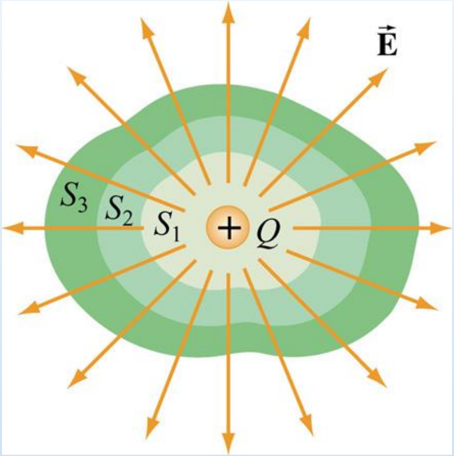


Negative Charge



Electric Dipole

Gauss's law



(Electric Flux)

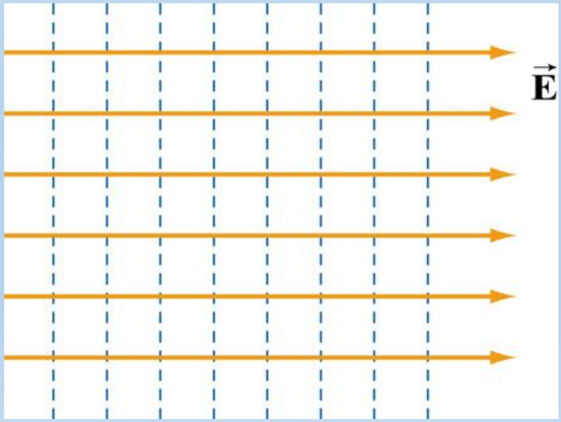
$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

(Gauss's law)

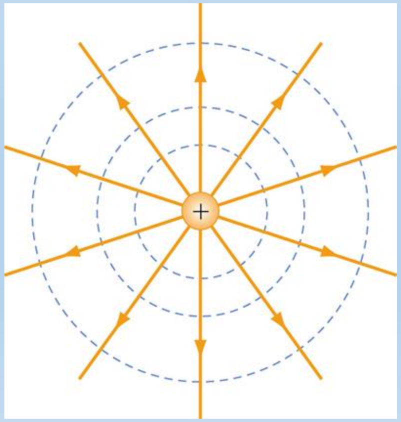
Electric Potential (φ or V):

$$\vec{E} = -\nabla \varphi$$

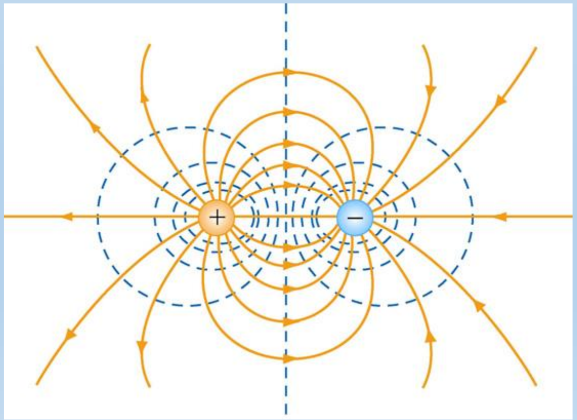
$$\Delta\varphi = \varphi_B - \varphi_A = -\int_A^B \vec{E} \cdot d\vec{s}$$



Constant E field



Positive Charge

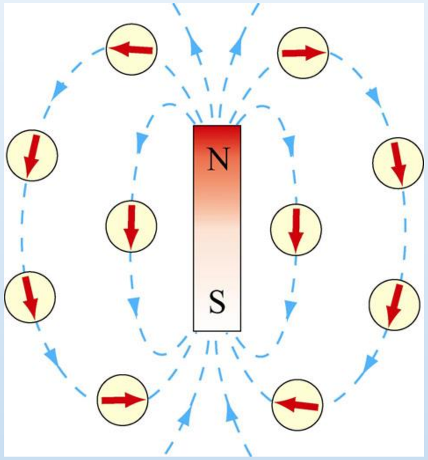


Electric Dipole

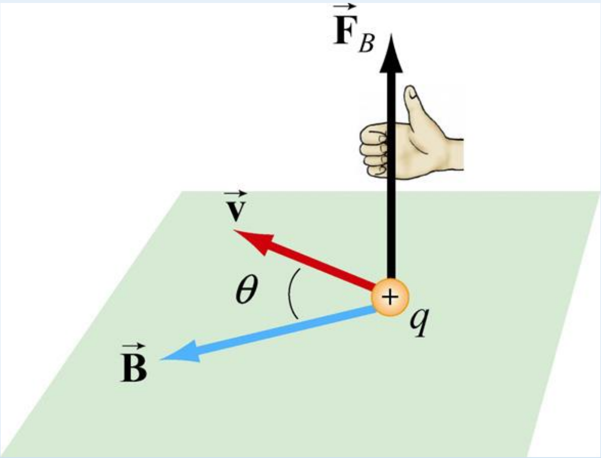
L2. Magnetostatics and Faraday's Law

Magnetostatics

Magnetic Field

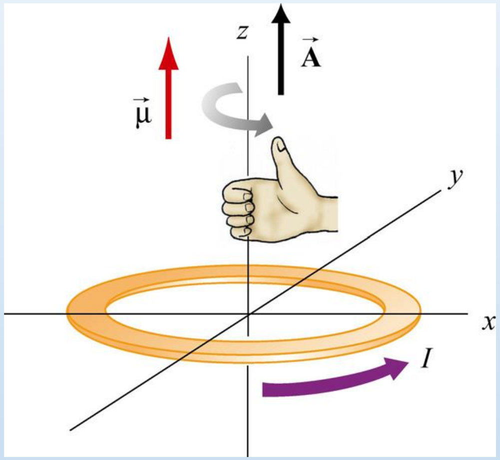


Magnetic Force

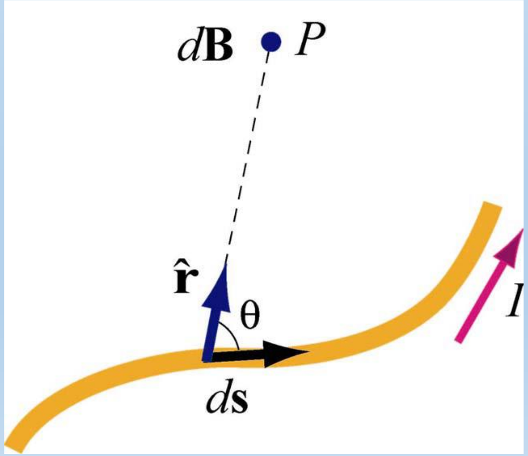


$$\overrightarrow{F}_B = q\vec{v} \times \overrightarrow{B}$$

Magnetic Dipole



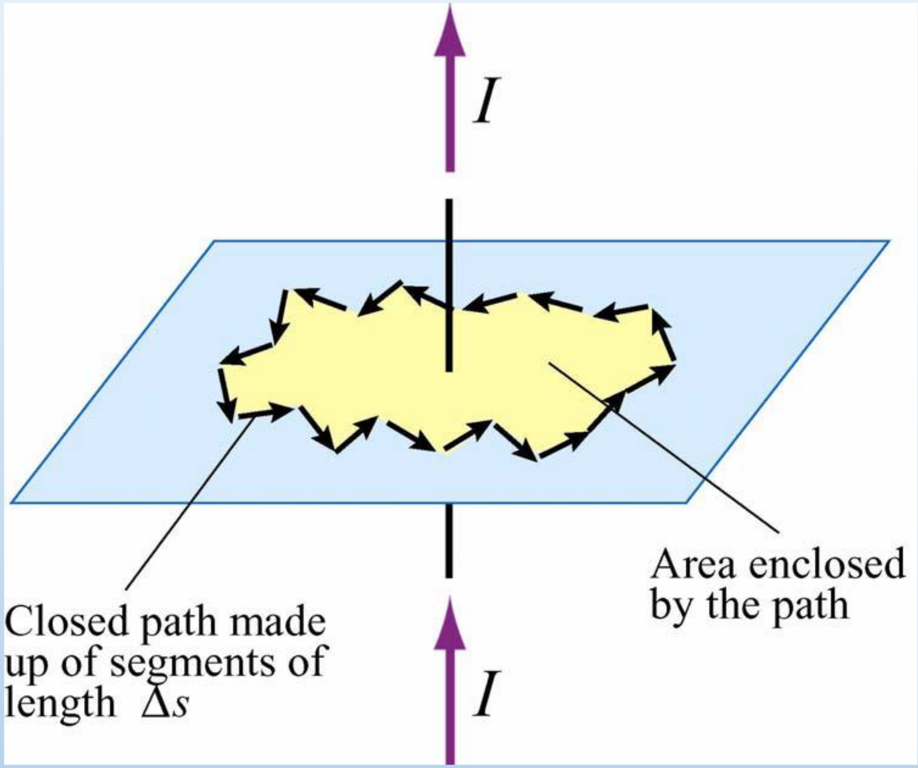
$$\vec{p}_m = I\vec{A} \qquad \vec{\tau} = \vec{p}_m \times \vec{B}$$



The Biot-Savart Law

$$d\overrightarrow{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \vec{r}}{r^2}$$

Ampere’s Law



$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

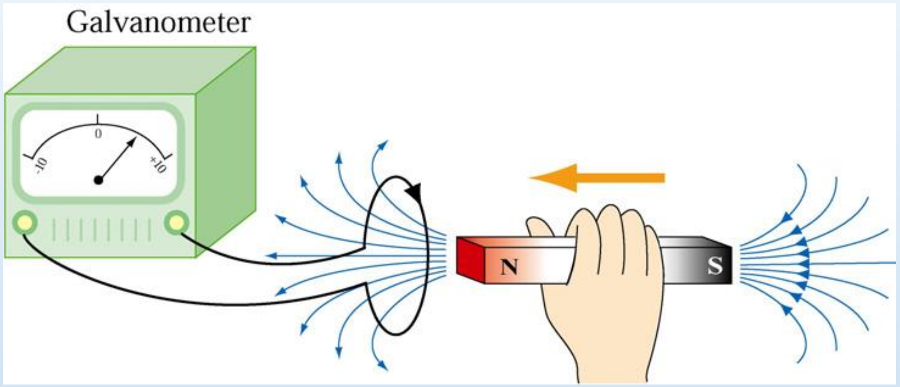
(Ampere’s Law)

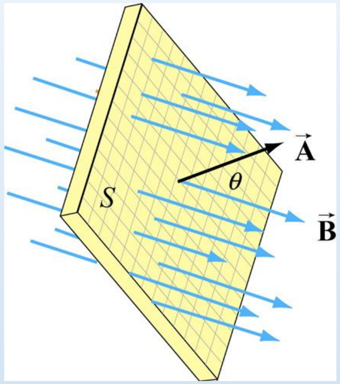
The line integral is around any closed contour bounding an open surface S.

I_{enc} is current through S:

$$I_{enc} = \oiint_S \vec{J} \cdot d\vec{A}$$

Faraday's Law



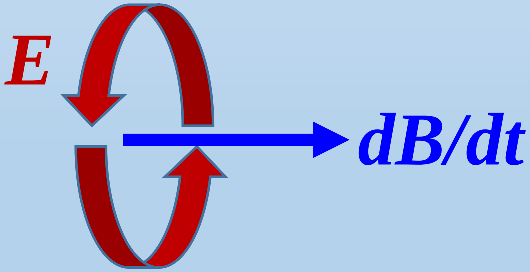


Magnetic Flux

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A}$$

Electromotive force: $$emf = -\frac{d\Phi_B}{dt}$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ **(Faraday's Law)**

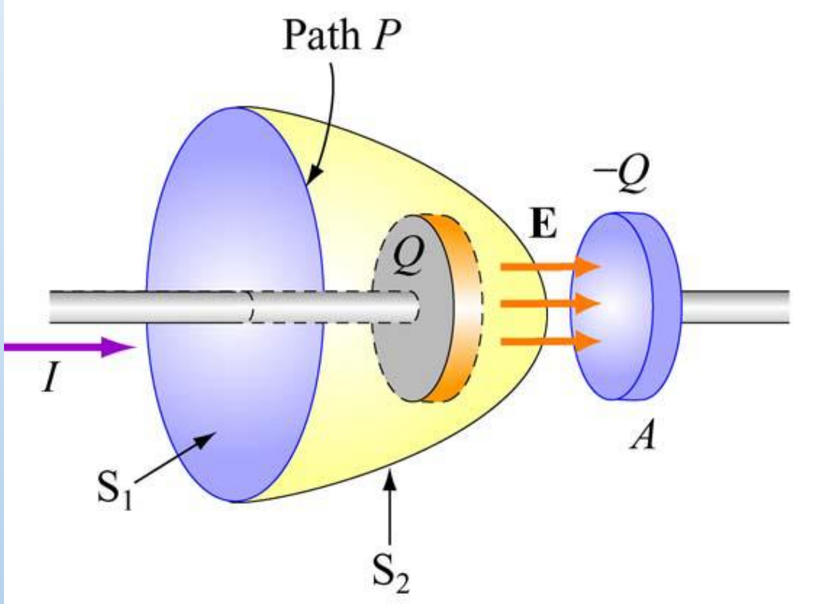


L3. Maxwell's Equations and Electromagnetic Waves

Ampere’s Law in Capacitor

Consider a charging capacitor:

Use Ampere’s Law to calculate the magnetic field



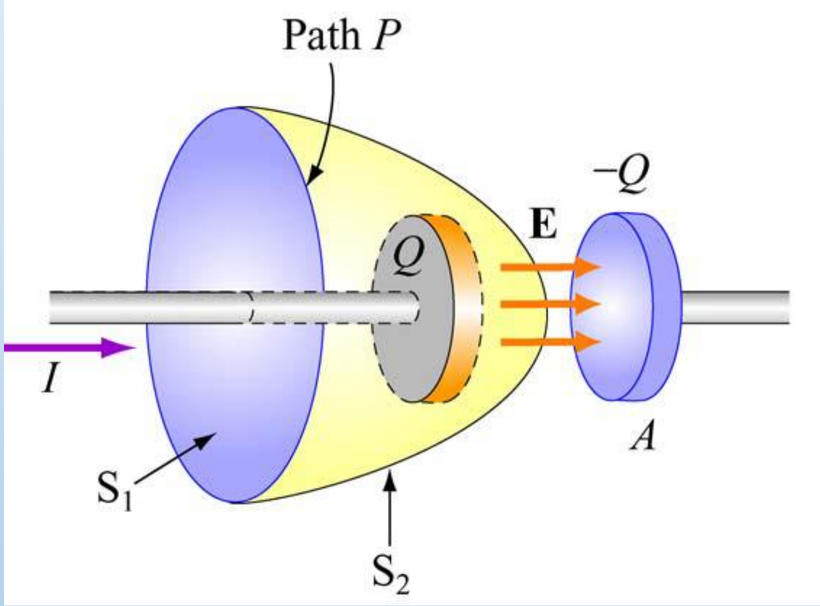
$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

1) Blue Amperian Area, $I_{enc} = I$
2) Yellow Amperian Area, $I_{enc} = 0$

What’s Going On?

Displacement Current

We don't have current between the capacitor plates but we do have a changing E field. Can we "make" a current out of that?



$$E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 EA = \varepsilon_0 \Phi_E$$

$$\boxed{\frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} = I_d}$$

This is called (for historic reasons) the **Displacement Current**

Maxwell-Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + I_d \right)$$

$$= \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

James Clerk Maxwell FRS FRSE (13 June 1831 – 5 November 1879) was a Scottish scientist in the field of mathematical physics. His most notable achievement was to formulate the classical theory of electromagnetic radiation, bringing together for the first time electricity, magnetism, and light as different manifestations of the same phenomenon.

Maxwell's Equations in Vacuum

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{in}}{\varepsilon_0}$$ (Gauss's Law)

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ (Faraday's Law)

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$ (Magnetic Gauss's Law)

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$ (Ampere's Law)

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$ (Lorentz force Law)

Prediction of Electromagnetic Waves

In vacuum and no source $Q_{in} = 0$ $I_{enc} = 0$

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oiint_S \vec{E} \cdot d\vec{S} = 0$$

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

Displacement current implies the possibility of electromagnetic waves

Differential Maxwell's Equations in Vacuum

Integral Form | **Differential Form**

$$\oiint_S \vec{E}\cdot d\vec{S} = \frac{Q_{in}}{\varepsilon_0} \Rightarrow \iiint_V \left(\nabla\cdot\vec{E}\right)dV = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV \Rightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \Rightarrow \iint_S \left(\nabla\times\vec{E}\right)\cdot d\vec{S} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S} \Rightarrow \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\oiint_S \vec{B}\cdot d\vec{S} = 0 \Rightarrow \iiint_V \left(\nabla\cdot\vec{B}\right)dV = 0 \Rightarrow \nabla\cdot\vec{B} = 0$$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d\Phi_E}{dt}$$

$$\iint_S \left(\nabla\times\vec{B}\right)\cdot d\vec{S} = \mu_0\iint_S\left(\vec{J} + \varepsilon_0\frac{d\vec{E}}{dt}\right)\cdot d\vec{S}$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Wave Equation

Source-free case in vacuum

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \left(\nabla \times \vec{E} \right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B} \right) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

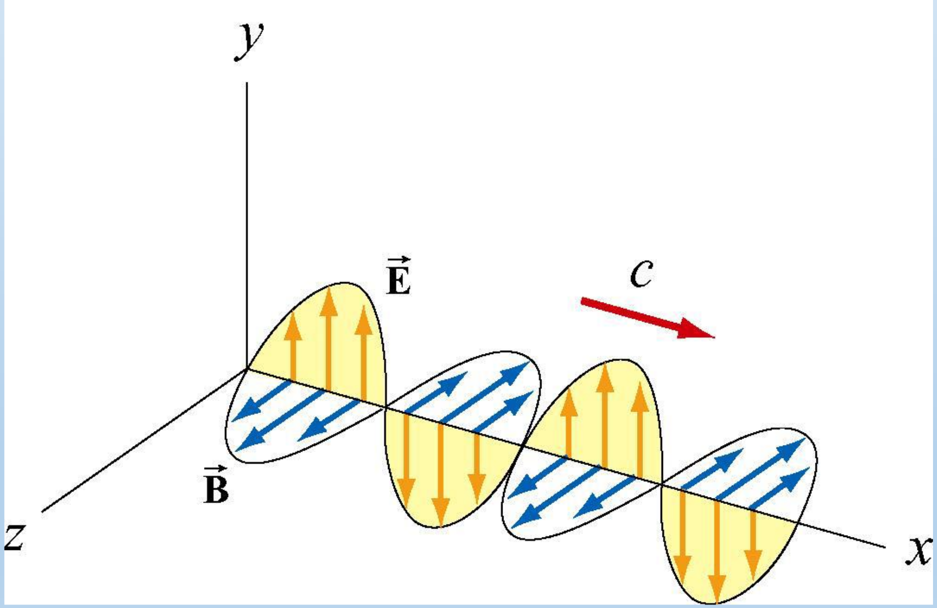
$$\boxed{\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} = 0}$$ **(Helmholtz Wave Equation)**

$$\nabla \times \left(\nabla \times \vec{A} \right) = \nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}$$

Sinusoidal EM Plane Wave

$$\vec{E} = \hat{y}E_y\left(x,t\right) = \hat{y}E_0 \cos k\left(x - vt\right) = \hat{y}E_0 \cos\left(kx - \omega t\right)$$

$$\vec{B} = \hat{z}B_z\left(x,t\right) = \hat{z}B_0 \cos k\left(x - vt\right) = \hat{z}B_0 \cos\left(kx - \omega t\right)$$



Traveling Sine Wave

- **Propagation direction:**

$$\vec{E} \times \vec{B}$$

- **The E and H fields are perpendicular to each other.**

$$\vec{E} \cdot \vec{B} = 0$$

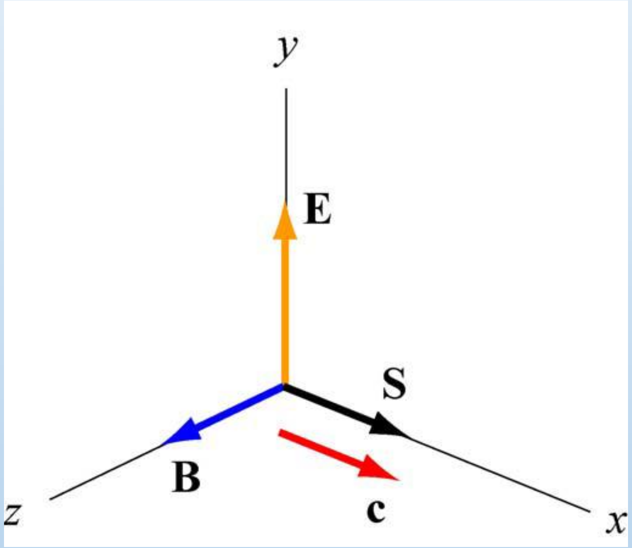
Properties of EM Waves

- **Wavelength (λ)**: Wavelength is the amount of distance per cycle. [m]
- **Period (T)**: Period is the amount of time per cycle. [s]
- **Frequency (f)**: Frequency is the number of cycles per amount of time. [1/s] or [Hz]
- **Angular frequency (ω)**: The angular frequency tells you how much angle the phase of the wave advances in a given amount of time. [rad/s]
- **Wavenumber (k)**: The wavenumber tells you how much the phase of the wave advances in a given amount of distance.
- **Wave speed (v)**: $v = 2.9979 \times 10^8$ [m/s]

$$k = \frac{2\pi}{\lambda} \qquad \omega = kv = 2\pi \frac{v}{\lambda} = 2\pi f$$

Poynting Vector and Intensity

Direction of energy flow = direction of wave propagation



$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \quad \textbf{(Poynting vector)}$$

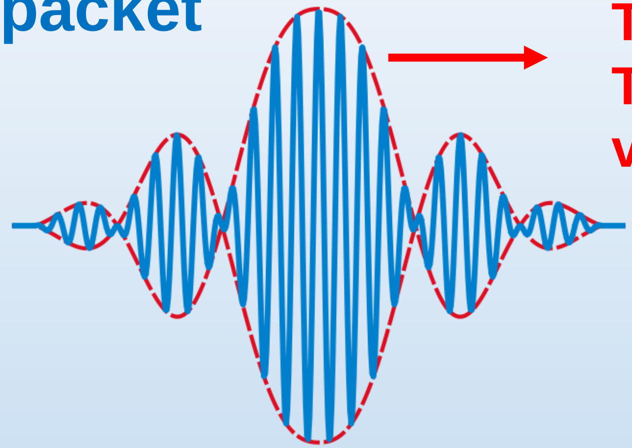
units: Joules per square meter per sec. [J/(m^{2}s)] or [W/m^2]

Intensity *I*:

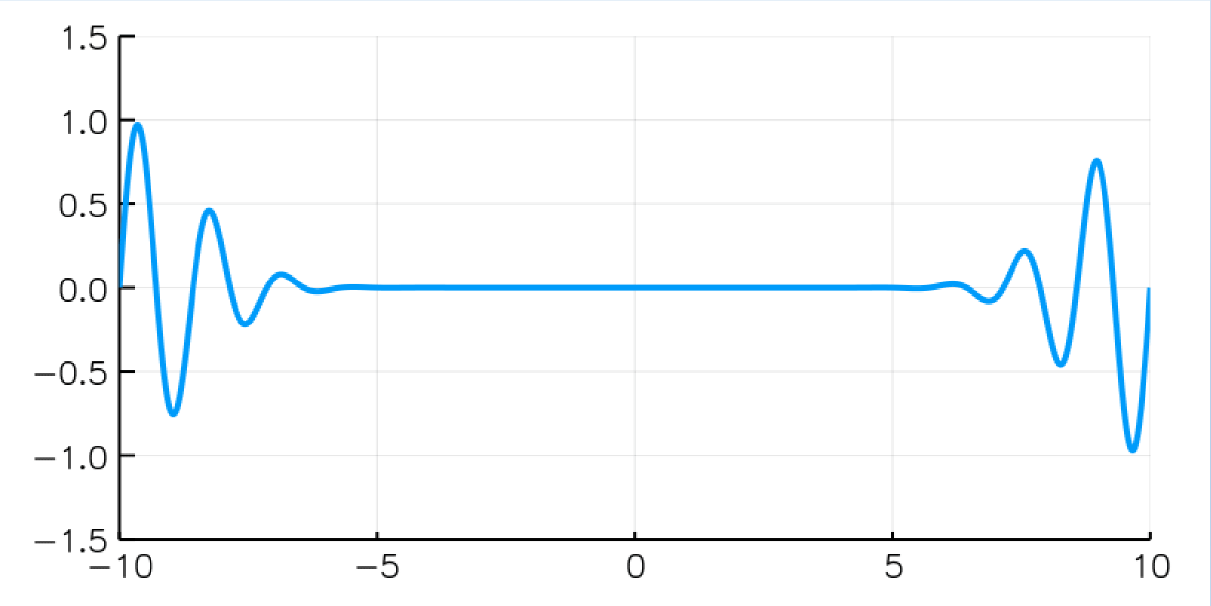
$$I \equiv \langle S \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0}$$

Wavepacket

A wave packet



The *envelope* of the wave packet. The envelope moves at the group velocity.



Phase Velocity vs. Group Velocity

$$E = E_1 + E_2 = \underbrace{E_0 \cos\left(k_1 x - \omega_1 t\right)}_{\text{Wave \#1}} + \underbrace{E_0 \cos\left(k_2 x - \omega_2 t\right)}_{\text{Wave \#2}}$$

$$\Delta\phi = \left(k_2 x - \omega_2 t\right) - \left(k_1 x - \omega_1 t\right) = \left(k_2 - k_1\right) x - \left(\omega_2 - \omega_1\right) t$$

$$\Downarrow$$

$$\left(k_2 - k_1\right) \Delta x = \left(\omega_2 - \omega_1\right) \Delta t$$

$$\Downarrow$$

$$\boxed{v_g = \frac{\Delta x}{\Delta x} = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{\partial \omega}{\partial k}}$$

(Group velocity)

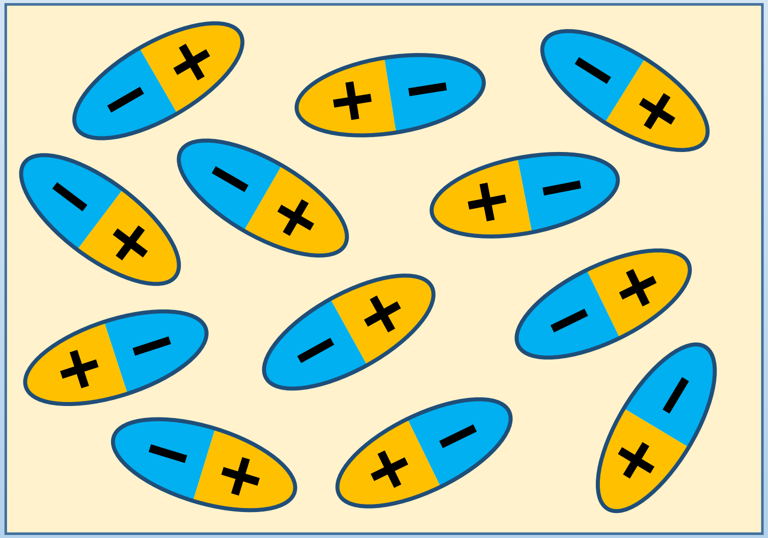
$$\boxed{v_p = \frac{\omega}{k}}$$

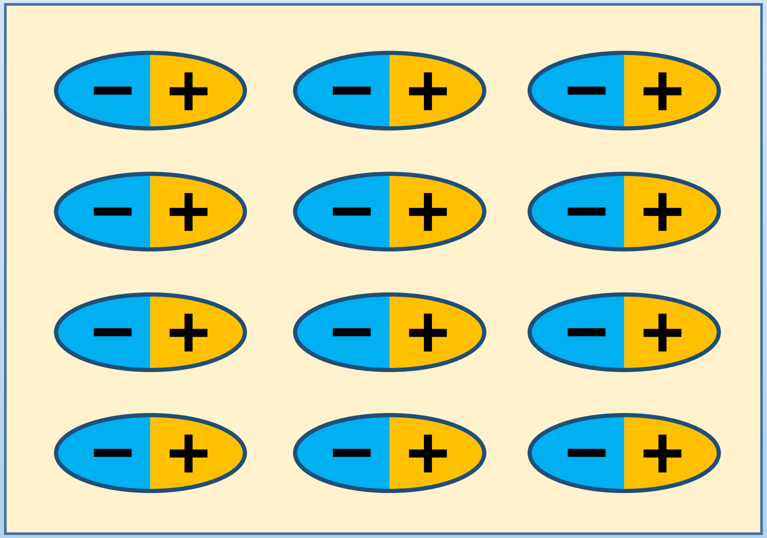
(Phase velocity)

L4. Boundary conditions

Molecular View of Dielectrics

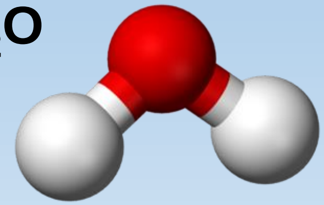
Polar Dielectrics:
Dielectrics permanent electric dipole moments with electric dipole moments.
Example: Water





$\vec{E}$

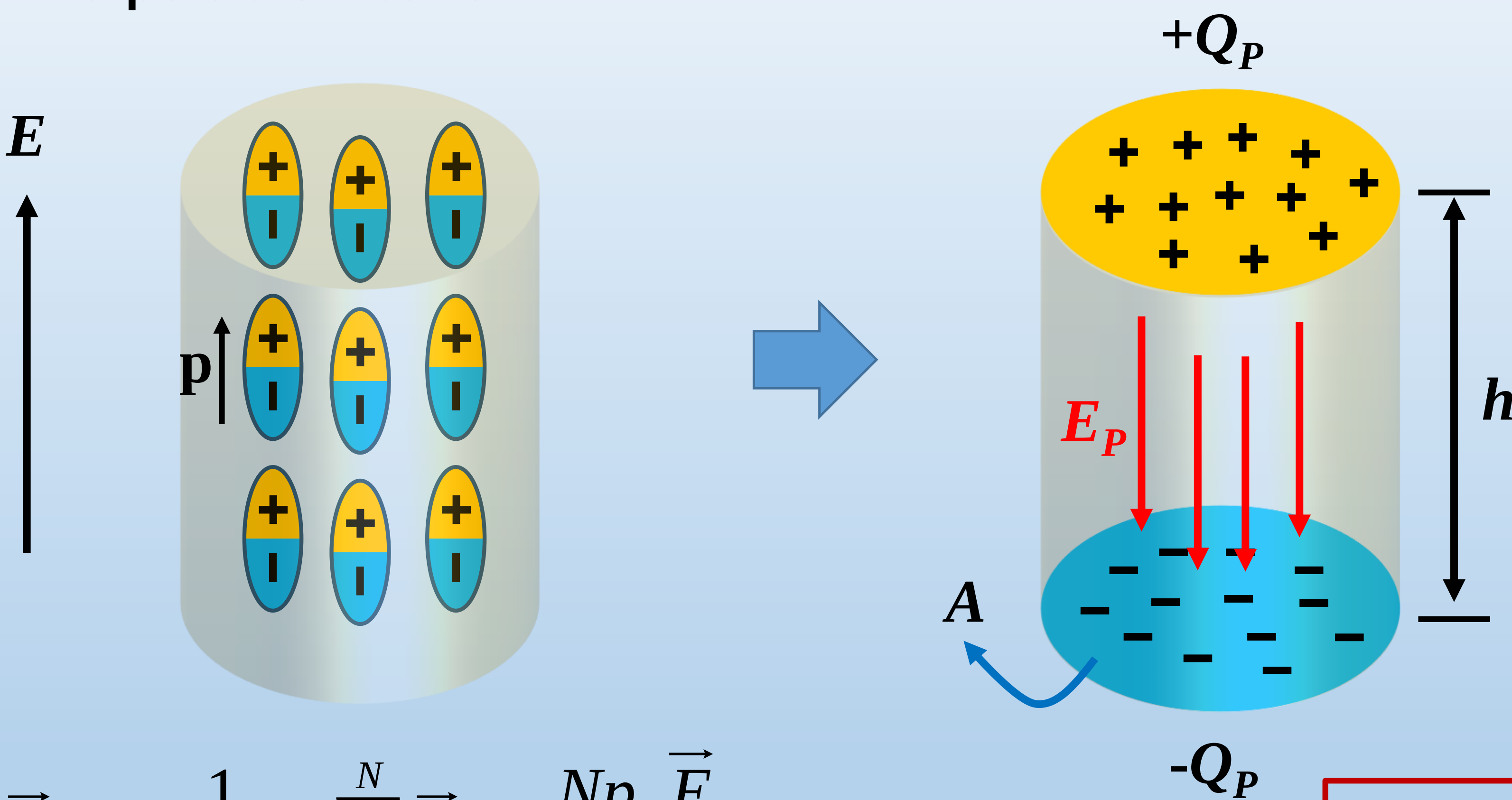
H_2O



Polarization

A cylinder with uniform dipole distribution

Equivalent charge distribution



$$\vec{P} = \frac{1}{volume}\sum_{i=1}^{N}\vec{p}_i = \frac{Np}{Ah}\frac{\vec{E}}{|E|}$$

(polarization density)

$$Q_P = \frac{Np}{h}$$

$$\boxed{\vec{E}_P = -\frac{\vec{P}}{\varepsilon_0}}$$

Displacement Fields

$$\oiint_S \vec{D}\cdot d\vec{S} = \oiint_S \left(\varepsilon_0 \vec{E} + \vec{P}\right)\cdot d\vec{S} = \iiint_V \rho_{free} dV$$

The electric displacement field is defined as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Displacement field D accounts for the effects of unbound ("free") charges within materials.

$$\oiint_S \varepsilon_0 \vec{E}\cdot d\vec{S} = \iiint_V \rho_{total} dV$$

Electric field E accounts for the effects of total charges (both "bound" and "free") within materials.

Integral Form

$$\oiint_S \vec{D} \cdot d\vec{S} = \oiint_S \left(\varepsilon_0 \vec{E} + \vec{P} \right) \cdot d\vec{S} = \iiint_V \rho_{free} dV$$

Divergence theorem

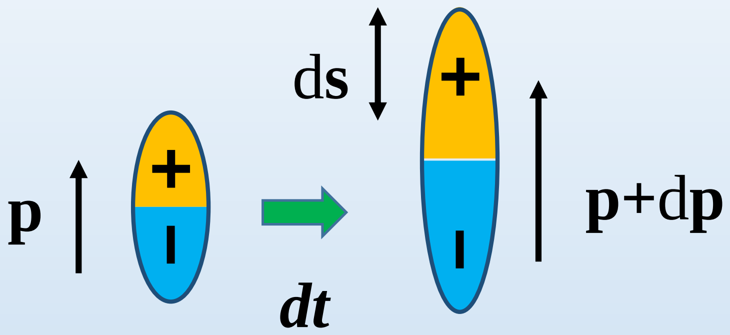
$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_V \left(\nabla \cdot \vec{F} \right) dV$$

$$\boxed{\nabla \cdot \vec{D} = \rho_{free}}$$ **(Gauss's Law)**

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_{total}$$

$$\nabla \cdot \vec{P} = -\rho_{bound}$$

Polarization in changing fields



$$\vec{J}_{bound} = \rho\vec{v} = Nq\frac{d\vec{s}}{dt} = N\frac{d\vec{p}}{dt} \;\Rightarrow\; \boxed{\vec{J}_{bound} = \frac{d\vec{P}}{dt}}$$

$$\nabla\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\frac{d\vec{P}}{dt} + \mu_0\vec{J}_{free}$$ **(Ampere's Law)**

Bound-charge current density

Free-charge current density

Ampere's Law for Dielectrics

Vacuum displacement current density

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{d\vec{P}}{dt} + \mu_0 \vec{J}_{free}$$

Bound-charge current density

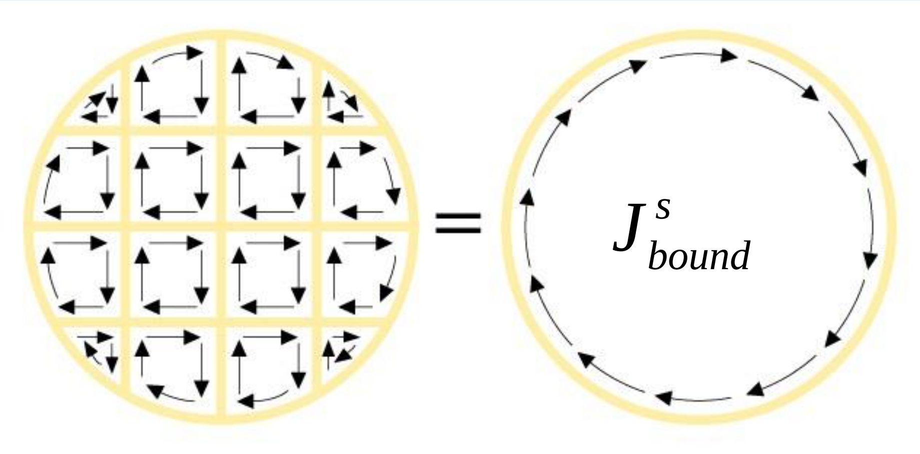
Free-charge current density

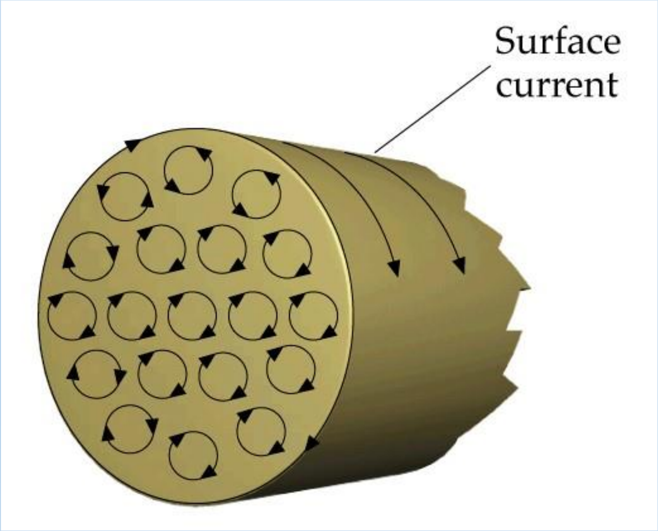
$$\boxed{\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} + \mu_0 \vec{J}_{free}}$$

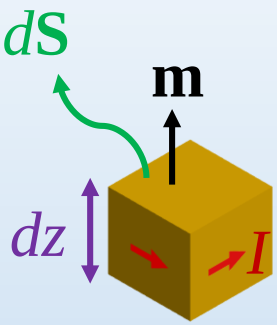
(Ampere's Law)

Magnetization and Bound Currents

Uniform magnetization:







$$\mathbf{m} = I\, d\mathbf{S}$$

$$I = M\, dz$$

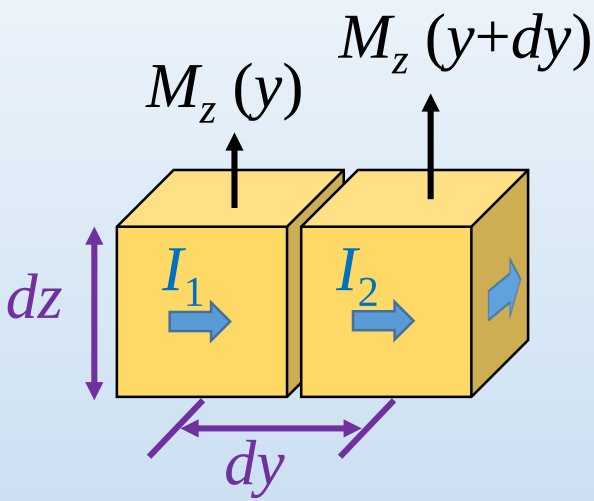
$$\vec{M} = \frac{1}{volume}\sum_{i=1}^{N}\vec{m}_i$$

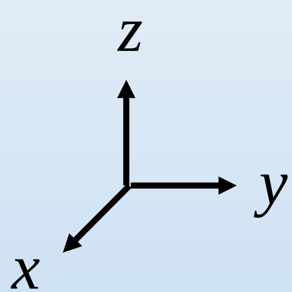
(Magnetization)

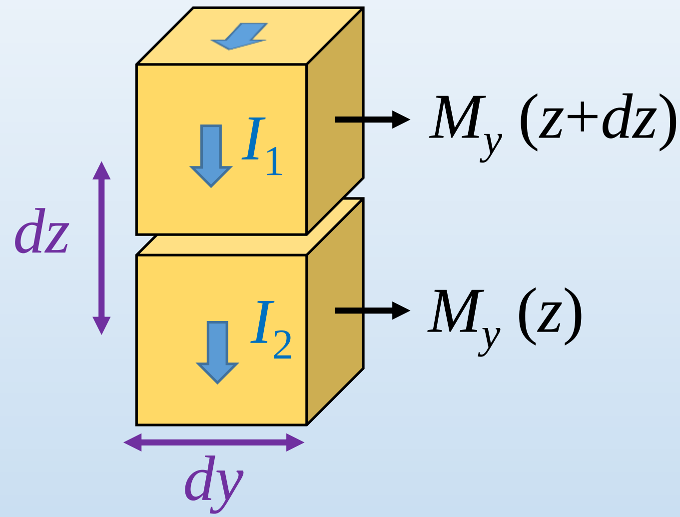
$$J^{S}_{bound} = \frac{I}{dz} = M$$

(surface bound current)

Nonuniform magnetization:







A net current in the *x* direction

$$I_x = \left[M_z\left(y+dy\right) - M_z\left(y\right) \right] dz$$

$$= \frac{\partial M_z}{\partial y} dydz$$

A net current in the *x* direction

$$I_x = -\left[M_y\left(z+dz\right) - M_y\left(z\right) \right] dy$$

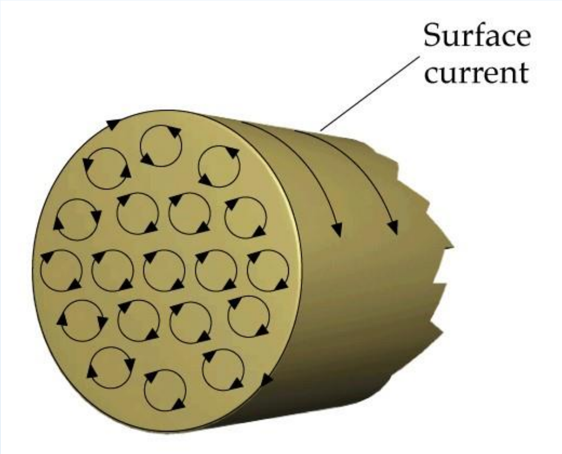
$$= -\frac{\partial M_y}{\partial z} dydz$$

$$\boxed{\left(J_{bound}\right)_x = \frac{I_x}{dydz} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}} \Rightarrow \boxed{\vec{J}_{bound} = \nabla \times \vec{M}}$$

(bound current)

Ampere's Law for Magnets

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} + \mu_0 \left(\vec{J}_{free} + \vec{J}_{bound} \right)$$



$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} + \mu_0 \left(\vec{J}_{free} + \nabla \times \vec{M} \right)$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free} \Rightarrow \boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free}}$$

(Ampere's Law)

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}}$$

(Magnetic field strength [A/m])

Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = \rho_{free}$$ **(Gauss's Law)**

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ **(Faraday's Law)**

$$\nabla \cdot \vec{B} = 0$$ **(Magnetic Gauss's Law)**

$$\nabla \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$ **(Ampere's Law)**

$$\nabla \cdot \vec{J} = -\partial \rho / \partial t$$ **(The continuity equation)**

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1 + \chi_e\right) \vec{E} = \varepsilon \vec{E}$$ **ε : permittivity**

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1 + \chi_m\right) \vec{H} = \mu \vec{H}$$ **μ : permeability**

Integral Form

$$\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_{free} dV$$ (Gauss's Law)

$$\oint_C \vec{E}\cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$ (Faraday's Law)

$$\oiint_S \vec{B}\cdot d\vec{S} = 0$$ (Magnetic Gauss's Law)

$$\oint_C \vec{H}\cdot d\vec{l} = \iint_S \left(\vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \right)\cdot d\vec{S}$$ (Ampere's Law)

$$\nabla \cdot \vec{J} = -\partial \rho / \partial t$$ (The continuity equation)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1+\chi_e\right)\vec{E} = \varepsilon \vec{E}$$ ε : permittivity

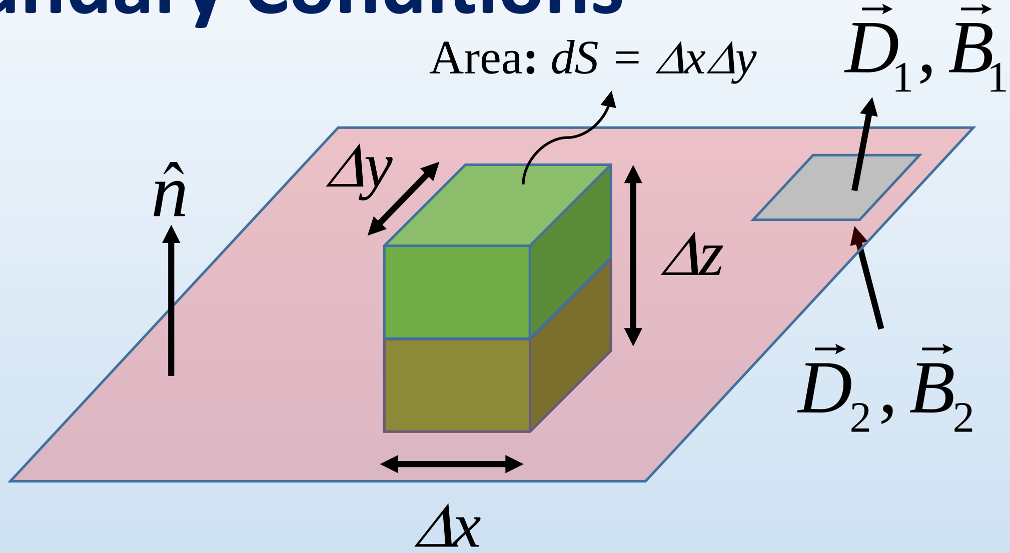
$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1+\chi_m\right)\vec{H} = \mu \vec{H}$$ μ : permeability

Electric Boundary Conditions

$$\oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_{free}\, dV$$

$$\left(D_{1\perp} - D_{2\perp}\right) dS = \rho_s dS$$

$$\boxed{\hat{n}\cdot\left(\vec{D}_1 - \vec{D}_2\right) = \rho_s}$$

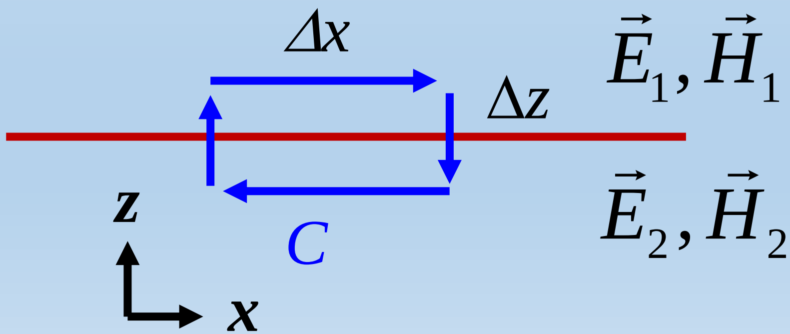


ρ_s **(surface charge density [C/m²])**

$$\iint_S \left(\nabla\times\vec{E}\right)\cdot d\vec{S} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

$$\vec{E}_1\cdot\hat{x}\Delta x - \vec{E}_2\cdot\hat{x}\Delta x = 0$$

$$\boxed{\hat{n}\times\left(\vec{E}_1 - \vec{E}_2\right) = 0}$$



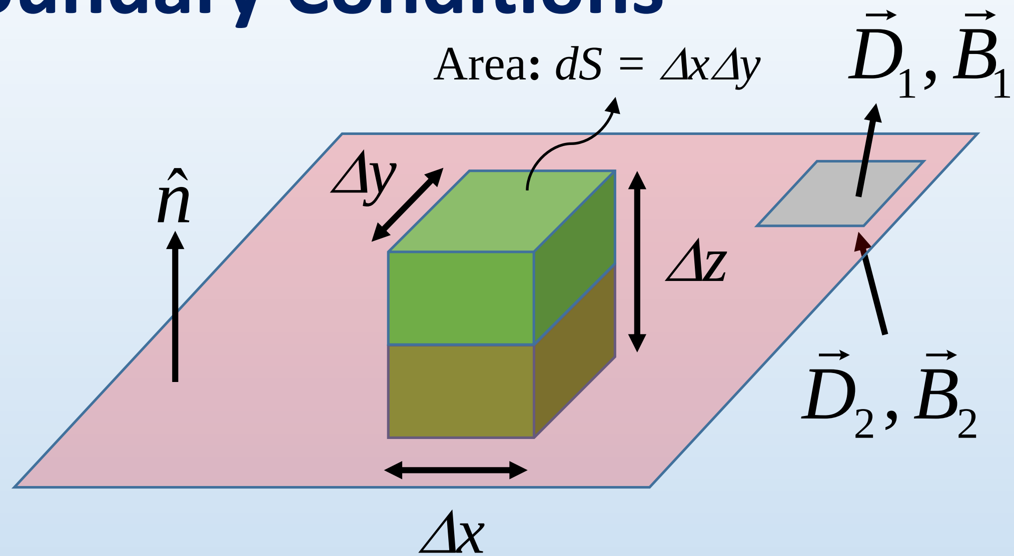
$\hat{n}$: **Points from region 2 to region 1**

Magnetic Boundary Conditions

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

$$\left(B_{1\perp} - B_{2\perp}\right) dS = 0$$

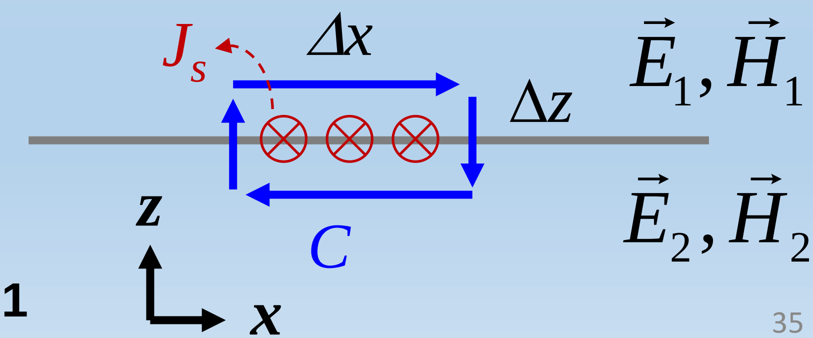
$$\boxed{\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0}$$



$$\iint_S \left(\nabla \times \vec{H}\right) \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{l} = \iint_S \left(\vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S}$$

$$\vec{H}_1 \cdot \hat{x} \Delta x - \vec{H}_2 \cdot \hat{x} \Delta x = \vec{J}_{free} \cdot \left(\hat{y}\right) \Delta x \Delta z$$

$$\boxed{\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s}$$



$\hat{n}$: **Points from region 2 to region 1**

General Boundary Conditions

(Electric)

$$\hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\rho_s$$

$$\hat{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0$$

(Magnetic)

$$\hat{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0$$

$$\hat{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{J}_s$$

ρ_s **(surface charge density [C/m²])**

$\vec{J}_s$ **(surface current density [A/m])**

$\hat{n}$ **(Points from region 2 to region 1)**

Scalar and Vector potentials in Static Fields

$$\nabla \cdot \vec{D} = \rho_{free} \qquad \boxed{\nabla \cdot \vec{B} = 0}$$

$$\boxed{\nabla \times \vec{E} = 0} \qquad \nabla \times \vec{H} = \vec{J}_{free}$$

Let $\vec{E} = -\nabla \underline{\varphi}$

(electric potential)

Let $\vec{B} = \nabla \times \underline{\vec{A}}$ **and** $\nabla \cdot \vec{A} = 0$

(magnetic vector potential)

$$\nabla \cdot \left(-\varepsilon \nabla \varphi\right) = \rho_{free}$$

$$\boxed{\nabla^2 \varphi = -\rho_{free} / \varepsilon}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \mu \vec{J}_{free}$$

$$\boxed{\nabla^2 \vec{A} = -\mu \vec{J}_{free}}$$

The problem turns into the solution of two Poisson's Equations

Boundary Conditions and Uniqueness Theorems

$$\boxed{\nabla^2 \varphi = -\rho_{free} / \varepsilon} \qquad \boxed{\nabla^2 \vec{A} = -\mu \vec{J}_{free}}$$

Poisson's Equations

Unique solution exists when the one of the following boundary equations is satisfied.

(1) Dirichlet boundary condition

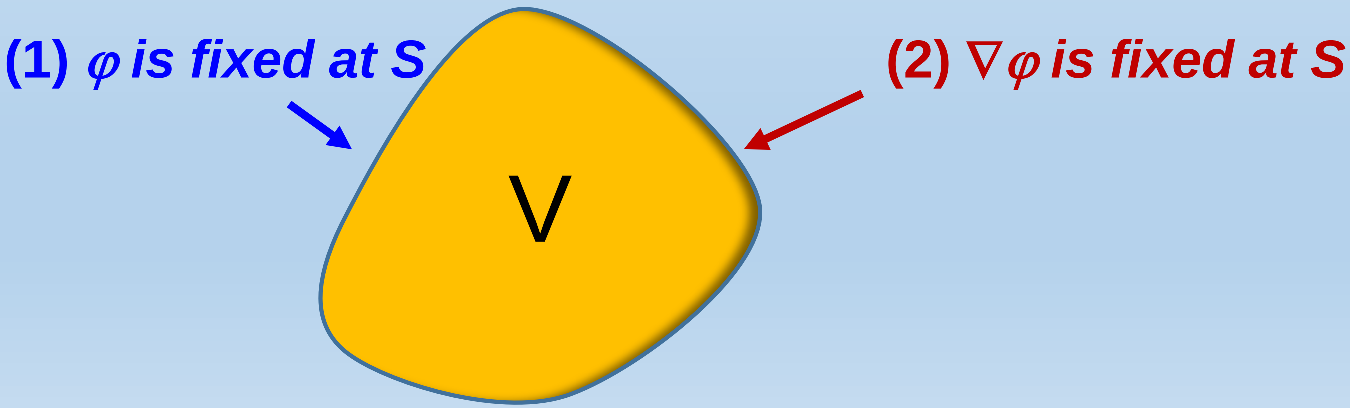
- **φ is well defined at the boundary S**

(2) Neumann boundary condition

- **$\nabla \varphi$ is well defined at the boundary S**

(3) Mixed boundary conditions

- **mix of 1 and 2**



The method of images

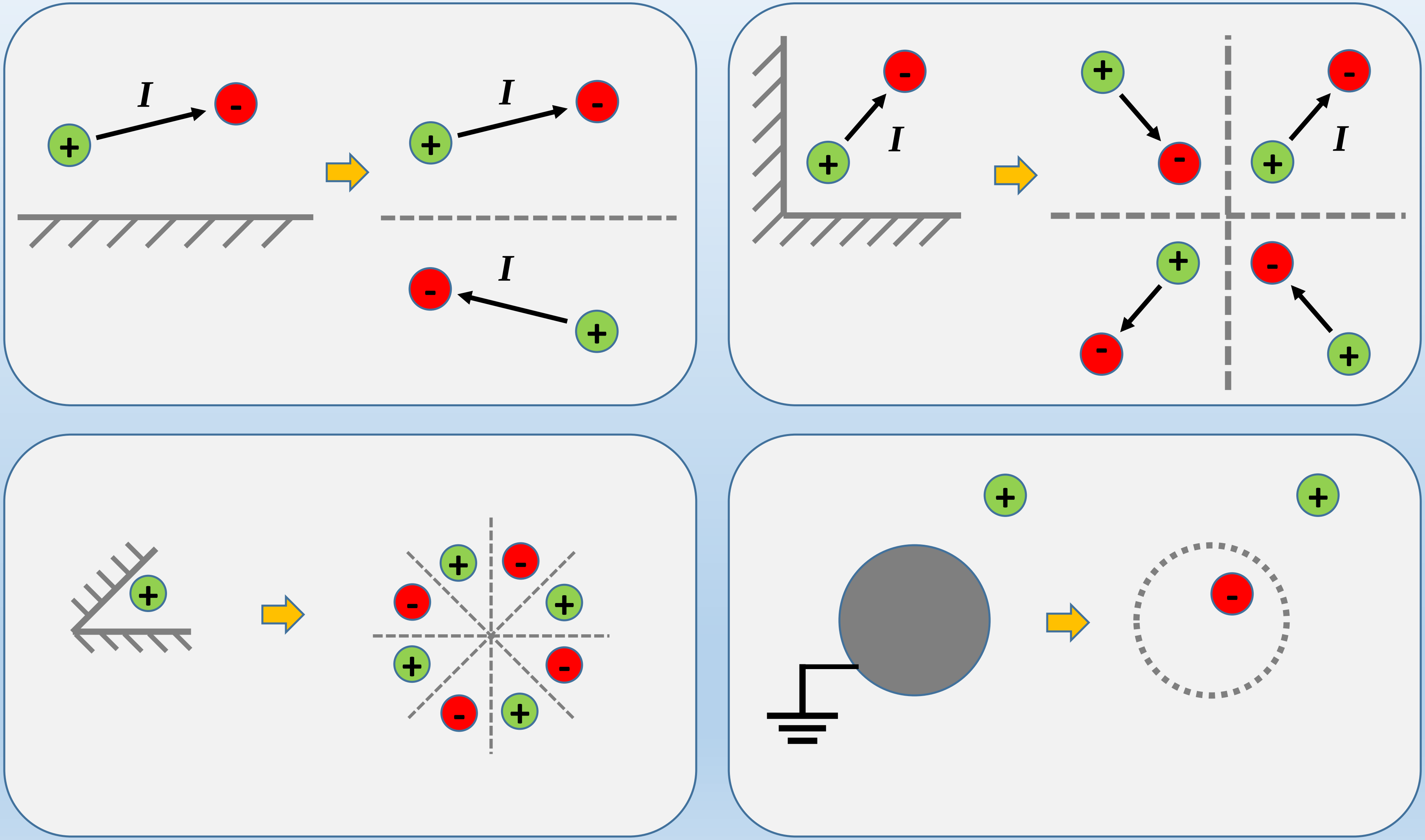
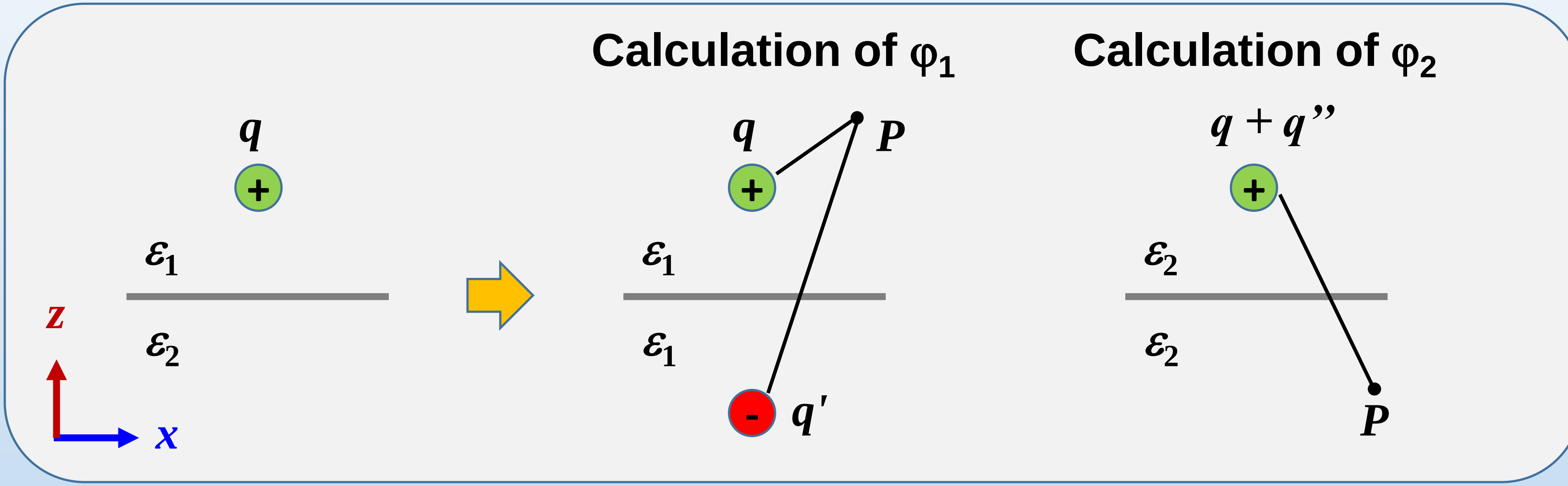


Image problems near dielectrics



$$\varphi_1(x,y,z)=\frac{1}{4\pi\varepsilon_1}\left[\frac{q}{\sqrt{x^2+y^2+(z-h)^2}}+\frac{q'}{\sqrt{x^2+y^2+(z+h)^2}}\right],\quad z\geq 0$$

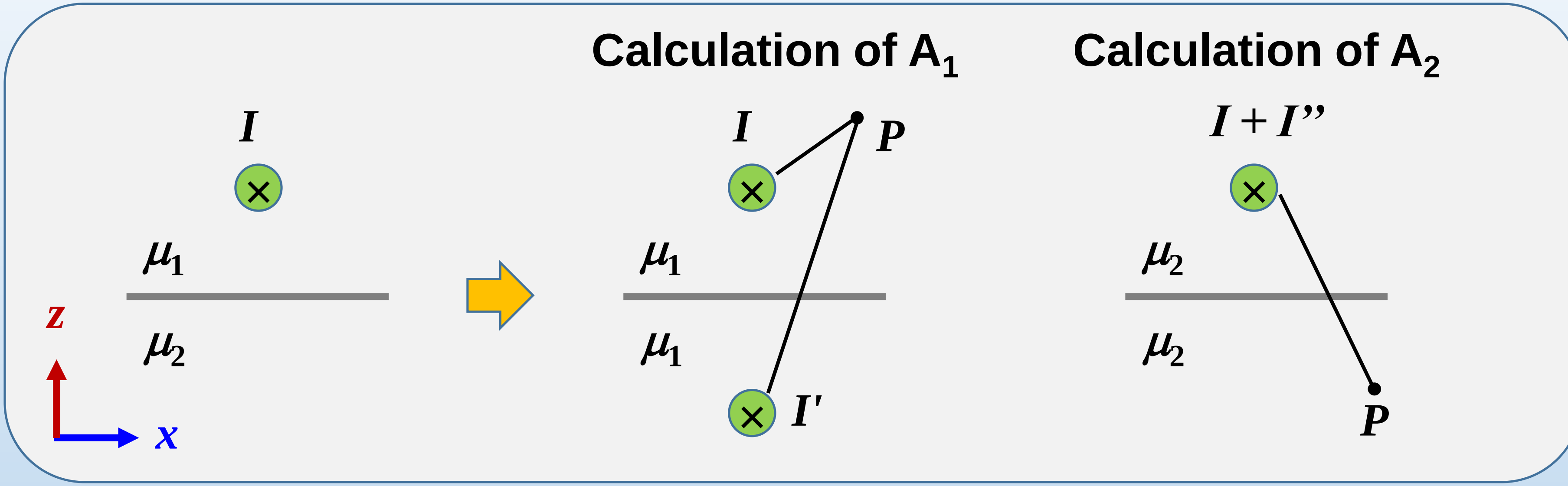
$$\varphi_2(x,y,z)=\frac{1}{4\pi\varepsilon_2}\left[\frac{q+q''}{\sqrt{x^2+y^2+(z-h)^2}}\right],\quad z\leq 0$$

$$q'=\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

$$q''=-\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

Boundary Conditions at z = 0: $\varepsilon_1\frac{\partial\varphi_1}{\partial z}=\varepsilon_2\frac{\partial\varphi_2}{\partial z}$ $\qquad \varphi_1=\varphi_2$

Image problems near magnets



$$A_{1y}(x,y,z) = \frac{\mu_1 I}{2\pi} \ln \frac{1}{\sqrt{x^2 + (z-h)^2}} + \frac{\mu_1 I'}{2\pi} \ln \frac{1}{\sqrt{x^2 + (z+h)^2}}, \quad z \geq 0$$

$$A_{2y}(x,y,z) = \frac{\mu_2 (I + I'')}{2\pi} \ln \frac{1}{\sqrt{x^2 + (z-h)^2}}, \quad z \leq 0$$

$$\boxed{I' = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} I \qquad I'' = -\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} I}$$

Boundary Conditions at z = 0: $\frac{1}{\mu_1} \frac{\partial A_1}{\partial z} = \frac{1}{\mu_2} \frac{\partial A_2}{\partial z} \qquad A_1 = A_2$

L5. Waves in Media

Time-Harmonic Fields

In many practical systems involving electromagnetic waves, the time variations are of cosinusoidal form and are referred to as *time-harmonic.*

For time-harmonic fields, we can relate the instantaneous fields, current density and charge (represented by script letters) to their complex forms (represented by roman letters) by

$$\vec{\mathcal{E}}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{E}\left(\vec{r}\right)e^{j\omega t}\right\} \qquad \vec{\mathcal{D}}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{D}\left(\vec{r}\right)e^{j\omega t}\right\}$$

$$\vec{\mathcal{B}}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{B}\left(\vec{r}\right)e^{j\omega t}\right\} \qquad \vec{\mathcal{H}}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{H}\left(\vec{r}\right)e^{j\omega t}\right\}$$

$$\vec{\mathcal{J}}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{J}\left(\vec{r}\right)e^{j\omega t}\right\} \qquad \vec{\rho}\left(\vec{r},t\right)=\mathrm{Re}\left\{\vec{\rho}\left(\vec{r}\right)e^{j\omega t}\right\}$$

Example: $\vec{E}\left(\vec{r}\right)=e^{-jkz} \longrightarrow \vec{\mathcal{E}}\left(\vec{r},t\right)=\mathrm{Re}\left\{e^{-jkz}e^{j\omega t}\right\}=\cos\left(kz-\omega t\right)$

Time-Harmonic Form of Maxwell's Equations

$$\nabla \cdot \vec{D} = \rho_{free}$$ **(Gauss's Law)**

$$\nabla \times \vec{E} = -j\omega\vec{B}$$ **(Faraday's Law)**

$$\nabla \cdot \vec{B} = 0$$ **(Magnetic Gauss's Law)**

$$\nabla \times \vec{H} = \vec{J}_{free} + j\omega\vec{D}$$ **(Ampere's Law)**

$$\partial/\partial t \rightarrow j\omega \quad \partial^2/\partial t^2 \rightarrow (j\omega)^2$$

Helmholtz wave equation (Source-Free):

$$\left(\nabla^2 + \omega^2\mu\varepsilon\right)\vec{E} = 0$$

Plane wave solution:

$$\vec{E}\left(\vec{r}\right) = \vec{E}_0 \exp\left(-j\vec{k}\cdot\vec{r}\right)$$

$$\vec{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$$

$$k_x^2 + k_y^2 + k_y^2 = \omega^2\mu\varepsilon = k^2$$

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

Time-Averaged Poynting Power Vector

$$\vec{S}(t)=\vec{E}(t)\times\vec{H}(t)=\mathrm{Re}\left\{\vec{E}e^{j\omega t}\right\}\times\mathrm{Re}\left\{\vec{H}e^{j\omega t}\right\}$$

$$=\frac{1}{2}\left\{\vec{E}e^{j\omega t}+\vec{E}^{*}e^{-j\omega t}\right\}\times\frac{1}{2}\left\{\vec{H}e^{j\omega t}+\vec{H}^{*}e^{-j\omega t}\right\}$$

$$=\frac{1}{4}\left\{\vec{E}\times\vec{H}e^{j2\omega t}+\vec{E}\times\vec{H}^{*}+\vec{E}^{*}\times\vec{H}+\vec{E}^{*}\vec{H}^{*}e^{-j2\omega t}\right\}$$

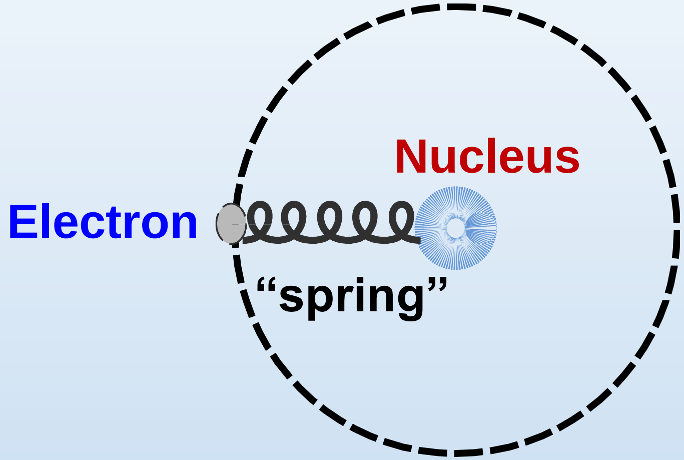
$$=\frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}^{*}\right\}+\frac{1}{2}\mathrm{Re}\left\{\vec{E}\times\vec{H}e^{j2\omega t}\right\}$$

$$\left\langle\vec{S}(t)\right\rangle=\frac{1}{T}\int_{0}^{T}\vec{S}(t)\,dt=\frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^{*}\right)$$

$$\boxed{\vec{S}=\vec{E}\times\vec{H}^{*}}$$

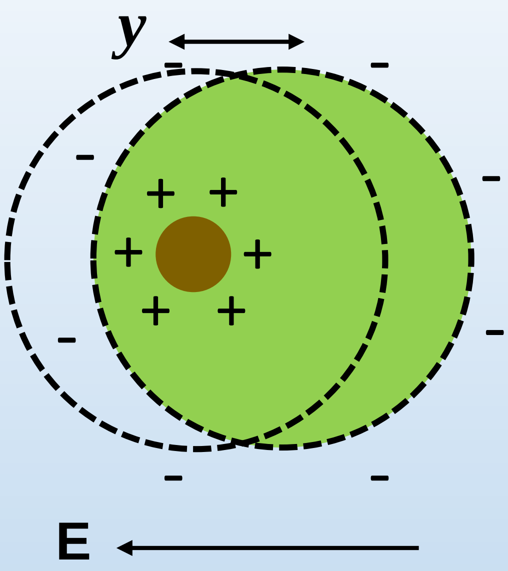
(Complex Poynting Vector)

Lorentz Oscillator Model



No external E Field

$$P = -Ney$$



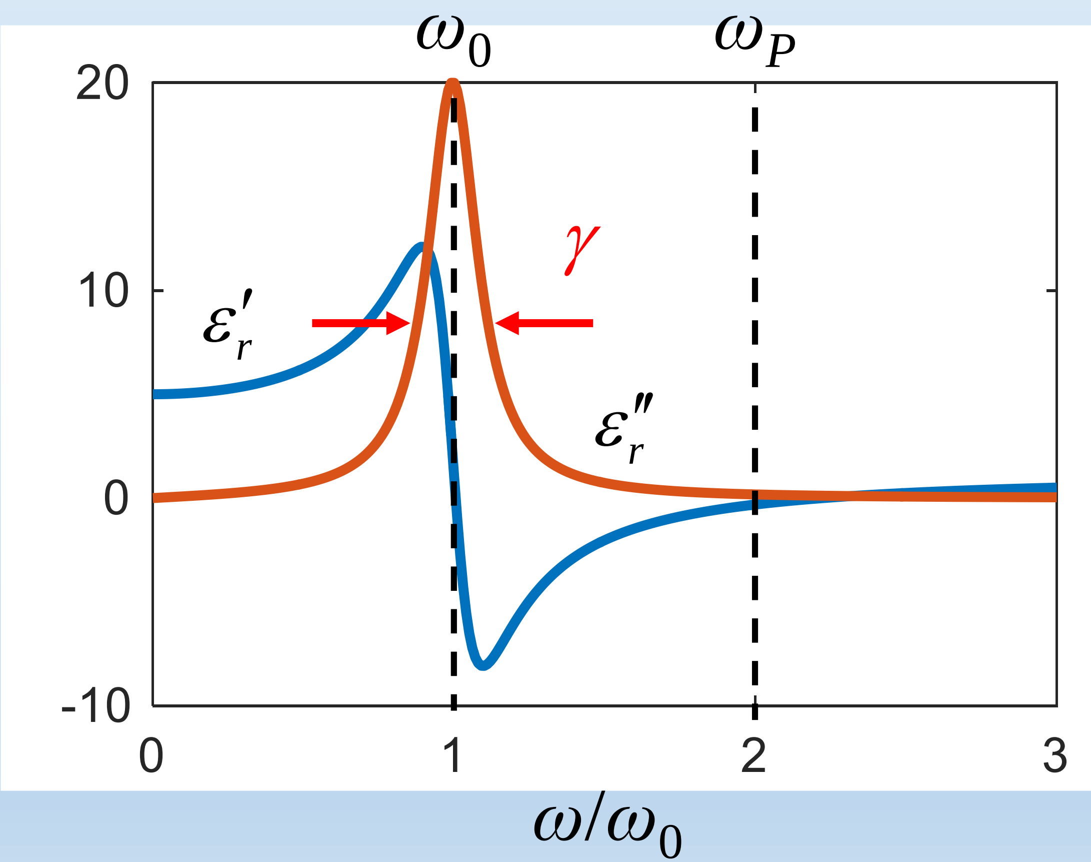
$$m\frac{d^2y}{dt^2} = -eE - m\gamma\frac{dy}{dt} - m\omega_0^2 y \Rightarrow m\left(\omega^2 - \gamma j\omega - \omega_0^2\right)y = eE$$

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{P}{\varepsilon_0 E}\right) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega}\right)}$$

$$\omega_P = \sqrt{Ne^2/m\varepsilon_0}$$ **(Plasma frequency)**

Complex permittivity

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega}\right) = \varepsilon_0\left(\varepsilon_r' - j\varepsilon_r''\right)$$



$$\varepsilon_r' - 1 = -\frac{\omega_p^2\left(\omega^2 - \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2}$$

$$\varepsilon'' = \frac{\omega_p^2\gamma\omega}{\left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2}$$

$$\tilde{n} = n - j\kappa = \sqrt{\varepsilon_r' - j\varepsilon_r''}$$

Around the resonance frequency ω_0, the magnitude of ε_r' has a drastic change and ε_r'' has a maximum value.

Dispersion

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega} \right)$$

(Lorentz Oscillator Model)

Prism

The above equation shows that permittivity depends on the frequency, besides the plasma frequency and damping (which are properties of the medium). A medium displaying such behavior (that is, whose permittivity depends on the frequency of the wave) is called dispersive, named after ”dispersion”, which is the phenomenon exhibited in a prism or raindrop that causes white light to be spread out into a rainbow of colors (white light is a mixture of beams of many different colors {all traveling at the same speed, but having different frequencies and wavelengths).

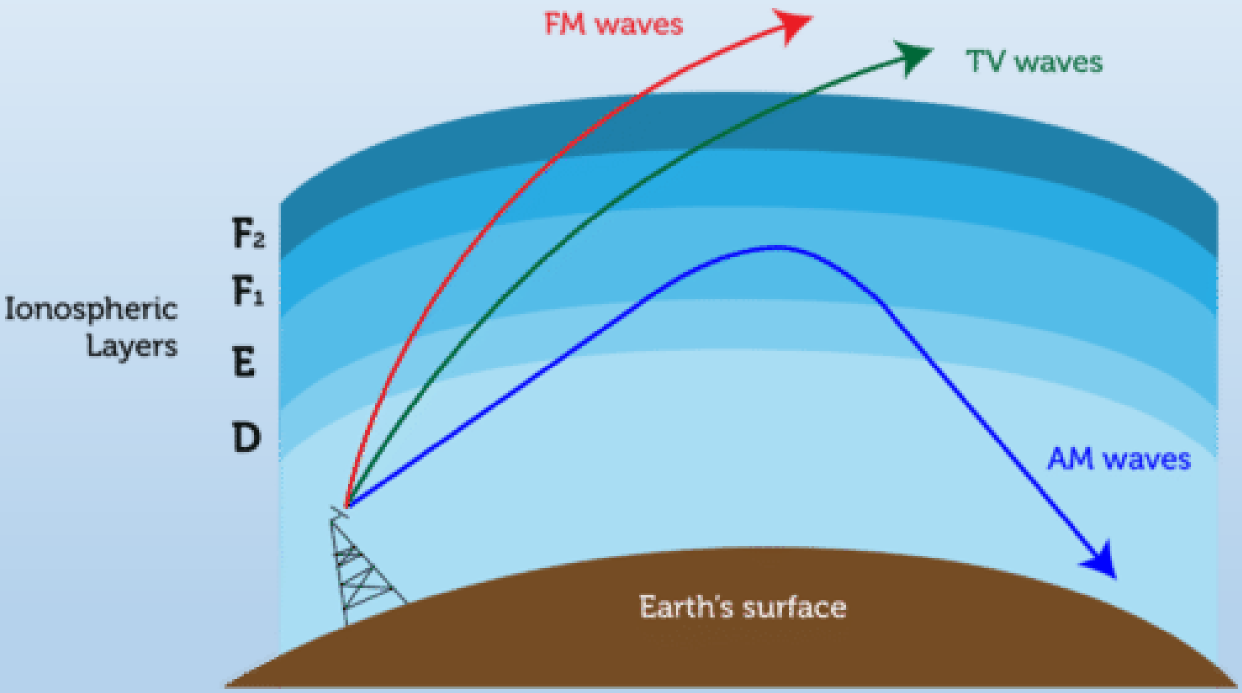
Waves in Plasma

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)}$$

$$\omega_P = \sqrt{Ne^2/m\varepsilon_0}$$

(Plasma frequency)

$$\boldsymbol{\omega < \omega_P \longrightarrow \varepsilon < 0}$$
$$\boldsymbol{\omega > \omega_P \longrightarrow \varepsilon > 0}$$



AM radio is in the range 520-1610 kHz **Reflected**

FM radio in in the range 87.5 to 108 MHz **Transmitted**

Conducting Media and Penetration Depth

Consider a conducting medium governed by Ohm's law

$$\vec{J}_c = \sigma \vec{E} \qquad \nabla \times \vec{H} = j\omega \vec{D} + \vec{J}_{free} + \vec{J}_c$$

$$\nabla \times \vec{H} = j\omega \left(\varepsilon + \frac{\sigma}{j\omega} \right) \vec{E} + \vec{J}_{free}$$

We can define a new permittivity for conducting media

$$\boxed{\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon - j\frac{\sigma}{\omega}}$$

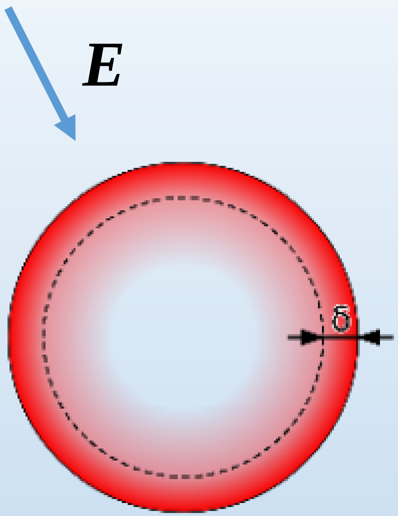
Conducting Media and Penetration Depth

$$\vec{J}_c = \sigma \vec{E}$$

$$\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon - j\frac{\sigma}{\omega}$$

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{1/2} = k' - jk''$$

$$E \propto \exp(-jkz) = \exp(-k''z - jk'z)$$



Penetration depth is defined as

$$\boxed{d_P = \frac{1}{k''}}$$

- **The wave amplitude attenuates by a factor of e^{-1} in a distance d_p**
- **d_p of Copper at 10 GHz is about 0.65 um**

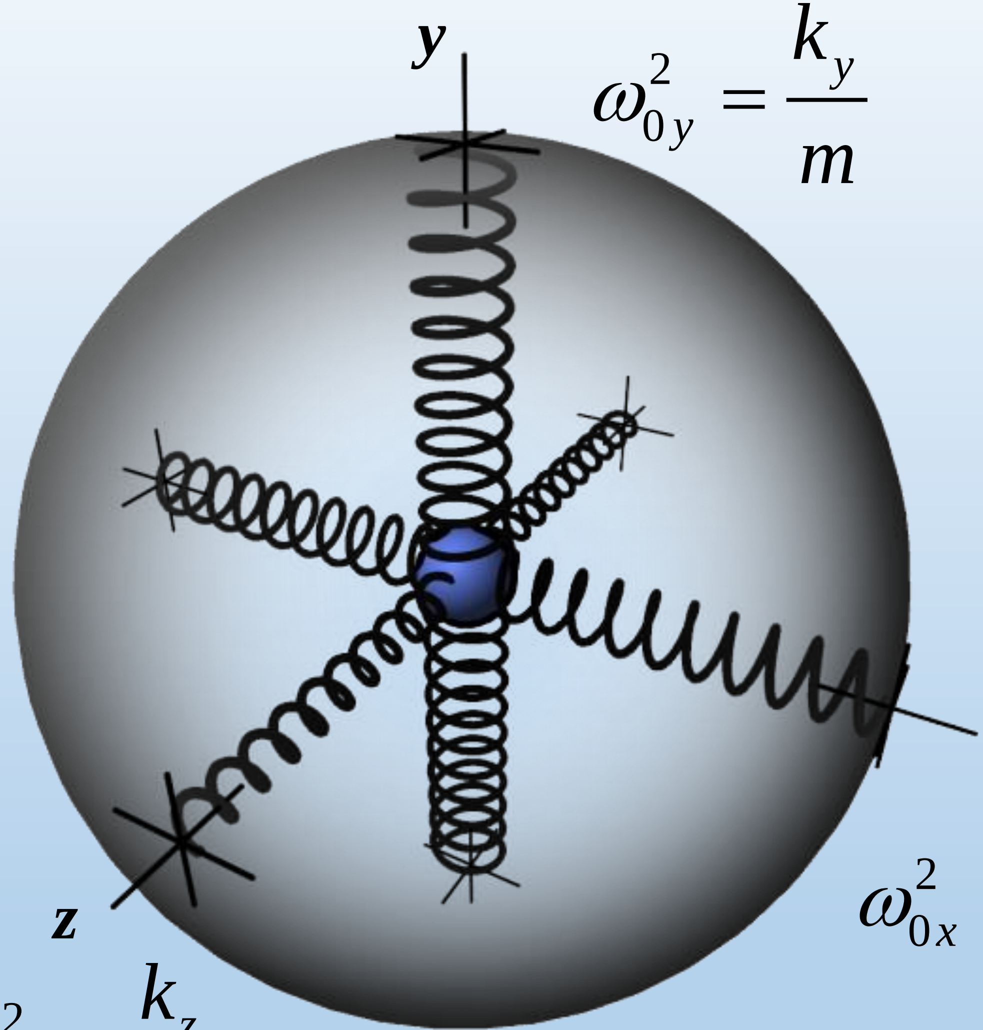
Anisotropic Material

The molecular "spring constant" can be different for different directions

If $\omega_{0x} = \omega_{0z}$, then the material has a single optics axis and is called uniaxial crystal

$$\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$$

$$n_x \neq n_y \neq n_z$$



$$\omega_{0y}^2 = \frac{k_y}{m}$$

$$\omega_{0x}^2 = \frac{k_x}{m}$$

$$\omega_{0z}^2 = \frac{k_z}{m}$$

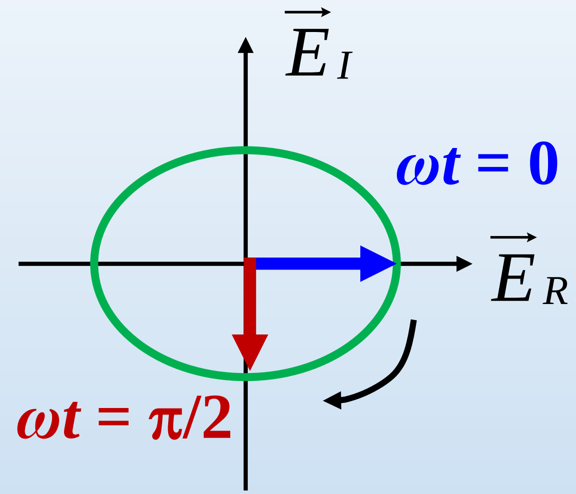
Polarization of Monochromatic Waves

At plane z = 0:

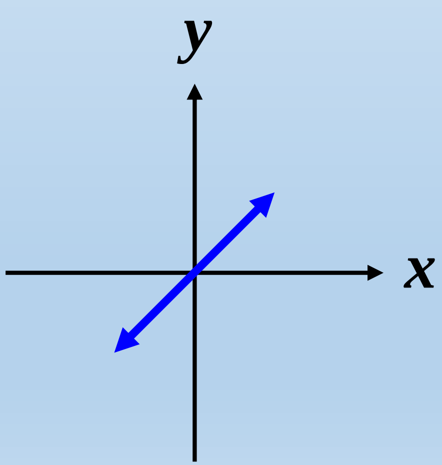
$$\vec{E} = \vec{E}_R + j\vec{E}_I$$

$$\vec{E}(t) = \mathrm{Re}\left\{\left(\vec{E}_R + j\vec{E}_I\right)e^{j\omega t}\right\}$$

$$= \vec{E}_R \cos\omega t - \vec{E}_I \sin\omega t$$

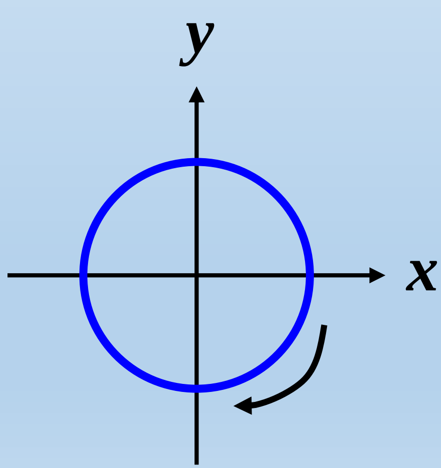


(a) Linear Polarization



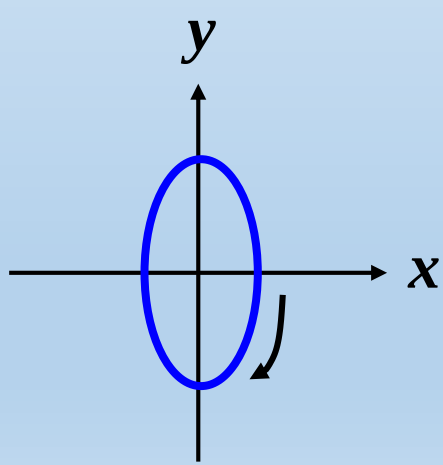
$$\vec{E} = \hat{x} + \hat{y}$$

(b) Circular Polarization



$$\vec{E} = \hat{x} + j\hat{y}$$

(c) Elliptical Polarization

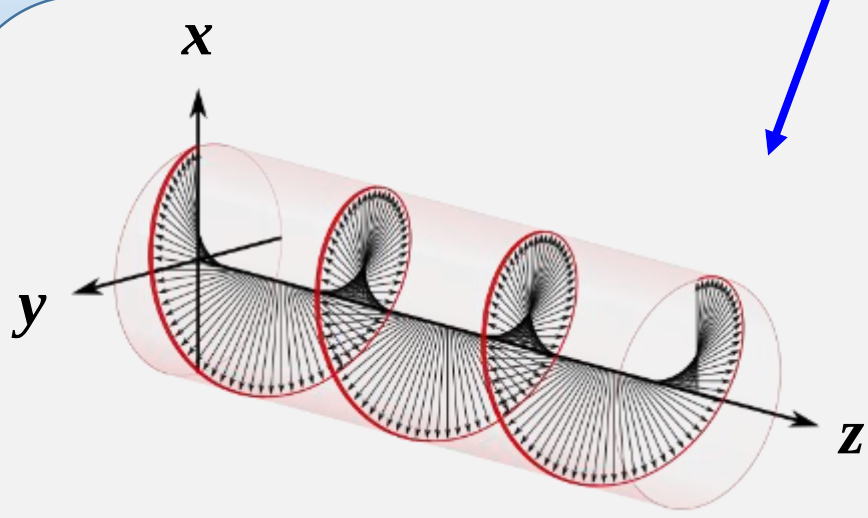


$$\vec{E} = 0.5\hat{x} + j\hat{y}$$

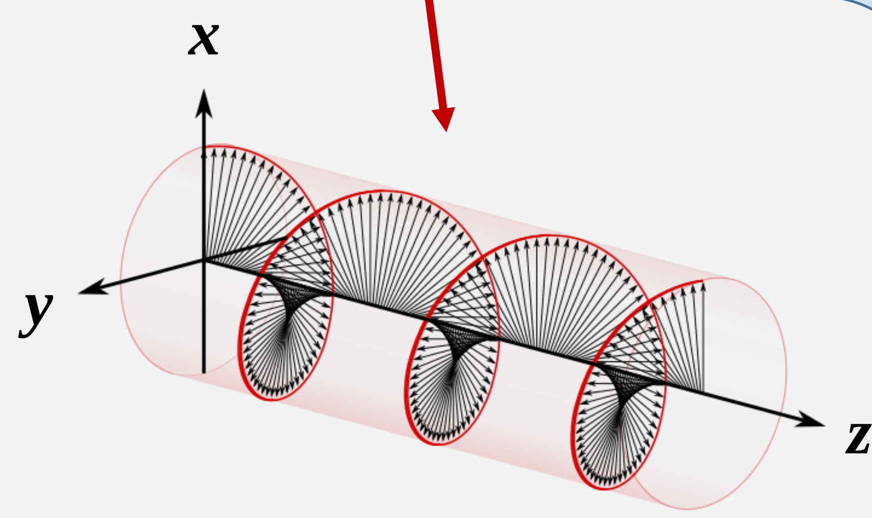
Circular Polarization

For any plane wave propagating along +z direction, the electric field can be decomposed into LCP and RCP components.

$$\vec{E} = \vec{E}_R + j\vec{E}_I = E_{LCP}\frac{\hat{x} + j\hat{y}}{\sqrt{2}} + E_{RCP}\frac{\hat{x} - j\hat{y}}{\sqrt{2}}$$



Left-handed Circularly Polarized Waves (LCP waves)



Right-handed Circularly Polarized Waves (RCP waves)