Chapter 8: Resonance

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8.1 Plane-parallel resonators

An electrical resonator such as a parallel inductor-capacitor (LC) circuit allows only electrical oscillations at the resonant frequency f_o (determined by L and C) within a narrow bandwidth around f_o. Such an LC circuit thereby stores energy at the same frequency. We know that it also acts as a filter at the resonant frequency f_o, which is how we tune in our favorite radio stations. An optical resonator is the optical counterpart of the electrical resonator, storing energy or filtering light only at certain frequencies (wavelengths).

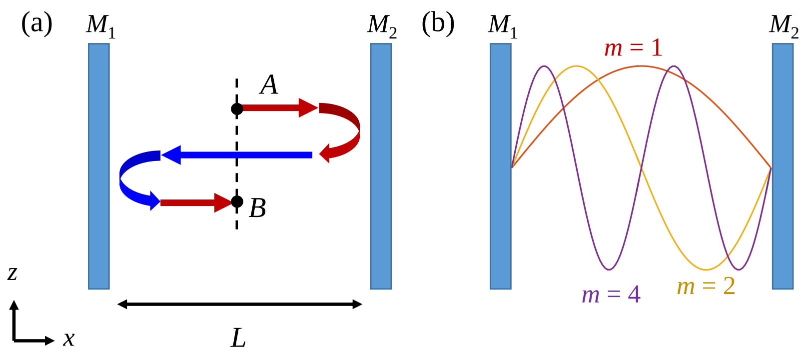

Figure 8.1 Schematic illustration of the Fabry–Perot optical cavity and its properties. (a) Reflected waves interfere. (b) Only standing EM waves, modes, of certain wavelengths are allowed in the cavity.

When two mirrors are perfectly aligned to be parallel as in Figure 8.1(a) with free space between them, electromagnetic wave reflections between the two mirrors M_1 and M_2 lead to constructive and destructive interference of these waves within the cavity. Waves reflected from M1 traveling toward the right interfere with waves reflected from M_2 traveling toward the left. The result is a series of allowed stationary or standing EM waves in the cavity as in Figure 8.1(b) (just like the stationary waves of a vibrating guitar string stretched between two fixed points). From chapter 7, when k_z = 0, we find that the standing fields in this parallel plates become

$$
\begin{aligned}
E_y(x,z) &= E_0 \sin(kx) \\
H_z(x,z) &= -\frac{k}{j\omega\mu} E_0 \cos(kx)
\end{aligned}
\tag{8.1}
$$

with the guidance condition changing to

$$kL = m\pi \tag{8.2}$$

It indicates that we can only fit in an integer number m of half-wavelengths, $\lambda/2$, into the cavity length L,

$$\boxed{L = m\frac{\lambda}{2}, \quad m = 1, 2, 3, \cdots} \tag{8.3}$$

Each particular allowed λ, labeled as λ_m, satisfying Eq. (8.3) for a given m defines a cavity mode as depicted in Figure 8.1(b). Inasmuch as light frequency f and wavelength λ are related by $f = c/\lambda$, the corresponding frequencies f_m of these modes are the resonant frequencies of the cavity

$$\boxed{f_m = m\frac{c}{2L}, \quad m = 1, 2, 3, \cdots} \tag{8.4}$$

The lowest resonant frequency is $f_1 = c/2L$ corresponding to $m = 1$. The frequency separation of two neighboring modes, $\Delta f = f_{m+1} - f_m = f_1$. It is known as the free spectral range. Figure 8.2 illustrates schematically the intensity of the allowed modes as a function of frequency. If there are no losses from the cavity, that is, the mirrors are perfectly reflecting, then the peaks at frequencies f_m defined by Eq. (8.4) would be sharp lines. If the mirrors are not perfectly reflecting so that some radiation escapes from the cavity, then the mode peaks are not as sharp and have a finite width, as indicated in Figure 8.2. It is apparent that this simple optical cavity serves to "store" radiation energy only at certain frequencies and it is called a *Fabry–Perot optical resonator*. The optical resonator does not have to have two mirrors and free space between them as shown in Figure 8.1. It can be a solid medium, such as a dielectric (e.g., glass) plate or a rod, whose ends are used to reflect light. The ends of the solid can even have thin film coatings, or dielectric mirrors, to enhance the reflection. Optical resonators are also called *etalons*.

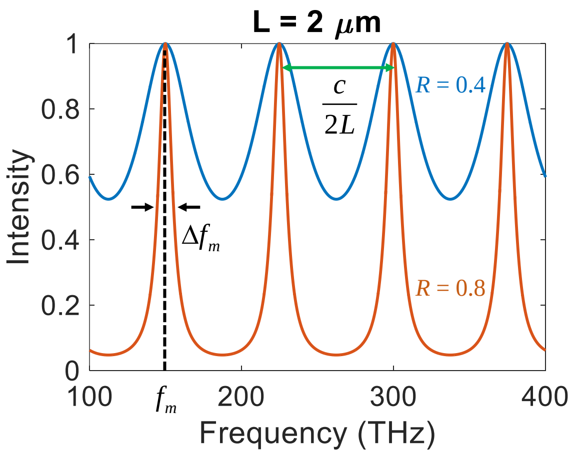

Figure 8.2 Intensity vs. frequency for various modes. R is the mirror reflectance and lower R means higher loss from the cavity.

Consider an arbitrary wave such as $A = E_0$ traveling toward the right at some instant as shown in Figure 8.1. After one round trip this wave would be again traveling toward the right but now as wave B, which has a phase difference and a different magnitude due to non-perfect reflections. If the mirrors M_1 and M_2 are identical with a reflection coefficient of magnitude R, then B has one round-trip phase difference of $k(2L)$ and a magnitude R^2 (two reflections) with respect to A. When A and B interfere, the result is

$$A+B=E_0+E_0R^2\exp(-j2kL) \tag{8.5}$$

Of course, just like A, B will continue on and will be reflected twice, and after one round trip it would be going toward the right again and we will now have three waves interfering and so on. After infinite round-trip reflections, the resultant field E_{cavity} is due to infinite such interferences.

$$E_{cavity}=E_0\left[1+R^2\exp(-j2kL)+R^4\exp(-j4kL)+\cdots\right]=\frac{E_0}{1-R^2\exp(-j2kL)} \tag{8.6}$$

Once we know the field in the cavity we can calculate the intensity $I_{cavity} \propto |E_{cavity}|^2$. Assume R is real number, the final result after algebraic manipulation is

$$I_{cavity}=\frac{I_0}{\left|1-R^2\exp(-j2kL)\right|^2}=\frac{I_0}{\left(1-R^2\right)^2+4R^2\sin^2(kL)} \tag{8.7}$$

in which $I_0 \propto E_0^2$ is the original intensity. The intensity in the cavity is maximum I_{max} whenever $\sin^2(kL)$ in the denominator of Eq. (8.7) is zero, which corresponds to (kL) being $m\pi$, in which m is an integer. Thus, the intensity vs. k, or equivalently, the intensity vs. frequency spectrum, peaks whenever $kL = m\pi$, as in Figure 8.2. These peaks are located at $k = k_m$ that satisfy $k_mL = m\pi$, which leads directly to Eqs. (8.3) and (8.4) that were derived intuitively. For those resonant k_m values, Eq. (8.7) gives

$$I_{\max}=\frac{I_0}{\left(1-R^2\right)^2},\quad k_mL=m\pi \tag{8.8}$$

A smaller mirror reflectance R means more radiation loss from the cavity, which affects the intensity distribution in the cavity. We can show from Eq. (8.7) that smaller R values result in broader mode peaks and a smaller difference between the minimum and maximum intensity in the cavity as schematically illustrated in Figure 8.2. The spectral width Δf_m of the Fabry–Perot etalon is the full width at half maximum (FWHM) of an individual mode intensity, as defined in Figure 8.2.

The quality factor Q for the optical resonant cavity in a similar fashion to defining a *Q-factor* for an LC oscillator, that is,

$$Q=\frac{\text{resonannt frequency}}{\text{specral width}}=\frac{f_m}{\Delta f_m} \tag{8.9}$$

The Q-factor is a measure of the frequency selectiveness of a resonator; the higher the Q-factor, the more selective the resonator, or narrower the spectral width. It is also a measure of the energy stored in the resonator per unit energy dissipated (due to losses such as from the reflecting surfaces) per cycle of oscillation. Figure 8.2 plots the intensity spectra for two reflection coefficients: R = 0.4 and R = 0.8, we can see that a higher reflection of mirror results in a smaller FWHM and a higher Q-factor.

The Fabry–Perot optical cavities are widely used in laser, interference filter, and spectroscopic applications. Consider a light beam that is incident on a Fabry–Perot cavity as in Figure 8.3. The optical cavity is formed by partially transmitting and reflecting plates. Part of the incident beam enters the cavity. We know that only special cavity modes are allowed to exist in the cavity since other wavelengths lead to destructive interference. Thus, if the incident beam has a wavelength corresponding to one of the cavity modes, it can sustain oscillations in the cavity and hence lead to a transmitted beam. The output light is a fraction of the light intensity in the cavity and is proportional to Eq. (8.7). Commercial interference filters are based on this principle except that they typically use two cavities in series formed by dielectric mirrors (a stack of quarter wavelength layers); the structure is more complicated than in Figure 8.3. Further, adjusting the cavity length L provides a "tuning capability" to scan different wavelengths.

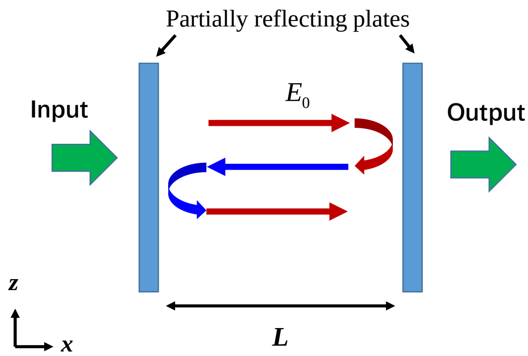

Figure 8.3 Transmitted light through a Fabry–Perot optical cavity.

Equation (1.11.3) describes the intensity of the radiation in the cavity. The intensity of the transmitted radiation in Figure 1.33 can be calculated, as above, by considering that each time a wave is reflected at the right mirror, a portion of it is transmitted, and that these transmitted waves can interfere only constructively to constitute a transmitted beam when $kL = m\pi$. Intuitively, if I_{input} is the incident light intensity, then a fraction $(1 - R^2)$ of this would enter the cavity to build up into Icavity in Eq. (1.11.3), and a fraction $(1 - R^2)$ of I_{cavity} would leave the cavity as the transmitted intensity $I_{tansmitted}$. Thus,

$$I_{transmitted} = \frac{\left(1-R^2\right)^2 I_{input}}{\left(1-R^2\right)^2 + 4R^2 \sin^2\left(kL\right)} \tag{8.10}$$

which is again maximum just as for I_{cavity} whenever $kL = m\pi$.

The ideas above can be readily extended to a medium with a refractive index n by using nk for k or λ/n for λ where k and λ are the free-space propagation constant and wavelength, respectively. Eqs. (8.3) and (8.4) become

$$L = m\frac{\lambda}{2n}, \quad m = 1, 2, 3, \cdots \tag{8.11}$$

$$f_m = m\frac{c}{2nL}, \quad m = 1, 2, 3, \cdots \tag{8.12}$$

Further, if the angle of incidence u at the etalon face is not normal, then we can resolve k to be along the cavity axis; that is use $k\cos\theta$ instead of k in the discussions above.

8.2 Rectangular Cavity Resonator

A resonator with uniform cross section in the z direction can be viewed as a waveguide with both ends closed. Instead of guided waves propagating along the z axis, the waves are standing in the z direction. The standing wave can be viewed as a superposition of a guided wave in the +z direction and a guided wave in the –z direction. The formulation for waveguides is also applicable to resonators. We have

$$\begin{aligned}
&\vec{E}_s = \frac{1}{\omega^2\mu\varepsilon - k_z^2}\left[\nabla_s\frac{\partial E_z}{\partial z} - j\omega\mu\nabla_s \times \vec{H}_z\right] \\
&\vec{H}_s = \frac{1}{\omega^2\mu\varepsilon - k_z^2}\left[\nabla_s\frac{\partial H_z}{\partial z} + j\omega\varepsilon\nabla_s \times \vec{E}_z\right] \\
&\left(\nabla^2 + \omega^2\mu\varepsilon\right)E_z = 0 \\
&\left(\nabla^2 + \omega^2\mu\varepsilon\right)H_z = 0
\end{aligned} \tag{8.13}$$

where the Laplacian operation ∇^2 is now a three-dimensional operator.

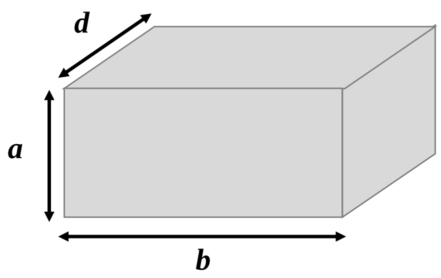

Figure 8.4 Rectangular cavity.

Consider the metallic rectangular cavity as shown in Figure 8.4. It is a waveguide closed with metallic walls at z = 0 and z = d. We find for TM modes

$$
\begin{aligned}
&E_z(x,y,z) = E_0 \sin(k_x x)\sin(k_y y)e^{-jk_z z} \\
&E_x = \frac{-k_x k_z}{\omega^2 \mu\varepsilon - k_z^2} E_0 \cos(k_x x)\sin(k_y y)e^{-jk_z z} \\
&E_y = \frac{-k_y k_z}{\omega^2 \mu\varepsilon - k_z^2} E_0 \sin(k_x x)\cos(k_y y)e^{-jk_z z} \\
&H_x = \frac{j\omega\varepsilon k_y}{\omega^2 \mu\varepsilon - k_z^2} E_0 \sin(k_x x)\cos(k_y y)e^{-jk_z z} \\
&H_y = \frac{-j\omega\varepsilon k_x}{\omega^2 \mu\varepsilon - k_z^2} E_0 \cos(k_x x)\sin(k_y y)e^{-jk_z z} \\
&H_z = 0
\end{aligned}
\tag{8.14}
$$

and for TE modes

$$
\begin{aligned}
&H_z = H_0 \cos(k_x x)\cos(k_y y)e^{-jk_z z} \\
&H_x = \frac{-k_x k_z}{\omega^2 \mu\varepsilon - k_z^2} H_0 \sin(k_x x)\cos(k_y y)e^{-jk_z z} \\
&H_y = \frac{-k_y k_z}{\omega^2 \mu\varepsilon - k_z^2} H_0 \cos(k_x x)\sin(k_y y)e^{-jk_z z} \\
&E_x = \frac{j\omega\mu k_y}{\omega^2 \mu\varepsilon - k_z^2} H_0 \cos(k_x x)\sin(k_y y)e^{-jk_z z} \\
&E_y = \frac{-j\omega\mu k_x}{\omega^2 \mu\varepsilon - k_z^2} H_0 \sin(k_x x)\cos(k_y y)e^{-jk_z z} \\
&E_z = 0
\end{aligned}
\tag{8.15}
$$

To satisfy the boundary conditions, we must have

$$
\boxed{
\begin{aligned}
&k_x a = m\pi \\
&k_y b = n\pi \\
&k_z d = p\pi
\end{aligned}
}
\tag{8.16}
$$

which is the resonance condition for the resonator.

The dispersion relation for both the TM and TE modes is

$$
k_r^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \tag{8.17}
$$

This gives the resonant spatial frequency

$$
\boxed{k_r = 2\pi\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{p}{2d}\right)^2}} \tag{8.18}
$$

The resonant spatial frequencies for TM_{mnp} modes and TE_{mnp} modes are identical. It is interesting to observe that TM_{mn0} modes corresponding to waveguide modes at cutoff, where $k_z = 0$.

8.3 Additional Problems

See References [2-7].

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