Chapter 7: Wave Guidance

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In this chapter, we study the wave guidance by conducting parallel plates, rectangular waveguides and dielectric slab waveguides. We also discuss the generic form of guided waves in arbitrary structures, in order to understand the basic physics and methods in optical waveguides.

7.1 Guidance by Conducting Parallel Plates

TE modes

Consider the guidance of electromagnetic waves by a pair of perfectly conducting plates at x = 0 and x = d (Figure 7.1). The medium between the plates is homogeneous and isotropic. The width of the waveguide along y is w and we assume that $w \gg d$, such that fringing fields can be neglected, and we have $\partial/\partial y = 0$. The Maxwell's equations can be decomposed into transverse electric (TE) and transverse magnetic (TM) components. We have

$$\begin{aligned} H_x &= \frac{1}{j\omega\mu}\frac{\partial}{\partial z}E_y \\ H_z &= -\frac{1}{j\omega\mu}\frac{\partial}{\partial x}E_y \\ \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}+\omega^2\mu\varepsilon\right)E_y &= 0 \end{aligned} \tag{7.1}$$

for TE waves and

$$\begin{aligned} E_x &= -\frac{1}{j\omega\varepsilon}\frac{\partial}{\partial z}H_y \\ E_z &= \frac{1}{j\omega\varepsilon}\frac{\partial}{\partial x}H_y \\ \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}+\omega^2\mu\varepsilon\right)H_y &= 0 \end{aligned} \tag{7.2}$$

For TM waves. The boundary conditions at the parallel plates require that the tangential electric field be zero at x = 0 and x = d.

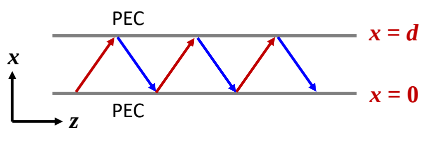

Figure 7.1 Parallel-plate waveguide.

The wave is guided along $\pm z$ directions. For waves propagating along $+z$ direction, the solution for TE waves comprises two plane wave components

$$E_y = A\exp\left(-jk_x x - jk_z z\right) + B\exp\left(jk_x x - jk_z z\right) \tag{7.3}$$

Substitute it into Eq. (7.1) we find the dispersion relation

$$k_z^2 + k_x^2 = \omega^2 \mu\varepsilon = k^2 \tag{7.4}$$

The boundary condition at $x = 0$ requires $E_y = 0$, namely,

$$\frac{A}{B} = -1 \tag{7.5}$$

which is the reflection coefficient at the boundary surface $x = 0$. The boundary conditions at $x = d$ requiring $E_y = 0$ gives

$$\frac{B}{A} = -\exp(-j2k_x d) \tag{7.6}$$

which is the reflection coefficient at the boundary surface $x = d$. The factor $\exp(-j2k_x d)$ is due to the fact that the coordinate origin is at $x = 0$. Multiplying Eq. (7.5) and Eq. (7.6), we obtain

$$\exp(-j2k_x d) = 1 = \exp(-j2m\pi) \tag{7.7}$$

We thus have

$$k_x d = m\pi \tag{7.8}$$

which states that the phase of the round trip of the plane wave in x direction must add up to integer numbers of 2π. Therefore, as a result of the boundary conditions at $x = 0$ and $x = d$, we must have

$$\boxed{k_z^2 + \left(\frac{m\pi}{d}\right)^2 = k^2 = \left(\frac{\omega}{c}\right)^2} \tag{7.9}$$

where m is any integer. Eq. (7.9) is called the *guidance condition* which is determined from the boundary conditions. Thus along the x direction, the number of periods of the spatial variation of a guided wave must be an integer in a distance of $2d$.

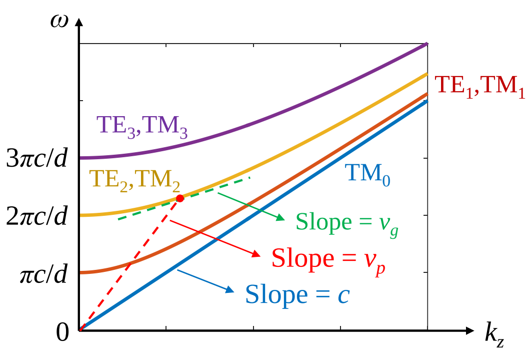

Figure 7.2 k_z-ω diagram for guided modes.

Substituting the guidance condition Eq. (7.9) in the dispersion relation of Eq. (7.4) we obtain

$$\boxed{k_x = \frac{m\pi}{d}} \tag{7.10}$$

This equation describes a family of hyperbolas for different values of m. In Figure 7.2 we plot the k_z-ω diagram. We see that for the m-th mode k_z will be imaginary if $k < m\pi/d$. The wave then becomes evanescent and attenuates exponentially in the z direction.

The spatial frequency at which $k_z = 0$ is called the cutoff spatial frequency k_{cm}

$$\boxed{k_{cm} = \frac{m\pi}{d}} \tag{7.11}$$

When $k < k_{cm}$, all modes with order higher than m will be evanescent. In order for the m-th order mode to propagate, the spatial frequency k must be larger than k_{cm}. Notice that if the m-th mode is propagating, then all l-th modes with $l < m$ can also propagate. The cutoff frequency for the TM_0 mode is zero and the TE_0 is zero. Thus for a given spatial frequency k such that $m\pi/d < k < (m+1)\ \pi/d$, there will be m TE modes and m+1 TM modes admissible inside the waveguide. The lowest order TE mode is TE_1 whose $k_{c1} = \pi/d$. For $k < \pi/d$, no TE mode can be excited. The single TE_1 mode operation range inside the guide is $\pi/d < k < 2\pi/d$.

For each mode, the *phase velocity* is calculated by

$$\boxed{v_p = \frac{\omega}{k_z}} \tag{7.12}$$

and the *group velocity* is

$$\boxed{v_g = \frac{d\omega}{dk_z}} \tag{7.13}$$

Substituting Eq. (7.5) into Eq. (7.3) we determine the electric field

$$E_y = E_0 \sin\left(k_x x\right) \exp\left(-jk_z z\right) \tag{7.14}$$

where $E_0 = -j2A$. We call the TE wave corresponding to each integer m the TE_m mode. There is no TE_0 mode because $E_y = 0$ for $m = 0$. The field patterns for E_y are plotted in Figure 7.3 for m = 1, 2, and 3.

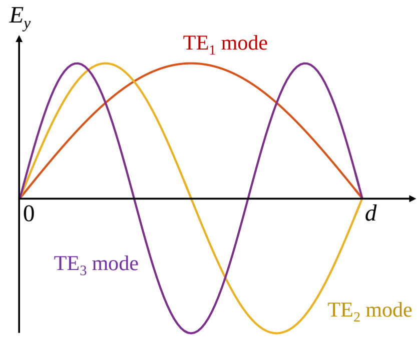

Figure 7.3 Field amplitudes for TE$_1$, TE$_2$, and TE$_3$ mode.

The magnetic field vector is obtained from Eq. (7.1)

$$\begin{aligned}\vec{H} &= -\frac{1}{j\omega\mu}\left(-\hat{x}\frac{\partial}{\partial z}E_y + \hat{z}\frac{\partial}{\partial x}E_y\right)\\ &= E_0\left[-\hat{x}\frac{k_z}{\omega\mu}\sin(k_x x) - \hat{z}\frac{k_x}{j\omega\mu}\cos(k_x x)\right]\exp(-jk_z z)\end{aligned} \tag{7.15}$$

We see that the magnetic field vector is perpendicular to both the electric field and the wave vector for the two bouncing plane waves.

We find the complex Poynting power from Eq. (7.14) and Eq. (7.15)

$$\begin{aligned}&\vec{S} = \vec{E}\times\vec{H}^*\\ &= \left[\hat{z}\frac{k_z^*}{\omega\mu}|E_0|^2\sin^2(k_x x) - \hat{x}\frac{k_x}{j\omega\mu}|E_0|^2\sin(k_x x)\cos(k_x x)\right]\exp(-jk_z z + jk_z^* z)\end{aligned} \tag{7.16}$$

The time-average Poynting power $< S_x >$ in the transverse direction is always zero. The time-average Poynting powver $< S_z >$ in the z direction is

$$\langle S_z\rangle = \frac{1}{2}\mathrm{Re}\left[\frac{k_z^*}{\omega\mu}|E_0|^2\sin^2(k_x x)\exp(-jk_z z + jk_z^* z)\right] \tag{7.17}$$

When $k_z = \sqrt{k^2 - k_x^2} = \sqrt{k^2 - (m\pi/d)^2}$ is imaginary, $k_z = -jk_z''$, the time-average Poynting power in the z direction $\langle S_z\rangle = \mathrm{Re}[S_z]/2 = 0$, which happens for higher order modes as $m\pi/d > k$ and those high order modes are evanescent in the z direction and carry no time-average power.

TM modes

For TM waves satisfying the boundary conditions, we find that the solution takes the form

$$\begin{aligned}\vec{H} &= \hat{y}H_0\cos(k_x x)\exp(-jk_z z)\\ \vec{E} &= \left[\hat{x}\frac{k_z}{\omega\varepsilon}\cos(k_x x) - \hat{z}\frac{k_x}{j\omega\varepsilon}\sin(k_x x)\right]H_0\exp(-jk_z z)\end{aligned} \tag{7.18}$$

The boundary condition of vanishing E_z at $x = 0$ and $x = d$ leads to the guidance condition

$$k_x = \frac{m\pi}{d} \tag{7.19}$$

which is identical to that of TE wave and is plotted in Figure 7.3.

There is one very important difference between the TM and TE cases. When $m = 0$, the TM field no longer vanishes as in the TE case and we now have the TM_0 mode which also called the TEM mode. The TM_0 mode has $k_x = 0$ and $k_z = k$ never becomes imaginary and the TEM wave propagates at all frequencies. The time-average Poynting power in the z direction is

$$\langle S_z \rangle = \frac{1}{2}\mathrm{Re}\left[\frac{k_z^*}{\omega\varepsilon}|H_0|^2\cos^2(k_x x)\exp(-jk_z z + jk_z^* z)\right] = \frac{k_z}{2\omega\varepsilon}|H_0|^2 \tag{7.20}$$

for the TM_0 mode. The TM_0 or TEM mode in a parallel-plate waveguide is termed fundamental or dominant mode.

Field solutions for the TM_0 mode follows Eq. (7.18) when we set $k_x = 0$ and $k_z = k$. We have

$$\begin{aligned}\vec{H} &= \hat{y}H_0\exp(-jk_z z)\\ \vec{E} &= \hat{x}\frac{k_z}{\omega\varepsilon}H_0\exp(-jk_z z)\end{aligned} \tag{7.21}$$

The electric field is seen to be perpendicular to the plates and the magnetic field parallel to the plates.

7.2 Rectangular Waveguide

TM modes

Consider a metallic rectangular waveguide having dimensions a along the x axis and b along the y axis (Figure 7.4). We first investigate transverse magnetic (TM) fields where all magnetic fields are transverse to the direction of propagation z. We have $H_z = 0$, and all field components can be derived from a single longitudinal component $E_z = \sin(k_x x)\sin(k_y y)\exp(-jk_z z)$. We obtain

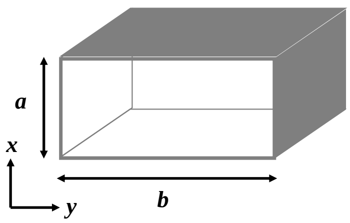

Figure 7.4 Metallic rectangular waveguides.

$$\begin{aligned}
E_z &= \sin(k_x x)\sin(k_y y)\exp(-jk_z z) \\
E_x &= \frac{-jk_x k_z}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\exp(-jk_z z) \\
E_y &= \frac{-jk_y k_z}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\exp(-jk_z z) \\
H_x &= \frac{j\omega\varepsilon k_y}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\exp(-jk_z z) \\
H_y &= \frac{-j\omega\varepsilon k_x}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\exp(-jk_z z) \\
H_z &= 0
\end{aligned} \tag{7.22}$$

We see that at x = 0 and a, Ez and Ey vanish, and at y = 0 and y = b, Ez and Ex vanish, provided that

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \end{aligned}} \tag{7.23}$$

where m and n are integer numbers. Eq. (7.23) gives the guidance conditions. For TM waves neither m nor n can be zero because then E_z will be zero, too. Substituting the guidance conditions in the field expressions, we see that for larger m there will be more variations for the fields as a function of *x*, and for larger *n* there will be more field variations along the *y* direction.

The guidance conditions and the dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon = (\omega/c)^2 = k^2 \tag{7.24}$$

combine to give the propagation constant

$$\boxed{k_z = \sqrt{(\omega/c)^2 - (m\pi/a)^2 - (n\pi/b)^2}} \tag{7.25}$$

According to particular values of m and n, the TM waves inside the rectangular waveguide are classified into TM_{mn} modes. The first index m is associated with the number of variations along the *x* direction and the second index with the number of variations along the *y* direction.

Cutoff occurs when k_z becomes imaginary such that the wave attenuates exponentially along the direction of propagation. For a TM_{mn} mode, the cutoff frequency is

$$\boxed{k_{cmn} = \sqrt{(m\pi/a)^2 + (n\pi/b)^2}} \tag{7.26}$$

TE modes

For TE fields, all field components are derivable from a single longitudinal component H_z with $E_z = 0$. Consider the boundary conditions of vanishing tangential electric fields on the metallic wall surfaces, we obtain

$$\begin{aligned}
H_z &= \cos(k_x x)\cos(k_y y)\exp(-jk_z z) \\
H_x &= \frac{jk_x k_z}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\exp(-jk_z z) \\
H_y &= \frac{jk_y k_z}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\exp(-jk_z z) \\
E_x &= \frac{j\omega\mu k_y}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\exp(-jk_z z) \\
E_y &= \frac{-j\omega\mu k_x}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\exp(-jk_z z) \\
E_z &= 0
\end{aligned} \tag{7.27}$$

The guidance conditions are obtained from the boundary conditions of Ex = 0 at y = 0 and b and Ey = 0 at x = 0 and a. The result is identical to Eq. (7.23)

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \end{aligned}} \tag{7.28}$$

The propagation constant k_z is again given as Eq. (7.25) and the cutoff spatial frequencies are found to be Eq. (7.26).

Notice that while neither m nor n can be zero for TM_{mn} modes, it is possible to have either m or n or both m and n equal to zero for the TE_{mn} modes. For $m = n = 0$, we find $H_z = \exp(-jkz)$. The equation $\nabla \cdot \vec{H} = 0$ implies $k = \omega\sqrt{\mu\varepsilon} = 0$ and consequently the TE_{00} mode is a static field solution in the waveguide.

7.3 Dielectric Slab Waveguide

Optical waveguides such as optical fibers typically trap and guide light within rectangular or cylindrical boundaries over useful distances. Rectangular shapes are easier to implement on integrated circuits, while cylindrical shapes are used for longer distances, up to 100 km or more. Exact wave solutions for such structures are beyond the scope of this text, but the same basic principles are evident in dielectric slab waveguides for which the derivations are simpler. Dielectric slab waveguides consist of an infinite flat dielectric slab of thickness $2d$ and permittivity ε_1 imbedded in an infinite medium of lower permittivity ε_2, as suggested in Figure 7.5 for a slab of finite width in the *y* direction. For simplicity we here assume $\mu = \mu_0$ everywhere, which is usually the case in practice too.

Uniform plane waves within the dielectric are perfectly reflected at the slab boundary if they are incident beyond the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$, where $n_1 = c_0\sqrt{\mu_0\varepsilon_1}$ and $n_2 = c_0\sqrt{\mu_0\varepsilon_2}$ are the refractive indices for two media. Such a wave and its perfect reflection propagate together along the *z* axis and form a standing wave in the orthogonal *x* direction. Outside the waveguide the waves are evanescent and decay exponentially away from the guide.

Consider a symmetric dielectric slab waveguide with boundaries at $x = -d$ and $x = d$. The fields inside a dielectric slab have the same form as those inside parallel-plate waveguides, although the boundary conditions are different.

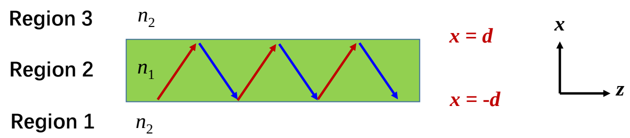

Figure 7.5 Slab dielectric waveguide.

We investigate transverse electric (TE) fields where all electric fields are transverse to the direction of propagation z. The electric fields in three regions can be written by

$$\begin{aligned}
\vec{E}_1 &= \hat{y}E_1 \exp\left(\alpha x - jk_z z\right), x \le -d \\
\vec{E}_2 &= \hat{y}\left[A\sin\left(k_x x\right) + B\cos\left(k_x x\right)\right]\exp\left(-jk_z z\right), -d < x < d \\
\vec{E}_3 &= \hat{y}E_3 \exp\left(-\alpha x - jk_z z\right), x \ge d
\end{aligned} \tag{7.29}$$

The first term ($\sin k_x x$) in $\vec{E}_2$ refers to the *anti-symmetric* mode and the second one ($\cos k_x x$) refers to the *symmetric* mode. Since the waves decay away from the slab, α is positive ($\alpha > 0$). Faraday's law in combination with Eq. (7.29) yields the corresponding magnetic field inside and outside the slab

$$\begin{aligned}
\vec{H}_1 &= -\frac{1}{j\omega\mu}\left(\hat{x}jk_z + \hat{z}\alpha\right)E_1 \exp\left(\alpha x - jk_z z\right), x \le -d \\
\vec{H}_2 &= -\frac{1}{j\omega\mu}\left\{\begin{aligned}&\hat{x}jk_z\left[A\sin\left(k_x x\right) + B\cos\left(k_x x\right)\right] \\ &+\hat{z}\left[k_x A\cos\left(k_x x\right) - k_x B\sin\left(k_x x\right)\right]\end{aligned}\right\}\exp\left(-jk_z z\right), -d < x < d \\
\vec{H}_3 &= -\frac{1}{j\omega\mu}\left[\hat{x}jk_z - \hat{z}\alpha\right]E_3 \exp\left(-\alpha x - jk_z z\right), x \ge d
\end{aligned} \tag{7.30}$$

Apply the boundary condition of continuous tangential electric and magnetic fields at $x = -d$ and $x = d$, we have

$$\begin{aligned}
E_1 \exp\left(-\alpha d\right) &= -A\sin\left(k_x d\right) + B\cos\left(k_x d\right) \\
E_3 \exp\left(-\alpha d\right) &= A\sin\left(k_x d\right) + B\cos\left(k_x d\right) \\
\alpha E_1 \exp\left(-\alpha d\right) &= k_x A\cos\left(k_x d\right) + k_x B\sin\left(k_x d\right) \\
-\alpha E_3 \exp\left(-\alpha d\right) &= k_x A\cos\left(k_x d\right) - k_x B\sin\left(k_x d\right)
\end{aligned} \tag{7.31}$$

Consequently, the *guidance condition* for the *anti-symmetric* TE dielectric slab waveguide modes (B = 0) is

$$-\alpha d = k_x d \cot\left(k_x d\right) \tag{7.32}$$

and the *guidance condition* for the *symmetric* TE dielectric slab waveguide modes (A = 0) is

$$\alpha d = k_x d \tan\left(k_x d\right) \tag{7.33}$$

where E_1 = -E_3 for the anti-symmetric mode and E_1 = E_3 for the symmetric mode.

Combine Eq. (7.32) and Eq. (7.33) together, we can obtain the *guidance condition*

$$\boxed{\alpha d = k_x d \tan\left(k_x d - \frac{m\pi}{2}\right)} \tag{7.34}$$

Here, m is an even number (m = 0, 2, 4, …) for a symmetric mode and an odd number (m = 1, 3, 5, …) for an anti-symmetric mode.

Combining the following two dispersion relations and eliminating k_z can provide the needed additional relation between k_x and α:

$$\begin{array}{l} k_x^2 + k_z^2 = n_1^2 k_0^2 \\ -\alpha^2 + k_z^2 = n_2^2 k_0^2 \end{array} \quad => \quad \boxed{k_x^2 + \alpha^2 = \left(n_1^2 - n_2^2\right) k_0^2} \tag{7.35}$$

By substituting into the guidance condition Eq. (7.34) the expression for α that follows from the slab dispersion relation Eq. (7.35) we obtain a transcendental guidance equation that can be solved numerically or graphically.

$$\boxed{\sqrt{\left(n_1^2 - n_2^2\right) k_0^2 / k_x^2 - 1} = \tan\left(k_x d - \frac{m\pi}{2}\right)} \tag{7.36}$$

Figure 7.6 plots the left- and right-hand sides of Eq. (7.36) separately (for the symmetric modes), so the modal solutions are those values of $k_x d$ for which the two families of curves intersect.

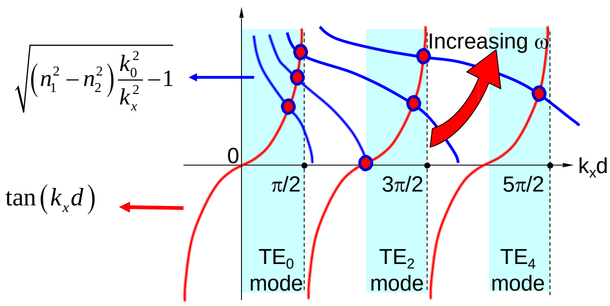

Figure 7.6 Symmetric TE modes for a dielectric slab waveguide

For the m-th TE mode the cutoff frequency occurs when α = 0, namely,

$$m\pi = k_x d = \sqrt{\left(n_1^2 - n_2^2\right) k_0^2 - \alpha^2}\, d = \sqrt{n_1^2 - n_2^2}\, k_0 d \tag{7.37}$$

We find the cutoff frequency for the m-th mode is

$$\boxed{\omega = c_0 k_0 = \frac{m\pi}{\sqrt{n_1^2 - n_2^2}\, d}} \tag{7.38}$$

At cutoff, the propagation constant in the *z* direction is

$$k_z = n_2 k_0 \tag{7.39}$$

It follows that the 0-th order mode possesses zero cutoff frequency.

7.4 Generic Form of Guided Waves

In treating guided waves along the *z* direction, the *z* dependence of all field vector is written as $\exp(\pm jk_z z)$ where k_z is the propagation constant and the $\pm$ signs indicate propagation along negative and positive *z* directions. With this dependence, we can replace $\partial^2/\partial z^2$ by $-k_z^2$. From the Maxwell's equations, we can express all field components parallel to the *z* axis. When all vectors are separated into their transverse and longitudinal components, Maxwell's two curl equations for isotropic media become, in vector notation,

$$\begin{aligned}\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\vec{E}_s + \vec{E}_z\right) &= -j\omega\mu\left(\vec{H}_s + \vec{H}_z\right)\\ \left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\vec{H}_s + \vec{H}_z\right) &= j\omega\varepsilon\left(\vec{E}_s + \vec{E}_z\right)\end{aligned} \tag{7.40}$$

where the subscript *s* denotes transverse components. Separating into transverse and longitudinal directions, we have

$$\begin{aligned}-j\omega\mu\vec{H}_s &= \nabla_s \times \vec{E}_z + \hat{z}\times\partial\vec{E}_s/\partial z\\ j\omega\varepsilon\vec{E}_s &= \nabla_s \times \vec{H}_z + \hat{z}\times\partial\vec{H}_s/\partial z\\ -j\omega\mu\vec{H}_z &= \nabla_s \times \vec{E}_s\\ j\omega\varepsilon\vec{E}_z &= \nabla_s \times \vec{H}_s\end{aligned} \tag{7.41}$$

Use the identities $\hat{z}\times\left(\nabla_s \times \vec{E}_z\right) = \nabla_s E_z$ and $\hat{z}\times\left(\hat{z}\times\vec{E}_s\right) = -\vec{E}_s$, we can express $\vec{E}_s$ and $\vec{H}_s$ in terms of E_z and H_z.

$$\boxed{\begin{aligned}\vec{E}_s &= \frac{1}{\omega^2\mu\varepsilon - k_z^2}\left[\nabla_s\frac{\partial E_z}{\partial z} - j\omega\mu\nabla_s \times \vec{H}_z\right]\\ \vec{H}_s &= \frac{1}{\omega^2\mu\varepsilon - k_z^2}\left[\nabla_s\frac{\partial H_z}{\partial z} + j\omega\varepsilon\nabla_s \times \vec{E}_z\right]\end{aligned}} \tag{7.42}$$

where we make use of the fact that $\partial^2/\partial z^2 = -k_z^2$ and $\vec{E}_z = \hat{z}E_z$ and $\vec{H}_z = \hat{z}H_z$. Substituting Eq. (7.42) in Eq. (7.41) we obtain

$$\boxed{\left[\nabla_s^2 + \omega^2\mu\varepsilon - k_z^2\right]\begin{pmatrix}E_z\\ H_z\end{pmatrix} = 0} \tag{7.43}$$

These are homogeneous Helmholtz equations for E_z and H_z. When the longitudinal components are solved from Eq. (7.43) and the transverse components determined from Eq. (7.42), we can proceed to match the appropriate boundary conditions imposed by the guiding structures. Figure 7.7 shows several kinds of optical integrated waveguides based on dielectrics, which are widely used in optical communications.

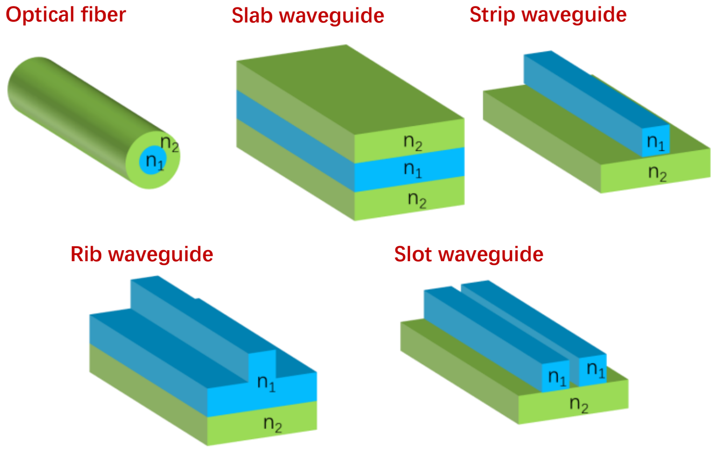

Figure 7.7 Different kinds of optical waveguides.

7.5 Additional Problems

See References [2-6].

Reference:

[1] MIT Physics 8.02: <<Electricity & Magnetism>>, by Sen-ben Liao, Peter Dourmashkin, and John W. Belcher.

[2] MIT Course 6.013: <<Electromagnetics and Applications>>, by David Staelin.

[3] <<Electromagnetic Wave Theory>>, J. A. Kong, Cambridge, 2008.

[4] << Introduction to Electrodynamics >>, David J. Griffiths, Fourth Edition, 2013.

[5] << Electricity and Magnetism >>, Edward M. Purcell, David J. Morin, Third Edition, 2013.

[6] << Theory and Computation of Electromagnetic Fields>>, Jian-Min Jin, Second Edition, 2015

Edited by Zuojia on 3 Nov 2019