Problem Solving 5: Waves in Media

OBJECTIVES:

1. To learn the use of the time-harmonic fields.

2. To learn the dispersion of medium, complex permittivity and penetration depth.

3. To learn the optical anisotropy/birefringence and the wave propagation in anisotropic media

REFERENCE: Chapter 5, Waves in Media

PROBLEM SOLVING STRATEGIES

A. Time-Harmonic Fields

In general, when the currents, charges, and fields oscillate at a single frequency, each quantity can be expressed as a sinusoidal/cosinusoidal function with an amplitude and a phase. For example, a monochromatic electric field can be written as

$$\vec{E}(\vec{r},t)=\vec{E}_0(\vec{r})\cos\left(\vec{k}\cdot\vec{r}-\omega t\right)$$

Define a complex quantity

$$\boxed{\vec{E}(\vec{r})=\vec{E}_0(\vec{r})e^{-j\vec{k}\cdot\vec{r}}}$$

which contains both the amplitude and phase of the field and is only a spatial function. The electric field can be written as

$$\boxed{\vec{E}(\vec{r},t)=\mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right]}$$

The complex quantity $\vec{E}(\vec{r})$ is called a *phasor*.

The time average value of the instantaneous Poynting's vector is

$$\boxed{\left\langle\vec{S}(\vec{r},t)\right\rangle=\frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right]=\frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]}$$

with the complex Poynting's vector

$$\boxed{\vec{S}=\vec{E}\times\vec{H}^*}$$

PROBLEM 1: Find the Maxwell's Equations for time-harmonic fields.

Solution:

The differential *Maxwell's Equations* in matter are:

$$\nabla \cdot \vec{D} = \rho_{free} \qquad (Gauss\text{'}s\ Law)$$
$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t \qquad (Faraday\text{'}s\ Law)$$
$$\nabla \cdot \vec{B} = 0 \qquad (Magnetic\ Gauss's\ Law)$$
$$\nabla \times \vec{H} = \vec{J}_{free} + \partial \vec{D} / \partial t \qquad (Ampere\text{'}s\ Law)$$

For time-harmonic fields, each quantity is replaced by a complex quantity, that is,

$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}(\vec{r})e^{j\omega t}\right], \quad \vec{D}(\vec{r},t) = \mathrm{Re}\left[\vec{D}(\vec{r})e^{j\omega t}\right]$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\left[\vec{H}(\vec{r})e^{j\omega t}\right], \quad \vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right]$$
$$\vec{J}(\vec{r},t) = \mathrm{Re}\left[\vec{J}(\vec{r})e^{j\omega t}\right], \quad \rho(\vec{r},t) = \mathrm{Re}\left[\rho(\vec{r})e^{j\omega t}\right]$$

Substitute them into the Maxwell's Equations, we have

$$\nabla \cdot \mathrm{Re}\left[\vec{D}(\vec{r})e^{j\omega t}\right] = \mathrm{Re}\left[\rho(\vec{r})e^{j\omega t}\right]$$
$$\nabla \times \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right] = -\frac{\partial}{\partial t}\mathrm{Re}\left[\vec{B}(\vec{r})e^{j\omega t}\right]$$
$$\nabla \cdot \mathrm{Re}\left[\vec{B}(\vec{r})e^{j\omega t}\right] = 0$$
$$\nabla \times \mathrm{Re}\left[\vec{H}(\vec{r})e^{j\omega t}\right] = \mathrm{Re}\left[\vec{J}(\vec{r})e^{j\omega t}\right] + \frac{\partial}{\partial t}\mathrm{Re}\left[\vec{D}(\vec{r})e^{j\omega t}\right]$$

=>

$$\mathrm{Re}\left\{\left[\nabla \cdot \vec{D}(\vec{r}) - \rho(\vec{r})\right]e^{j\omega t}\right\} = 0$$
$$\mathrm{Re}\left\{\left[\nabla \times \vec{E}(\vec{r}) + j\omega\vec{B}(\vec{r})\right]e^{j\omega t}\right\} = 0$$
$$\mathrm{Re}\left\{\left[\nabla \cdot \vec{B}(\vec{r})\right]e^{j\omega t}\right\} = 0$$
$$\mathrm{Re}\left\{\left[\nabla \times \vec{H}(\vec{r}) - \vec{J}(\vec{r}) - j\omega\vec{D}(\vec{r})\right]e^{j\omega t}\right\} = 0$$

for all time t. Therefore, we obtain the Maxwell's Equations for time-harmonic fields:

$$\nabla \cdot \vec{D}(\vec{r}) = \rho(\vec{r})$$
$$\nabla \times \vec{E}(\vec{r}) = -j\omega\vec{B}(\vec{r})$$
$$\nabla \cdot \vec{B}(\vec{r}) = 0$$
$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\vec{D}(\vec{r})$$

Usually, we omit writing the argument $\vec{r}$, that is,

$$\boxed{\begin{aligned} &\nabla \cdot \vec{D} = \rho \\ &\nabla \times \vec{E} = -j\omega\vec{B} \\ &\nabla \cdot \vec{B} = 0 \\ &\nabla \times \vec{H} = \vec{J} + j\omega\vec{D} \end{aligned}}$$

PROBLEM 2:

Obtain the phasor notation of the following time-harmonic functions (if possible):

(a) $V(t) = 6\cos(\omega t + \pi/4)$

(b) $I(t) = -8\sin(\omega t)$

(c) $A(t) = 3\sin(\omega t) - 2\cos(\omega t)$

(d) $C(t) = 6\cos(120\pi t - \pi/2)$

(e) $D(t) = 1 - \cos(\omega t)$

(f) $U(t) = \sin(\omega t + \pi/3)\sin(\omega t + \pi/6)$

Solutions:

(g) $V(t) = 6\cos(\omega t + \pi/4) = \mathrm{Re}\left[\left(6e^{j\pi/4}\right)\mathrm{e}^{j\omega t}\right]$ => $V = 6e^{j\pi/4}$

(h) $I(t) = -8\sin(\omega t) = \mathrm{Re}\left[\left(8e^{j\pi/2}\right)\mathrm{e}^{j\omega t}\right]$ => $I = 8e^{j\pi/2}$

(i) $A(t) = 3\sin(\omega t) - 2\cos(\omega t) = \mathrm{Re}\left[\left(-3e^{j\pi/2} - 2\right)\mathrm{e}^{j\omega t}\right]$ => $A = -3e^{j\pi/2} - 2 = -3j - 2$

(j) $C(t) = 6\cos(120\pi t - \pi/2) = \mathrm{Re}\left[\left(6e^{-j\pi/2}\right)\mathrm{e}^{j120\pi t}\right]$ => $C = 6e^{-j\pi/2} = -6j$

(k) None

(l) None

PROBLEM 3: Find the time-harmonic source-free Maxwell's Equations for plane wave solutions.

Solution:

The time-harmonic source-free *Maxwell's Equations* in matter are:

$$\begin{aligned}
&\nabla \cdot \vec{D} = 0 \\
&\nabla \times \vec{E} = -j\omega \vec{B} \\
&\nabla \cdot \vec{B} = 0 \\
&\nabla \times \vec{H} = j\omega \vec{D}
\end{aligned}$$

When plane wave solutions of the form $\exp\left(-j\vec{k}\cdot\vec{r}\right)$ is considered (wave vector $\vec{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$),

the source-free Maxwell's Equations become

$$\boxed{\begin{aligned}
&\vec{k} \cdot \vec{D} = 0 \\
&\vec{k} \times \vec{E} = \omega \vec{B} \\
&\vec{k} \cdot \vec{B} = 0 \\
&\vec{k} \times \vec{H} = -\omega \vec{D}
\end{aligned}}$$

These are the time-harmonic source-free Maxwell's Equations for plane wave solutions. We can see that $\vec{D}$ and $\vec{B}$ are always perpendicular to the wave vector $\vec{k}$.

B. Complex Permittivity and the Penetration Depth in Conducting Media

Complex relative permittivity: $\boxed{\varepsilon_r(\omega) = \varepsilon_r' - j\varepsilon_r''}$

Complex propagation constant or wave number: $\boxed{k = \omega\sqrt{\mu\varepsilon} = k' - jk''}$

The real part k' describes the propagation characteristics (e.g., phase velocity $v = \omega/k'$). The imaginary part k'' describes the rate of attenuation.

For conducting media, the permittivity can be defined as

$$\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}}$$

The complex wave number is

$$k = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = k' - jk''$$

where

$$k' = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2}\left(\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}+1\right)\right]^{1/2}$$

$$k'' = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2}\left(\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}-1\right)\right]^{1/2}$$

The penetration depth is defined as

$$\boxed{d_P = \frac{1}{k''}}$$

(i) For a highly conducting medium with $1 \ll \sigma/\omega\varepsilon$, the penetration depth is

$$d_P = \sqrt{\frac{2}{\omega\mu\sigma}}$$

(ii) For a highly conducting medium with $\sigma/\omega\varepsilon \ll 1$, the penetration depth is

$$d_P = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

PROBLEM 4:

(a) The complex permittivity for bottom round steak is about $\varepsilon = 40(1-0.3j)\varepsilon_0$ at the operating frequency (2.5 GHz) of a microwave oven. What is the penetration depth?

(b) Calculate the skin depths for sea water at frequencies 60 Hz and 10 MHz. Sea water can be characterized by conductivity σ = 4 mho/m, permittivity $\varepsilon = 80\varepsilon_0$, and permeability $\mu = \mu_0$ at those frequencies.

(c) A 100-Hz electromagnetic wave is propagating down into the sea water with an electric field intensity E of 1 V/m just beneath the sea surface. What is the intensity of E at a depth of 100 m?

Solution:

(a)

$$k = \omega\sqrt{\mu_0 40(1-0.3j)\varepsilon_0} = \frac{\omega}{c}\sqrt{40(1-0.3j)} = \frac{2\pi \times 2.5\times 10^9 Hz}{3\times 10^8 m/s}(6.40-0.94j)$$
$$= 335.1 - 49.2j \ (\mathrm{m}^{-1})$$

The penetration depth is $d_P = \frac{1}{k''} = \frac{1}{49.2\,(\mathrm{m}^{-1})} = 2\,\mathrm{cm}$.

(b)

at 60 Hz, $\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times 60\times 80\times 8.85\times 10^{-12}} = 1.5\times 10^7 \gg 1$, thus the penetration depth is

$$d_P = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi\times 60\times 4\pi\times 10^{-7}\times 4}} = \frac{10^3}{4\pi}\sqrt{\frac{1}{6}} = 32.5\,\mathrm{m}$$

at 10 MHz, $\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times 10\times 10^6\times 80\times 8.85\times 10^{-12}} = \frac{10^4}{\pi\times 4\times 8.85} = 90 \gg 1$, , thus the penetration depth is

$$d_P = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi\times 10\times 10^6\times 4\pi\times 10^{-7}\times 4}} = \frac{1}{4\pi} = 0.0796\,\mathrm{m}$$

(c)

at 100 Hz, $\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times 100\times 80\times 8.85\times 10^{-12}} \gg 1$, thus the penetration depth is

$$d_P = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi\times 100\times 4\pi\times 10^{-7}\times 4}} = \frac{10^3}{4\pi}\sqrt{\frac{1}{10}} = 25.2\,\mathrm{m}$$

$$k'' = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\times 100\times 4\pi\times 10^{-7}\times 4}{2}} = \frac{4\pi}{10^3}\sqrt{10} = 39.74\times 10^{-3}\ \mathrm{m}^{-1}$$

$$E = E_0\exp(-k''d) = \exp\left(39.74\times 10^{-3}\ \mathrm{m}^{-1}\times 100\,\mathrm{m}\right) = \exp(-3.9) = 0.02\,\mathrm{V/m}$$

C. Optical Anisotropy and Birefringence

The constitutive relations for anisotropic media are

$$\vec{D} = \overline{\overline{\varepsilon}}\vec{E}\ , \quad \vec{B} = \overline{\overline{\mu}}\vec{H}$$

with the permittivity and permeability tensor

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}, \quad \overline{\overline{\mu}} = \begin{bmatrix} \mu_x & & \\ & \mu_y & \\ & & \mu_z \end{bmatrix}$$

The wave properties in anisotropic media depend on both the propagating direction and the polarizations, which can be summarized in Table 5.1.

Table 5.1 Wave propagation in anisotropic media

k vector	electric field	magnetic field	wave number
$\vec{k} = [k_x, 0, 0]^T$	$\vec{E} = [0, E_y, 0]^T$	$\vec{H} = [0, 0, H_z]^T$	$k_x = \omega\sqrt{\mu_z \varepsilon_y}$
	$\vec{E} = [0, 0, E_z]^T$	$\vec{H} = [0, H_y, 0]^T$	$k_x = \omega\sqrt{\mu_y \varepsilon_z}$
$\vec{k} = [0, k_y, 0]^T$	$\vec{E} = [E_x, 0, 0]^T$	$\vec{H} = [0, 0, H_z]^T$	$k_y = \omega\sqrt{\mu_z \varepsilon_x}$
	$\vec{E} = [0, 0, E_z]^T$	$\vec{H} = [H_x, 0, 0]^T$	$k_y = \omega\sqrt{\mu_x \varepsilon_z}$
$\vec{k} = [0, 0, k_z]^T$	$\vec{E} = [E_x, 0, 0]^T$	$\vec{H} = [0, H_y, 0]^T$	$k_z = \omega\sqrt{\mu_y \varepsilon_x}$
	$\vec{E} = [0, E_y, 0]^T$	$\vec{H} = [H_x, 0, 0]^T$	$k_z = \omega\sqrt{\mu_x \varepsilon_y}$

PROBLEM 5:

For an anisotropic media with

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}, \quad \overline{\overline{\mu}} = \begin{bmatrix} \mu_x & & \\ & \mu_y & \\ & & \mu_z \end{bmatrix}$$

(1) assume a plane wave propagates in the *z* direction, the electric field is polarized in the *x* direction and the magnetic field is in the *y* direction, find the wave number for this plane wave.

(2) assume a plane wave propagates in the z direction, the electric field is polarized in the y direction and the magnetic field is in the x direction, find the wave number for this plane wave.

Solution:

For time-harmonic fields, the source-free Maxwell's Equations of plane waves are

$$\vec{k} \times \vec{E} = \omega \vec{B}$$
$$\vec{k} \times \vec{H} = -\omega \vec{D}$$

(1) Substitute $\vec{k} = \hat{z}k_z$, $\vec{E} = \hat{x}E_0$ and $\vec{H} = \hat{y}H_0$ into above equations, we have

$$\vec{B} = \frac{\hat{z}k_z \times \hat{x}E_0}{\omega} = \hat{y}\frac{k_z}{\omega}E_0$$
$$\vec{D} = \frac{\hat{z}k_z \times \hat{y}H_0}{-\omega} = \hat{x}\frac{k_z}{\omega}H_0$$

Then apply the constitutive relations

$$\vec{D} = \overline{\overline{\varepsilon}}\vec{E}, \quad \vec{B} = \overline{\overline{\mu}}\vec{H}$$

we obtain

$$\begin{bmatrix} \mu_x & & \\ & \mu_y & \\ & & \mu_z \end{bmatrix}\vec{H} = \hat{y}\frac{k_z}{\omega}E_0, \quad \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}\vec{E} = \hat{x}\frac{k_z}{\omega}H_0$$

=>

$$\mu_y H_0 = \frac{k_z}{\omega}E_0$$
$$\varepsilon_x E_0 = \frac{k_z}{\omega}H_0$$

=>

$$\left(\frac{k_z}{\omega}\right)^2 = \mu_y\varepsilon_x, \text{ or } \boxed{k_z = \omega\sqrt{\mu_y\varepsilon_x}}$$

(2) Substitute $\vec{k} = \hat{z}k_z$, $\vec{E} = \hat{y}E_0$ and $\vec{H} = \hat{x}H_0$ into above equations, we have

$$\vec{B} = \frac{\hat{z}k_z \times \hat{y}E_0}{\omega} = -\hat{x}\frac{k_z}{\omega}E_0$$
$$\vec{D} = \frac{\hat{z}k_z \times \hat{x}H_0}{-\omega} = -\hat{y}\frac{k_z}{\omega}H_0$$

Then apply the constitutive relations

$$\vec{D} = \overline{\overline{\varepsilon}}\vec{E}\ , \quad \vec{B} = \overline{\overline{\mu}}\vec{H}$$

we obtain

$$\begin{bmatrix} \mu_x & & \\ & \mu_y & \\ & & \mu_z \end{bmatrix} \vec{H} = -\hat{x}\frac{k_z}{\omega}E_0, \qquad \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix} \vec{E} = -\hat{y}\frac{k_z}{\omega}H_0$$

=>

$$\mu_x H_0 = \frac{k_z}{\omega}E_0$$
$$\varepsilon_y E_0 = \frac{k_z}{\omega}H_0$$

=>

$$\left(\frac{k_z}{\omega}\right)^2 = \mu_x \varepsilon_y \text{ , or } \boxed{k_z = \omega\sqrt{\mu_x \varepsilon_y}}$$

As a result, two polarizations have different wave numbers.