Chapter 4: Boundary Conditions

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In this chapter we study how electric/magnetic fields are affected by matter. We concern ourselves with insulators, or *dielectrics*, characterized by a *dielectric constant or a permittivity*. We also concern on magnetization in magnetic materials, characterized by a *permeability*. The boundary conditions adjacent to different media are discussed. The scalar and vector potentials are introduced to transform the vector Maxwell's equations into scalar Poisson's equations, so as to reduce the complexity in calculation of electric/magnetic fields. The method of image and uniqueness theorem are introduced in the last section of this chapter.

4.1 Dielectrics and Polarization

4.1.1 Dielectrics

We consider a capacitor consisted of two conductors, insulated from one another, with nothing in between. The system of two conductors was characterized by a certain capacitance C_0, a constant relating the magnitude of the charge Q on the capacitor (positive charge Q on one plate, equal negative charge on the other) to the difference in electric potential between the two conductors, V_1-V_2. Let's denote the potential difference by V_{12}. For the parallel-plate capacitor, two flat plates each of area A and separated by a distance d, we found that the capacitance is given by

$$C_0 = \frac{Q}{V_{12}} = \frac{\varepsilon_0 A}{d} \tag{4.1}$$

Capacitors like this can be found in some electrical apparatus. They are called vacuum capacitors and consist of plates enclosed in a highly evacuated bottle.

Far more common, however, are capacitors in which the space between the plates is filled with some nonconducting solid or liquid substance. Experimentally it was found that capacitance C increases when the space between the conductors is filled with dielectrics. When a dielectric material is inserted to completely fill the space between the plates, the capacitance increases to

$$C = \kappa_e C_0 \tag{4.2}$$

where is κ_e called the dielectric constant. In the Table below, we show some dielectric materials with their dielectric constant. Experiments indicate that all dielectric materials have. Note that every dielectric material has a characteristic dielectric strength which is the maximum value of electric field before breakdown occurs and charges begin to flow.

Material	Air	Paper	Glass	Water
κ_e	1.00059	3.7	4-6	80

The fact that capacitance increases in the presence of a dielectric can be explained from molecular point of view. We shall show that κ_e is a measure of the dielectric response to an external electric field. There are two types of dielectrics. The first type is polar dielectrics, which are dielectrics that have permanent electric dipole moments. An example of this type of dielectric is water.

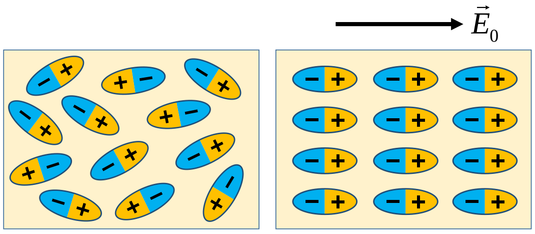

Figure 4.1 Orientations of polar molecules under an external electric field.

As depicted in Figure 4.1, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\vec{E}_0$ is present, a torque is set up and causes the molecules to align with $\vec{E}_0$. However, the alignment is not complete due to random thermal motion. The aligned molecules then generate an electric field that is opposite to the applied field but smaller in magnitude.

The second type of dielectrics is the non-polar dielectrics, which are dielectrics that do not possess permanent electric dipole moment. Electric dipole moments can be induced by placing the materials in an externally applied electric field.

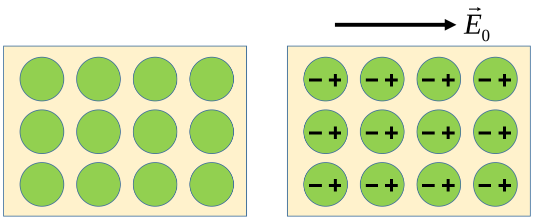

Figure 4.2 Orientations of non-polar molecules under an external electric field.

Figure 4.2 illustrates the orientation of non-polar molecules with and without an external field $\vec{E}_0$. The induced surface charges on the faces produces an electric field $\vec{E}_P$ in the direction opposite to $\vec{E}_0$, leading to $\vec{E} = \vec{E}_0 + \vec{E}_P$, with $|\vec{E}| < |\vec{E}_0|$. Below we show how the induced electric field $\vec{E}_P$ is calculated.

4.1.2 Polarization

We have shown that dielectric materials consist of many permanent or induced electric dipoles. One of the concepts crucial to the understanding of dielectric materials is the average electric field produced by many little electric dipoles which are all aligned. Suppose we have a piece of material in the form of a cylinder with area A and height h, as shown in Figure 4.3a, and that it consists of N electric dipoles, each with electric dipole moment $\vec{p}$ spread uniformly throughout the volume of the cylinder.

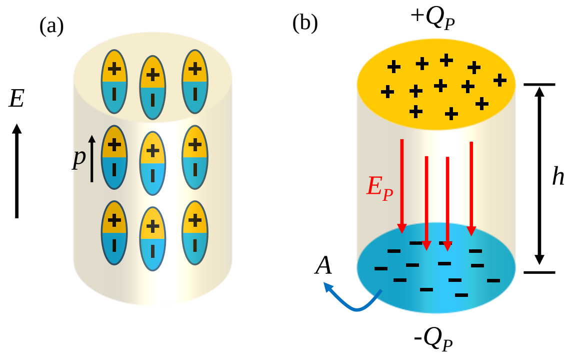

Figure 4.3 (a) A cylinder with uniform dipole distribution. (b) Equivalent charge distribution.

We furthermore assume for the moment that all of the electric dipole moments $\vec{p}$ are aligned with the axis of the cylinder. Since each electric dipole has its own electric field associated with it, in the absence of any external electric field, if we average over all the individual fields produced by the dipole, what is the average electric field just due to the presence of the aligned dipoles?

To answer this question, let us define the polarization vector $\vec{P}$ to be the net electric dipole moment vector per unit volume:

$$\vec{P} = \frac{1}{volume}\sum_{i=1}^{N}\vec{p}_i = \frac{Np}{Ah}\frac{\vec{E}}{|E|} \tag{4.3}$$

In the case of our cylinder, where all the dipoles are perfectly aligned, the magnitude of $\vec{P}$ is equal to

$$P = \frac{Np}{Ah} \tag{4.4}$$

and its direction is parallel to the aligned dipoles.

Now, what is the average electric field these dipoles produce? The key to figuring this out is realizing that the situation shown in Figure 4.3a is equivalent that shown in Figure 4.3b, where all the little ± charges associated with the electric dipoles in the interior of the cylinder are replaced with two equivalent charges, $\pm Q_P$, on the top and bottom of the cylinder, respectively.

The equivalence can be seen by noting that in the interior of the cylinder, positive charge at the top of any one of the electric dipoles is canceled on average by the negative charge of the dipole just above it. The only place where cancellation does not take place is for electric dipoles at the top of the cylinder, since there are no adjacent dipoles further up. Thus the interior of the cylinder appears uncharged in an average sense (averaging over many dipoles), whereas the top surface of the cylinder appears to carry a net positive charge. Similarly, the bottom surface of the cylinder will appear to carry a net negative charge.

How do we find an expression for the equivalent charge Q_P in terms of quantities we know? The simplest way is to require that the electric dipole moment Q_P produces, $Q_P h$, is equal to the total electric dipole moment of all the little electric dipoles. This gives $Q_P h = Np$, or

$$Q_P = \frac{Np}{h} \tag{4.5}$$

To compute the electric field produced by Q_P, we note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_P that is equal to the magnitude of the polarization:

$$\sigma_P = \frac{Q_P}{A} = \frac{Np}{Ah} = P \tag{4.6}$$

Note that the SI units of P are (C·m)/m^3, or C/m^2, which is the same as the surface charge density. In general if the polarization vector makes an angle θ with $\hat{n}$, the outward normal vector of the surface, the surface charge density would be

$$\sigma_P = \vec{P} \cdot \hat{n} = P\cos\theta \tag{4.7}$$

Thus, our equivalent charge system will produce an average electric field of magnitude $E_P = P/\varepsilon_0$. Since the direction of this electric field is opposite to the direction of $\vec{P}$, in vector notation, we have

$$\vec{E}_P = -\vec{P}/\varepsilon_0 \tag{4.8}$$

Thus, the average electric field of all these dipoles is opposite to the direction of the dipoles themselves. It is important to realize that this is just the average field due to all the dipoles. If we go close to any individual dipole, we will see a very different field.

We have assumed here that all our electric dipoles are aligned. In general, if these dipoles are randomly oriented, then the polarization $\vec{P}$ given in Eq. (4.3) will be zero, and there will be no average field due to their presence. If the dipoles have some tendency toward a preferred orientation, then $\vec{P} \neq \vec{0}$, leading to a non-vanishing average field $\vec{E}_P$.

Let us now examine the effects of introducing dielectric material into a system. We shall first assume that the atoms or molecules comprising the dielectric material have a permanent electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field, $\vec{P} = \vec{0}$ due to the random alignment of dipoles, and the average electric field $\vec{E}_P$ is zero as well. However, when we place the dielectric material in an external field $\vec{E}_0$, the dipoles will experience a torque $\vec{\tau} = \vec{p} \times \vec{E}_0$ that tends to align the dipole vectors $\vec{p}$ with $\vec{E}_0$. The effect is a net polarization $\vec{P}$ parallel to $\vec{E}_0$, and therefore an average electric field of the dipoles $\vec{E}_P$ anti-parallel to $\vec{E}_0$, i.e., that will tend to reduce the total electric field strength below $\vec{E}_0$. The total electric field $\vec{E}$ is the sum of these two fields:

$$\vec{E} = \vec{E}_0 + \vec{E}_P = \vec{E}_0 - \vec{P}/\varepsilon_0 \tag{4.9}$$

In most cases, the polarization $\vec{P}$ is not only in the same direction as $\vec{E}_0$, but also linearly proportional to $\vec{E}_0$ (and hence $\vec{E}$.) This is reasonable because without the external field $\vec{E}_0$ there would be no alignment of dipoles and no polarization $\vec{P}$. We write the linear relation between $\vec{P}$ and $\vec{E}$ as

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{4.10}$$

where χ_e is called the *electric susceptibility*. Materials they obey this relation are *linear dielectrics*. Combing Eqs. (4.9) and (4.10) gives

$$\vec{E}_0 = \left(1 + \chi_e\right)\vec{E} = \kappa_e \vec{E} \tag{4.11}$$

where

$$\kappa_e = 1 + \chi_e \tag{4.12}$$

is the dielectric constant. The dielectric constant κ_e is always greater than one since $\chi_e > 0$. This implies

$$E = \frac{E_0}{\kappa_e} < E_0 \tag{4.13}$$

Thus, we see that the effect of dielectric materials is always to decrease the electric field below what it would otherwise be.

In the case of dielectric material where there are no permanent electric dipoles, a similar effect is observed because the presence of an external field $\vec{E}_0$ induces electric dipole moments in the atoms or molecules. These induced electric dipoles are parallel to $\vec{E}_0$ again leading to a polarization $\vec{P}$ parallel to $\vec{E}_0$, and a reduction of the total electric field strength.

4.1.3 Gauss's Law for Dielectrics

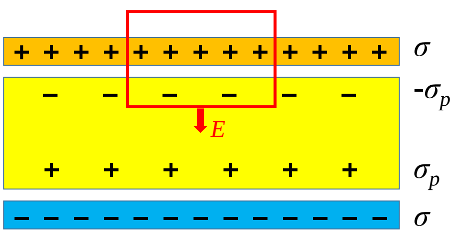

Figure 4.4 Gaussian surface in the presence of a dielectric.

Consider again a parallel-plate capacitor shown in Figure 4.4. When no dielectric is present, the electric field $\vec{E}_0$ in the region between the plates can be found by using Gauss's law:

$$\oiint_S \vec{E}\cdot d\vec{S} = E_0 A = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}, \;\; => \;\; E_0 = \frac{\sigma}{\varepsilon_0} \tag{4.14}$$

where Q is the free charge on the plate. When a dielectric is inserted, there is an induced charge Q_P of opposite sign on the surface, and the net charge enclosed by the Gaussian surface is $Q - Q_P$.

Gauss's law becomes

$$\oiint_S \vec{E}\cdot d\vec{S} = EA = \frac{Q_{inside}}{\varepsilon_0} = \frac{(\sigma - \sigma_P)A}{\varepsilon_0}, \;\; => \;\; E = \frac{(\sigma - \sigma_P)}{\varepsilon_0} \tag{4.15}$$

where σ_P is the surface charge density of the equivalent charge.

From Eq. (4.13), we find that

$$E = \frac{E_0}{\kappa_e} = \frac{\sigma}{\kappa_e \varepsilon_0} = \frac{(\sigma - \sigma_P)}{\varepsilon_0}, \;\; => \;\; \sigma_P = \sigma\left(1 - \frac{1}{\kappa_e}\right) \tag{4.16}$$

Substitute Eq. (4.16) into Eq. (4.15), we can see that Gauss's law with dielectric field can be rewritten as

$$\oiint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\kappa_e \varepsilon_0} \tag{4.17}$$

or

$$\boxed{\oiint_S \vec{D}\cdot d\vec{S} = Q_{free} = \iiint_V \rho_{free} dV} \tag{4.18}$$

where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \kappa_e \varepsilon_0 \vec{E} = \varepsilon \vec{E}$ is called the *electric displacement vector*, and $\varepsilon = \kappa_e \varepsilon_0$ is called the dielectric *permittivity*. The subscript '*free*' in Q_{free} is used to emphasis that, in the modified Gauss's law, the surface integration of the electric displacement vector over a closed surface is equal to the *free charge* in the enclosed volume.

Displacement field vector $\vec{D}$ accounts for the effects of unbound ("free") charges within materials. Electric field $\vec{E}$ accounts for the effects of total charges (both "bound" and "free") within materials:

$$\oiint_S \varepsilon_0 \vec{E}\cdot d\vec{S} = \iiint_V \rho_{total} dV . \tag{4.19}$$

Applying the divergence theorem ($\oiint_S \vec{F}\cdot d\vec{S} = \iiint_V \left(\nabla\cdot\vec{F}\right) dV$), we can get

$$\boxed{\nabla\cdot\vec{D} = \rho_{free}} \tag{4.20}$$

which is the differential form of the rewritten Gauss's law. The total charge density ρ_{total} is a sum of the free charge density and the bound charge density:

$$\rho_{total} = \rho_{free} + \rho_{bound}$$
$$\rho_{bound} = -\nabla \cdot \vec{P} \tag{4.21}$$

4.1.4 Polarization in changing field

So far we have considered only electrostatic fields in matter. We need to look at the effects of electric fields that are varying in time, like the field in a capacitor used in an alternating-current circuit. The important question is, will the changes in polarization keep up with the changes in the field? Will the ratio of $\vec{P}$ to $\vec{E}$, at any instant, be the same as in a static electric field? For very slow changes we should expect no difference but, as always, the criterion for slowness depends on the particular physical process. It turns out that induced polarization and the orientation of permanent dipoles are two processes with quite different response times.

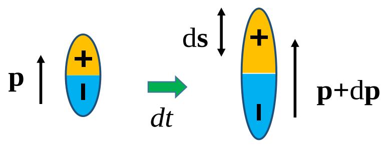

Figure 4.5 The dipole in matter changes with time.

Wherever the polarization in matter changes with time, there is an electric current, a genuine motion of charge. Suppose there are *N* dipoles per cubic meter of dielectric, and that in the time interval *dt* each changes $\vec{p}$ to $\vec{p}+d\vec{p}$. Then the macroscopic polarization density $\vec{P}$ changes from $\vec{P} = N\vec{p}$ to $\vec{P}+d\vec{P} = N\left(\vec{p}+d\vec{p}\right)$. Suppose the change $d\vec{p}$ was effected by moving a charge *q* through a distance $d\vec{s}$, in each atom: $qd\vec{s} = d\vec{p}$. Then during the time *dt* there was actually a charge cloud of density $\rho = Nq$, moving with velocity $\vec{v} = d\vec{s}/dt$. This is a conduction current of a certain density $\vec{J}$ in coulombs per second per square meter:

$$\vec{J} = \rho\vec{v} = Nq\frac{d\vec{s}}{dt} = N\frac{d\vec{p}}{dt}, \;\; => \;\; \boxed{\vec{J} = \frac{d\vec{P}}{dt}} \tag{4.22}$$

The connection between rate of change of polarization and current density, $\vec{J} = d\vec{P}/dt$, is independent of the details of the model. A changing polarization is a conduction current, not essentially different from any other. Note that if we take the divergence of both sides of Eq. (4.22) and use Eq. (4.21), we obtain

$$\nabla \cdot \vec{J} = \frac{d\left(\nabla \cdot \vec{P}\right)}{dt} = -\frac{d\rho_{bound}}{dt} \tag{4.23}$$

which is consistent with the continuity equation of current.

Naturally, such a current is a source of magnetic field. We should rewrite the Ampere's Law in Maxwell's equations as

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{d\vec{P}}{dt} + \mu_0 \vec{J}_{free} \tag{4.24}$$

The only difference between an "ordinary" conduction current density and the current density $d\vec{P}/dt$ is that one involves *free* charge in motion, the other *bound* charge in motion. There is one rather obvious practical distinction – you can't have a steady bound-charge current, one that goes on forever unchanged. Usually we prefer to keep account separately of the bound-charge current ($d\vec{P}/dt$) and the free-charge current ($\vec{J}_{free}$).

In a linear dielectric medium, we can write $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, allowing a shorter version of Eq. (4.24)

$$\boxed{\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} + \mu_0 \vec{J}_{free}} \tag{4.25}$$

The term $\partial\vec{D}/\partial t$ is usually referred to as the displacement current. Actually, the part of it that involves $\partial\vec{P}/\partial t$ represents, as we have seen, an honest conduction current, real charges in motion. The only part of the total current density that is not simply charge in motion is the $\partial\vec{E}/\partial t$ part, the true *vacuum displacement current*.

4.2 Magnetic Materials and Magnetization

The introduction of material media into the study of magnetism has very different consequences as compared to the introduction of material media into the study of electrostatics. When we dealt with dielectric materials in electrostatics, their effect was always to reduce $\vec{E}$ below what it would otherwise be, for a given amount of "free" electric charge. In contrast, when we deal with magnetic materials, their effect can be one of the following:

(1) reduce $\vec{B}$ below what it would otherwise be, for the same amount of "free" electric current (diamagnetic materials);

(2) increase $\vec{B}$ a little above what it would otherwise be (paramagnetic materials);

(3) increase $\vec{B}$ a lot above what it would otherwise be (ferromagnetic materials).

4.2.1 Magnetization

Magnetic materials consist of many permanent or induced magnetic dipoles. One of the concepts crucial to the understanding of magnetic materials is the average magnetic field produced by many magnetic dipoles which are all aligned. Suppose we have a piece of material in the form of a long cylinder with area *S* and height *L*, and that it consists of *N* magnetic dipoles, each with magnetic dipole moment $\vec{m}$, spread uniformly throughout the volume of the cylinder, as shown in Figure 4.6a.

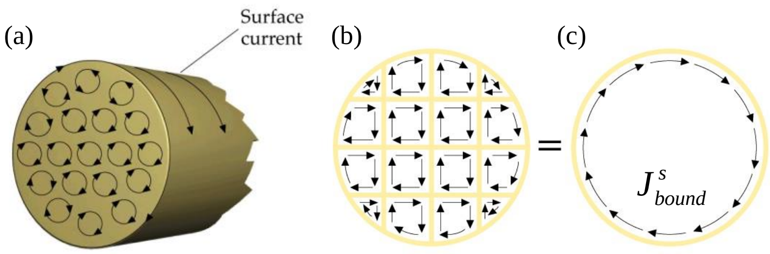

Figure 4.6 Uniform magnetization of a magnetic cylinder and the equivalent bound current.

We also assume that all of the magnetic dipole moments $\vec{m}$ are aligned with the axis of the cylinder. In the absence of any external magnetic field, what is the average magnetic field due to these dipoles alone?

To answer this question, we note that each magnetic dipole has its own magnetic field associated with it. Let's define the magnetization vector $\vec{M}$ to be the net magnetic dipole moment vector per unit volume:

$$\vec{M} = \frac{1}{volume}\sum_i \vec{m}_i \tag{4.26}$$

In the case of our cylinder, where all the dipoles are aligned, the magnitude of $\vec{M}$ is simply $\vec{M} = Nm/SL$.

Now, what is the average magnetic field produced by all the dipoles in the cylinder?

Figure 4.6b depicts the small current loops associated with the dipole moments and the direction of the currents, as seen from above. We see that in the interior, currents flow in a given direction will be cancelled out by currents flowing in the opposite direction in neighboring loops. The only place where cancellation does not take place is near the edge of the cylinder where there are no adjacent loops further out. Thus, the average current in the interior of the cylinder vanishes, whereas the sides of the cylinder appear to carry a net current. The equivalent situation is shown in Figure 4.6c, where there is an equivalent surface bound current $J^s_{bound} = M$ on the sides.

Nonuniform magnetization

If the magnetization $\vec{M}$ within a volume of material is not uniform but instead varies with position as $\vec{M}(x, y, \mathrm{z})$, the equivalent current distribution is given simply by

$$\boxed{\vec{J}_{bound} = \nabla \times \vec{M}} \tag{4.27}$$

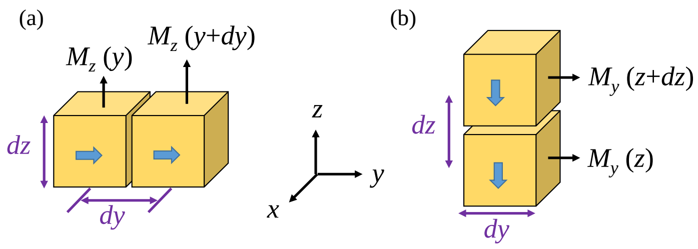

Figure 4.7 Nonuniform magnetization is equivalent to a volume current density.

Let's see how this comes about in one situation. Figure 4.7 shows two adjacent chunks of magnetized material, with a larger arrow on the one to the right suggesting greater magnetization at that point. On the surface where they join, there is a net current in the x direction, given by

$$I_x = \left[M_z\left(y+dy\right) - M_z\left(y\right)\right]dz = \frac{\partial M_z}{\partial y}dydz \tag{4.28}$$

The corresponding volume current density is therefore

$$\left(J_{bound}\right)_x = \frac{I_x}{dydz} = \frac{\partial M_z}{\partial y} \tag{4.29}$$

By the same token, a nonuniform magnetization in the y direction would contribute an amount $-\partial M_y/\partial z$, so

$$\left(J_{bound}\right)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \tag{4.30}$$

In general, then,

$$\vec{J}_{bound} = \nabla \times \vec{M} \tag{4.31}$$

Example 4.1: Show that the $J^s_{bound} = M$ follows from $\vec{J}_{bound} = \nabla \times \vec{M}$ in Figure 4.8.

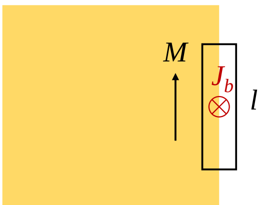

Figure 4.8 The surface integral of the bound current over a thin rectangle at the boundary.

Solutions:

Consider a thin rectangle, with one of its long sides inside the material and the other outside. If we integrate $\vec{J}_{bound} = \nabla \times \vec{M}$ over the surface S of this rectangle, we can use Stokes' theorem to write

$$\iint_S \vec{J}_{bound} \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{M}\right) \cdot d\vec{S} = \oint_C \vec{M} \cdot d\vec{l}$$

This gives

$$I_s = \iint_S \vec{J}_{bound} \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{M}\right) \cdot d\vec{S} = \oint_C \vec{M} \cdot d\vec{l}$$

where I_S is the current passing through S. This current can be written as $J^s_{bound} l$, where l is the height of the rectangle. This is true because we can make the rectangle arbitrary thin, so any current, passing through it must arise from a surface current density J^s_{bound} . The right integral simply equals Ml, because $\vec{M}$ is nonzero only along the left side of the rectangle. It gives $J^s_{bound} l = Ml$, and thus $J^s_{bound} = M$ as desired.

Base on above analysis, we have found that the effect of magnetization is to establish bound currents $\vec{J}_{bound} = \nabla \times \vec{M}$ within the material and on the surface. The field due to magnetization of the medium is just the field produced by these bound currents. We are now ready to put everything together: the field attributable to bound currents, plus the field due to everything else—which I shall call the free current. The free current might flow through wires imbedded in the magnetized substance or, if the latter is a conductor, through the material itself. In any event, the total current can be written as

$$\vec{J} = \vec{J}_{free} + \vec{J}_{bound} = \vec{J}_{free} + \nabla \times \vec{M} \tag{4.32}$$

The Ampere's Law for magnetic materials can be rewritten as

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} + \mu_0 \left(\vec{J}_{free} + \nabla \times \vec{M}\right) \tag{4.33}$$

or, collecting together the two curls:

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free} \tag{4.34}$$

The quantity in parentheses is designated by the letter $\vec{H}$, called *magnetic field strength* (with the unit A/m)

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}} \tag{4.35}$$

In terms of $\vec{H}$, then, Ampère's law reads

$$\boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free}} \tag{4.36}$$

4.2.2 Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when B is removed, M disappears. In fact, for most substances the magnetization is proportional to the field, provided the field is not too strong. For notational consistency with the electrical case, I *should* express the proportionality thus:

$$\vec{M} = \frac{1}{\mu_0}\chi_m \vec{B} \quad \text{(incorrect!)} \tag{4.37}$$

But custom dictates that it be written in terms of *H*, instead of *B*:

$$\boxed{\vec{M} = \chi_m \vec{H}} \tag{4.38}$$

The constant of proportionality χ_m is called the *magnetic susceptibility*. It is a dimensionless quantity that varies from one substance to another, positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5}.

In view of Eq. (4.38),

$$\vec{B} = \mu \vec{H} \tag{4.39}$$

where

$$\mu \equiv \mu_0 \left(1 + \chi_m\right) \tag{4.40}$$

μ is called the *permeability* of the material. In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the permeability of free space. If you factor out μ_0, what's left is called the *relative permeability* $\mu_r = 1 + \chi_m$.

4.3 Maxwell's Equations in Matter

During the analysis on the polarization and magnetization, we have encountered the differential *Maxwell's Equations* in matter

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{D} = \rho_{free} & (Gauss's\ Law) \\ \nabla \times \vec{E} = -\partial \vec{B}/\partial t & (Faraday's\ Law) \\ \nabla \cdot \vec{B} = 0 & (Magnetic\ Gauss's\ Law) \\ \nabla \times \vec{H} = \vec{J}_{free} + \partial \vec{D}/\partial t & (Ampere's\ Law) \end{array}} \tag{4.41}$$

and the *continuity equation*

$$\boxed{\nabla \cdot \vec{J} = -\partial \rho_{free}/\partial t} \tag{4.42}$$

with the *constitutive relations*

$$\boxed{\begin{array}{l} \vec{D} = \varepsilon \vec{E} \\ \vec{B} = \mu \vec{H} \end{array}} \tag{4.43}$$

The integral forms of the Maxwell's equations in matters are

$$\boxed{\begin{array}{ll} \oiint_S \vec{D}\cdot d\vec{S} = \iiint_V \rho_{free} dV & (Gauss\text{'s } Law) \\ \oint_C \vec{E}\cdot d\vec{l} = -\iint_S \left(\partial\vec{B}/\partial t\right)\cdot d\vec{S} & (Faraday\text{'s } Law) \\ \oiint_S \vec{B}\cdot d\vec{S} = 0 & (Magnetic\ Gauss\text{'s } Law) \\ \oint_C \vec{H}\cdot d\vec{l} = \iint_S \left(\vec{J}_{free} + \partial\vec{D}/\partial t\right)\cdot d\vec{S} & (Ampere\text{'s } Law) \end{array}} \tag{4.44}$$

4.4 Boundary Conditions

Maxwell's equations characterize macroscopic matter by means of its permittivity ε, permeability μ, and conductivity σ, where these properties are usually represented by scalars and can vary among media. This section discusses how Maxwell's equations strongly constrain the behavior of electromagnetic fields at boundaries between two media having different properties, where these constraint equations are called boundary conditions. One result of these boundary conditions is that waves at boundaries are generally partially transmitted and partially reflected with directions and amplitudes that depend on the two media and the incident angles and polarizations. Static fields also generally have different amplitudes and directions on the two sides of a boundary. Some boundaries in both static and dynamic situations also possess surface charge or carry surface currents that further affect the adjacent fields. The boundary conditions governing the perpendicular components of $\vec{E}$ and $\vec{H}$ follow from the integral forms of Gauss's laws, while the boundary conditions governing the parallel components of $\vec{E}$ and $\vec{H}$ follow from Faraday's and Ampere's laws.

4.4.1 Boundary conditions for perpendicular field components

Beginning with the boundary condition for the perpendicular component $D_\perp$, we integrate electric Gauss's law in Eq. (4.44) over the surface S and volume $V=\Delta x\Delta y\Delta z$ of the thin infinitesimal pillbox illustrated in Figure 4.9.

$$\left(D_{1\perp} - D_{2\perp}\right)\Delta x\Delta y = \rho_s \Delta x\Delta y \tag{4.45}$$

where ρ_s is the surface charge density [C/m^2]. The subscript s for surface charge ρ_s distinguishes it from the volume charge density ρ [C/m^3]. The pillbox is so thin ($\Delta z\to 0$) that: 1) the contribution to the surface integral of the sides of the pillbox vanishes in comparison to the rest of the integral, and 2) only a surface charge q can be contained within it, where where $\rho_s = q/\Delta x\Delta y = \lim \rho\Delta z$ as the charge density $\rho\to\infty$, and $\Delta z\to 0$. Thus Eq. (4.45) becomes $D_{1\perp} - D_{2\perp} = \rho_s$, which can be written as

$$\hat{n}\cdot\left(\vec{D}_1 - \vec{D}_2\right) = \rho_s \tag{4.46}$$

where $\hat{n}$ is the unit vector normal to the boundary and points into medium 1. Thus the perpendicular component of the electric displacement vector $\vec{D}$ changes value at a boundary in accord with the surface charge density ρ_s.

Because Gauss's laws are the same for electric and magnetic fields, except that there are no magnetic charges, the same analysis for the magnetic flux density $\vec{B}$ yields a similar boundary condition

$$\hat{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0 \tag{4.47}$$

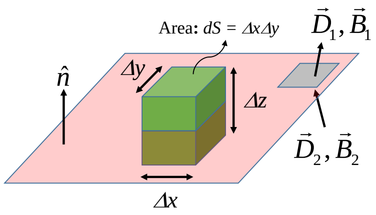

Figure 4.9 Elemental volume for deriving boundary conditions for perpendicular field components.

4.4.2 Boundary conditions for parallelfield components

The boundary conditions governing the parallel components of $\vec{E}$ and $\vec{H}$ follow from Faraday's and Ampere's laws in Eq. (4.44). We can integrate these equations around the elongated rectangular contour C that straddles the boundary and has infinitesimal area $dS = \Delta x\Delta z$, as illustrated in Figure 4.10. We assume the total height Δz of the rectangle is much less than its length Δx, and circle C in a right-hand sense relative to the surface normal $\hat{n}=\hat{y}$.

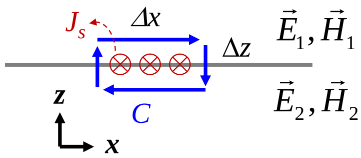

Figure 4.10 Elemental volume for deriving boundary conditions for parallel field components.

Beginning with Faraday's law, we find

$$\vec{E}_1\cdot\hat{x}\Delta x-\vec{E}_2\cdot\hat{x}\Delta x=-\iint_S\left(\partial\vec{B}/\partial t\right)\cdot d\vec{S}\rightarrow 0 \tag{4.48}$$

where the integral of B over area dS approaches zero in the limit where Δz approaches zero too, because there can be no impulses in B. Since $\Delta x\neq 0$, it follows from Eq. (4.48) that $E_{1x}-E_{2x}=0$, or more generally:

$$\hat{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \tag{4.49}$$

Thus the parallel component of E must be continuous across any boundary.

A similar integration of Ampere's law, under the assumption that the contour C is chosen to lie in a plane perpendicular to the surface current J_s and is traversed in the right-hand sense, yields:

$$\vec{H}_1 \cdot \hat{x}\Delta x - \vec{H}_2 \cdot \hat{x}\Delta x = \iint_S \left(\vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = \vec{J}_{free} \cdot (\hat{y}) \Delta x \Delta z \tag{4.50}$$

where we note that the area integral of $\partial D/\partial t$ approaches zero as $\Delta z \to 0$, but not the integral over the surface current J_s, which occupies a surface layer thin compared to Δz. Thus, $H_{1x} - H_{2x} = \hat{y} \cdot \vec{J}_s$, or more generally:

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2 \right) = \vec{J}_s \tag{4.51}$$

where $\hat{n}$ is defined as pointing from medium 2 into medium 1. If the medium is non-conducting, $\vec{J}_s = \vec{0}$.

4.4.3 General boundary conditions

Combine together the boundary conditions for perpendicular and parallel fields, we now get the general *boundary conditions*:

$$\boxed{\begin{aligned} &\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2 \right) = \rho_s \\ &\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2 \right) = 0 \\ &\hat{n} \times \left(\vec{E}_1 - \vec{E}_2 \right) = 0 \\ &\hat{n} \times \left(\vec{H}_1 - \vec{H}_2 \right) = \vec{J}_s \end{aligned}} \tag{4.52}$$

Boundary conditions adjacent to source-free media

If there are no free charges and currents, the boundary conditions adjacent to ideal media (dielectrics or magnetic media) become

$$\boxed{\begin{aligned} &\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2 \right) = 0 \\ &\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2 \right) = 0 \\ &\hat{n} \times \left(\vec{E}_1 - \vec{E}_2 \right) = 0 \\ &\hat{n} \times \left(\vec{H}_1 - \vec{H}_2 \right) = 0 \end{aligned}} \tag{4.53}$$

Example 4.2: Two insulating planar dielectric slabs having ε_1 and ε_2 ($\varepsilon_2 > \varepsilon_1$) are bonded together (Figure 4.11). Slab 1 has $\vec{E}_1$ at angle θ_1 to the surface normal. What are $\vec{E}_2$ and θ_2 if we assume the surface charge at the boundary $\rho_s = 0$? What are the components of $\vec{E}_2$ if $\rho_s \neq 0$?

Solutions: $E_{||}$ is continuous across any boundary, and if $\rho_s = 0$, then $D_\perp$ is continuous too, which implies $E_{2\perp} = (\varepsilon_1/\varepsilon_2) E_{1\perp} = 0.5 E_{1\perp}$. Also, $\theta_1 = \arctan\left(E_{||}/E_{1\perp}\right)$, and $\theta_2 = \arctan\left(E_{||}/E_{2\perp}\right)$. It follows that $\theta_2 = \arctan\left[(\varepsilon_1/\varepsilon_2)\tan\theta_1\right]$. If $\rho_s \neq 0$ then $E_{||}$ is unaffected and $D_{2\perp} = D_{1\perp} + \rho_s$, so that $E_{2\perp} = D_{2\perp}/\varepsilon_2 = (\varepsilon_1/\varepsilon_2) E_{1\perp} + \rho_s/\varepsilon_2$.

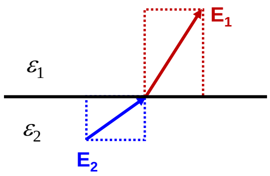

Figure 4.11 Boundary conditions of source-free dielectric media

4.4.4 Boundary conditions adjacent to perfect electric conductors

The four boundary conditions in Eq. (4.52) are simplified when one medium is a perfect conductor (σ= ∞) because electric and magnetic fields must be zero inside it. The electric field is zero because otherwise it would produce enormous $\vec{J} = \sigma\vec{E} \to \infty$, then $\vec{H} \to \infty$, and $W_m = \mu H^2/2 \to \infty$. Physically, the magnetic energy cannot be infinitely large, thus we must have $\vec{E} = 0$ inside perfect conductors. According to the Gauss's law, $\rho = \nabla \cdot \varepsilon\vec{E} = 0$ inside perfect conductors as well.

Because $\oint_C \vec{E} \cdot d\vec{l} = -\iint_S \partial\vec{B}/\partial t \cdot d\vec{S}$, then $\partial\vec{B}/\partial t = 0$, and thus $\vec{H} = 0$ inside perfect conductors.

The boundary conditions adjacent to perfect conductors can be summarized as

$$\boxed{\begin{aligned} &\hat{n} \cdot \vec{D}_1 = \rho_s \\ &\hat{n} \cdot \vec{B}_1 = 0 \\ &\hat{n} \times \vec{E}_1 = 0 \\ &\hat{n} \times \vec{H}_1 = \vec{J}_s \end{aligned}} \tag{4.54}$$

These four boundary conditions state that magnetic fields can only be parallel to perfect conductors, while electric fields can only be perpendicular. Moreover, the magnetic fields are always associated with surface currents flowing in an orthogonal direction; these currents have a numerical value equal to $\vec{H}$. The perpendicular electric fields are always associated with a surface charge ρ_s numerically equal to $\vec{D}$; the sign of σ is positive when $\vec{D}$ points away from the conductor, as illustrated in Figure 4.13.

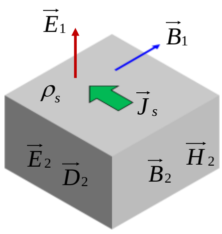

Figure 4.13 Boundary conditions for perfect conductors

Example 4.3: Two infinite perfect electric conductors are placed at z = 0 and z = d, with the electric field between them:

$$\vec{E}(x, y, t) = \hat{y}E_0 \sin\left(\frac{\pi z}{d}\right)\cos(k_x x - \omega t) \ [\text{V/m}] \tag{4.55}$$

(a) What is the magnetic field?

(b) What is surface current density?

(c) What is the surface charge density?

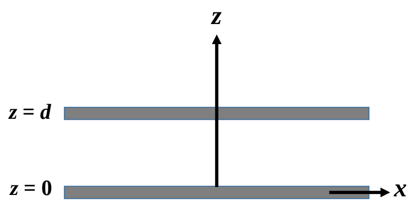

Figure 4.13 Boundary conditions for perfect conductors

Solutions:

(a) Use Faraday's law we can find the magnetic field between two plates

$$\begin{aligned}\frac{\partial \vec{H}}{\partial t} &= -\frac{1}{\mu_0}\nabla\times\vec{E} = -\frac{1}{\mu_0}\left(\hat{z}\frac{\partial}{\partial x} - \hat{x}\frac{\partial}{\partial z}\right)E_y\left(x,z,t\right) \\ &= \hat{z}\frac{E_0 k_x}{\mu_0}\sin\left(\frac{\pi z}{d}\right)\sin\left(k_x x - \omega t\right) + \hat{x}\frac{E_0\pi}{\mu_0 d}\cos\left(\frac{\pi z}{d}\right)\cos\left(k_x x - \omega t\right)\end{aligned}$$

Integrating over time, we can get

$$\vec{H} = \int\frac{\partial \vec{H}}{\partial t}dt = \hat{z}\frac{k_x E_0}{\omega\mu_0}\sin\left(\frac{\pi z}{d}\right)\cos\left(k_x x - \omega t\right) - \hat{x}\frac{\pi E_0}{\omega\mu_0 d}\cos\left(\frac{\pi z}{d}\right)\sin\left(k_x x - \omega t\right)$$

(b) Use the boundary conditions for perfect conductors

at $z = 0$: $\vec{J}_s = \hat{z}\times\vec{H}\left(z=0\right) = -\hat{y}\frac{\pi E_0}{\omega\mu_0 d}\sin\left(k_x x - \omega t\right)$ [A/m].

at $z = d$: $\vec{J}_s = -\hat{z}\times\vec{H}\left(z=d\right) = -\hat{y}\frac{\pi E_0}{\omega\mu_0 d}\sin\left(k_x x - \omega t\right)$ [A/m]

(c) Use the boundary conditions for perfect conductors

at $z = 0$: $\rho_s = \hat{z}\cdot\vec{D}\left(z=0\right) = 0$.

at $z = d$: $\rho_s = -\hat{z}\cdot\vec{D}\left(z=0\right) = 0$

4.5 Scalar and Vector potentials in Static Fields

For static fields ($\partial/\partial t \to 0$), the Maxwell's equations are written as

$$\begin{aligned}&\nabla\cdot\vec{D}=\rho_{free}\\&\nabla\times\vec{E}=0\\&\nabla\cdot\vec{B}=0\\&\nabla\times\vec{H}=\vec{J}_{free}\end{aligned}\tag{4.56}$$

Using the vector identities $\nabla\times(\nabla\varphi)=0$ and $\nabla\cdot\left(\nabla\times\vec{A}\right)=0$, we can define a *scalar potential* φ and a *vector potential* $\vec{A}$ to express the electric and magnetic fields:

$$\begin{aligned}&\vec{E}=-\nabla\varphi\\&\vec{B}=\nabla\times\vec{A}\end{aligned}\tag{4.57}$$

with $\nabla\cdot\vec{A}$ set to zero. The scalar potential φ is also known as electric potential (to distinguish it from the volume, we use φ instead of *V*).

Substitute Eq. (4.57) into Eq. (4.56), we find the Poisson's equations for static field problems

$$\boxed{\begin{aligned}&\nabla^2\varphi=-\rho_{free}/\varepsilon\\&\nabla^2\vec{A}=-\mu\vec{J}_{free}\end{aligned}}\tag{4.58}$$

Usually, by introducing the scalar and vector potentials, we can transform *vector equations* into four separate *scalar equations*, so as to reduce the complexity in solution.

Uniqueness Theorems

For a general Poisson's equation (e.g., $\nabla^2\varphi=-\rho_{free}/\varepsilon$), uniqueness theorems state that unique solution exists when the one of the following boundary equations is satisfied (Figure 4.14):

(1) Dirichlet boundary condition: φ is well defined at the boundary *S* of a region *V*;

(2) Neumann boundary condition: $\nabla\varphi$ is well defined at the boundary *S* of a region *V*;

(3) Mixed boundary conditions: mix of Dirichlet and Neumann boundary condition.

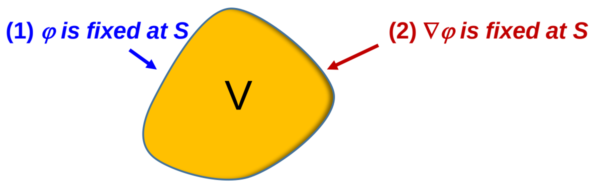

Figure 4.14 Boundary conditions for perfect conductors

4.6 The method of images

One very useful problem solving technique is to change the problem definition to one that is easier to solve but is known to have the same answer. An excellent example of this approach is the use of mirror-image charges and currents, which also works for wave problems.

Point charges near perfect conductors

Consider the problem of finding the fields produced by a charge q located a distance h above an infinite perfectly conducting plane, as illustrated in Figure 4.15. Boundary conditions at the conductor require only that the electric field lines be perpendicular to its surface. Any other set of boundary conditions that imposes the same constraint must yield the same unique solution by virtue of the uniqueness theorem.

Here, we need to solve the Poisson's Equation with two boundary conditions: (1) $\varphi = 0$ at $z = 0$; (2) $\varphi \to 0$ far from the charge. To satisfy such boundary conditions, we assume an image charge $q' = -q$ at $z = -h$, then the electric potential in the region of interest ($z \geq 0$) is the superposition of two charges:

$$\varphi(x, y, z) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+h)^2}}\right], \quad z \geq 0 \tag{4.59}$$

The induced surface charge density at $z = 0$ is calculated by

$$\rho_s = \hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \hat{n} \cdot \left(-\varepsilon \nabla \varphi\right) = -\varepsilon \frac{\partial \varphi}{\partial z}\Big|_{z=0} = -\frac{qh}{2\pi\left(x^2 + y^2 + h^2\right)^{3/2}} \tag{4.60}$$

The induced total charge at $z = 0$ is given by

$$q_{in} = \int_S \rho_s dS = -\frac{qh}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dxdy}{\left(x^2 + y^2 + h^2\right)^{3/2}} = -\frac{qh}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\frac{\rho d\rho d\phi}{\left(\rho^2 + h^2\right)^{3/2}} = -q \tag{4.61}$$

The total charge induced on the plane is $-q$, as expected, with the amount identical to that of the positive charge.

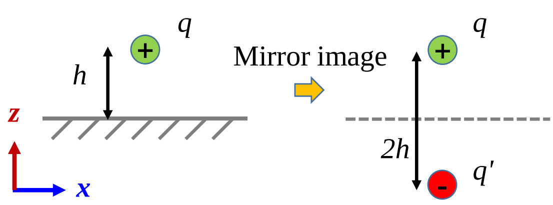

Figure 4.15 Point charges near perfect conductors

Other image problems near perfect conductors

This equivalence applies for multiple charges or for a charge distribution, as illustrated in Figure 4.16. In fact, the mirror image method remains valid so long as the charges change value or position slowly with respect to the relaxation time of the conductor. The relaxation time is the $1/e$ time constant required for the charges within the conductor to approach new equilibrium positions after the source charge distribution outside changes.

Because the mirror image method works for varying or moving charges, it works for the currents that must be associated with them by conservation of charge, as suggested in Figure 4.16a. The mirror image method continues to work if the upper half plane contains a conductor, as illustrated in Figure 4.16b; the conductor must be imaged too. These conductors can even be at angles, as suggested in Figure 4.16(c). The region over which the deduced fields are valid is naturally restricted to the original opening between the conductors. Still more complex image configurations, such as sphere (Figure 4.16d) can be used for other conductor placements, and may even involve an infinite series of progressively smaller image charges and currents.

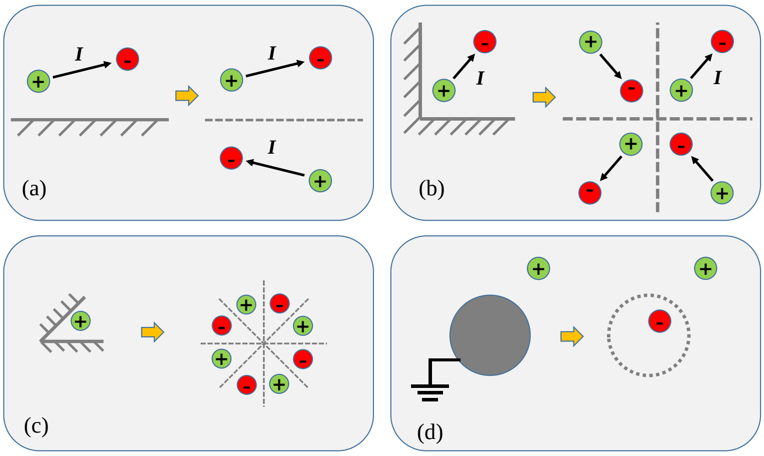

Figure 4.16 Image charges and currents for different shapes of conductor.

Image problems near dielectrics

The method of image can also be applied for charges near dielectrics. Consider the problem of finding the fields produced by a charge q located a distance h above an interface of two dielectrics ε_1 and ε_2, as illustrated in Figure 4.17. The point charge q can generate induced charge distribution at the boundary, and the total field can be considered as a superposition of the fields from the point charge and the induced charge.

The method of image for this case includes two steps: (1) To calculate the field in region 1, an additional image charge q' is assumed at $z = -h$ in region 2, to replace the induced charge distribution at the boundary. The whole space is considered to be full of dielectric ε_1. (2) To calculate the field in region 2, an additional image charge q'' is assumed at $z = h$ in region 1, also to replace the induced charge distribution at the boundary. The whole space is considered to be full of dielectric ε_2.

Base on above image approach, the electric potentials in two dielectrics can be written by

$$\begin{aligned} \varphi_1(x,y,z) &= \frac{1}{4\pi\varepsilon_1}\left[\frac{q}{\sqrt{x^2+y^2+(z-h)^2}}+\frac{q'}{\sqrt{x^2+y^2+(z+h)^2}}\right], \quad z\geq 0 \\ \varphi_2(x,y,z) &= \frac{1}{4\pi\varepsilon_2}\left[\frac{q+q''}{\sqrt{x^2+y^2+(z-h)^2}}\right], \quad z\leq 0 \end{aligned} \tag{4.62}$$

Then we use boundary conditions to find the value of the image charges q' and q''.

At $z = 0$, boundary condition requires

$$\begin{aligned} \hat{z}\cdot(-\varepsilon_1\nabla\varphi_1+\varepsilon_2\nabla\varphi_2) &= 0 \\ \hat{z}\times(-\nabla\varphi_1+\nabla\varphi_2) &= 0 \end{aligned} \quad => \quad \begin{cases} \varepsilon_1\dfrac{\partial\varphi_1}{\partial z} = \varepsilon_2\dfrac{\partial\varphi_2}{\partial z} \\ \varphi_1 = \varphi_2 \end{cases} \tag{4.63}$$

Substitute the potentials into Eq. (4.63), then we get

$$\begin{cases} \dfrac{q+q'}{\varepsilon_1} = \dfrac{q+q''}{\varepsilon_2} \\ q-q' = q+q'' \end{cases} \tag{4.64}$$

The image charges are then expressed as

$$q' = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q, \quad q'' = -\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q \tag{4.65}$$

The corresponding total electric potentials in two dielectrics can then be found from Eq. (4.62). The electric field can be derived by $\vec{E} = -\nabla\varphi$.

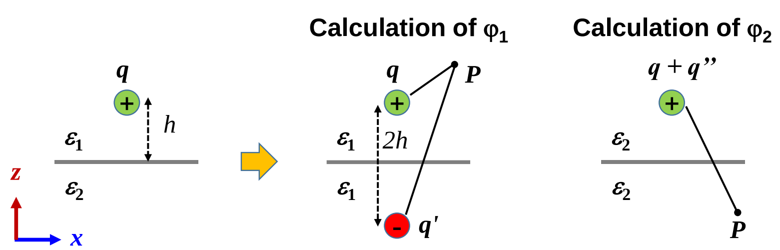

Figure 4.17 Image charges near dielectrics

Image problems near magnets

Similar to electrostatics, we can also use the method of image to calculate the magnetic field of a wire current near magnet boundaries. Consider the problem of finding the fields produced by a current I located a distance h above an interface of two magnets μ_1 and μ_2, as illustrated in Figure 4.18. The magnetic field of the current induces bound currents at the boundary, and the total magnetic field can be considered as a superposition of the fields from the current I and the induced bound current.

The method of image for this case includes also two steps: (1) To calculate the field in region 1, an additional current I' is assumed at $z = -h$ in region 2, to replace the induced bound current at the boundary. The whole space is considered to be full of dielectric μ_1. (2) To calculate the field in region 2, an additional current I'' is assumed at $z = h$ in region 1, also to replace the induced bound current at the boundary. The whole space is considered to be full of dielectric μ_2.

The y-components of the magnetic vector potentials in two magnets can be written by

$$\begin{aligned} A_{1y}(x,y,z) &= \frac{\mu_1 I}{2\pi}\ln\frac{1}{\sqrt{x^2+(z-h)^2}} + \frac{\mu_1 I'}{2\pi}\ln\frac{1}{\sqrt{x^2+(z+h)^2}}, \quad z \ge 0 \\ A_{2y}(x,y,z) &= \frac{\mu_2 (I+I'')}{2\pi}\ln\frac{1}{\sqrt{x^2+(z-h)^2}}, \quad z \le 0 \end{aligned} \tag{4.66}$$

Then we use boundary conditions to find the value of the image currents I' and I''.

At $z = 0$, boundary condition requires

$$\begin{aligned} &\hat{z}\cdot\left(\nabla\times\vec{A}_1 - \nabla\times\vec{A}_2\right) = 0 \\ &\hat{z}\times\left(\frac{1}{\mu_1}\nabla\times\vec{A}_1 - \frac{1}{\mu_2}\nabla\times\vec{A}_2\right) = 0 \end{aligned} \quad \Rightarrow \quad \begin{cases} A_{1y} = A_{2y} \\ \dfrac{1}{\mu_1}\dfrac{\partial A_{1y}}{\partial z} = \dfrac{1}{\mu_2}\dfrac{\partial A_{2y}}{\partial z} \end{cases} \tag{4.67}$$

Substitute the potentials into Eq. (4.63), then we get

$$\begin{cases} \mu_1 (I + I') = \mu_2 (I + I'') \\ I - I' = I + I'' \end{cases} \tag{4.68}$$

The image currents are then expressed as

$$I' = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} I, \quad I'' = -\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} I \tag{4.69}$$

The corresponding total vector potentials in two dielectrics can then be found from Eq. (4.66). The magnetic field can be derived by $\vec{B} = \nabla \times \vec{A}$.

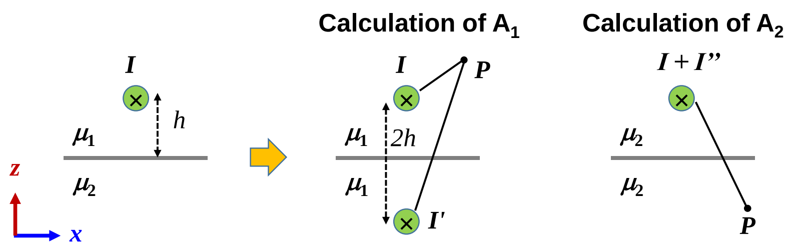

Figure 4.18 Image charges near magnets

4.7 Additional Problems

See References [2-4].

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