Problem Solving 3: Maxwell's Equations and Electromagnetic Waves

OBJECTIVES:

1. To calculate the vector identity of del operator.
2. To learn how to calculate the information of a travelling electromagnetic wave.
3. To learn how to calculate the polarization of a wave.

REFERENCE: Chapter 3, Maxwell's Equations and Electromagnetic Waves

PROBLEM SOLVING STRATEGIES

A. Vector identity

PROBLEM 1: Calculation of del operator

Prove that $\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$

Solution:

$$
\begin{aligned}
\nabla\times\left(\nabla\times\vec{E}\right) &= \nabla\times\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ E_x & E_y & E_z\end{vmatrix}\\
&=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left[\hat{x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\right]\\
&=\hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\right]\\
&\quad+\hat{y}\left[-\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\right]\\
&\quad+\hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\right]\\
&=\hat{x}\left[\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x\right]\\
&\quad+\hat{y}\left[\frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_y\right]\\
&\quad+\hat{z}\left[\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z\right]\\
&=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}
\end{aligned}
$$

B. Traveling Electromagnetic Waves

This chapter explores various properties of the electromagnetic waves. The electric and the magnetic fields of the wave obey the wave equation. Once the functional form of either one of the fields is given, the other can be determined from Maxwell's equations. As an example, let's consider a sinusoidal electromagnetic wave with

$$\vec{E}(z,t) = \hat{x}E_0 \sin(kz - \omega t).$$

The equation above contains the complete information about the electromagnetic waves:

1. Direction of wave propagation: $(kz - \omega t) = k(z - vt)$, which indicates that the wave is propagating in the +z-direction.

2. Wavelength: The wavelength λ is related to the wave number k by $\lambda = 2\pi / k$

3. Frequency: The frequency of the wave, f , is related to the angular frequency ω by $f = \omega / 2\pi$.

4. Speed of propagation: The speed of the wave is given by

$$v = \lambda f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

   In vacuum, the speed of the electromagnetic wave is equal to the speed of light, c.

5. Magnetic field $\vec{B}$: The magnetic field $\vec{B}$ is perpendicular to both $\vec{E}$ which points in the +x direction, and $\hat{z}$, the unit vector along the +z axis, which is the direction of propagation, as we have found. In addition, since the wave propagates in the same direction as the cross product $\vec{E} \times \vec{B}$, we conclude that $\vec{B}$ must point in the +y direction.

   Since $\vec{B}$ is always in phase with $\vec{E}$, the two fields have the same functional form. Thus, we may write the magnetic field as

$$\vec{B}(z,t) = \hat{y}B_0 \sin(kz - \omega t).$$

   where B_0 is the amplitude. Using Maxwell's equations one may show that $B_0 = E_0(k/\omega) = E_0/c$ in vacuum.

6. The Poytning vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{\mu_0}\left[\hat{x}E_0 \sin(kz - \omega t) \times \hat{y}B_0 \sin(kz - \omega t)\right] = \frac{E_0 B_0 \sin^2(kz - \omega t)}{\mu_0}\hat{z}$$

7. Intensity: The intensity of the wave is equal to the average of S

$$I = \langle S \rangle = \frac{E_0 B_0}{\mu_0}\left\langle \sin^2(kz - \omega t)\right\rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$

PROBLEM 2: Plane Electromagnetic Wave

Suppose the electric field of a plane electromagnetic wave is given by

$$\vec{E}(z,t) = \hat{x}E_0 \cos(kz - \omega t)$$

Find the following quantities:

(a) The direction of wave propagation

(b) The corresponding magnetic field $\vec{B}$

Solution:

(a) By writing the argument of the cosine function as $kz - \omega t = k(z - ct)$ where $\omega = ck$, we see that the wave is traveling in the + z direction.

(b) The direction of propagation of the electromagnetic waves coincides with the direction of the Poynting vector which is given by $\vec{S} = \vec{E} \times \vec{B}/\mu_0$, the magnetic field is given by

$$\boxed{B} \qquad \cancel{\vec{E}}(z,t) = \hat{y}B_0 \cos(kz - \omega t)$$

The magnitude is determined by the Faraday's law:

Poynting vector which is given by $\vec{S} = \vec{E} \times \vec{B}/\mu_0$, the magnetic field is given by

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

and results $B_0 = cE_0$.

PROBLEM 3: Poynting Vector of a Charging Capacitor

A parallel-plate capacitor with circular plates of radius R and separated by a distance h is charged through a straight wire carrying current I, as shown in the Figure 1.

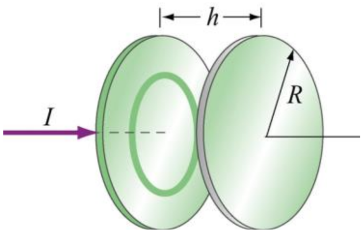

Figure 1 Parallel plate capacitor

(a) Show that as the capacitor is being charged, the Poynting vector $\vec{S}$ points radially inward toward the center of the capacitor.

(b) By integrating $\vec{S}$ over the cylindrical boundary, show that the rate at which energy enters the capacitor is equal to the rate at which electrostatic energy is being stored in the electric field.

Solution:

(a) Let the axis of the circular plates be the z-axis, with current flowing in the +z direction. Suppose at some instant the amount of charge accumulated on the positive plate is +Q. The electric field is

$$\vec{E} = \frac{Q}{\pi R^2 \varepsilon_0} \hat{z}$$

According to the Ampere-Maxwell's equation, a magnetic field is induced by changing electric flux:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \frac{d}{dt}\iint_S \vec{E}\cdot d\vec{A}$$

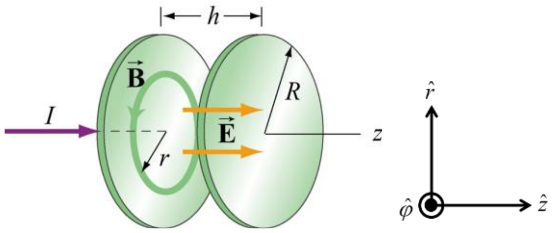

Figure 2

From the cylindrical symmetry of the system, we see that the magnetic field will be circular, centered on the z-axis, i.e., $\vec{B} = B\hat{\varphi}$ (Figure 2).

Consider a circular path of radius $r < R$ between the plates. Using the above formula, we obtain

$$B(2\pi r) = 0 + \mu_0\varepsilon_0 \frac{d}{dt}\left(\frac{Q}{\pi R^2\varepsilon_0}\pi r^2\right) = \mu_0 \frac{r^2}{R^2}\frac{dQ}{dt}$$

or

$$\vec{B} = \mu_0 \frac{r}{2\pi R^2}\frac{dQ}{dt}\hat{\varphi}$$

The Poynting vector can then be written as

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(\frac{Q}{\pi R^2\varepsilon_0}\hat{z}\right)\times\left(\frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt}\hat{\varphi}\right)$$
$$= -\left(\frac{Qr}{2\pi^2 R^4\varepsilon_0}\right)\left(\frac{dQ}{dt}\right)\hat{r}$$

Note that for points in the $dQ/dt > 0$, $\vec{S}$ points in the $-\hat{r}$ direction, or radially inward toward the center of the capacitor.

(b) The energy per unit volume carried by the electric field is $u_E = \varepsilon_0 E^2/2$. The total energy stored in the electric field then becomes

$$U_E = u_E V = \frac{\varepsilon_0}{2}E^2\left(\pi R^2 h\right) = \frac{\varepsilon_0}{2}\left(\frac{Q}{\pi R^2\varepsilon_0}\right)\pi R^2 h = \frac{Q^2 h}{2\pi R^2\varepsilon_0}$$

Differentiating the above expression with respect to t, we obtain the rate at which this energy is being stored:

$$\frac{dU_E}{dt} = \frac{d}{dt}\left(\frac{Q^2 h}{2\pi R^2 \varepsilon_0}\right) = \frac{Qh}{\pi R^2 \varepsilon_0}\frac{dQ}{dt}$$

On the other hand, the rate at which energy flows into the capacitor through the cylinder at $r = R$ can be obtained by integrating $\vec{S}$ over the surface area:

$$\oint \vec{S}\cdot d\vec{A} = SA_R = \left(\frac{Qr}{2\pi^2 R^4 \varepsilon_0}\frac{dQ}{dt}\right)(2\pi Rh) = \frac{Qh}{\varepsilon_0 \pi R^2}\frac{dQ}{dt}$$

which is equal to the rate at which energy stored in the electric field is changing.

PROBLEM 4: Poynting Vector of a Conductor

A cylindrical conductor of radius a and conductivity σ carries a steady current I which is distributed uniformly over its cross-section, as shown in Figure 3.

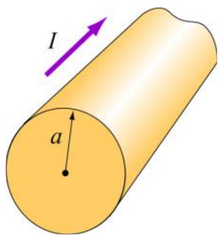

Figure 3

(a) Compute the electric field $\vec{E}$ inside the conductor.

(b) Compute the magnetic field $\vec{B}$ just outside the conductor.

(c) Compute the Poynting vector $\vec{S}$ at the surface of the conductor. In which direction does $\vec{S}$ point?

(d) By integrating $\vec{S}$ over the surface area of the conductor, show that the rate at which electromagnetic energy enters the surface of the conductor is equal to the rate at which energy is dissipated.

Solution:

(a) Let the direction of the current be along the z-axis. The electric field is given by

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{\vec{I}}{\sigma\pi a^2}\hat{z}$$

~~where R is the resistance and l is the length of the conductor.~~

(b) The magnetic field can be computed using Ampere's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

Choosing the Amperian loop to be a circle of radius r, we have $B(2\pi r) = \mu_0 I$, or

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

(c) The Poynting vector on the surface of the wire ($r = a$) is

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(\frac{I}{\sigma\pi a^2}\hat{z}\right)\times\left(\frac{\mu_0 I}{2\pi r}\hat{\varphi}\right) = -\frac{I^2}{2\sigma\pi^2 a^3}\hat{r}$$

Notice that $\vec{S}$ points radially inward toward the center of the conductor.

(d) The rate at which electromagnetic energy flows into the conductor is given by

$$P = \frac{dU}{dt} = \oiint_S \vec{S}\cdot d\vec{A} = \left(\frac{I^2}{2\sigma\pi^2 a^3}\right)2\pi a l = \frac{I^2 l}{\sigma\pi a^2}$$

However, since the conductivity σ is related to the resistance R by

$$\sigma = \frac{1}{\rho} = \frac{l}{AR} = \frac{l}{\pi a^2 R}$$

The above expression becomes

$$P = I^2 R$$

which is equal to the rate of energy dissipation in a resistor with resistance R.

C. Polarizations

PROBLEM 5: Determine the polarization of an electromagnetic wave

Consider an electromagnetic wave propagating in the +z-direction with

$$\vec{E} = \hat{x}A_x\cos(kz-\omega t+\varphi_x) + \hat{y}A_y\cos(kz-\omega t+\varphi_y)$$

where A_x, A_y, φ_x, and φ_y are all real numbers.

(a) Let $A_x = 2$, $A_y = 1$, $\varphi_x = \pi/2$, $\varphi_y = \pi/4$. What is the polarization?

(b) Let $A_x = 1$, $A_y = 0$, $\varphi_x = 0$. This is a linearly polarized wave. Prove that it can be expressed as the superposition of a right-hand circularly polarized wave and a left-hand circularly polarized wave.

(c) Let $A_x = 1$, $A_y = 1$, $\varphi_x = \pi/4$, and $\varphi_y = -\pi/4$. This is a circularly polarized wave. Prove that it can be decomposed into two linearly polarized waves.

Solution:

(a) $\vec{E} = \hat{x}2\cos\left(kz-\omega t+\frac{\pi}{2}\right) + \hat{y}\cos\left(kz-\omega t+\frac{\pi}{4}\right)$

At the point z = z0, $\vec{E}$ will trace a trajectory

$$x = 2\cos\left(kz_0-\omega t+\frac{\pi}{2}\right) = -2\sin(kz_0-\omega t)$$
$$y = \cos\left(kz_0-\omega t+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\cos(kz_0-\omega t) - \frac{1}{\sqrt{2}}\sin(kz_0-\omega t)$$

The equation describing this trajectory is

$$\frac{x^2}{2} - \sqrt{2}xy + 2y^2 = 1$$

which is the equation of an ellipse. Therefore, the wave is elliptically polarized.

(b) Consider the following linearly polarized plane wave:

$$\begin{aligned}\vec{E} &= \hat{x}\cos(kz - \omega t) \\ &= \left[\frac{1}{2}\hat{x}\cos(kz-\omega t) + \frac{1}{2}\hat{y}\cos(kz-\omega t)\right] + \left[\frac{1}{2}\hat{x}\cos(kz-\omega t) - \frac{1}{2}\hat{y}\cos(kz-\omega t)\right]\end{aligned}$$

This is the superposition of two circularly polarized waves.

(c) Consider the following circularly polarized plane wave:

$$\vec{E} = \hat{x}\cos(kz - \omega t + \pi/4) + \hat{y}\cos(kz - \omega t - \pi/4)$$

This is the superposition of two linearly polarized waves.