Chapter 9: Radiation

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9.1 Retarded Potentials

In the static case, we have built the Poisson's equations in Chapter 4

$$\begin{aligned} \nabla^2\varphi &= -\rho/\varepsilon \\ \nabla^2\vec{A} &= -\mu\vec{J} \end{aligned} \tag{9.1}$$

where φ and $\vec{A}$ are the scalar and vector potentials, respectively. Mathematically, the solutions to above equations are

$$\begin{aligned} \varphi(\vec{r}) &= \frac{1}{4\pi\varepsilon}\iiint\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV' \\ \vec{A}(\vec{r}) &= \frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV' \end{aligned} \tag{9.2}$$

where $|\vec{r}-\vec{r}\,'|$ is the distance from the source point $\vec{r}\,'$ to the field point $\vec{r}$ (Figure 9.1). Now, electromagnetic "news" travels at the speed of light. In the *non-static* case, therefore, it's not the status of the source right now that matters, but rather its condition at some earlier time tr (called the retarded time) when the "message" left. Since this message must travel a distance $|\vec{r}-\vec{r}\,'|$, the delay is $|\vec{r}-\vec{r}\,'|/c$:

$$\boxed{t_r \equiv t - \frac{|\vec{r}-\vec{r}\,'|}{c}} \tag{9.3}$$

The natural generalization of Eq. (9.2) for *non-static* sources is therefore

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{1}{4\pi\varepsilon}\iiint\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV' \\ \vec{A}(\vec{r},t) &= \frac{\mu}{4\pi}\iiint\frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV' \end{aligned}} \tag{9.4}$$

Here $\rho(\vec{r}\,',t_r)$ is the charge density that prevailed at point $\vec{r}\,'$ at the retarded time t_r. Because the integrands are evaluated at the retarded time, these are called *retarded potentials*. (It is just like the night sky: The light we see now left each star at the retarded time corresponding to that star's distance from the earth). Note that the retarded potentials reduce properly to Eq. (9.2) in the static case, for which ρ and J are independent of time.

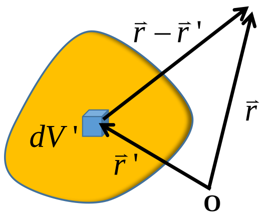

Figure 9.1

We then switch to the time-harmonic representations because radiation is frequency dependent. The *time-harmonic forms* of the retarded potentials are

$$\boxed{\begin{aligned}\varphi(\vec{r},\omega) &= \frac{1}{4\pi\varepsilon}\iiint \frac{\rho(\vec{r}\,',\omega)}{|\vec{r}-\vec{r}\,'|}e^{-jk|\vec{r}-\vec{r}\,'|}dV' \\ \vec{A}(\vec{r},\omega) &= \frac{\mu}{4\pi}\iiint \frac{\vec{J}(\vec{r}\,',\omega)}{|\vec{r}-\vec{r}\,'|}e^{-jk|\vec{r}-\vec{r}\,'|}dV'\end{aligned}} \tag{9.5}$$

where $k=\omega\sqrt{\mu\varepsilon}=2\pi/\lambda$ is the wave number.

The radiated fields can then be derived from the potentials:

$$\boxed{\begin{aligned}&\vec{B}=\nabla\times\vec{A} \\ &\vec{E}=-\nabla\varphi-j\omega\vec{A}\end{aligned}} \tag{9.6}$$

9.2 Electric Dipole Radiation

Picture two tiny metal spheres separated by a distance l and connected by a fine wire (Fig. 9.2); at time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω:

$$q(t)=q\cos(\omega t)=\mathrm{Re}\left[qe^{j\omega t}\right] \tag{9.7}$$

The result is an oscillating electric dipole:

$$\vec{p}_e(t)=\hat{z}ql\cos(\omega t)=\mathrm{Re}\left[\hat{z}qle^{j\omega t}\right] \tag{9.8}$$

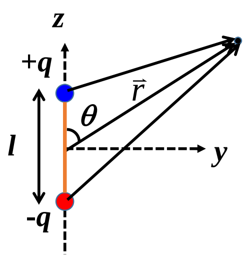

Figure 9.2 Electric dipole radiation

According to the definition of current:

$$i(t) = \frac{dq(t)}{dt} = -q\omega \sin(\omega t) = \mathrm{Re}\left[j\omega q e^{j\omega t} \right] \tag{9.9}$$

we find the time-harmonic current as

$$I = j\omega q = j\omega \frac{p_e}{l} \tag{9.10}$$

and the current element as

$$\vec{J}(r')dV' = \hat{z}\frac{I}{\Delta S}\Delta S dz' = \hat{z} I dz', \quad \left(-\frac{l}{2} < z' < \frac{l}{2} \right) \tag{9.11}$$

When the distance of two spheres approaches infinitely small ($l \to 0$), the current distribution can be expressed by a delta function:

$$\boxed{\vec{J}(r') = \hat{z} I l \delta(\vec{r}')} \tag{9.12}$$

For electrically small dipoles ($l \ll r$), the propagation term in Eq. (9.5) can be approximate to

$$\frac{e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{e^{-jkr}}{r} \tag{9.13}$$

The vector potential of an electric dipole can be written by

$$\vec{A}(\vec{r},\omega) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint \vec{J}(\vec{r}',\omega)dV' = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2}\hat{z} I dz' = \hat{z}\frac{\mu I l}{4\pi r}e^{-jkr} \tag{9.14}$$

Using the coordinate transformation method into the spherical coordinates (Figure 9.3), we find the three components of the potential $\vec{A}(\vec{r},\omega) = \hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$:

$$\boxed{\begin{aligned} A_r &= \vec{A}\cdot\hat{r} = A_z\cos\theta = \frac{\mu I l}{4\pi r}\cos\theta e^{-jkr} \\ A_\theta &= \vec{A}\cdot\hat{\theta} = -A_z\sin\theta = -\frac{\mu I l}{4\pi r}\sin\theta e^{-jkr} \\ A_\phi &= \vec{A}\cdot\hat{\phi} = 0 \end{aligned}} \tag{9.15}$$

The radiated electric and magnetic fields can be calculated by

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\nabla\times\vec{A} = \frac{1}{\mu r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}, \quad \vec{E} = \frac{1}{j\omega\varepsilon}\nabla\times\vec{H} \tag{9.16}$$

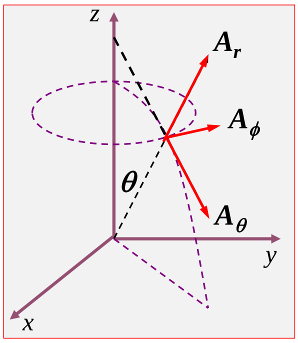

Figure 9.3 Electric dipole radiation

According to Eq. (9.15) and Eq. (9.16), the electric and magnetic field distributions of the electric dipole radiation are

$$\boxed{\begin{aligned} E_r &= \frac{2Ilk^3\cos\theta}{4\pi\omega\varepsilon}\left(\frac{1}{(kr)^2}-\frac{j}{(kr)^3}\right)e^{-jkr} \\ E_\theta &= \frac{Ilk^3\sin\theta}{4\pi\omega\varepsilon}\left(\frac{j}{kr}+\frac{1}{(kr)^2}-\frac{j}{(kr)^3}\right)e^{-jkr} \\ H_\phi &= \frac{k^2 Il\sin\theta}{4\pi}\left(\frac{j}{kr}+\frac{1}{(kr)^2}\right)e^{-jkr} \end{aligned}} \quad (9.17)$$

with $E_\phi = 0$, $H_r = 0$, and $H_\theta = 0$.

The radiation of an electric dipole can be generalized into two cases: (i) near field radiation with $kr << 1$ and (ii) far field region with $kr >> 1$.

Near Field Radiation ($kr << 1$)

In the near field region, we have

$$\frac{1}{kr} \ll \frac{1}{(kr)^2} \ll \frac{1}{(kr)^3} \quad \text{and} \quad e^{-jkr} \approx 1 \quad (9.18)$$

the radiated fields become

$$\boxed{\begin{aligned} E_r &= -j\frac{Il\cos\theta}{2\pi\omega\varepsilon r^3} \\ E_\theta &= -j\frac{Il\sin\theta}{4\pi\omega\varepsilon r^3} \\ H_\phi &= \frac{Il\sin\theta}{4\pi r^2} \end{aligned}} \quad (9.19)$$

Substituting $I = j\omega q$ and $p_e = ql$, we have

$$\boxed{\begin{aligned} E_r &= \frac{p_e \cos\theta}{2\pi\varepsilon r^3} \\ E_\theta &= \frac{p_e \sin\theta}{4\pi\varepsilon r^3} \\ H_\phi &= j\omega \frac{p_e \sin\theta}{4\pi r^2} \end{aligned}} \tag{9.20}$$

We find that the electric and magnetic fields show π/2 phase difference. They are quasi-static fields similar to those generated by static electric dipoles.

The time-average Poynting vector of the radiation field is

$$\left\langle \vec{S}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] = 0 \tag{9.21}$$

Because $\vec{S} = \vec{E}\times\vec{H}^*$ is purely negative imaginary in the near fields, these fields correspond to reactive power and stored electric energy.

Far Field Radiation (*kr* >> 1)

In the far field region, we have

$$\frac{1}{kr} \gg \frac{1}{(kr)^2} \gg \frac{1}{(kr)^3} \tag{9.22}$$

the radiated fields of Eq.(9.17) become

$$\boxed{\begin{aligned} E_\theta &= j\frac{Il\sin\theta}{2\lambda r}\eta e^{-jkr} \\ H_\phi &= j\frac{Il\sin\theta}{2\lambda r}e^{-jkr} = \frac{E_\theta}{\eta} \end{aligned}} \tag{9.23}$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance. For free space, $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 120\pi$.

The far fields radiated by an electric dipole antenna are thus radially propagating θ-polarized plane waves with ϕ-directed magnetic fields H_ϕ of magnitude $|E_\theta|/\eta_0$. The time-average power density of these radial waves is given by the Poynting's vector

$$\boxed{\begin{aligned} &\left\langle \vec{S}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\theta}E_\theta \times \hat{\phi}H_\phi^*\right] = \frac{1}{2}\mathrm{Re}\left[\hat{r}E_\theta H_\phi^*\right] \\ &= \hat{r}\frac{|E_\theta|^2}{2\eta} = \hat{r}\frac{\eta|H_\phi|^2}{2} = \hat{r}\frac{\eta}{2}\left(\frac{Il\sin\theta}{2\lambda r}\right)^2 \end{aligned}} \tag{9.24}$$

This angular distribution of radiated power is illustrated in Figure 9.4.

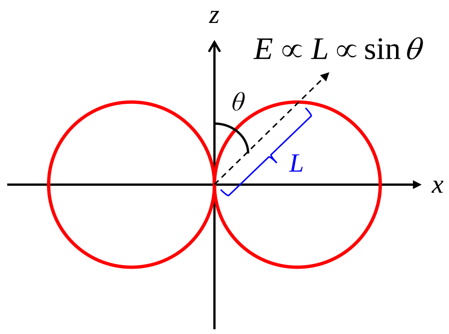

Figure 9.4 Angular distribution of the electric dipole radiation

The total power radiated is the integral of this intensity over 4π steradians:

$$\boxed{\begin{aligned} P_{\text{rad}} &= \oiint_S \left\langle \vec{S}(t) \right\rangle \cdot d\vec{S} = \oiint_S \hat{r}\frac{\eta}{2}\left(\frac{Il\sin\theta}{2\lambda r}\right)^2 \cdot \hat{r}r^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} 15\pi\left(\frac{Il}{\lambda}\right)^2 \sin^3\theta d\theta = 40\pi^2 I^2 \left(\frac{l}{\lambda}\right)^2 \end{aligned}} \tag{9.25}$$

We can see that the radiation power is related to the length of the electric dipole l/λ. From above equation, it is demonstrated that waves of smaller wavelength will have higher radiation power for a fixed current I and a fixed length l. This explains the blue sky in our natural life: sunlight reaches Earth's atmosphere and is scattered in all directions by all the gases and particles in the air. Blue light is scattered in all directions by the tiny molecules of air in Earth's atmosphere. Blue is scattered more than other colors because it travels as shorter, smaller waves. This is why we see a blue sky most of the time. Similarly, for a red sunset: as the sun gets lower in the sky, its light is passing through more of the atmosphere to reach you. Even more of the blue light is scattered, allowing the reds and yellows to pass straight through to your eyes. Other phenomena including Lunar eclipse can also be explained by the radiation of dipoles.

Radiation Intensity

Radiation intensity in a given direction is defined as "the power radiated from an antenna per unit solid angle." The radiation intensity is a far-field parameter, and it can be obtained by simply multiplying the radiation density by the square of the distance. In mathematical form it is expressed as

$$U = r^2 W_{\text{rad}} \tag{9.26}$$

where

U = radiation intensity (W/unit solid angle)

$W_{\text{rad}} = \left\langle \vec{S}(t) \right\rangle$ = radiation density (W/m^2)

Directivity

In the 1983 version of the *IEEE Standard Definitions of Terms for Antennas*, there has been a substantive change in the definition of directivity, compared to the definition of the 1973 version. Basically the term *directivity* in the new 1983 version has been used to replace the term *directive gain* of the old 1973 version. In the new 1983 version the term directive gain has been deprecated. According to the authors of the new 1983 standards, "this change brings this standard in line with common usage among antenna engineers and with other international standards, notably those of the International Electrotechnical Commission (IEC)." Therefore, *directivity of an antenna* defined as "the ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions. The average radiation intensity is equal to the total power radiated by the antenna divided by 4π. If the direction is not specified, the direction of maximum radiation intensity is implied." Stated more simply, the directivity of a nonisotropic source is equal to the ratio of its radiation intensity in a given direction over that of an isotropic source. In mathematical form, the *directivity* can be written as

$$D = \frac{4\pi U}{P_{\mathrm{rad}}} \tag{9.27}$$

If the direction is not specified, it implies the direction of maximum radiation intensity (*maximum directivity*) expressed as

$$D_{\max} = \frac{4\pi U_{\max}}{P_{\mathrm{rad}}} \tag{9.28}$$

9.3 Electric-Magnetic Duality

Although we haven't found magnetic charges in nature, it is straightforward to include magnetic sources in Maxwell's equations for completeness. The total fields can be decomposed into two parts: (i) fields ($\vec{E}_e, \vec{H}_e$) generated by electric sources ($\vec{J}_e, \rho_e$); (ii) fields ($\vec{E}_m, \vec{H}_m$) generated by magnetic sources ($\vec{J}_m$, ρ_m). That is

$$\vec{E} = \vec{E}_e + \vec{E}_m \quad \text{and} \quad \vec{H} = \vec{H}_e + \vec{H}_m \tag{9.29}$$

The Maxwell's equations including magnetic sources become

$$\boxed{\begin{aligned} &\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}_e \\ &\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} - \vec{J}_m \\ &\nabla \cdot \vec{H} = \frac{\rho_m}{\mu} \\ &\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon} \end{aligned}} \tag{9.30}$$

The fields generated by electric sources yields to the equations

$$\nabla\times\vec{H}_e=\varepsilon\frac{\partial\vec{E}_e}{\partial t}+\vec{J}_e,\qquad \nabla\times\vec{E}_e=-\mu\frac{\partial\vec{H}_e}{\partial t}$$
$$\nabla\cdot\vec{H}_e=\frac{\rho_m}{\mu},\qquad \nabla\cdot\vec{E}_e=0 \tag{9.31}$$

The fields generated by magnetic sources yield to

$$\nabla\times\vec{H}_m=\varepsilon\frac{\partial\vec{E}_m}{\partial t},\qquad \nabla\times\vec{E}_m=-\mu\frac{\partial\vec{H}_m}{\partial t}-\vec{J}_m$$
$$\nabla\cdot\vec{H}_m=0,\qquad \nabla\cdot\vec{E}_m=\frac{\rho_e}{\varepsilon} \tag{9.32}$$

Duality works because Maxwell's equations without charges or currents are duals of themselves. That is, by transforming $\vec{E}_e\to\vec{H}_m$, $\vec{H}_e\to-\vec{E}_m$, $\vec{J}_e\to\vec{J}_m$, $\rho_e\to\rho_m$, $\mu\to\varepsilon$ and $\varepsilon\to\mu$, the set of Maxwell's equations is unchanged. In other words, Eq. (9.31) is transformed to Eq. (9.32). The transformed set of equations is the same as the original, although sequenced differently. As a result, any solution to Maxwell's equations is also a solution to the dual problem where the variables and boundary conditions are all transformed as indicated above.

9.4 Magnetic dipole radiation

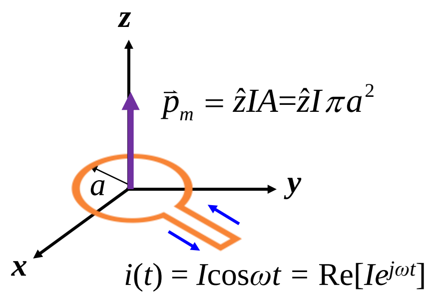

Figure 9.5 Magnetic dipole radiation

Suppose now that we have a wire loop of radius a (Figure 9.5), around which we drive an alternating current:

$$i(t)=I\cos(\omega t)=\mathrm{Re}\left[Ie^{j\omega t}\right] \tag{9.33}$$

This is a model for an oscillating magnetic dipole (equivalent to $\vec{m}$ in Chapter 4),

$$\vec{p}_m=\hat{z}IA=\hat{z}I\pi a^2 \tag{9.34}$$

where $A=\pi a^2$ is the area of the magnetic dipole. The magnetization is then The magnetic dipole can be considered as combination of positive and negative magnetic charges ($+q_m$ and $-q_m$), with the distance of l. Then we have the magnetization

$$\vec{M} = \frac{\vec{p}_m}{volume} = \frac{\vec{p}_m}{lA} = \hat{z}\frac{I}{l} \tag{9.35}$$

Compare the constitutive relations of $\vec{D} = \varepsilon\vec{E} + \vec{P}$ and $\vec{B} = \mu\vec{H} + \mu\vec{M}$, we can find that the magnetic current density is related to the magnetization ($\vec{J}_m(\vec{r},t) = \mu\, d\vec{M}(\vec{r},t)/dt$) . Thus the time-harmonic magnetic current element is

$$\vec{J}_m dV' = j\omega\mu\vec{M}dV' = \hat{z}\frac{j\omega\mu IA}{l}dz' \quad \left(-\frac{l}{2} < z' < \frac{l}{2}\right) \tag{9.36}$$

with the magnetic current density written by

$$\vec{J}_m(\vec{r}\,') = \hat{z}j\omega\mu IA\delta(\vec{r}\,') \tag{9.37}$$

Since the current density for an electric dipole is $\vec{J}_e(r') = \hat{z}Il\delta(\vec{r}\,')$ (from Eq. (9.12)), we can use the following transformation to find the dual radiation field of a magnetic dipole from electric counterpart:

$$\boxed{\begin{aligned} &Il \to j\omega\mu IA \\ &E \to H \\ &H \to -E \\ &\varepsilon \leftrightarrow \mu \end{aligned}} \tag{9.38}$$

As a result, the radiated far fields (kr >> 1) of the magnetic dipole are

$$\boxed{\begin{aligned} H_\theta &= -\frac{\omega\mu IA\sin\theta}{2\lambda r}\frac{1}{\eta}e^{-jkr} \\ E_\phi &= \frac{\omega\mu IA\sin\theta}{2\lambda r}e^{-jkr} = -\eta H_\theta \end{aligned}} \tag{9.39}$$

9.5 Additional Problems

See References [2-7].

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