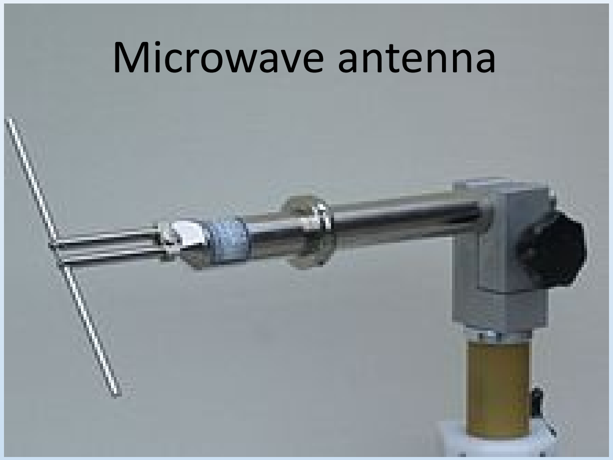
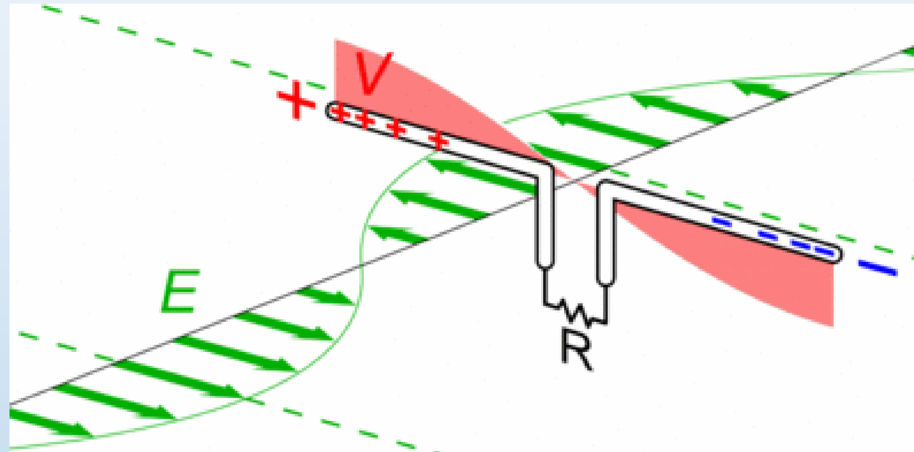
Lecture 9: Radiation

Outline
- Retarded Potentials
- Electric Dipole Radiation
- Electric-Magnetic Duality
- Magnetic Dipole Radiation

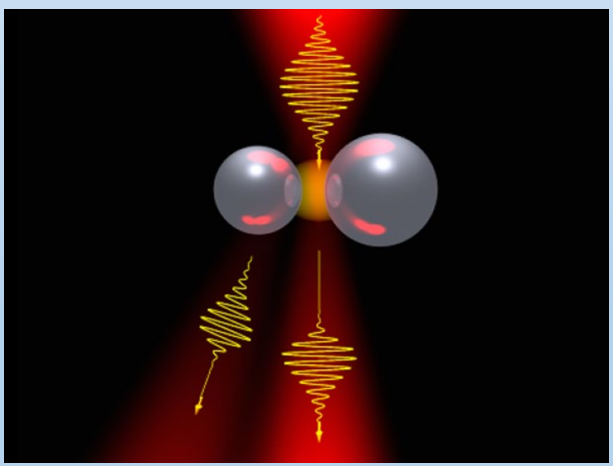
Electromagnetic radiation from antennas



Microwave antenna



A half-wave dipole antenna receiving a radio signal. The incoming radio wave (whose electric field is shown as **E, green arrows**) causes an oscillating electric current within the antenna elements (**black arrows**), alternately charging the two sides of the antenna positively (+) and negatively (−).



Optical antenna

Maxwell's Equations

$$\nabla \cdot \vec{D} = \rho$$ **(Gauss's Law)**

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ **(Faraday's Law)**

$$\nabla \cdot \vec{B} = 0$$ **(Magnetic Gauss's Law)**

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ **(Ampere's Law)**

$$\nabla \cdot \vec{J} = -\partial \rho / \partial t$$ **(The continuity equation)**

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \left(1 + \chi_e\right) \vec{E} = \varepsilon \vec{E}$$ **ε : permittivity**

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right) = \mu_0 \left(1 + \chi_m\right) \vec{H} = \mu \vec{H}$$ **μ : permeability**

Scalar and Vector potentials

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \textbf{Let} \quad \vec{B} = \nabla \times \underline{\vec{A}}$$ **(magnetic vector potential)**

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad \Rightarrow \quad \textbf{Let} \quad \vec{E} = -\nabla \underline{\varphi} - \frac{\partial \vec{A}}{\partial t}$$

(electric potential)

The Lorenz gauge: $\nabla \cdot \vec{A} = -\mu\varepsilon \dfrac{\partial \varphi}{\partial t}$

Substitute potentials into Maxwell's equations:

$$\boxed{\nabla^2 \varphi - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}} \qquad \boxed{\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}}$$

d'Alembert operator: $\nabla^2 - \mu\varepsilon\, \partial^2 \big/ \partial t^2$

The Case of Static Fields

$$\boxed{\nabla^2 \varphi = -\rho/\varepsilon}$$

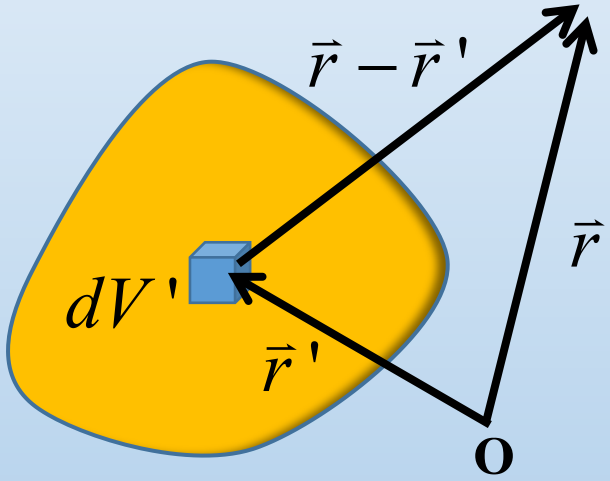
$$\boxed{\nabla^2 \vec{A} = -\mu \vec{J}}$$

Poisson's Equations

The solutions:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV'$$

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV'$$



Example (A point charge *q* at origin):

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV' = \frac{q}{4\pi\varepsilon r}$$

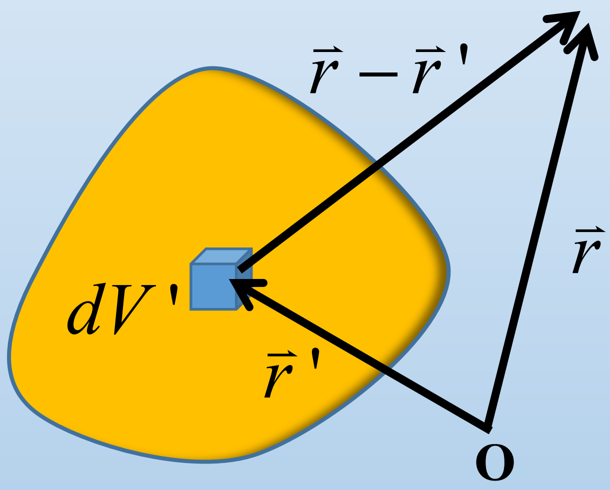
$$\vec{E} = -\nabla\varphi = \hat{r}\frac{q}{4\pi\varepsilon r^2}$$

Retarded Potentials

Now, electromagnetic "news" travels at the speed of light. In the nonstatic case, therefore, it's not the status of the source right now that matters, but rather its condition at some earlier time t_r (called the retarded time) when the "message" left.

Retarded Potentials:

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV' \\ \vec{A}(\vec{r},t) &= \frac{\mu}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV' \end{aligned}}$$



$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{v}$$ **(retarded time)**

The light we see now left each star at the retarded time corresponding to that star's distance from the earth.

Retarded Potentials (time-harmonic)

$$\nabla^2\varphi+\omega^2\mu\varepsilon\varphi=-\frac{\rho}{\varepsilon}$$

$$\nabla^2\vec{A}+\omega^2\mu\varepsilon\vec{A}=-\mu\vec{J}$$

$$\nabla\cdot\vec{A}+j\omega\mu\varepsilon\varphi=0$$

$$\vec{B}=\nabla\times\vec{A}$$

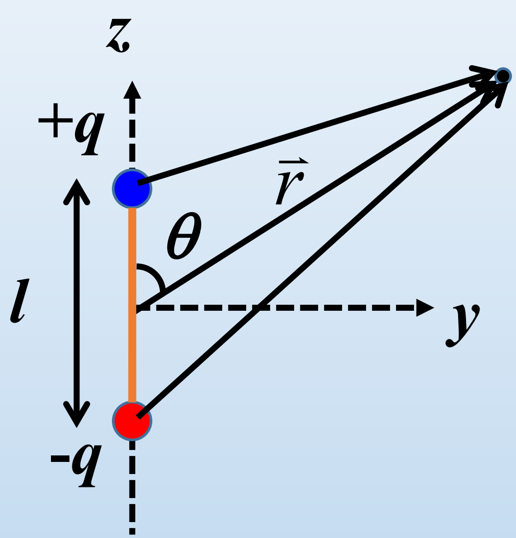
$$\vec{E}=-\nabla\varphi-j\omega\vec{A}$$

$$\varphi(\vec{r})=\frac{1}{4\pi\varepsilon}\int\frac{\rho(\vec{r}\,')e^{-jk|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}dV'$$

$$\vec{A}(\vec{r})=\frac{\mu}{4\pi}\int\frac{\vec{J}(\vec{r}\,')e^{-jk|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}dV'$$

$$k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}$$

Electric Dipole Radiation



Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω

$$q(t) = q\cos(\omega t) = \mathrm{Re}\left[qe^{j\omega t}\right]$$

Oscillating electric dipole:

$$\vec{p}(t) = \hat{z}ql\cos(\omega t) = \mathrm{Re}\left[\hat{z}qle^{j\omega t}\right]$$

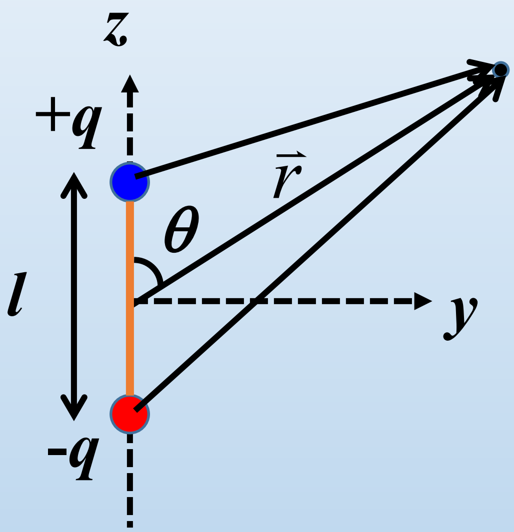
$$i(t) = \frac{dq(t)}{dt} = -q\omega\sin(\omega t) = \mathrm{Re}\left[j\omega qe^{j\omega t}\right]$$

Time-harmonic current: $I = j\omega q = j\omega\dfrac{p}{l}$ $\quad\boxed{\vec{I}l = j\omega\vec{p}}$

Current element: $\vec{J}(r')dV' = \hat{z}\dfrac{I}{\Delta S}\Delta S dz' = \hat{z}Idz' \quad \left(-\dfrac{l}{2} < z' < \dfrac{l}{2}\right)$

$l \to 0$ $\quad$ **Hertzian dipole at $\vec{r}\,'$:** $\vec{J}(\vec{r}\,') = \hat{z}Il\delta(\vec{r}\,')$

Electric Dipole Radiation



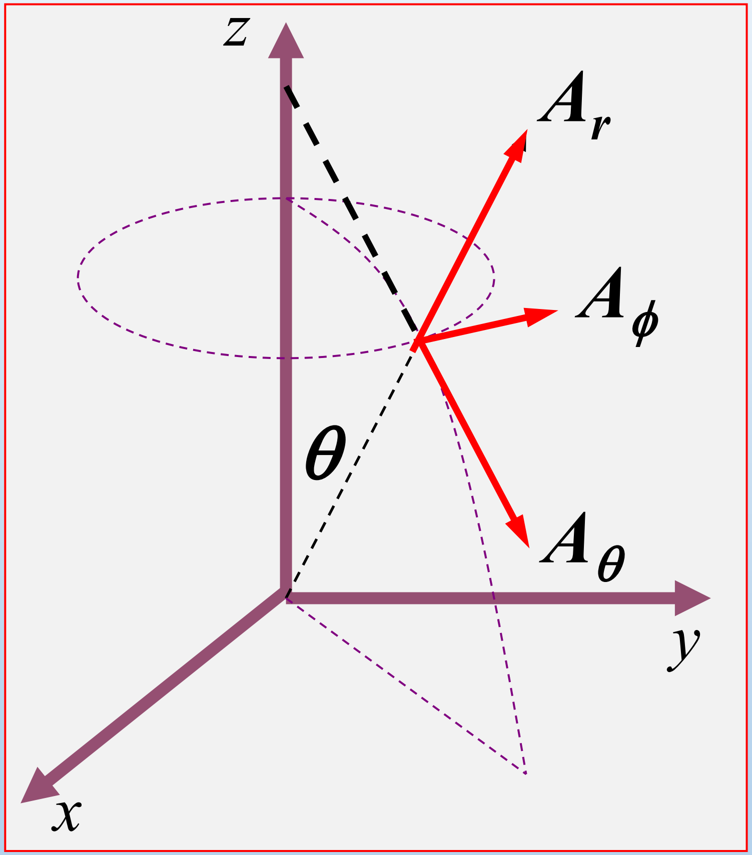
For $l \ll r$ ⇨ $$\frac{e^{-jk|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \approx \frac{e^{-jkr}}{r}$$

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int \frac{\vec{J}(\vec{r}\,')e^{-jk|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}dV'$$

$$= \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkr}}{r}\hat{z}Idz' = \hat{z}\frac{\mu Il}{4\pi r}e^{-jkr}$$

Or $$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int \underbrace{\hat{z}Il\delta(\vec{r}\,')}_{\vec{J}(\vec{r}\,')}dV' = \hat{z}\frac{\mu Il}{4\pi r}e^{-jkr}$$

Transform into spherical coordinates



$$A_r = \vec{A}\cdot\hat{r} = A_z\cos\theta$$

$$= \frac{\mu I l}{4\pi r}\cos\theta e^{-jkr}$$

$$A_\theta = \vec{A}\cdot\hat{\theta} = -A_z\sin\theta$$

$$= -\frac{\mu I l}{4\pi r}\sin\theta e^{-jkr}$$

$$A_\phi = \vec{A}\cdot\hat{\phi} = 0$$

Electric Dipole Radiation

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\nabla \times \vec{A} = \frac{1}{\mu r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon}\nabla \times \vec{H}$$

Electric Dipole Radiation

$$E_r = \frac{2Ilk^3\cos\theta}{4\pi\omega\varepsilon}\left(\frac{1}{(kr)^2} - \frac{j}{(kr)^3}\right)e^{-jkr}$$

$$E_\theta = \frac{Ilk^3\sin\theta}{4\pi\omega\varepsilon}\left(\frac{j}{kr} + \frac{1}{(kr)^2} - \frac{j}{(kr)^3}\right)e^{-jkr}$$

$$E_\phi = 0$$

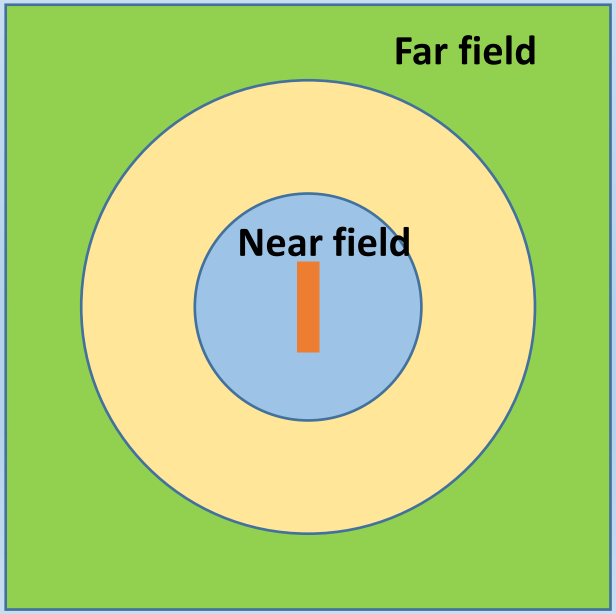
$$H_r = 0$$

$$H_\theta = 0$$

$$H_\phi = \frac{k^2 Il\sin\theta}{4\pi}\left(\frac{j}{kr} + \frac{1}{(kr)^2}\right)e^{-jkr}$$

Near field region: $kr \ll 1$

Far field region: $kr \gg 1$



Electric Dipole Radiation - Near field

$$kr \ll 1 \Rightarrow \frac{1}{kr} \ll \frac{1}{(kr)^2} \ll \frac{1}{(kr)^3} \qquad e^{-jkr} \approx 1$$

$$E_r = -j\frac{Il\cos\theta}{2\pi\omega\varepsilon r^3}$$

$$E_\theta = -j\frac{Il\sin\theta}{4\pi\omega\varepsilon r^3}$$

$$H_\phi = \frac{Il\sin\theta}{4\pi r^2}$$

$$I = j\omega q$$

$$p = ql$$

$$E_r = \frac{ql\cos\theta}{2\pi\varepsilon r^3}$$

$$E_\theta = \frac{ql\sin\theta}{4\pi\varepsilon r^3}$$

$$H_\phi = \frac{Il\sin\theta}{4\pi r^2}$$

Quasi-static

- **π/2 phase difference between E and H**
- **No radiation power**
- **Quasi-static field**

$$\left\langle \vec{S}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] = 0$$

Electric Dipole Radiation - Far field

$$E_r = \frac{2Ilk^3 \cos\theta}{4\pi\omega\varepsilon}\left(\frac{1}{(kr)^2} - \frac{j}{(kr)^3}\right)e^{-jkr}$$

$$E_\theta = \frac{Ilk^3 \sin\theta}{4\pi\omega\varepsilon}\left(\frac{j}{kr} + \frac{1}{(kr)^2} - \frac{j}{(kr)^3}\right)e^{-jkr}$$

$$H_\phi = \frac{k^2 Il \sin\theta}{4\pi}\left(\frac{j}{kr} + \frac{1}{(kr)^2}\right)e^{-jkr}$$

$$kr \gg 1$$

$$E_\theta = j\frac{Ilk^2 \sin\theta}{4\pi\omega\varepsilon r}e^{-jkr}$$

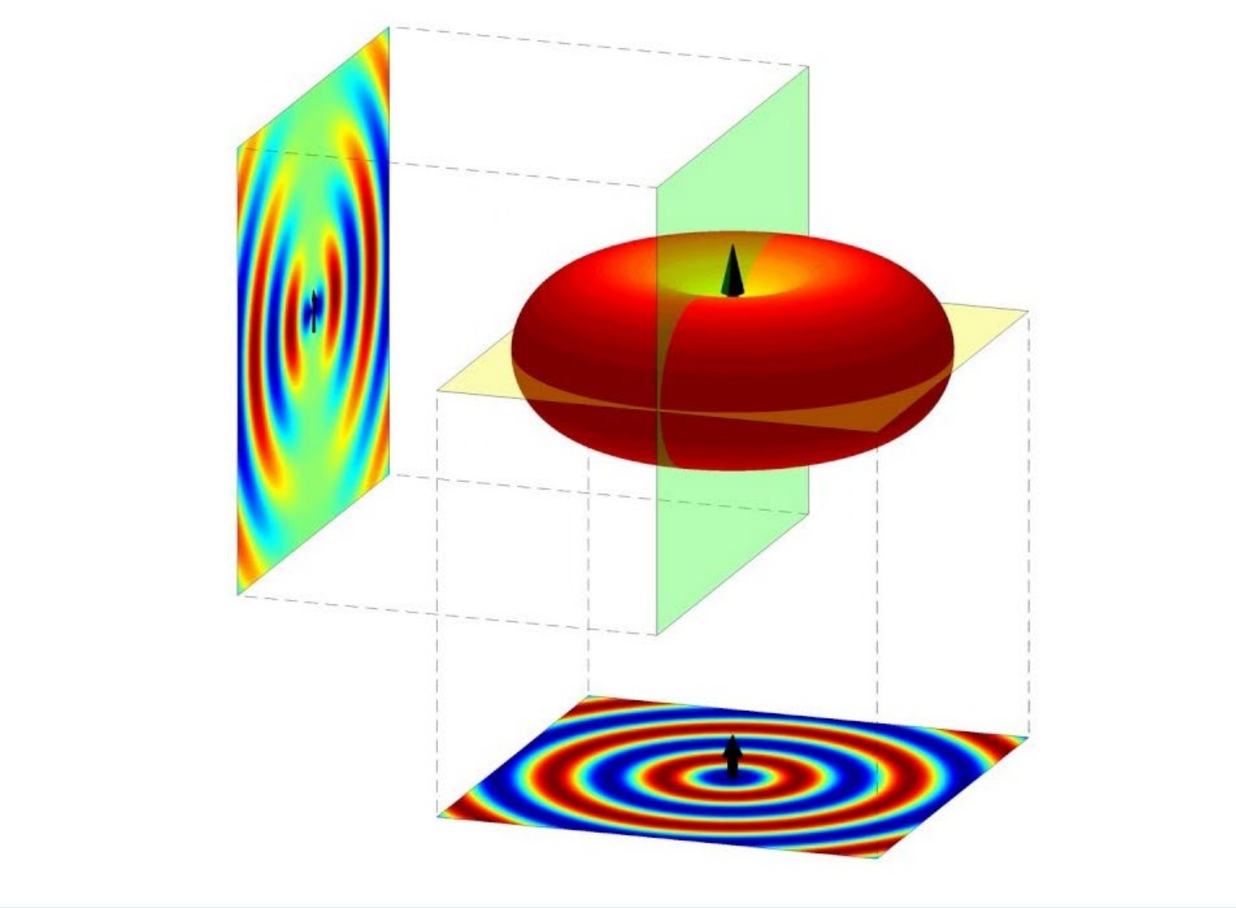
$$H_\phi = j\frac{kIl \sin\theta}{4\pi r}e^{-jkr}$$

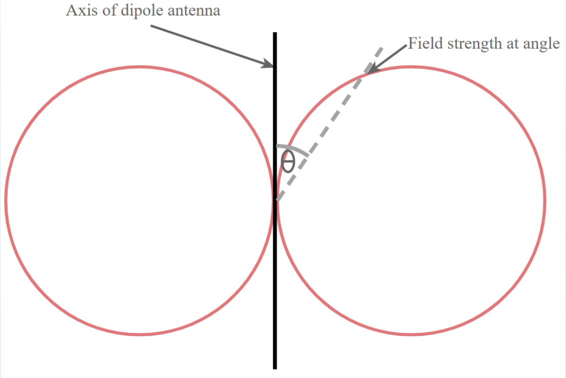
$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$E_\theta = j\frac{Il \sin\theta}{2\lambda r}\eta e^{-jkr} \qquad H_\phi = j\frac{Il \sin\theta}{2\lambda r}e^{-jkr} = \frac{E_\theta}{\eta}$$

Far-field radiation pattern





$$F(\theta,\phi)=\frac{\left|\vec{E}(\theta,\phi)\right|}{\left|\vec{E}_{\max}\right|}$$

$$E_\theta = j\frac{Il\sin\theta}{2\lambda r}\eta e^{-jkr} \qquad H_\phi = j\frac{Il\sin\theta}{2\lambda r}e^{-jkr}=\frac{E_\theta}{\eta}$$

Radiation Power

Time-average power density:

$$\left\langle \vec{S}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\theta}E_\theta \times \hat{\varphi}H_\phi^*\right] = \frac{1}{2}\mathrm{Re}\left[\hat{r}E_\theta H_\phi^*\right] = \hat{r}\frac{\left|E_\theta\right|^2}{2\eta} = \hat{r}\frac{\eta\left|H_\phi\right|^2}{2}$$

$$= \hat{r}\frac{\eta}{2}\left(\frac{Il\sin\theta}{2\lambda r}\right)^2$$

Radiation power:

$$P_r = \oint_S \left\langle \vec{S}(t) \right\rangle \cdot d\vec{S} = \oint_S \hat{r}\frac{\eta}{2}\left(\frac{Il\sin\theta}{2\lambda r}\right)^2 \cdot \hat{r}r^2\sin\theta d\theta d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} 15\pi\left(\frac{Il}{\lambda}\right)^2 \sin^3\theta d\theta = 40\pi^2 I^2\left(\frac{l}{\lambda}\right)^2$$

Radiation resistance: $R_r = \frac{2P_r}{I^2} = 80\pi^2\left(\frac{l}{\lambda}\right)^2$

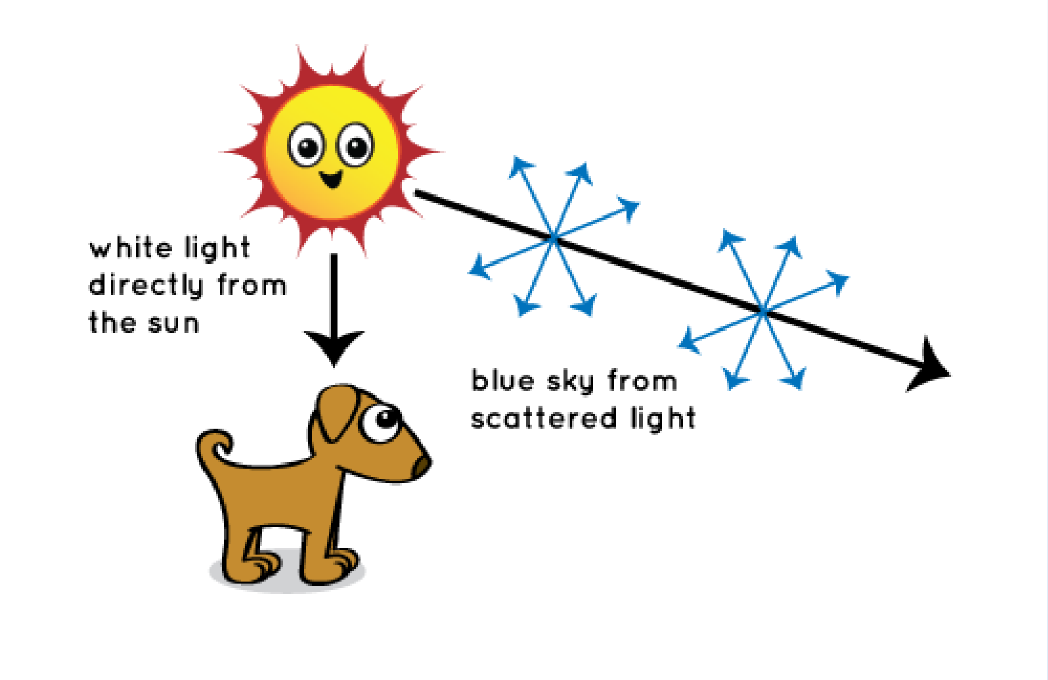
Why is the sky blue?

Radiation power:

$$P_r = 40\pi^2 I^2 \left(\frac{l}{\lambda}\right)^2$$

Small wavelength

high radiation



Sunlight reaches Earth's atmosphere and is **scattered** in all directions by all the gases and particles in the air. Blue light is scattered in all directions by the tiny molecules of air in Earth's atmosphere. Blue is scattered more than other colors because it travels as shorter, smaller waves. This is why we see a blue sky most of the time.

What makes a red sunset?

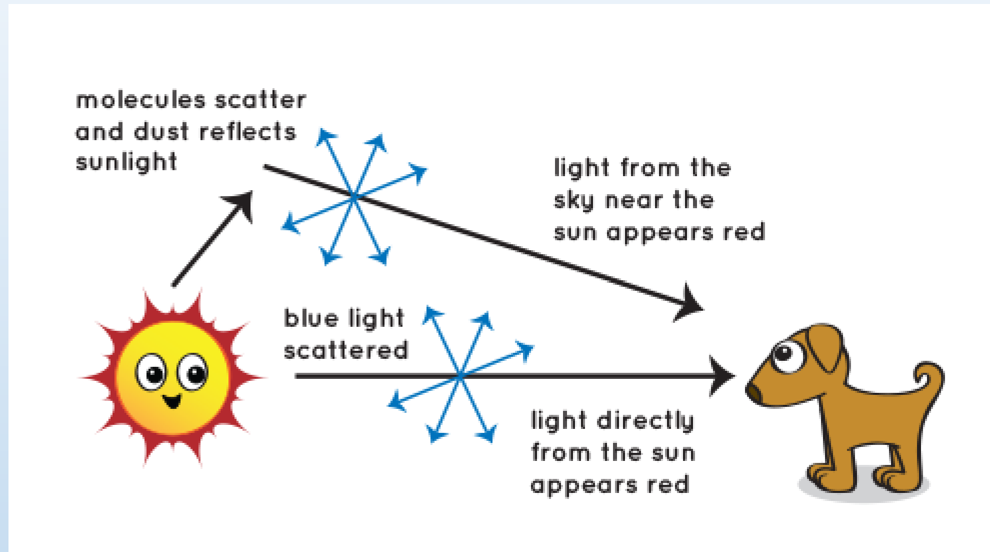
Radiation power:

$$P_r = 40\pi^2 I^2 \left(\frac{l}{\lambda}\right)^2$$

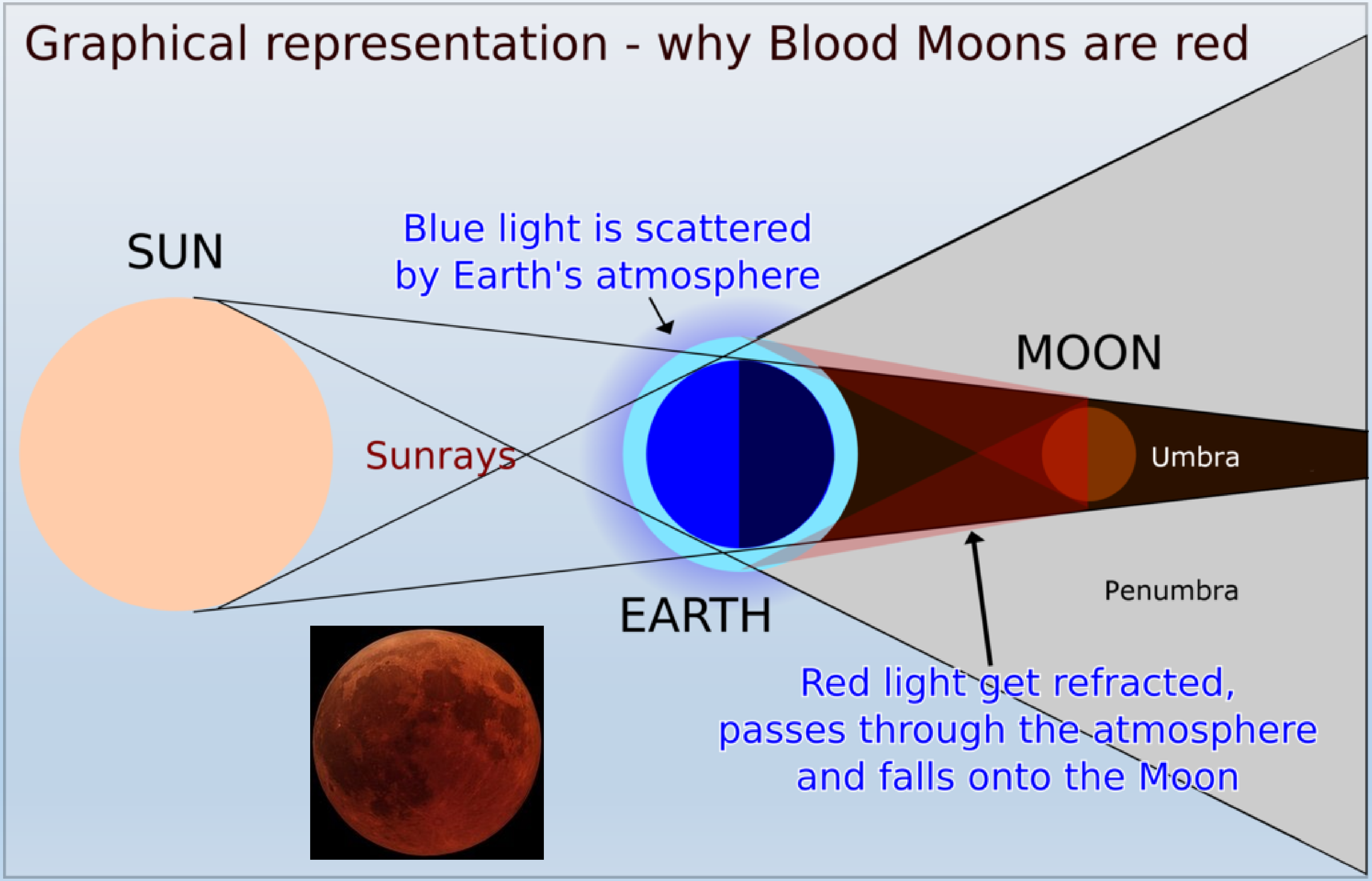
Small wavelength

high radiation

As the sun gets lower in the sky, its light is passing through more of the atmosphere to reach you. Even more of the blue light is scattered, allowing the reds and yellows to pass straight through to your eyes.



Lunar eclipse



A totally eclipsed Moon is sometimes called a **blood moon**

Maxwell's Equations

Total fields can be decomposed into two parts:

- **Fields (E_e, H_e) generated by electric sources (J_e, ρ_e)**
- **Fields (E_m, H_m) generated by magnetic sources (J_m, ρ_m)**

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}_e$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} - J_m$$

$$\nabla \cdot \vec{H} = \frac{\rho_m}{\mu}$$

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon}$$

$$\vec{E} = \vec{E}_e + \vec{E}_m$$

$$\vec{H} = \vec{H}_e + \vec{H}_m$$

$$\left(\vec{J}_e, \rho_e\right) \Rightarrow \left(\vec{E}_e, \vec{H}_e\right)$$

$$\left(\vec{J}_m, \rho_m\right) \Rightarrow \left(\vec{E}_m, \vec{H}_m\right)$$

Electric-Magnetic Duality

$$\nabla \times \vec{H}_e = \varepsilon \frac{\partial \vec{E}_e}{\partial t} + \vec{J}_e$$

$$\nabla \times \vec{E}_e = -\mu \frac{\partial \vec{H}_e}{\partial t}$$

$$\nabla \cdot \vec{H}_e = 0$$

$$\nabla \cdot \vec{E}_e = \frac{\rho_e}{\varepsilon}$$

$$\overline{E}_e \rightarrow \overline{H}_m$$

$$\overline{H}_e \rightarrow -\overline{E}_m$$

$$\overline{J}_e \rightarrow \overline{J}_m$$

$$\rho_e \rightarrow \rho_m$$

$$\mu \rightarrow \varepsilon$$

$$\varepsilon \rightarrow \mu$$

$$\nabla \times \vec{E}_m = -\mu \frac{\partial \vec{H}_m}{\partial t} - \vec{J}_m$$

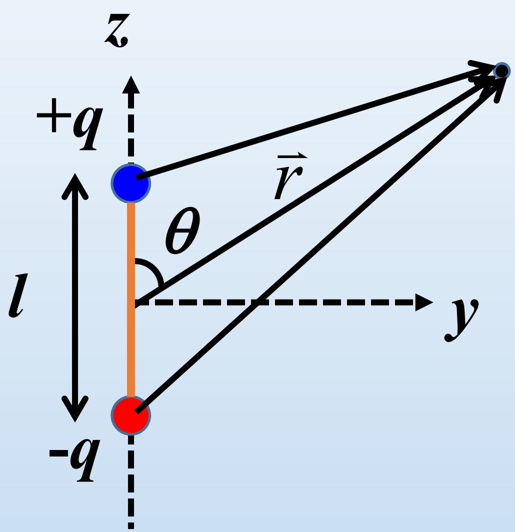
$$\nabla \times \vec{H}_m = \varepsilon \frac{\partial \vec{E}_m}{\partial t}$$

$$\nabla \cdot \vec{E}_m = 0$$

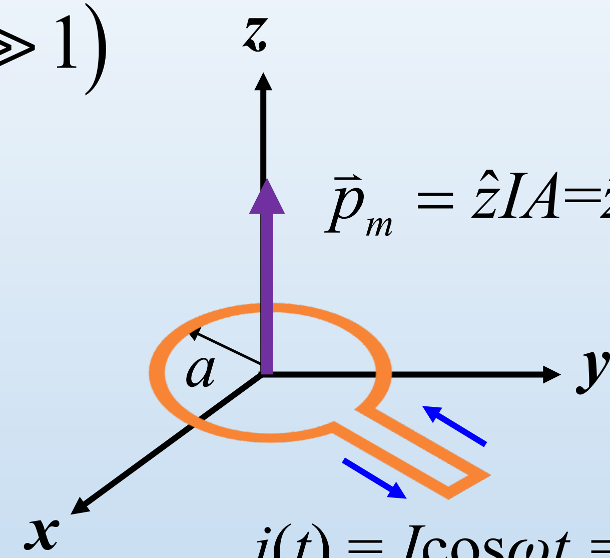
$$\nabla \cdot \vec{H}_m = \frac{\rho_m}{\mu}$$

The formulation for the field due to the magnetic source can be obtained directly using the transformation.

Magnetic dipole radiation



$$\left(kr \gg 1\right)$$



$$\vec{p}_m = \hat{z}IA = \hat{z}I\pi a^2$$

$$i(t) = I\cos\omega t = \mathrm{Re}[Ie^{j\omega t}]$$

$$\vec{J}_e\left(\vec{r}\,'\right) = j\omega\vec{p}_e = \hat{z}Il\delta\left(\vec{r}\,'\right)$$

$$\vec{J}_m\left(\vec{r}\,'\right) = j\omega\mu\vec{p}_m = \hat{z}j\omega\mu IA\delta\left(\vec{r}\,'\right)$$

$$E_\theta = j\frac{Il\sin\theta}{2\lambda r}\eta e^{-jkr}$$

$$H_\phi = j\frac{Il\sin\theta}{2\lambda r}e^{-jkr} = \frac{E_\theta}{\eta}$$

Electric dipole radiation

$$Il \rightarrow j\omega\mu IA$$

$$E \rightarrow H$$

$$H \rightarrow -E$$

$$\varepsilon \leftrightarrow \mu$$

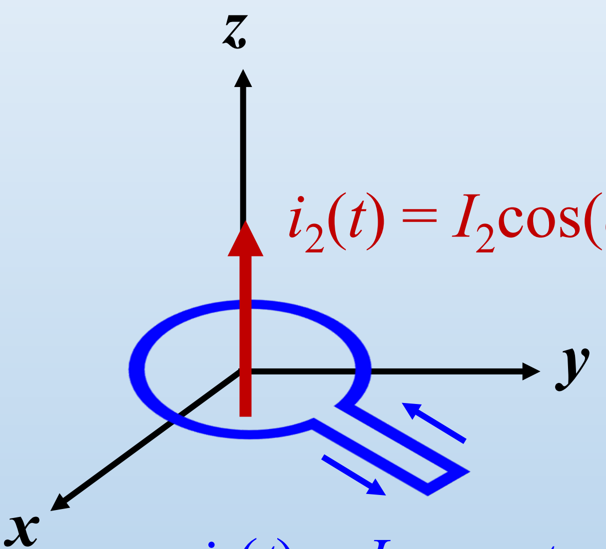
$$H_\theta = -\frac{\omega\mu IA\sin\theta}{2\lambda r}\frac{1}{\eta}e^{-jkr}$$

$$E_\phi = \frac{\omega\mu IA\sin\theta}{2\lambda r}e^{-jkr}$$

Magnetic dipole radiation

After-Class Question

What's the far-field radiation from a combination of electric and magnetic dipoles?



$$i_2(t) = I_2\cos(\omega t + \varphi) = \mathrm{Re}[I_2 e^{j\varphi} e^{j\omega t}]$$

$$i_1(t) = I_1\cos\omega t = \mathrm{Re}[I_1 e^{j\omega t}]$$

Hint: (1) electric dipole: $I \rightarrow I_2 \exp\left(j\varphi\right)$

(2) magnetic dipole: $I \rightarrow I_1$