Chapter 5: Waves in Media

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History: *René Descartes had seen light separated into the colors of the rainbow by glass or water (Figure 5.1), though the source of the color was unknown. Isaac Newton's 1666 experiment of bending white light through a prism demonstrated that all the colors already existed in the light, with different color "corpuscles" fanning out and traveling with different speeds through the prism. It was only later that Young and Fresnel combined Newton's particle theory with Huygens' wave theory to explain how color arises from the spectrum of light.*

Figure 5.1 Light is separated into the colors of the rainbow by glass or water.

In this chapter, we study the wave propagation in different kinds of media and learn the underlying physics of various phenomena in natural life, such as prism, rainbow and radio communication. We concern ourselves with time-harmonic fields, Lorentz oscillator model of an atom and dispersion of complex permittivity. We also concern on wave propagation in plasma media, conducting media, uniaxial media and chiral media.

5.1 Time-Harmonic Fields

Continuous Monochromatic Waves

For electromagnetic waves of a particular frequency in the steady state, the fields are time-harmonic and known as monochromatic waves or continuous waves (CW). The CW cases are important for three reasons: (i) the CW assumption can be used to eliminate the time dependence in the Maxwell equations and thus considerably simplify the mathematics; (ii) once the CW wave is solved and a sound understanding is developed for the frequency-domain phenomena, Fourier theory can be applied to study the time-domain phenomena; (iii) CW representation covers the whole spectrum of electromagnetic waves. Clearly, a thorough understanding of CW or the time-harmonic case is essential in the study of all electromagnetic wave phenomena.

In general, when the currents, charges, and fields oscillate at a single frequency, each quantity can be expressed as a sinusoidal/cosinusoidal function with an amplitude and a phase. For example, a monochromatic electric field can be written as

$$\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r})\cos\left(\vec{k}\cdot\vec{r} - \omega t\right) \tag{5.1}$$

where $\vec{k}$ and ω are the wavevector and the angular frequency, respectively. Using Euler's formula, this can be written as

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_0(\vec{r})e^{-j\vec{k}\cdot\vec{r}}e^{j\omega t}\right] \tag{5.2}$$

where $j = \sqrt{-1}$ and Re stands for the real part. Now, define a complex quantity

$$\boxed{\vec{E}(\vec{r}) = \vec{E}_0(\vec{r})e^{-j\vec{k}\cdot\vec{r}}} \tag{5.3}$$

which contains both the amplitude and phase of the field and is only a spatial function. The electric field in Eq. (5.1) can be written as

$$\boxed{\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right]} \tag{5.4}$$

The complex quantity defined in Eq. (5.3) is called a *phasor*. By expressing each of the source and field quantities in the form of Eq. (5.4) and substituting them into differential Maxwell's equations, we obtain the time-harmonic form of Maxwell's equations

$$\boxed{\begin{aligned} &\nabla\cdot\vec{D} = \rho_{free} \\ &\nabla\times\vec{E} = -j\omega\vec{B} \\ &\nabla\cdot\vec{B} = 0 \\ &\nabla\times\vec{H} = \vec{J}_{free} + j\omega\vec{D} \end{aligned}} \tag{5.5}$$

Clearly, in this conversion, all one has to do is to replace the time derivative $\partial/\partial t$ with $j\omega$. We can do the same to Maxwell's equations in integral form and boundary conditions.

We see that the Maxwell equations for time-harmonic fields no longer have time dependence, thus mathematically reduced the four independent variables x, y, z, t to x, y, z. The electromagnetic field quantities are now dependent on space only. However, all field quantities are now complex. To recover the real space-time dependent field vector, we simply multiply the complex quantity by $e^{j\omega t}$ and take its real part as shown in Eq. (5.4).

Polarization of Monochromatic Waves

Considering the time evolution of $\vec{E}(\vec{r},t)$ at $r = 0$ by letting

$$\vec{E} = \vec{E}_R + j\vec{E}_I \tag{5.6}$$

The plane defined by the two vector $\vec{E}_R$ and $\vec{E}_I$ is called the polarization plane. For the time dependent electric field vector, we find from Eq. (5.4)

$$\vec{E}(t) = \mathrm{Re}\left[\left(\vec{E}_R + j\vec{E}_I\right)e^{j\omega t}\right] = \vec{E}_R\cos(\omega t) - \vec{E}_I\sin(\omega t) \tag{5.7}$$

In Figure 5.2 we sketch the two vectors $\vec{E}_R$ and $\vec{E}_I$. At t = 0, $\vec{E}(t)$ coincides with $\vec{E}_R$. At $\omega t = \pi/2$, $\vec{E}(t)$ coincides with $-\vec{E}_I$. The tip of $\vec{E}$ as a function of time traces out an ellipse. When $\vec{E}_R$ is perpendicular to $\vec{E}_I$, one represents the major axis and the other the minor axis of the ellipse.

The time derivative of $\vec{E}(t)$ gives

$$\frac{\partial \vec{E}(t)}{\partial t} = -\omega\left[\vec{E}_R \sin(\omega t) + \vec{E}_I \cos(\omega t)\right] \tag{5.8}$$

At t = 0, the time rate of change of the electric field vector is in the direction of $-\vec{E}_I$. As $\omega t = \pi/2$, the time rate of change of $\vec{E}$ is in the direction of $-\vec{E}_R$. Thus as time increases, the vector $\vec{E}$ moves from $\vec{E}_R$ to $-\vec{E}_I$.

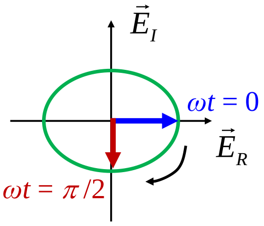

Figure 5.2 The polarization plane.

When $\vec{E}_R$ and $\vec{E}_I$ are parallel or anti-parallel to each other, the total electric field vector represents a linearly polarized wave. For an electromagnetic wave propagating out of the paper in the direction of the thumb, the motion of the tip of $\vec{E}(\vec{r},t)$ follows the right-hand finger. The wave is right-handed elliptically polarized. When $\vec{E}_R$ and $\vec{E}_I$ are perpendicular to each other and have the same magnitudes, the wave will be circularly polarized. Examples of different polarizations are illustrated in Figure 5.3

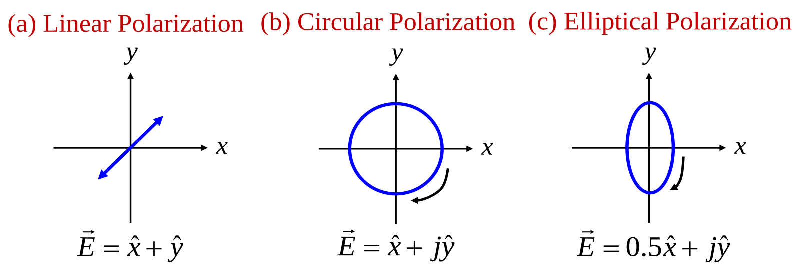

Figure 5.3 Different polarization states.

Time-average Poynting Power Vector

The complex Poynting's theorem is derived by dot-multiplying $\nabla \times \vec{E} = -j\omega\vec{B}$ by $\vec{H}^*$ and substracting the complex conjugate of $\nabla \times \vec{H} = \vec{J}_{free} + j\omega\vec{D}$ dot multiplied by $\vec{E}$. Making use of the identity $\vec{H}^* \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}^* = \nabla \cdot \left(\vec{E} \times \vec{H}^*\right)$, we obtain

$$\nabla \cdot \left(\vec{E} \times \vec{H}^*\right) = -j\omega\left[\vec{H}^* \cdot \vec{B} - \vec{E} \cdot \vec{D}^*\right] - \vec{E} \cdot \vec{J}^* \tag{5.9}$$

The complex Poynting's vector $\vec{S}$ is defined to be

$$\vec{S} = \vec{E} \times \vec{H}^* \tag{5.10}$$

However, it is noted that, mathematically $\vec{E} \times \vec{H}^*$ is not a uniquely defined quantity as far as Poynting's theorem is concerned. An arbitrary curl field $\nabla \times \vec{A}$ can be added to $\vec{E} \times \vec{H}^*$ without changing Eq. (5.9). Physically the complex vector $\vec{S}$ as defined in Eq. (5.10) has been identified as a complex power density vector.

The term $\vec{E} \cdot \vec{J}^* = \vec{E} \cdot \left(\vec{J}_c^* + \vec{J}_f^* \right)$ consists of two parts: one part due to the ohmic current $\vec{J}^*$ and the other due to the free current $\vec{J}_f$. Eq. (5.9) can be rearranged to read

$$-\vec{E} \cdot \vec{J}_f^* = \nabla \cdot \left(\vec{E} \times \vec{H}^* \right) + \vec{E} \cdot \vec{J}_c^* - j\omega \left[\vec{E} \cdot \vec{D}^* - \vec{B} \cdot \vec{H}^* \right] \tag{5.11}$$

Consider a small volume element *V*. Eq. (5.11) states that the complex power supplied to *V* by $\vec{J}_f$, $-\vec{E} \cdot \vec{J}_f^*$, is equal to the divergence of the complex Poynting power flow out of *V*, $\nabla \cdot \left(\vec{E} \times \vec{H}^* \right)$, plus the complex power dissipated in *V*, $\vec{E} \cdot \vec{J}_c^*$, plus the last term related to the stored complex electromagnetic energy in *V*.

We let a complex field vector be represented by two real vectors. We write

$$\vec{E}(\vec{r}) = \vec{E}_R(\vec{r}) + j\vec{E}_I(\vec{r}) \tag{5.12}$$

where $\vec{E}_R$ and $\vec{E}_I$ are both real vectors representing the real and imaginary parts of the complex vector $\vec{E}$. Similarly,

$$\vec{H}(\vec{r}) = \vec{H}_R(\vec{r}) + j\vec{H}_I(\vec{r}) \tag{5.13}$$

The instantaneous values for the field vectors are

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\left[\vec{E}(\vec{r}) e^{j\omega t} \right] = \vec{E}_R \cos \omega t - \vec{E}_I \sin \omega t \tag{5.14}$$

and

$$\vec{H}(\vec{r}, t) = \mathrm{Re}\left[\vec{H}(\vec{r}) e^{j\omega t} \right] = \vec{H}_R \cos \omega t - \vec{H}_I \sin \omega t \tag{5.15}$$

The complex Poynting's vector is

$$\vec{S} = \vec{E} \times \vec{H}^* = \vec{E}_R \times \vec{H}_R + \vec{E}_I \times \vec{H}_I + j\left(\vec{E}_I \times \vec{H}_R - \vec{E}_R \times \vec{H}_I \right) \tag{5.16}$$

In previous chapters, we have defined the instantaneous Poynting's vector as

$$\vec{S}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) \tag{5.17}$$

In view of Eq. (5.14) and Eq. (5.15), we have

$$\begin{aligned}\vec{S}(\vec{r},t) &= \left[\vec{E}_R\cos\omega t - \vec{E}_I\sin\omega t\right]\times\left[\vec{H}_R\cos\omega t - \vec{H}_I\sin\omega t\right] \\ &= \vec{E}_R\times\vec{H}_R\cos^2\omega t + \vec{E}_I\times\vec{H}_I\sin^2\omega t - \left(\vec{E}_R\times\vec{H}_I + \vec{E}_I\times\vec{H}_R\right)\sin\omega t\cos\omega t\end{aligned} \tag{5.18}$$

The instantaneous Poynting's vector $\vec{S}(\vec{r},t)$ is a real vector and is time-dependent. To relate $\vec{S}(\vec{r})$ to $\vec{S}(\vec{r},t)$ we must eliminate the time dependence in $\vec{S}(\vec{r},t)$. This is accomplished by a time averaging process. We find

$$\left\langle\vec{S}(\vec{r},t)\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} d(\omega t)\vec{S}(\vec{r},t) = \frac{1}{2}\left(\vec{E}_R\times\vec{H}_R + \vec{E}_I\times\vec{H}_I\right) = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right] \tag{5.19}$$

It is noticeable that, when the complex Poynting's power vector $\vec{S} = \vec{E}\times\vec{H}^*$ is known, taking half of its real part yields the time average value of the instantaneous Poynting's vector:

$$\left\langle\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] \tag{5.20}$$

This rule, in general, applied to the product of any two field quantities. That is, the time average of the product of two field quantities is equal to half of the real part of the product of one complex field quantity and the complex conjugate of the other complex field quantity. The time average of Poynting vector is also known as *intensity* (or *irradiance*), labelled by *I*.

5.2 Lorentz Oscillator Model of an Atom

Brief Background

In 1900, Max Planck presented his "purely formal assumption" that the energy of electromagnetic waves must be a multiple of some elementary unit and therefore could be described as consisting of small packets of energy. The term "quantum" comes from the Latin "quantus", meaning "how much", and was used by Planck in this context to represent "counting" of these elementary units. This idea was exploited by Albert Einstein, who in 1905 showed that EM waves could be equivalently treated as corpuscles - later named "photons" - with discrete, "quantized" energy, which was dependent on the frequency of the wave.

Prior to the advent of quantum mechanics in the 1900s, the most well-known attempt by a classical physicist to describe the interaction of light with matter in terms of Maxwell's equations was carried out by a Hendrik Lorentz. Despite being a purely classical description, the Lorentz oscillator model was adapted to quantum mechanics in the 1900s and is still of considerable use today.

Hendrik Antoon Lorentz was a Dutch physicist in the late 19th century, responsible for the derivation of the electromagnetic Lorentz force and the Lorentz transformations, later used by Einstein in the development of Special Relativity. Lorentz shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect (splitting of spectral lines when a static magnetic field is applied). In his attempt to describe the interaction between atoms and electric fields in classical terms, Lorentz proposed that the electron (a particle with some small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law - that is, a spring-like force. An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring, which would set the electron into oscillating motion. This is the so-called *Lorentz oscillator model*.

Lorentz Model of Atomic Oscillator

If we assume the nucleus of the atom is much more massive than the electron, we can treat the problem as if the electron-spring system is connected to an infinite mass (Figure 5.4), which does not move, allowing us to use the mass of the electron, $m = 9.11\times10^{-31}$kg. Depending on the case, this value may be substituted by the reduced or effective electron mass to account for deviations from this assumption.

Moreover, the assumption that the binding force behaves like a spring is a justified approximation for *any* kind of binding, given that the displacement is small enough (meaning that only the constant and linear terms in the Taylor expansion are relevant). The damping term comes from internal collisions in the solid and radiation emitted by the electron.

For the spring oscillator shown in Figure 5.4, from the Newton's second law, we have

$$m\frac{d^2y}{dt^2} = F_{driving} + F_{damping} + F_{spring} = F_{driving} - \gamma m\frac{dy}{dt} - m\omega_0^2 y \tag{5.21}$$

where $F_{damping} = -\gamma m\, dy/dt$ is the damping force, $F_{driving} = -m\omega_0^2 y$ is the restoring force from the spring, ω_0 is the resonant frequency of the spring. .

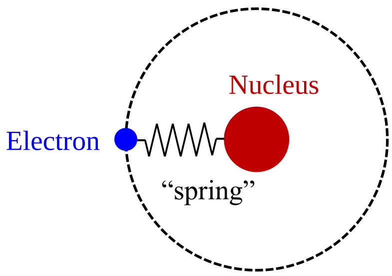

Figure 5.4 Classical model of an atom

All there is left for us to complete the Lorentz oscillator equation is to determine the driving force. Under an applied electromagnetic wave field, an electron is subject to the Lorentz force $\vec{F} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) \approx -e\vec{E}$. The second term is negligible for $v/c \ll 1$, as $\left|\vec{B}\right| = \left|\vec{E}\right|/c$ for a plane wave in free space, and $\left|\vec{v}\times\vec{B}\right| \approx (v/c)\left|\vec{E}\right| \ll \left|\vec{E}\right|$, although $\vec{v}\times\vec{B}$ and $\vec{E}$ are in different directions. We have learned the formula for the polarization:

$$\vec{P} = -Ne\vec{y} \tag{5.22}$$

The polarization vector is nothing but the density (per volume) of dipole moments, which in turn are defined simply as the product of the charge and the displacement vector from the negative to the positive charge, that is, from the electron to the nucleus.

Substitute $y(t) = -P(t)/Ne$ and $F_{driving} = -eE$ into Eq. (5.21), we find

$$\frac{d^2P}{dt^2} + \gamma\frac{dP}{dt} + \omega_0^2 P = \frac{Ne^2}{m}E \tag{5.23}$$

For time-harmonic fields, we can replace d/dt by $j\omega$, so as to obtain

$$P = -\frac{Ne^2/m}{\omega^2 - j\omega\gamma - \omega_0^2}E\,. \tag{5.24}$$

As a result, the *complex permittivity* of a general dispersive medium is

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{P}{\varepsilon_0 E}\right) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega}\right)}. \tag{5.25}$$

where $\omega_P = \sqrt{Ne^2/\varepsilon_0 m}$ is the plasma frequency.

5.3 Complex Permittivity

From Eq. (5.25), we can find a medium can be described in terms of a *complex relative permittivity* (or dielectric constant), that is,

$$\varepsilon_r(\omega) = \varepsilon_r' - j\varepsilon_r''\,. \tag{5.26}$$

where the real part ε_r' determines the polarization of the medium with losses ignored and the imaginary part must be negative (ε_r'' positive) due to energy conservation, which describes the losses in the medium. For a lossless medium, obviously $\varepsilon_r = \varepsilon_r'$. The loss ε_r'' depends on the frequency of the wave and usually peaks at certain natural (resonant) frequencies involved in the absorption process.

An EM wave that is traveling in a medium and experiencing attenuation due to absorption can be generally described by a *complex propagation constant k*, that is,

$$k = k' - jk''\,. \tag{5.27}$$

where k_r' and k_r'' are the real and imaginary parts. If we put Eq. (5.27) (assume $\vec{k} = k\hat{z}$, $\vec{r} = z\hat{z}$) into Eqs. (5.3) and (5.4) we will find:

$$E = E_0 \exp(-k''z)\exp\left[j(\omega t - k'z)\right]. \tag{5.28}$$

The amplitude decays exponentially while the wave propagates along *z*. The real part k' of the complex propagation constant (wave vector) describes the propagation characteristics (e.g., phase velocity $v = \omega/k'$). The imaginary part k'' describes the rate of attenuation along *z*. The intensity *I* at any point along *z* is

$$I \propto |E|^2 \propto= \exp(-2k''z)\,. \tag{5.29}$$

so that the rate of change in the intensity with distance is

$$dI/dz = -2k''I\,. \tag{5.30}$$

where the negative sign represents attenuation. The *attenuation coefficient α* is defined as the fractional decrease in the intensity *I* of a wave per unit distance along the direction of propagation *z*:

$$\alpha = -\frac{dI}{Idz}\,. \tag{5.31}$$

When the irradiance decreases, dI/dz is negative, and the attenuation coefficient is a positive number. If the attenuation of the wave is due to absorption only, then α is the *absorption coefficient*. Combine Eq. (5.30) and Eq. (5.31) together, we can easily find $\alpha = 2k''$.

Suppose that k_0 is the propagation constant (or wave number) in vacuum. This is a real quantity as a plane wave suffers no loss in free space. The complex refractive index $\tilde{n}$ with a real part *n* and imaginary part κ is defined as the ratio of the complex propagation constant in a medium to propagation constant in free space.

$$\tilde{n} = n - j\kappa = k/k_0 = (1/k_0)[k' - jk'']\,. \tag{5.32}$$

that is,

$$n = k'/k_0 \quad \text{and} \quad \kappa = k''/k_0\,. \tag{5.33}$$

The real part *n* is simply and generally called the refractive index and κ is called the *extinction coefficient*. In the absence of attenuation, $k'' = 0$, $k = k'$ and $\tilde{n} = n = k/k_0 = k'/k_0$.

We know that in the absence of loss, the relationship between the refractive index *n* and the relative permittivity ε_r is $n = (\varepsilon_r)^{1/2}$. This relationship is also valid in the presence of loss except that we must use a complex refractive index and complex relative permittivity, that is

$$\tilde{n} = n - j\kappa = \sqrt{\varepsilon_r' - j\varepsilon_r''}\,. \tag{5.34}$$

By squaring both sides we can relate *n* and κ directly to ε_r' and ε_r''. The final result is

$$n^2 - \kappa^2 = \varepsilon_r' \quad \text{and} \quad 2n\kappa = \varepsilon_r''\,. \tag{5.35}$$

Optical properties of materials are typically reported either by showing the frequency dependences of *n* and κ or ε_r' and ε_r''. Clearly we can use Eq. (5.35) to obtain one set of properties from the other.

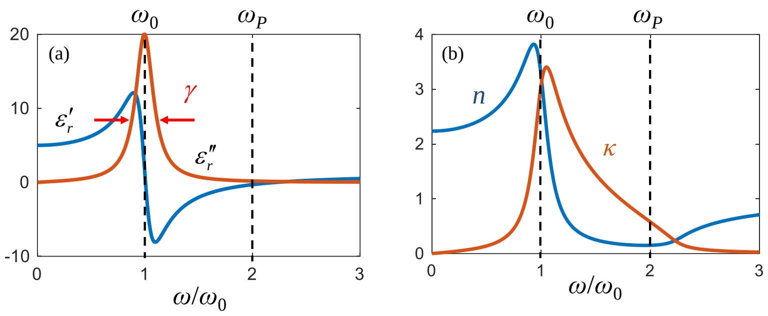

Figure 5.5 (a) The complex permittivity and (b) refractive index from a Lorentz model. $\omega_P = 2\omega_0$, $\gamma = 0.2\ \omega_0$.

Figure 5.5 shows an example of $\omega_P = 2\omega_0$, $\gamma = 0.2\ \omega_0$. We can find that Around the resonance frequency ω_0, the magnitude of ε_r' has a drastic change and ε_r'' has a maximum value.

Dispersion

From Lorentz oscillator model, we find that permittivity depends on the frequency, besides the plasma frequency and damping (which are properties of the medium). A medium displaying such behavior (that is, whose permittivity depends on the frequency of the wave) is called dispersive, named after “dispersion”, which is the phenomenon exhibited in a *prism* or *raindrop* that causes white light to be spread out into a rainbow of colors (white light is a mixture of beams of many different colors, all traveling at the same speed, but having different frequencies and wavelengths).

5.4 Plasma

Simply stated, a plasma is an ionized, electrically conducting gas of charged particles, usually occurring under conditions of very high temperature and/or very low particle density. Plasmas exhibit many cool effects, as you probably have seen, in aurora (polar light) or in a plasma ball (Figure 5.6).

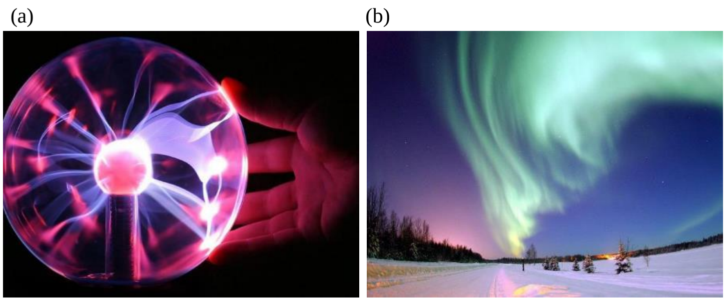

Figure 5.6 (a) The effect of a conducting object (a hand) touching the plasma globe. (b) An aurora seen above Bear Lake, Alaska, USA

Many of these effects take place collectively. One of the most fundamental collective effects of a plasma is the plasma oscillation.

In equilibrium, the electric fields of the electrons and the ionized nuclei cancel each other out, but this equilibrium is hardly maintained.

Instead of dealing with the individual (chaotic!) motion of electrons and nuclei, consider the center of mass of the nuclei and the center of mass of the electrons. In equilibrium, they coincide. However, when they shift with respect to each other, a Coulomb force arises trying to restore their position, initiating an oscillatory behavior (think of a blob of fluid floating at zero gravity). The frequency at which these oscillations resonate is called the plasma frequency.

The magnitude of this frequency has highly significant implications with respect to the propagation of electromagnetic waves through the plasma.

Plasma exists naturally in what we call the ionosphere (80 km - 120 km above the surface of the Earth). There, UV light from the Sun ionizes air molecules.

For plasma, we will assume $\gamma = 0$ (lossless) and $\omega_0 = 0$ (electrons can free to move), then permittivity of plasma can be written as

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \tag{5.36}$$

The wave propagation in plasma media can be summarized in two cases:

(1) When $\omega < \omega_P$, the permittivity is negative ($\varepsilon < 0$). The propagation constant k becomes an imaginary number ($k = \omega\sqrt{\mu\varepsilon} = -jk''$). As a results, field decays exponentially in plasma media and thus electromagnetic waves cannot pass through below the plasma frequency.

(2) When $\omega > \omega_P$, the permittivity is positive ($\varepsilon > 0$). Waves can propagate through the plasma media.

This natural resonance of a plasma has some interesting effects. For example, if one tries to propagate a radio wave through the ionosphere, one finds that it can penetrate only if its frequency is higher than the plasma frequency. Otherwise the signal is reflected back. We must use high frequencies if we wish to communicate with a satellite in space. On the other hand, if we wish to communicate with a radio station beyond the horizon, we must use frequencies lower than the plasma frequency, so that the signal will be reflected back to the earth.

Metals

For metal, we will assume $\gamma \neq 0$ (due to the loss) and $\omega_0 = 0$ (no restoring force and electrons can free to move). The permittivity of metal from Lorentz model is written by

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - j\gamma\omega}\right) = \varepsilon_0\left(\varepsilon_r' - j\varepsilon_r''\right)} \tag{5.37}$$

with $\varepsilon_r' = 1-\omega_p^2 \big/ \left(\omega^2+\gamma^2\right)$, $\varepsilon_r'' = 1-\gamma\omega_p^2 \big/ \left[\omega\left(\omega^2+\gamma^2\right)\right]$.

5.5 Penetration Depth in Conducting Media

There is another way to find the permittivity of metal in the low frequency limit. We consider a source-free conducting medium governed by Ohm's law

$$\vec{J}_c = \sigma \vec{E} \tag{5.38}$$

From the Maxwell equation

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}_c \tag{5.39}$$

where $\vec{J}_c$ is the conducting current. We can absorb $\vec{J}_c$ in $\vec{D}$ by noting that $\vec{D} = \varepsilon\vec{E}$. We find

$$\nabla \times \vec{H} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\vec{E} \tag{5.40}$$

Thus we can define a new permittivity for conducting media

$$\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}} \tag{5.41}$$

Let $\varepsilon = \varepsilon_0$, and compared with the Lorentz model in Eq. (5.37),we find in the low frequency limit

$$\sigma = \varepsilon_0 \frac{\omega_p^2}{\gamma + j\omega} \approx \varepsilon_0 \frac{\omega_p^2}{\gamma} = \frac{Ne^2}{m\gamma} \tag{5.42}$$

For copper, $\sigma \approx 7\times10^7\ \mathrm{S/m}$ and $N = 8\times10^{28}\,\mathrm{m^{-3}}$, we find $\gamma \approx 3.2\times10^{13}\,\mathrm{Hz}$.

The Maxwell equation for the conducting medium in source-free regions are simplified as

$$\begin{aligned} &\nabla\cdot\vec{D} = 0 \\ &\nabla\times\vec{E} = -j\omega\vec{H} \\ &\nabla\cdot\vec{H} = 0 \\ &\nabla\times\vec{H} = j\omega\varepsilon_c\vec{E} \end{aligned} \tag{5.43}$$

The wave equation for $\vec{E}$ is

$$\left(\nabla^2 + \omega^2\mu\varepsilon_c\right)\vec{E} = 0 \tag{5.44}$$

Wave solution $\vec{E} = \hat{x}E_x = \hat{x}E_0\exp\left(-jkz\right)$ has the dispersion relation

$$k^2 = \omega^2\mu\varepsilon_c = \omega^2\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right) \tag{5.45}$$

Similarly, the propagation constant (spatial frequency) is complex $k = k' - jk''$. The wave propagating in the z direction is

$$\begin{cases} \vec{E} = \hat{x}E_0 \exp(-k''z - jk'z) \\ \vec{H} = \hat{y}\dfrac{k' - jk''}{\omega\mu} E_0 \exp(-k''z - jk'z) \\ \vec{S} = \hat{z}\dfrac{k' + jk''}{\omega\mu} |E_0|^2 \exp(-2k''z) \\ \langle \vec{S} \rangle = \hat{z}\dfrac{k'}{2\omega\mu} |E_0|^2 \exp(-2k''z) \end{cases} \tag{5.46}$$

The electric field in the time-domain is

$$\begin{aligned} \vec{E}(z,t) &= \hat{x}E_x(z,t) = \mathrm{Re}\left[\hat{x}E_0 \exp(-k''z - jk'z)\exp(j\omega t)\right] \\ &= \hat{x}E_0 \exp(-k''z)\cos(\omega t - k'z) \end{aligned} \tag{5.47}$$

The wave propagates and attenuates in the z direction. The spatial variation of the electric field is shown in Figure 5.7.

The penetration depth is defined as

$$\boxed{d_P = \frac{1}{k''}} \tag{5.48}$$

such that the wave amplitude attenuates by a factor of $1/e$ in a distance d_P.

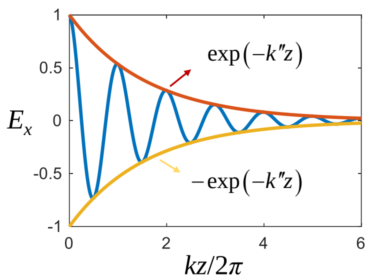

Figure 5.7 Wave propagation in conducting media at $t = 0$.

We now consider the two limiting cases of very high conductivity and very small conductivity:

(i) For a highly conducting medium with $1 \ll \sigma/\omega\varepsilon$, we approximate

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{1/2} \approx \omega\sqrt{\mu\varepsilon}\left[-j\frac{\sigma}{\omega\varepsilon}\right]^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}}[1 - j] \tag{5.49}$$

We find the penetration depth

$$d_P = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{5.50}$$

which is usually a very small number known as the *skin depth*.

(ii) For a highly conducting medium with $\sigma/\omega\varepsilon \ll 1$, we can approximate

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{1/2} \approx \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{2\omega\varepsilon}\right] = \omega\sqrt{\mu\varepsilon} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \tag{5.51}$$

We find the penetration depth

$$d_P = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} \tag{5.52}$$

It is interesting to note that the penetration depth in Eq. (5.52) is independent of frequency. However, we have assumed a homogeneous medium here, at high frequencies there will be large attenuation due to scattering.

To find a general solution for k' and k'' in Eq. (5.45), we obtain from the identity

$$\sqrt{1 - jA} = \sqrt{0.5\left(\sqrt{1 + A^2} + 1\right)} - j\sqrt{0.5\left(\sqrt{1 + A^2} - 1\right)}$$

and find

$$k' = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2}\left(\sqrt{1 + \frac{\sigma^2}{\varepsilon^2\omega^2}} + 1\right)\right]^{1/2} \tag{5.53}$$

$$k'' = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2}\left(\sqrt{1 + \frac{\sigma^2}{\varepsilon^2\omega^2}} - 1\right)\right]^{1/2} \tag{5.54}$$

5.6 Optical Anisotropy and Birefringence

In some crystals, the molecular "spring constant" of Lorentz model can be different in different directions, as illustrated in Figure 5.8. Different restoring forces of the springs give the resonant frequencies $\omega_{0x} \neq \omega_{0y} \neq \omega_{0z}$ in three directions. As a results, the permittivity and refractive index is different in three directions. Such phenomenon is called *optical anisotropy*, with the constitutive parameters given in a tensor form

$$\vec{D} = \overline{\overline{\varepsilon}}\vec{E}\ , \quad \vec{B} = \overline{\overline{\mu}}\vec{H} \tag{5.55}$$

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}, \quad \overline{\overline{\mu}} = \begin{bmatrix} \mu_x & & \\ & \mu_y & \\ & & \mu_z \end{bmatrix} \tag{5.56}$$

The wave properties in anisotropic media depend on both the propagating direction and the polarizations. We summarize the behaviors in Table 5.1.

Table 5.1 Wave propagation in anisotropic media

k vector	electric field	magnetic field	wave number
$\vec{k}=\left[k_x,0,0\right]^T$	$\vec{E}=\left[0,E_y,0\right]^T$	$\vec{H}=\left[0,0,H_z\right]^T$	$k_x=\omega\sqrt{\mu_z\varepsilon_y}$
	$\vec{E}=\left[0,0,E_z\right]^T$	$\vec{H}=\left[0,H_y,0\right]^T$	$k_x=\omega\sqrt{\mu_y\varepsilon_z}$
$\vec{k}=\left[0,k_y,0\right]^T$	$\vec{E}=\left[E_x,0,0\right]^T$	$\vec{H}=\left[0,0,H_z\right]^T$	$k_y=\omega\sqrt{\mu_z\varepsilon_x}$
	$\vec{E}=\left[0,0,E_z\right]^T$	$\vec{H}=\left[H_x,0,0\right]^T$	$k_y=\omega\sqrt{\mu_x\varepsilon_z}$
$\vec{k}=\left[0,0,k_z\right]^T$	$\vec{E}=\left[E_x,0,0\right]^T$	$\vec{H}=\left[0,H_y,0\right]^T$	$k_z=\omega\sqrt{\mu_y\varepsilon_x}$
	$\vec{E}=\left[0,E_y,0\right]^T$	$\vec{H}=\left[H_x,0,0\right]^T$	$k_z=\omega\sqrt{\mu_x\varepsilon_y}$

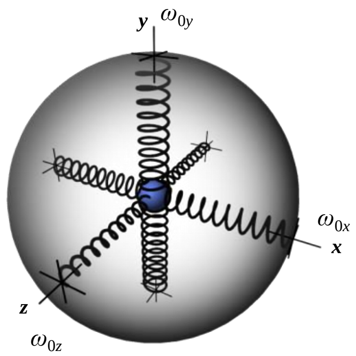

Figure 5.8 Lorentz model of anisotropic medium.

Uniaxial Media

Uniaxial medium is one kind of anisotropic media in which two permittivity components are same. For instance, let $\varepsilon_x=\varepsilon_y=\varepsilon$, and $\mu_x=\mu_y=\mu_z=\mu$. Then permittivity tensor is given by

$$\overline{\overline{\varepsilon}}=\begin{bmatrix}\varepsilon & & \\ & \varepsilon & \\ & & \varepsilon_z\end{bmatrix} \tag{5.57}$$

Consider a plane wave, whose electric field can be expressed as $\vec{E}=\vec{E}_0\exp\left(-j\vec{k}\cdot\vec{r}\right)$, propagating in a uniaxial medium. For such a plane wave, the source-free Maxwell's equations become

$$\begin{aligned} \nabla\times\vec{E} &= -j\omega\vec{H} \\ \nabla\times\vec{H} &= j\omega\overline{\overline{\varepsilon}}\vec{E} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \vec{k}\times\vec{E} &= \omega\vec{H} \\ \vec{k}\times\vec{H} &= -\omega\varepsilon_c\vec{E} \end{aligned} \tag{5.58}$$

From which we can obtain

$$\vec{k}\times\left(\vec{k}\times\vec{E}\right) = -\omega^2\mu\overline{\overline{\varepsilon}}\vec{H} \tag{5.59}$$

To simplify the analysis, we assume that the wave propagates in the *x*-direction such that $\vec{k} = \hat{x}k_x$. Under this assumption, Eq. (5.59) becomes

$$\begin{bmatrix} -\omega^2\mu\varepsilon & 0 & 0 \\ 0 & k_x^2-\omega^2\mu\varepsilon & 0 \\ 0 & 0 & k_x^2-\omega^2\mu\varepsilon_z \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \tag{5.60}$$

It is immediately observed that $E_x = 0$ and Eq. (5.60) can be reduced to

$$\begin{bmatrix} k_x^2-\omega^2\mu\varepsilon & 0 \\ 0 & k_x^2-\omega^2\mu\varepsilon_z \end{bmatrix}\begin{bmatrix} E_y \\ E_z \end{bmatrix} = 0 \tag{5.61}$$

For this equation to have a nontrivial solution, the determinant of its coefficient matrix must vanish, which yields

$$\left(k_x^2-\omega^2\mu\varepsilon\right)\left(k_x^2-\omega^2\mu\varepsilon_z\right) = 0 \tag{5.62}$$

Eq. (5.62) has two solutions. The first solution is given by

$$k_x^2-\omega^2\mu\varepsilon = 0 \quad \text{or} \quad k_x = \omega\sqrt{\mu\varepsilon} = k_o \tag{5.63}$$

With this, $E_y \neq 0$ and $E_z = 0$; hence, the electric field intensity and flux density can be written as

$$\vec{E} = \hat{y}E_0\exp\left(-jk_ox\right), \quad \vec{D} = \hat{y}\varepsilon E_0\exp\left(-jk_ox\right) \tag{5.64}$$

from which the magnetic field intensity and flux density can be found as

$$\vec{H} = \hat{z}\sqrt{\frac{\varepsilon}{\mu}}E_0\exp\left(-jk_ox\right), \quad \vec{B} = \mu\vec{H} \tag{5.65}$$

It is clear that in this solution, ε_z does not have any effect. The dispersion relation given in Eq. (5.63) and the fields given in Eqs. (5.64) and (5.65) are the same as those in an isotropic medium with a permittivity of ε. For this reason, the wave corresponding to this solution is called an *ordinary wave*.

The second solution to Eq. (5.62) is given by

$$k_x^2 - \omega^2 \mu \varepsilon_z = 0 \quad \text{or} \quad k_x = \omega\sqrt{\mu\varepsilon_z} = k_e \tag{5.66}$$

Under this solution, $E_y = 0$ and $E_z \neq 0$; hence, the electric field intensity and flux density can be written as

$$\vec{E} = \hat{z}E_0 \exp(-jk_e x), \quad \vec{D} = \hat{z}\varepsilon_z E_0 \exp(-jk_e x) \tag{5.67}$$

and the magnetic field intensity and flux density are given by

$$\vec{H} = \hat{y}\sqrt{\frac{\varepsilon_z}{\mu}} E_0 \exp(-jk_e x), \quad \vec{B} = \mu\vec{H} \tag{5.68}$$

The effect of ε_z is clearly seen in both the phase constant and wave impedance. In fact, the dispersion relation given in Eq. (5.66) and the fields given in Eqs. (5.67) and (5.68) are the same as those in an isotropic medium with a permittivity of ε_z. For this reason, the wave corresponding to this solution is called an *extraordinary wave*.

Clearly, the ordinary wave is affected only by ε because its electric field has only a y-component and the permittivity in the y-direction is ε. Similarly, the extraordinary wave is affected only by ε_z because its electric field has only a z-component and the permittivity in the z-direction is ε_z. The permittivity affects a plane wave through the electric field, which can actually be seen very easily from the constitutive relation $\vec{D} = \bar{\bar{\varepsilon}}\vec{E}$.

A typical application of the uniaxial medium is the quarter-wave plate. As shown in Figure 5.9, consider the plane wave

$$\vec{E} = (\hat{y} + \hat{z}) E_0 \exp(-jk_x x) \tag{5.69}$$

passing through a dielectric slab made of the uniaxial medium characterized by Eq. (5.57) Once it enters the uniaxial medium, the y- and z-components propagate at two different phase velocities, which are determined by their wavenumbers. Right after passing through the slab, the electric field becomes

$$\begin{aligned}\vec{E} &= \hat{y}E_0 \exp(-jk_o d) + \hat{z}E_0 \exp(-jk_e d) \\ &= \left[\hat{y} + \hat{z}\exp(jk_o d - jk_e d)\right] E_0 \exp(-jk_o d)\end{aligned} \tag{5.70}$$

where d denotes the thickness of the slab. Here we ignore the wave reflection at the surfaces of the dielectric slab. Eq. (5.70) represents an elliptically polarized plane wave. In particular, if the thickness is chosen such that $(k_o - k_e)d = \pm(n + 1/2)\pi$ with n being a integer, Eq. (5.70) becomes (assuming that $n = 1$)

$$\vec{E} = [\hat{y} \pm j\hat{z}] E_0 \exp(-jk_o d) \tag{5.71}$$

which represents a circularly polarized plane wave. Such a dielectric slab is called a quarter-wave plate, which can convert a linearly polarized plane wave into a circularly polarized wave. The direction of rotation of the electric field depends on the values of ε and ε_z.

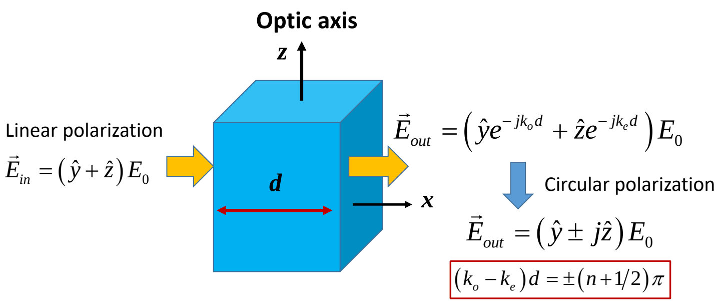

Figure 5.9 Transmission through a quarter-wave plate.

***Example 5.1** A general case of wave propagation in the uniaxial medium

Consider the propagation vector lies in the *xz*-plane such that $\vec{k} = \hat{x}k_x + \hat{z}k_z$. Then Eq. (5.59) becomes

$$\begin{bmatrix} k_z^2 - \omega^2\mu\varepsilon & 0 & -k_x k_z \\ 0 & k_x^2 + k_z^2 - \omega^2\mu\varepsilon & 0 \\ -k_x k_z & 0 & k_x^2 - \omega^2\mu\varepsilon_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \tag{5.72}$$

For this equation to have a nontrivial solution, the determinant of its coefficient matrix must vanish, which yields

$$\left(k_x^2 + k_z^2 - \omega^2\mu\varepsilon\right)\left[\left(k_x^2 - \omega^2\mu\varepsilon_z\right)\left(k_z^2 - \omega^2\mu\varepsilon\right) - k_x^2 k_z^2\right] = 0 \tag{5.73}$$

This equation has two solutions. The first solution is given by

$$k_x^2 + k_z^2 - \omega^2\mu\varepsilon = 0 \tag{5.74}$$

With this solution $E_y \neq 0$ and $E_x = E_z = 0$; hence, the electric field intensity and flux density can be written as

$$\vec{E} = \hat{y}E_0 \exp\left(-j\vec{k}\cdot\vec{r}\right), \quad \vec{D} = \hat{y}\varepsilon E_0 \exp\left(-j\vec{k}\cdot\vec{r}\right) \tag{5.75}$$

from which the magnetic field intensity and flux density can be found as

$$\vec{H} = \frac{1}{\omega\mu}\left(-\hat{x}k_z + \hat{z}k_x\right)E_0 \exp\left(-j\vec{k}\cdot\vec{r}\right), \quad \vec{B} = \mu\vec{H} \tag{5.76}$$

This solution is not affected by ε_z and is the same as the ordinary-wave solution discussed earlier.

The second solution to is given by

$$\left(k_x^2-\omega^2\mu\varepsilon_z\right)\left(k_z^2-\omega^2\mu\varepsilon\right)-k_x^2k_z^2=0 \tag{5.77}$$

which can also be written as

$$\frac{k_x^2}{\omega^2\mu\varepsilon_z}+\frac{k_z^2}{\omega^2\mu\varepsilon}=1 \tag{5.78}$$

Under this solution, Ey = 0 and $\left(k_z^2-\omega^2\mu\varepsilon\right)E_x-k_xk_zE_z=0$, which can also be written as

$$\varepsilon k_xE_x+\varepsilon_zk_zE_z=0 \tag{5.79}$$

This equation is nothing but $\vec{k}\cdot\vec{D}=0$. With these, the electric field intensity and flux density are given by

$$\vec{E}=\left(\hat{x}-\hat{z}\frac{k_x\varepsilon}{k_z\varepsilon_z}\right)E_{0x}\exp\left(-j\vec{k}\cdot\vec{r}\right),\quad \vec{D}=\left(\hat{x}-\hat{z}\frac{k_x}{k_z}\right)\varepsilon E_{0x}\exp\left(-j\vec{k}\cdot\vec{r}\right) \tag{5.80}$$

and the magnetic field intensity and flux density are given by

$$\vec{H}=\hat{y}\frac{\omega\varepsilon}{k_z}E_{0x}\exp\left(-j\vec{k}\cdot\vec{r}\right),\quad \vec{B}=\mu\vec{H} \tag{5.81}$$

In this case, both ε and ε_z affect the wave propagation and the degree of the effect depends on the propagation direction. If thewave propagates in the *x*-direction, the effect of ε disappears and the wave reduces to the extraordinary wave discussed earlier. If the wave propagates in the *z*-direction, the effect of ε_z disappears and the wave becomes an ordinary wave. Other than these two special directions, the phase constants k_x and k_y depend on the propagation direction. More interesting, since $\vec{E}$ is not parallel to $\vec{D}$, the Poynting vector $\vec{E}\times\vec{H}^*/2$ is not in the direction of $\vec{k}$; hence, the power flows in a direction different from the propagation direction. This wave is called a *general extraordinary wave*. Therefore, even if the ordinary and extraordinary waves propagate in the same direction, their energy propagates in different directions, exhibiting again the phenomenon of birefringence.

5.7 Chiral Media

A chiral medium is a bi-isotropic medium, in which the field intensities and flux densities are related by the constitutive relations

$$\vec{D}=\varepsilon\vec{E}-j\chi\vec{H},\quad \vec{B}=\mu\vec{H}+j\chi\vec{E} \tag{5.82}$$

where χ is called the *chirality parameter*. In such a chiral medium, right-handed (+) and left-handed (-) waves can propagate with different wave numbers:

$$k_\pm=\omega\left(\sqrt{\varepsilon\mu}\pm\chi\right) \tag{5.83}$$

For a linear polarized wave $\vec{E} = \hat{x}E_0 \exp(-jkz)$ passing through a chiral slab with a thickness *d*. Two circular polarizations will experience different phase accumulation and the transmitted wave becomes

$$\begin{aligned}\vec{E}_{out} &= \left[(\hat{x} - j\hat{y})\exp(-jk_+d) + (\hat{x} + j\hat{y})\exp(-jk_-d)\right]\frac{1}{2}E_0 \\ &= \left[\hat{x}\cos\left(\frac{k_+ - k_-}{2}d\right) - \hat{y}\sin\left(\frac{k_+ - k_-}{2}d\right)\right]\exp\left(\frac{-jk_+d - jk_-d}{2}\right)E_0\end{aligned} \tag{5.84}$$

We can find that the polarization orientation of the transmitted wave rotates by an angle of

$$\theta = \frac{k_+ - k_-}{2}d \tag{5.85}$$

This phenomenon is called *optical rotation* or *optical activity* in chiral medium.

5.8 Additional Problems

See References [2-6].

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