Physics I: Introduction to Wave Theory
SDU Course Number: sd01232810 (Fall 2023)

Lecture 8: Resonance

Outline

- Atomic Oscillators and Resonance
- Fabry-Perot Resonance
- Rectangular Cavity Resonator

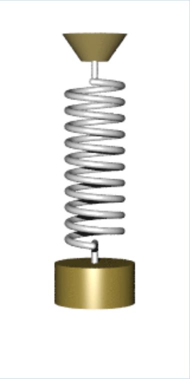
The Phenomenon of Resonance

In physics, **resonance** is a phenomenon in which a vibrating system or external force drives another system to oscillate with greater amplitude at specific frequencies.

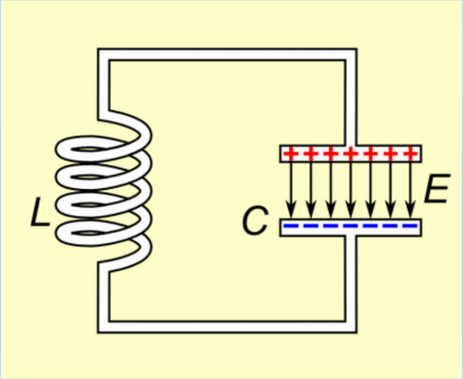
Frequencies at which the response amplitude is a relative maximum are known as the system's **resonant frequencies** or resonance frequencies.

A **resonator** is a device or system that exhibits resonance or resonant behavior, that is, it naturally oscillates at some frequencies, called its resonant frequencies, with greater amplitude than at others.

Swing

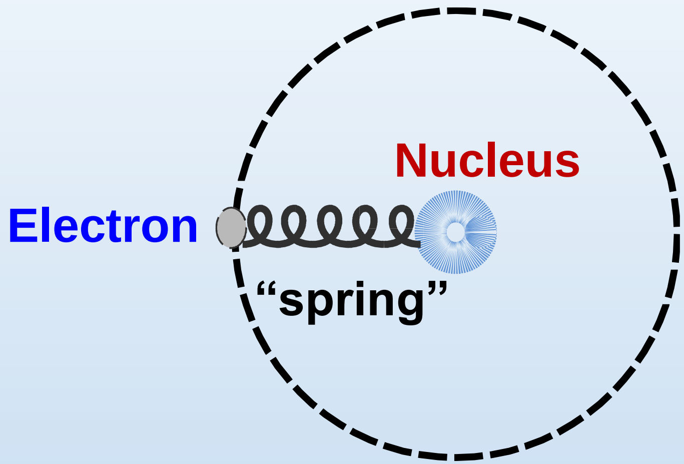


Spring



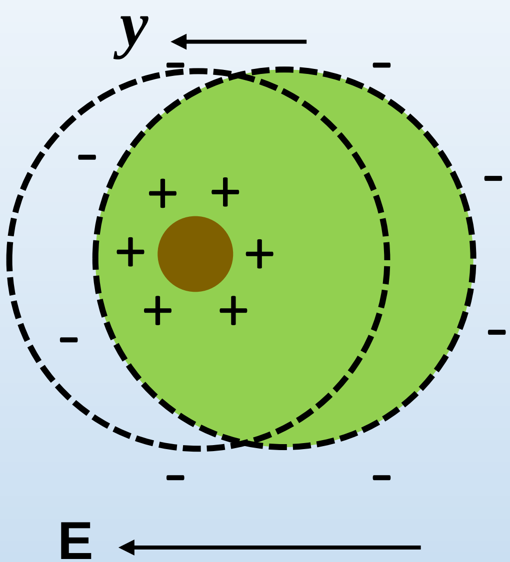
LC circuit

Atomic Oscillators and Resonance



No external E Field

$$P = -Ney$$



(Lorentz Oscillator Model)

$$\boxed{\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{P}{\varepsilon_0 E}\right) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega}\right)}$$

$$\omega_P = \sqrt{Ne^2/m\varepsilon_0}$$

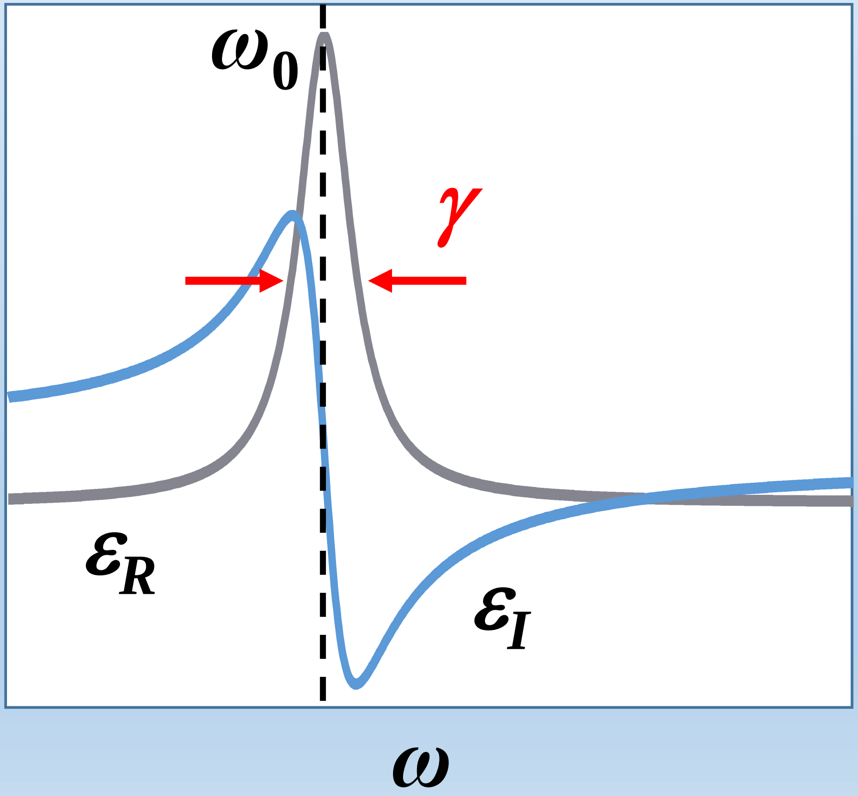
(Plasma frequency)

$$\omega_0 = \sqrt{k/m}$$

Resonant frequency
(or natural frequency)

Complex permittivity

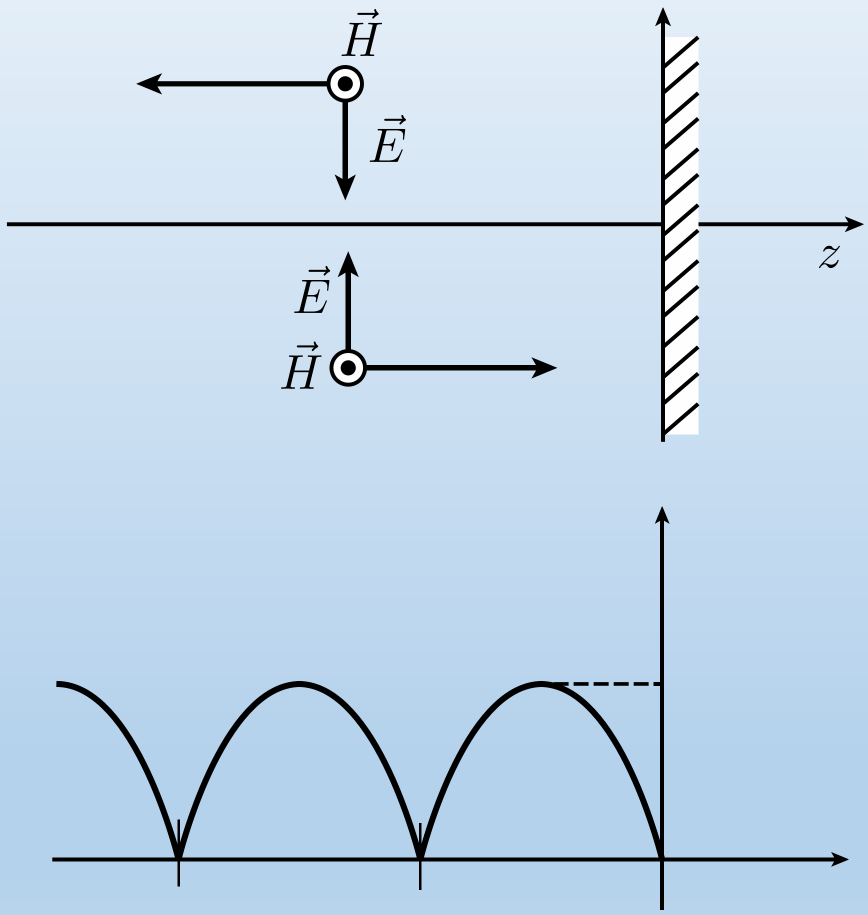
$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega} \right) = \varepsilon_R - j\varepsilon_I$$



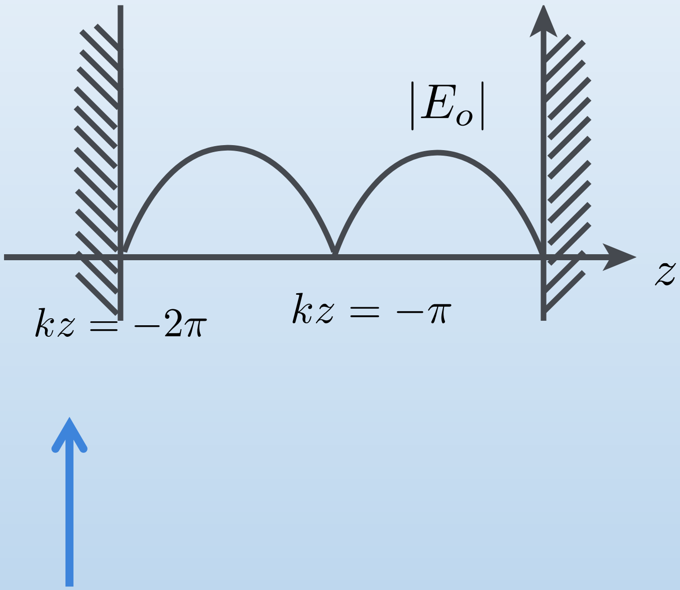
Around the resonance frequency ω_0, the magnitude of ε_R has a drastic change and ε_I has the maximum value.

Resonators

STANDING WAVE

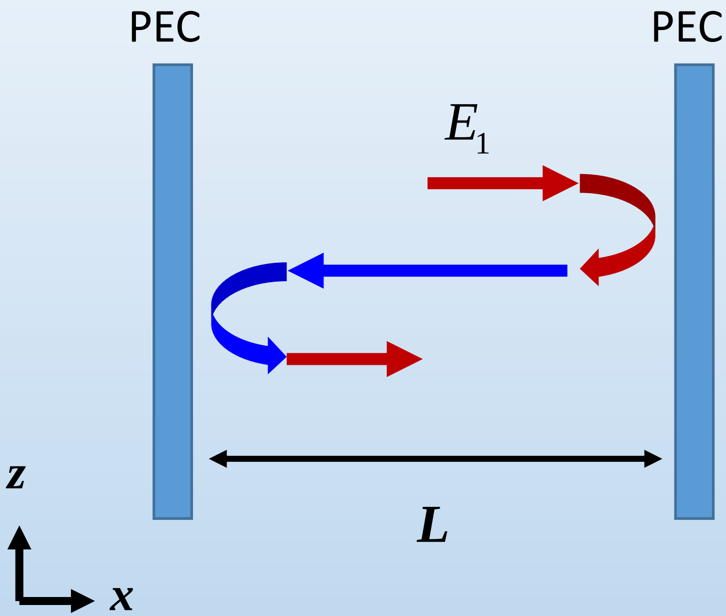


RESONATORS



Terminate the standing wave with a second wall to form a resonator

Resonator (Parallel PEC mirrors)



TE Waveguide Mode (kz = 0)

$$E_y(x,z) = E_0 \sin(kx)$$

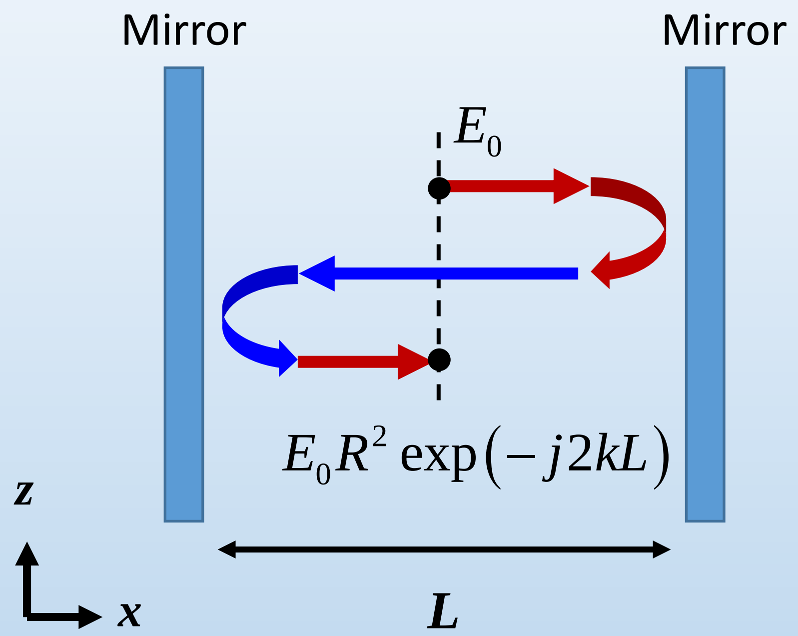
$$H_z(x,z) = -\frac{k}{j\omega\mu} E_0 \cos(kx)$$

$$kL = m\pi$$ **(Guidance Condition)**

$$\Rightarrow \frac{2\pi f}{c} L = m\pi \Rightarrow \boxed{f = m\frac{c}{2L}} \quad m = 1, 2, 3, \cdots$$

(Resonant frequency)

Resonator (general mirrors)



$$E_1 = E_0$$

$$E_2 = E_0 R^2 \exp(-j2kL)$$

$$E_3 = E_0 R^4 \exp(-j4kL)$$

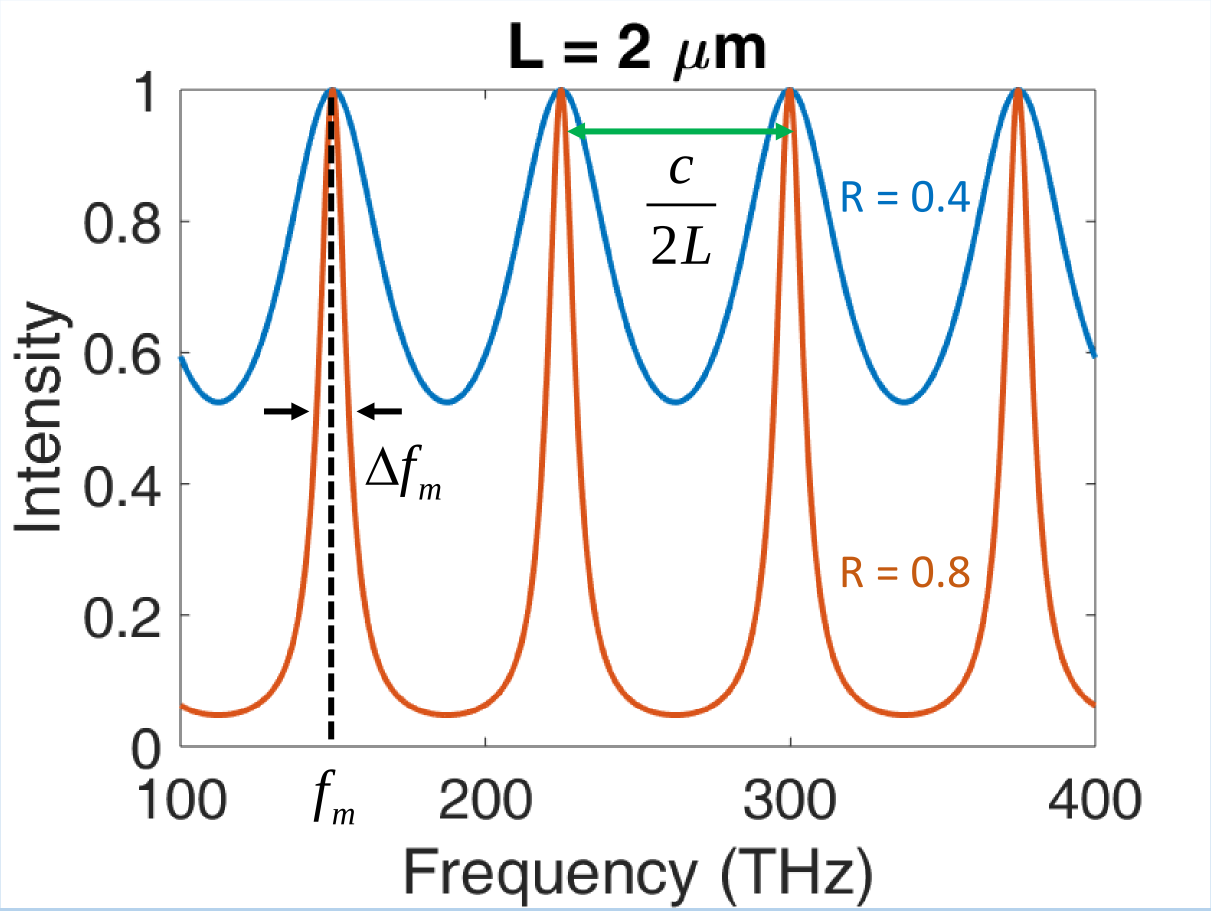
$$\vdots$$

$$E_{cavity} = E_1\left[1 + R^2 \exp(-j2kL) + R^4 \exp(-j4kL) + \cdots\right] = \frac{E_1}{1 - R^2 \exp(-j2kL)}$$

$$I_{cavity} = \frac{I_0}{\left|1 - R^2 \exp(-j2kL)\right|^2} = \frac{I_0}{\left(1 - R^2\right)^2 + 4R^2 \sin^2(kL)}$$

Peaks of I_{cavity}: $\boxed{kL = m\pi}$ **(Fabry-Perot resonator)**

Fabry-Perot resonator



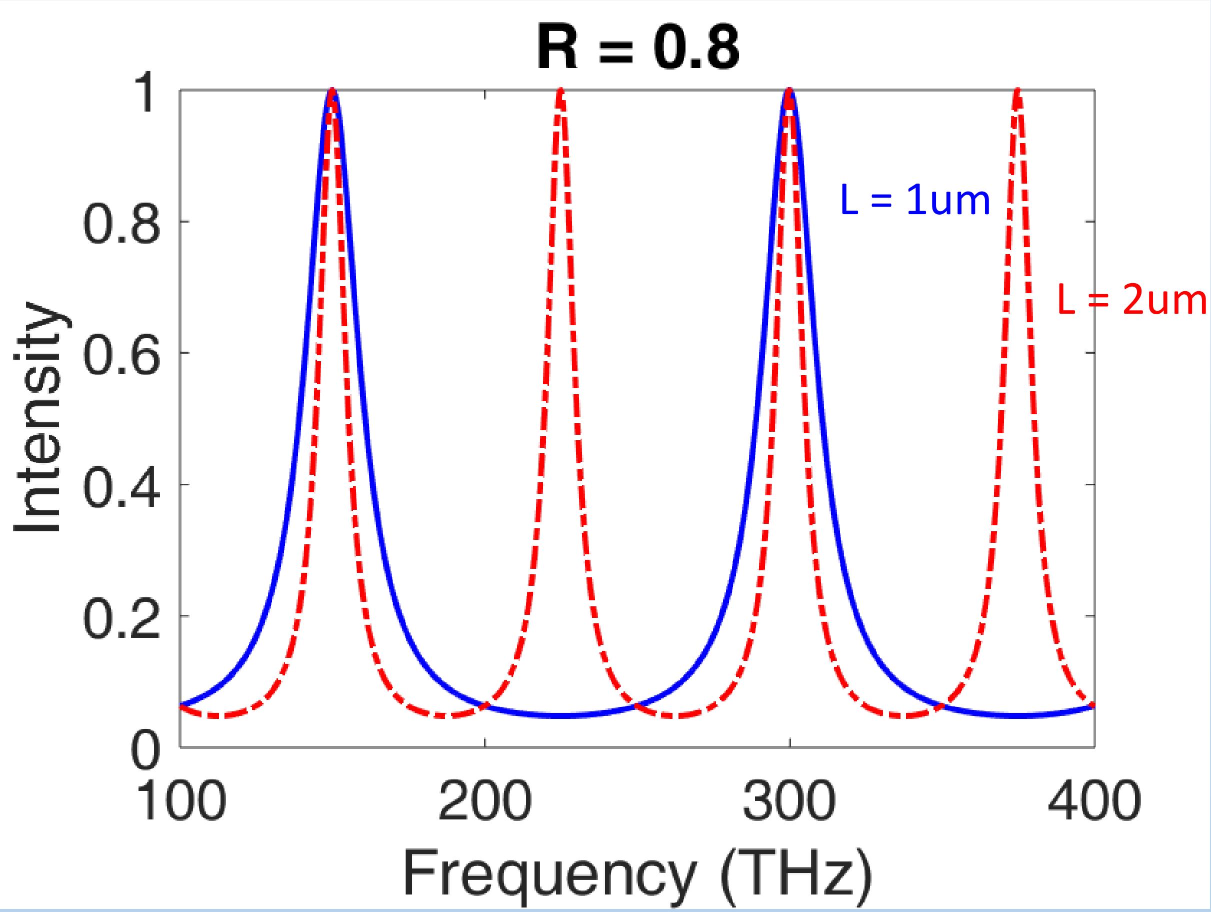
$$f_m = m\frac{c}{2L}$$

$$I_{\max} = \frac{I_0}{\left(1-R^2\right)^2}$$

Full width at half maximum (FWHM): Δf_m

Quality Factor: $Q = f_m / \Delta f_m$

Fabry-Perot resonator



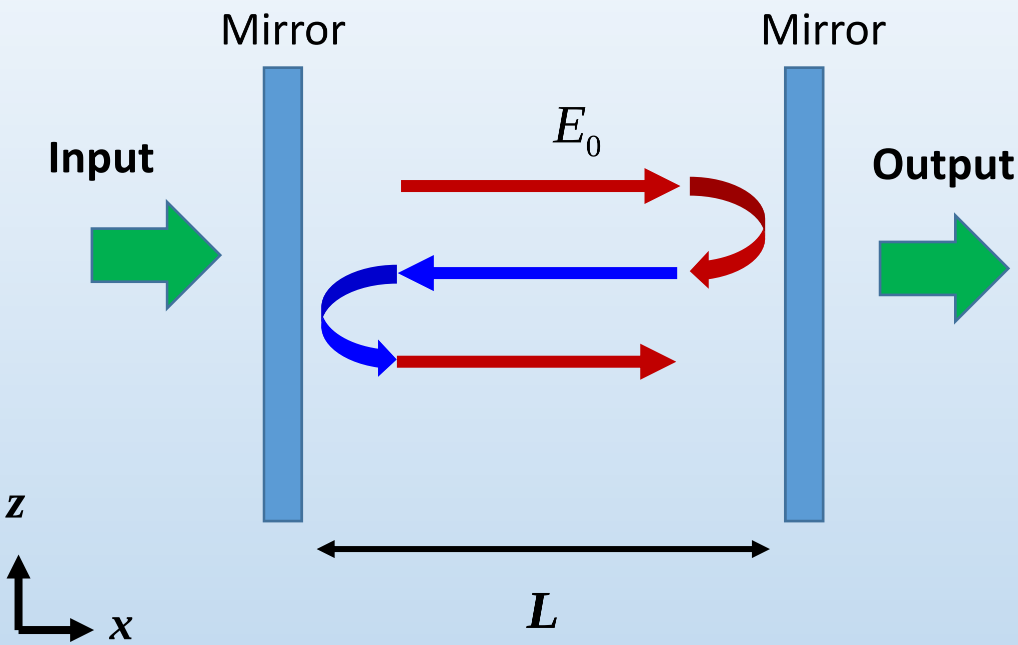
$$f_m = m\frac{c}{2L}$$

$$I_{\max} = \frac{I_0}{\left(1-R^2\right)^2}$$

Full width at half maximum (FWHM): Δf_m

Quality Factor: $Q = f_m / \Delta f_m$

Transmitted light through a FP cavity



$$E_{t1} = T^2 E_{input} \exp(-jkL)$$

$$E_{t2} = T^2 E_{input} R^2 \exp(-j3kL)$$

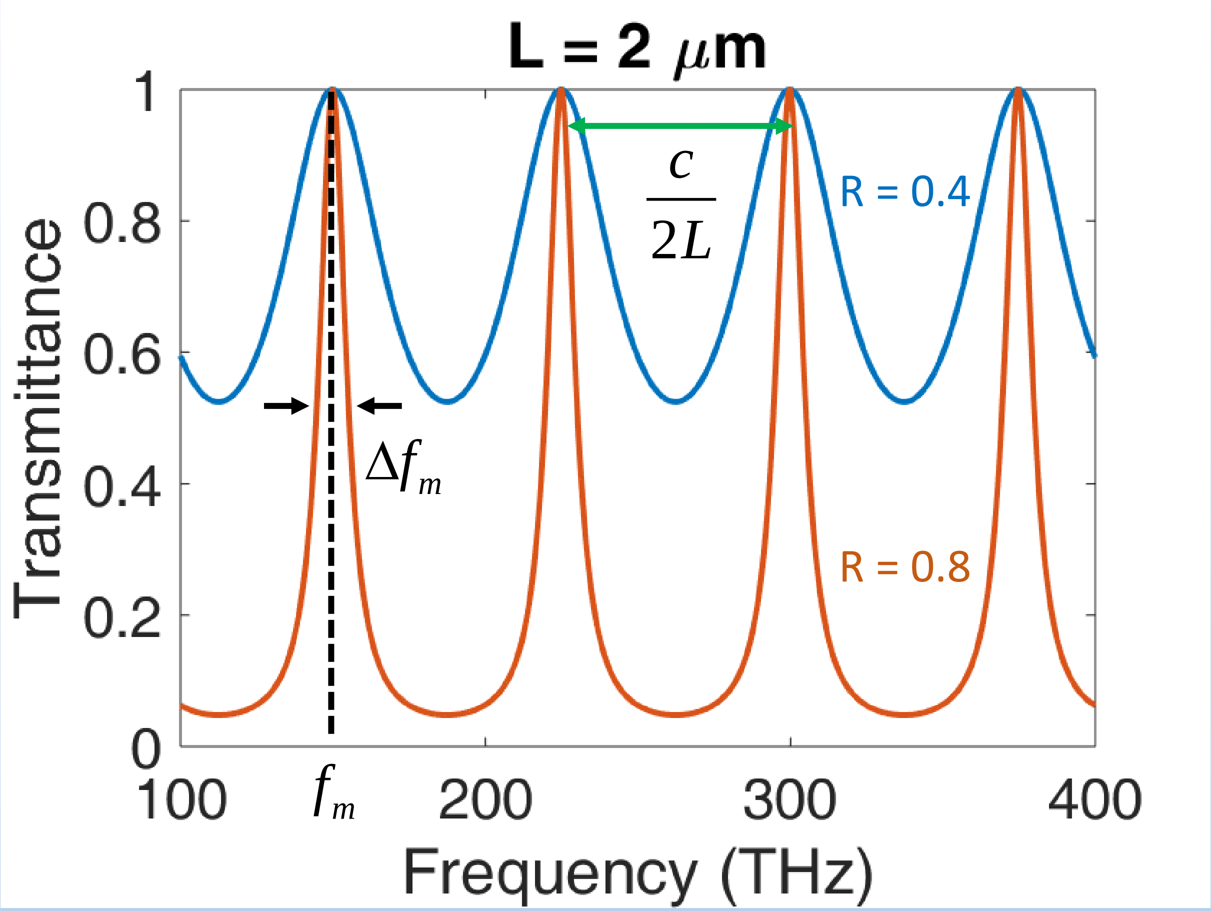
$$E_{t3} = T^2 E_{input} R^4 \exp(-j5kL)$$

$$\vdots$$

$$E_{transmitted} = \sum_n E_{tn} = \frac{T^2 \exp(-jkL)}{1 - R^2 \exp(-j2kL)} E_{input}$$

$$I_{transmitted} = \frac{T^4 I_{input}}{\left|1 - R^2 \exp(-j2kL)\right|^2} = \frac{\left(1 - R^2\right)^2 I_{input}}{\left(1 - R^2\right)^2 + 4R^2 \sin^2(kL)}$$

Transmitted light through a FP cavity



Maximum:

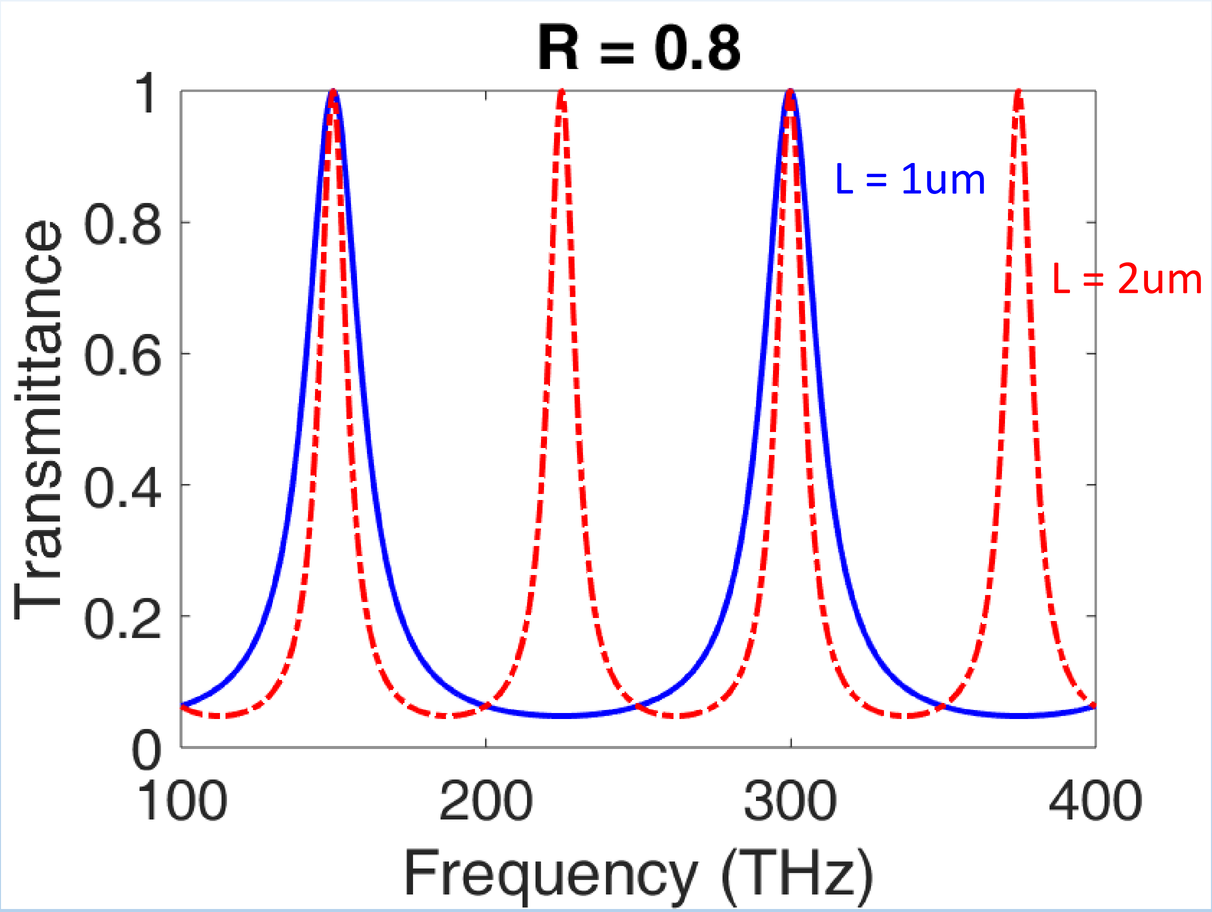
$$kL = m\pi$$

Minimum:

$$kL = \left(m + \frac{1}{2}\right)\pi$$

$$t = \frac{I_{transmitted}}{I_{input}} = \frac{\left(1-R^2\right)^2}{\left(1-R^2\right)^2 + 4R^2\sin^2\left(kL\right)}$$

Fabry-Perot resonator



Maximum:

$$kL = m\pi$$

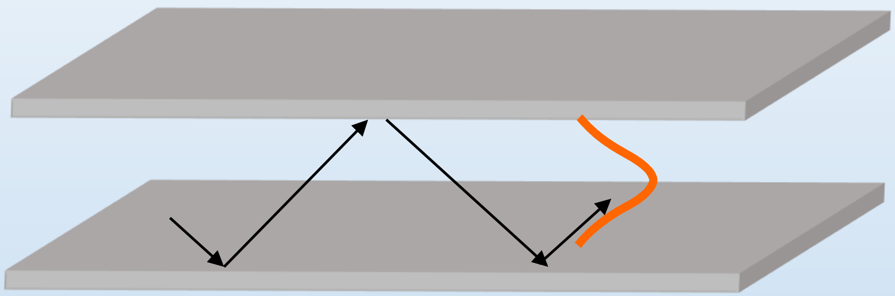
Minimum:

$$kL = \left(m + \frac{1}{2}\right)\pi$$

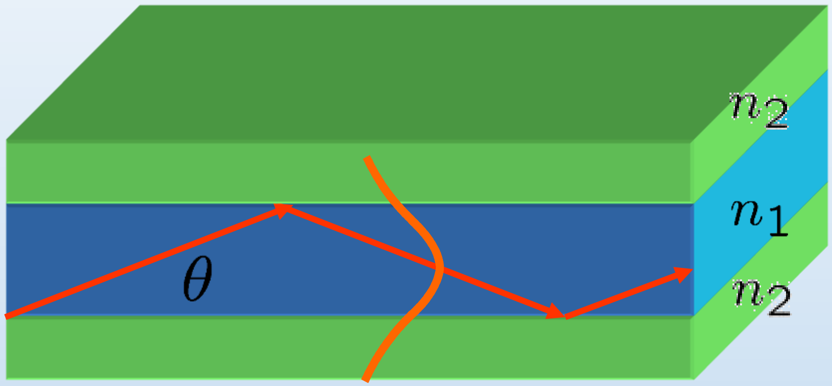
$$t = \frac{I_{transmitted}}{I_{input}} = \frac{\left(1 - R^2\right)^2}{\left(1 - R^2\right)^2 + 4R^2 \sin^2\left(kL\right)}$$

FP resonance in dielectric slab

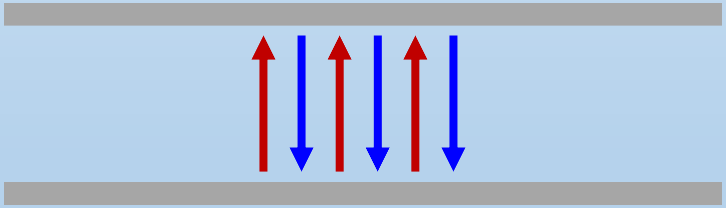
Metal waveguides

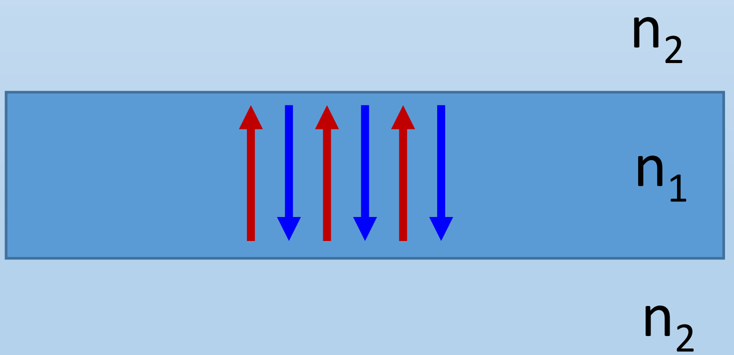


Dielectric waveguides



Resonance



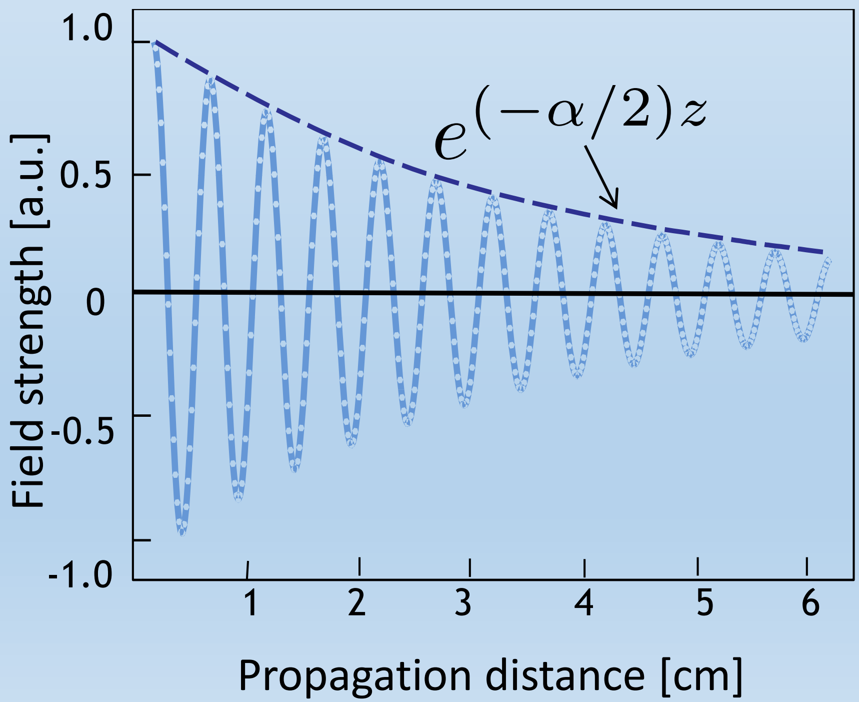


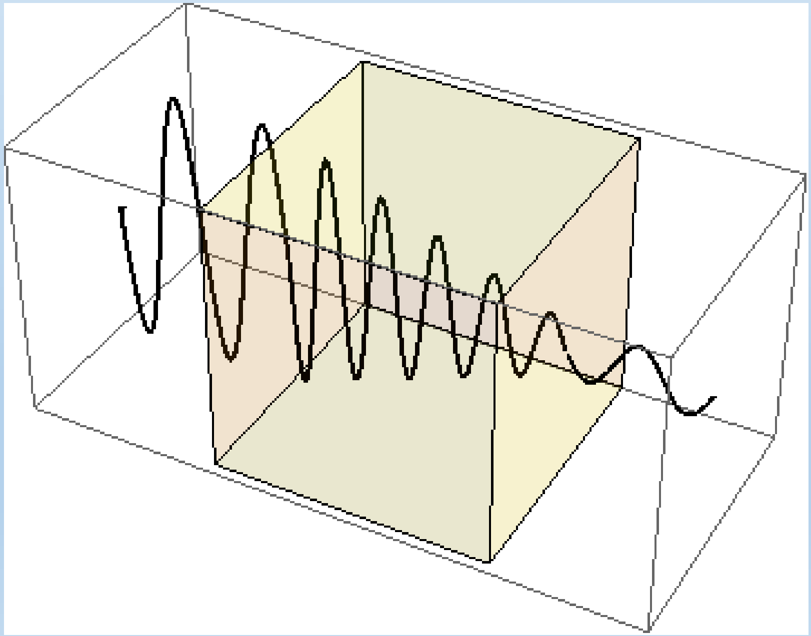
Plane Waves in Lossy Materials

Complex refractive index: $\tilde{n} = n - j\kappa$ $\qquad \kappa > 0$

Complex wave vector: $\tilde{k} = (n - j\kappa)k_0$ $\qquad \alpha = 2\kappa k_0$

$$E_y(z,t) = \mathrm{Re}\left\{e^{-j(n-j\kappa)k_0 z} e^{j\omega t}\right\} = E_0 e^{-\alpha z/2} \cos(\omega t - nk_0 z)$$





Resonators with Internal Loss

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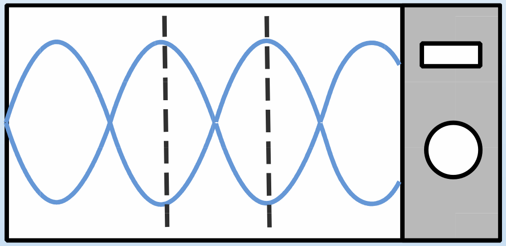
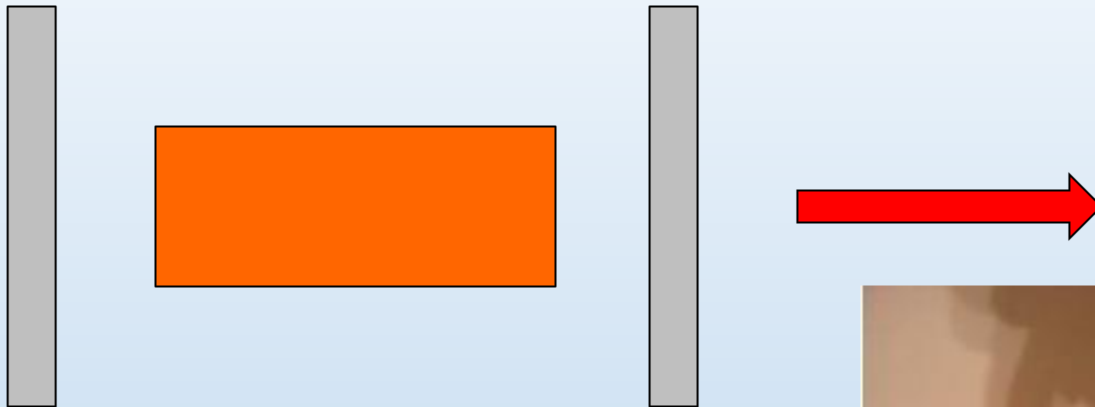
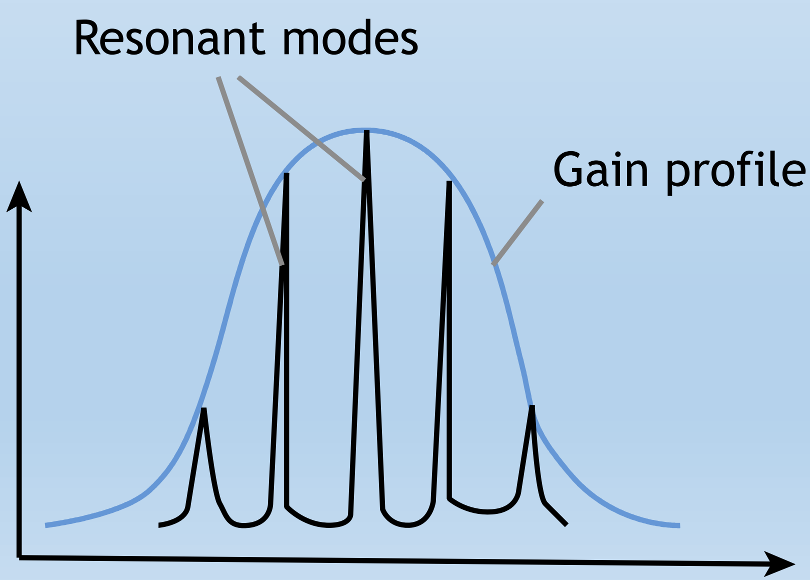


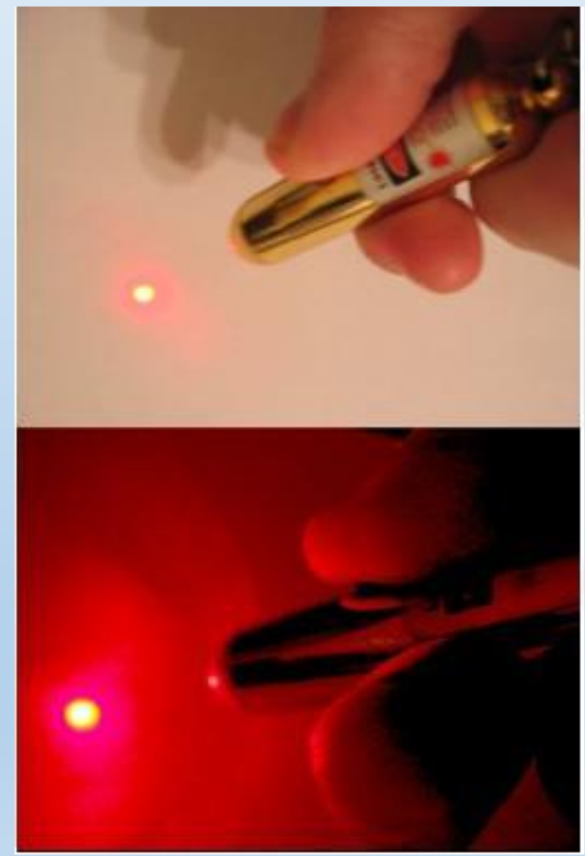
$$\frac{E_t}{E_i} = \frac{T^2 \exp(-jkL)}{1 - R^2 \exp(-j2kL)} = \frac{T^2 \exp(-\kappa k_0 L)\exp(-jnk_0 L)}{1 - R^2 \exp(-2\kappa k_0 L)\exp(-j2nk_0 L)}$$

...the EM wave loss is what heats the water inside the food

Laser Using Fabre-Perot Cavity







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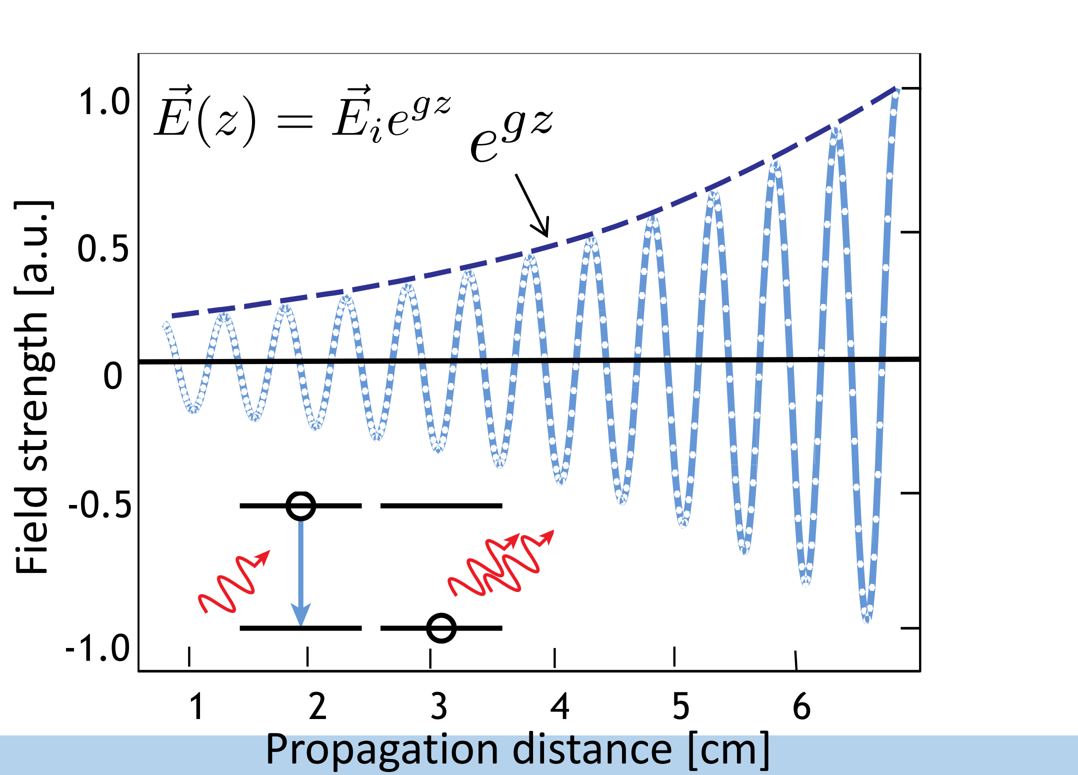
Resonators with Internal Gain

What if it was possible to make a material with “negative absorption” so the field grew in magnitude as it passed through a material?

$$\tilde{n} = n - j\kappa$$

$$\kappa < 0$$



$$t = \frac{E_t}{E_i} = \frac{T^2 \exp(-\kappa k_0 L)\exp(-jnk_0 L)}{1 - R^2 \exp(-2\kappa k_0 L)\exp(-j2nk_0 L)}$$

Resonance:

$$e^{j2nk_0 L} = 1$$

Lasers: Something for Nothing (almost)

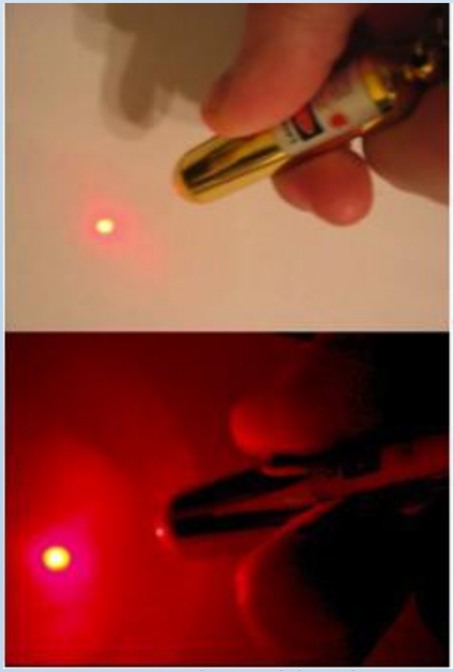
at resonance $e^{j2nk_0L} = 1$

$$\frac{E_t}{E_i} = \frac{T^2 \exp(-\kappa k_0 L)\exp(-jnk_0 L)}{1 - R^2 \exp(-2\kappa k_0 L)\exp(-j2nk_0 L)}$$

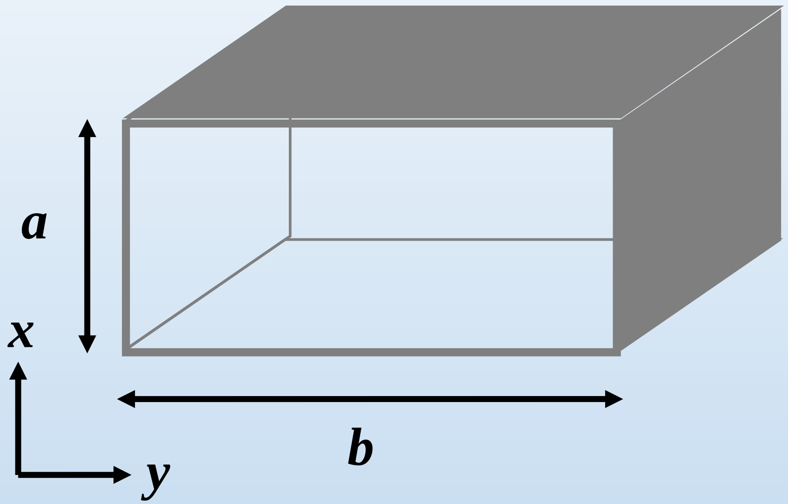
singularity at

$$1 = R^2 \exp(-2\kappa k_0 L)\exp(-j2nk_0 L)$$

$$\frac{E_t}{E_i} \to \infty$$



Rectangular Waveguide - TE Modes



$$H_z = \cos(k_x x)\cos(k_y y)e^{-jk_z z}$$

$$H_x = \frac{jk_x k_z}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)e^{-jk_z z}$$

$$H_y = \frac{jk_y k_z}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)e^{-jk_z z}$$

$$E_x = \frac{j\omega\mu k_y}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)e^{-jk_z z}$$

$$E_y = \frac{-j\omega\mu k_x}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)e^{-jk_z z}$$

Boundary Conditions:

(1) Ex = 0 at y = 0 and b

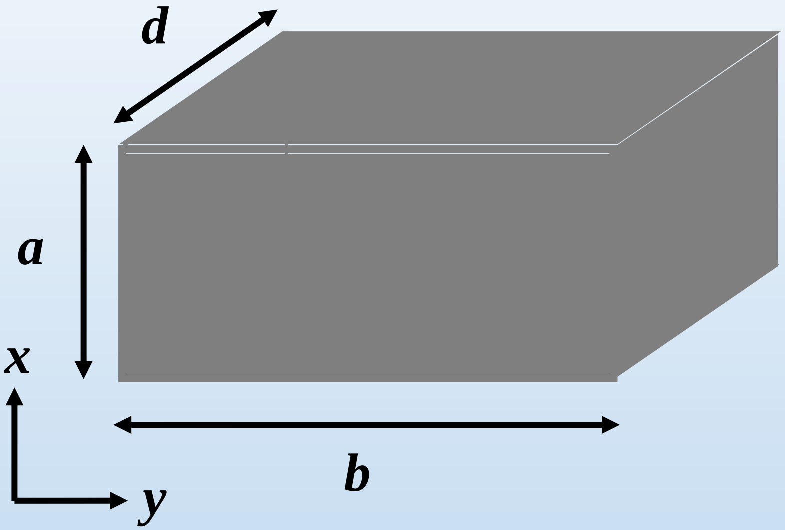
(2) Ey = 0 at x = 0 and a

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \end{aligned}}$$

(Guidance Condition)

$$\boxed{k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Rectangular Cavity Resonator - TE Modes



$$H_z = \cos(k_x x)\cos(k_y y)\sin(k_z z)$$

$$H_x = \frac{-k_x k_z}{\omega^2 \mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)$$

$$H_y = \frac{-k_y k_z}{\omega^2 \mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\cos(k_z z)$$

$$E_x = \frac{j\omega\mu k_y}{\omega^2 \mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)$$

$$E_y = \frac{-j\omega\mu k_x}{\omega^2 \mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)$$

Boundary Conditions:

(1) Ex = 0 at y = 0 and b

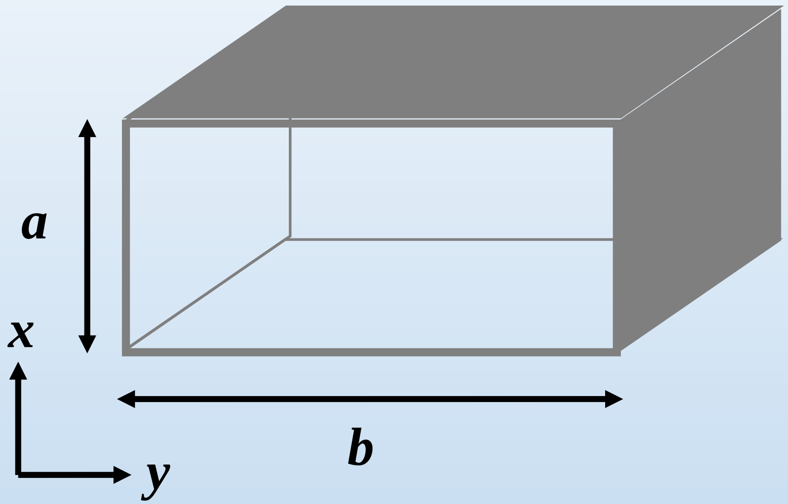
(2) Ey = 0 at x = 0 and a

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \\ k_z d &= p\pi \end{aligned}}$$

(Resonance Condition)

$$\boxed{k_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Rectangular Waveguide - TM Modes



$$E_z(x,y,z) = \sin(k_x x)\sin(k_y y)e^{-jk_z z}$$

$$E_x = \frac{-jk_x k_z}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)e^{-jk_z z}$$

$$E_y = \frac{-jk_y k_z}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)e^{-jk_z z}$$

$$H_x = \frac{j\omega\varepsilon k_y}{\omega^2\mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)e^{-jk_z z}$$

$$H_y = \frac{-j\omega\varepsilon k_x}{\omega^2\mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)e^{-jk_z z}$$

Boundary Conditions:

(1) Ex = 0 at y = 0 and b

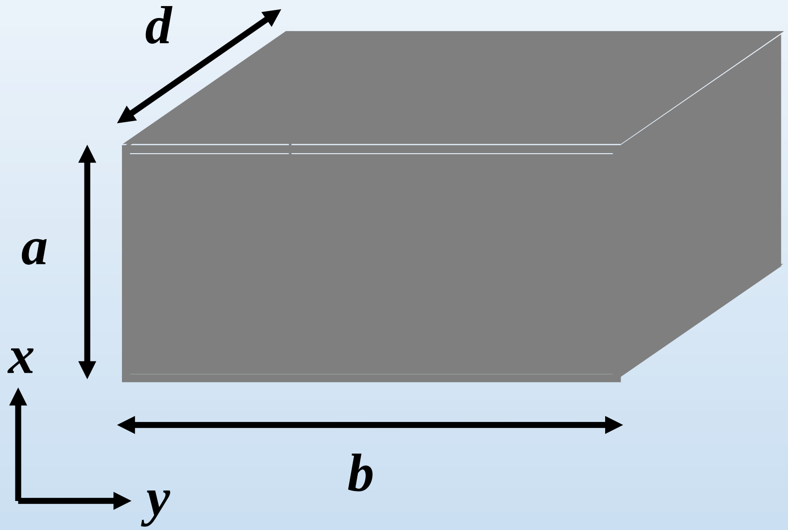
(2) Ey = 0 at x = 0 and a

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \end{aligned}}$$

(Guidance Condition)

$$\boxed{k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Rectangular Cavity Resonator - TM Modes



$$E_z(x,y,z) = \sin(k_x x)\sin(k_y y)\cos(k_z z)$$

$$E_x = \frac{-k_x k_z}{\omega^2 \mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)$$

$$E_y = \frac{-k_y k_z}{\omega^2 \mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)$$

$$H_x = \frac{j\omega\varepsilon k_y}{\omega^2 \mu\varepsilon - k_z^2}\sin(k_x x)\cos(k_y y)\cos(k_z z)$$

$$H_y = \frac{-j\omega\varepsilon k_x}{\omega^2 \mu\varepsilon - k_z^2}\cos(k_x x)\sin(k_y y)\cos(k_z z)$$

Boundary Conditions:

(1) Ex = 0 at y = 0 and b

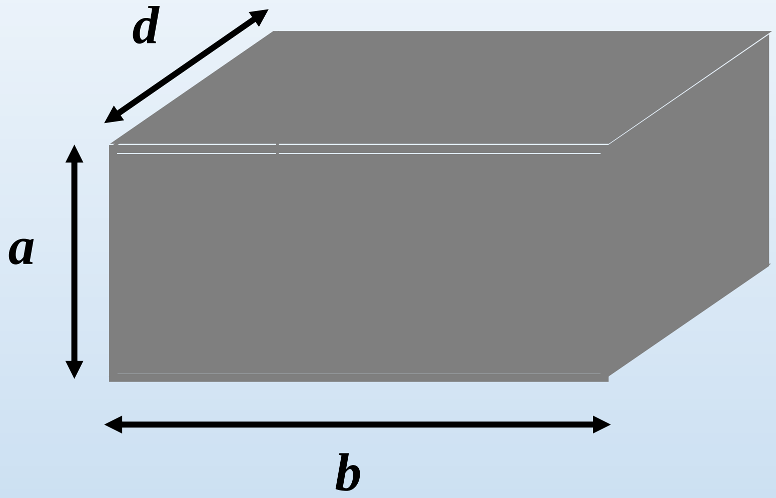
(2) Ey = 0 at x = 0 and a

$$\boxed{\begin{aligned} k_x a &= m\pi \\ k_y b &= n\pi \\ k_z d &= p\pi \end{aligned}}$$

(Resonance Condition)

$$\boxed{k_r = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Example



When the resonator dimensions are such that a>b>d, the lowest resonant spatial frequency (k) is

$$k_r = \sqrt{\left(\pi/a\right)^2 + \left(\pi/b\right)^2}$$

(m = n = 1, p = 0)

TM_{110} mode:

$$E_z\left(x,y,z\right) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

$$H_x = \frac{j\pi k_y}{\omega\mu b}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$$

$$E_x = E_y = 0$$

$$H_y = \frac{-j\pi}{\omega\mu a}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

TM$_{110}$ mode:

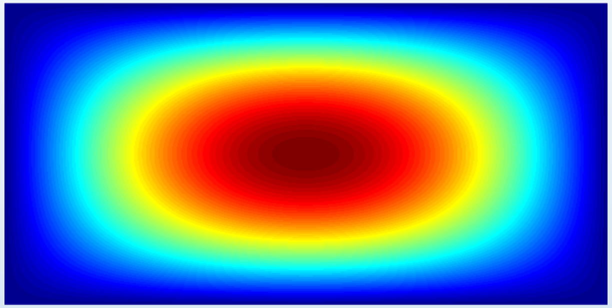
$$k_r = \sqrt{(\pi/a)^2 + (\pi/b)^2}$$

$$E_z(x,y,z) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

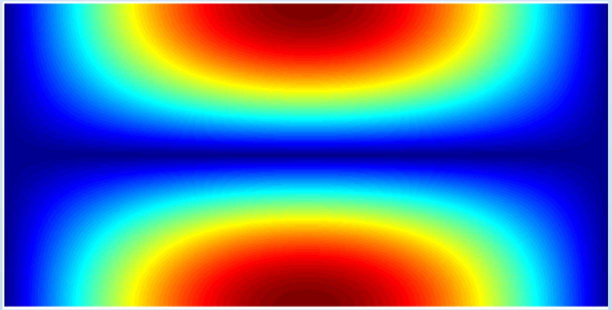
$$H_x = \frac{j\pi k_y}{\omega\mu b}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)$$

$$H_y = \frac{-j\pi}{\omega\mu a}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

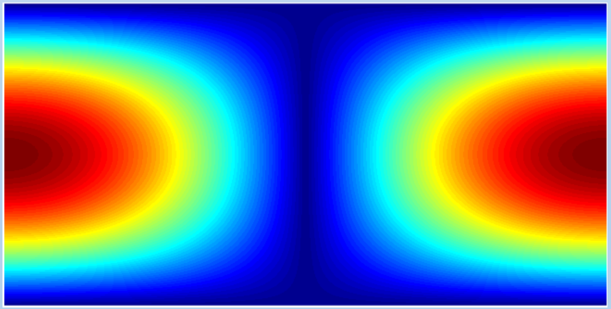
$$f_r = \frac{c_0 k_r}{2\pi n} = \frac{1}{2n}\sqrt{(c_0/a)^2 + (c_0/b)^2}$$



$\boldsymbol{E_z}$



$\boldsymbol{H_x}$



$\boldsymbol{H_y}$