第三章习题《基础物理 I 波动理论导引》

习题 3.1： 平面波必须满足麦克斯韦方程。在真空中，假设有下列电场矢量分布

$$\begin{aligned}
\vec{E}_1 &= \hat{x}\cos(\omega t - kz) \\
\vec{E}_2 &= \hat{z}\cos(\omega t - kz) \\
\vec{E}_3 &= (\hat{x}+\hat{z})\cos(\omega t + ky) \\
\vec{E}_4 &= (\hat{x}+\hat{z})\cos\left(\omega t + k|x-z|/\sqrt{2}\right)
\end{aligned}$$

试问：（1）这些电场矢量场是否满足波动方程

$$\left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

（2） ω 与 k 之间的关系是什么？

（3）求对应的磁场 $\vec{B}$

（4）上述哪些场是电磁波？如果不是电磁波，试说明违反哪一条麦克斯韦方程。对于电磁波，证明电场矢量和磁场矢量与波的传播方向垂直。

解：（1） $\vec{E}_1$、 $\vec{E}_2$ 和 $\vec{E}_3$ 满足波动方程。

（2） $k = \omega\sqrt{\mu_0\varepsilon_0}$

（3）利用法拉第定律： $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$

$$-\partial\vec{B}_1/\partial t = \nabla\times\vec{E}_1 = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left[\hat{x}\cos(\omega t-kz)\right] = \hat{y}k\sin(\omega t-kz)$$

$$\Rightarrow \quad \vec{B}_1 = \hat{y}\frac{k}{\omega}\cos(\omega t-kz)$$

$$-\partial\vec{B}_2/\partial t = \nabla\times\vec{E}_2 = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left[\hat{z}\cos(\omega t-kz)\right] = 0$$

$$\Rightarrow \quad \vec{B}_2 = 0$$

$$-\partial\vec{B}_3/\partial t = \nabla\times\vec{E}_3 = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left[(\hat{x}+\hat{z})\cos(\omega t+ky)\right] = (\hat{z}-\hat{x})k\sin(\omega t+ky)$$

$$\Rightarrow \quad \vec{B}_3 = (\hat{z}-\hat{x})\frac{k}{\omega}\cos(\omega t+ky)$$

$$-\partial\vec{B}_4/\partial t = \nabla\times\vec{E}_4 = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\begin{cases}(\hat{x}+\hat{z})\cos\left(\omega t+k(x-z)/\sqrt{2}\right), x>z \\ (\hat{x}+\hat{z})\cos\left(\omega t-k(x-z)/\sqrt{2}\right), x<z\end{cases}$$

$$= \begin{cases}\sqrt{2}\hat{y}k\sin\left(\omega t+k(x-z)/\sqrt{2}\right), x>z \\ -\sqrt{2}\hat{y}k\sin\left(\omega t-k(x-z)/\sqrt{2}\right), x<z\end{cases}$$

$$\Rightarrow \quad \vec{B}_4=\begin{cases}\sqrt{2}\hat{y}\dfrac{k}{\omega}\cos\left(\omega t+k\left(x-z\right)/\sqrt{2}\right), x>z\\ -\sqrt{2}\hat{y}\dfrac{k}{\omega}\cos\left(\omega t-k\left(x-z\right)/\sqrt{2}\right), x<z\end{cases}$$

（4）$\vec{E}_1$ 和 $\vec{E}_3$ 是电磁波。$\vec{E}_2$ 违反高斯定律（$\nabla\cdot\vec{E}=0$）；$\vec{E}_4$ 和 $\vec{B}_4$ 的导数在 $z=x$ 处不存在；所以 $\vec{E}_2$ 和 $\vec{E}_4$ 不是电磁波。

习题 3.2：已知的电磁波频谱覆盖了很宽的频率范围。电磁现象都是由麦克斯韦方程描述的，按照惯例，通常根据波长或频率进行分类。无线电波，电视信号，雷达波束，可见光，X 射线和伽马射线都是电磁波的例子。

（1）用波长来表示下列电磁波频率：60 Hz、AM radio (535–1605 kHz)、FM radio (88–108 MHz)、C- band (4–6 GHz)、Visible light (~ 10^{14} Hz)、X-rays (~ 10^{18} Hz)

（2）用时间频率来表示下列波长：1km、1m、1mm、1μm。

解：（1） 60Hz：$\lambda = c/f = 5\times10^6$ (m)

AM radio：$\lambda = 186.9 \sim 560.8$ (m)

FM radio (88–108 MHz): $\lambda = 2.778 \sim 3.409$ (m)

C- band (4–6 GHz): $\lambda = 0.05 \sim 0.075$ (m)

Visible light (~ 10^{14} Hz): $\lambda =\sim 3\times10^{-6}$ (m)

X-rays (~10^{18} Hz): $\lambda = \sim 3\times10^{-10}$ (m)

（2） 1 km: $f = c/\lambda = 3\times10^5$ (Hz)

1 m: $f = 3\times10^8$ (Hz)

1 mm: $f = 3\times10^{11}$ (Hz)

1 μm: $f = 3\times10^{14}$ (Hz)

习题 3.3：证明下列矢量恒等式。

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$$

$$\nabla\cdot\left(\vec{E}\times\vec{H}\right)=\vec{H}\cdot\left(\nabla\times\vec{E}\right)-\vec{E}\cdot\left(\nabla\times\vec{H}\right)$$

$$\nabla\cdot\left(\nabla\times\vec{A}\right)=0$$

$$\nabla\times\left(\nabla\Phi\right)=0$$

解：

$$
\begin{aligned}
\nabla\times\left(\nabla\times\vec{E}\right) &= \nabla\times\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} \\
&= \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left[\hat{x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\right] \\
&= \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\right] \\
&\quad+\hat{y}\left[-\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\right] \\
&\quad+\hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\right] \\
&= \hat{x}\left[\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x\right] \\
&\quad+\hat{y}\left[\frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_y\right] \\
&\quad+\hat{z}\left[\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z\right] \\
&= \nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\left(\vec{E}\times\vec{H}\right) &= \nabla\cdot\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{bmatrix} \\
&= \frac{\partial}{\partial x}\left(E_yH_z-E_zH_y\right)+\frac{\partial}{\partial y}\left(E_zH_x-E_xH_z\right)+\frac{\partial}{\partial z}\left(E_xH_y-E_yH_x\right) \\
&= H_x\left(\frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y\right)+H_y\left(\frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z\right)+H_z\left(\frac{\partial}{\partial x}E_y-\frac{\partial}{\partial y}E_x\right) \\
&\quad-E_x\left(\frac{\partial}{\partial y}H_z-\frac{\partial}{\partial z}H_y\right)-E_y\left(\frac{\partial}{\partial z}H_x-\frac{\partial}{\partial x}H_z\right)-E_z\left(\frac{\partial}{\partial x}H_y-\frac{\partial}{\partial y}H_x\right) \\
&= \vec{H}\cdot\left(\nabla\times\vec{E}\right)-\vec{E}\cdot\left(\nabla\times\vec{H}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\left(\nabla\times\vec{A}\right) &= \nabla\cdot\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{bmatrix} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right) \\
&= 0
\end{aligned}
$$

$$\nabla\times(\nabla\Phi)=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial\Phi/\partial x & \partial\Phi/\partial y & \partial\Phi/\partial z\end{bmatrix}$$
$$=\hat{x}\left(\frac{\partial^2\Phi}{\partial y\partial z}-\frac{\partial^2\Phi}{\partial y\partial z}\right)+\hat{y}\left(\frac{\partial^2\Phi}{\partial x\partial z}-\frac{\partial^2\Phi}{\partial x\partial z}\right)+\hat{z}\left(\frac{\partial^2\Phi}{\partial x\partial y}-\frac{\partial^2\Phi}{\partial x\partial y}\right)$$
$$=0$$

习题 3.4：考虑一个沿着 z 方向传播的电磁波：

$$\vec{E}=\hat{x}A_x\cos\left(kz-\omega t+\varphi_x\right)+\hat{y}A_y\cos\left(kz-\omega t+\varphi_y\right)$$

其中 A_x、A_y、φ_x 和 φ_y 都是实数。

（1） 令 $A_x = 2$，$A_y = 1$，$\varphi_x = \pi/2$，$\varphi_y = \pi/4$，求该电磁波的极化状态。

（2） 令 $A_x = 1$，$A_y = 0$，$\varphi_x = 0$，此时这是一个线极化波，证明该电磁波可以分解成一个左旋圆极化波和一个右旋圆极化波的叠加。

（3） 令 $A_x = 1$，$A_y = 1$，$\varphi_x = \pi/4$，$\varphi_y = -\pi/4$，此时这是一个圆极化波，证明该电磁波可以分解成两个线极化波的叠加。

解：（1）取 $z = 0$，则 $\vec{E}=\hat{x}2\cos\left(-\omega t+\frac{\pi}{2}\right)+\hat{y}\cos\left(-\omega t+\frac{\pi}{4}\right)$

⇨ $E_x=2\sin(\omega t)$，$E_y=\cos\left(\omega t-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\cos(\omega t)+\frac{1}{\sqrt{2}}\sin(\omega t)$

⇨ 有电场矢量的轨迹满足方程：$\frac{E_x^2}{2}-\sqrt{2}E_xE_y+2E_y^2=1$

⇨ 所以是椭圆极化波。

（2）$\vec{E}=\hat{x}\cos(kz-\omega t)=\frac{1}{2}\left[\hat{x}\cos(kz-\omega t)+\hat{y}\sin(kz-\omega t)\right]+\frac{1}{2}\left[\hat{x}\cos(kz-\omega t)-\hat{y}\sin(kz-\omega t)\right]$

所以能分解成一个左旋圆极化波和一个右旋圆极化波的叠加。

（3）$\vec{E}=\hat{x}\cos\left(kz-\omega t+\frac{\pi}{4}\right)+\hat{y}\cos\left(kz-\omega t-\frac{\pi}{4}\right)$

这就是两个线极化波的叠加。

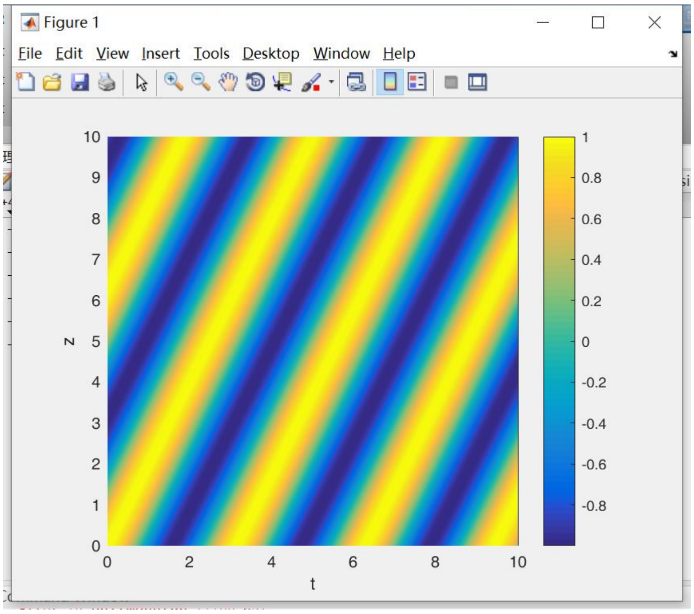
习题 3.5：用 matlab 运行下列代码，观察平面波的传播。跟踪与原点具有相同相位的线。线的斜率代表波速。增加 k 的值，观察电磁波会如何变化。

```
k =1;
omega = 2;
[z,t] = meshgrid(0:0.01:10, 0:0.01:10);
wave = cos(k*z - omega*t);
```

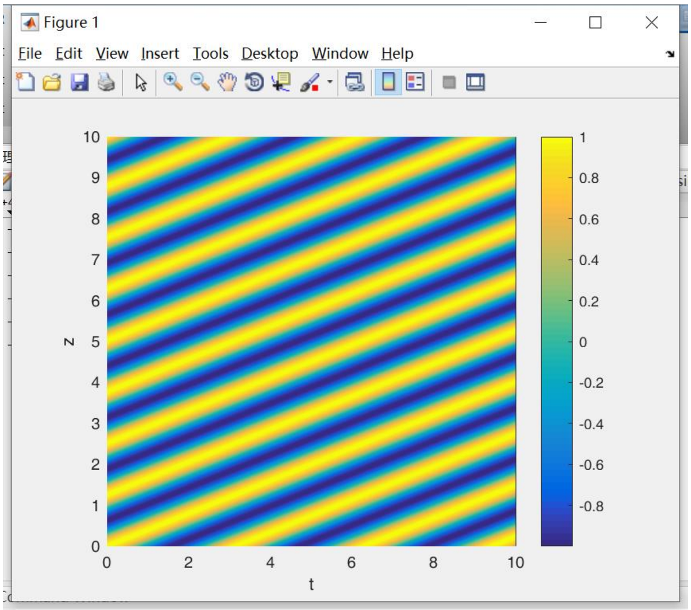
```
pcolor(t,z,wave);shading flat;colorbar;
xlabel('t');ylabel('z');axis square;
```

解：

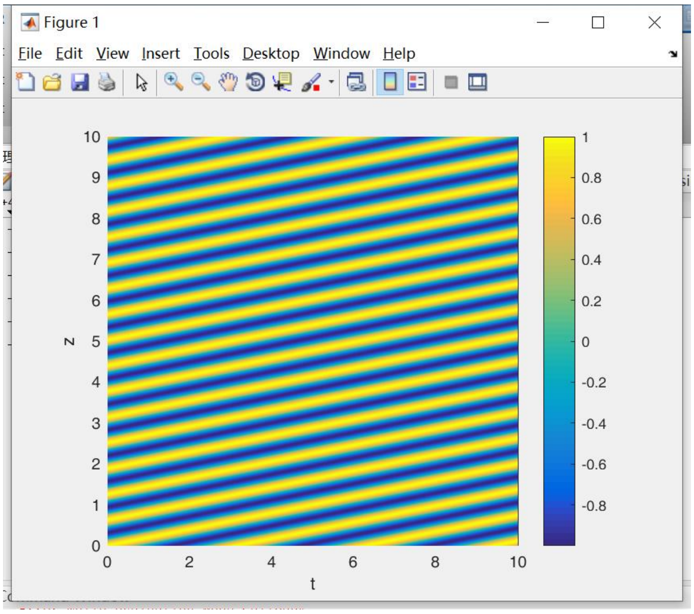
k =1



k =5



k =10



可以看出，k 越大，波速越小，这与波速公式$v = \omega / k$一致。