Chapter 2: Magnetostatics and Faraday's Law

2.1 Introduction to Magnetic Fields

We have seen that a charged object produces an electric field $\vec{E}$ at all points in space. In a similar manner, a bar magnet is a source of a magnetic field $\vec{B}$. This can be readily demonstrated by moving a compass near the magnet. The compass needle will line up along the direction of the magnetic field produced by the magnet, as depicted in Figure 2.1

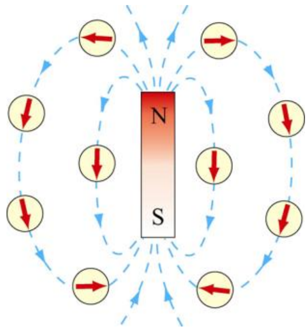


Figure 2.1 Magnetic field produced by a bar magnet

Notice that the bar magnet consists of two poles, which are designated as the north (N) and the south (S). Magnetic fields are strongest at the poles. The magnetic field lines leave from the north pole and enter the south pole. When holding two bar magnets close to each other, the like poles will repel each other while the opposite poles attract (Figure 2.2).

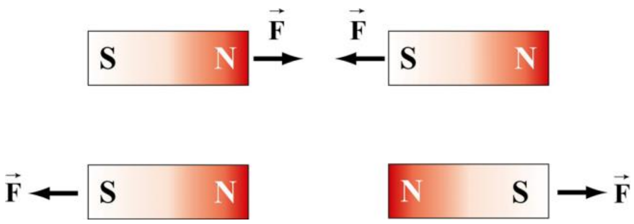


Figure 2.2 Magnets attracting and repelling

Unlike electric charges which can be isolated, the two magnetic poles always come in a pair. When you break the bar magnet, two new bar magnets are obtained, each with a north pole and a south pole (Figure 2.3). In other words, magnetic “monopoles” do not exist in isolation, although they are of theoretical interest.

Figure 2.3 Magnetic monopoles do not exist in isolation

How do we define the magnetic field $\vec{B}$? In the case of an electric field $\vec{E}$, we have already seen that the field is defined as the force per unit charge:

$$\vec{E} = \frac{\vec{F}_e}{q} \tag{2.1}$$

However, due to the absence of magnetic monopoles, $\vec{E}$ must be defined in a different way

2.1.1 The Definition of a Magnetic Field

To define the magnetic field at a point, consider a particle of charge q and moving at a velocity $\vec{v}$. Experimentally we have the following observations:

(1) The magnitude of the magnetic force $\vec{F}_B$ exerted on the charged particle is proportional to both v and q.

(2) The magnitude and direction of $\vec{F}_B$ depends on $\vec{v}$ and $\vec{B}$.

(3) The magnetic force $\vec{F}_B$ vanishes when $\vec{v}$ is parallel to $\vec{B}$. However, when $\vec{v}$ makes an angle θ with $\vec{B}$, the direction of $\vec{F}_B$ is perpendicular to the plane formed by $\vec{v}$ and $\vec{B}$, and the magnitude of $\vec{F}_B$ is proportional to $\sin\theta$.

(4) When the sign of the charge of the particle is switched from positive to negative (or vice versa), the direction of the magnetic force also reverses.

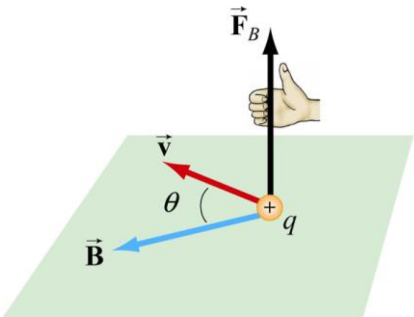


Figure 2.4 The direction of the magnetic force

The above observations can be summarized with the following equation:

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{2.2}$$

The above expression can be taken as the working definition of the magnetic field at a point in space. The magnitude of $\vec{F}_B$ is given by

$$F_B = |q| vB\sin\theta \tag{2.3}$$

The SI unit of magnetic field is the tesla (T):

$$1\,\text{tesla} = 1\,\text{T} = 1\frac{\text{Newton}}{(\text{Coulomb})(\text{meter}/\text{second})} = 1\frac{\text{N}}{\text{C}\cdot\text{m/s}} = 1\frac{\text{N}}{\text{A}\cdot\text{m}} \tag{2.4}$$

Another commonly used non-SI unit for $\vec{B}$ is the gauss (G), where 1T $=10^4$G.

Note that $\vec{F}_B$ is always perpendicular to $\vec{v}$ and $\vec{B}$, and cannot change the particle's speed v (and thus the kinetic energy). In other words, magnetic force cannot speed up or slow down a charged particle. Consequently, $\vec{F}_B$ can do no work on the particle:

$$dW = \vec{F}_B \cdot d\vec{s} = q\left(\vec{v}\times\vec{B}\right)\cdot\vec{v}dt = q\left(\vec{v}\times\vec{v}\right)\cdot\vec{B}dt = 0 \tag{2.5}$$

The direction of $\vec{v}$, however, can be altered by the magnetic force, as we shall see below.

2.1.2 Magnetic Force on a Current-Carrying Wire

We have just seen that a charged particle moving through a magnetic field experiences a magnetic force $\vec{F}_B$. Since electric current consists of a collection of charged particles in motion, when placed in a magnetic field, a current-carrying wire will also experience a magnetic force.

Consider a long straight wire suspended in the region between the two magnetic poles. The magnetic field points out the page and is represented with dots (•). It can be readily demonstrated that when a downward current passes through, the wire is deflected to the left. However, when the current is upward, the deflection is rightward, as shown in Figure 2.4.

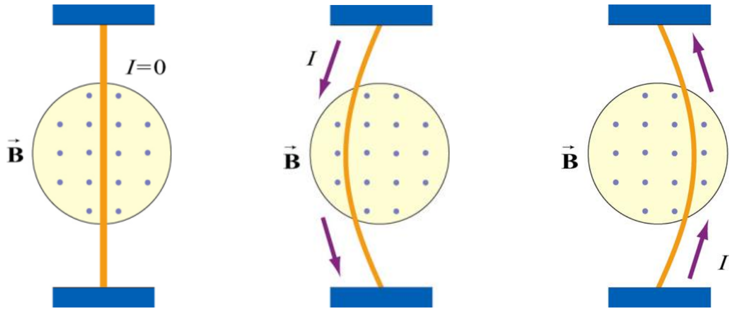


Figure 2.4 Deflection of current-carrying wire by magnetic force

To calculate the force exerted on the wire, consider a segment of wire of length l and cross-sectional area A, as shown in Figure 2.5. The magnetic field points into the page, and is represented with crosses (×).

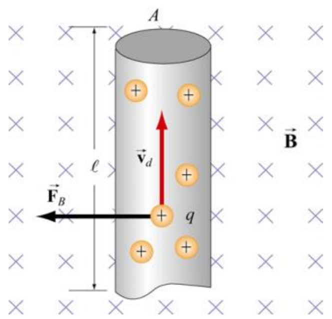


Figure 2.5 Magnetic force on a conducting wire

The charges move at an average drift velocity V_d. Since the total amount of charge in this segment is $Q_{tot} = q\left(nAl\right)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

$$\vec{F}_B = Q_{tot}\vec{v}_d \times \vec{B} = qnAl\left(\vec{v}_d \times \vec{B}\right) = I\left(\vec{l}\times\vec{B}\right) \tag{2.6}$$

Where $I = nqv_d A$, and $\vec{l}$ is a length vector with a magnitude l and directed along the direction of the electric current.

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{s}$ (Figure 2.6).

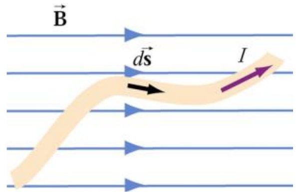


Figure 2.6 Current-carrying wire placed in a magnetic field

The magnetic force acting on the segment is

$$d\vec{F}_B = I\left(d\vec{s} \times \vec{B}\right) \tag{2.7}$$

Thus, the total force is

$$\boxed{\vec{F}_B = I\int_a^b d\vec{s} \times \vec{B}} \tag{2.8}$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{B}$, as shown in Figure 2.7.

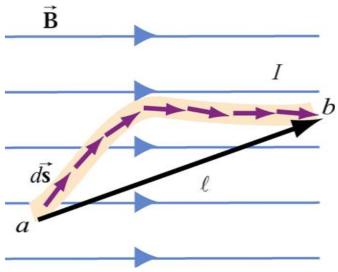


Figure 2.7 A curved wire carrying a current I

Using Eq. (2.8), the magnetic force on the wire is given by

$$\vec{F}_B = I\left(\int_a^b d\vec{s}\right) \times \vec{B} = I\vec{l} \times \vec{B} \tag{2.9}$$

where $\vec{l}$ is the length vector directed from a to b. However, if the wire forms a closed loop of arbitrary shape (Figure 2.8), then the force on the loop becomes

$$\vec{F}_B = I\left(\oint d\vec{s}\right) \times \vec{B} \tag{2.10}$$

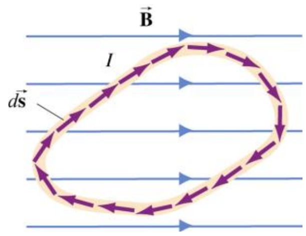


Figure 2.8 A closed loop carrying a current I in a uniform magnetic field.

Since the set of differential length elements $d\vec{s}$ form a closed polygon, and their vector sum is zero, i.e., $\oint d\vec{s} = 0$. The net magnetic force on a closed loop is $\vec{F}_B = 0$.

Example 2.1 Magnetic Force on a Semi-Circular Loop

Consider a closed semi-circular loop lying in the *xy* plane carrying a current *I* in the counterclockwise direction, as shown in Figure 2.9. A uniform magnetic field pointing in the +*y* direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.

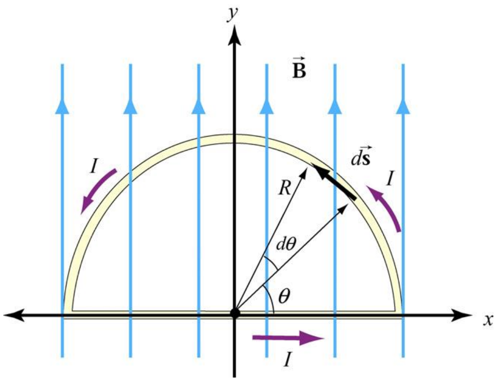


Figure 2.9 Semi-circular loop carrying a current I

Solution:

Let $\vec{B} = B\hat{y}$, $\vec{F}_1$ and $\vec{F}_1$ the forces acting on the straight segment and the semicircular parts, respectively. Using Eq. (2.8) and noting that the length of the straight segment is 2*R*, the magnetic force is

$$\vec{F}_1 = I\left(2R\hat{x}\right)\times\left(B\hat{y}\right) = 2IRB\hat{z}$$

where $\hat{z}$ is directed out of the page.

To evaluate $\vec{F}_2$, we first note that the differential length element $d\vec{s}$ on the semicircle can be written as $d\vec{s} = \hat{\theta}ds = Rd\theta\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right)$. The force acting on the length element $d\vec{s}$ is

$$d\vec{F}_2 = Id\vec{s}\times\vec{B} = IRd\theta\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right)\times\left(B\hat{y}\right) = -IBR\sin\theta d\theta\hat{z}$$

Here we see that $d\vec{F}_2$ points into the page. Integrating over the entire semi-circle arc, we have

$$\vec{F}_2 = -IBR\hat{z}\int_0^{\pi} \sin\theta d\theta = -2IBR\hat{z}$$

Thus, the net force is

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = \vec{0}$$

This is consistent from our previous claim that the net magnetic force acting on a closed current-carrying loop must be zero.

2.1.3 Torque on a Current Loop

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{B} = \hat{x}B$ which runs parallel to the plane of the loop, as shown in Figure 2.10(a)

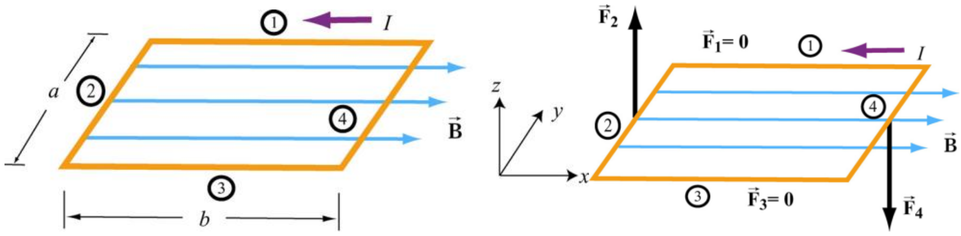


Figure 2.10 (a) A rectangular current loop placed in a uniform magnetic field. (b) The magnetic forces acting on sides 2 and 4.

From Eq. (2.11), we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{l}_1 = -b\hat{x}$ and $\vec{l}_3 = b\hat{x}$ are parallel and anti-parallel to $\vec{B}$ and their cross products vanish. On the other hand, the magnetic forces acting on segments 2 and 4 are non-vanishing:

$$\begin{cases} \vec{F}_2 = I(-a\hat{y}) \times (B\hat{x}) = IaB\hat{z} \\ \vec{F}_4 = I(a\hat{y}) \times (B\hat{x}) = -IaB\hat{z} \end{cases} \tag{2.11}$$

with $\vec{F}_2$ pointing out of the page and $\vec{F}_4$ into the page. Thus, the net force on the rectangular loop is

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0} \tag{2.12}$$

as expected. Even though the net force on the loop vanishes, the forces $\vec{F}_2$ and $\vec{F}_4$ will produce a torque which causes the loop to rotate about the y-axis (Figure 2.11). The torque with respect to the center of the loop is

$$\begin{aligned} \vec{\tau} &= \left(-\frac{b}{2}\hat{x}\right) \times \vec{F}_2 + \left(\frac{b}{2}\hat{x}\right) \times \vec{F}_4 = \left(-\frac{b}{2}\hat{x}\right) \times (IaB\hat{z}) + \left(\frac{b}{2}\hat{x}\right) \times (-IaB\hat{z}) \\ &= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{y} = IabB\hat{y} = IAB\hat{y} \end{aligned} \tag{2.13}$$

where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y-axis. It is convenient to introduce the area vector $\vec{A} = A\hat{n}$ where $\hat{n}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{n}$ is set by the conventional right-hand rule. In our case, we have $\hat{n} = +\hat{z}$. The above expression for torque can then be rewritten as

$$\vec{\tau} = I\vec{A} \times \vec{B} \tag{2.14}$$

Notice that the magnitude of the torque is at a maximum when $\vec{B}$ is parallel to the plane of the loop (or perpendicular to $\vec{A}$).

Consider now the more general situation where the loop (or the area vector $\vec{A}$) makes an angle θ with respect to the magnetic field.

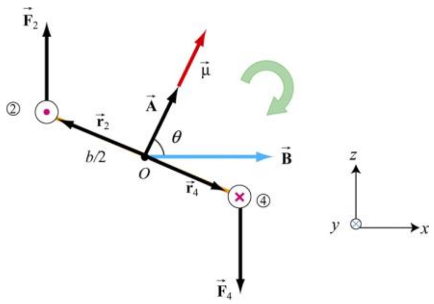


Figure 2.11 Rotation of a rectangular current loop

From Figure 2.11, the lever arms and can be expressed as:

$$\vec{r}_2 = \frac{b}{2}(-\sin\theta\hat{x} + \cos\theta\hat{z}) = -\vec{r}_4 \tag{2.15}$$

and the net torque becomes

$$\begin{aligned}\vec{\tau} &= \vec{r}_2 \times \vec{F}_2 + \vec{r}_4 \times \vec{F}_4 = 2\vec{r}_2 \times \vec{F}_2 = 2 \cdot \frac{b}{2}(-\sin\theta\hat{x} + \cos\theta\hat{z}) \times (IaB\hat{z}) \\ &= IabB\sin\theta\hat{y} = I\vec{A} \times \vec{B}\end{aligned} \tag{2.16}$$

For a loop consisting of N turns, the magnitude of the toque is

$$\tau = NIAB\sin\theta \tag{2.17}$$

The quantity $NI\vec{A}$ is called the magnetic dipole moment $\vec{\mu}$

$$\boxed{\vec{\mu} = NI\vec{A}} \tag{2.18}$$

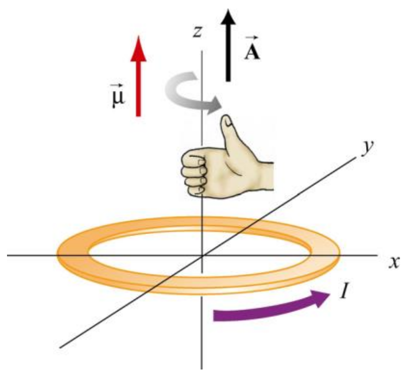


Figure 2.12 Right-hand rule for determining the direction of $\vec{\mu}$

The direction of $\vec{\mu}$ is the same as the area vector $\vec{A}$ (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure 2.12). The SI unit for the magnetic dipole moment is ampere-meter^2 ($A \cdot m^2$). Using the expression for $\vec{\mu}$, the torque exerted on a current-carrying loop can be rewritten as

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}} \tag{2.19}$$

The above equation is analogous to $\vec{\tau} = \vec{p} \times \vec{E}$. the torque exerted on an electric dipole moment $\vec{p}$ in the presence of an electric field $\vec{E}$. Recalling that the potential energy for an electric dipole is $U = -\vec{p} \cdot \vec{E}$, a similar form is expected for the magnetic case. The work done by an external agent to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$\begin{aligned} W_{\text{ext}} &= \int_{\theta_0}^{\theta} \tau d\theta' = \int_{\theta_0}^{\theta} (\mu B \sin\theta') d\theta' = \mu B (\cos\theta_0 - \cos\theta) \\ &= \Delta U = U - U_0 \end{aligned} \tag{2.20}$$

Once again, $W_{\text{ext}} = -W$, where W is the work done by the magnetic field. Choosing $U_0 = 0$ at $\theta_0 = \pi/2$, the dipole in the presence of an external field then has a potential energy of

$$U = -\mu B \cos\theta = -\vec{\mu} \cdot \vec{B} \tag{2.21}$$

The configuration is at a stable equilibrium when $\vec{\mu}$ is aligned parallel to $\vec{B}$, making U a minimum with $U_{\min} = -\mu B$. On the other hand, when $\vec{\mu}$ and $\vec{B}$ are anti-parallel, $U_{\max} = +\mu B$ is a maximum and the system is unstable.

2.1.4 Charged Particles in a Uniform Magnetic Field

If a particle of mass m moves in a circle of radius r at a constant speed v, what acts on the particle is a radial force of magnitude $F = mv^2/r$ that always points toward the center and is perpendicular to the velocity of the particle.

In previous section, we have also shown that the magnetic force $\vec{F}_B$ always points in the direction perpendicular to the velocity $\vec{v}$ of the charged particle and the magnetic field $\vec{B}$. Since $\vec{F}_B$ can do not work, it can only change the direction of $\vec{v}$ but not its magnitude. What would happen if a charged particle moves through a uniform magnetic field $\vec{B}$ with its initial velocity $\vec{v}$ at a right angle to $\vec{B}$? For simplicity, let the charge be $+q$ and the direction of $\vec{B}$ be into the page. It turns out that $\vec{F}_B$ will play the role of a centripetal force and the charged particle will move in a circular path in a counterclockwise direction, as shown in Figure 2.13.

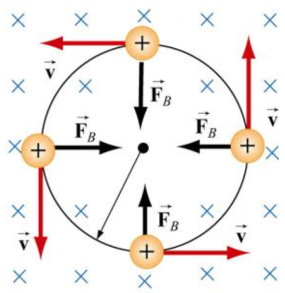


Figure 2.13 Path of a charge particle moving in a uniform $\vec{B}$ field with velocity $\vec{v}$ initially perpendicular to $\vec{B}$.

With

$$qvB = \frac{mv^2}{r} \tag{2.22}$$

the radius of the circle is found to be

$$r = \frac{mv}{qB} \tag{2.23}$$

The period T (time required for one complete revolution) is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB} \tag{2.24}$$

Similarly, the angular speed (cyclotron frequency) ω of the particle can be obtained as

$$\omega = 2\pi f = \frac{v}{r} = \frac{qB}{m} \tag{2.25}$$

If the initial velocity of the charged particle has a component parallel to the magnetic field $\vec{B}$, instead of a circle, the resulting trajectory will be a helical path, as shown in Figure 2.14:

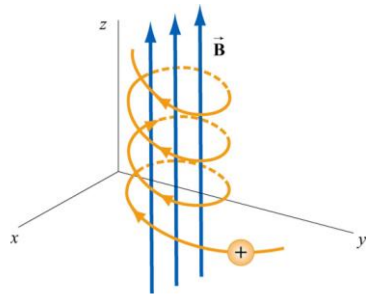


Figure 2.14 Helical path of a charged particle in an external magnetic field. The velocity of the particle

has a non-zero component along the direction of $\vec{B}$.

2.1.5 Magnetic force on a dipole

As we have shown above, the force experienced by a current-carrying rectangular loop (i.e., a magnetic dipole) placed in a uniform magnetic field is zero. What happens if the magnetic field is non-uniform? In this case, there will be a net force acting on the dipole.

Consider the situation where a small dipole $\vec{\mu}$ is placed along the symmetric axis of a bar magnet, as shown in Figure 2.15.

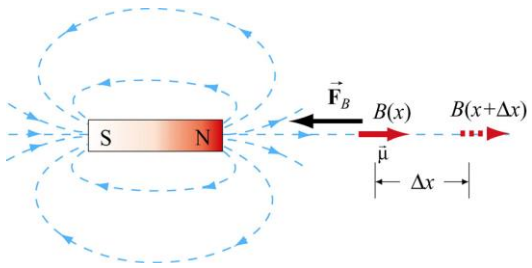


Figure 2.15 A magnetic dipole near a bar magnet.

The dipole experiences an attractive force by the bar magnet whose magnetic field is non-uniform in space. Thus, an external force must be applied to move the dipole to the right. The amount of force exerted by an external agent to move the dipole by a distance Δx is given by

$$F_{ext}\Delta x = W_{ext} = \Delta U = -\mu B(x+\Delta x) + \mu B(x) = -\mu\left[B(x+\Delta x) - B(x)\right] \tag{2.26}$$

where we have used Eq. (2.21). For small Δx, the external force may be obtained as

$$F_{ext} = -\mu\frac{\left[B(x+\Delta x) - B(x)\right]}{\Delta x} = -\mu\frac{dB}{dx} \tag{2.27}$$

which is a positive quantity since $dB/dx < 0$, i.e., the magnetic field decreases with increasing x. This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have

$$F_B = \mu\frac{dB}{dx} = \frac{d}{dx}\left(\vec{\mu}\cdot\vec{B}\right) \tag{2.28}$$

More generally, the magnetic force experienced by a dipole $\vec{\mu}$ placed in a non-uniform magnetic field $\vec{B}$ can be written as

$$F_B = \nabla\left(\vec{\mu}\cdot\vec{B}\right) \tag{2.29}$$

where

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{2.30}$$

is the gradient operator.

2.1.6 Applications

There are many applications involving charged particles moving through a uniform magnetic field.

Velocity Selector

In the presence of both electric field $\vec{E}$ and magnetic field $\vec{B}$, the total force on a charged particle is

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2.31}$$

This is known as the Lorentz force. By combining the two fields, particles which move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons. In Figure 2.16 the schematic diagram of Thomson's apparatus is depicted.

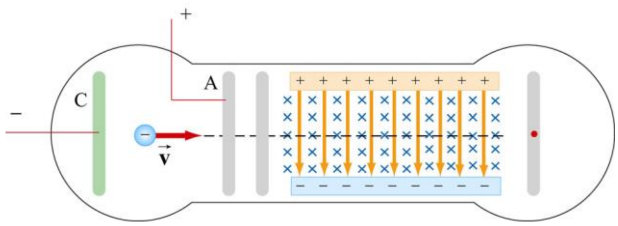


Figure 2.16 Thomson's apparatus

The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be $V_A - V_C = \Delta V$. The change in potential energy is equal to the external work done in accelerating the electrons: $\Delta U = W_{ext} = q\Delta V = -e\Delta V$. By energy conservation, the kinetic energy gained is $\Delta K = -\Delta U = mv^2/2$. Thus, the speed of the electrons is given by

$$v = \sqrt{\frac{2e\Delta V}{m}} \tag{2.32}$$

If the electrons further pass through a region where there exists a downward uniform electric field, the electrons, being negatively charged, will be deflected upward. However, if in addition to the electric field, a magnetic field directed into the page is also applied, then the electrons will experience an additional downward magnetic force $-e\vec{v} \times \vec{B}$. When the two forces exactly cancel, the electrons will move in a straight path. From Eq. (2.28), we see that when the condition for the cancellation of the two forces is given by $eE = evB$ which implies

$$v = \frac{E}{B} \tag{2.33}$$

In other words, only those particles with speed $v = E/B$ will be able to move in a straight line. Combining the two equations, we obtain

$$\frac{e}{m}=\frac{E^2}{2(\Delta V)B^2} \tag{2.34}$$

By measuring E, ΔV and B, the charge-to-mass ratio can be readily determined. The most precise measurement to date is $e/m = 1.758820174(71)\times 10^{3}\ \mathrm{C/kg}$.

Mass Spectrometer

Various methods can be used to measure the mass of an atom. One possibility is through the use of a mass spectrometer. The basic feature of a Bainbridge mass spectrometer is illustrated in Figure 2.17. A particle carrying a charge $+q$ is first sent through a velocity selector.

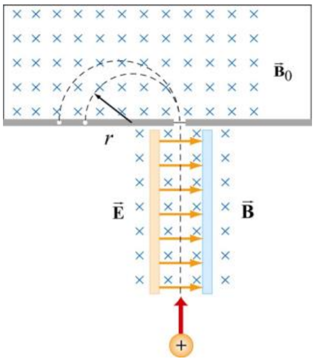


Figure 2.17 A Bainbridge mass spectrometer

The applied electric and magnetic fields satisfy the relation $E = vB$ so that the trajectory of the particle is a straight line. Upon entering a region where a second magnetic field $\vec{B}_0$ pointing into the page has been applied, the particle will move in a circular path with radius r and eventually strike the photographic plate. We have

$$r=\frac{mv}{qB_0} \tag{2.35}$$

Since $v = E/B$, the mass of the particle can be written as

$$m=\frac{qB_0 r}{v}=\frac{qB_0 Br}{E} \tag{2.36}$$

2.2 Sources of Magnetic Fields

2.2.1 Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current *I*, the magnetic field at any point *P* due to the current can be calculated by adding up the magnetic field contributions, $d\vec{B}$, from small segments of the wire $d\vec{s}$, (Figure 2.18).

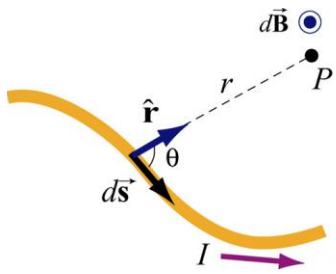


Figure 2.18 Magnetic field $d\vec{B}$ at point P due to a current-carrying element $Id\vec{s}$.

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $Id\vec{s}$.

Let *r* denote as the distance from the current source to the field point P, and $\hat{r}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{B}$, from the current source, $Id\vec{s}$,

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2}} \tag{2.37}$$

Where μ_0 is a constant called the permeability of free space:

$$\mu_0 = 4\pi\times10^{-7}\ \mathrm{T\cdot m/A} \tag{2.38}$$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element *dq*:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \tag{2.39}$$

Adding up these contributions to find the magnetic field at the point *P* requires integrating over the current source,

$$\vec{B} = \int_{wire} d\vec{B} = \frac{\mu_0 I}{4\pi}\int_{wire}\frac{d\vec{s}\times\hat{r}}{r^2} \tag{2.40}$$

The integral is a vector integral, which means that the expression for $\vec{B}$ is really three integrals, one for each component of $\vec{B}$. The vector nature of this integral appears in the cross product $Id\vec{s}\times\hat{r}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Example 2.2: Magnetic Field due to a Finite Straight Wire

A thin, straight wire carrying a current *I* is placed along the x-axis, as shown in Figure 2.19. Evaluate the magnetic field at point *P*. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point *P*.

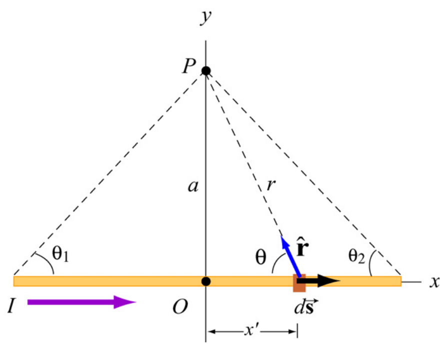


Figure 2.19 A thin straight wire carrying a current I.

Solutions:

This is a typical example involving the use of the Biot-Savart law. We solve the problem using the methodology summarized in the following.

(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{s}=\hat{x}dx'$ carrying current *I* in the *x*-direction. The location of this source is represented by $\vec{r}\,'=x'\hat{x}$.

(2) Field point (coordinates denoted with a subscript "*P*")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing *P* is $\vec{r}_P=a\hat{y}$.

(3) Relative position vector

The vector $\vec{r}=\vec{r}_P-\vec{r}\,'$ is a relative position vector which points from the source point to the field point. In this case, $\vec{r}=a\hat{y}-x'\hat{x}$, and the magnitude $r=|\vec{r}|=\sqrt{a^2+x'^2}$ is the distance between the source and *P*. The corresponding unit vector is given by

$$\hat{r}=\frac{\vec{r}}{r}=\frac{a\hat{y}-x'\hat{x}}{\sqrt{a^2+x'^2}}=\sin\theta\hat{y}-\cos\theta\hat{x}$$

(4) The cross product $d\vec{s}\times\hat{r}$

The cross product is given by

$$d\vec{s}\times\hat{r}=\left(dx'\hat{x}\right)\times\left(\sin\theta\hat{y}-\cos\theta\hat{x}\right)=\left(dx'\sin\theta\right)\hat{z}$$

(5) Write down the contribution to the magnetic field due to $Id\vec{s}$

The expression is

$$d\vec{B}=\frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\hat{r}}{r^2}=\frac{\mu_0 I}{4\pi}\frac{dx\sin\theta}{r^2}\hat{z}$$

which shows that the magnetic field at P will point in the $+\hat{z}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure 2.19, we have

$$\begin{cases} r=a/\sin\theta=a\csc\theta \\ x=a\cot\theta\Rightarrow dx=-a\csc^2\theta d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB=\frac{\mu_0 I}{4\pi}\frac{\left(-a\csc^2\theta d\theta\right)\sin\theta}{\left(a\csc\theta\right)^2}=-\frac{\mu_0 I}{4\pi a}\sin\theta d\theta$$

Integrating over all angles subtended from -θ_1 to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we get

$$B=-\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta d\theta=\frac{\mu_0 I}{4\pi a}\left(\cos\theta_2+\cos\theta_1\right)$$

The first term involving θ_2 accounts for the contribution from the portion along the +x axis, while the second θ_1 term involving contains the contribution from the portion along the –x axis. The two terms add!

Example 2.3: Magnetic Field due to a Circular Current Loop

A circular loop of radius R in the xy plane carries a steady current I, as shown in Figure 2.20.

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\mu}=\mu_z\hat{z}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e. $\vec{\mu}=-\mu_z\hat{z}$.

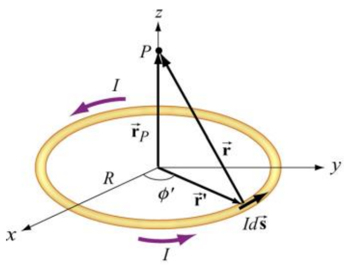


Figure 2.20 Magnetic field due to a circular loop carrying a steady current.

Solution:

(a) This is another example that involves the application of the Biot-Savart law.

(1) Source point

In Cartesian coordinates, the differential current element located at $\vec{r}\,'=R(\cos\phi'\hat{x}+\sin\phi'\hat{y})$ can be written as $Id\vec{s}=I\left(d\vec{r}\,'/d\phi'\right)d\phi'=IRd\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})$.

(2) Field point

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{r}_P=z\hat{z}$.

(3) Relative position vector $\vec{r}=\vec{r}_P-\vec{r}\,'$

The relative position vector is given by

$$\vec{r}=\vec{r}_P-\vec{r}\,'=-R\cos\phi'\hat{x}-R\sin\phi'\hat{y}+z\hat{z}$$

and its magnitude

$$r=|\vec{r}|=\sqrt{(-R\cos\phi')^2+(-R\sin\phi')^2+z^2}=\sqrt{R^2+z^2}$$

is the distance between the differential current element and P. Thus, the corresponding unit vector from $Id\vec{s}$ to P can be written as

$$\hat{r}=\frac{\vec{r}}{r}=\frac{\vec{r}_P-\vec{r}}{|\vec{r}_P-\vec{r}|}$$

(4) Simplifying the cross product

The cross product $d\vec{s}\times(\vec{r}_P-\vec{r}\,')$ can be simplified as

$$\begin{aligned}&d\vec{s}\times(\vec{r}_P-\vec{r}\,')=Rd\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})\times[-R\cos\phi'\hat{x}-R\sin\phi'\hat{y}+z\hat{z}]\\&=Rd\phi'[z\cos\phi'\hat{x}+z\sin\phi'\hat{y}+R\hat{z}]\end{aligned}$$

(5) Writing down $d\vec{B}$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times(\vec{r}_P - \vec{r}\,')\vec{r}}{|\vec{r}_P - \vec{r}\,'|^3}$$
$$= \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\hat{x} + z\sin\phi'\hat{y} + R\hat{z}}{(R^2+z^2)^{3/2}}d\phi'$$

(6) Carrying out the integration

Using the result obtained above, the magnetic field at P is

$$\vec{B} = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\hat{x} + z\sin\phi'\hat{y} + R\hat{z}}{(R^2+z^2)^{3/2}}d\phi'$$

The x and the y components can be readily shown to be zero:

$$B_x = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\cos\phi'd\phi' = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\sin\phi'\Big|_0^{2\pi} = 0$$

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\phi'd\phi' = -\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\cos\phi'\Big|_0^{2\pi} = 0$$

On the other hand, the z component is

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi}d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$$

Thus, we see that along the symmetric axis, Bz is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments.

The behavior of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown in Figure 2.21.

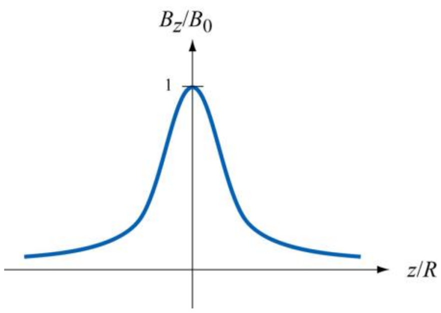


Figure 2.21 The ratio of the magnetic field, B_z / B_0, as a function of z / R

(b) If we place a magnetic dipole $\vec{\mu} = \mu_z\hat{z}$ at the point P, due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{F}_B = \nabla\left(\vec{\mu}\cdot\vec{B}\right) = \nabla\left(\mu_z B_z\right) = \mu_z\left(\frac{dB_z}{dz}\right)\hat{z}$$

Upon differentiating the field expression, we obtain

$$\vec{F}_B = -\frac{3\mu_z\mu_0 IR^2 z}{2\left(R^2+z^2\right)^{5/2}}\hat{z}$$

Thus the dipole is attracted toward the current-carrying ring. On the other hand, if the direction of the dipole is reversed, $\vec{\mu} = -\mu_z\hat{z}$, the resulting force will be repulsive.

2.2.2 Force Between Two Parallel Wires

We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert force on each other.

Consider two parallel wires separated by a distance a and carrying currents I_1 and I_2 in the $+x$-direction, as shown in Figure 2.21.

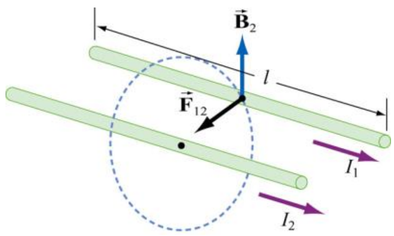


Figure 2.21 Force between two parallel wires

The magnetic force, $\vec{F}_{12}$, exerted on wire 1 by wire 2 may be computed as follows: Using the result from the previous example, the magnetic field lines due to I_2 going in the $+x$-direction are circles concentric with wire 2, with the field $\vec{B}_2$ pointing in the tangential direction. Thus, at an arbitrary point P on wire 1, we have $\vec{B}_2 = -\left(\mu_0 I_2/2\pi a\right)\hat{y}$, which points in the direction perpendicular to wire 1, as depicted in Figure 2.21. Therefore,

$$\vec{F}_{12} = I_1\vec{l}\times\vec{B} = I_1\left(l\hat{x}\right)\times\left(-\frac{\mu_0 I_2}{2\pi a}\hat{y}\right) = -\frac{\mu_0 I_1 I_2 l}{2\pi a}\hat{z} \qquad (2.41)$$

Clearly $\vec{F}_{12}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive.

2.2.3 Ampere's Law

Let us now divide a circular path of radius r into a large number of small length vectors $\Delta\vec{s} = \Delta s\hat{\varphi}$, that point along the tangential direction with magnitude Δs (Figure 2.22).

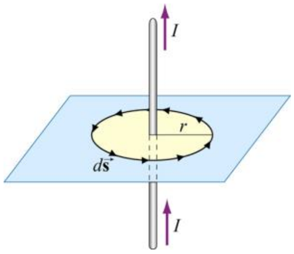


Figure 2.22 Amperian loop

In the limit $\Delta\vec{s} \to \vec{0}$, we obtain

$$\oint \vec{B} \cdot d\vec{s} = B\oint ds = \left(\frac{\mu_0 I}{2\pi r}\right)(2\pi r) = \mu_0 I \tag{2.42}$$

The result above is obtained by choosing a closed path, or an "Amperian loop" that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, as that shown in Figure 2.23

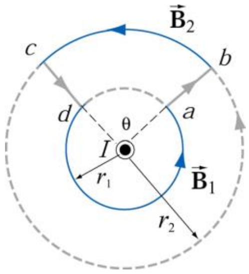


Figure 2.23 An Amperian loop involving two field lines

The line integral of the magnetic field around the contour *abcda* is

$$\begin{aligned} \oint_{abcda} \vec{B} \cdot d\vec{s} &= \oint_{ab} \vec{B} \cdot d\vec{s} + \oint_{bc} \vec{B} \cdot d\vec{s} + \oint_{cd} \vec{B} \cdot d\vec{s} + \oint_{da} \vec{B} \cdot d\vec{s} \\ &= 0 + B_2(r_2\theta) + 0 + B_1\left[r_1(2\pi - \theta)\right] \end{aligned} \tag{2.43}$$

where the length of arc *bc* is $r_2\theta$, and $r_1(2\pi - \theta)$ for arc *da*. The first and the third integrals vanish since the magnetic field is perpendicular to the paths of integration. With $B_1 = \mu_0 I / 2\pi r_1$ and $B_2 = \mu_0 I / 2\pi r_2$, the above expression becomes

$$\oint_{abcda} \vec{B} \cdot d\vec{s} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1}\left[r_1(2\pi - \theta)\right] = \mu_0 I \tag{2.44}$$

We see that the same result is obtained whether the closed path involves one or two magnetic field lines.

As shown in Example 2.2, in cylindrical coordinates (r, z, ϕ) with current flowing in the +z-axis, the magnetic field is given by $\vec{B} = (\mu_0 I / 2\pi r)\hat{\varphi}$. An arbitrary length element in the cylindrical coordinates can be written as

$$d\vec{s} = \hat{r}dr + \hat{\varphi}rd\varphi + \hat{z}dz \qquad (2.45)$$

which implies

$$\oint_{closed\ path} \vec{B}\cdot d\vec{s} = \oint_{closed\ path} \left(\frac{\mu_0 I}{2\pi r}\right) rd\varphi = \frac{\mu_0 I}{2\pi}\oint_{closed\ path} d\varphi = \mu_0 I \qquad (2.44)$$

In other words, the line integral of $\oint \vec{B}\cdot d\vec{s}$ around any closed Amperian loop is proportional to I_{enc}, the current encircled by the loop.

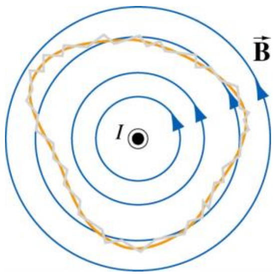


Figure 2.24 An Amperian loop of arbitrary shape.

The generalization to any closed loop of arbitrary shape (see for example, Figure 2.24) that involves many magnetic field lines is known as Ampere's law:

$$\boxed{\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}} \qquad (2.45)$$

Ampere's law in magnetism is analogous to Gauss's law in electrostatics.

Example 2.4: Field Inside and Outside a Current-Carrying Wire

Consider a long straight wire of radius *R* carrying a current *I* of uniform current density, as shown in Figure2.25. Find the magnetic field everywhere.

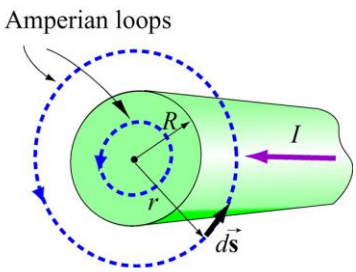


Figure 2.25 Amperian loops for calculating the $\vec{B}$ field of a conducting wire of radius *R*.

Solution:

(i) Outside the wire where $r \geq R$, the Amperian loop (circle 1) completely encircles the current, i.e., $I_{enc} = I$. Applying Ampere's law yields

$$\oint \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r) = \mu_0 I$$

which implies

$$B = \frac{\mu_0 I}{2\pi r}$$

(ii) Inside the wire where $r < R$, the amount of current encircled by the Amperian loop (circle 2) is proportional to the area enclosed, i.e.,

$$I_{enc} = \left(\frac{\pi r^2}{\pi R^2}\right) I$$

Thus, we have

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r) = \mu_0 I \left(\frac{\pi r^2}{\pi R^2}\right) \quad => \quad \boxed{B = \frac{\mu_0 I r}{2\pi R}}$$

We see that the magnetic field is zero at the center of the wire and increases linearly with r until $r = R$. Outside the wire, the field falls off as $1/r$. The qualitative behavior of the field is depicted in Figure 2.26 below:

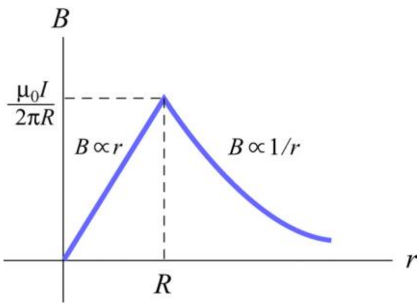


Figure 2.26 Magnetic field of a conducting wire of radius R carrying a steady current *I*.

2.2.4 Solenoid

A solenoid is a long coil of wire tightly wound in the helical form. Figure 2.27 shows the magnetic field lines of a solenoid carrying a steady current *I*. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter. For an "ideal" solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.

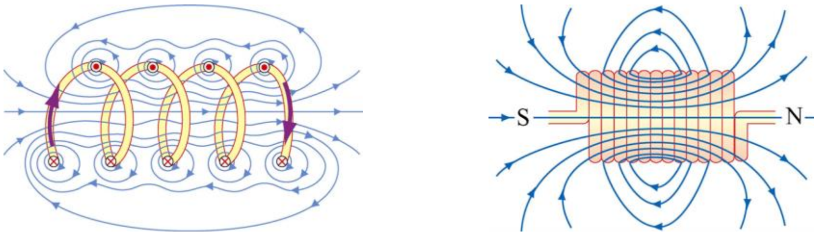


Figure 2.27 Magnetic field lines of a solenoid

We can use Ampere's law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure 2.28 To compute $\vec{B}$ we consider a rectangular path of length l and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{B}$ along this loop is

$$\begin{aligned}\oint \vec{B}\cdot d\vec{s} &= \oint_1 \vec{B}\cdot d\vec{s} + \oint_2 \vec{B}\cdot d\vec{s} + \oint_3 \vec{B}\cdot d\vec{s} + \oint_4 \vec{B}\cdot d\vec{s} \\ &= 0+0+Bl+0\end{aligned} \tag{2.46}$$

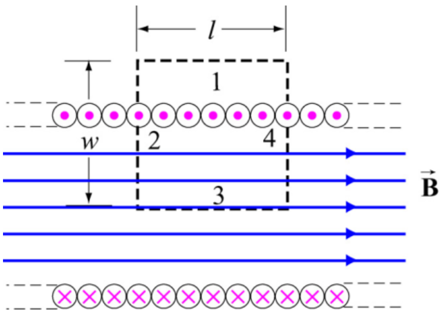


Figure 2.28 Amperian loop for calculating the magnetic field of an ideal solenoid.

In the above, the contributions along sides 2 and 4 are zero because $\vec{B}$ is perpendicular to $d\vec{s}$. In addition, $\vec{B}=\vec{0}$ along side 1 because the magnetic field is non-zero only inside the solenoid. On the other hand, the total current enclosed by the Amperian loop is $I_{enc} = NI$, where N is the total number of turns. Applying Ampere's law yields

$$\oint \vec{B}\cdot d\vec{s} = Bl = \mu_0 NI \tag{2.47}$$

or

$$B = \frac{\mu_0 NI}{l} = \mu_0 nI \tag{2.48}$$

where $n = N/l$ represents the number of turns per unit length.

2.3 Faraday's Law of Induction

The electric fields and magnetic fields considered up to now have been produced by stationary charges and moving charges (currents), respectively. Imposing an electric field on a conductor gives rise to a current which in turn generates a magnetic field. One could then inquire whether or not an electric field could be produced by a magnetic field. In 1831, Michael Faraday discovered that, by varying magnetic field with time, an electric field could be generated. The phenomenon is known as electromagnetic induction. Figure 2.29 illustrates one of Faraday's experiments.

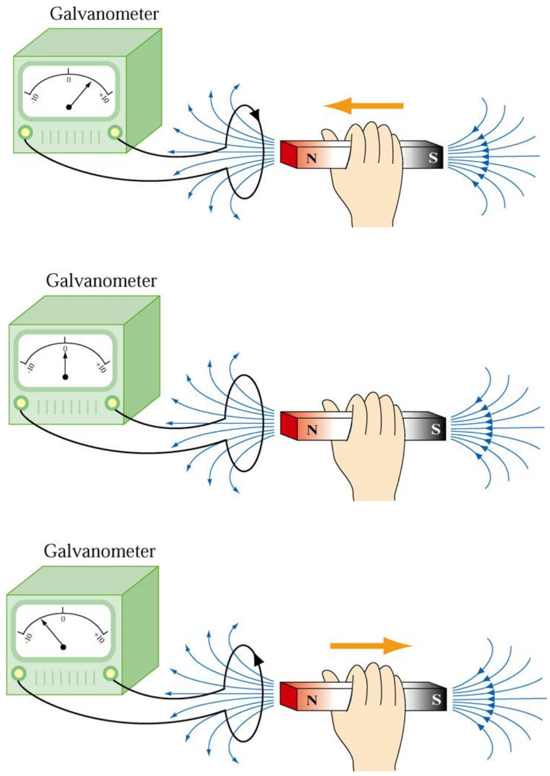


Figure 2.29 Electromagnetic induction

Faraday showed that no current is registered in the galvanometer when bar magnet is stationary with respect to the loop. However, a current is induced in the loop when a relative motion exists between the bar magnet and the loop. In particular, the galvanometer deflects in one direction as the magnet approaches the loop, and the opposite direction as it moves away. Faraday's experiment demonstrates that an electric current is induced in the loop by changing the magnetic field. The coil behaves as if it were connected to an *emf* source. Experimentally it is found that the induced *emf* depends on the rate of change of magnetic flux through the coil.

2.3.1 Magnetic Flux

Consider a uniform magnetic field passing through a surface S, as shown in Figure 2.30 below:

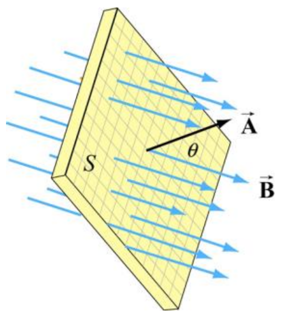


Figure 2.30 Magnetic flux through a surface

Let the area vector be $\vec{A} = A\hat{n}$, where A is the area of the surface and $\hat{n}$ its unit normal. The magnetic flux through the surface is given by

$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta \tag{2.49}$$

where θ is the angle between $\vec{B}$ and $\hat{n}$. If the field is non-uniform, Φ_B then becomes

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A} \tag{2.50}$$

The SI unit of magnetic flux is the weber (Wb):

$$1\,\mathrm{Wb} = 1\,\mathrm{T} \cdot \mathrm{m}^2$$

Faraday's law of induction may be stated as follows:

The induced *emf* in a coil is proportional to the negative of the rate of change of magnetic flux:

$$emf = -\frac{d\Phi_B}{dt} \tag{2.51}$$

For a coil that consists of N loops, the total induced *emf* would be *N* times as large:

$$emf = -N\frac{d\Phi_B}{dt} \tag{2.52}$$

Combining them together, we obtain, for a spatially uniform field $\vec{B}$,

$$emf = -\frac{d}{dt}\left(BA\cos\theta\right) = -\left(\frac{dB}{dt}\right)A\cos\theta - B\left(\frac{dA}{dt}\right)\cos\theta + BA\sin\theta\left(\frac{d\theta}{dt}\right) \tag{2.53}$$

Thus, we see that an emf may be induced in the following ways:

(i) by varying the magnitude of $\vec{B}$ with time (illustrated in Figure 2.31)

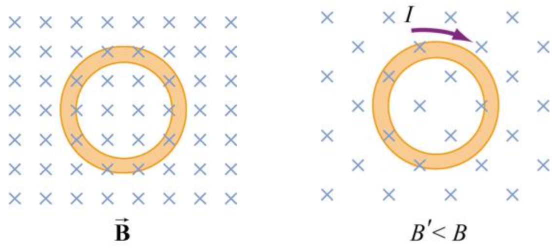


Figure 2.31 Inducing *emf* by varying the magnetic field strength

(ii) by varying the magnitude of $\vec{A}$, i.e., the area enclosed by the loop with time (illustrated in Figure 2.32)

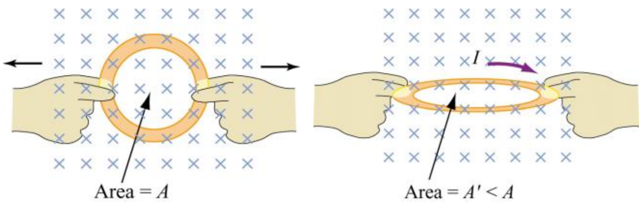


Figure 2.32 Inducing *emf* by changing the area of the loop

(iii) varying the angle between $\vec{B}$ and the area vector $\vec{A}$ with time (illustrated in Figure 2.33)

Figure 2.33 Inducing *emf* by varying the angle between $\vec{B}$ and $\vec{A}$.

2.3.2 Lenz's Law

The direction of the induced current is determined by Lenz's law:

The induced current produces magnetic fields which tend to oppose the change in magnetic flux that induces such currents.

To illustrate how Lenz's law works, let's consider a conducting loop placed in a magnetic field. We follow the procedure below:

1. Define a positive direction for the area vector $\vec{A}$.

2. Assuming that $\vec{B}$ is uniform, take the dot product of $\vec{B}$ and $\vec{A}$. This allows for the determination of the sign of the magnetic flux Φ_B.

3. Obtain the rate of flux change $d\Phi_B/dt$ by differentiation. There are three possibilities:

$$\frac{d\Phi_B}{dt}:\begin{cases} >0 \Rightarrow emf < 0 \\ <0 \Rightarrow emf > 0 \\ =0 \Rightarrow emf = 0 \end{cases}$$

4. Determine the direction of the induced current using the right-hand rule. With your thumb pointing in the direction of $\vec{A}$, curl the fingers around the closed loop. The induced current flows in the same direction as the way your fingers curl if $emf > 0$, and the opposite direction if $emf < 0$, as shown in Figure 2.34.

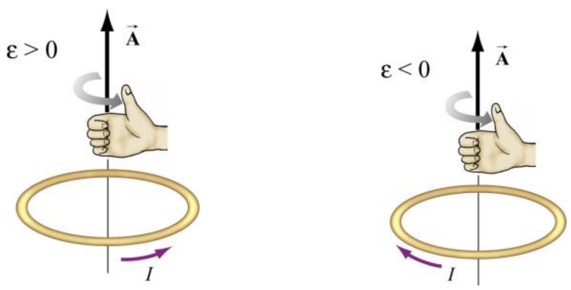


Figure 2.34 Determination of the direction of induced current by the right-hand rule

2.3.3 Motional EMF

Consider a conducting bar of length l moving through a uniform magnetic field which points into the page, as shown in Figure 2.35. Particles with charge q > 0 inside experience a magnetic force $\vec{F}_B = q\vec{v}\times\vec{B}$ which tends to push them upward, leaving negative charges on the lower end.

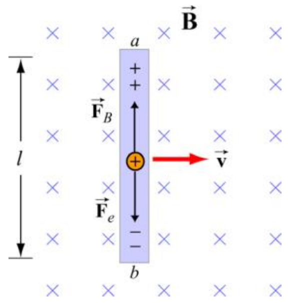


Figure 2.35 A conducting bar moving through a uniform magnetic field

The separation of charge gives rise to an electric field $\vec{E}$ inside the bar, which in turn produces a downward electric force $\vec{F}_e = q\vec{E}$. At equilibrium where the two forces cancel, we have $qvB = qE$, or $E = vB$. Between the two ends of the conductor, there exists apotential difference given by

$$V_{ab} = V_a - V_b = emf = El = Blv \tag{2.54}$$

Since *emf* arises from the motion of the conductor, this potential difference is called the motional *emf*. In general, motional *emf* around a closed conducting loop can be written as

$$emf = \oint \left(\vec{v}\times\vec{B}\right)\cdot d\vec{s} \tag{2.55}$$

where $d\vec{s}$ is a differential length element.

2.3.4 Induced Electric Field

we have seen that the electric potential difference between two points A and B in an electric field $\vec{E}$ can be written as

$$\Delta V = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{s} \tag{2.56}$$

When the electric field is conservative, as is the case of electrostatics, the line integral of $\vec{E}\cdot d\vec{s}$ is path-independent, which implies $\oint \vec{E}\cdot d\vec{s}$.

Faraday's law shows that as magnetic flux changes with time, an induced current begins to flow. What causes the charges to move? It is the induced *emf* which is the work done per unit charge. However, since magnetic field can do not work, the work done on the mobile charges must be electric, and the electric field in this situation cannot be conservative because the line integral of a conservative field must vanish. Therefore, we conclude that there is a non-conservative electric field $\vec{E}_{nc}$ associated with an induced *emf*:

$$emf = \oint \vec{E}_{nc}\cdot d\vec{s} \tag{2.57}$$

Combining with Faraday's law then yields

$$\boxed{\oint \vec{E}_{nc}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}} \tag{2.58}$$

The above expression implies that a changing magnetic flux will induce a non-conservative electric field which can vary with time. It is important to distinguish between the induced, non-conservative electric field and the conservative electric field which arises from electric charges.

As an example, let's consider a uniform magnetic field which points *into* the page and is confined to a circular region with radius *R*, as shown in Figure 2.36. Suppose the magnitude of $\vec{B}$ increases with time, *i.e.*, $dB/dt > 0$. Let's find the induced electric field everywhere due to the changing magnetic field.

Since the magnetic field is confined to a circular region, from symmetry arguments we choose the integration path to be a circle of radius *r*. The magnitude of the induced field $\vec{E}_{nc}$ at all points on a circle is the same. According to Lenz's law, the direction of $\vec{E}_{nc}$ must be such that it would drive the induced current to produce a magnetic field opposing the change in magnetic flux. With the area vector $\vec{A}$ pointing out of the page, the magnetic flux is negative or inward. With $dB/dt > 0$, the inward magnetic flux is increasing. Therefore, to counteract this change the induced current must flow counterclockwise to produce more outward flux. The direction of $\vec{E}_{nc}$ is shown in Figure 2.36.

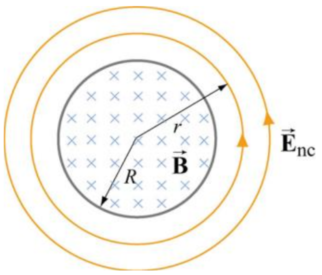


Figure 2.36 Induced electric field due to changing magnetic flux

Let's proceed to find the magnitude of $\vec{E}_{nc}$. In the region $r < R$, the rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}\left(\vec{B}\cdot\vec{A}\right) = \frac{d}{dt}(-BA) = -\left(\frac{dB}{dt}\right)\pi r^2 \tag{2.59}$$

Using Eq. (2.58), then we obtain

$$\oint \vec{E}_{nc}\cdot d\vec{s} = E_{nc}\left(2\pi r\right) = -\frac{d\Phi_B}{dt} = \left(\frac{dB}{dt}\right)\pi r^2 \tag{2.60}$$

which implies

$$E_{nc} = \frac{r}{2}\frac{dB}{dt} \tag{2.61}$$

Similarly, for $r > R$, the induced electric field may be obtained as

$$E_{nc} = \frac{R^2}{2r}\frac{dB}{dt} \tag{2.62}$$

A plot of E_{nc} as a function of r is shown in Figure 2.37.

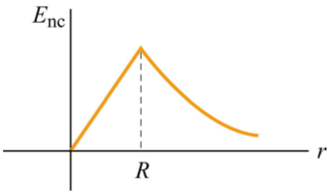
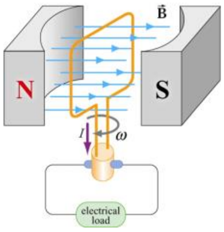


Figure 2.37 Induced electric field as a function of r

2.3.5 Generators

One of the most important applications of Faraday's law of induction is to generators and motors. A generator converts mechanical energy into electric energy, while a motor converts electrical energy into mechanical energy.



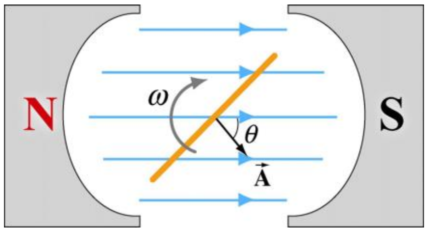


Figure 2.38 (a) A simple generator. (b) The rotating loop as seen from above.

Figure 2.38(a) is a simple illustration of a generator. It consists of an *N*-turn loop rotating in a magnetic field which is assumed to be uniform. The magnetic flux varies with time, thereby inducing an *emf*. From Figure 2.38(b), we see that the magnetic flux through the loop may be written as

$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta = BA\cos\omega t \tag{2.63}$$

The rate of change of magnetic flux is

$$\frac{d\Phi_B}{dt} = -BA\omega\sin\omega t \tag{2.64}$$

Since there are *N* turns in the loop, the total induced *emf* across the two ends of the loop is

$$emf = -N\frac{d\Phi_B}{dt} = NBA\omega\sin\omega t \tag{2.65}$$

If we connect the generator to a circuit which has a resistance R, then the current generated in the circuit is given by

$$I = \frac{|emf|}{R} = \frac{NBA\omega}{R}\sin\omega t \tag{2.66}$$

The current is an alternating current which oscillates in sign and has an amplitude $I_0 = NBA\omega/R$. The power delivered to this circuit is

$$P = I|emf| = \frac{(NBA\omega)^2}{R}\sin^2\omega t \tag{2.67}$$

On the other hand, the torque exerted on the loop is

$$\tau = \mu B\sin\theta = \mu B\sin\omega t \tag{2.68}$$

Thus, the mechanical power supplied to rotate the loop is

$$P_m = \tau\omega = \mu B\omega\sin\omega t \tag{2.69}$$

Since the dipole moment for the *N*-turn current loop is

$$\mu = NIA = \frac{N^2A^2B\omega}{R}\sin\omega t \tag{2.70}$$

the above expression becomes

$$P_m = \left(\frac{N^2 A^2 B\omega}{R} \sin \omega t \right) B\omega \sin \omega t = \frac{(NAB\omega)^2}{R} \sin^2 \omega t \tag{2.71}$$

As expected, the mechanical power put in is equal to the electrical power output.

2.3.6 Eddy Currents

We have seen that when a conducting loop moves through a magnetic field, current is induced as the result of changing magnetic flux. If a solid conductor were used instead of a loop, as shown in Figure 2.39, current can also be induced. The induced current appears to be circulating and is called an eddy current.

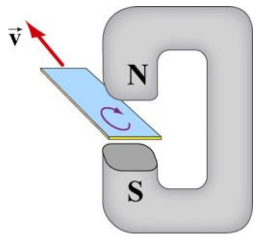


Figure 2.39 Appearance of an eddy current when a solid conductor moves through a magnetic field.

The induced eddy currents also generate a magnetic force that opposes the motion, making it more difficult to move the conductor across the magnetic field (Figure 2.40).

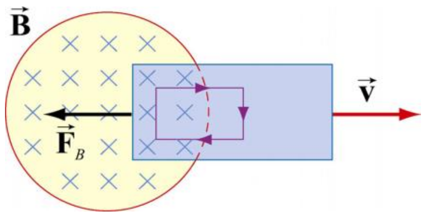


Figure 2.40 Magnetic force arising from the eddy current that opposes the motion of the conducting slab.

Since the conductor has non-vanishing resistance *R*, Joule heating causes a loss of power by an amount $P = emf^2/R$. Therefore, by increasing the value of *R*, power loss can be reduced. One way to increase *R* is to laminate the conducting slab, or construct the slab by using gluing together thin strips that are insulated from one another (see Figure 2.41(a)). Another way is to make cuts in the slab, thereby disrupting the conducting path (Figure 2.41(b)).

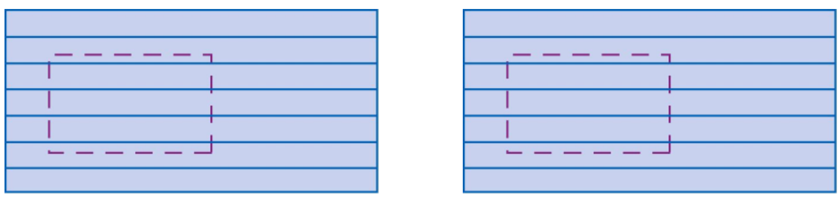


Figure 2.3.5.3 Eddy currents can be reduced by (a) laminating the slab, or (b) making cuts on the slab.

There are important applications of eddy currents. For example, the currents can be used to suppress unwanted mechanical oscillations. Another application is the magnetic braking systems in high-speed transit cars.

2.4 Additional Problems

See Reference [1].

Reference:

[1] MIT Physics 8.02 : <<Electricity & Magnetism>>, by Sen-ben Liao, Peter Dourmashkin, and John W. Belcher.

Finished by Zuojia on 24 Aug 2019