Problem Solving 4: Boundary Conditions

OBJECTIVES:

1. Understand the general boundary conditions for electromagnetic fields.

2. To learn the application of boundary conditions to electrostatic problems.

3. To learn how to use the method of image and the method of variable separation

REFERENCE: Chapter 4, Boundary Conditions

PROBLEM SOLVING STRATEGIES

A. Constitutive relations and boundary conditions

The differential *Maxwell's Equations* in matter:

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{D}=\rho_{free} & (Gauss\text{'s } Law) \\ \nabla\times\vec{E}=-\partial\vec{B}/\partial t & (Faraday\text{'s } Law) \\ \nabla\cdot\vec{B}=0 & (Magnetic\ Gauss\text{'s } Law) \\ \nabla\times\vec{H}=\vec{J}_{free}+\partial\vec{D}/\partial t & (Ampere\text{'s } Law) \end{array}}$$

with the *constitutive relations*

$$\boxed{\begin{array}{l} \vec{D}=\varepsilon\vec{E} \\ \vec{B}=\mu\vec{H} \end{array}}$$

The general *boundary conditions*:

$$\boxed{\begin{array}{l} \hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\rho_s \\ \hat{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0 \\ \hat{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \\ \hat{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{J}_s \end{array}}$$

B. Boundary conditions for static fields

For static fields ($\partial/\partial t\to 0$), we can define a *scalar potential* φ and a *vector potential* $\vec{A}$ to express the electric and magnetic fields:

$$\boxed{\begin{array}{l} \vec{E}=-\nabla\varphi \\ \vec{B}=\nabla\times\vec{A} \end{array}}$$

with $\nabla\cdot\vec{A}$ set to zero. The Maxwell's Equations are then transformed to Poisson's equations

$$\boxed{\begin{array}{l}\nabla^2\varphi = -\rho_{free}/\varepsilon \\ \nabla^2\vec{A} = -\mu\vec{J}_{free}\end{array}}$$

Usually, by introducing the scalar and vector potentials, we can transform *vector equations* into four separate *scalar equations*, so as to reduce the complexity in solution.

Table S1. The boundary conditions of static fields in media

	Source-free dielectrics	**Source-free magnets**
Boundary conditions	$E_{1t} = E_{2t}$ $D_{1n} = D_{2n}$	$H_{1t} = H_{2t}$ $B_{1n} = B_{2n}$
	$\varphi_1 = \varphi_2$ $\varepsilon_1 \dfrac{\partial\varphi_1}{\partial n} = \varepsilon_2 \dfrac{\partial\varphi_2}{\partial n}$	$\hat{z}\cdot\left(\nabla\times\vec{A}_1 - \nabla\times\vec{A}_2\right) = 0$ $\hat{z}\times\left(\dfrac{1}{\mu_1}\nabla\times\vec{A}_1 - \dfrac{1}{\mu_2}\nabla\times\vec{A}_2\right) = 0$

Here, the subscripts 't' and 'n' represent the transverse and normal components, respectively.

C. The method of images

The method of images for perfect electric conductor is illustrated in Figure S1.

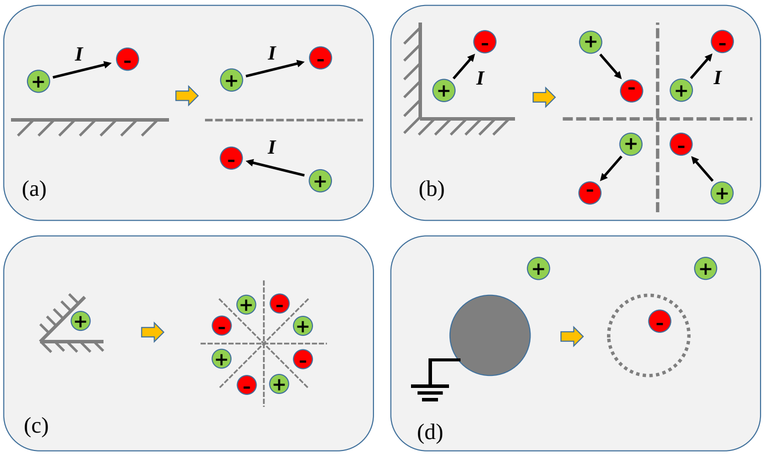

Figure S1 Image charges and currents for different shapes of conductor.

The method of images near dielectric is illustrated in Figure S2.

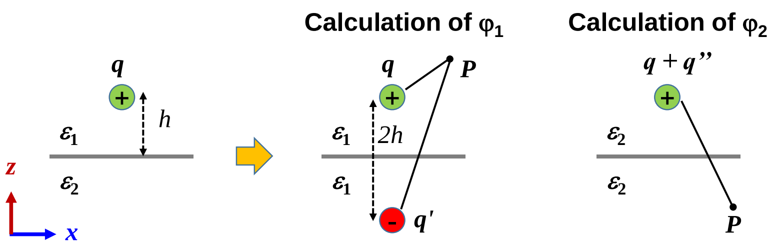

Figure S2 Image charges near dielectrics

The method of images near magnets is illustrated in Figure S3.

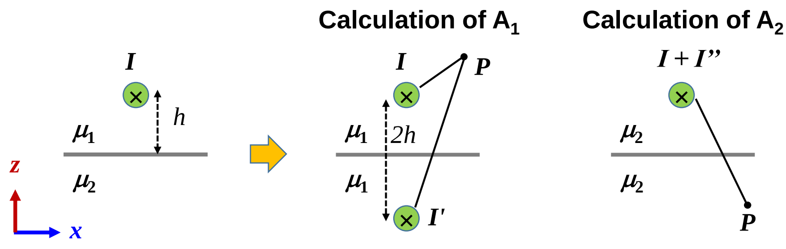

Figure S3 Image charges near magnets

PROBLEM 1: The method of image for conducting sphere.

A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.

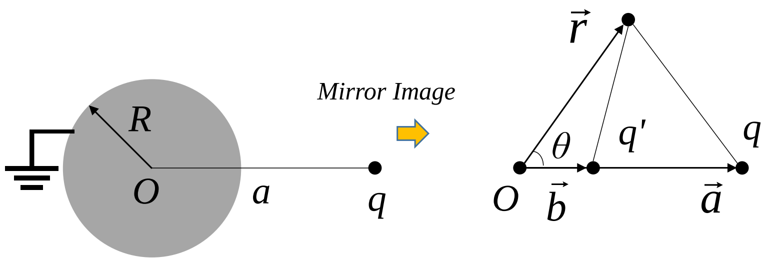

Figure S4 Image charge near a conducting sphere

Solution:

The potential due to the charges q and the image charge q' is:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{r}-\vec{a}|}+\frac{q'}{|\vec{r}-\vec{b}|}\right)$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|r\hat{r}-a\hat{x}|}+\frac{q'}{|r\hat{r}-b\hat{x}|}\right)$$

At $r = R$:

$$\varphi(r=R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|R\hat{r}-a\hat{x}|}+\frac{q'}{|R\hat{r}-b\hat{x}|}\right)$$

$$\varphi(r=R)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R\left|\hat{r}-\frac{a}{R}\hat{x}\right|}+\frac{q'}{b\left|\frac{R}{b}\hat{r}-\hat{x}\right|}\right)$$

From above equation it will be seen that the choices:

$$\frac{q}{R}=-\frac{q'}{b},\quad \frac{a}{R}=\frac{R}{b}$$

make $\varphi(r=R)=0$, for all possible values of $\hat{r}\cdot\hat{x}$. Hence the magnitude and position of the image charge are

$$q'=-\frac{R}{a}q,\quad b=\frac{R^2}{a}$$

The total potential outside the sphere is then written by:

$$\begin{aligned}\varphi&=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}+\frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right)\\&=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{R}{a\sqrt{r^2+\left(R^2/a\right)^2-2r\left(R^2/a\right)\cos\theta}}\right)\end{aligned}$$

The induced charge density on the surface of the sphere can be calculated from the normal derivative of φ at the surface (r = R):

$$\rho_s=\hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\hat{r}\cdot\vec{D}_1=-\hat{r}\cdot\varepsilon_0\nabla\varphi=-\varepsilon_0\frac{\partial\varphi}{\partial r}$$

$$\rho_s=\hat{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\hat{r}\cdot\vec{D}_1=-\hat{r}\cdot\varepsilon_0\nabla\varphi=-\varepsilon_0\frac{\partial\varphi}{\partial r}$$

$$\frac{\partial\varphi}{\partial r}=\frac{q}{4\pi\varepsilon_0}\left(\frac{-\left(r-a\cos\theta\right)}{\left(r^2+a^2-2ra\cos\theta\right)^{3/2}}+\frac{R\left(r-\left(R^2/a\right)\cos\theta\right)}{a\left(r^2+\left(R^2/a\right)^2-2r\left(R^2/a\right)\cos\theta\right)^{3/2}}\right)$$

$$\frac{\partial\varphi}{\partial r}(r=R)=\frac{q}{4\pi\varepsilon_0}\left(\frac{-(R-a\cos\theta)}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}+\frac{R\left(R-\left(R^2/a\right)\cos\theta\right)}{a\left(R^2+\left(R^2/a\right)^2-2R\left(R^2/a\right)\cos\theta\right)^{3/2}}\right)$$

$$==\frac{q}{4\pi\varepsilon_0}\left(\frac{-(R-a\cos\theta)}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}+\frac{\left(1-(R/a)\cos\theta\right)}{Ra\left(1+(R/a)^2-2R(1/a)\cos\theta\right)^{3/2}}\right)$$

$$=\frac{q}{4\pi\varepsilon_0 R}\left(\frac{-\left(R^2-Ra\cos\theta\right)}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}+\frac{\left(a^2-Ra\cos\theta\right)}{\left(a^2+R^2-2Ra\cos\theta\right)^{3/2}}\right)$$

$$=\frac{q}{4\pi\varepsilon_0 R}\left(\frac{a^2-R^2}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}\right)$$

$$\rho_s=-\varepsilon_0\frac{\partial\varphi}{\partial r}(r=R)=-\frac{q\left(a^2-R^2\right)}{4\pi R\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}$$

The total charge on the surface is:

$$Q=\oint_S\rho_s dS=-\int_0^{2\pi}\int_0^{\pi}\frac{q\left(a^2-R^2\right)R^2\sin\theta d\theta d\phi}{4\pi R\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}$$

$$=\frac{q\left(a^2-R^2\right)R}{2}\int_0^{\pi}\frac{1}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}d\cos\theta$$

$$=\frac{q\left(a^2-R^2\right)R}{2}\int_1^{-1}\frac{1}{\left(R^2+a^2-2Rax\right)^{3/2}}dx$$

$$=\frac{q\left(a^2-R^2\right)R}{2}\int_1^{-1}\frac{1}{Ra}d\left(\frac{1}{\sqrt{R^2+a^2-2Rax}}\right)$$

$$=\frac{q\left(a^2-R^2\right)}{2a}\left(\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right)$$

$$=\frac{q\left(a^2-R^2\right)}{2a}\left(\frac{1}{a+R}-\frac{1}{a-R}\right)$$

$$=\frac{q\left(a^2-R^2\right)}{2a}\left(\frac{-2R}{a^2-R^2}\right)$$

$$=-\frac{qR}{a}$$

D. Separation of variables in Cartesian coordinates

The method of separation of variables is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (φ) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: We look for solutions that are products of functions, each of which depends on only one of the coordinates. The algebraic details, however, can be formidable, so I'm going to develop the method through a sequence of examples. We'll deal with two-dimensional problems in Cartesian coordinates.

For the two-dimensional static field problem in Cartesian coordinates, the Laplace's equation is written by

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

The first step is to separate the variables:

$$\varphi(x,y)=X(x)Y(y)$$

Then we have

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2}=-k^2$$

The general solution is

$$\boxed{\varphi(x,y)=(A_0x+B_0)(C_0y+D_0)+\sum_{n=1}^{\infty}(A_n\sin k_nx+B_n\cos k_nx)(C_n\sinh k_ny+D_n\cosh k_ny)}$$

If we replace k^2 with $-k^2$, then the general solution is

$$\boxed{\varphi(x,y)=(A_0x+B_0)(C_0y+D_0)+\sum_{n=1}^{\infty}(A_n\sinh k_nx+B_n\cosh k_nx)(C_n\sin k_ny+D_n\cos k_ny)}$$

where $\sinh x=\left(e^x-e^{-x}\right)/2$ and $\cosh x=\left(e^x+e^{-x}\right)/2$.

PROBLEM 2.

Four infinitely-long grounded metal plates are along *z* direction. The potentials at the boundaries are illustrated in Figure S5. Find the potential distribution inside the cavity.

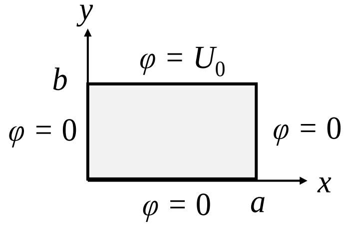

Figure S5

Solution:

Since the metallic plate is infinitely-long in the z direction, so we can use the variable separation in two-dimensional Cartesian coordinates.

The boundary conditions are

$$\begin{aligned}
&\varphi(0,y)=0, && (0\le y<b)\\
&\varphi(a,y)=0, && (0\le y<b)\\
&\varphi(x,0)=0, && (0\le x<a)\\
&\varphi(x,b)=U_0, && (0\le x<a)
\end{aligned}$$

Consider that $\varphi(x,y)=0$ at $x=0$ and $x=b$, then we choice the general solution form:

$$\varphi(x,y)=(A_0x+B_0)(C_0y+D_0)+\sum_{n=1}^{\infty}(A_n\sin k_nx+B_n\cos k_nx)(C_n\sinh k_ny+D_n\cosh k_ny)$$

Substitute the first boundary condition, we then obtain

$$\varphi(0,y)=B_0(C_0y+D_0)+\sum_{n=1}^{\infty}B_n(C_n\sinh k_ny+D_n\cosh k_ny)=0$$

all over the range of $0\le y<b$. So we need to set $B_n=0$ $(n=0,1,2,\cdots)$, then the solution is

$$\varphi(x,y)=A_0x(C_0y+D_0)+\sum_{n=1}^{\infty}(A_n\sin k_nx)(C_n\sinh k_ny+D_n\cosh k_ny)$$

Substitute the second boundary condition, we then obtain

$$\varphi(a,y)=A_0a(C_0y+D_0)+\sum_{n=1}^{\infty}(A_n\sin k_na)(C_n\sinh k_ny+D_n\cosh k_ny)=0$$

all over the range of $0\le y<b$. So we have $A_0=0$, and $A_n\sin k_na=0$. As a result

$$k_na=n\pi,\qquad (n=1,2,\cdots)$$

Then we have

$$\varphi(x,y)=\sum_{n=1}^{\infty}\left(A_n\sin\frac{n\pi}{a}x\right)\left(C_n\sinh\frac{n\pi}{a}y+D_n\cosh\frac{n\pi}{a}y\right)$$

Substitute the third boundary condition, we then obtain

$$\varphi(x,0)=\sum_{n=1}^{\infty}\left(A_n\sin\frac{n\pi}{a}x\right)(D_n)=0$$

all over the range of $0\le x<a$ So we must have $D_n=0$. Then the potential function becomes

$$\varphi(x,y)=\sum_{n=1}^{\infty}\left(A_n\sin\frac{n\pi}{a}x\right)\left(C_n\sinh\frac{n\pi}{a}y\right)=\sum_{n=1}^{\infty}\left(A_n'\sin\frac{n\pi}{a}x\right)\left(\sinh\frac{n\pi}{a}y\right)$$

Substitute the fourth boundary condition, we then obtain

$$\varphi(x,b)=\sum_{n=1}^{\infty}\left(A_n'\sin\frac{n\pi}{a}x\right)\left(\sinh\frac{n\pi}{a}b\right)=U_0$$

all over the range of $0\le x<a$. Then multiply both sides by $\sin\frac{m\pi}{a}x$ and integrate over $0\le x<a$, we have

$$\int_0^a\sin\frac{m\pi}{a}x\sum_{n=1}^{\infty}\left(A_n'\sin\frac{n\pi}{a}x\right)\left(\sinh\frac{n\pi}{a}b\right)dx=U_0\int_0^a\sin\frac{m\pi}{a}xdx$$

and

$$\frac{a}{2}A_m'\left(\sinh\frac{m\pi}{a}b\right)=U_0\int_0^a\sin\frac{m\pi}{a}xdx=-U_0\frac{a}{m\pi}\int_0^a d\cos\frac{m\pi}{a}x=\begin{cases}\frac{2a}{m\pi}U_0, & m\text{ is }odd\\ 0, & m\text{ is }even\end{cases}$$

Thus we get the coefficients

$$A_n'=\begin{cases}\frac{4U_0}{n\pi\sinh(n\pi b/a)}, & m\text{ is }odd\\ 0, & m\text{ is }even\end{cases}$$

As a result, the solution is

$$\varphi(x,y)=\frac{4U_0}{\pi}\sum_{n=1,3,\cdots}^{\infty}\left(\frac{4U_0}{n\sinh(n\pi b/a)}\sin\frac{n\pi}{a}x\right)\left(\sinh\frac{n\pi}{a}y\right)$$

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx=\begin{cases}0, & m\ne n\\ \pi, & m=n\end{cases}$$

$$\int_{-a}^{a}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)d\frac{\pi}{a}x=\begin{cases}0, & m\ne n\\ \pi, & m=n\end{cases}$$

$$\int_{-a}^{a}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx=\begin{cases}0, & m\ne n\\ a, & m=n\end{cases}$$

$$\int_0^a\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx=\frac{1}{2}\int_{-a}^{a}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx=\begin{cases}0, & m\ne n\\ \frac{a}{2}, & m=n\end{cases}$$