Problem Solving 2: Magnetostatics and Faraday's Law

OBJECTIVES:

1. To learn how to calculate the magnetic force & Torque.
2. To learn how to use Biot-Savart Law and Ampere's Law for calculating magnetic fields.
3. To calculate the rate of change of magnetic flux and the induced current by Faraday's Law and Lenz's Law

REFERENCE: Chapter 2, Magnetostatics and Faraday's Law

PROBLEM SOLVING STRATEGIES

A. Magnetic Force & Torque

The magnetic force acting on a charge q traveling at a velocity v in a magnetic field $\vec{B}$ is given by

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

The magnetic force acting on a wire of length $\vec{l}$ carrying a steady current I in a magnetic field $\vec{B}$ is

$$\vec{F}_B = I\vec{l} \times \vec{B}$$

The magnetic force $d\vec{F}_B$ generated by a small portion of current I of length $d\vec{s}$ in a magnetic field $\vec{B}$ is

$$d\vec{F}_B = Id\vec{s} \times \vec{B}$$

The total force is calculated as

$$\vec{F}_B = I\int_a^b d\vec{s} \times \vec{B}$$

PROBLEM 1: Bar Magnet in Non-Uniform Magnetic Field

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current I, as shown in the Figure 1. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

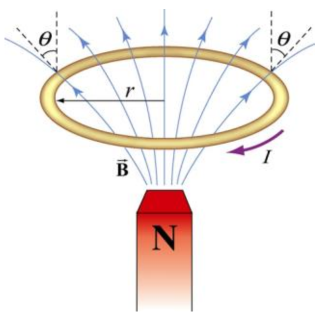

Figure 1 A bar magnet approaching a conducting ring

Solutions:

The magnetic force acting on a small differential current-carrying element $Id\vec{s}$ on the ring is given by $d\vec{F}_B = Id\vec{s} \times \vec{B}$, where $\vec{B}$ is the magnetic field due to the bar magnet. Using cylindrical coordinates ($\hat{r}$, $\hat{\varphi}$, $\hat{z}$) as shown in Figure 6, we have

$$d\vec{F}_B = I\left(-ds\hat{\varphi}\right) \times \left(B\sin\theta\hat{r} + B\cos\theta\hat{z}\right) = \left(IBds\right)\sin\theta\hat{z} - \left(IBds\right)\cos\theta\hat{r}$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the *z*-component.

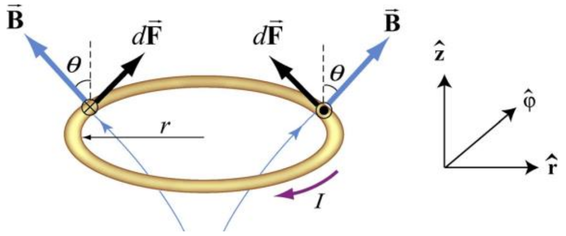

Figure 2 Magnetic force acting on the conducting ring

The total force acting on the ring then becomes

$$\vec{F}_B = \left(IB\sin\theta\right)\hat{z}\oint ds = \left(2\pi rIB\sin\theta\right)\hat{z}$$

The force points in the +*z* direction and therefore is repulsive.

B. Biot-Savart Law:

The law states that the magnetic field at a point P due to a length element $d\vec{s}$ carrying a steady current *I* located at $\vec{r}$ away is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{Id\vec{s} \times \vec{r}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $Id\vec{s}$, and the vector $\vec{r}\,'$ describing the position of $Id\vec{s}$.The magnitude $r' = \left|\vec{r}\,'\right|$ is the distance between $Id\vec{s}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point *P* is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{r}_P$ for the field point *P*. The quantity $r_P = \left|\vec{r}_P\right|$ is the distance between the origin and *P*.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{r} = \vec{r}_P - \vec{r}\,'$. The corresponding unit vector is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r}_P - \vec{r}\,'}{\left|\vec{r}_P - \vec{r}\,'\right|}$$

where $r = |\vec{r}| = |\vec{r}_P - \vec{r}\,'|$ is the distance between the source and the field point *P*.

(4) Calculate the cross product $d\vec{s} \times \hat{r}$ or $d\vec{s} \times \vec{r}$. The resultant vector gives the direction of the magnetic field $\vec{B}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained $d\vec{B}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{B}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

PROBLEM 2: Magnetic Field due to a Finite Straight Wire

A thin, straight wire carrying a current *I* is placed along the x-axis, as shown in Figure 3. Evaluate the magnetic field at point *P*. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point *P*.

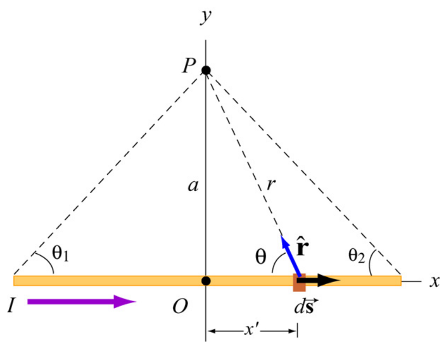

Figure 3 A thin straight wire carrying a current I.

Solutions:

This is a typical example involving the use of the Biot-Savart law. We solve the problem using the methodology summarized in the following.

(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{s} = \hat{x}dx'$ carrying current *I* in the *x*-direction. The location of this source is represented by $\vec{r}\,' = x'\hat{x}$.

(2) Field point (coordinates denoted with a subscript "*P*")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing *P* is $\vec{r}_P = a\hat{y}$.

(3) Relative position vector

The vector $\vec{r} = \vec{r}_P - \vec{r}\,'$ is a relative position vector which points from the source point to the field point. In this case, $\vec{r} = a\hat{y} - x'\hat{x}$, and the magnitude $r = |\vec{r}| = \sqrt{a^2 + x'^2}$ is the distance between the source and *P*. The corresponding unit vector is given by

$$\hat{r} = \frac{\vec{r}}{r} = \frac{a\hat{y} - x'\hat{x}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{y} - \cos\theta\,\hat{x}$$

(4) The cross product $d\vec{s} \times \hat{r}$

The cross product is given by

$$d\vec{s} \times \hat{r} = (dx'\,\hat{x}) \times (\sin\theta\,\hat{y} - \cos\theta\,\hat{x}) = (dx'\sin\theta)\,\hat{z}$$

(5) Write down the contribution to the magnetic field due to $Id\vec{s}$

The expression is

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\sin\theta}{r^2}\,\hat{z}$$

which shows that the magnetic field at *P* will point in the $+\hat{z}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables *θ*, *x* and *r* are not independent of each other. In order to complete the integration, let us rewrite the variables *x* and *r* in terms of *θ*. From Figure 3, we have

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \Rightarrow dx = -a\csc^2\theta\, d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi}\frac{(-a\csc^2\theta\, d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we get

$$B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second θ_1 term involving contains the contribution from the portion along the $-x$ axis. The two terms add!

C. Ampere's law:

Ampere's law states that the line integral of $\vec{B}\cdot d\vec{s}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{B}\cdot d\vec{s}$ around the closed loop.

(4) Equate $\oint \vec{B}\cdot d\vec{s}$ with $\mu_0 I_{enc}$ and solve for $\vec{B}$.

PROBLEM 3: Two Infinitely Long Wires

Consider two infinitely long wires carrying currents are in the –*x* direction.

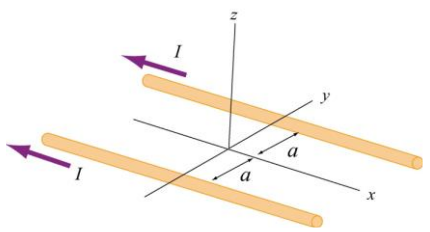

Figure 4 Two infinitely long wires

(a) Plot the magnetic field pattern in the *yz*-plane.

(b) Find the distance d along the *z*-axis where the magnetic field is a maximum.

Solutions:

(a) The magnetic field lines are shown in Figure 5. Notice that the directions of both currents are into the page.

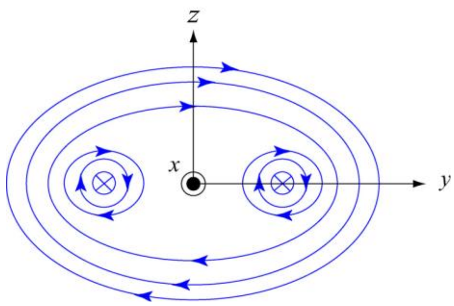

Figure 5 Magnetic field lines of two wires carrying current in the same direction

(b) The magnetic field at (0, 0, z) due to wire 1 on the left is, using Ampere's law:

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}$$

Since the current is flowing in the –x-direction, the magnetic field points in the direction of the cross product

$$(-\hat{x})\times\hat{r}_1 = (-\hat{x})\times(\cos\theta\hat{y}+\sin\theta\hat{z}) = \sin\theta\hat{y}-\cos\theta\hat{z}$$

Thus, we have

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}(\sin\theta\hat{y}-\cos\theta\hat{z})$$

For wire 2 on the right, the magnetic field strength is the same as the left one: $B_1 = B_2$. However, its direction is given by

$$\vec{B} = \vec{B}_1+\vec{B}_2 = \frac{\mu_0 I\sin\theta}{\pi\sqrt{a^2+z^2}}\hat{y} = \frac{\mu_0 Iz}{\pi(a^2+z^2)}\hat{y}$$

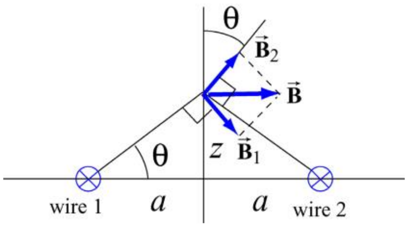

Figure 6 Superposition of magnetic fields due to two current sources

To locate the maximum of B, we set $dB / dz = 0$ and find

$$\frac{dB}{dz} = \frac{\mu_0 I}{\pi}\left(\frac{1}{a^2+z^2}-\frac{2z^2}{(a^2+z^2)^2}\right) = \frac{\mu_0 I}{\pi}\frac{a^2-z^2}{(a^2+z^2)^2}$$

which gives

$$z = a$$

Thus, at $z = a$, the magnetic field strength is a maximum, with a magnitude

$$B_{\max} = \frac{\mu_0 I}{2\pi a}$$

D. Faraday's Law and Lenz's Law

A changing magnetic flux induces an *emf*

$$emf = -N\frac{d\Phi_B}{dt}$$

according to the Faraday's law of induction. For a conductor which forms a closed loop, the *emf* sets up an induced current $I = |emf| / R$, where R is the resistance of the loop. To compute the induced current and its direction, we follow the procedure below:

(1) For the closed loop of area on a plane, define an area vector $\vec{A}$ and let it point in the direction of your thumb, for the convenience of applying the right-hand rule later. Compute the magnetic flux through the loop using

$$\Phi_B = \begin{cases} \vec{B}\cdot\vec{A} & (\vec{B} \text{ is uniform}) \\ \iint \vec{B}\cdot d\vec{A} & (\vec{B} \text{ is non-uniform}) \end{cases}$$

Determine the sign of Φ_B.

(2) Evaluate the rate of change of magnetic flux $d\Phi_B/dt$. Keep in mind that change could be caused by

(i) changing the magnetic field $dB/dt \neq 0$,

(ii) changing the loop area if the conductor is moving ($dA/dt \neq 0$), or

(iii) changing the orientation of the loop with respect to the magnetic field ($d\theta/dt \neq 0$).

Determine the sign of $d\Phi_B/dt$.

(3) The sign of the induced *emf* is the opposite of that of $d\Phi_B/dt$. The direction of the induced current can be found by using Lenz's law.

(4)

PROBLEM 4: Rectangular Loop Near a Wire

An infinite straight wire carries a current I is placed to the left of a rectangular loop of wire with width w and length l, as shown in Figure 7

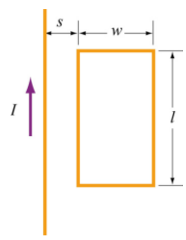

Figure 7 Rectangular loop near a wire

(a) Determine the magnetic flux through the rectangular loop due to the current *I*.

(b) Suppose that the current is a function of time with $I(t) = a + bt$, where a and b are positive constants. What is the induced emf in the loop and the direction of the induced current?

Solutions:

(a) Using Ampere's law:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$$

the magnetic field due to a current-carrying wire at a distance *r* away is

$$B = \frac{\mu_0 I}{2\pi r}$$

The total magnetic flux Φ_B through the loop can be obtained by summing over contributions from all differential area elements $dA = l dr$:

$$\Phi_B = \int d\Phi_B = \int \vec{B}\cdot d\vec{A} = \frac{\mu_0 I l}{2\pi}\int_s^{s+w}\frac{dr}{r} = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{s+w}{s}\right)$$

Note that we have chosen the area vector to point into the page, so that $\Phi_B > 0$.

(b) According to Faraday's law, the induced *emf* is

$$emf = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\frac{\mu_0 I l}{2\pi}\ln\left(\frac{s+w}{s}\right)\right] = \frac{\mu_0 l}{2\pi}\ln\left(\frac{s+w}{s}\right)\frac{dI}{dt} = -\frac{\mu_0 b l}{2\pi}\ln\left(\frac{s+w}{s}\right)$$

where we have used $dI/dt = b$.

The straight wire carrying a current *I* produces a magnetic flux into the page through the rectangular loop. By Lenz's law, the induced current in the loop must be flowing *counterclockwise* in order to produce a magnetic field out of the page to counteract the increase in inward flux.