Problem Solving 1: Electrostatics

OBJECTIVES:

1. To look at the meaning of linear, area, and volume charge densities.

2. To calculate the electric field and the electric potential from Continuous Charge Distributions.

3. To calculate the electric field from Gauss's law.

REFERENCE: Chapter 1, Electrostatics

PROBLEM SOLVING STRATEGIES

A. Calculating Electric field (Continuous Charge Distributions)

In order to calculate the electric field created by a continuous charge distribution we must break the charge into a number of small pieces *dq*, each of which create an electric field $d\vec{E}$. For example, if the charge is to be broken into point charges, we can write:

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}$$

where r is the distance from dq to P and $\hat{r}$ is the corresponding unit vector. In general use the following steps

(1) Break your charge distribution into small pieces *dq*.

(2) Write out the appropriate $d\vec{E}$ for the dq. For example, for point charges we will use $d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$.

(3) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda dl & \text{(length)} \\ \sigma dA & \text{(area)} \\ \rho dV & \text{(volume)} \end{cases}$$

depending on whether the charge is distributed over a length, an area, or a volume.

(4) Substitute dq into the expression for $d\vec{E}$

(5) Specify an appropriate coordinate system (Cartesian, cylindrical or spherical) and express the differential element (dl, dA or dV) and r in terms of the coordinates.

	Cartesian (x, y, z)	Cylindrical (ρ, φ, z)	Spherical (r, θ, φ)
dl	dx, dy, dz	$d\rho$, $\rho d\varphi$, dz	dr, $rd\theta$, $r\sin\theta d\varphi$

dA	*dydz*, *dzdx*, *dxdy*	*ρdφdz*, *dρdz*, *ρdφdρ*	r^2*sinθdθdφ*, *rsinθdφdr*, *rdrdθ*,
dV	*dxdydz*	*ρdρdφdz*	r^2*sinθdrdθdφ*

Table 1. Different elements of length, area and volume in different coordinates

(6) Rewrite $d\vec{E}$ in terms of the integration variable, and apply symmetry argument to identify non-vanishing component(s) of the electric field.

(7) Complete the integration to obtain $\vec{E}$

PROBLEM 1: Electric Field of an Arc

A thin rod with a uniform charge per unit length λ is bent into the shape of an arc of a circle of radius R. The arc subtends a total angle $2\theta_0$. What's the electric field at the origin O?

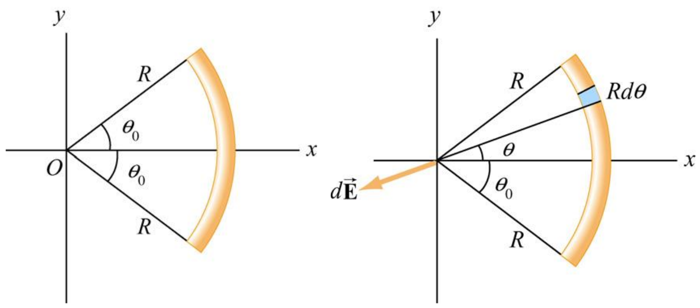

Figure 1 (a) Geometry of charged source. (b) Charge element *dq*

Solution:

Consider a differential element of length $dl = Rd\theta$, which makes an angle θ with the x-axis. The amount it carries is $dq = \lambda dl = \lambda R d\theta$.

Write out the appropriate $d\vec{E}$:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\left(-\cos\theta\hat{x} - \sin\theta\hat{y}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\theta}{R}\left(-\cos\theta\hat{x} - \sin\theta\hat{y}\right)$$

Integrating over the angle from $-\theta_0$ to $+\theta_0$, we have

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}\int_{-\theta_0}^{+\theta_0} d\theta\left(-\cos\theta\hat{x} - \sin\theta\hat{y}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right)\Big|_{-\theta_0}^{+\theta_0} = -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\sin\theta_0}{R}\hat{x}$$

We see that the electric field only has the x-component, as required by a symmetry argument.

(i) $\theta_0 = \pi$ (a circular ring). In this case, $\vec{E} = 0$

(ii) $\theta_0 = \pi/2$ (a semicircle). $\vec{E} = -\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{R}\hat{x} = -\frac{Q}{2\pi^2\varepsilon_0 R^2}\hat{x}$,

where $Q = \lambda\pi R$ is the total charge on the semicircle.

(iii) For very small θ_0, $\sin\theta_0 \approx \theta_0$, we recover the point-charge limit:

$$\vec{E} \approx -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\theta_0}{R}\hat{x} = -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda\theta_0 R}{R^2}\hat{x} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\hat{x}\,, \quad Q = \lambda\left(2R\theta_0\right)$$

B. Calculating Electric Potential

Unlike electric field, electric potential is a scalar quantity. For the discrete distribution, we apply the superposition principle and sum over individual contributions:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

For the continuous distribution, we must evaluate the integral

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

In analogy to the case of computing the electric field, we use the following steps to complete the integration:

(1) Start with $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda dl & \text{(length)} \\ \sigma dA & \text{(area)} \\ \rho dV & \text{(volume)} \end{cases}$$

(3) Substitute dq into the expression for dV

(4) Specify an appropriate coordinate system (Cartesian, cylindrical or spherical) and express the differential element (dl, dA or dV) and r in terms of the coordinates.

(5) Rewrite dV in terms of the integration variable.

(6) Complete the integration to obtain V.

Using the result obtained for V, one may calculate the electric field by $\vec{E} = -\nabla V$.

C. Gauss's Law

The electric field can be computed by the Gauss's law:

$$\Phi_E = \oiint_S \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

The procedures are summarized as following:

(1) Identify the symmetry of the problem (cylindrical, planar, or spherical)

(2) Determine the direction of $\vec{E}$.

(3) Divide the space into different regions

(4) Choose Gaussian surface

(5) Calculate electric flux

(6) Calculate enclosed charge q_{in}

(7) Apply Gauss's law $\Phi_E = q_{in}/\varepsilon_0$ to find $\vec{E}$

PROBLEM 2: Non-Conducting Solid Sphere with a Cavity

A sphere of radius $2R$ is made of a non-conducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius R is then carved out from the sphere, as shown in the figure below. Compute the electric field within the cavity.

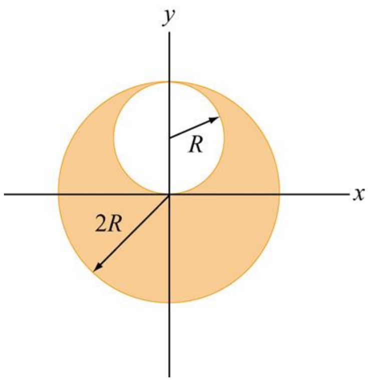

Figure 2 Non-conducting solid sphere with a cavity

Solution:

(1) Consider the problem as a superposition of a positively charged sphere (with the radius $2R$) and a negatively charge sphere (with the radius R).

(2) For the spherical symmetry, we apply the Gauss's law within the positively charged sphere:

$$\oiint_S \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{1}{\varepsilon_0} \rho \cdot \frac{4\pi}{3} r^3$$

then we get the electric field generated

$$\vec{E} = \frac{\rho}{3\varepsilon_0} r\hat{r} = \frac{\rho}{3\varepsilon_0} \vec{r}$$

(3) Similarly, we get the electric field within the negatively-charged sphere (the cavity with the charge density of $-\rho$)

$$\vec{E}' = -\frac{\rho}{3\varepsilon_0} \vec{r}'$$

where $\vec{r}\,'$ is the vector from the center of the cavity to the space position in consideration.

(4) Combine two electric field together by using the superposition principle.

$$\vec{E}_{total} = \vec{E} + \vec{E}\,' = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - \vec{r}\,'\right)$$

In our case, $\vec{r} - \vec{r}\,' = R\hat{y}$, thus we the total electric field within the cavity is

$$\vec{E}_{total} = \frac{\rho}{3\varepsilon_0} R\hat{y}$$