Chapter 1: Electrostatics

1.1 Fields

"… In order therefore to appreciate the requirements of the science [of electromagnetism], the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress …"

James Clerk Maxwell [1855]

Classical electromagnetic field theory emerged in more or less complete form in 1873 in James Clerk Maxwell's A Treatise on Electricity and Magnetism. In this theory, electromagnetic fields are the mediators of the interaction between material objects.

In the field theory view, although the two objects are not in direct contact with one another, they are in direct contact with a medium or mechanism that exists between them. The force between the objects is transmitted (at a finite speed) by stresses induced in the intervening space by the presence of the objects. This is the concept of "action by continuous contact" in "field theory." The "contact" is provided by a stress, or "field," induced in the space between the objects by their presence.

This is the essence of field theory. In this first chapter of your introduction to field theory, we discuss what a field is, and how we represent fields. We begin with scalar fields.

1.1.1 Scalar Fields

A scalar field is a function that gives us a single value of some variable for every point in space. As an example, the image in Figure 1.1 shows the nighttime temperatures measured by the Thermal Emission Spectrometer instrument on the Mars Global Surveyor (MGS).

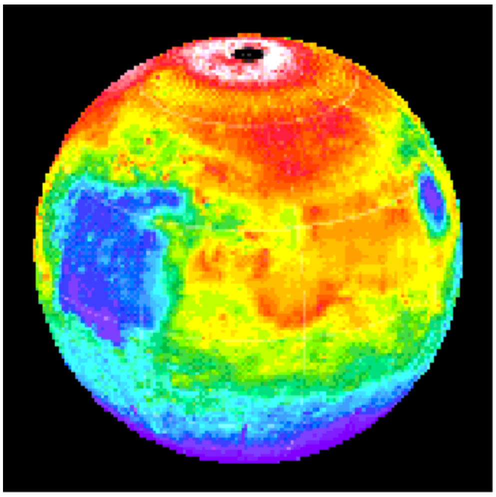


Figure 1.1 Nighttime temperature map for Mars

This map, however, is limited to representing only the temperature on a two-dimensional surface and thus, it does not show how temperature varies as a function of altitude.

Figure 1.2 illustrates the variation of temperature as a function of height above the surface of the Earth.

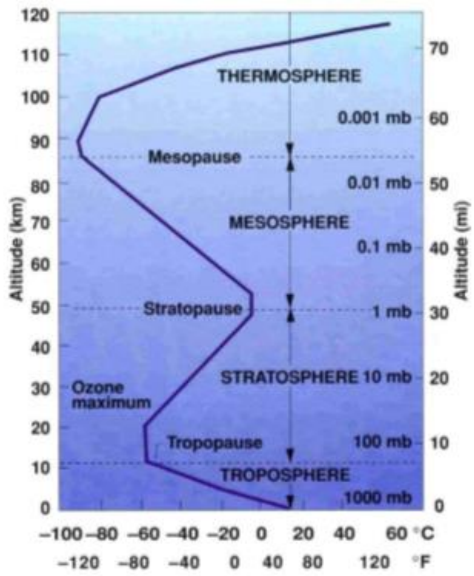


Figure 1.2 Atmospheric temperature variation as a function of altitude above the Earth's surface

We can simply represent the temperature variation by a mathematical function. For the Earth we shall use spherical coordinates (r, θ, φ) with the origin chosen to coincide with the center of the Earth. The temperature function T (r, θ, φ) is an example of a "scalar field."

1.1.2 Vector Fields

A vector is a quantity which has both a magnitude and a direction in space. Vectors are used to describe physical quantities such as velocity, momentum, acceleration and force, associated with an object.

If we try to analyze the motion of continuous bodies such as fluids, a velocity vector then needs to be assigned to every point in the fluid at any instant in time. Each vector describes the direction and magnitude of the velocity at a particular point and time. Figure 1.3 depicts a scenario of the variation of the jet stream, which is the wind velocity as a function of position.

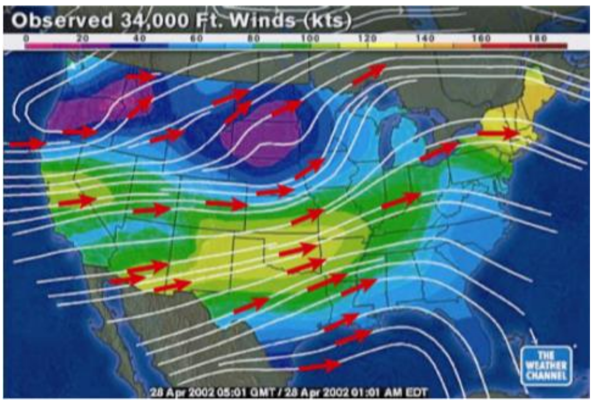


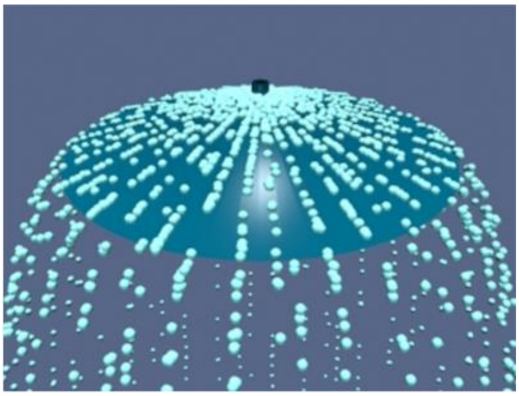
Figure 1.3 Jet stream with arrows indicating flow velocity. The "streamlines" are formed by joining arrows from head to tail.

In general, a vector field $\vec{F}(x,y,z)$ can be written as

$$\vec{F}(x,y,z) = \hat{x}F_x(x,y,z) + \hat{y}F_y(x,y,z) + \hat{z}F_z(x,y,z) \tag{1.1}$$

Below we use fluids to examine the properties associated with a vector field since fluid flows are the easiest vector fields to visualize.

In Figure 1.4, we represent the fluid by a finite number of particles. In Figure1.4(a), particles (fluid elements) appear at the center of a cone (a "source") and then flow downward under the effect of gravity. We also call this a diverging flow, since the particles appear to "diverge" from the creation point. Figure 1.4(b) is the converse of this, a converging flow, or a "sink" of particles.



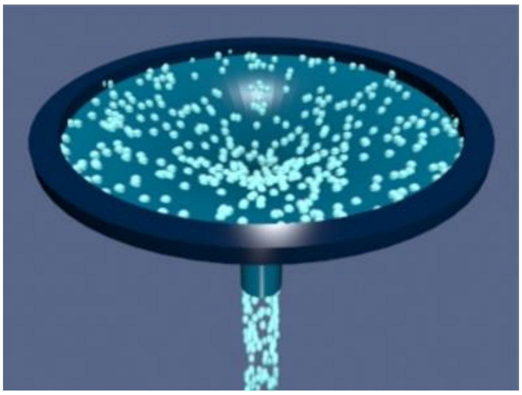


Figure 1.4 (a) An example of a source of particles and the flow associated with a source, (b) An example of a sink of particles and the flow associated with a sink.

Another representation of a diverging flow is in depicted in Figure 1.5.

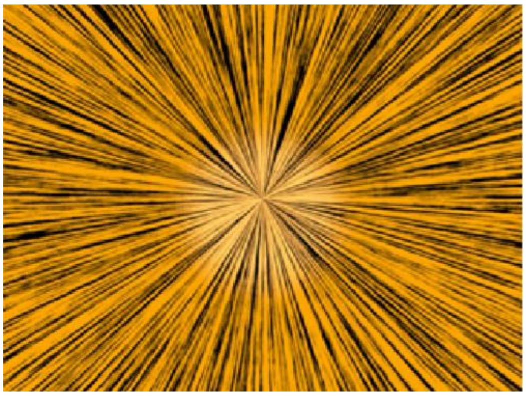


Figure 1.5 Representing the flow field associated with a source using textures.

1.1.3 Gravitational Field

The gravitational field of the Earth is another example of a vector field. According to Newton's universal law of gravitation, the gravitational force between two masses m and M is given by

$$\vec{F}_g = -G\frac{Mm}{r^2}\hat{r} \tag{1.2}$$

where r is the distance between the two masses and $\hat{r}$ is the unit vector located at the position of m that points from M towards m. The constant of proportionality is the gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2 / kg^2}$.

1.1.4 Electric Field

The interaction between electric charges at rest is called the electrostatic force. Consider an object which has charge Q. A "test charge" that is placed at a point P, a distance r from Q will experience a Coulomb force:

$$\vec{F}_e = k_e \frac{Qq}{r^2}\hat{r} \tag{1.3}$$

where $\hat{r}$ is the unit vector that points from Q to q. The constant of proportionality $k_e = 9.0 \times 10^9\ \mathrm{N \cdot m^2 / C^2}$ is called the Coulomb constant. The electric field at P is defined as

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}_e}{q} = k_e \frac{Q}{r^2}\hat{r} \tag{1.4}$$

In terms of the field concept, we may say that the charge Q creates an electric field $\vec{E}$ which exerts a force $\vec{F}_e = q\vec{E}$ on q.

1.1.5 Magnetic Field

Magnetic field is another example of a vector field. If we place some compasses near a bar magnet, the needles will align themselves along the direction of the magnetic field, as shown in Figure 1.6

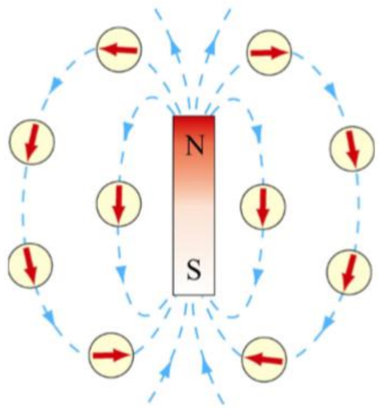


Figure 1.6 Magnets attracting and repelling

The Earth's magnetic field behaves as if there were a bar magnet in it (Figure 1.7). Note that the south pole of the magnet is located in the northern hemisphere.

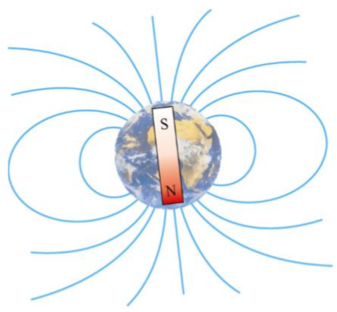


Figure 1.7 Magnetic field of the Earth

1.2 Coulomb's Law

1.2.1 Electric Charge

There are two types of observed electric charge, which we designate as positive and negative. Like charges repel and opposite charges attract each other. The unit of charge is called the Coulomb (C).

The smallest unit of "free" charge known in nature is the negative charge of an electron or the positive charge of proton, which has a magnitude of

$$e = 1.602 \times 10^{-19}\ \mathrm{C} \tag{1.5}$$

Charge can neither be created nor destroyed, but it can be transferred from one body to another.

1.2.2 Coulomb's Law

Consider a system of two point charges, q_1 and q_2, separated by a distance r in vacuum. The force exerted by q_1 on q_2 is given by Coulomb's law:

$$\vec{F}_{12} = k_e \frac{q_1 q_2}{r^2}\hat{r} \tag{1.6}$$

where k_e is the Coulomb constant, and $\hat{r} = \vec{r}/r$ is a unit vector directed from q_1 to q_2, as illustrated in Figure 1.8.

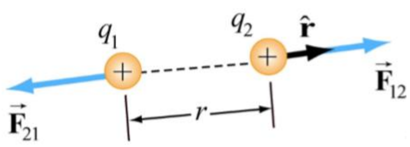


Figure 1.8 Coulomb interaction between two charges

the Coulomb constant k_e is given by

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.9875 \times 10^9\ \mathrm{N \cdot m^2 / C^2} \tag{1.7}$$

where

$$\varepsilon_0 = \frac{1}{4\pi\left(8.99 \times 10^9\ \mathrm{N \cdot m^2 / C^2}\right)} = 8.85 \times 10^{-12}\ \mathrm{C^2 / N \cdot m^2} \tag{1.8}$$

is known as the "permittivity of free space." Similarly, the force on q_1 due to q_2 is given by $\vec{F}_{12} = -\vec{F}_{21}$. This is consistent with Newton's third law.

As an example, consider a hydrogen atom in which the proton (nucleus) and the electron are separated by a distance r = 5.3 × 10^{-11} m. The electrostatic force between the two particles is approximately $F_e = k_e e^2 / r^2 = 8.2\times10^{-8}$ N. On the other hand, one may show that the gravitational force is only $F_g = 3.6\times10^{-47}$ N. Thus, gravitational effect can be neglected when dealing with electrostatic forces!

1.2.3 Principle of Superposition

Coulomb's law applies to any pair of point charges. When more than two charges are present, the net force on any one charge is simply the vector sum of the forces exerted on it by the other charges. For example, if three charges are present, the resultant force experienced by q_3 due to q_1 and q_2 will be

$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23} \tag{1.9}$$

Example 1.1: Three Charges

Three charges are arranged as shown in Figure 1.9. Find the force on the charge q_3 assuming that q_1 = +6.0×10^{-6} C, q_2 = $-q_2$ = –6.0×10^{-6} C, q_3 = +3.0×10^{-6} C and a = 2.0×10^{-2} m.

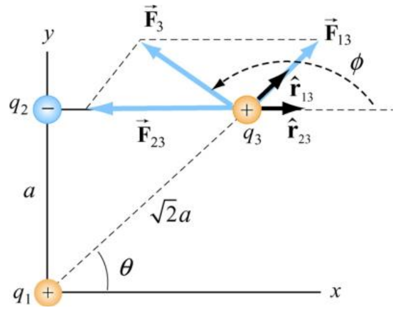


Figure 1.9 A system of three charges

Solution:

Using the superposition principle, the force on q_3 is

$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}^2}\hat{r}_{13} + \frac{q_2 q_3}{r_{23}^2}\hat{r}_{23}\right) \tag{1.10}$$

In this case the second term will have a negative coefficient, since q_2 is negative.

From the figure, we see that the unit vector $\hat{r}_{13}$ which points from q_1 to q_3 can be written as

$$\hat{r}_{13} = \hat{x}\cos\theta + \hat{y}\sin\theta = \frac{\sqrt{2}}{2}\left(\hat{x} + \hat{y}\right) \tag{1.11}$$

Similarly, the unit vector $\hat{r}_{23} = \hat{x}$ points from q_2 to q_3. Therefore, the total force is

$$\vec{F}_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_3}{r_{13}^2}\hat{r}_{13} + \frac{q_2q_3}{r_{23}^2}\hat{r}_{23}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{a^2}\left[\left(\frac{\sqrt{2}}{4}-1\right)\hat{x} + \frac{\sqrt{2}}{4}\hat{y}\right] \tag{1.12}$$

upon adding the components. The magnitude of the total force is

$$\begin{aligned}\vec{F}_3 &= \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{a^2}\left[\left(\frac{\sqrt{2}}{4}-1\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2\right]^{1/2} \\ &= \left(9.0\times10^9\ \mathrm{N\cdot m^2/C^2}\right)\frac{\left(6.0\times10^{-6}\ \mathrm{C}\right)\left(3.0\times10^{-6}\ \mathrm{C}\right)}{\left(2.0\times10^{-2}\ \mathrm{m}\right)^2}(0.74) = 3.0\ \mathrm{N}\end{aligned} \tag{1.13}$$

1.2.4 Electric Field

The electrostatic force is a force that acts at a distance, even when the objects are not in contact with one another. We say that one charge creates a field which in turn acts on the other charge.

The electric field $\vec{E}$ is defined as

$$\vec{E} = \lim_{q_0\to 0}\frac{\vec{F}_e}{q_0} \tag{1.14}$$

The analogy between the electric field and the gravitational field is depicted in Figure 1.10.

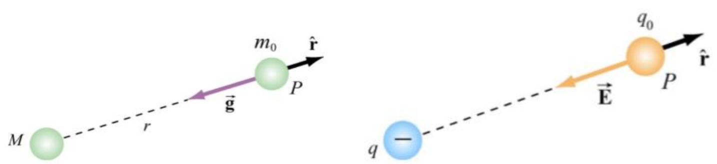


Figure 1.10 Analogy between the gravitational field $\vec{g}$ and the electric field $\vec{E}$.

Using the definition of electric field given in Eq. (1.14) and the Coulomb's law, the electric field at a distance *r* from a point charge *q* is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \tag{1.15}$$

Using the superposition principle, the total electric field due to a group of charges is equal to the vector sum of the electric fields of individual charges:

$$\vec{E} = \sum_i \vec{E}_i = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i^2}\hat{r} \tag{1.16}$$

1.2.5 Electric Field Lines

The field lines for a positive and a negative charge are shown in Figure 1.11.

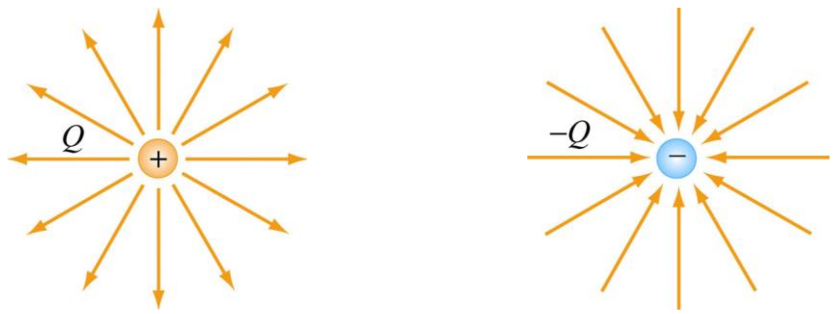


Figure 1.11 Field lines for (a) positive and (b) negative charges.

The properties of electric field lines may be summarized as follows:

- The direction of the electric field vector E at a point is tangent to the field lines.

- The field lines must begin on positive charges (or at infinity) and then terminate on negative charges (or at infinity).

- No two field lines can cross each other; otherwise the field would be pointing in two different directions at the same point

1.2.6 Electric Dipole

An electric dipole consists of two equal but opposite charges, $+q$ and $-q$, separated by a distance $2a$, as shown in Figure 1.12.

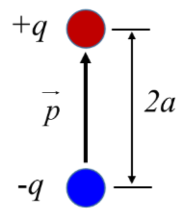


Figure 1.12 Electric dipole

The dipole moment vector $\vec{p}$ which points from $-q$ to $+q$ (in the $+y$ -direction) is given by

$$\vec{p} = 2aq\hat{y} \tag{1.17}$$

In Figure 1.13, we can conclude that the distance between P and $\pm q$ ($r_{\pm}$) is

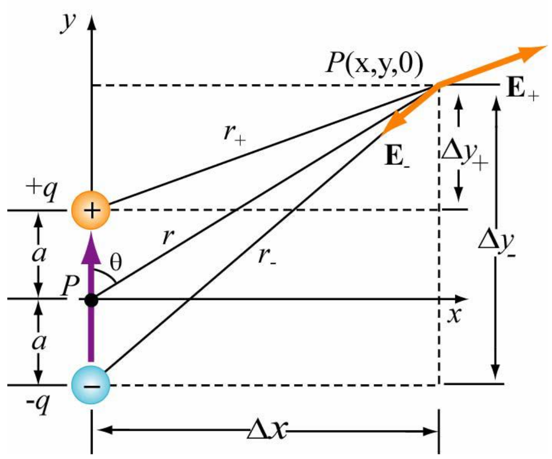


Figure 1.13 The electric field of a dipole

$$r_{\pm} = r^2 + a^2 \mp 2ra\cos\theta = x^2 + (y \mp a)^2 \tag{1.18}$$

So that the *x*-component of the electric field strength at the point *P* is

$$E_x = \frac{q}{4\pi\varepsilon_0}\left(\frac{\cos\theta_+}{r_+^2} - \frac{\cos\theta_-}{r_-^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{x}{\left[x^2+(y-a)^2\right]^{3/2}} - \frac{x}{\left[x^2+(y+a)^2\right]^{3/2}}\right) \tag{1.19}$$

Similarly, the *y*-component is

$$E_y = \frac{q}{4\pi\varepsilon_0}\left(\frac{\sin\theta_+}{r_+^2} - \frac{\sin\theta_-}{r_-^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{y-a}{\left[x^2+(y-a)^2\right]^{3/2}} - \frac{y+a}{\left[x^2+(y+a)^2\right]^{3/2}}\right) \tag{1.20}$$

In the "point-dipole" limit where $r \gg a$, one may verify that the above expressions reduce to

$$E_x = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta \tag{1.21}$$

and

$$E_y = \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos^2\theta - 1\right) \tag{1.22}$$

where $\sin\theta = x/r$ and $\cos\theta = y/r$. With $3pr\cos\theta = 3\vec{p}\cdot\vec{r}$ and some algebra, the electric field may be written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(-\frac{\vec{p}}{r^3} + \frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5}\right) \tag{1.23}$$

The equation indicates that the electric field $\vec{E}$ due to a dipole decreases with r as ,$1/r^3$,unlike the $1/r^2$ behavior for a point charge. This is to be expected since the net charge of a dipole is zero and therefore must fall off more rapidly than at large distance. The electric field lines due to a finite electric dipole and a point dipole are shown in Figure1.14.

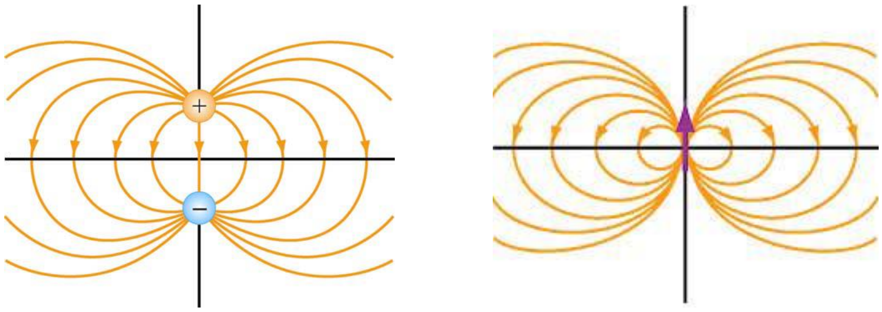


Figure 1.14 The electric field of a dipole

1.2.7 Dipole in Electric Field

What happens when we place an electric dipole in a uniform field $\vec{E} = \hat{x}E$, with the dipole moment vector $\vec{p}$ making an angle with the x-axis? From Figure 1.15, we see that the unit vector which points in the direction of $\vec{p}$ is $\hat{x}\cos\theta + \hat{y}\sin\theta$. Thus, we have

$$\vec{p} = 2qa\left(\hat{x}\cos\theta + \hat{y}\sin\theta\right) \tag{1.24}$$

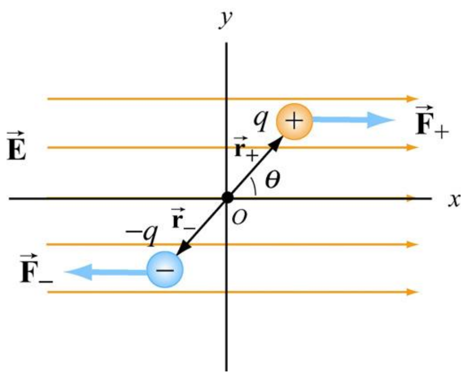


Figure 1.15 Electric dipole placed in a uniform field.

The net force on the dipole is net $\vec{F}_{net} = \vec{F}_+ + \vec{F}_- = 0$. Even though the net force vanishes, the field exerts a torque a toque on the dipole. The torque about the midpoint O of the dipole is

$$\begin{aligned} \vec{\tau} &= \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \\ &= \left(\hat{x}a\cos\theta + \hat{y}a\sin\theta\right)\times\vec{F}_+ + \left(-\hat{x}a\cos\theta - \hat{y}a\sin\theta\right)\times\vec{F}_+ \\ &= -\hat{z}2aF\sin\theta \end{aligned} \tag{1.25}$$

The direction of the torque is $-\hat{z}$, or into the page. The effect of the torque $\vec{\tau}$ is to rotate the dipole clockwise so that the dipole moment $\vec{p}$ becomes aligned with the electric field $\vec{E}$. With $F = qE$, the general expression for toque becomes

$$\vec{\tau} = \vec{p} \times \vec{E} \tag{1.26}$$

1.2.8 Charge Density

If we have a very large number of charges distributed in some region in space, how could we compute it? Let's consider the system shown in Figure 1.16:

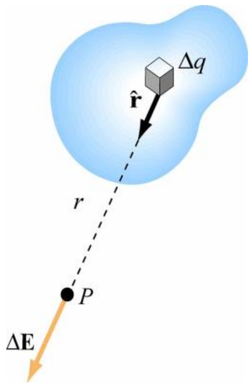


Figure 1.16 Electric field due to a small charge element Δq

The total amount of charge within the entire volume V is

$$Q = \sum_i \Delta q_i \rightarrow \int_V dq \tag{1.27}$$

The electric field at a point P due to each charge element dq is given by Coulomb's law:

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\Delta q}{r^2}\hat{r} \rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \tag{1.28}$$

Using the superposition principle, the total electric field $\vec{E}$ is the vector sum (integral) of all these infinitesimal contributions:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{dq}{r^2}\hat{r} \tag{1.29}$$

If there's a small volume element ΔV_i which contains an amount of charge Δq_i , in the limit where ΔV_i becomes infinitesimally small, we may define a volume charge density $\rho(\vec{r})$ as

$$\rho(\vec{r}) = \lim_{\Delta V_i \to 0}\frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV} \tag{1.30}$$

The dimension of $\rho(\vec{r})$ is charge/unit volume (C/m^3) in SI units.

In a similar manner, the charge can be distributed over a surface S of area A with a surface charge density σ (lowercase Greek letter sigma):

$$\sigma(\vec{r}) = \frac{dq}{dA} \tag{1.31}$$

The dimension of σ is charge/unit area (C/m^2) in SI units. The total charge on the entire surface is:

$$Q = \iint_S \sigma(\vec{r})dA \tag{1.32}$$

If the charge is distributed over a line of length l, then the linear charge density λ is

$$\lambda(\vec{r}) = \frac{dq}{dl} \tag{1.33}$$

The dimension of σ is charge/unit length (C/m) in SI units. The total charge on the entire surface is:

$$Q = \int_{line} \sigma(\vec{r})dl \tag{1.34}$$

1.3 Electrical Potential

1.3.1 Potential and Potential Energy

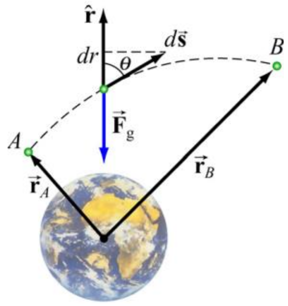


Figure 1.17 Gravitational potential

Consider moving a particle of mass m under the influence of gravity (Figure 1.17). The work done by gravity in moving m from A to B is

$$W_g = \int \vec{F}_g \cdot d\vec{s} = \int_{r_A}^{r_B} \left(-\frac{GMm}{r^2} \right) dr = GMm\left(\frac{1}{r_B} - \frac{1}{r_A} \right) \tag{1.35}$$

The result shows that W_g depends only on the endpoints A and B. It is important to draw distinction between W_g, the work done by the field and W_{ext}, the work done by an external agent such as you. They simply differ by a negative sign: $W_g = -W_{ext}$.

The change in potential energy associated with a conservative force acting on an object as it moves from A to B is defined as:

$$\Delta U = U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{s} = -W \tag{1.36}$$

A concept which is closely related to potential energy is “potential.” The gravitational potential can be obtained as

$$\Delta V_g = \frac{\Delta U_g}{m} = -\int_A^B \frac{\vec{F}_g}{m} \cdot d\vec{s} = -\int_A^B \vec{g} \cdot d\vec{s} \tag{1.37}$$

Physically ΔV_g represents the negative of the work done per unit mass by gravity to move a particle from A to B.

1.3.2 Electric Potential and Energy

Our treatment of electrostatics is remarkably similar to gravitation.

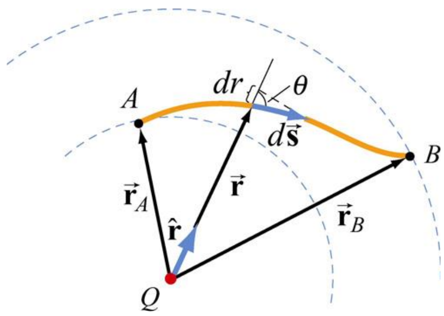


Figure 1.18 Electric Potential difference between two points A and B

In the presence of an electric field $\vec{E}$ shown in Figure 1.18, we define the electric potential difference between two points A and B as

$$\Delta V = -\int_A^B \frac{\vec{F}_e}{q_0} \cdot d\vec{s} = -\int_A^B \vec{E} \cdot d\vec{s} \tag{1.38}$$

The SI unit of electric potential is volt (V):

$$1\ \text{volt} = 1\ \text{joule / coulomb}\ (1\ \text{V} = 1\ \text{J / C}) \tag{1.39}$$

Since the work done to move q from A to B is $W_{ext} = \Delta U = U_B - U_A = q\Delta V_e$, the potential created by point charge is

$$\Delta V = -\int_A^B \frac{1}{4\pi\varepsilon_0} Q \frac{\hat{r}}{r^2} \cdot d\vec{s} = -\frac{1}{4\pi\varepsilon_0} Q \int_A^B \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0} Q \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \tag{1.40}$$

1.3.3 Electric Potential due to Point Charges

As in the case of gravity, only the difference in electrical potential is physically meaningful, and one may choose a reference point and set the potential there to be zero. In practice, we can choose the reference point to be at infinity, so that the electric potential at a point P becomes

$$V_P = -\int_\infty^P \vec{E} \cdot d\vec{s} \tag{1.41}$$

With this reference, the electric potential at a distance r away from a point charge Q becomes

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \tag{1.42}$$

1.3.4 Continuous Charge Distribution

If the charge distribution is continuous, the potential at a point P can be found by summing over the contributions from individual differential elements of charge dq as shown in Figure 1.16.

Taking infinity as our reference point with zero potential, the electric potential at P due to dq is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \tag{1.43}$$

Summing over contributions from all differential elements, we have

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r} \tag{1.44}$$

1.3.5 Deriving Electric Field from the Electric Potential

In Eq. (1.38) we established the relation between $\vec{E}$ and V. If we consider two points which are separated by a small $d\vec{s}$, the following differential form is obtained:

$$dV = -\vec{E}\cdot d\vec{s} \tag{1.45}$$

In Cartesian coordinates, $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$ and $d\vec{s} = \hat{x}dx + \hat{y}dy + \hat{z}dz$, we have

$$dV = -\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)\cdot\left(\hat{x}dx + \hat{y}dy + \hat{z}dz\right) = -E_x dx - E_y dy - E_z dz \tag{1.46}$$

which implies

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z} \tag{1.47}$$

By introducing a differential quantity called the "del (gradient) operator"

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{1.48}$$

the electric field can be written as

$$\begin{gathered}\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z = -(\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}) = -(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})V = -\nabla V \\ \vec{E} = -\nabla V\end{gathered} \tag{1.49}$$

1.4 Gauss's Law

1.4.1 Electric Flux

The number of electric field lines that penetrates a given surface is called an "electric flux," which we denote as Φ_E. The electric field can therefore be thought of as the number of lines per unit area.

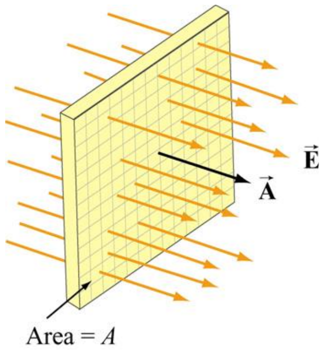


Figure 1.19 Electric field lines passing through a surface of area A

The flux through the surface in Figure 1.19 is

$$\Phi_E = \vec{E} \cdot \vec{A} = \vec{E} \cdot \hat{n} A = EA \tag{1.50}$$

On the other hand, if the electric field $\vec{E}$ makes an angle θ with $\hat{n}$ (Figure 1.20).

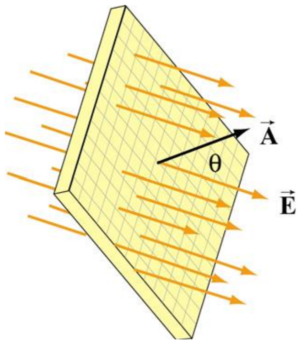


Figure 1.20 Electric field lines passing through a surface of area A whose normal makes an angle θ with the field.

The electric flux becomes

$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta \tag{1.51}$$

Note that the electric flux is positive if the electric field lines are leaving the surface, and negative if entering the surface.

In order to compute the electric flux in a closed surface, we divide the surface into a large number of infinitesimal area elements, as shown in Figure 1.21. Note that for a closed surface the unit vector is chosen to point in the outward normal direction.

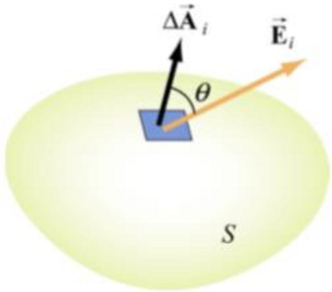


Figure 1.21 Electric field passing through an area element ΔA_i

The total flux through the entire surface can be obtained by summing over all the area elements. Taking the limit $\Delta A_i \to 0$ and the number of elements to infinity, we have

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum \vec{E}_i \cdot d\vec{A}_i \tag{1.52}$$

1.4.2 Gauss's Law

Consider a positive point charge Q located at the center of a sphere of radius r, as shown in Figure 1.22. We enclose the charge by an imaginary sphere of radius r called the "Gaussian surface."

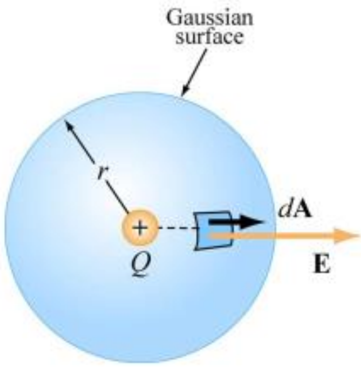


Figure 1.22 A spherical Gaussian surface enclosing a charge Q

The $\vec{E}$ field at surface can be written as $\vec{E} = (Q/4\pi\varepsilon_0 r^2)\hat{r}$ according to previous derivation.

$$\Phi_E = \oint\!\!\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{4\pi\varepsilon_0 r^2} \oint\!\!\oint_S dA = \frac{Q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{Q}{\varepsilon_0} \tag{1.53}$$

In the above, we have chosen a sphere to be the Gaussian surface. However, it turns out that the shape of the closed surface can be arbitrarily chosen. For the surfaces shown in Figure 1.23, the same result ($\Phi_E = Q/\varepsilon_0$) is obtained, whether the choice is S_1, S_2 or S_3.

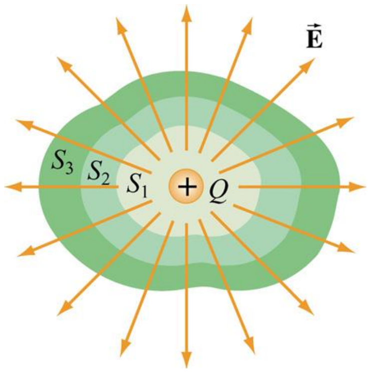


Figure 1.23 Different Gaussian surfaces with the same outward electric flux.

The statement that the net flux through any closed surface is proportional to the net charge enclosed is known as Gauss's law. Mathematically, Gauss's law is expressed as

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \tag{1.54}$$

where q_{in} is the net charge inside the surface.

Example 1.2: Infinite Plane of Charge

Consider an infinitely large non-conducting plane (Figure 1.24) in the *xy*-plane with uniform surface charge density σ. Determine the electric field everywhere in space.

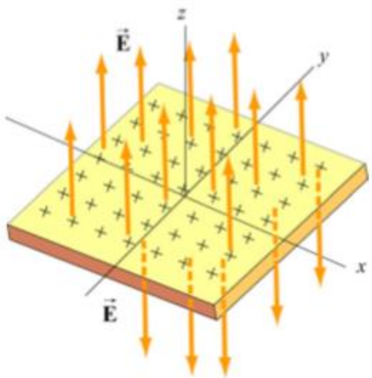


Figure 1.24 Electric field for uniform plane of charge

Solution:

(1) An infinitely large plane possesses a planar symmetry.

(2) Since the charge is uniformly distributed on the surface, the electric field $\vec{\mathbf{E}}$ must point perpendicularly away from the plane.

We choose our Gaussian surface to be a cylinder (Figure 1.25).

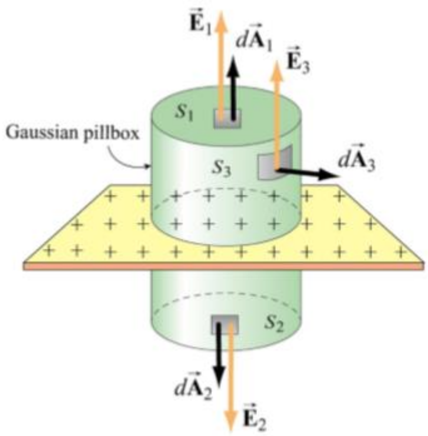


Figure 1.25 A Gaussian surface for calculating the electric field due to a large plane.

(3) Since the surface charge distribution on is uniform, the charge enclosed by the Gaussian surface is $q_{in} = \sigma A$, where $A = A_1 = A_2$ is the area of the end-caps.

(4) The total flux through the Gaussian surface is

$$\begin{aligned} \Phi_E &= \oiint_S \vec{E} \cdot d\vec{A} = \iint_{S_1} \vec{E}_1 \cdot d\vec{A}_1 + \iint_{S_2} \vec{E}_2 \cdot d\vec{A}_2 \iint_{S_3} \vec{E}_3 \cdot d\vec{A}_3 \\ &= E_1 A_1 + E_2 A_2 + 0 = 2EA = \frac{\sigma A}{\varepsilon_0} \end{aligned} \tag{1.55}$$

Which gives

$$E = \frac{\sigma}{2\varepsilon_0} \tag{1.56}$$

Example 1.3: Infinitely Long Rod of Uniform Charge Density

An infinitely long rod of negligible radius has a uniform charge density λ (Figure 1.26). Calculate the electric field at a distance *r* from the wire.

Solution:

(1) An infinitely long rod possesses cylindrical symmetry.

(2) The electric field $\vec{E}$ is point radially away from the symmetry axis of the rod. Therefore, we choose a coaxial cylinder as our Gaussian surface.

(3) The amount of charge enclosed by the Gaussian surface, a cylinder of radius *r* and length *l* (Figure 1.25), is $q_{in} = \lambda l$.

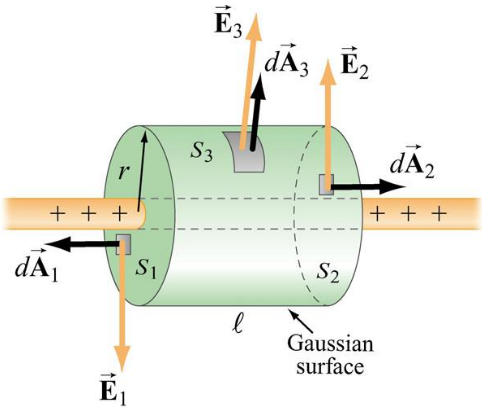


Figure 1.26 Gaussian surface for a uniformly charged rod.

(4) The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{A} = E 2\pi r l = \frac{\lambda l}{\varepsilon_0} \tag{1.57}$$

Which gives

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_0} r \tag{1.58}$$

Example 1.4: Non-Conducting Solid Sphere

An electric charge $+Q$ is uniformly distributed throughout a non-conducting solid sphere of radius a (Figure1.27). Determine the electric field everywhere inside and outside the sphere.

Solution:

The charge distribution is spherically symmetric with the charge density given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi a^3} \tag{1.59}$$

where V is the volume of the sphere.

Case 1: $r \le a$

We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 1.27(a).

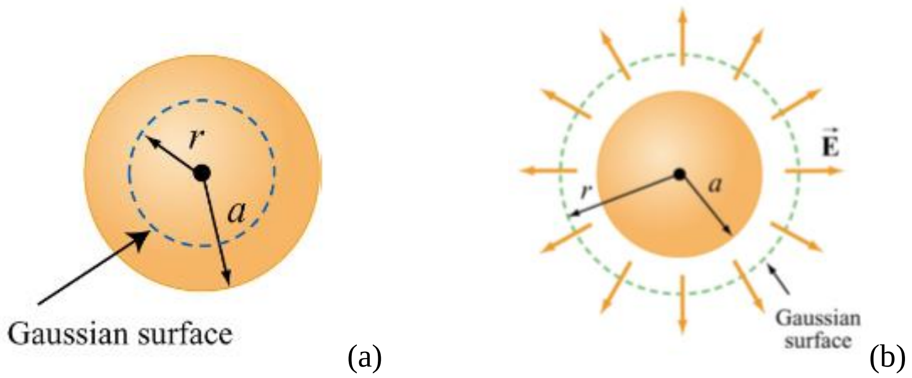


Figure 1.27 Gaussian surface for uniformly charged solid sphere, for (a) $r \le a$, and (b) $r > a$.

The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{A} = EA = E(4\pi r^2) \tag{1.60}$$

The charge enclosed is

$$q_{in} = \int_V \rho dV = \rho V = \rho\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r^3}{a^3}\right) \tag{1.61}$$

Applying Gauss's law $\Phi_E = q_{in} / \varepsilon_0$,we obtain

$$E(4\pi r^2) = \frac{\rho}{\varepsilon_0}\left(\frac{4}{3}\pi r^3\right) = \frac{Q}{\varepsilon_0}\left(\frac{r^3}{a^3}\right) \tag{1.62}$$

or

$$E = \frac{\rho r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 a^3}, r \le a \tag{1.63}$$

Case 2: $r > a$

In this case, our Gaussian surface is a sphere of radius $r > a$, as shown in Figure 1.27(b).

With the electric flux through the Gaussian surface given by $\Phi_E = E(4\pi r^2)$, upon applying Gauss's law, we obtain

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, r > a \tag{1.64}$$

1.5 Additional Problems

See Reference [1].

Reference:

[1] MIT Physics 8.02 : <<Electricity & Magnetism>>, by Sen-ben Liao, Peter Dourmashkin, and John W. Belcher.

Finished by Zuojia on 23 Aug 2019