Applications of Linear Programming

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Abstract

In this paper, we aim to introduce the concept of linear programming. Linear programming primarily deals with attempting to maximize or minimize a single function whose variables are subject to linear constraints. Linear programming itself is a much deeper topic than what we will be exploring in this paper, and it has a very wide range of applications. However, The most common application for linear programming is to maximize profits and minimize costs for a business. All of the concepts of we aim to explain will be explored in the context of linear programming's business applications. Linear programs can be solved by hand using the Simplex Method. In this paper, we will introduce a minimization problem that can be solved using the Simplex Method. We will then provide some basic definitions and really simple concepts that are specific to the application of simplex method to a linear programming problem. We will cover a brief history of Dantzig's Simplex Algorithm, and who Dantzig actually was. Finally we will use the Simplex Method to solve the problem we initially presented in the paper.

What does a Linear Programming problem look like?

We want to provide some context for what a linear program actually looks like when written formally. Here we will introduce a very basic linear program that we want to minimize. First, consider a linear function z:

$$-z = -8x_1 - 10x_2 - 7x_3$$

Where each of the variables in z are subject to the given constraints:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 \leq& 10 \\ -x_1 - 5x_2 - x_3 \geq& -8 \\ x_1, x_2, x_3 \leq& 0 \end{aligned}$$

Now that you have an idea of what a linear programming problem may look like, we will discuss some key elements that arise in a linear program when using the Simplex Method.

1. Elements of Linear Programming

There are a few key ideas to keep in mind when attempting to solve a linear program using the Simplex Method. These key ideas are constraints, the objective function, and slack variables. First, lets talk about constraints.

A constraint is basically the limitation of a thing, whatever that thing may be. Constraints in the context of solving a problem then, are limits to how we may go about solving the problem. Consider that the most common application of linear programming is maximizing the profit of a business. The profit of a business depends on a number of different factors, but the primary factor that influences profit, is cost; Since profit is the difference between a business' gross income and the cost of running the business. There are different kinds of costs to consider when running a business. Such as how much time it takes to complete a task, or create a product; How much capital the business has to spend, or how much it costs to create their product. These are just a few examples of the what it costs to run a business. For the sake of generalization, think of costs as the spending of resources. The finite amount of each resource we have to spend, is something we then have to account for when considering the cost of the business, and subsequently must be also be accounted for when maximizing profit. In this context then, the constraints would be the amount of resources we have to spend, since we will only have a finite amount. In linear programming, constraints are written as inequalities:

$$C_1x_1 + \cdots + C_nx_n \leq k$$

In the context of a business problem, k represents the finite amount of resources we have, and the coefficients C_n represent the amount of each variable that can be produced per 1 unit of resource. In the constraints, the amount of each resource we have to spend is represented by k. This should provide a good understanding of constraints in the context of a linear program.

Now, let's talk about the objective function. The objective function is the function that we want to maximize or minimize in a linear programming problem. The objective function represents how much each variable contributes to the value we are attempting to minimize or maximize. Generally, it can be written as follows:

$$Z = \sum_{i=1}^{n} c_i X_i$$

With Z representing the value we want minimized or maximized, c representing the coefficients, and X representing our variables. In the context of maximizing the profit of a business then, think of each variable in the objective function as representing a specific product the business sells. The coefficients in the objective function then, would represent the price of each product. The constraints are how much of each resource we have to spend producing each product. Remember that the coefficients in the constraints in the context of maximizing profit represent how much of each product can be produced per one unit of a given resource k. A very simple example of a resource is time. Say our manufacturing plant is able to produce five computers an hour. Well, the maximum amount of hours we can have our employees work is 40 hours. So this constraint can be written as follows:

$$5_{x_1} \leq 40$$

Our profit then depends on how many computers we are able to make given the finite amount of time per week we have to make computers. So the objective function represents our profitability, the coefficient of the variable in the objective function represents the price of the computer, and the variable itself in the objective function is the maximum amount of computers we can make. The constraint we listed previously must be considered when trying to maximize the amount of computers we make, since we can only make a certain number of computers per hour, and our employees can only work at maximum 40 hours a week. This should give you a better understanding of the real world applications of linear programming, and some common themes that arise in the context of solving a real world linear programming problem.

Finally, let us introduce the idea of slack variables. The formal mathematical definition of slack variables is not what will be exploring in this paper. For now, we will define slack variables only in the context of business applications of linear programming. Slack variables specifically arise in the constraints of a linear programming problem. Remember that the constraints of a linear programming problem are generally written as follows with scalars C_n and variables x_n.

$$C_1x_1 + \cdots + C_nx_n \leq k$$

Slack variables are variables we introduce into these constraints to "pick up the slack" for possible negative values of the variables in our objective function, since our variables cannot assume negative values in the context of a real-world business problem. This introduces another constraint $x_i \geq 0$, $i = 1 \cdots n$ that must be included in each linear programming problem. Our slack variables are introduced into each constraint with the purpose of transforming a linear system of constraints, into a linear system of equations. This new linear system of equations can that satisfy the given constraint $x_i \geq 0$ due to the introduction of a slack variable. These new equations are written as follows:

$$C_1x_1 + \cdots + C_nx_n + s_1 = k$$

Slack variables are an important concept to understand in the context of linear programming, because they allow the variables in the objective function to assume values greater than or equal to 0. The value of a slack variable for a given constraint also provides additional information about the constraint, but we will get there in a moment. For now, think of a linear system of equations that is represented as a plane in any dimension. A plane has two dimensions, and each dimension can be written as a linear equation. The area of the plane will represent profitability, just to provide some real world context. So we would want to maximize the area of the plane. Maximizing the area of the plane depends on how much we can change each dimension. Remember that the each of plane's two dimensions are represented by the constraints we have turned into linear equations with the introduction of a slack variable. If the value of a slack variable for one of the linear equations is 0, that dimension cannot be changed. If the slack variable is positive, then that dimension can be changed, and further maximization can occur since we can scale one of the dimensions of the plane. Please note that this is not a very informal explanation of what a slack variable actually is in formal mathematics, but for the sake of simplicity and in the context of business applications of linear programming, it will suffice.

2. A Brief History of Linear programming

Figure 1: Leonid Kantorovich

The founding father of linear programming is widely recognized as Leonid Kantorovich. Kantorovich was a Russian mathematician born in St. Petersburg, Russia on January 19, 1912. At the age of 14, he began his studies at Leningrad University. At age 18, he graduated from the University. By age 22, he had become a full-time professor at the university. Later Kantorovich began working for the Soviet government. One of his first tasks was optimizing the production of plywood. It was in solving this problem where he developed some early methods of linear programming. Linear programming was very effective for optimizing the allocation of resources, and his methods were able to further be developed while being applied for uses in military planning, given that World War II was occurring around this time. Kantorovich was eventually given the Nobel Memorial Prize for "his contributions to the theory of optimum allocation of resources." He died on April 7, 1986, in Moscow, Russia.

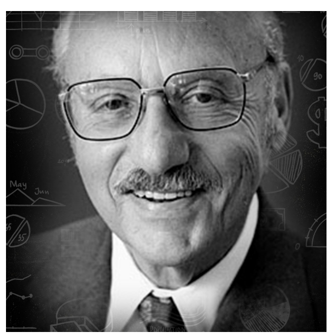

Figure 2: George Bernard Dantzig

Although Kantorovich is recognized as the father of linear programming, It was George Bernard Dantzig who developed the Simplex Method, an efficient algorithm that can be used to solve linear programs. Dantzig was born on November 8, 1914, in Portland, Oregon. Dantzig eventually attended the University of Maryland, where he majored in mathematics and physics. In 1946, during World War II, Dantzig was appointed as a Mathematical Advisor to the U.S. Department of Defense. His work at the Department of Defense dealt with solving logistical military planning and scheduling problems. In fact, at the time, the method of solving these military logistics and scheduling problems was referred to within the military as "programming". Dantzig further standardized the programming process by using a linear structure to solve programs. The Simplex Method was the method he developed to solve these linear programs. In 1948, Dantzig visited the RAND corporation to explain his development of the Simplex Method to a colleague of his, TJ

Koopmans. Koopmans is actually the one who coined the term "Linear Programming" during this visit with Dantzig. Dantzig's first application of The Simplex Method was to find the minimum cost of a healthy diet. The linear system in this initial application consisted of nine equations with 77 unknowns. It took 120 days of hand calculations to solve this particular problem. The ability to solve programs using the Simplex Method was something that always awed Dantzig; he is quoted as saying that "The tremendous power of the Simplex Method [was] always a surprise to [him]". The Simplex Method is still widely used as a way to solve linear programs. The general application of Linear Programming is optimizing the allocation of resources, which means that it has a ton of real-world applications; Especially in the modern world. Almost all schedule and resource management software utilizes some method of Linear Programming. Linear Programming is a very powerful optimization tool whose development is largely due to Kantorovich and Dantzig's individual contributions to the subject.

3. Dantzig's Simplex Algorithm

The Simplex Algorithm is one of many methods that can be used to solve a linear programming problem. The key characteristic of the Simplex Algorithm is that we can solve linear programs by hand, regardless of the number of variables or equations.

1. Constructing the Standard Form

   In order to solve a linear program, all of the linear constraints in the program must be transformed into linear equations, or what's also known as the standard form equation. This is an essential step of the Simplex Method, In this step, there are three requirements to achieve.

   - The objective function must be written as a maximization problem
   - All linear constraints should be written in a less-than-or-equal-to format.
   - All variables must be non-negative

   Therefore, we have to optimize our problem to the standard form.

   Minimize:

   $$-z = -8x_1 - 10x_2 - 7x_3$$

   We can change this to a maximization problem by multiplying each side of the objective function by -1.

   $$\begin{aligned} (-1) - z &= (-8x_1 - 10x_2 - 7x_3)(-1) \\ z &= 8x_1 + 10x_2 + 7x_3 \end{aligned}$$

   All linear constraints must be written in less-than-or-equal to format as well. We can change any constraints that are not in this form to a less-than-or-equal to format by multiplying each side of the constraint by -1.

   $$\begin{aligned} (-1)(-x_1 - 5x_2 - x_3 \geq &- 8) \\ x_1 + 5x_2 + x_3 \leq &8 \end{aligned}$$

   We have fulfilled the first two requirements of the Simplex Method. Our objective function is written as a maximization problem, and the constraints are in less-than-or-equal to format. We must now fulfill the third requirement: that all variables are greater-than-or-equal-to 0.

We do this, by simply writing an additional constraint for all variables at the bottom of our linear program.

2. Introducing slack variables

   Slack variables are additional variables that we introduce into our linear constraints to transform them from linear inequalities into linear equations. Slack variables typically represent an unused resource and therefore they contribute nothing to the objective function value.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + s_1 &= 10 \\ x_1 + 5x_2 + x_3 + s_2 &= 8 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

3. Creating the tableau

   In order to solve the linear program, We are going to use something called a "tableau". A tableau can be thought of as a modified augmented matrix. Because a tableau is still a matrix, we can perform row operations on the tableau. The ability to perform row operations on a tableau is a key part of the Simplex Method. We will now write our linear program as a tableau.

$$Maximize : z = 8x_1 + 10x_2 + 7x_3$$

$$\begin{aligned} subject\ to : x_1 + 3x_2 + 2x_3 + s_1 &= 10 \\ x_1 + 5x_2 + x_3 + s_2 &= 8 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0

The bolded variables in this paper doesn't indicate they are vectors

The top row of the tableau represents all of the variables in the linear program. The first and the second row each represent the linear constraints. Finally, the last row represents the objective function with all of it's variables on one side: $-8x_1 - 10x_2 - 7x_3 + z = 0$.

Each column represents the coefficients of the variable in the column's top row for all of the constraints, as well as the objective function.

Out of all the rows, the last row is the most important row in the tableau since it determines whether the objective function is optimized or not. Since the linear programming model is subject to

$$x_1, x_2, x_3, s_1, s_2 \geq 0,$$

When all values in the last row are greater-than-or-equal-to 0, we have optimized the objective function.

4. Identifying the pivot

   There are 3 negative values in the tableau. Therefore, this is not the optimal solution. Similar to using matrix row operations to find a *RREF* form of a matrix, we can also use matrix row operations on the tableau. In addition to row operations, we use something that's known as an indicator, to reduce the tableau.

   To identify the pivot value, we need to see the row that represents the optimal function; the last row. The column that contains the smallest value of the last row will be the pivot column.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0

   In this case, the column that contains -10 is the pivot column.

   The pivot variable can be found by using the indicator. To find an indicator, divide **b** by the numbers in the previously identified pivot column for each corresponding row. The row with the lowest indicator indicates which row of the column contains the pivot for that column.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0

   Thus, the indicator for the row

$$1 \quad 3 \quad 2 \quad 1 \quad 0 \quad 0 \quad 10$$

   is 10/3,

   and the indicator for the row

$$1 \quad 5 \quad 1 \quad 0 \quad 1 \quad 0 \quad 8$$

   is 8/5.

   Since $\frac{8}{5} < \frac{10}{3}$, the pivot variable is 5.

5. Optimizing the Rows

   Optimizing the tableau is somewhat similar to the reducing a matrix to a *RREF* form. First, the pivot variable must be transformed into 1. It can be achieved by scaling the row.

$$\frac{1}{5} \times \{1 \quad 5 \quad 1 \quad 0 \quad 1 \quad 0 \quad 8\} = \tfrac{1}{5} \quad 1 \quad \tfrac{1}{5} \quad 0 \quad \tfrac{1}{5} \quad 0 \quad \tfrac{8}{5}$$

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
1	3	2	1	0	0	10
$\frac{1}{5}$	1	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{8}{5}$
-8	-10	-7	0	0	1	0

After the pivot variable is transformed into 1, all of the other variables in the pivot column should be transformed into 0. This can also be done by the basic row operations.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$1+(-3*\frac{1}{5})$	$3+(-3*1)$	$2+(-3*\frac{1}{5})$	$1+(-3*0)$	$0(-3*\frac{1}{5})$	$0(-3*0)$	$10(-3*\frac{8}{5})$
$\frac{1}{5}$	1	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{8}{5}$
$-8+(10*\frac{1}{5})$	$-10+(10*1)$	$-7+(10*\frac{1}{5})$	$0+(10*0)$	$0+(10*\frac{1}{5})$	$1+(10*0)$	$0+(10*\frac{8}{5})$

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$\frac{2}{5}$	0	$\frac{7}{5}$	1	$-\frac{3}{5}$	0	$\frac{26}{5}$
$\frac{1}{5}$	1	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{8}{5}$
-6	0	-5	0	2	1	16

Although the tableau is arranged for the pivot column to be $\begin{array}{c}\mathbf{x_2}\\0\\1\\0\end{array}$, there are still negative values in the last row. Thus, the tableau is not optimized yet. We must then continue to reduce the tableau.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$\frac{2}{5}$	0	$\frac{7}{5}$	1	$\frac{-3}{5}$	0	$\frac{26}{5}$
$\frac{1}{5}$	1	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{8}{5}$
-6	0	-5	0	2	1	0

The next pivot column is the first column, as -6 is the smallest value in the last row. Thus, the indicators for the 2nd and 3rd row are:

$$\frac{\frac{26}{5}}{\frac{2}{5}} = 13$$

And

$$\frac{\frac{8}{5}}{\frac{1}{5}} = 8,$$

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$\frac{2}{5}$	0	$\frac{7}{5}$	1	$-\frac{3}{5}$	0	$\frac{26}{5}$
$\frac{1}{5}$	1	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{8}{5}$
-6	0	-5	0	2	1	16

Therefore, the pivot for the first column is $\frac{1}{5}$, because the row with the smallest indicator, 8, intersects with our previously identified pivot column.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$\frac{2}{5}$	0	$\frac{7}{5}$	1	$-\frac{3}{5}$	o	$\frac{26}{5}$
1	5	1	0	1	0	8
-6	0	-5	0	2	1	16

Next, scale the row with the pivot variable by 5 to make the pivot variable into 1, as the tableau above shows.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
$\frac{2}{5}+(-\frac{2}{5}*1)$	$0+(-\frac{2}{5}*5)$	$\frac{7}{5}+(-\frac{2}{5}*1)$	$1+(-\frac{2}{5}*0)$	$-\frac{3}{5}+(-\frac{2}{5}*1)$	$0+(-\frac{2}{5}*0)$	$\frac{26}{5}+(-\frac{2}{5}*8)$
1	5	1	0	1	0	8
$-6+(6*1)$	$0+(6*5)$	$-5+(6*1)$	$0+(6*0)$	$2+(6*1)$	$1+(6*0)$	$16+(6*8)$

We must then reduce each pivot column so that each pivot is the only non-zero entry in the column.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

This newly created tableau has reached its optimality because all values in the bottom row are zero or greater than zero.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

Therefore, the optimal values can be acquired.

$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{z}$	$\mathbf{b}$
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

The basic variables are the variables with the column that only contains a single 1 and one or more 0. In order to find the value of a basic variable,

- locate 1 under a basic variable
- the number under the column of **b** that is in the same row of the 1 is the value for the basic variable

The value for non-basic variables are all 0.

Therefore, the optimal values for the tableau is

$$\begin{aligned} x_1 &= 8 \\ x_1 &= 0 \\ x_3 &= 0 \\ z &= 64 \end{aligned}.$$

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4. Conclusion

Linear programming is one of the most important problem solving tools of the modern world. The Simplex Method is an extremely powerful optimization algorithm due to it's ability to optimally allocate resources. The work done by Dantzig and Kantorovich is responsible for ensuring that a lot of the logistics and operations in the modern world are maintained and continue to run smoothly. Its applications range from optimizing shipping routes, controlling air traffic and managing flight schedules, to maximizing the profitability of a business. Laszlo Lovasz, the current president of the Hungarian Academy of Sciences, and a world renowned Mathematician, is quoted as saying that if one were to statistically analyze what uses up most of the computer time in the world, [..] the answer [is] linear programming."

References