Taller #2 - Calculo

- Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades):

a) $e^{-a|t|}$, $a \in \mathbb{R}^+$

$$\text{des } e^{-a|t|} \begin{cases} e^{at} & \text{para } t<0 \\ e^{-at} & \text{para } t>0 \end{cases}$$

$$\mathcal{F}\{e^{-a|t|}\} = \int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{-jwt+at} dt + \int_{0}^{\infty} e^{-jwt-at} dt$$

$$= \int_{-\infty}^{0} e^{(-jw+a)t} dt + \int_{0}^{\infty} e^{(-jw-a)t} dt$$

$$= \left[\frac{e^{(-jw+a)t}}{-jw+a}\right]_{-\infty}^{0} + \left[\frac{e^{(-jw-a)t}}{-jw-a}\right]_{0}^{\infty}$$

$$= \left[\frac{e^{-jw(0)} e^{a(0)}}{-jw+a} - \frac{e^{-jw(-\infty)} e^{a(-\infty)}}{-jw+a}\right] + \left[\frac{e^{-jw(\infty)} e^{-a(\infty)}}{-jw-a} - \frac{e^{-jw(0)} e^{-a(0)}}{-jw-a}\right]$$

$$= \frac{1}{-jw+a} + \frac{1}{jw+a}$$

b) $\cos(w_c t)$, $w_c \in \mathbb{R}$

$$\mathcal{F}\{\cos(w_c t)\} = \int_{-\infty}^{\infty} \cos(w_c t) e^{-jwt} dt$$

$$\therefore \cos(w_c t) = \frac{e^{jw_c t} + e^{-jw_c t}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{e^{jw_c t} + e^{-jw_c t}}{2}\right) e^{-jwt} dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{jw_c t} e^{-jwt} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-jw_c t} e^{-jwt} dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w-w_c)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w+w_c)} dt$$

$$\therefore \int_{-\infty}^{\infty} e^{-j\alpha t} dt = 2\pi\delta(\alpha)$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\pi\delta(w-w_c)\right] + \frac{1}{\cancel{2}}\left[\cancel{2}\pi\delta(w+w_c)\right]$$

$$= \pi\left[\delta(w-w_c) + \delta(w+w_c)\right]$$

c) $\mathrm{sen}(w_s t)$, $w_s \in \mathbb{R}$

$$\mathcal{F}\{\mathrm{sen}(w_s t)\} = \int_{-\infty}^{\infty} \mathrm{sen}(w_s t) e^{-jwt} dt$$

$$\therefore \mathrm{sen}(w_s t) = \frac{e^{jw_s t} - e^{-jw_s t}}{2j}$$

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$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{e^{jw_s t}-e^{-jw_s t}}{2j}\right)e^{-jwt}\,dt$$

$$=\frac{1}{2j}\int_{-\infty}^{\infty}e^{jw_s t}e^{-jwt}\,dt-\frac{1}{2j}\int_{-\infty}^{\infty}e^{-jw_s t}e^{-jwt}\,dt$$

$$=\frac{1}{2j}\int_{-\infty}^{\infty}e^{-j(w-w_s)t}\,dt-\frac{1}{2j}\int_{-\infty}^{\infty}e^{-j(w+w_s)t}\,dt$$

$$=\frac{1}{\cancel{2}j}\left[\cancel{2}\pi\delta(w-w_s)-\cancel{2}\pi\delta(w+w_s)\right]$$

$$=\frac{\pi}{j}\left(\delta(w-w_s)-\delta(w+w_s)\right)$$

d) $f(t)\cos(w_c t)$, $w_c \in \mathbb{R}$

$$\mathcal{F}\{f(t)\cos(w_c t)\}=\int_{-\infty}^{\infty}f(t)\cos(w_c t)e^{-jwt}\,dt$$

$$=\int_{-\infty}^{\infty}f(t)\left(\frac{e^{jw_c t}+e^{-jw_c t}}{2}\right)e^{-jwt}\,dt$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{jw_c t}e^{-jwt}\,dt+\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-jw_c t}e^{-jwt}\,dt$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-j(w-w_c)t}\,dt+\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-j(w+w_c)t}\,dt$$

e) $e^{-a|t|^2}$, $a\in\mathbb{R}^+$

$e^{-at^2} \Rightarrow |t|^2 \geq 0$

$$\mathcal{F}\{e^{-at^2}\}=\int_{-\infty}^{\infty}e^{-at^2}e^{-jwt}\,dt$$

$$=\sqrt{\frac{\pi}{a}}\cdot e^{-\frac{(\pi f)^2}{a}}$$

f) $A\,rect_d(t)$, $A, d \in \mathbb{R}$

$$A\,rect_d(t) = \begin{cases} A, & -\frac{d}{2} < t < \frac{d}{2} \\ 0, & cc \end{cases}$$

$$\mathcal{F}\{A\,rect_d(t)\} = \int_{-d/2}^{d/2} A e^{-jwt}\, dt = A \int_{-d/2}^{d/2} e^{-jwt}\, dt$$

$$= A\left[\frac{e^{-jwt}}{-jw}\right]_{-d/2}^{d/2} = -\frac{A}{jw}\left[e^{\frac{-jwd}{2}} - e^{\frac{jwd}{2}}\right]$$

$$sen(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$\Rightarrow \frac{A\,(2)}{jw\,(2)}\left[e^{\frac{jwd}{2}} - e^{\frac{-jwd}{2}}\right]$$

$$= \frac{2A}{w}\, sen\left(\frac{wd}{2}\right)$$

Pero $sinc(x) = \frac{sen\,x}{x}$

$$\Rightarrow \frac{2A}{w}\frac{\left(\frac{d}{2}\right)}{\left(\frac{d}{2}\right)} sen\left(\frac{wd}{2}\right)$$

$$= \boxed{Ad\, sinc\left(\frac{wd}{2}\right)}$$

- Aplique las propiedades de la transformada de Fourier para resolver

a) $\mathcal{F}\{e^{-jw_1 t}\cos(w_c t)\}$, $w_2, w_1 \in \mathbb{R}$

Como $\cos(w_1 t) = \frac{e^{jw_1 t} + e^{-jw_1 t}}{2}$

$$e^{-jw_2 t}\cos(w_1 t) = e^{-jw_2 t}\left(\frac{e^{jw_1 t} + e^{-jw_1 t}}{2}\right)$$

$$= \frac{1}{2}e^{-jw_2 t}e^{jw_1 t} + \frac{1}{2}e^{-jw_2 t}e^{-jw_1 t}$$

$$= \frac{1}{2}\left(e^{-j(w_2 - w_1)t} + e^{-j(w_2 + w_1)t}\right)$$

Y como $\mathcal{F}\{e^{-jw_0 t}\} = 2\pi\delta(w - w_0)$

$$\mathcal{F}\{e^{-jw_1 t}\cos(w_1 t)\} = \frac{1}{2}\left[\mathcal{F}\{e^{-j(\ldots)t}\} + \mathcal{F}\{e^{-j(w_2 + w_1)t}\}\right]$$

$$\mathcal{F}\{e^{-jw_2 t}\cos(w_1 t)\} = \frac{1}{2}\left[2\pi\delta(w-(w_2-w_1)) + 2\pi\delta(w-(w_2+w_1))\right]$$

$$= \pi\left(\delta(w-(w_2-w_1)) + \delta(w-(w_2+w_1))\right)$$

b) $\mathcal{F}\{u(t)\cos^2(w_1 t)\}$, $w_1 \in \mathbb{R}$

Como $\cos^2(w_1 t) = \dfrac{1+\cos(2w_1 t)}{2}$

$$u(t)\cos^2(w_1 t) = \frac{1}{2}u(t) + \frac{1}{2}u(t)\cos(2w_1 t)$$

Ahora $\mathcal{F}\{u(t)\} = \pi\delta(w) + \dfrac{1}{jw} = U(w)$

Sabemos que

$$\mathcal{F}\{u(t)\cos(2w_1 t)\} = \frac{1}{2}\left[\mathcal{F}\{u(t)e^{j2w_1 t}\} + \mathcal{F}\{u(t)e^{-j2w_1 t}\}\right]$$

Usando modulación

$$\mathcal{F}\{u(t)e^{\pm jw_0 t}\} = U(w \mp w_0)$$

$$\mathcal{F}\{u(t)\cos(2w_1 t)\} = \frac{1}{2}\left[U(w-2w_1) + U(w+2w_1)\right]$$

$$\mathcal{F}\{u(t)\cos^2(w_1 t)\} = \frac{1}{2}\mathcal{F}\{u(t)\} + \frac{1}{2}\mathcal{F}\{u(t)\cos(2w_1 t)\}$$

$$= \frac{1}{2}U(w) + \frac{1}{2}\left[\frac{1}{2}(U(w-2w_1) + U(w+2w_1))\right]$$

$$= \frac{1}{2}U(w) + \frac{1}{4}(U(w-2w_1) + U(w+2w_1))$$ ✓

c) $\mathcal{F}^{-1}\left\{\dfrac{1}{w^2+6w+45} * \dfrac{10}{(8+jw/3)^2}\right\}$

Sabemos que:

$$x(t) * y(t) \longleftrightarrow X(jw)\cdot Y(jw)$$

Entonces

$$\mathcal{F}^{-1}\{X(jw)\} * \mathcal{F}^{-1}\{Y(jw)\} = \mathcal{F}^{-1}\{X(jw) * Y(jw)\}$$

$$X(jw) = \frac{1}{w^2+6w+45} \Rightarrow \frac{1}{(w+5)^2+6^2}$$

Y como la $\mathcal{F}^{-1}\left\{\frac{1}{(w-w_0)^2+a^2}\right\} = \pi e^{jw_0 t} e^{-a|t|}$, para $t \in \mathbb{R}$

$$\mathcal{F}^{-1}\left\{\frac{7}{(w+3)^2+6^2}\right\} = 7\pi e^{-3jt} e^{-6|t|}$$

Ahora para $Y(jw) = \frac{10}{(8+jw/3)^2}$

Sea $s = jw$

$$Y(s) = \frac{10}{(8+s/3)^2} = \frac{10 \cdot 9}{(3 \cdot 8+s)^2} = \frac{90}{(s+24)^2}$$

Y como

$$\mathcal{F}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = t e^{-at} u(t)$$

$$\therefore \mathcal{F}^{-1}\{Y(jw)\} = 90 t e^{-24t} u(t)$$

$$\mathcal{F}^{-1}\{X(jw) \cdot Y(jw)\} = \left(7\pi e^{-3jt} e^{-6|t|}\right)\left(90 t e^{-24t} u(t)\right)$$

$$= 630\pi t e^{-3jt - 6|t| - 24t} u(t)$$

d) $\mathcal{F}\{3t^3\}$

Como $\mathcal{F}\{t^n\} = 2\pi j^n \delta^{(n)}(w)$

$$\mathcal{F}\{3t^3\} = 3(2\pi j)^3 \delta^{(3)}(w)$$

$$= -6\pi j \delta^{(3)}(w)$$

e) $\frac{B}{T}\sum_{n=-\infty}^{\infty}\left(\frac{1}{a^2+(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)}\right)$; $n \in \{0, \pm1, \pm2, \pm3, \dots\}$, $w_0 = \frac{2\pi}{T}$, $B, T \in \mathbb{R}^+$

Analizamos cada termino

$$\mathcal{F}^{-1}\left\{\frac{1}{a^2+(w-nw_0)^2}\right\} = \pi e^{jnw_0 t} e^{-a|t|}$$ → Lorentziana

$$\mathcal{F}^{-1}\left\{\frac{1}{a+j(w-nw_0)}\right\} = e^{-at} u(t) \cdot e^{jnw_0 t}$$ → Modulacion de un exponencial buscado

$$x(t) = \frac{B}{T}\sum_{n=-\infty}^{\infty}\left(\pi e^{jnw_0 t} e^{-a|t|} + e^{-at} u(t) e^{jnw_0 t}\right)$$

Norma

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