NOTES ON MEAN FIELD GAMES //2

Guangyu Hou

Contents

1 Static Mean Field Games

1.1 Problem setting

- *Action Space*: There are n agents, and each agent chooses an action at the same time. An action is an element of a given set A, called the *action space*. We will assume throughout that A is a compact metric space (while in applications it's often finite).
- *Strategy Profile*: A *strategy profile* is a vector $(a_1, ...a_n) \in A^n$ (actions taken by each player).
- *Objective function*: In the n-player game, player i has an *objective function* $J_i^n : A^n \to \mathbb{R}$, which assigns a "reward" to every possible strategy profile.

The goal of player i is to choose $a_i \in A$ to maximize the reward J_i^n. But J_i^n depends on all of the other agents' actions. So to resolve such inter-dependent optimization problems, we use **Nash equilibrium**.

A *Nash equilibrium* (for the n-player game, or 0-Nash equilibrium) is a strategy profile $(a_1, ...a_n) \in A^n$ such that, for every $i = 1, ..., n$ and every $\tilde{a} \in A$, we have

$$J_i^n(a_1, ...a_n) \geq J_i^n(a_1, ..., a_{i-1}, \tilde{a}, a_{i+1}, ...a_n)$$

Similarly, given $\epsilon \geq 0$, an ϵ-*Nash equilibrium* is a strategy profile $(a_1, ...a_n) \in A^n$ such that, for every $i = 1, ..., n$ and every $\tilde{a} \in A$, we have

$$J_i^n(a_1, ...a_n) \geq J_i^n(a_1, ..., a_{i-1}, \tilde{a}, a_{i+1}, ...a_n) - \epsilon.$$

Intuitively, in *Nash equilibrium*, each player i is choosing a_i optimally, *given the other agents' choices*. In an ϵ-Nash equilibrium, each player could improve their reward, but by no more than ϵ.
Nash equilibria can be difficult to compute when n is large, so it's often simpler to work with the $n \to \infty$ limit and study a game with a *continuum of players* in terms of approximate equilibria of games with a finite number of players (Such limiting analysis is possible for those in which the objective functions are suitably symmetric)
Games with a *continuum of players* were introduced in order to illustrate situations, where the number of agents is large enough to make a single player negligible—when we consider the impact of his/her action on aggregate variables, while joint action of the whole set of such negligible players is not negligible. (i.e. In the presence of infinitely many agents, no single agent can influence μ by changing actions.)

Assume a single common payoff function of the form $F : A \times \mathcal{P}(A) \to \mathbb{R}$, and in the n-player game the objective function for player i is given by

$$J_i^n(a_1, ...a_n) = F(a_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{a_k}).$$

Actually, $F(a,m)$ represents the reward to a player choosing the action a when the distribution of actions chosen by other players is m. The objective of player i depends on the actions of the other agents only through their empirical distribution. In particular, the "names" or "labels" assigned to the players are irrelevant. With a sense of *anonymous*, only the distribution of actions matters. Such cost structure renders the game symmetric (equal for any permutation).
When n is very large, agent does not contribute much to the empirical measure. Then the Nash equilibrium property will be reflected in a consistency between the limiting distribution obtained from the empirical measure and the action of a typical player

NOTE. it is arguably more natural to use the empirical measure $\frac{1}{n-1}\sum_{k\neq i}\delta_{a_k}$ of the other agents, not including agent i.
Now we will always assume the following statement: **Standing Assumption:** A is a compact metric space, and $F: A\times\mathcal{P}(A)\to\mathbb{R}$ is jointly continuous, using the weak convergence topology on $\mathcal{P}(A)$.

1.2 mean field equilibrium(MFE)

Theorem 1.
Assume F is jointly continuous, using the weak convergence topology on $\mathcal{P}(A)$. Suppose that for each n we are given $\epsilon_n \geq 0$ and an ϵ_n-Nash equilibrium $(a_1, ...a_n)$. Suppose $lim_n \epsilon_n = 0$, and let

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{a_i^n}.$$

Then $(\mu_n)\subset\mathcal{P}(A)$ is tight, and, for every weak limit point $\mu\in\mathcal{P}(A)$, we have that μ is supported on the set $\{a\in A : F(a,\mu) = sup_{b\in A}F(b,\mu)\}$.
We say a probability measure m is supported on a set B if $m(B)=1$

- tightness is easy to prove where

$$\mu_n(K_\epsilon^c) = \frac{1}{n}\sum_{i=1}^{n}1_{K_\epsilon^c}(X_i)\to P(K_\epsilon^c)$$

  by law of large numbers

- Prokhorov's theorem ensures that (μ_n) admits a weak limit along a **subsequence** (μ_{n_k}).

- we will compare an alternative action $b\in A$ to use the ϵ_n-Nash equilibrium. $F(a_i^n,\mu_n)\geq F(b,\mu_n^i[b])-\epsilon_n$,then we find

$$\int_A F(a,\mu_n)\mu_n(da) = \frac{1}{n}\sum_{i=1}^{n}F(a_i^n,\mu_n)\geq\frac{1}{n}\sum_{i=1}^{n}F(b,\mu_n^i[b])-\epsilon_n.$$

- A is compact, so is $\mathcal{P}(A)$, and so the continuous function F is in fact uniformly continuous. Thus

$$\lim_{n\to\infty} \sup_{b\in A} \max_{i=1,\dots n} |F(b, \mu_n^i[b]) - F(b, \mu_n)| = 0$$

- We conclude that $\int_A F(a,\mu)\mu(da) \geq F(b,\mu)$, so as the supremum of $F(b,\mu)$. Therefore, we find the inequality can only happen if

$$\mu\{a \in A : F(a,\mu) = \sup_{b\in A} F(b,\mu)\} = 1.$$

What we're doing above takes for granted the existence of (ϵ-)Nash equilibria for the n-player games. But this is not necessarily possible. We can only ensure it's possible if we work with *mixed strategies* by Nash's work.

- *mixed strategies*: there are probability distributions over pure strategies and the pay-off functions are the expectations of the players
- *n-tuple of strategies:* one for each player, regarded as a point in the product space obtained by multiplying the n strategy spaces of the players.
- *counter*: One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n1$ strategies of the other players in the countered n-tuple.
- A self-countering n-tuple is called an equilibrium point.
- There is a a one-to-many mapping of the product space into itself describing that countering relationship.
- By using the continuity of the pay-off functions we see that the graph of the mapping is closed + the image of each point under the mapping is convex. From Kakutani's theorem, the mapping has a fixed point.

A probability measure $\mu \in \mathcal{P}(A)$ is called a mean field equilibrium (MFE) if

$$\mu\{a \in A : F(a,\mu) = \sup_{b\in A} F(b,\mu)\} = 1.$$

In other words, μ is an MFE if it is supported on the set of maximizers of the function $F(,\mu)$.

1.3 Uniqueness

? It is not enough just to know that the function $a \mapsto F(a,m)$ has a unique maximizer for each a. That is, if $\hat{a}(m)$ is the unique maximizer of $F(\cdot,m)$ for each $m \in \mathcal{P}(A)$, then any MFE μ must satisfy $\mu = \delta_{\hat{a}(\mu)}$. But this does not mean the MFE is unique.**?**

Theorem 2.
Suppose the objective function F satisfies the *monotonicity condition*,

$$\int_A \left(F\left(a, m_1\right)-F\left(a, m_2\right)\right)\left(m_1-m_2\right)(da)<0,$$

for all $m_1, m_2 \in \mathcal{P}(A)$ with $m_1 \neq m_2$. Then there is at most one MFE.

For an example of a function F satisfying the monotonicity condition, suppose the space A is finite, with cardinality $|A|=d$. Then $\mathcal{P}(A)$ can be identified with the simplex in $\mathbb{R}^d$, namely, the set Δ^d of $(m_1, \ldots, m_n) \in \mathbb{R}^d$ with $m_i \geq 0$ and $\sum_{i=1}^d m_i=1$. For $i \in A$, write $F_i(m)=F(i, m)$. We call the game a potential game if there exists a function $G: \Delta^d \rightarrow \mathbb{R}$, such that $\nabla G=(F_1, \ldots, F_d)$, and we call G the potential function. Suppose that we have a strictly concave potential G. One of the many characterizations of strict concavity reads as

$$\left(\nabla G(m)-\nabla G\left(m^{\prime}\right)\right) \cdot\left(m-m^{\prime}\right)<0, \quad \text{for all } m_1, m_2 \in \Delta^d, m_1 \neq m_2.$$

This inequality may be written as

$$\sum_{i=1}^d\left(F_i(m)-F_i\left(m^{\prime}\right)\right)\left(m_i-m_i^{\prime}\right)<0,$$

which is exactly the assumption. We can go a bit further with this idea. Suppose our game admits a potential function G, not necessarily concave. Then the directional derivative of G at $m \in \Delta^d$ in the direction of $m^{\prime} \in \Delta^d$ is given by

$$\begin{aligned}
D_{m^{\prime}} G(m) &:=\frac{d}{d \epsilon} G\left(m+\epsilon\left(m^{\prime}-m\right)\right)=\nabla G(m) \cdot\left(m^{\prime}-m\right) \\
&=\sum_{i=1}^d F_i(m)\left(m_i^{\prime}-m_i\right) \\
&=\int_A F(a, m) m^{\prime}(da)-\int_A F(a, m) m(da).
\end{aligned}$$

Now, $m \in \Delta^d$ is a MFE iff $\int_A F(a, m) m^{\prime}(da) \leq \int_A F(a, m) m(da)$ iff $D_{m^{\prime}} G(m) \leq 0$ for every $m^{\prime} \in \Delta^d$.
In other words, m is a mean field equilibrium if and only if it locally maximizes the potential function G. If G is assumed concave, we conclude that m is a mean field equilibrium if and only if it maximizes G globally

1.4 A converse to Theorem 1

Granted a continuous objective function, we know limit points of n-player equilibria are always MFE. But does every MFE arise as the limit of some sequence of n-player Nash equilibria?

Theorem 3. Suppose $\mu \in \mathcal{P}(A)$ is an MFE. Then there exist $\epsilon_n \geq 0$ a sequence of strategy profiles $(a_i)_{i\in\mathbb{N}}$ such that

$$\lim_n \frac{1}{n}\sum_{i=1}^{n} \delta_{a_i} = \mu, \ \lim_n \epsilon_n = 0.$$

- Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. A-valued r.v.s, μ_n be its empirical measure.
- $\epsilon_n = \max_{i=1,\ldots,n}(\sup_{a\in A} F(a, \mu_n^i[a]) - F(X_i, \mu_n))$
- $\tilde{\epsilon_n} = \max_{i=1,\ldots,n}(\sup_{a\in A} F(a, \mu_n) - F(X_i, \mu_n))$
- μ is an MFE and X_i is a sample from μ

However, there can exist MFE μ for which there is no sequence of Nash equilibria converging to μ.

1.5 Existence of MFE

Theorem 1. states that any convergent subsequence of n-player approximate equilibria converges to a MFE, and Theorem 2. argues that conversely, every MFE is the limit of some sequence of approximate equilibria. (Not cover the existence of n-player equilibria)
Existence of MFE, however, holds automatically under our standing assumptions.
We use the famous fixed point theorem for set function of Kakutani.

Fixed Point Theorem.
Suppose K is a convex compact subset of a locally convex topological vector space. Suppose $\Gamma : K \to 2^K$ is a set-valued function (where 2^K is the set of subsets of K) satisfying the following conditions:

- (i) $\Gamma(x)$ is nonempty and convex for every $x \in K$.
- (ii) The graph $\mathrm{Gr}(\Gamma) = \{(x, y) \in K \times K : y \in \Gamma(x)\}$ is closed.

Then there exists a fixed point, i.e., a point $x \in K$ such that $x \in \Gamma(x)$.

Theorem 4.
There exists a MFE

Proof. Define a map $\Gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ by letting $\Gamma(\mu)$ denote the set of probability measures which are supported on the set of maximizers of $F(\cdot, \mu)$. That is,

$$\Gamma(\mu) = \left\{ m \in \mathcal{P}(A) : m \left(\left\{ a \in A : F(a, \mu) = \sup_{b\in A} F(b, \mu) \right\} \right) = 1 \right\}.$$

We see from the definition that $\mu \in \mathcal{P}(A)$ is an MFE if and only if it is a fixed point of Γ, i.e., $\mu \in \Gamma(\mu)$. Note that we may also write

$$\Gamma(\mu) = \left\{ m \in \mathcal{P}(A) : \int_A F(a, \mu)m(da) \geq F(b, \mu), \forall b \in A \right\}.$$

We now check the conditions of Kakutani's theorem.
Recall from the Riesz representation that the topological dual of the space $C(A)$ of continuous functions on A is precisely the set $M(A)$ of signed measures on A of bounded variation. The corresponding weak * topology is precisely the weak convergence topology, and $\mathcal{P}(A)$ is a convex compact subset of $M(A)$ with this topology. So we can take $K = \mathcal{P}(A)$ in Kakutani's theorem. Let us check the two required properties of the map Γ :
(i) Fix $\mu \in \mathcal{P}(A)$. Let $S \subset A$ denote the set of maximizers of $F(\cdot, \mu)$, defined by

$$\left\{ a \in A : F(a, \mu) = \sup_{b \in A} F(b, \mu) \right\}.$$

We can then write $\Gamma(\mu) = \{m \in \mathcal{P}(A) : m(S) = 1\}$. Because F is continuous, the set S is nonempty, and we conclude that $\Gamma(\mu)$ is also nonempty. Moreover, $\Gamma(\mu)$ is clearly convex: If $m_1, m_2 \in \Gamma(\mu)$ and $t \in (0, 1)$, then setting $m = tm_1 + (1-t)m_2$ we have

$$m(S) = tm_1(S) + (1-t)m_2(S) = t + (1-t) = 1,$$

so $m \in \Gamma(\mu)$.
(ii) The graph of Γ can be written as

$$\mathrm{Gr}(\Gamma) = \bigcap_{b \in A} K_b,$$

where we define, for $b \in A$,

$$K_b = \left\{ (\mu, m) \in \mathcal{P}(A) \times \mathcal{P}(A) : \int_A F(a, \mu) m(da) \geq F(b, \mu) \right\}$$

If we show that K_b is closed for each $b \in A$, then it will follow that $\mathrm{Gr}(\Gamma)$ is closed. □

Interestingly, our n-player game may fail to have a Nash equilibrium (in pure strategies), but Theorem 4. ensures that there still exists an MFE. Using Theorem 3, we conclude that there exist $\epsilon_n \to 0$ such that for each n there exists an ϵ_n-Nash equilibrium for the n-player game! So, even though there are not necessarily Nash equilibria, the mean field structure lets us construct *approximate equilibria* for large n.

1.6 Multiple types of agents

The model studied in this section is admittedly unrealistic in the sense that the agents are extremely homogeneous. A much more versatile framework is obtained by introducing different *types of agents*, with the essential ideas behind the analysis being the same.

1.6.1 Set-up

- *action space* A
- $\mathcal{T}$ be a complete, separable metric space, which we will call the *type space*. (In practice, both A and $\mathcal{T}$ are often finite.)
- The *payoff function* is now $F : \mathcal{T} \times A \times \mathcal{P}(\mathcal{T} \times A) \to \mathbb{R}$, acting on a type, an action, and a type-action distribution. If each agent i in the n-player game is assigned a type

t_i, then the reward for agent i is

$$F\left(t_i, a_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{(t_k,a_k)}\right)$$

when agents choose actions $a_1, \ldots, a_n$.

- An important additional feature we can incorporate is a *dependence of the set of allowable actions on the type parameter*. Suppose $C : \mathcal{T} \to 2^A$ is a set-valued map, with the interpretation that $C(t)$ is the set of admissible actions available to an agent of type t. We call C the *constraint map*. let

$$\mathrm{Gr}(C) = \{(t, a) \in \mathcal{T} \times A : a \in C(t)\}$$

denote the graph of C.

We now define an ϵ-Nash equilibrium associated with types $(t_1, \ldots, t_n)$ as a vector $(a_1, \ldots, a_n) \in A$, with $a_i \in C(t_i)$ for each i, such that

$$F\left(t_i, a_i, \frac{1}{n}\sum_{k=1}^{n} \delta_{(t_k,a_k)}\right) \geq F\left(t_i, b, \frac{1}{n}\sum_{k\neq i} \delta_{(t_k,a_k)} + \frac{1}{n}\delta_{(t_i,b)}\right) - \epsilon,$$

for each $b \in C(t_i)$, for each $i = 1, \ldots, n$.

This more general setup retains the essential symmetric features of the previous setup, the idea being that in a large- n limit we can still obtain distributional limits if we know something about the distribution of types.

1.6.2 Extension of Theorem 1

Theorem 1′. Assume that F and C satisfy the following assumptions:

- F is jointly continuous on $\mathrm{Gr}(C) \times \mathcal{P}(\mathrm{Gr}(C))$, where $\mathcal{P}(\mathrm{Gr}(C))$ is shorthand for the set of $\mu \in \mathcal{P}(\mathcal{T} \times A)$ with $\mu(\mathrm{Gr}(C)) = 1$.
- $C(t)$ is nonempty for each $t \in \mathcal{T}$.
- The graph $\mathrm{Gr}(C)$ is closed.
- C is lower hemicontinuous, which means: If $t_k \to t$ in $\mathcal{T}$ and $a \in C(t)$, then there exist a subsequence (k_j) and some $a_j \in C\left(t_{k_j}\right)$ such that $a_j \to a$.

Suppose for each n we are given $(t_1^n, \ldots, t_n^n) \in \mathcal{T}^n$, as well as an ϵ_n- Nash equilibrium $(a_1^n, \ldots, a_n^n) \in A^n$ for the corresponding game, where $\epsilon_n \to 0$. Let

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\left(t_k^n, a_k^n\right)}$$

denote the empirical *type-action distribution.* Suppose finally that the empirical type distribution converges weakly to some $\lambda \in \mathcal{P}(\mathcal{T})$, i.e.,

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{t_k^n} \to \lambda$$

Then (μ_n) is tight, and every weak limit point $\mu \in \mathcal{P}(\mathcal{T} \times A)$ is supported on the set

$$\left\{(t, a) \in \mathcal{T} \times A : F(t, a, \mu) = \sup_{b \in A} F(t, a, \mu)\right\}$$

The proofs of Theorem and of the existence of mean field equilibria are very similar to those of Theorems 1 and 4. The key missing ingredient is (a special case of) Berge's theorem:

Berge's theorem.
Suppose C satisfies the assumptions of Theorem 4.11. For each $m \in \mathcal{P}(\mathrm{Gr}(C))$ and $t \in \mathcal{T}$, define $C^*(t, m)$ to be the set of admissible maximizers of $F(t, \cdot, m)$, i.e.,

$$C^*(t,m) = \left\{ a \in C(t) : F(t,a,m) = \sup_{b \in C(t)} F(t,b,m) \right\}$$

Define also the maximum value

$$F^*(t,m) = \sup_{b \in C(t)} F(t,b,m)$$

Then F^* is (jointly) continuous, and the graph

$$\mathrm{Gr}\,(C^*) = \{(t,a,m) \in \mathcal{T} \times A \times \mathcal{P}(\mathrm{Gr}(C)) : a \in C^*(t,m)\}$$

is closed.

1.6.3 Congestion games

These are standard models of network traffic, which we will think of in terms of their original application in road networks for the sake of intuition, though recent applications focus on different kinds of networks.
We have a finite directed graph $G = (V, E)$, the vertices represent locations in the network, and a directed edge (u, v) is a road from u to v.
Each edge is assigned an increasing cost function $c_e : [0,1] \to \mathbb{R}_+$, with $c_e(u)$ representing the speed or efficiency of the road e when the load on the road is u, which means that the road is utilized by a fraction u of the population.
The type space $\mathcal{T}$ is a subset of $V \times V$, with an element $(u, v) \in \mathcal{T}$ representing a source-destination pair. An agent of type (u, v) starts from the location u and must get to v.
The action space A (a subset of 2^E) is the set of all Hamiltonian paths in the graph G. A Hamiltonian path is simply a subset of E which can be arranged into a path connecting two vertices.
Finally, the admissible actions to an agent of type $t = (u, v)$ is the set $C(t)$ consisting of all paths connecting u to v.
Therefore, the cost function $F : \mathcal{T} \times A \times \mathcal{P}(\mathcal{T} \times A) \to \mathbb{R}$ is defined by setting

$$F(t,a,m) = \sum_{e \in a} c_e\left(\ell_e(m)\right), \quad \text{where} \quad \ell_e(m) = m\left\{(t', a') \in \mathcal{T} \times A : e \in a'\right\}$$

Given a distribution of type-action pairs m, the value $\ell_e(m)$ is the fraction of agents who use the road e in their path, and thus it is called the *load of* e. The travel time faced by an agent of type t choosing path a is then calculated by adding, over every road used ($e \in a$), the cost incurred on that road, which is a function (c_e) of the load.

Note that agents are now seeking to minimize travel time by taking $-F$ to be our reward function.

One nice feature of congestion games is that they are always potential games. Indeed, the function

$$U(m) := \sum_{e \in E} \int_0^{\ell_e(m)} c_e(s)ds$$

can be shown to be a convex potential function. This means that minimizers of U correspond to mean field equilibria. In fact, U is sometimes even strictly convex.

An important question studied in this context pertains to the effect of network topology on the *efficiency of Nash equilibria*. A common measure of efficiency is the so-called *price of anarchy*, defined as the worst-case ratio of average travel time in equilibrium to the minimal possible travel time achievable by a central planner. To be somewhat more: The average travel times over a continuum of agents with type-action distribution m is given by

$$A(m) = \int_{\mathcal{T} \times A} F(t, a, m) m(dt, da)$$

If we let $\mathcal{M}$ denote the set of mean field equilibria with a given type distribution, then the price of anarchy is defines as

$$\mathrm{PoA} = \frac{\sup_{m \in \mathcal{M}} A(m)}{\inf_{m \in \mathcal{P}(\mathrm{Gr}(C))} A(m)}$$

Of course, the price of anarchy is always at least 1 (and we should be careful to avoid dividing by zero). Various upper bounds are known for different kinds of networks and different restrictions on the cost functions c_e.

References

Professor Daniel Lacker, IEOR at Columbia University.Rough lecture notes from the Spring 2018 PhD course (IEOR E8100) on mean field games and interacting diffusion models.