[Definition]Definition
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Production of an exhaustible resource and Mexican wave

MFE in applications

Guangyu Hou

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Outline

Oil production

Background

- A large number of oil producers so that one can apply simple hypotheses like the continuum (mean field games modeling) and perfect competition (price-taker behavior of agents-accept the equilibrium price at which it sells goods)
- Initial reserve R_0, distributed according to an initial distribution $m(0, \cdot)$
- Each reserve will contribute to production q, we have $dR(t) = -q(t)dt + \nu R(t)dW_t$ where the Brownian motion is specific to each specific agent

Definition

Profit criterion (the same for all agents)

$$\max_{(q(t))_t} \mathbb{E} \int_0^{\infty} (p(t)q(t) - C(q(t)))e^{-rt} ds, \text{ with } q(t), R(t) \geq 0$$

Background

- C is the cost function (Here $C(q) = \alpha q + \beta \frac{q^2}{2}$)
- the prices p are determined according to the **supply/demand** equilibrium on the market at each moment
- demand is given by a function $D(t, p)$ at instant t. It can be written as $We^{\rho t} p^{-\sigma}$
- $We^{\rho t} p^{-\sigma}$ denotes the total wealth affected by a constant growth rate to model economic growth and σ is the elasticity of demand (elasticity of substitution between oil and any other good)
- supply is given by the total oil production of the agents.

Equilibrium of the Deterministic Case

Characterization

Here $\nu = 0$.
Equilibrium, which is characterized by the following equations where p, q and λ are unknown functions and R_0 the level of initial oil reserve.

$$D(s, p(s)) = \int q\left(s, R_0\right) m_0\left(R_0\right) dR_0$$

$$q\left(s, R_0\right) = \frac{1}{\beta}\left[p(s) - \alpha - \lambda\left(R_0\right) e^{rs}\right]_+$$

$$\int_0^{\infty} q\left(s, R_0\right) ds = R_0$$

Let's consider the problem of an oil producer with an oil reserve equal to R_0. The optimal production levels can be found using a Lagrangian:

$$\mathcal{L} = \int_0^{\infty} (p(s)q(s) - C(q(s)))e^{-rs} ds + \lambda\left(R_0 - \int_0^{\infty} q(s) ds\right)$$

Proof

The first order condition is:

$$p(s) = C'(q(s)) + \lambda e^{rs}$$

where λe^{rs} is the Hotelling rent (the opportunity coast of depleting oil reserves over time when sold in the future). In this equation λ depends on the initial oil stock (or reserve) which measures the strength of the constraint associated to the exhaustible nature of oil, and it will be denoted $\lambda(R_0)$, which equalizes the whole stream of production and the initial oil reserve:

$$\int_0^\infty q(s, R_0)\, ds = \frac{1}{\beta}\int_0^\infty \left(p(s) - \alpha - \lambda(R_0)\, e^{rs}\right)_+ ds = R_0$$

Now, we need to find the prices that were left unknown. This simply is given by the demand/supply equality.

$$D(s, p(s)) = \int q(s, R_0)\, m_0(R_0)\, dR_0$$

Computation

Since q only depends on $\lambda(\cdot)$ and $p(\cdot)$ we can totally separate the variables t and R_0. We consider a dynamical system indexed by the variable θ like the following

$$\partial_\theta p(t,\theta) = D(t, p(t,\theta)) - \int q\left(t, R_0\right) m_0\left(R_0\right) dR_0$$

$$\partial_\theta \lambda\left(R_0,\theta\right) = \int_0^\infty q\left(t, R_0\right) dt - R_0$$

where

$$q\left(t, R_0\right) = \frac{1}{\beta}\left[p(t,\theta) - \alpha - \lambda\left(R_0,\theta\right) e^{rt}\right]_+$$

Once a dynamical system is chosen, we will have

$$\lim_{\theta\to+\infty} p(t,\theta) = p(t)$$

$$\lim_{\theta\to+\infty} \lambda\left(R_0,\theta\right) = \lambda\left(R_0\right)$$

Stochastic case

The mean field games PDEs

With noise and interference in the model, we will develop completely coupled PDEs.
To start writing the equations, let's introduce $u(t, R)$ the Bellman function of the problem, namely:

$$u(t, R) = \max_{(q(s))_{s\geq t}, q\geq 0} \mathbb{E} \int_t^{\infty} (p(s)q(s) - C(q(s)))e^{-r(s-t)} ds$$
$$\text{s.t. } dR(s) = -q(s)ds + \nu R(s)dW_s, R(t) = R$$

The Hamilton Jacobi Bellman equation associated to this optimal control problem is:

$$\partial_t u(t, R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t, R) - ru(t, R) + \max_{q\geq 0} (p(t)q - C(q) - q\partial_R u(t, R)) = 0$$

Transport Equation

Now, let's denote $m(t, R)$ the distribution of oil reserves at time t. This distribution is transported by the optimal production decisions of the agents $q^*(t, R)$ where, now, R is the reserve at time t and not the initial reserve as in the deterministic case.
The transport equation is:

$$\text{(Kolmogorov)} \qquad \partial_t m(t, R) + \partial_R \left(-q^*(t, R) m(t, R)\right) = \frac{\nu^2}{2} \partial^2_{RR} \left[R^2 m(t, R)\right]$$

with $m(0, \cdot)$ given.

Interdependence

Now, let's discuss the interdependence between u and m. m is linked to u quite naturally since m is transported by the optimal decisions of the agents determined by the optimal control in the HJB equation. This optimal control is given by :

$$q^*(t,R) = \left[\frac{p(t) - \alpha - \partial_R u(t,R)}{\beta}\right]_+$$

Now, u depends on m through the price $p(t)$ and this price can be seen as a function of m. Indeed, because $p(t)$ is fixed so that supply and demand are equal, $p(t)$ is given by:

$$p(t) = D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right)$$

Coupled Equations

Therefore, we can update to have these coupled equations

$$\partial_t u(t,R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t,R) - ru(t,R)$$

$$+\frac{1}{2\beta}\left[\left(D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right) - \alpha - \partial_R u(t,R)\right)_+\right]^2 = 0$$

$$\partial_t m(t,R) + \partial_R\left(-\left[\frac{D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right) - \alpha - \partial_R u(t,R)}{\beta}\right]_+ m(t,R)\right)$$

$$= \frac{\nu^2}{2}\partial^2_{RR}\left(R^2 m(t,R)\right)$$

Mexican wave

Introduction

It is set to understand how a Mexican wave can be one of the solution of a mean field game involving a (infinite) set of supporters and a taste for mimicry.
To simplify our study, we regard our stadium as a circle of length L, thus each one of the of the continuum of individuals is referenced by a coordinate $x \in [0, L)$.
They are free to behave and can be either seated $(z = 0)$ or standing $(z = 1)$ or in an intermediate position $z \in (0, 1)$.
We model this using a utility function u. Typically, u will be defined as $u(z) = -Kz^{\alpha}(1 - z)^{\beta}$ to express being standing or seated is more comfortable than in an intermediate position.
The optimization function for any agent:

- pays a price $h(a)dt$ to change his position from z to $z + adt$. Here it's $\frac{a^2}{2}$
- an agent wants to behave as his neighbors. Then an agent in x maximizes

$$-\frac{1}{\epsilon^2}\int (z(t,x) - z(t,x-y))^2 \frac{1}{\epsilon} g\left(\frac{y}{\epsilon}\right) dy$$

  where g is a Gaussian kernel and y is the distance between the agent and his neighbors.
- An agent maximizes his comfort described by u.

Optimization criterion

The optimization criterion for an agent localized at x is then

$$\sup_{z(\cdot,x)} \operatorname{liminf}_{T\to+\infty} \frac{1}{T}\int_0^T \left\{ \left[-\frac{1}{\epsilon^2}\int (z(t,x)-z(t,x-y))^2 \frac{1}{\epsilon} g\left(\frac{y}{\epsilon}\right) dy \right] + u(z(t,x)) -$$

This ergodic control problem can be formally transformed in a differential way and we get:

$$-\frac{2}{\epsilon^2}\int (z(t,x)-z(t,x-y))\frac{1}{\epsilon} g\left(\frac{y}{\epsilon}\right) dy + u'(z(t,x)) = -\partial_{tt}^2 z(t,x)$$

If we let ϵ tends to 0 , we get in the distribution sense that our problem is to solve the equation:

$$\partial_{tt}^2 z(t,x) + \partial_{xx}^2 z(t,x) = -u'(z(t,x))$$

Mean Field Expression

This equation doesn't seem to be of the mean field type but we can write the associated mean field equations. Let's consider that agents are indexed by x. For each x, the Bellman function associated to the problem of an agent in x can be written as $J(x;\cdot)$ solving the Hamilton-Jacobi equation:

$$0 = \partial_t J(x;t,z) + \frac{1}{2}\left(\partial_z J(x;t,z)\right)^2 + u(z) - \frac{1}{\epsilon^2}\int (z-\bar{z})^2 m(\bar{x};t,\bar{z})\frac{1}{\epsilon} g\left(\frac{x-\bar{x}}{\epsilon}\right) d\bar{z}d\bar{x}$$

where $m(x;t,\cdot)$ is the probability distribution function of the position z of an agent situated in x. $m(x;\cdot,\cdot)$ solves a Kolmogorov equation that is:

$$\partial_t m(x;t,z) + \operatorname{div}\left(\partial_z J(x;t,z) m(x;t,z)\right) = 0$$

with $m(x;0,z) = \delta_{z(0,x)}(z)$. Hence, the problem can be written as a set of Hamilton-Jacobi equations indexed by x with the associated Kolmogorov equations.

As a Solution

A Mexican wave is, by definition, a wave. Hence we are going to look for a solution of the form $z(t,x) = \phi(x - vt)$ where v is the speed of the wave. But what we call Mexican wave is usually a specific form of wave and we want to call Mexican wave a function ϕ with a compact support on $(0, L)$. If we look for such a function ϕ, we can easily see that it must solve:

$$\left(1 + v^2\right)\phi'' = -u'(\phi)$$

Existence of Mexican waves for $\alpha, \beta \in (1;2)$). Suppose that $\alpha, \beta \in (1;2)$. Then, for any v verifying

$$\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)\Gamma\left(1 - \frac{\beta}{2}\right)}{\Gamma\left(2 - \frac{\alpha+\beta}{2}\right)} < \sqrt{\frac{K}{2\left(1 + v^2\right)}}L$$

there exists a Mexican wave ϕ solution of $\left(1 + v^2\right)\phi'' = -u'(\phi)$.

As a Solution

Using an "energy method". Consequently, we have

$$\phi' = \pm\sqrt{\frac{2K}{1+v^2}}\phi^{\alpha/2}(1-\phi)^{\beta/2}$$

Reference

Olivier Guéant, Jean-Michel Lasry, Pierre-Louis Lions. Mean field games and applications

Thank you!