[Definition]Definition
[Corollary]Corollary
[Theorem]Theroem

An Initial Simple Model

better understanding for MFE

Guangyu Hou

2024/08/29

Outline

Background

Background

- A meeting is scheduled for a certain time t which only starts several minutes after the scheduled time.
- The actual time T when the meeting starts depends on the arrival of its participants.
- A rule sets the start of the meeting at the point when a certain quorum is reached. (which is a form of strategic interaction)

Definition

Three kinds of time.

- the scheduled time of the meeting $\tilde{t}$.
- the time agent i planned to arrive τ^i, and the actual arrival time $\tilde{\tau}^i = \tau^i + \sigma^i \tilde{\epsilon}^i$ where $\tilde{\epsilon}^i$ is a normal noise with variance 1, specific to each agent.
- note m_0 the distribution of σ^i in the population.
- the actual time the meeting will start T

Total time cost

To decide one's intended arrival time τ^i, each will optimize a total cost $c(\tilde{t}, T, \tilde{\tau})$ which is assumed to be made of three components: reputation effect + personal inconvenience + waiting time cost. (simple version, convex)

- *reputation effect*: a cost of lateness in relation to scheduled time t, $c_1(\tilde{t}, T, \tilde{\tau}) = \alpha[\tilde{\tau} - \tilde{t}]_+$
- *personal inconvenience*: a cost of lateness in relation to the actual starting time of the meeting T, $c_2(\tilde{t}, T, \tilde{\tau}) = \beta[\tilde{\tau} - T]_+$
- *waiting time cost*: time lost waiting to reach time T, $c_3(t, T, \tilde{\tau}) = \gamma[T - \tilde{\tau}]_+$

Resolution

Individual choices

Each agent aims to minimize his expected total cost by assuming T to be known. T is a *priori* a random variable. But we are considering an infinite number of players, the "law of large numbers" will imply T (the mean field) is deterministic. For agent i, the problem is therefore:

$$\tau^i = \arg\min_{\tau^i} \mathbb{E}[c(\tilde{t}, T, \tilde{\tau}^i)], \; \tilde{\tau}^i = \tau^i + \sigma^i \tilde{\epsilon}^i$$

We want to show that individual choices fully generate the realization of this time T supposing T is known, i.e. we're going to show the existence of a fixed point and in this case, the point is T.
The problem starts with examine agents' individual choices (to minimize the expected total cost). By the first-order condition and a strictly monotonic cumulative distribution function, we prove the existence and uniqueness of τ^i, a function of $(\tilde{t}, T, \sigma^i)$ and the distribution is transported from $\sigma^i \mapsto \tilde{\tau}^i$.

$$\alpha \mathcal{N}\left(\frac{\tau^i - \tilde{t}}{\sigma^i}\right) + (\beta + \gamma)\mathcal{N}\left(\frac{\tau^i - T}{\sigma^i}\right) = \gamma$$

Scheme

If we note F the deterministic CDF of the agents' real arrival times, we can establish a rule on the real starting time T, depending on the function $F(\cdot)$. Then we reached the following statement.

- Starting from a value T, we obtain agents' optimal strategies $(\tau^i(\cdot;T))_i$ $T \mapsto (\tau^i(\cdot;T))_i$
- Actual arrival time is based on the optimal strategies but affected by noise $(\tau^i(\cdot;T))_i \mapsto (\tilde{\tau}^i(\cdot;T))_i$
- From the law of large numbers and the hypothesis of the independence of agents' uncertainties, these arrival times are distributed according to a deterministic F representing the proportion (or probability) of agents who have arrived by a particular time $(\tilde{\tau}^i(\cdot;T))_i \mapsto F$
- T is deduced from F by the meeting starting time rule $F \mapsto T^*(F)$

We reach the scheme

$$T^{**} : T \mapsto (\tau^i(\cdot;T))_i \mapsto (\tilde{\tau}^i(\cdot;T))_i \mapsto F = F(\cdot,T) \mapsto T^*(F)$$

Existence and Uniqueness of a fixed point

Theorem

If $\alpha, \beta, \gamma > 0$ and if $0 \notin \overline{supp(m_0)}$, then T^{**} is a contraction mapping of $[t; +\infty]$, and there's a unique solution T to our problem.

Differentiate with respect to T the FOC defines τ^i,

$$\frac{d\tau^i}{dT}\left[\alpha\mathcal{N}'\left(\frac{\tau^i - \tilde{t}}{\sigma^i}\right) + (\beta+\gamma)\mathcal{N}'\left(\frac{\tau^i - T}{\sigma^i}\right)\right] = (\beta+\gamma)\mathcal{N}'\left(\frac{\tau^i - T}{\sigma^i}\right)$$

Since 0 is not in the support of m_0, we know $\frac{d}{dT}\tau(t,\sigma;T) \leq k < 1$. Hence, $\forall T, s, h > 0$,

$$\begin{aligned} F(s; T+h) &= \mathbb{P}(\tau^i(\sigma^i); T+h) + \sigma^i\epsilon^i \leq s) && (1) \\ &\geq \mathbb{P}(\tau^i(\sigma^i); T) + kh + \sigma^i\epsilon^i \leq s) && (2) \\ &= F(s - kh; T) && (3) \end{aligned}$$

Existence and Uniqueness of a fixed point

Therefore,

$$
\begin{aligned}
T^*(F(\cdot; T+h)) &\leq T^*(F(\cdot - kh; T)) && (4)\\
&\leq T^*(F(\cdot; T)) + kh && (5)
\end{aligned}
$$

and this proves the result through the contraction mapping theorem.
It's not difficult but necessary to prove the following properties of the setting rule $T^* : F(\cdot) \mapsto T$ for the proceeding proof.

- $\forall F(\cdot)$, $T^*(F(\cdot)) \geq \tilde{t}$; the meeting never starts before $\tilde{t}$
- **(Monotony)** For two cumulative distribution functions $F(\cdot)$ and $G(\cdot)$, if $F(\cdot) \leq G(\cdot)$, then $T^*(F(\cdot)) \geq T^*(G(\cdot))$
- **(Sub-additivity)** $\forall s > 0,\ T^*(F(\cdot - s)) - T^*(F(\cdot)) \leq s$

Variants

"Geographical" Model

We will talk about a "geographical" model, i.e. the agents are initially distributed in different places and must come to where the meeting is being held.
The interest of this variant is that it will show how *coupled forward/backward* PDEs emerge.
More specifically, suppose that the agents are distributed on the negative half-line according to distribution function $m_0(\cdot)$ (with compact support and such that $m_0(0) = 0$) and that they must go to the meeting held at 0.
Suppose that in order to get to 0, an agent i moves according to the process $dX_t^i = a_t^i dt + \sigma dW_t^i$ where drift a is controlled for a quadratic cost $\frac{1}{2}a^2$.
Therefore, the optimization problem each agent is faced can be written as :

$$\min_{a(\cdot)} \mathbb{E}\left[c(\tilde{t}, T, \tilde{\tau}^i) + \frac{1}{2}\int_0^{\tilde{\tau}^i} a^2(t)dt\right]$$

with $X_0^i = x_0$ and the time to reach 0 is given by $\tilde{\tau}^i = \min\{s : X_s^i = 0\}$.

HJB Equation

When looking for a Nash-MFG equilibrium, given T, it's a problem of stochastic control for each agent. Here the reward functional is defined as $u = u(t, x, a) = \mathbb{E}\left[c(\tilde{t}, T, \tilde{\tau}^i) + \frac{1}{2}\int_t^{\tilde{\tau}^i} a^2(s)ds | X_0 = x\right]$ where t is the departure time, x is the initial position, a is the drift or action taken along the time of moving with a quadratic cost.

For DPP, we can now define the value function as $V(t, x) = \inf_a u(t, x, a)$. By Markov property, one can derive that fix $0 \leq t < r \leq \tilde{\tau}^i, V(t, x) = \inf_a \left[\mathbb{E}[\frac{1}{2}\int_t^r a^2(s)ds | X_0 = x] + V(r, X_r^{t,x})\right]$ where $X_r^{t,x}$ stands for the position of an agent at time r when he departs at time t at place x.

Definition

Infinitesimal generator.

For a smooth function $\psi : \mathbb{R}^d \to \mathbb{R}$, and for $(x, a) \in \mathbb{R}^d \times A$, define

$$L^a\psi(x) = b(x, a) \cdot \nabla_x \psi(x) + \frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^t(x, a)\nabla^2\psi(x)\right]$$

HJB Equation

By Verification Theorem ($u(t, x)$ is $C^{1,2}([0, T] \times \mathbb{R}^d)$, $u(\tilde{\tau}^i, x) = u(\tilde{\tau}^i, 0) = \{c(\tilde{t}, T, \tilde{\tau}^i) : X_0 = x_0, X_{\tilde{\tau}^i} = x = 0\}$), we have the following Hamilton Jacobi Bellman equation: (here $b(x, a) = a_t^i$ and $\sigma(x, a) = \sigma$ independent from a lead to a *reduced Hamiltonian*)

$$0 = \partial_t u + \min_a \left(a\partial_x u + \frac{1}{2}a^2 \right) + \frac{\sigma^2}{2}\partial_{xx}^2 u$$

It's easy to know the minimum of the expression, then we can rewrite it:

$$\partial_t u - \frac{1}{2}(\partial_x u)^2 + \frac{\sigma^2}{2}\partial_{xx}^2 u = 0$$

with the condition at the limit is $\forall \tau, u(\tau, 0) = c(t, T, \tau)$ for a fixed and deterministic T. (Note that c here has the same shape as the preceding setup but we impose it not to be piecewise-linear but twice continuously differentiable.)

Kolmogorov Forward Equation

Moreover, the solution here is Markovian, i.e. the optimal drift $a(s,x) = -\partial_x u(s,x)$ (minimum point of the quadratic function) therefore depends only on the place x and the time s.
Consider a stochastic process X_t governed by the following stochastic differential equation (SDE): $dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$. Let $p(x,t)$ be the PDF of the stochastic process X_t at time t, where x is the value of X_t. The PDF $p(t,x)$ describes the likelihood that the process is at a particular state x at time t.
Then the Kolmogorov Forward Equation describes how the PDF $p(x,t)$ evolves over time:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x,t)p(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x,t)p(x,t)\right]$$

Kolmogorov Forward Equation

This equation consists of two main parts:

- Drift Term $-\frac{\partial}{\partial x}[\mu(x,t)p(x,t)]$: the effect of the deterministic drift $\mu(x,t)$ on the distribution of the process and describes the flow of probability mass due to the deterministic part of the dynamics.
- Diffusion Term $\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x,t)p(x,t)\right]$: the effect of the stochastic diffusion $\sigma(x,t)$ on the distribution, describes the spread of the probability density due to random fluctuations.

To solve the Kolmogorov forward equation, appropriate boundary and initial conditions are required

Distribution of agents

The distribution $m(t,x)$ of agents corresponds to the distribution of players who have not yet arrived at 0 and m loses mass through 0. Replace μ to be the optimal drift, we derive the dynamics of m:

$$\partial_t m + \partial_x((-\partial_x u)m) = \frac{\sigma^2}{2}\partial^2_{xx} m$$

$m(0,\cdot) = m_0(\cdot)$ is fixed and we will try to find a solution with the condition: $m(\cdot, 0) = 0$

As we have chosen to model the problem by the dynamics of Brownian diffusion, the model must be complemented and restricted to a compact domain, $[0, T_{max}] \times [-X_{max}, 0]$ and the boundary conditions are

$$u(T_{max}, \cdot) = c(\tilde{t}, T, T_{max}),\ u(\cdot, -X_{max}) = c(\tilde{t}, T, T_{max}),\ m(\cdot, -X_{max}) = 0$$

The flow reaching 0 (when the agents reach the meeting place) is $s \mapsto -\partial_x m(s,0)$. Thus the cumulative distribution function F of arrival times is defined by

$$F(s) = -\int_0^s \partial_x m(v,0)dv$$

Distribution of agents

Now, T is fixed by the quorum rule (with let's say $\theta = 90\%$) but we impose that it must be in the interval $[\tilde{t}, T_{\max}]$. In other words:

$$T = \begin{cases} \tilde{t}, & \text{if } F^{-1}(\theta) \leq \tilde{t} \\ T_{\max}, & \text{if } F\left(T_{\max}\right) \leq \theta \\ F^{-1}(\theta), & \text{otherwise} \end{cases}$$

Existence of an equilibrium for the meeting starting time

What we need to do is to prove that there is a time T coherent with the (rational) expectations of the agents with a fixed point theorem. One starts from a given T and deduces u. The Kolmogorov equation then gives us m and therefore the arrival flow at 0. Since the time T in our example is given by the arrival of a proportion θ of all the agents, it clearly is a matter of fixed point.
Before going deeply in the mathematics, let's introduce some hypotheses:

- $T \mapsto c(\tilde{t}, T, \tau)$ is a continuous function
- $\tau \mapsto c(\tilde{t}, T, \tau)$ is a C^2 function.
- $m_0(0) = m_0(-X_{\max}) = 0$. Also, we suppose that $|m_0'(0)| > 0$ and $|m_0'(-X_{\max})| > 0$

Similarly, we have the following scheme

$$T \mapsto c(t, T, \cdot) \in C^2 \mapsto u \in C^2 \mapsto \partial_x u \in C^1 \mapsto m \in C^1 \mapsto -\partial_x m(\cdot, 0) \in C^0 (\mapsto F) \mapsto$$

Since the scheme is from $[t, T_{\max}]$ to $[t, T_{\max}]$, to obtain a fixed point result, we just need to prove that the scheme is continuous.

Continuity

- We will start from the second part of the scheme $(c(\tilde{t}, T, \cdot) \in C^2 \mapsto u \in C^2)$. Let's consider the following PDE: $\partial_t u - \frac{1}{2}(\partial_x u)^2 + \frac{\sigma^2}{2}\partial_{xx}^2 u = 0$ with the boundary conditions

$$u(\cdot, 0) = c(\tilde{t}, T, \cdot) \quad u(T_{\max}, \cdot) = c(\tilde{t}, T_{\max}, T_{\max}), \quad u(\cdot, -X_{\max}) = c(\tilde{t}, T_{\max},$$

The solution u is in $C^2([0, T_{\max}] \times [-X_{\max}, 0])$ and $\exists K, \forall T \in [t, T_{\max}], \partial_x u$ is a K-Lipschitz function. Moreover the mapping $c(\tilde{t}, T, \cdot) \in C^2 \mapsto u \in C^2$ is continuous.

- Then consider the following PDE:

$$\text{(Kolmogorov)} \quad \partial_t m + \partial_x(am) = \frac{\sigma^2}{2}\partial_{xx}^2 m$$

with $a \in C^1$ (and hence Lipschitz) and the boundary conditions $m(0, \cdot) = m_0(\cdot), \quad m(\cdot, 0) = 0, \quad m(\cdot, -X_{\max}) = 0$ where m_0 is supposed to verify the above hypotheses. Then the solution m is in $C^1((0, T_{\max}) \times (-X_{\max}, 0))$ and $\exists \epsilon > 0, \inf |\partial_x m(\cdot, 0)| \geq \epsilon$. Moreover ϵ only depends on the Lipschitz constant of the function a. Also the mapping $a \mapsto m \in C^1$ is continuous.

Continuity

- From the above two lemmas, we can deduce a third one adapted to our problem. Indeed, since u is a C^2 function, $a = -\partial_x u$ is a Lipschitz function and hence we have a lower bound to the flow arriving at the meeting: $\exists \epsilon > 0, \forall T \in [t, T_{\max}]$, $\inf |\partial_x m(\cdot, 0)| \geq \epsilon$
- Now, let's consider the mapping $\Psi : -\partial_x m(\cdot, 0) \in C^0 \mapsto T$, then Ψ is a Lipschitz function on $C^0\left([0, T_{\max}], \mathbb{R}_+^*\right)$.

N-player Game

Approximation

We will approximate the N-player game through a first order expansion "$G_0 + \frac{1}{N} G_1 + ...$", where G_0 is the mean field game and G_1 the first order correction coefficient. The solution of "$G_0 + \frac{1}{N} G_1$" reflects a strategic space which agents do not care about other agents individually at first. Only when taking into account the "granularity of the game" and the imperfectness of the continuum hypothesis, we shall care more about the population dynamics and the number of players.

Nash Equilibrium

To simplify, let us say that the number of players is $N = 10k(k = 1, 2, 3, \ldots)$ and thus that the meeting begins with the arrival of the $9k^{th}$ player. A given player (let us say player 1) will aim for an arrival time τ^* which should verify (symmetrical Nash equation):

$$\tau^* = \text{argmin}_{\tau^1} \mathbb{E}\left[C\left(\tau^1 + \sigma\tilde{\epsilon}^1, \tau^* + \sigma\tilde{\epsilon}^2, \ldots, \tau^* + \sigma\tilde{\epsilon}^N\right)\right]$$

This function C does not really depend on all the components of $\left(\tau^* + \sigma\tilde{\epsilon}^2, \ldots, \tau^* + \sigma\tilde{\epsilon}^N\right)$ but only on two statistics of order $\tau^* + \sigma\tilde{\epsilon}_{(9k-1)}$ and $\tau^* + \sigma\tilde{\epsilon}_{(9k)}$ where one has noted $\tilde{\epsilon}_{(r)}$ the r^{th} element, in the order, in $\left\{\tilde{\epsilon}^2, \ldots, \tilde{\epsilon}^N\right\}$. We shall only take into account that $\tau^1 + \sigma\tilde{\epsilon}^1, \tau^* + \sigma\tilde{\epsilon}_{(9k-1)}$ and $\tau^* + \sigma\tilde{\epsilon}_{(9k)}$ as 90^{th} percentile. Thus the Nash equilibrium is characterized by:

$$\tau^* = \text{argmin}_{\tau^1} \mathbb{E}\left[G\left(\tau^1 + \sigma\tilde{\epsilon}^1, \tau^* + \sigma\tilde{y}, \tau^* + \sigma\tilde{z}\right)\right]$$

where $(\tilde{y}, \tilde{z})$ are statistics of order corresponding to the $(9k-1)^{\text{th}}$ and $9k^{\text{th}}$ ordered elements of $\left\{\tilde{\epsilon}^2, \ldots, \tilde{\epsilon}^N\right\}$. Hence, the variables $(\tilde{y}, \tilde{z})$ are independent of $\tilde{\epsilon}^1$.

Solution

Taking up the initial model, the function G (which is continuous, piecewise linear and convex) is defined by:

$$\forall a, \forall b, \forall c \geq b, \quad G(a, b, c) = G\left(a, t \vee b \wedge T_{\max}, t \vee c \wedge T_{\max}\right)$$

$$\forall b \leq c \in [t, T_{\max}], G(a, b, c) = \begin{cases} -\gamma(a-b) & a \leq t \\ -\gamma(a-b) + \alpha(a-t) & a \in (t, b] \\ \alpha(a-t) & a \in (b, c] \\ \alpha(a-t) + \beta(a-c) & a > c \end{cases}$$

While G is not practical for optimization purposes. Let's introduce H the function $\left(\tau^1, b, c\right) \mapsto \int_{-\infty}^{\infty} G\left(\tau^1 + \sigma x, b, c\right) \mathcal{N}'(x) dx$ where $\mathcal{N}$ still is the cumulative distribution function of a normal variable with variance 1. Thus $\forall b \leq c, H$ is a strictly convex function of τ^1 that decreases and then increases with following derivatives:

Solution

$$\partial_1 H\left(\tau^1, b, c\right) = \left[-\gamma \mathcal{N}\left(\frac{b-\tau^1}{\sigma}\right) + \alpha\left(1-\mathcal{N}\left(\frac{t-\tau^1}{\sigma}\right)\right) + \beta\left(1-\mathcal{N}\left(\frac{c-\tau^1}{\sigma}\right)\right.\right.$$

$$\partial_{11}^2 H\left(\tau^1, b, c\right) = \frac{1}{\sigma}\left[\gamma \mathcal{N}'\left(\frac{b-\tau^1}{\sigma}\right) + \alpha \mathcal{N}'\left(\frac{t-\tau^1}{\sigma}\right) + \beta \mathcal{N}'\left(\frac{c-\tau^1}{\sigma}\right)\right]$$

$$\partial_{12}^2 H\left(\tau^1, b, c\right) = -\frac{1}{\sigma}\gamma \mathcal{N}'\left(\frac{b-\tau^1}{\sigma}\right)$$

$$\partial_{13}^3 H\left(\tau^1, b, c\right) = -\frac{1}{\sigma}\alpha \mathcal{N}'\left(\frac{c-\tau^1}{\sigma}\right)$$

Let's now recall that we want to find a symmetrical Nash equilibrium and the condition is given by:

$$\tau^* = \text{argmin}_{\tau^1} \mathbb{E}\left[G\left(\tau^1 + \sigma\tilde{\epsilon}^1, \tau^* + \sigma\tilde{y}, \tau^* + \sigma\tilde{z}\right)\right]$$

Clearly this can be rewritten using the function H and we get:

$$\tau^* = \text{argmin}_{\tau^1} \mathbb{E}\left[H\left(\tau^1, \tau^* + \sigma\tilde{y}, \tau^* + \sigma\tilde{z}\right)\right]$$

The following lemma will be helpful for introducing compactness

Solution

Lemma

$$B = \left\{ \tau^1 \mid \exists b \leq c, \partial_1 H \left(\tau^1, b, c \right) = 0 \right\}$$

is a bounded set.

Let's introduce now the best response function of agent 1. This function Γ is defined as:

$$\Gamma \left(\tau^* \right) = \operatorname{argmin}_{\tau^1} \mathbb{E} \left[H \left(\tau^1, \tau^* + \sigma \tilde{y}, \tau^* + \sigma \bar{z} \right) \right]$$

Another (though implicit) definition of this function is based on the first order condition:

$$\mathrm{E} \left[\partial_1 H \left(\Gamma \left(\tau^* \right), \tau^* + \sigma \tilde{y}, \tau^* + \sigma \tilde{z} \right) \right] = 0$$

Contraction mapping

Then we can deduce that $\forall \tau^*$, $\inf B \leq \Gamma(\tau^*) \leq \sup B$, i.e. we can restrict Γ to the set $K = [\inf B, \sup B]$. If we define $\Gamma_{|K} : \tau^* \in K \mapsto \Gamma(\tau^*)$, we see that any symmetrical Nash equilibrium must be a fixed point of $\Gamma_{|K}$.
We will then prove $\Gamma_{|K}$ is a contraction mapping from K to K.

Proof.

Given $(*)$, using the implicit function theorem we have,

$$\Gamma'(\tau^*) = -\frac{\mathbb{E}\left[\partial^2_{12} H\left(\Gamma\left(\tau^*\right), \tau^* + \sigma\tilde{y}, \tau^* + \sigma\tilde{z}\right) + \partial^2_{13} H\left(\Gamma\left(\tau^*\right), \tau^* + \sigma\tilde{y}, \tau^* + \sigma\tilde{z}\right)\right]}{\mathbb{E}\left[\partial^2_{11} H\left(\Gamma\left(\tau^*\right), \tau^* + \sigma\tilde{y}, \tau^* + \sigma\bar{z}\right)\right]}$$

Since $0 < -\partial^2_{12} H - \partial^2_{13} H < \partial^2_{11} H$, we have $0 \leq \Gamma^{\nu}\left(\tau^*\right) < 1$. Now because K is compact, there exists a constant $\varepsilon > 0$ so that $\forall \tau^* \in K, \Gamma'_{|K}\left(\tau^*\right) \leq 1 - \epsilon$. $\square$

Now using a classical fixed point result we know the existence and uniqueness of symmetrical Nash equilibrium for the game with N players.

Approximation in $1/N$

Before beginning the analysis, recall that the equilibrium is a Dirac measure in the mean field game case since all individuals have the same σ. We note this equilibrium τ^*_{MFG}, and the starting time for the meeting will be (except when a limit is reached) $\tau^*_{MFG} + \sigma F^{-1}(\theta)$ where F is here the cumulative distribution function of a normal distribution. Thus, rather than being defined by:

$$\mathbb{E}\left[\partial_1 H\left(\tau^*_N, \tau^*_N + \sigma\tilde{y}, \tau^*_N + \sigma\bar{z}\right)\right] = 0$$

the mean field games equilibrium is defined by:

$$\partial_1 H\left(\tau^*_{MFG}, \tau^*_{MFG} + \sigma F^{-1}(\theta), \tau^*_{MFG} + \sigma F^{-1}(\theta)\right) = 0$$

Error

Define $J(t,y,z) = \partial_1 H(t, t+\sigma y, t+\sigma z)$ and use Taylor expansion for $\mathbb{E}J\left(\tau_N^*, \tilde{y}, \tilde{z}\right)$ and the properties of order statistics. Let

$$
\begin{aligned}
\xi &= \lim_{N\to\infty} N\mathbb{E}\left(\tilde{y} - F^{-1}(\theta)\right) \in \mathbb{R} \\
\zeta &= \lim_{N\to\infty} N\mathbb{E}\left(\tilde{z} - F^{-1}(\theta)\right) \in \mathbb{R} \\
v &= \lim_{N\to\infty} N\mathbb{E}\left(\tilde{y} - F^{-1}(\theta)\right)^2 \\
&= \lim_{N\to\infty} N\mathbb{E}\left(\tilde{z} - F^{-1}(\theta)\right)^2 = \lim_{N\to\infty} N\mathrm{NE}\left(\tilde{z} - F^{-1}(\theta)\right)^2 \in \mathbb{R}
\end{aligned}
$$

then we can derive that

$$
\tau_N^* = \tau_{MFG}^* - \frac{1}{N}\frac{1}{\partial_1 J}\left[\xi\partial_2 J + \zeta\partial_3 J + \frac{v^2}{2}\left(\partial_{22}J + \partial_{33}J + 2\partial_{23}J\right)\right] + o\left(\frac{1}{N}\right)
$$

Reference

Olivier Gu´eant, Jean-Michel Lasry, Pierre-Louis Lions. Mean field games and applications

Thank you!