Tutorial - III

(1)

i) $\frac{1}{(n+1)!}$

$= \cancel{\sum_{n=0}^{\infty}} \quad \cancel{+} \qquad Z\left\{\frac{1}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$

$\because Z\left\{\frac{1}{(n+r)!}\right\} = z^{r}\left[e^{1/z} - 1 - \frac{1}{z} - \frac{1}{2!z^2} - \cdots - \frac{1}{(r-1)!\,z^{r-1}}\right]$

$\therefore Z\left\{\frac{1}{(n+1)!}\right\} = z^{1}\left[e^{1/z} - 1\right]$

ii) $\cosh n\theta$

$Z\{\cosh n\theta\} = Z\left[\frac{e^{n\theta} + e^{-n\theta}}{2}\right]$

$= \frac{1}{2}\left[Z(e^{n\theta}) + Z(e^{-n\theta})\right]$

$\cancel{\frac{1}{2}\left[Z\{(e^{\theta})^{-n}\}\right]}$

$= \frac{1}{2}\left\{Z\left[(e^{-\theta})^{-n}\right] + Z\left[(e^{\theta})^{-n}\right]\right\}$

$Z(k^{-n} U_n) = U(kz)$

Let $U_n = 1$, $k_1 = e^{-\theta}$, $k_2 = e^{\theta}$

$$\therefore z\{\cosh n\theta\} = \frac{1}{2}\left[z(k_1^{-n}\cdot 1) + z(k_2^{-n}\cdot 1)\right]$$

$$= \frac{1}{2}\left[U(k_1 z) + U(k_2 z)\right]$$

$$\because z(1) = \frac{z}{z-1} = U(z)$$

$$U(kz) = \frac{kz}{kz-1}$$

$$\therefore z(\cosh n\theta) = \frac{1}{2}\left[\frac{k_1 z}{k_1 z - 1} + \frac{k_2 z}{k_2 z - 1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-\theta}z}{e^{-\theta}z-1} + \frac{e^{\theta}z}{e^{\theta}z-1}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - e^{-\theta}z + z^2 - e^{\theta}z}{z^2 - e^{-\theta}z - e^{\theta}z + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z(e^{-\theta}+e^{\theta})}{z^2 - z(e^{-\theta}+e^{\theta}) + 1}\right]$$

$$= \frac{z^2 - z\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)}{z^2 - z(e^{\theta}+e^{-\theta}) + 1}$$

$$= \frac{z^2 - z\cosh\theta}{z^2 - z(e^{\theta}+e^{-\theta}) + 1}$$

$$= \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(2)

i) $\cancel{a^n}$ $\cancel{\sin}$ $z\{a^n \sinh n\theta\}$

$$z\{\sinh n\theta\} = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

Replace z by z/a

$$z\{a^n \sinh n\theta\} = \frac{z/a \sinh\theta}{(z/a)^2 - 2(z/a)\cosh\theta + 1}$$

$$= \frac{z\sinh\theta}{\cancel{a}} \times \frac{a^{\cancel{2}}}{z^2 - 2za\cosh\theta + a^2}$$

$$z\{a^n \sinh n\theta\} = \frac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$$

ii) $a^{|n|}$

$$z\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}$$

$$\therefore z\{a^{|n|}\} = \sum_{-\infty}^{-1} a^{-n} z^{-n} + \sum_{0}^{\infty} a^n z^{-n}$$

$$= \left[\cdots + a^3z^3 + a^2z^2 + az\right] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-a/z}$$

$|az| < 1$ and $\left|\frac{a}{z}\right| < 1$

$|z| < \frac{1}{|a|}$ and $|z| > |a|$

$\therefore Z\{a^{|n|}\} = \frac{az}{1-az} + \frac{z}{z-a}$

$= \frac{z(1-az)}{(1-az)(z-a)}, \quad |a| < |z| < \frac{1}{|a|}$

$\therefore Z\{a^{|n|}\} = \frac{z}{(z-a)}, \quad |a| < |z| < \frac{1}{|a|}$

(3) Find the Z-transform of $2n + \sin\frac{n\pi}{4} - 3a^4$. And region of convergence of $\frac{1}{n!}$.

$Z\{2n + \cancel{3}\sin\frac{n\pi}{4} - 3a^4\}$

$= 2Z\{n\} + \cancel{3}Z\left\{\sin\frac{n\pi}{4}\right\} - 3Z\{a^4\}$

$= 2Z\{n\} + \cancel{3}Z\left\{\sin\frac{n\pi}{4}\right\} - 3a^4 Z\{1\}$

$= \frac{2z}{(z-1)^2} + \frac{\cancel{3}z\sin \cancel{n}\pi/4}{z^2 - 2z\cos\frac{\pi}{4} + 1} - \frac{3a^4 z}{z-1}$

$\cancel{= \frac{2z}{(z-1)^2} +}$

FOR EDUCATIONAL USE

(4) Find the z-transform of

i) $u_k = {}^{k+n}C_n \, a^k$

ii) $u_n = \dfrac{1}{(n-r)!}$

$$Z\{u_n\} = Z\left\{\frac{1}{(n-r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n-r)!} z^{-n}$$

$$Z\left\{\frac{1}{n!}\right\} = e^{1/z}$$

From Right Shifting property,

$$Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$$

$$\therefore Z\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{1/z}$$

$$\cancel{\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{1/z}}$$

(5) $$Z^{-1}\left\{\frac{4z^2-2z}{z^3-5z^2+8z-4}\right\}$$

$$U(z) = \frac{4z^2-2z}{(z-1)(z-2)^2}$$

$$\frac{U(z)}{z} = \frac{4z-2}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$4z - 2 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

Put, $z = 1 \Rightarrow 4 - 2 = A \Rightarrow \boxed{A = 2}$

$z = 2 \Rightarrow 8 - 2 = C \Rightarrow \boxed{C = 6}$

$z = 0 \Rightarrow -2 = 4A + B(-1)(-2) + C(-1)$

$$-2 = 4A + 2B - C$$

$$-2 = 8 + 2B - 6$$

$$-2 = 2(1 + B)$$

$$\boxed{B = -2}$$

$$\frac{U(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$U(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{6z}{(z-2)^2}$$

$$z\{n\} = \frac{z}{(z-1)^2}$$

Replace z by z/a

$$z\{na^n\} = \frac{az}{(z-a)^2}$$

$$z^{-1}\{U(z)\} = 2(1)^n - 2(2)^n + 3z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$$= 2(1)^n - 2(2)^n + 3n(2)^n$$

(6) Use convolution theorem to find $z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$

$z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$; $z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$\therefore z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$= \sum_{m=0}^{n} \frac{a^m b^n}{b^m}$$

$$= \sum_{m=0}^{n} b^n \left(\frac{a}{b}\right)^m$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$a = 1$, $r = \frac{a}{b}$, no. of terms $= n+1$

$$= b^n \left[\frac{(a/b)^{n+1} - 1}{a/b - 1}\right]$$

$$= b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}\right]$$

$$= \cancel{b^n}\left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n}\cdot\cancel{b}} \times \frac{\cancel{b}}{a-b}\right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

(7) $z^{-1}\left\{\frac{z^3}{(z-1)^3}\right\}$

$$U(z) = \frac{z^3}{(z-1)^3} \cancel{= (z-1)^{-3}}$$

$$= \left(\frac{z-1}{z}\right)^{3}$$

$$= \left(1 - \frac{1}{z}\right)^{-3}$$

$$\therefore U(z) = 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \cdots$$

$$\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$$

Comparing with $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, we get

$$u_n = \frac{(n+1)(n+2)}{2}$$

(8) $Z^{-1}\left\{ 2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{z-1} \right\}$

$$U(z) = 2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{z-1}$$

$$\therefore u_n = 2Z^{-1}\{z^{-2}\cdot 1\} - Z^{-1}\left\{z^{-4}\cdot\frac{z}{z-1}\right\} + Z^{-1}\left\{z^{-6}\cdot\frac{z}{z-1}\right\}$$

~~u_n =~~ By Right shifting property $Z^{-1}\{z^{-k}U(z)\} = u_{n-k}$

$$u_n = 2u(n-2) - u(n-4) + u(n-6)$$

(9) Use convolution theorem to find $Z^{-1}\left\{\frac{z^2}{(z-1)(2z-1)}\right\}$

$$U(z) = Z\{u_n\} = \frac{z}{z-1}$$

$$V(z) = Z\{v_n\} = \frac{z}{2z-1} = \frac{1}{2}\left(\frac{z}{z-1/2}\right)$$

~~$u_n = (1)^n$~~ $Z^{-1}\{u_n\} = (1)^n$, $Z^{-1}\{v_n\} = \frac{1}{2}\left(\frac{1}{2}\right)^n$

$u_n = (1)^n$ and $v_n = \frac{1}{2}\left(\frac{1}{2}\right)^n$ $\quad\because Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$

By convolution theorem

$$z^{-1}\left\{ \frac{z^2}{(z-1)(2z-1)} \right\} = (1)^n * \left(\frac{1}{2}\right)^{n+1}$$

$$= \sum_{m=0}^{n} (1)^m \left(\frac{1}{2}\right)^{n+1-m}$$

$$= \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n-1} + \cdots + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \right]$$

$$= \frac{1}{2}\left[\frac{1}{1-1/2} \left(1 - \left(\frac{1}{2}\right)^{n+1} \right) \right]$$

$$= \frac{1}{2}\left[2\left(1 - \left(\frac{1}{2}\right)^{n+1} \right) \right]$$

$$= 1 - \left(\frac{1}{2}\right)^{n+1}$$