Homework 5

Due Wednesday, March 6 at 11:59 pm.

Problem 1. *(20 points)* Let $A = \{a_1, a_2, \ldots, a_n\}$ be an array of n distinct numbers. If $i < j$ and $a_i > a_j$, then the pair (i, j) is called an *inversion* of A.

a) List the five inversions of the array $\{2, 3, 8, 6, 1\}$.

b) What array with elements from the set $\{1, 2, \ldots, n\}$ has the most inversions? How many does it have?

c) What is the relationship between the running time of insertion sort and the number of inversions in the input array? Justify your answer.

The insertion sort algorithm is reproduced here for your reference.

0 **procedure** insertion-sort $(a_1, a_2, \ldots, a_n)$ $\triangleright$ $a_1, a_2, \ldots, a_n$ are n distinct integers
1   **for** $j := 2$ **to** n **do**
2         $key := a_j$
3         $i = j - 1$
4   **while** $i > 0$ and $a_i > key$ **do** $\triangleright$ insert a_j into the sorted sequence $a_1, \ldots, a_{j-1}$
5             $a_{i+1} := a_i$
6             $i := i - 1$
7         $a_{i+1} := key$
8     $\triangleright$input now sorted in ascending order

Hint: How many inversions does each iteration of the **while** loop in lines $4 - 6$ eliminate?

Problem 2. In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a 4×4 grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

$\rightarrow$

A	B	C	D
E	F	G	H
I	J	K	L
M	O		**N**

$\rightarrow$

A	B	C	D
E	F	G	H
I	J		L
M	O	**K**	N

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove the answer is "no".

Theorem 1.1. *No sequence of moves transforms the board below on the left into the board below on the right.*

A	B	C	D
E	F	G	H
I	J	K	L
M	**O**	**N**	

A	B	C	D
E	F	G	H
I	J	K	L
M	**N**	**O**	

a) *(4 pts)* We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is A, B, C, D, E, etc.

   Can a row move change the order of the tiles? Prove your answer.

b) *(4 pts)* How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters L_1 and L_2 will L_1 appear earlier in the order of the tiles than L_2 before the column move and later in the order after the column move? Prove your answer correct.

c) *(4 pts)* We define an *inversion* to be a pair of letters L_1 and L_2 for which L_1 precedes L_2 in the alphabet, but L_1 appears after L_2 in the order of the tiles. For example, consider the following configuration:

A	B	C	E
D	H	G	F
I	J	K	L
M	N	O	

   There are exactly four inversions in the above configuration: E and D, H and G, H and F, and G and F.

   What effect does a row move have on the parity of the number of inversions? Prove your answer.

d) *(8 pts)* What effect does a column move have on the parity of the number of inversions? Prove your answer.

e) *(16 pts)* The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an odd number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an *invariant*, a property of the puzzle that never changes, no matter how you slide the tiles around.

Lemma 1.2. *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square* [1]*.*

row 1	A	B	C	D
row 2	E	F	G	H
row 3	I	J	K	L
row 4	M	O	N	

Prove this lemma.

f) *(4 pts)* Prove the theorem that we originally set out to prove.

[1] So the configuration shown here has one (an odd number) inversion, (O, N), and the blank square is in row 4 (an even number). The parity of the number of inversions is therefore different from the parity of the row containing the blank square.

Problem 3. *(20 points)* The plane is divided into regions by straight lines. Show by mathematical induction that it is always possible to color the regions with 2 colors so that adjacent regions are never the same color.

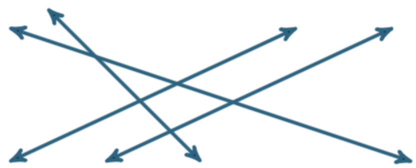

Figure 1: Lines drawn in plane. The regions in the plane created by these lines can be colored with two colors in such a way that any two regions sharing a common border have different colors.

Hint 1: Use induction on the number of lines.

Suggestion: Approach this problem by drawing a few lines and coloring the regions.

Hint 2: Mathematical induction is a proof method closely related to algorithmic proofs. Suppose the plane is divided into regions by k lines and is colored with two colors as the problem asks. Now suppose you draw a $k+1$-st line in the plane. This new line will create a few new regions. Can you think of a procedure that assigns colors to the regions so that adjacent regions are still colored with two colors?

Note: In fact, it is also possible to give a neat non-inductive proof of this coloring theorem. You can read this proof in the Induction Handout posted on Canvas.

Problem 4. *(20 points)* Lines in a plane are said to be in a general position if no 2 are parallel and no 3 meet in a point. If 10 lines are drawn in general position in the plane, into how many regions do they divide the plane?

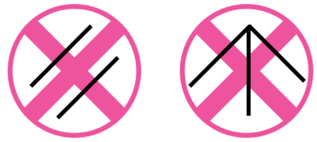

Figure 2: No parallel lines. No 3 lines meet in a point.

Explain clearly how you found the solution.

Hints and suggestions.

It would be a nightmare to draw 10 lines in a plane and try to count the number of regions. The problem is implicitly asking you to carry out an inductive investigation. Start by drawing 1, 2, 3, 4 ... lines and count the number of regions created by the lines. You may be able to spot an emerging pattern. Is there a recurrence relation that can be extracted from the pattern? Can you conjecture a formula for the number of regions created by n lines in a plane?