Exercise 19

If the X_n are integrable and $\lim \int_A X_n$ exists and is finite for every A, then the $\int |X_n|$ are uniformly bounded, $\int_A |X_n| \to 0$ uniformly in n as $\mu A \to 0$ and as $A \downarrow \emptyset$, and there exists an integrable X, determined up to an equivalence, such that $\int_A X_n \to \int_A X$ for every A. (Use 18.)

Solution

Is it about uniformly integrability?

Using the conclusion of 14, we can define $\varphi_n(A) = \int_A X_n$. With the condition $\mu A \to 0$ and $A \downarrow$, we have that $\varphi_n \to 0$. Conditioned on $A \downarrow 0$, $\varphi_n A \to 0$ as $\mu A \to 0$. Then, with the same logic for exercise 18 of chapter 1, we know that $\varphi_n A \to 0$ uniformly as $\mu A \to 0$ and $A \downarrow \emptyset$ and $\varphi = \lim \varphi_n$ is μ-continuous and σ-additive.

The question here is why we need $A \downarrow \emptyset$. Maybe because we try to avoid things like Dirac function? But in the next question, it seems that this condition could be suppressed when μ is finite.

The fact that $\varphi_n A \to 0$ uniformly as $\mu A \to 0$ and $A \downarrow \emptyset$, plus the conclusion in exercise 15 implies that $\int_A |X_n| \to 0$ uniformly in n as $\mu A \to 0$ and as $A \downarrow \emptyset$ (since φ_n's are all finite, then by exercise 9 of chapter 1 it is true). The μ-continuity and the σ-additivity of φ, along with the Radon-Nikodym theorem tells us that φ is an indefinite integral of some X up to an equivalence and

$$\varphi_n \to \varphi \implies \text{ for every A, } \int_A X_n \to \int_A X.$$

The only thing that remains to prove is that $\int |X_n|$ are uniformly bounded.

We can show that based on the given condition, $X_n \xrightarrow{\text{a.e.}} X$. If not, then we know that

$$\mu[X_n \not\to X] \neq 0$$

Therefore, we know that there is a set, say, A_0 with $\mu A_0 = \delta_0 > 0$ such that $X_n \not\to X$ on A_0. This means that there exists some $\epsilon_0 > 0$, and for any $N_0 > 0$ there exists some $n > N_0$ such that on A_0

$$|X_n - X| \geq \epsilon_0$$

If it is possible, then we could then find a infinite subsequence $X_{nj}, j = 1, 2, ...$ such that $|X_{nj} - X| \geq \epsilon_0$ on A_0 but $\int_B X_{nj} \to \int_B X$ for all $B \subset A_0$. We need to argue then it is impossible.

Consider the subsequence of X_{nj}, say X_{njk} where we have $\mu[X_{njk}-X \geq \epsilon_0] \geq \frac{\delta}{2}$. Without loss of generality, assume this subsubsequence is infinite. Notice that we still have $\int_B X_{njk} \to \int_B X$ for all $B \subset A_0$. There must exists some $B_0 \subset A_0$ with $\mu B_0 \geq \frac{\delta}{2}$ such that on B_0, there are infinitely many X_{njk}'s such taht $X_{njk} - X \geq \epsilon_0$. Otherwise, there would exists a $B_1 \subset A_1$ such that there exists K_1 such that when $k > K_1$, $X_{njk} - X < \epsilon_0$ on B_1. But this means that for all $k > K_1$, $X_{njk} - X < \epsilon_0$ on $A_0 - B_1$ where $\mu(A_0 - B_1) < \frac{\delta}{2}$ which is contradictory.

Now we obtain B_0 with $\mu B_0 \geq \frac{\delta}{2}$ and $X_{njk} - X \geq \epsilon_0$ on B_0. This means that for any $K_2 > 0$, there exists some $k > K_2$ such that,

$$\int_{B_0} (X_{njk} - X) d\mu \geq \frac{\delta\epsilon_0}{2}.$$

However, the given condition tells us that for $\frac{\delta\epsilon_0}{2} > 0$, there exists an K such that when $k > K$,

$$\int_{B_0} X_n - X < \frac{\delta\epsilon_0}{2}.$$

We have found a contradiction. This means that we cannot have subsubsequence to be infinite. Then the subsubsequence for $\mu[X_{njk} - X < -\epsilon_0] \geq \frac{\delta}{2}$ cannot be infinite for the same reason. Then the subsequence of X_{nj} cannot be infinite. And so the whole assumption that $X_n \not\to X$ on A_0 is invalid. In summary, we must have

$$X_n \xrightarrow{\text{a.e.}} X.$$

With a similar argument in exercise 17, we could use dominated convergence theorem to control $(X - X_n)^+$ and $(X - X_n)^-$ by $|X|$ integrable to get

$$I_n := \int_\Omega |X_n - X| \to 0.$$

which means that there exists some N_1 such that when $n > N_1$,

$$I_n < \int |X|$$

Denote $M = \max\{I_1, I_2, \dots, I_{N_1}, \int |X|\}$. Then

$$\int |X_n| \leq \int |X_n - X| + \int |X| \leq 2M,$$

which means that $\int |X_n|$ uniformly bounded.