Mathematics is the shortest path to independent thinking.  
V. Kaverin

Continuum mechanics is a section of mechanics and theoretical physics, or rather the continuation of theoretical mechanics that deals with analysis of deformable bodies. However, mathematics in continuum mechanics represents the main constructive tool.

One of the goals that I pursued while writing this textbook was to produce a Textbook on continuum mechanics and thermodynamics that would be as "simple" as possible, or rather a tutorial on basic equations in continuum mechanics.  
The so-called simplicity in science, as well as the absolute truth, is unattainable, however we still have to strive for them and follow the call of a mathematician and mechanic L.V. Ovsyannikov to get rid of "rubbish" that prevents us from grasping the truth in the reports, lectures, articles, monographs and textbooks.  
I would be glad if some readers when they familiarize with some sections of this book would get an enthusiastic thought flash through their minds that would be: “Oh, it turned out to be clear and simple!”

At the end of this book there is a list of all textbooks on Continuum mechanics known to me that are worth at least a glance through for everyone studying the subject. Some of those textbooks caused me to adopt a very critical attitude, others, in contrast, stimulated interest and desire to study them in greater detail. I believe that this book will also find her readers among students, teachers and researchers.

Continuum mechanics allows to demonstrate the power of logic and mathematical thinking. Based on a few fundamental postulates and principles, using the mathematical apparatus can reveal non-trivial, and even striking results. The following are just a few illustrative examples.

Only on the basis of the laws of conservation of mass and momentum, and the introduced concept of discontinuity surface used to simulate the shock wave remarkable equations were produced. For example, it follows from these equations that by relatively simple measurements of the speed of the shock wave in the target and the impactor (projectile) velocity, triggering a shock wave in the target, we can "accurately" determine the value of the pressure and density of matter at high speed impact created by an explosion when the entire process lasts millionths of a second (microseconds). Hence it becomes possible to explore the properties of matter at the enormous pressures of about 106 bar = 1011 Pa, which can be created by a projectile flying at a speed of 1 - 5 km/s. Under such impact even metals are compressed by 1.5 - 2 times within microseconds. It was this method that was the experimental basis for derivation of the equations of state for condensed matter at high pressures (105 - 106 bars). Similar equations were obtained during the development of the atomic bomb. But later they proved to be necessary in astrophysics, geology, explosive technology, etc.

Is it not incredible that, based solely on mathematical considerations and equations, it is possible to prove that any deformation, including "pure shear", can be reduced to compression or tension along three mutually perpendicular directions, called principal. It goes in line with the following from the "ordinary" laws of Newtonian mechanics beautiful mathematical result: the number of stresses at all possible elementary areas, passing through any point of the medium is determined by only six numbers or values of three tensile or compressive stresses on three mutually orthogonal elementary areas.

It is striking that on the basis of simple considerations, including considerations and dimensions of physical quantities, it is established that the distance traveled by a strong shock wave in air from the site of the explosion of a powerful bomb grows with time according to law . This nontrivial exponent  shows how deep the connections between the defining parameters in physics are.

These examples confirm the words of the eminent physicist Heinrich Hertz, who was the first to experimentally prove the existence of electromagnetic waves, "Equations are smarter than their creators."

The words of E.M. Lifshitz in the preface to the latest edition of "Continuum Mechanics», which has become especially popular among those who were studying in the physics departments: "Among the books I had a chance to write together with Lev Davidovich Landau, this book holds a special place. He put a part of his soul into it», indicate the special place that continuum mechanics courses occupy in theoretical physics.

In the years between 1950 and 1980 most of the mechanics, physics and technology graduates were engaged in the problems of aerospace engineering and nuclear weapons. The formulation and theoretical and experimental methods of solving of the problems in these areas have already been established. In recent decades more and more demand for the study of natural processes in the solid, liquid and gas phases of our planet is formed.

The paradigm that is used in modern mechanics is based on conservation laws that in mechanics are reduced to balances of mass, momentum and energy. Similar theoretical paradigm is demanded for analyzing socio-economic processes. And these processes are multivariable and involve the interaction of a large number of processes, which also have the laws of conservation and balances reflected in them[[1]](#footnote-1). Modern methods of computing calculations allow you to use complex models to identify the different possible scenarios. And the researchers should be able to make adequate mathematical models in multivariable systems and using them to conduct their research of real objects. ……

A few words about the features of this course: while teaching theoretical subjects it is necessary to thoroughly expound the core elements (by presenting articulate mathematical formulations, striving to show consistency in the use of concepts and their logic) thus ensuring a correct understanding of the basics by the audience is reached. One has to pay more attention to the (basic/core) tenets of the theory, relying on them in the proof of the subsequent formulas and theorems. It is important to strive for consistency in the notations.

Foundation of continuum mechanics consists of:

1) material continuum model in the form of a deformable (with mechanical stresses and other macroparameters) continuous medium, described via several piecewise continuous differentiable functions. Building such a model is carried out by averaging the parameters of real materials that have a discrete atomic and molecular structure;

2) differential, integral, tensor calculus and the theory of dimensions with the fundamental idea of invariance under transformation of coordinate systems and systems dimension;

3) The laws of conservation of mass, momentum, angular momentum and energy, the laws of thermodynamics, expressed in terms of macroscopic parameters of the material continuum;

4) mechanical (rheological), thermal and electrical experiments that allow us to find connections between macroparameters of different substances at different mechanical, thermal, electromagnetic and physical-chemical processes.

These representations constitute, in particular, the mathematical theory of thermo-electro-magneto-mechanical field.

This book includes sections on the methods of large-scale averaging of those continuum mechanics equations that are already averaged on a smaller scale.   
This double averaging is needed for derivation of the equations of turbulent motion, as well as the derivation of the averaged equations of two-phase media with two velocity, temperature and pressure fields. It is also needed for the calculation of macroscopic closure equations based on the schematization of micromovements that are described by equations of the single-phase continuous medium on a small scale.

One of the important sections of especially continuum mechanics is the kinetic theory of gases, which is based on continuous (continuum) differentiable functions of distribution that obey the Boltzmann equation. This section is often erroneously interpreted as a section of discrete mechanics.

In chapters 4, 6 and 8 you are given examples of specific closed mathematical models, based on the considered continuum mechanics equations. These examples are not chosen with a view to provide a framework of hydrodynamics, gas dynamics, elasticity and plasticity theory or multiphase systems, but in order to clarify the physical nature of the derived continuum mechanics equations.

This course uses tensor representations in the Cartesian coordinate system of the observer. Whereas § 8 of ch.3 describes in detail how to write the continuum mechanics equations in the arbitrary curvilinear coordinate system. This way the common link is not lost, but the exposition becomes easier and clearer.

At the end of the book you will find the fundamentals of tensor calculus in a Cartesian coordinate system in the form of Appendix.

Continuum mechanics was formed as a separate branch of mechanics and theoretical physics after the formation of hydrodynamics, elasticity theory and thermodynamics. However, tuition at universities goes in reverse order: first goes the study of Continuum mechanics and then hydrodynamics, elasticity theory, etc.

The presence of these opposing trends in scientific research that comes from the particular to the general, i.e. by induction, and in the teaching of theoretical subjects that comes from the general to the particular, i.e. by deduction, is quite typical. Scientists learn about the world, based on the particular facts, and only then create generalizing rules and theoretical justification. Theories allow to navigate the "sea of ​​facts", to memorize those facts in accordance with the logic of the theory, to predict the "new facts" and, finally, to teach young people without wasting time on "diving" into the details and routine of all initial facts and experiments. It is much easier and "more economical" to teach on the basis of general theories, i.e. deductively. It also allows you to avoid duplication. But often these advances in ease and time savings are overestimated by the teachers.

Firstly, a popular saying which goes as follows "repetition is the mother of learning" is often ignored by the compilers of the curriculum who are seeking to "save" on the duplicating. After all, it is very important to consider many of the concepts from different angles and in different forms.

Secondly, general theory long mastered by the teacher appears to him to be very simple and clear. The teacher then gets the false impression that it appears the same way to the students and they will quickly master the theory. In fact, the mastering of general theories, especially of those that are abstract and use complex mathematical apparatus, requires some time and effort. At the same time many people are not interested in the theory, which is not connected to the natural sciences. This is expressed very well by a Russian Academician S.P. Novikov[[2]](#footnote-2): "For any mentally normal person this question (the question of the possibility of applying the ideas that we are now developing) is natural and even essential.

Thirdly, excessive passion for deductive methods and theories can lead to a loss of interest in the subject. It also leads to the neglect of "the prose of life," an inability to develop new theories, proceeding from particular facts and to a lack of skills to solve specific tasks. This results in a loss of rational creativity, inability to "chew over" the real facts and a scholastic penchant for theorizing even in simple situations. Therefore it is very important to find a "middle ground" between the specificity and generality, between clarity and rigor between induction and deduction.

The same applies to the teaching of mathematical analysis, probability theory, mathematical physics equations and other mathematical disciplines. Over the past twenty years the teachers have become very enthusiastic about abstract set-theoretic concepts in mathematical analysis. Now proceeding from such abstract and highly generalized ideas mathematical analysis is taught in isolation from the mathematical analysis on the set of real numbers from the first year of the course. It caused severe damage to the training of specialists on Russian Mechanics and Mathematics schools. Apparently, the greatest "contribution" to this process was made by the Mechanics and Mathematics School of Moscow State University. named by M.V. Lomonosov (see. the above mentioned article by S. Novikov).

Teaching experience has convinced me that most of the students in whose minds they have not yet formed the "building of the studied branch of science", it is better to learn the subject from the particular to the general, from simple to complex. This is closer to how cognition happened for the creators of scientific knowledge. It is much more convenient for the teachers to go from the general to particular in their explanations, because in their minds they already have the "building of the subject" and they lead a tour through this "building" that they know very well. However, the students have yet to create this "building" in their heads. Teaching from general to specific can be very "economical" for the "teacher-guide", but is not always effective in helping the students to construct the "building" of the subject in their minds.

The author while being a student first studied hydrodynamics, gas dynamics, thermodynamics and the theory of elasticity, and only then attended a course in continuum mechanics, read by L.I. Sedov. The general outlook on the already built "building of the subject" inherent to the course of continuum mechanics, aroused enthusiasm and admiration. But when such a "building of the subject" is not present, the listening to the course in continuum mechanics reminds of hasty swallowing of large and "non-chewable" pieces of valuable food, the taste of which remains unfelt for the student.

I paid great attention to the flowcharts that promote the assimilation of the material. Teaching mechanics and physics without them is simply absurd because flowcharts are an important method of learning.

It is important to start teaching the ability to evaluate the different terms in the differential equations and to simplify the equations by discarding small quantities.

Similarly to many other authors of courses on Continuum mechanics I was not able to overcome all the excess deductive reasoning and excess abstractness. Illustrative in this sense were the wonderful lectures that also included some on continuum mechanics read on the Mechanics and Mathematics Faculty of Moscow University by Kh.A. Rakhmatulin that could uniquely develop and present a mechanical theory, based on the simple and individual, but rather illustrative examples.

The course read by G.I. Barenblatt in the same 1960-70-ies was very interesting. This course used inductive logic and outlined solutions to very important problems from different areas of hydrodynamics, the bang theory, filtration, turbulence and fracture mechanics. Particular attention was paid to the dimension theory, asymptotic behavior, self-similar solutions, which are the asymptotics of more detailed solutions, and the transitions from one to the other intermediate asymptotics.

In the preparation of this course, I proceeded from the fact that the lectures on hydrodynamics and gas dynamics, elasticity and plasticity theory, mechanics of multiphase media are read as individual titles, following the course on continuum mechanics in the syllabus. Therefore, this book does not address the initial and boundary value problems (except §6 Ch. 8), as well as methods for their solution, as the author believes that they will be considered in special courses. The book contains only a few of the most model examples and formulations of the problems from these sections, which, firstly, allow to understand the derived equations and, secondly, these productions must remain in memory as illustration of an applied character. In particular, the book contains some of the productions used in different sections of mechanics and physics, namely concept of self-similarity and self-similar solutions, the concept of linear and nonlinear shock waves, etc.

An important branch of continuum mechanics is the theoretical thermodynamics (Ch. 5). This section is very difficult to learn for the students.  
In contrast to all the courses in the Continuum mechanics, this course includes sections on the basics of mechanics of heterogeneous (multiphase) environments (Chap. 6) and the kinetic theory of gases (Chap. 7). The equations of multiphase media are based on averaging of already averaged continuum mechanics equations that are describing the small-scale motion on the scale of bubble droplets and other inhomogeneities. Kinetic theory of gases is based on the Boltzmann equation for a continuous (continuum) differentiable distribution function of the time, the coordinates and velocities of the molecules.

Physical quantities have a fundamental property of invariance under transformation of coordinate systems and the use of different dimensions. Invariance under the transformation of coordinate systems leads to the theory of tensors. Invariance under transformation of systems of dimensions leads to the theory of dimensions, to which I dedicated Ch. 8. When writing it, I referred to books by LI Sedov and GI Barenblatt, as well as chapter 10 in the book "Continuum Mechanics in tasks" (eds. M.E. Èglit) written by V.P. Karlikov.

The concept of this book began to develop while I was reading a course in continuum mechanics to the students of the Faculty of Physics at Tyumen State University from 1986 to 1992. Materials on the mentioned course were issued in small quantities in 1987. When I returned to MSU Schools of Mechanics and Mathematics in 2009, I read a continuum mechanics course for students of the Department of Mechanics, which mainly consisted of the material presented in Chap. 1 - 3 and a part of Ch. 4 of this book. The material in this book is expanded so as to help the readers get an idea on all sections of continuum mechanics, based on the presentation of the three fundamental chapters.

Only a few of university graduates are only concerned with those sections of Mechanics and Mathematics that they worked on in student days. Therefore, I sought to ensure that this book could help master different branches of mechanics and thermodynamics of continuous media for many years.

I wish to thank Professor A. V. Zvyagin for preparing the material of §9, chapter 4, dedicated to the theory of plasticity, the docent V. F. Nikitin, who prepared the material in & 5, chapter 6, dedicated to turbulence and Professor V.L. Kovalev for preparing the material §§2-9, ch.7 about rarefied gas dynamics. I thank the senior researcher V.M. Prostakishin for editing the book, the docent V. R. Dushin for remarks on ch.1, senior researcher B.L. Kantsyrev for editing chapters 5 & 6, and also senior researchers V.A. Belokon and I.N. Sibgatullin, who made a number of useful remarks in chapter 3.

A special thanks to the critic Professor M.E. Eglit for thoroughly reading the manuscript and making substantial remarks, which enabled me to prevent some inaccurate mistakes and improve sentence structure in chapters 1 – 4. The comments made by academician A. G. Kulikovsky in chapter 5, dedicated to thermodynamics, by academician A. S. Sarkisjan, the corresponding member of RAS (Russian Academy of Sciences) R. A. Ibraev, the professors P.O. Zavialov and G.M. Reznik on §9, chap 6, dedicated to ocean hydrodynamics equations, professor G. A. Tyrsky on chap 7, dedicated to gas dynamics of rarefied gas, and professor V.P. Karlikov on chap 8, dedicated to the theory of dimensions and similarity were very helpful.

Using the chance of presenting this textbook, I want to remember the teachers who had the greatest influence on me as a science worker and lecturer. One of them is my father, professor in the energetics field – Iskander Nigmatulovich Nigmatulin. For example, he explained to me elements of engineering thermodynamics, combined engine theory and nuclear energy basics.

I remember professors of Bauman Moscow State Technical University (MSTU). Among them were a mathematics lecturer Igor Nikolaevich Miroslavlev and professor Nikolaj Nikolaevich Malinin, specializing on resistance of materials and turbine strength.

My prominent teacher was Vladimir Vladimirovich Uvarov, a professor of hydrodynamics. He was the head of the “Turbine Construction» department in Bauman MSTU and he was among the pioneers in developing gas turbines for energetics. Sadly, in 1960 -1980s USSR academy of Sciences was researching in a deadlock direction, which was magnetohydrodynamics, thus did not support the development of gas turbine themes. USA and Germany, however, based on the outstanding achievements in creation of super alloys and methods of cooling the high temperature elements of turbines, made powerful gas turbines. Nowadays the 350 MW power gas turbines exist and those of the 500 MW power are being designed. In the past 20 years gas turbines allowed us to make a revolution in energetics. They have brought the efficiency ratio of power plants with steam gas cycle to 53 – 55%, which leads to the saving of the natural gas (fuel) by 30 – 40%, and that lowers the need of building the nuclear power plants significantly.

Here I mention the docent Vsevolod Evgenievich Mihalzev, who supervised my diploma project dedicated to the aircraft gas turbine engine with a nuclear reactor. In 43 years I began working with his younger brother – Doctor of Technical Sciences, Igor Evgenievich Mihalzev, prominent specialist in ocean underwater technology, who led the work in the development of the well-known underwater devices (apparatus) MIR-1 and MIR-2.

Furthermore I want to distinguish my professors on Mechanics and Mathematics department of Lomonosov Moscow State University. They are the docent (later academician) Andrei Alexandrovich Gonchar and postgraduate student Alexander Moiseevich Olevsky (at present he’s a professor at University of Jerusalem.) Their lectures and seminars became remarkable phenomenon to my classmates and me, and later to my brother Bulat Iskanderovich in the understanding of mathematical analysis. They are followed by gas and wave dynamics department lecturers headed by my main scientific teacher Halil Akhmedovich Rakhmatulin. I listened to the gas dynamics on lectures led by Arthur Yakovlevich Sagomonjan and theoretic dynamics – under the lecturer Igor Nicholaevich Zverev. I listened to the dynamics of the rarefied air under the docent (later professor) Abraham Isaakovich Bunimovich and to the lectures about waves in continuous media under docent Anatoly Leontievich Pavlenko. Remarkable was the seminar of the department where we objectively discussed numerous scientific problems in mechanics, technical physics and applied mathematics.

I worked in Lomonosov University of Mechanics for 23 years, from 1963 to 1986. These years were, obviously, decisive for my future scientific and teaching practice. Here I was taught by Samvel Samvelovich Grigorjan, a candidate of physical and mathematical sciences at first, than a doctor of science and now an academician. I always tried to report Samvel Samvelovich about my scientific “sufferings” and progress and then listen to his critical evaluation. I also remember the seminar by the academician Leonid Ivanovich Sedov, which appeared to be the strictest to me.

The Institute of Mechanics was superintended by a member - correspondent of USSR Academy of Sciences and later an academician Gorimir Gorimirovich Cherniy. He was an example of a demanding scientific leader and a very proper director of the Institute. In fact he was my only director and I remember not only his scientific but directorship lessons as well.

Professors Gregory Aleksandrovich Tirsky, Leo Vladimirovich Altshuler and Leo Gerasimovich Lojtsyansky had a great influence on me. They were my opponents in the doctoral dissertation, and their respect to my work inspired me. L.G. Loizansky and my father I.N. Nigmatulin incited me to write a three tome book dedicated to mechanics of multiphase media, yet Leo Gerasimovich initiated its publishing in “Nauka”.

In 1980s I was inspired by the attention of Jacob Borisovich Zeldovich to some of my works, and I often remember his kind words, which he found time to say before his sudden death.

I remember the meetings and discussions with a salient Siberian Thermo physicist academician Samson Semenovich Kutateladze, who possessed a remarkable scientific intuition.

With gratitude I think about an academician Vladimir Eliferevich Nakorjakov because his friendship and cooperation with me determined my career in Academy of Sciences to a large extent.

With big gratitude I think about my colleagues, whom I have supervised and who in their turn taught me a great deal. A lot of them became prominent scientists and work in numerous countries. I am proud of them. I am not going to list all of the names, however I would just like to note the very first one – Alexey Ivanovich Ivandaev. Together we had been working in the Institute of Mechanics in the Lomonosov MSU for twenty years and then moved to Tyumen to set up a Scientific center, an institute dedicated to mechanics of multiphase media and develop The Tyumen State University where he worked as a rector for the very difficult five years of “perestroika”.

I thank Academician Vladimir Pavlovich Melnikov, a chairman of Tyumen Scientific Center of Siberian Branch of the Russian Academy of Sciences, for cooperation and friendship during my “Tyumen years” (1983-1993).  
 And finally I thank my wife Venera Abdrahmanovna for her patience and continued support.

With gratitude to all of those mentioned and not mentioned, who were involved in my research and teaching activities, I release a textbook in the scientific and educational space.

1. The corresponding method named as Input-Output method and was proposed by V.V. Leontief in 1930-th. This famous method is used in all countries for analyzing of national economic systems consisting with many branches. In fact the proper equations are analogues of mass conservation equations for multicomponent media. These equations that are ignored in modern Russian economic projects. [↑](#footnote-ref-1)
2. S.P. Novikov, The second half of XX-th century and its [↑](#footnote-ref-2)