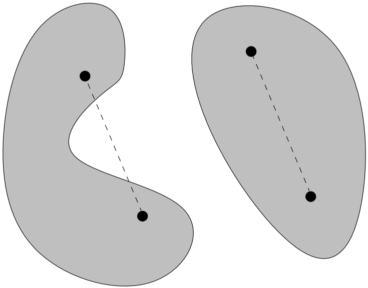
Convex Sets

2nd November, 2023

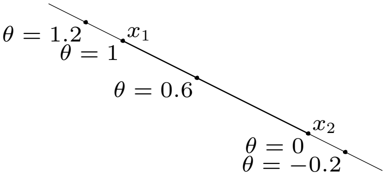
Convex Set



Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

Question: Can every affine set be expressed as solution set of system of linear equations?

Affine set

Given x_1, x_2 and x_3 three points in an affine set C, then

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \quad \in C$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Affine set

Given x_1, x_2 and x_3 three points in an affine set C, then

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \quad \in C$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$.
Proof: Let y be affine combination of x_1 and x_2 then $y \in C$

$$y = \frac{\alpha_1}{\alpha_1 + \alpha_2} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} x_2 \quad \in C$$

x is affine combination of y and x_3 hence $x \in C$

$$\begin{aligned} x &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \\ &= (1 - \alpha_3) y + \alpha_3 x_3 \quad \in C \end{aligned}$$

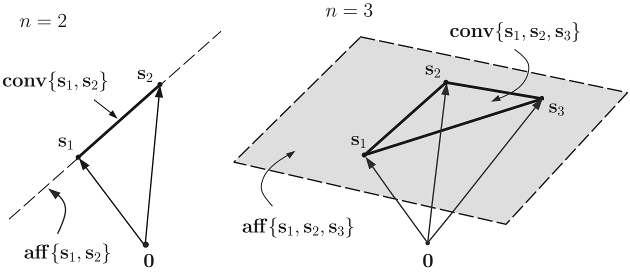
where $1 - \alpha_3 = \alpha_1 + \alpha_2$

Affine set

Subspace C is an affine set and $x_0 \in C$, then the set $V = C - x_0 = \{x - x_0 | x \in C\}$ is a subspace.
Hint: $y \in V$ and $z \in V$ then $\lambda y \in V$ and $y + z \in V$

Affine hull of a set of vectors $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is defined as

$$\textbf{aff}\{\mathbf{s}_1, \ldots, \mathbf{s}_n\} = \left\{ x = \sum_i \theta_i \mathbf{s}_i \mid \sum_i \theta_i = 1 \right\}$$



Affine set I

Subspace C is an affine set and $x_0 \in C$, then the set
$V = C - x_0 = \{x - x_0 | x \in C\}$ is a subspace.
Hint: $y \in V$ and $z \in V$ then $\lambda y \in V$ and $y + z \in V$

If C is an affine set, then there exist a matrix $\mathbf{A}$ and a vector $\boldsymbol{b}$ such that $C = \{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$

- Let C be an affine set in R^n, and $\boldsymbol{x}_0$ be any point in C. Then we can define the set $V = C - \boldsymbol{x}_0 = \{\boldsymbol{x} - \boldsymbol{x}_0 | x \in C\}$, which is a subspace of R^n.
- Let k be the dimension of V, and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a basis of V. Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-k}\}$ be orthogonal complement to V, and define a matrix $\mathbf{A}$ of size $(n - k) \times n$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_{n-k} \end{bmatrix}^T$$

- Then null space of $\mathbf{A}$ is equal to V. To see this, for any $\boldsymbol{x}$ in R^n and $\mathbf{A}\boldsymbol{x} = 0$, x must belong to V.

Affine set II

- Conversely, if $\boldsymbol{x}$ belongs to V, then $\boldsymbol{x}$ can be written as a linear combination of $\mathbf{V} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$: $\boldsymbol{x} = \mathbf{V}\boldsymbol{c}$.

$$\mathbf{A}\boldsymbol{x} = \mathbf{AV}\boldsymbol{c} = 0$$

- This implies that $C = \{\boldsymbol{x} | \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$, where $\boldsymbol{b} = \mathbf{A}\boldsymbol{x}_0$.

Convex set

line segment between x_1 and x_2: all points

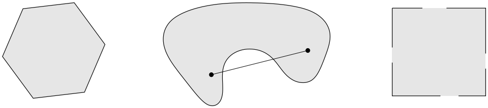
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



Convex set

Example Let C be a convex set, with $x_1, x_2, \ldots, x_k \in C$
and $\theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0$.
Show that $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$.

Convex set

Example Let C be a convex set, with $x_1, x_2, \dots, x_k \in C$
and $\theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0$.
Show that $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$.

Proof.

Let $\alpha_i = \frac{\theta_i}{1-\theta_k}$. It is obvious that $\alpha_i \geq 0$ and $\sum_{i=1}^{k-1} \alpha_i = 1$. By induction, we can assume that $y_{k-1} = \sum_{i=1}^{k-1} \alpha_i x_i \in C$.
Hence

$$\theta_k x_k + (1 - \theta_k) y_{k-1} \in C$$

$\square$

Convex set

Example Prove that the set $C = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq a\}$ is convex.

Convex set

Example Prove that the set $C = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq a\}$ is convex.

Proof.

Let $0 \leq \theta \leq 1$. Denote two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in C. We show that $z = \theta x + (1-\theta)y$ is also in C

$$\begin{aligned}
z_1 z_2 &= (\theta x_1 + (1-\theta)y_1)(\theta x_2 + (1-\theta)y_2) \\
&= \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \\
&\geq \theta^2 a + (1-\theta)^2 a + \theta(1-\theta)\, 2\sqrt{x_1 y_2\, x_2 y_1} \\
&\geq a(\theta^2 + (1-\theta)^2 + 2\theta(1-\theta)) \\
&= a
\end{aligned}$$

□

Convex set: Example

Example Prove that the set $C_n = \{x \in \mathbb{R}^n_+ \mid x_1 x_2 \cdots x_n \geq a\}$ is convex.

Proof.

(By induction). Assume that C_{n-1} is convex for $n > 2$, we need to show that C_n is also convex.

- For two distint points, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in C_n$, if $y \geq x$, then for any $0 \leq \theta \leq 1$, the new point $z = (1-\theta)x + \theta y = x + \theta(y - x) \geq x$. Hence $\prod_i z_i \geq \prod_i x_i \geq a$. Thus $z \in C$. The case, $x > y$, is proved similarly.
- When neither y nor x is dominant, we can always choose $i \neq j$ such that $(x_i - y_i)(x_j - y_j) \leq 0$. Wolg, assume $(i, j) = (n-1, n)$.

$$
\begin{aligned}
&(\theta x_{n-1} + (1-\theta)y_{n-1})(\theta x_n + (1-\theta)y_n) - (\theta x_{n-1}x_n + (1-\theta)y_{n-1}y_n) \\
&= -\theta(1-\theta)(x_{n-1} - y_{n-1})(x_n - y_n) \geq 0
\end{aligned}
$$

Since $\tilde{x} = (x_1, \ldots, x_{n-2}, x_{n-1}x_n)$ and $\tilde{y} = (y_1, \ldots, y_{n-2}, y_{n-1}y_n) \in C_{n-1}$, $z = \theta\tilde{x} + (1-\theta)\tilde{y} \in C_{n-1}$, for $0 \leq \theta \leq 1$, and

$$
\begin{aligned}
a &\leq z_1 \cdots z_{n-2} z_{n-1} \\
&= (\theta x_1 + (1-\theta)y_1) \cdots (\theta x_{n-2} + (1-\theta)y_{n-2})(\theta x_{n-1}x_n + (1-\theta)y_{n-1}y_n) \\
&\leq (\theta x_1 + (1-\theta)y_1) \cdots (\theta x_{n-1} + (1-\theta)y_{n-1})(\theta x_n + (1-\theta)y_n)
\end{aligned}
$$

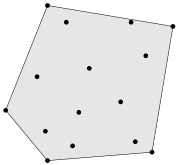
Convex combination and convex hull

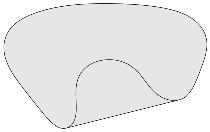
convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



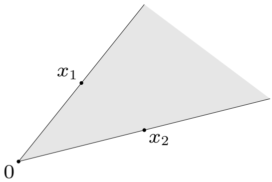


Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

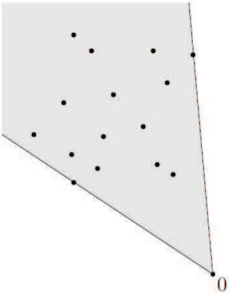
$$x = \theta_1 x_1 + \theta_2 x_2$$

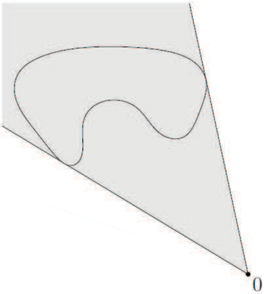
with $\theta_1 \geq 0$, $\theta_2 \geq 0$



convex cone: set that contains all conic combinations of points in the set

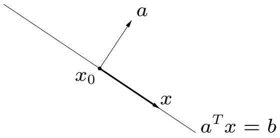
Conic hull: examples



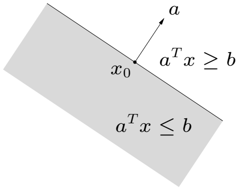


Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Hyperplanes and halfspaces

Example two distint points a and $b \in \mathbb{R}^n$.
Set of all points x which are closer to a than b, i.e.

$$\{x \mid \|x - a\| \leq \|x - b\|\}$$

is a halfspace.

Hint The halfspace $S = \{x \in \mathbb{R}^n \mid c^T x \leq d\}$
where $c = 2(b - a)$ and $d = b^T b - a^T a$.

Hyperplanes and halfspaces

Question When a halfspace $\{x \,|\, a^T x \leq b\}$ contains another halfspace $\{x \,|\, c^T x \leq d\}$?

Answer if there exists a $\lambda > 0$ such that $a = \lambda c$ and $b \geq \lambda d$.

Euclidean balls and ellipsoids

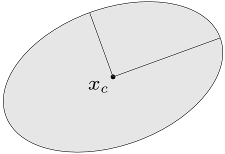
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Question Is Euclidean ball a convex set?
Question prove that $A = P^{1/2}$.

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

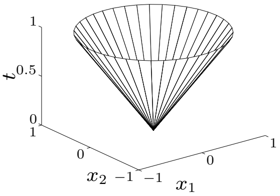
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

Norm cones are epi-graphs of norm functions

Examples of norms

- ℓ_p-norm on $\mathbb{R}^n$: $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$
  - ℓ_∞-norm on $\mathbb{R}^n$: $\|x\|_\infty = \max_i |x_i|$
- Quadratic norms: for $P \in \mathbf{S}^n_{++}$,
  the P-quadratic norm of x

$$\|x\|_P = (x^T P x)^{1/2} = \|P^{1/2} x\|_2$$

- **Question** Find the set of points whose distance to a does not exceed a fixed fraction $0 \leq \theta \leq 1$ of the distance to b, i.e.

$$S = \{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$$

  Hint

Examples of norms

- ℓ_p-norm on $\mathbb{R}^n$: $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$
  - ℓ_∞-norm on $\mathbb{R}^n$: $\|x\|_\infty = \max_i |x_i|$
- Quadratic norms: for $P \in \mathbf{S}^n_{++}$,
  the P-quadratic norm of x

$$\|x\|_P = (x^T P x)^{1/2} = \|P^{1/2} x\|_2$$

- **Question** Find the set of points whose distance to a does not exceed a fixed fraction $0 \leq \theta \leq 1$ of the distance to b, i.e.

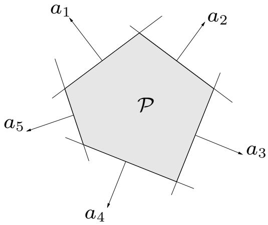
$$S = \{x | \|x - a\|_2 \leq \theta \|x - b\|_2\}$$

Hint

$$S = \{x \,|\, (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\}$$

If $\theta = 1$, S is a halfspace.
If $\theta < 1$, S is a ball

Polyhedra



Set of finitely many linear inequalities and equalities

$$S = \{x \mid Ax \preceq b, Cx = d\}$$

($\preceq$ is componentwise inequality)

- Every polyhedral set is convex (because it is the intersection of finite number of halfspaces and hyperplanes)
- Linear Programming problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Bx \leq b, Dx = d \end{aligned}$$

Polyhedra: example

Verifying that the following set is polyhedron

$$S = \{x \in \mathbb{R}^n | x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^{n} |y_i| = 1\}$$

Hint $S = \{x \,|\, |x_k| \leq 1, k = 1, \ldots, n\}$.
Select $y = \pm e_k$, then

$$\pm x^T e_k = \pm x_k \leq 1$$

Next with any x such that $|x_k| \leq 1$,

$$x^T y \leq \sum_i |x_i||y_i| \leq \sum_i |y_i| = 1.$$

Example

$$S = \{x \in \mathbb{R}^n | x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$$

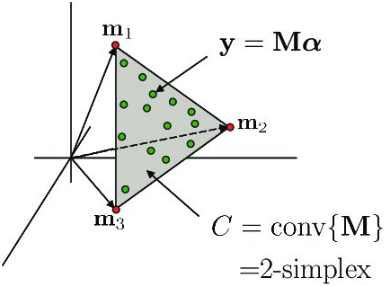
Hint $S = \{x \mid \|x\|_2 \leq 1\}$.

Simplex

A simplex is a set given as a convex combination of a finite collection of vectors $v_0, v_1, \dots, v_k$

$$C = \textbf{conv}\{v_0, v_1, \dots, v_k\} = \{\theta_0 v0 + \cdots + \theta_k v_k \mid \theta \geq 0, \mathbf{1}^T\theta = 1\}$$

The dimension of the simplex C is equal to the maximum number of linearly independent vectors among $v_1 - v_0, \dots, v_m - v_0$.



Probability simplex
$P = \{x \in \mathbb{R}^{n+1} \quad | \quad \mathbf{1}^T x = 1, x \geq 0\}$
vertices are the standard unit vectors
Question: Projection of a point x onto a probability simplex?

Example: Projection onto a probability simplex I

$$\begin{aligned} \min_x \quad & \|x-y\|_2^2 \\ \text{s.t.} \quad & \mathbf{1}^T x = 1, x \geq 0 \end{aligned}$$

Lagrangian function

$$L(x,\lambda,\nu) = \frac{1}{2}\|x-y\|_2^2 - \nu(\mathbf{1}^T x - 1) - \lambda^T x$$
$$\nabla_x L = x - y - \mathbf{1}\nu - \lambda = 0$$

gives the optimal $x^\star = y + \mathbf{1}\nu + \lambda$, where $\lambda \geq 0$.
For positive elements $x_i > 0$, $i \in \mathcal{I}$, we have $\lambda_i = 0$, hence
$\nu = \frac{1}{I}(\sum_{i\in\mathcal{I}} x_i - y_i) = \frac{1}{I}(1 - \sum_{i\in\mathcal{I}} y_i)$.
$x_i = y_i - \nu > 0$ indicates that for all $y_i > -\nu$ we get $x_i > 0$, or $\mathcal{I} = \{i : y_i > -\nu\}$.
This suggests an algorithm which

- Sort $y_1 \geq y_2 \geq \cdots \geq y_n$
- Check and find the largest I such that $y_I \geq \frac{1}{I}(1 - \sum_{i=1}^{I} y_i)$.

Example: LP I

Solve the LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \mathbf{1}^T x = 1, x \geq 0 \end{aligned}$$

Lagrangian function

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x - \nu(\mathbf{1}^T x - 1) - \lambda^T x \\ \nabla_x L &= c - \mathbf{1}\nu - \lambda = 0 \end{aligned}$$

For the optimal solution, $x^\star$, denote by $\mathcal{I} = \{i : x_i^\star > 0\}$. From the complementary slackness condition, we have $\lambda_i = 0$ for $i \in \mathcal{I}$, implying that

$$c_i = \nu\,, \quad \text{for all}\, i \in \mathcal{I}$$

The loss function is rewritten as

$$f(x^\star) = c^T x^\star = \sum_{i \in \mathcal{I}} c_i x_i^\star = \nu \sum_{i \in \mathcal{I}} x_i^\star = \nu$$

Example: LP II

It means ν is the minimum element of c: $f(x^\star) = \nu = c_{min}$ and the optimal $x^\star_{\mathcal{I}} = \frac{1}{|\mathcal{I}|}$
The Dual problem

$$\begin{aligned} \max \quad & y \\ \text{s.t.} \quad & \mathbf{1}y \leq c \end{aligned}$$

has the optimal dual solution $y^\star = c_{min}$.

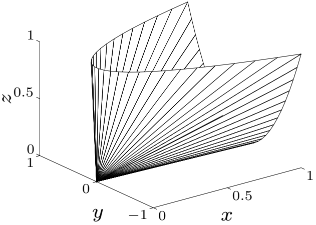
Positive semidefinite cone

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}_+^n \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}_+^n$ is a convex cone
- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



Equivalent to $\{x \geq 0, z \geq 0, xz \geq y^2\}$
Problem prove $\mathbf{S}_+^n$ is a convex cone
Question: find a nearest psd matrix to a given matrix?

Example: find a nearest psd matrix to a given matrix I

Given a symmetrix matrix, $\mathbf{A} = \mathbf{A}^T$, and a skew-symmetric matrix, $\mathbf{B} = -\mathbf{B}^T$, then

$$\|\mathbf{B} + \mathbf{A}\|_F^2 = \|\mathbf{B}\|_F^2 + \|\mathbf{A}\|_F^2$$

since

$$\begin{aligned} 2\,\mathrm{trace}(\mathbf{A}\mathbf{B}^T) &= \mathrm{trace}(\mathbf{A}\mathbf{B}^T) + \mathrm{trace}(\mathbf{A}\mathbf{B}^T) \\ &= \mathrm{trace}(\mathbf{A}^T\mathbf{B}^T) + \mathrm{trace}(\mathbf{A}(-\mathbf{B})) \\ &= \mathrm{trace}((\mathbf{B}\mathbf{A})^T) - \mathrm{trace}(\mathbf{A}\mathbf{B}) \\ &= \mathrm{trace}(\mathbf{B}\mathbf{A}) - \mathrm{trace}(\mathbf{A}\mathbf{B}) \\ &= \mathrm{trace}(\mathbf{A}\mathbf{B}) - \mathrm{trace}(\mathbf{A}\mathbf{B}) = 0 \end{aligned}$$

Nearest symmetrix matrix to a matrix $\mathbf{Y}$: $\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T)$

$$\begin{aligned} \|\mathbf{Y} - \mathbf{X}\|_F^2 &= \|\frac{\mathbf{Y} - \mathbf{Y}^T}{2} + \frac{\mathbf{Y} + \mathbf{Y}^T}{2} - \mathbf{X}\|_F^2 \\ &= \|\frac{\mathbf{Y} - \mathbf{Y}^T}{2}\|_F^2 + \|\frac{\mathbf{Y} + \mathbf{Y}^T}{2} - \mathbf{X}\|_F^2 \end{aligned}$$

Example: find a nearest psd matrix to a given matrix II

Nearest psd matrix to a symmetric matrix $\mathbf{Y}$
Denote EVD of $\mathbf{Y} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$ and $\mathbf{Z} = \mathbf{U}^T\mathbf{X}\mathbf{U}$

$$
\begin{aligned}
\|\mathbf{Y} - \mathbf{X}\|_F^2 = \|\mathbf{\Lambda} - \mathbf{Z}\|_F^2 &= \sum_{i \neq j} z_{i,j}^2 + \sum_i (\lambda_i - z_{i,i})^2 \\
&\geq \sum_{\lambda_i < 0} (\lambda_i - z_{i,i})^2 \geq \sum_{\lambda_i < 0} \lambda_i^2
\end{aligned}
$$

Note that $z_{ii} \geq 0$ since $\mathbf{Z}$ is psd. The lower bound is attained when
$\mathbf{Z} = \mathrm{diag}([\lambda]_+)$, or $\mathbf{X} = \mathbf{U}\,\mathrm{diag}([\lambda]_+)\mathbf{U}^T$

Example

Let $C = \{x \in \mathbb{R}^n \mid f(x) = x^T A x + b^T x + c \leq 0\}$.
Show that C is convex if $A \succeq 0$.

Example

Let $C = \{x \in \mathbb{R}^n \mid f(x) = x^T Ax + b^T x + c \leq 0\}$.
Show that C is convex if $A \succeq 0$.

Proof.

Let $x_1 \in C$ and $x_2 \in C$ and $z = \theta x_1 + (1-\theta)x_2$ for $0 \leq \theta \leq 1$.

$$\begin{aligned}
f(z) &= z^T Az + b^T z + c \\
&= (\theta x_1 + (1-\theta)x_2)^T A(\theta x_1 + (1-\theta)x_2) + b^T(\theta x_1 + (1-\theta)x_2) + c \\
&= \theta(x_1^T Ax_1 + b^T x_1 + c) + (1-\theta)(x_2^T Ax_2 + b^T x_2 + c) \\
&\quad - \theta(1-\theta)(x_1 - x_2)^T A(x_1 - x_2) \\
&= \theta f(x_1) + (1-\theta)f(x_2) - \theta(1-\theta)(x_1 - x_2)^T A(x_1 - x_2) \\
&\leq 0
\end{aligned}$$

Since $x_1 \in C$ and $x_2 \in C$, $f(x_1) \leq 0$, $f(x_2) \leq 0$
Since $A \succeq 0$, $(x_1 - x_2)^T A(x_1 - x_2) \geq 0$ for all x_1 and x_2.
Hence $z \in C$. □

Convexity preserving operations I

Prove that a set is convex

Build it up from simple convex sets using convexity preserving operations.

- **Intersection** If C and D are convex sets, then $C \cap D$ is also convex.
- **Affine function** If $C \subset \mathbb{R}^n$ is a convex set, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, then $AC + b = \{Ax + b | x \in C\}$ is also convex.
  Example: Elipsoid $\{x \,|\, (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$ is an affine tranform $Ax + x_c$ of the unit ball $B = \{x \,|\, \|x\|_2 \leq 1\}$, where $A = P^{1/2}$.

Convexity preserving operations II

- **Perspective transform** $C \in \mathbb{R}^n \times \mathbb{R}_{++}$ is a convex set, then the perpestive transform $P(C)$ is also convex

$$P(x \in C) = P(x_1, x_2, \ldots, x_n, t) = (x_1/t, x_2/t, ..., x_n/t) \in \mathbb{R}^n \,.$$

Proof: Assume (x, t_1) and (z, t_2) in C and $u = x/t_1$ and $v = z/t_2$ are their perpestive transform, respectively.
We need to show that a new point $y = \theta u + (1-\theta)v$ is in perpestive transform $P(C)$, where $0 \le \theta \le 1$
i.e., $\exists\, (x', t') \in C$, $y = \frac{x'}{t'}$
Actually we can find $0 \le \alpha \le 1$ such that

$$\frac{x'}{t'} = \frac{\alpha x + (1-\alpha)z}{\alpha t_1 + (1-\alpha)t_2} = \theta \frac{x}{t_1} + (1-\theta)\frac{z}{t_2}$$

$$\alpha = \frac{\theta t_2}{(1-\theta)t_1 + \theta t_2}$$

Recognizing a convex set

- Using the definition of a convex set
- Writing C as the convex hull of a set of points X, or the intersection of a set of halfspaces
- Building it up from convex sets using convexity preserving operations

Intersection

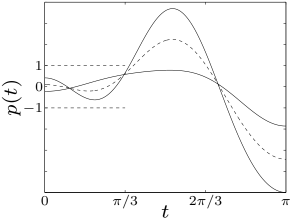
the intersection of (any number of) convex sets is convex

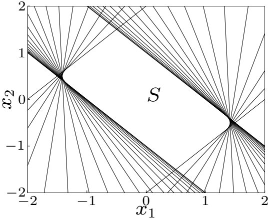
example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





Intersection: example

Show that the positive semidefinite cone $S^n_+ = \{P \in S^n | P \succeq 0\}$ is convex.

by definition of convexity.

Let $A \in S^n_+$ and $B \in S^n_+$ and $\theta \in [0,1]$. For all x,
$x^T(\theta A + (1-\theta)B)x = \theta x^T Ax + (1-\theta)x^T Bx \geq 0$. □

based on intersection.

S^n_+ can be expressed as

$$S^n_+ = \cap_{z \neq 0}\{X \in S^n | z^T Xz \geq 0\}$$

Since the set $\{X \in S^n | z^T Xz = (z \otimes z)^T \text{vec}(X) \geq 0\}$ is a halfspace in S^n, it is convex.
S^n_+ is the intersection of an infinite number of halfspaces,
so it is convex. □

Intersection: example

The set of points closer to a given point than a given set, i.e.,

$$\{x \| \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subset \mathbb{R}^n$.
Hint: This set is an intersection of halfspaces

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

Line Restriction: example

$C = \{x \in \mathbb{R}^n \mid f(x) = x^T A x + b^T x + c \leq 0\}$
C is convex if $A \geq 0$.
Hint: C is convex if its intersection with an arbitrary line $\{x + tv \mid t \in R\}$ is convex.
Let $\hat{x}$ be a point in the intersection of C and a line $\{\hat{x} + tv \mid t \in R\}$

$$\begin{aligned} C \cap \{\hat{x} + tv\} &= \{\hat{x} + tv | f(\hat{x} + tv) \leq 0\} \\ &= \{\hat{x} + tv | \alpha t^2 + \beta t + \gamma \leq 0\} \end{aligned}$$

where $\alpha = v^T A v$, $\beta = b^T v + 2\hat{x}^T A v$, $\gamma = c + b^T \hat{x} + \hat{x}^T A \hat{x}$.
The set is convex if $\alpha = v^T A v \geq 0$.

Line Restriction: Example

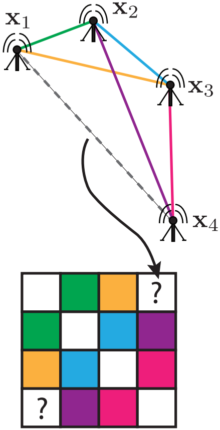
Prove the set $S = \{(x, y)|x^2 + y^2 \leq 1\}$ is convex in $\mathbb{R}^2$.

We need to show that for any line $L = \{(x, y) : ax + by + c = 0\}$ in $\mathbb{R}^2$, the intersection $S \cap L$ is a convex set.
Let $p = [p_x, p_y]$ and $q = [q_x, q_y]$ be any two points in the intersection of S and L. We need to show that for any $\theta \in [0, 1]$, the point $r = \theta p + (1 - \theta)q$ is also in $S \cap L$.
It is obvious that $r \in L$. Since the norm function is convex, we have

$$||r|| = ||\theta p + (1 - \theta)q|| \leq \theta||p|| + (1 - \theta)||q|| \leq \theta + (1 - \theta) = 1$$

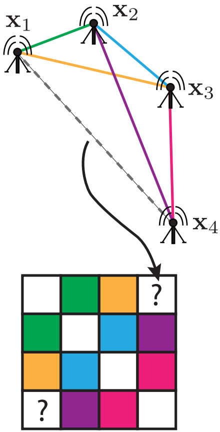
This means that r is in the set S, as required.

Euclidean distance matrix



- The matrix $D \in S^n$, where $D_{i,j} = \|x_i - x_j\|_2^2$, is an Euclidean distance matrix (EDM) if and only if $D_{ii} = 0$ and $x^T D x \leq 0$ for all x with $1^T x = 0$.
  $D = \text{diag}(X^T X)\mathbf{1}^T + \mathbf{1}\text{diag}(X^T X)^T - 2X^T X$ where $X = [x_1, \ldots, x_n] \in \mathbb{R}^{k \times n}$.
- Show that the set of EDMs D is a convex cone.

Euclidean distance matrix



- The matrix $D \in S^n$, where $D_{i,j} = \|x_i - x_j\|_2^2$, is an Euclidean distance matrix (EDM) if and only if $D_{ii} = 0$ and $x^T D x \le 0$ for all x with $1^T x = 0$. $D = \text{diag}(X^T X)\mathbf{1}^T + \mathbf{1}\text{diag}(X^T X)^T - 2X^T X$ where $X = [x_1, \ldots, x_n] \in \mathbb{R}^{k \times n}$.
- Show that the set of EDMs D is a convex cone. **Hint:** The set of EDMs in S^n is the intersection of (infinitely many) halfspaces defined by the following homogeneous inequalities

$$e_i^T D\, e_i \le 0, \qquad e_i^T D\, e_j \ge 0,$$
$$x^T D\, x = \sum_{j,k} x_j x_k D_{jk} \le 0$$

for all $i = 1, \ldots, n$, and all x with $1^T x = 1$.

Affine functions I

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$ (with $P \in \mathbf{S}^n_+$)

Affine functions II

Example
Show that the hyperbolic cone $S = \{x \in R^n \mid x^T P x \le (c^T x)^2, c^T x \ge 0$ where $P \in S^n_+$, is convex.
Proof Define an affine function $f : R^n \to S^{n+1}$

$$f(x) = (P^{1/2} x, c^T x)$$

The S is the inverse image of the second-order cone

$$\{(z, t) \mid \|z\|_2 \le t, t \ge 0\}$$

Hence it is convex.

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Perspective: example

The polyhedron $C = \texttt{conv}\{(v_1, t_1), \ldots, (v_K, t_K)\}$ where $v_i \in \mathbb{R}^n$ and $t_i > 0$. Show that image of C

$$P(C) = \{v/t \mid (v, t) \in C, t > 0\}$$

is also a convex hull.

Perspective: example

The polyhedron $C = \mathbf{conv}\{(v_1, t_1), \ldots, (v_K, t_K)\}$ where $v_i \in \mathbb{R}^n$ and $t_i > 0$.
Show that image of C

$$P(C) = \{v/t \mid (v, t) \in C, t > 0\}$$

is also a convex hull.
Hint

- For $\theta \succeq 0$ and $\mathbf{1}^T\theta = 1$, let $v = \sum_{i=1}^K \theta_i v_i$ and $t = \sum_{i=1}^K \theta_i t_i$,

$$P(v, t) = v/t = \sum_{i=1}^{K} \frac{\theta_i t_i}{t} \frac{v_i}{t_i}.$$

  Since $\sum_{i=1}^K \frac{\theta_i t_i}{t} = 1$, $P(v, t) \in \mathbf{conv}\{v_1/t_1, \ldots, v_K/t_K\}$.
- Consider a point $z = \sum_{i=1}^K \mu_i \frac{v_i}{t_i}$ in $\mathbf{conv}\{v_1/t_1, \ldots, v_K/t_K\}$ for some $\mu \succeq 0$ and $\mathbf{1}^T\mu = 1$. We need to show $z \in P(C)$.
  Define $\theta_i = \frac{\mu_i}{t_i \sum_{j=1}^K \mu_j/t_j}$, $t = \sum_i \theta_i t_i = \frac{1}{\sum_j \mu_j/t_j}$ and $v = \sum_i \theta_i v_i$, i.e., $(v, t) \in C$.
  It can be shown that $z = P(v, t) \in P(C)$.

Perspective: example

Find perspective image of $C = \{(v,t) | f^T v + gt = h, t > 0\}$.
Hint

$$
\begin{aligned}
P(C) &= \{z \mid f^T z + g = h/t, t > 0\} \\
&= \begin{cases} \{z \mid f^T z + g = 0\}, & h = 0 \\ \{z \mid f^T z + g > 0\}, & h > 0 \\ \{z \mid f^T z + g < 0\}, & h < 0 \end{cases}
\end{aligned}
$$

Linear-fractional function: example

Find the inverse image of $C = \{y | g^T y \leq h\}$ after the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad \textbf{dom} f = \{x | c^T x + d > 0\}$$

Hint

$$\begin{aligned} f^{-1}(C) &= \{x \in \textbf{dom} f \mid g^T f(x) \leq h\} \\ &= \{x \mid g^T (Ax + b)/(c^T x + d) \leq h, c^T x + d > 0\} \\ &= \{x \mid (A^T g - hc)^T x \leq hd - g^T b, c^T x + d > 0\} \end{aligned}$$

Generalized inequalities

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Positive Semidefinite Cone $\mathbf{S}^n_+$ I

$\mathbf{S}^n_+$ is a closed pointed convex cone in $\mathbf{S}^n$ with interior $\mathbf{S}^n_{++}$.

- If X and $-X \in \mathbf{S}^n_+$, then $X = 0$, all eigen values of X are zeros, hence, $\mathbf{S}^n_+$ contains no line.
- $\mathbf{S}^n_+ = \{A \in \mathbf{S}^n : x^T A x \geq 0, \forall x \in \mathbb{R}^n\} = \bigcup_{x \in \mathbb{R}^n} \{A \in \mathbf{S}^n : x^T A x \geq 0\}$
  where half space $H_x = \{A \in \mathbf{S}^n : x^T A x \geq 0\}$ is a closed convex set in $\mathbf{S}^n$ for any fixed x. $\mathbf{S}^n_+$ is intersection of closed halfspaces, H_x, hence $\mathbf{S}^n_+$ is closed.
- int$(\mathbf{S}^n_+) = \mathbf{S}^n_{++}$. We need to show
  - $\mathbf{S}^n_{++}$ is contained in $\mathbf{S}^n_+$ and
  - if $X \in \mathbf{S}^n_+ \setminus \mathbf{S}^n_{++}$, X is not in the interior of $\mathbf{S}^n_+$

Positive Semidefinite Cone S_+^n II

- $\mathbf{S}_{++}^n \subseteq \text{int}(\mathbf{S}_+^n)$
  Let $X \in \mathbf{S}_{++}^n$, then $\lambda_n(X) > 0$. We need to prove that there exists a ball centered at X that is contained in $\mathbf{S}_+^n$
  Define spectral norm ball $B = \{Y \in \mathbf{S}_+^n : \|Y - X\|_2 < \lambda_n(X)\}$. Then for every $x \in \mathbb{R}^n$

  $$x^T Y x = x^T X x + x^T (Y - X) x \geq \lambda_n(X) \|x\|^2 - \|Y - X\| \, \|x\|^2 > 0$$

  implying that $\mathbf{S}_{++}^n$ is contained in the interior of $\mathbf{S}_+^n$

  $$x^T (Y - X) x = \|x\|^2 u^T (Y - X) u \geq -\|x\|^2 \|Y - X\|$$

  where $u = \frac{x}{\|x\|}$ is a unit lenth vector.
- Assume $X \in \mathbf{S}_+^n \setminus \mathbf{S}_{++}^n$, hense its EVD $X = U \operatorname{diag}(\lambda(X)) U^T$ has $\lambda_n(X) = 0$.
  Define $X_k = U \operatorname{diag}(\lambda(X) - \frac{1}{k}) U^T$. Then $X_k \to X$.
  If X is in the interior of $\mathbf{S}_+^n$, any ball centered at X contained in $\mathbf{S}_+^n$ would contain some of the matrices X_k for large enough k. However, $X_k \notin \mathbf{S}_+^n$ since its has negative eigenvalues, $\lambda_n(X_k) < 0$. This contradicts.
  Therefore, X must be on the boundary of $\mathbf{S}_+^n$, and hence not in its interior.

Generalized inequalities

generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Minimum and minimal elements I

$\preceq_K$ is not in general a linear ordering: we can have $x \npreceq y$ and $y \npreceq x$
$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

$$\forall y \in S \quad \Longrightarrow \quad x \preceq_K y$$

$S \subseteq x + K$
$x \in S$ is **the minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \Longrightarrow \quad x = y$$

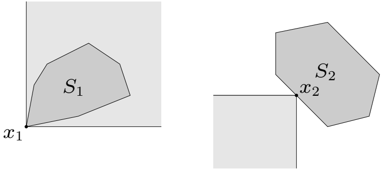
$S \cap (x - K) = x$

Minimum and minimal elements II

A minimum element of S is always minimal, but minimal elements need not be unique.

example $(K = \mathbf{R}^2_+)$

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2



example: S is the ball $S = \{x \in \mathbb{R}^2 : \|x - 1\| \leq 1\}$ and $K = \mathbf{S}^n_+$, U is a given matrix of size $m \times n$.
example: S is the set of symmetric matrices
$S = \{U \operatorname{diag}(v) U^T : v \in \mathbb{R}^n_+ : 1^T v = 1, v \geq 0\}$ and $K = \mathbf{S}^n_+$

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

Problem Prove dual cones of $\mathbf{R}_+$ and $\mathbf{S}^n_+$.

Dual cones: example

Find the dual cone of $\{Ax \mid x \succeq 0\}$ where $A \in \mathbb{R}^{m \times n}$.

Hint $K^* = \{y \mid A^T y \succeq 0\}$

Dual of positive semidefinite cone

The positive semidefinite cone S_+^n is self-dual.

Proof.

Let Y in dual cone K^* of S_+^n. Suppose $Y \notin S_+^n$, then $\exists q$ with $q^T Y q = \mathrm{tr}((qq^T)Y) < 0$, which contradicts $Y \in K^*$.
If $Y \in S_+^n$, it is obvious that $\mathrm{tr}(XY) \geq 0$ for all $X \in S_+^n$. $\square$

Dual of ℓ_1-norm cone

The dual of the cone $K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\|_1 \leq t\}$ is the cone

$$K^* = \{(u, s) \in \mathbb{R}^{n+1} \mid \|u\|_\infty \leq s\}$$

Proof.

Let $y = (u, s)$. Denote $C = \{(u, s) \in \mathbb{R}^{n+1} \mid \|u\|_\infty \leq s\}$

1. For a point $(u, s) \in K^*$, then $u^T x + st \geq 0$ for all $(x, t) \in K$. Select $x = \pm t\, e_n$. $u^T x + st = \pm u_n t + s\, t \geq 0$. Hence $|u_n| \leq s$, for all n, or $\|u\|_\infty \leq s$, or $K^\star \subseteq C$
2. For a point (u, s) in the norm cone C: $\|u\|_\infty \leq s$

$$\begin{aligned}(u, s)^T(x, t) = u^T x + st &\geq \sum_n - \max(|u_n|)|x_n| + s\, t \\ &= -\sum_n \|u\|_\infty |x_n| + s\, t \\ &\geq s(t - \sum_n - \max(|u_n|)|x_n|) \geq 0\end{aligned}$$

implying that $(u, s) \in K^\star$ or $C \subseteq K^\star$

□

Dual of a norm cone

The dual of the cone $K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$ is the cone defined by the dual norm $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$

$$K^* = \{(u, s) \in \mathbb{R}^{n+1} | \|u\|_* \leq s\}$$

Proof.

Assume that $(u, s) \in K^*$. We need to prove

$$x^T u + ts \geq 0 \quad \forall \|x\| \leq t \iff \|u\|_* \leq s$$

1. Suppose $\|u\|_* > s$. From definition of the dual norm, $\exists x$ with $\|x\| \leq 1$ and $u^T x \geq s$, or $u^T(-x) + s < 0$. This is a contradiction.
2. Suppose $\|u\|_* \leq s$. Let $\bar{x} = x/\|x\|$, i.e., $\|\bar{x}\| = 1$.

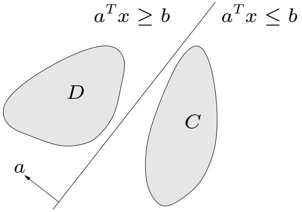
$$\begin{aligned} x^T u + ts &\geq \|x\| \bar{x}^T u + \|x\| \|u\|_* \\ &= \|x\| (\bar{x}^T u + \|u\|_*) \geq 0 \end{aligned}$$

□

Separating hyperplane theorem I

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Separating hyperplane theorem II

Proof: Denote by $c \in C$ and $d \in D$ two closest points between C and D, i.e.,

$$(c, d) = \underset{x\in C,\, y\in D}{\arg\min} \quad \|x - y\|_2^2$$

Define $a = c - d$, and the hyperplane $f(x) = a^T(x - (c + d)/2)$. We have

$$f(c) = \frac{1}{2}\|c - d\|_2^2 \geq 0\,, \quad f(d) = \frac{-1}{2}\|c - d\|_2^2 \leq 0$$

We need to prove (by contradiction) that $\forall\, x \in C$: $f(x) \geq 0$
Assume that $\exists x \in C$ such that

$$0 > f(x) = (c - d)^T(x - \frac{c + d}{2}) = (c - d)^T(x - c) + \frac{\|c - d\|^2}{2}$$

Note that $(x - c)$ is a descent direction for $g(x) = \|x - d\|^2$ at c

$$(\nabla g(c))^T(x - c) = 2(c - d)^T(x - c) < -\|c - d\|^2$$

Hence $\exists \varepsilon > 0$ s.t. $\forall t \in (0, \varepsilon)$

$$g(c + t(c - x)) < g(c)$$

this contradicts since c is the closest point to d.

System of strict linear inequalities

Let $A \in R^{m \times n}$.

$$\{x \mid Ax \leq b\} \text{ is empty } \iff \exists \lambda \geq 0 \text{ s.t. } \lambda^T A = 0, \lambda^T b < 0.$$

Hint
Let $C = \{b - Ax | x \in \mathbb{R}^n\}$ and $D = R^n_{++}$. Since D is open and C is an affine set, the two sets are disjoint if there exists a separating hyperplane $\lambda^T y \leq \mu$ on C and $\lambda^T y \geq \mu$ on D.
For C, it means that $\lambda^T (b - Ax) \leq \mu$ for all x, implying that $A^T \lambda = 0$ and $\lambda^T b \leq \mu$.
For D, the condition $\lambda^T y \geq \mu$ for all $y \succ 0$, implies that $\mu \leq 0$. All together

$$\lambda \neq 0, \lambda \geq 0, A^T \lambda = 0, \lambda^T b \leq 0$$

Example Check feasibility of the inequality system $Ax < 0$ where

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -3 \\ 4 & -1 & 10 \end{bmatrix}$$

Hint: $\lambda = [1, 5/3, 1, 1/3]^T$ is a solution of $A^T \lambda = 0$. Hence the set is infeasible.

Farkas's Lemma

Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$,

$$(S)\begin{cases} Ax \geq 0 \\ c^T x < 0 \end{cases} \qquad \text{and } (S^*)\begin{cases} A^T y = c \\ y \geq 0 \end{cases}$$

The system (S) has a solution if and only if the dual system (S^*) has no solution.

Example The system

$$(S)\begin{cases} x_1 - x_2 + 2x_3 \geq 0 \\ -x_1 + x_2 - x_3 \geq 0 \\ 2x_1 - x_2 + 3x_3 \geq 0 \\ 4x_1 - x_2 + 10x_3 < 0 \end{cases}$$

has no solution, because the dual system

$$\begin{cases} y_1 - y_2 + 2y_3 = 4 \\ -y_1 + y_2 - y_3 = -1 \\ 2y_1 - y_2 + 3y_3 = 10 \end{cases}$$

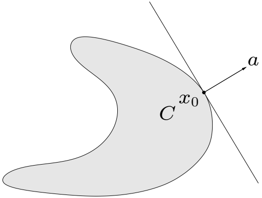
has a nonnegative solution $y = (3, 5, 3)$.

Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C