Convex Optimization Problems and Techniques, Linear Programming and Norm Minimization

By

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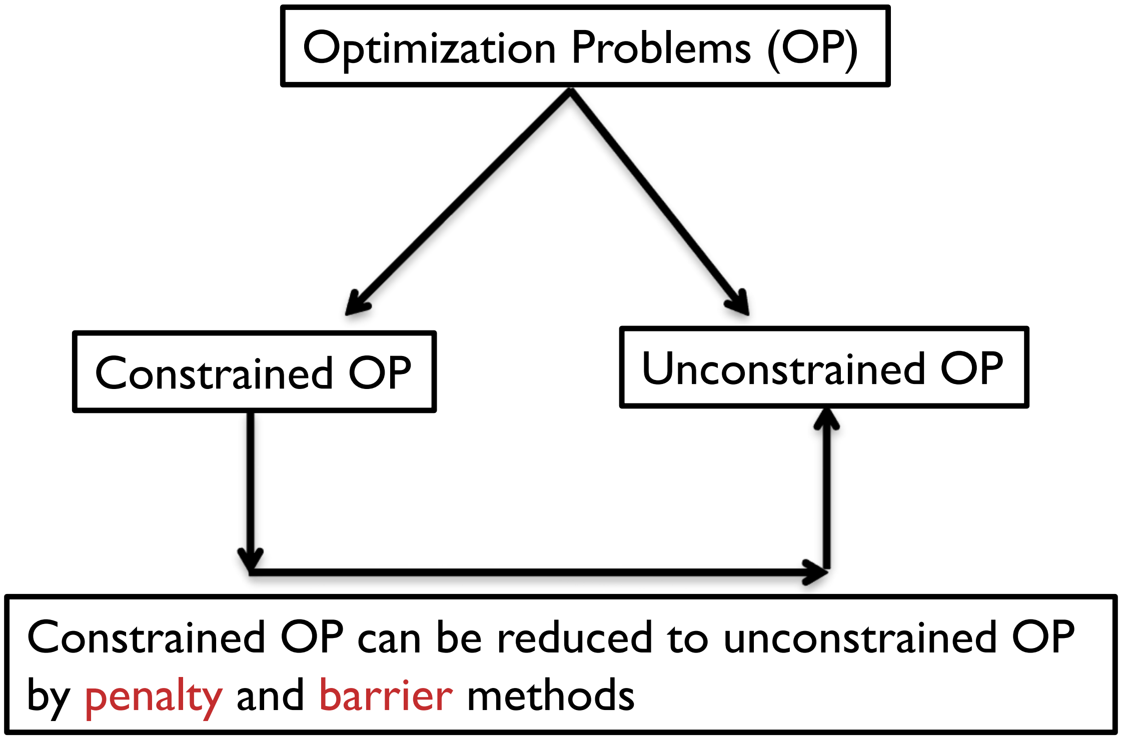
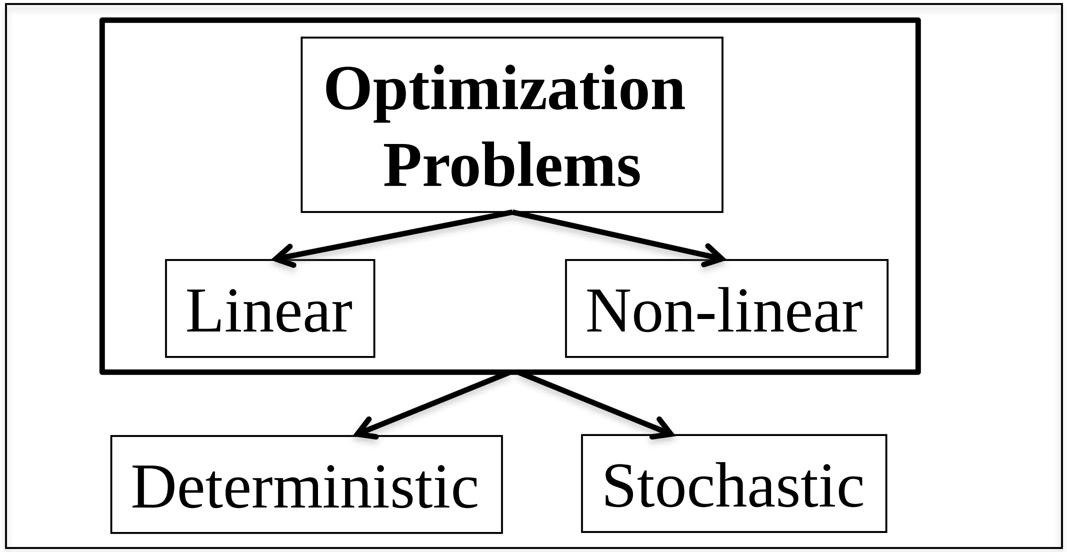
Outline

- **Convex Optimization Problems and Techniques**
- **Linear Programming: Definitions and Concepts**
- **Duality**
- **Linear Programming Solvers**
- **Norm Minimization Problems**

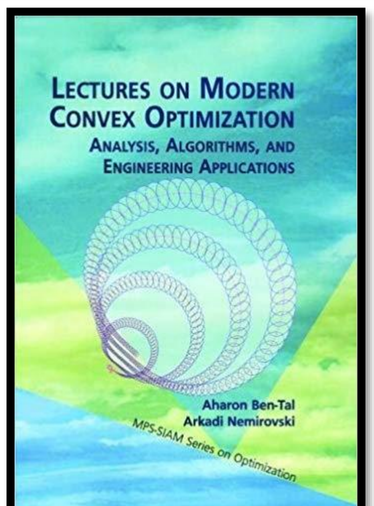
Optimization Problem Categories

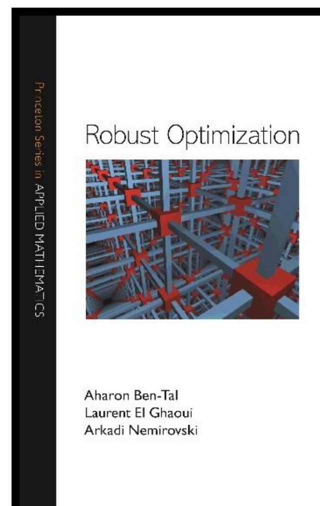




! **Robust optimization (variables with uncertainty)** !

Arkadi Nemirovski



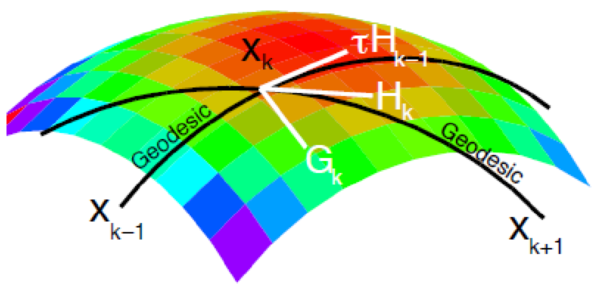


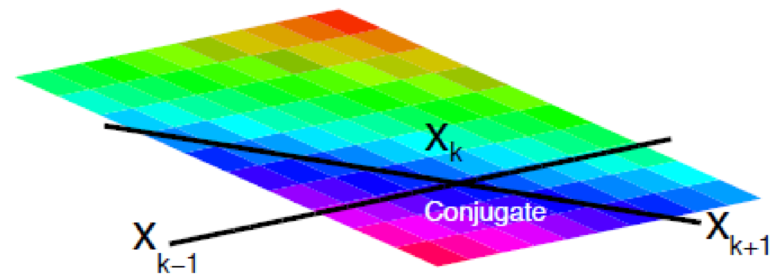
Aharon Ben-Tal

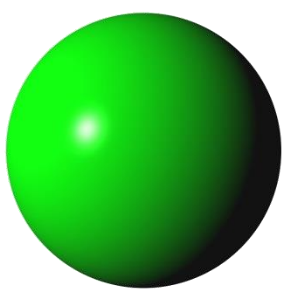
Optimization problems

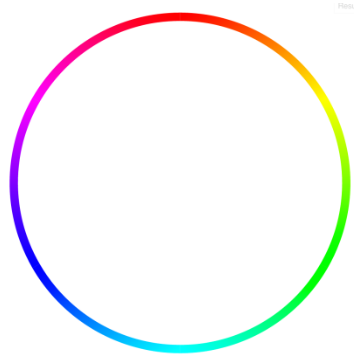
Riemannian optimization

Classical optimization



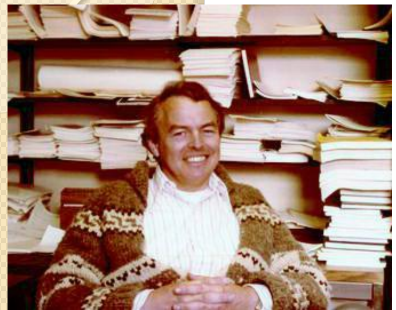






$$\min f_0(x)$$
$$s.t.\quad Ax = b$$

$$\min f_0(x)$$
$$s.t.\quad \|x\|_2 = 1$$



"...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

\- R. Tyrrell Rockafellar, in *SIAM Review*, 1993

Optimization Problems (OP)

Convex Optimization

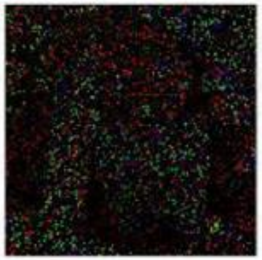
Nonconvex Optimization

It is of interest to convert nonconvex optimization problems to the convex ones taking advantage of convex optimization techniques and their properties.

Polynomial time complexity & Global optimization property

Matrix and tensor completion

Incomplete image with
10% observations



Can we recover the
original image?

Original image

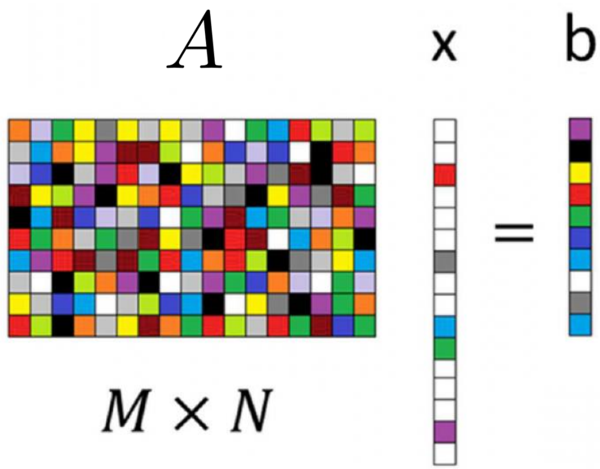
$$\underset{\mathbf{X}}{\text{minimize}} \quad \text{rank}(\mathbf{X}) \qquad \text{subject to} \quad \mathbf{X}_{\mathbb{O}} = \mathbf{T}_{\mathbb{O}},$$

Nonconvex optimization

$$\underset{\mathbf{X}}{\text{minimize}} \quad \|\mathbf{X}\|_* \qquad \text{subject to} \quad \mathbf{X}_{\mathbb{O}} = \mathbf{T}_{\mathbb{O}},$$

Convex optimization

Compress sensing



With the smallest **possible number** of **nonzero** (positive)

Sparse solution

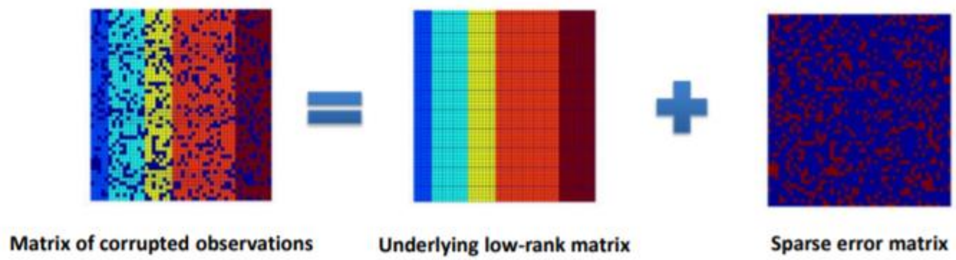
$$\min_{x} \|x\|_0 \quad \textbf{subj.to} \quad b = Ax$$

Nonconvex optimization

$$\min_{x} \|x\|_1 \quad \textbf{subj.to} \quad b = Ax$$

Convex optimization

Robust PCA


Matrix of corrupted observations
Underlying low-rank matrix
Sparse error matrix

Basic definitions and concepts in optimization

Objective function

Inequality constraint functions

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

Equality constraints

The set of points for which the objective and all constraint functions are defined,

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \ \cap \ \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

is called the *domain* of the optimization problem

Maximization → Utility or satisfaction level

Minimization → Loss or risk (in finance)

Basic definitions and concepts in optimization

The set of all points satisfying the constraints of an optimization problem is called its feasible set.

! Note that feasible set and domain for a given optimization problem are not necessarily the same. !

If x is feasible and $f_i(x) = 0$, we say the ith inequality constraint $f_i(x) \leq 0$ is *active* at x. If $f_i(x) < 0$, we say the constraint $f_i(x) \leq 0$ is *inactive*. (The equality constraints are active at all feasible points.) We say that a constraint is *redundant* if deleting it does not change the feasible set.

Basic definitions and concepts in optimization

The optimal value q^* of the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

$$q^* = inf\{ f_0(x) |\ f_i(x) \leq 0, i = 1,2, \cdots, m,\ h_i(x) = 0, i = 1,2, \cdots, p\}$$

We allow q^* to take on the extended values $\pm\infty$. If the problem is infeasible, we have $q^* = \infty$ (following the standard convention that the infimum of the empty set is ∞.)

If there are feasible points x_k with $f_0(x_k) \to -\infty$ as $k \to \infty$, then $q^* = -\infty$

unbounded below

Basic definitions and concepts in optimization

We say $x^\star$ is an *optimal point*, or solves the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

if x^* is feasible and $f_0(x^*) = q^*$.

The set of all optimal points is the *optimal set*, denoted

$$X_{opt} = \{x \mid f_i(x) \le 0, i = 1,2, \dots, m, \;\; h_i(x) = 0, i = 1,2, \dots, p, \;\; f_0(x) = q^*\}$$

Feasibility problem

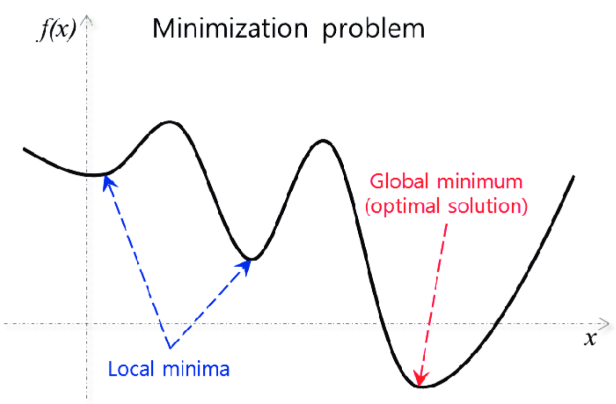
$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{array}$$

Basic definitions and concepts in optimization

We say a feasible point x is *locally optimal* if there is an $R > 0$ such that

$$\begin{aligned} f(x) = \inf\{f_0(z) \mid\ & f_i(z) \le 0,\ i = 1, \ldots, m, \\ & h_i(z) = 0,\ i = 1, \ldots, p,\ \|z - x\|_2 \le R\}, \end{aligned}$$

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & f_i(z) \le 0, \quad i = 1, \ldots, m \\ & h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \le R \end{array}$$



Basic definitions and concepts in optimization

Examples.

- $f_0(x) = \frac{1}{x}, q^* = 0, but\ the\ optimal\ value\ is\ not\ achived.$
- $f_0(x) = -\log(x)\ ;\ q^* = -\infty, so\ this\ problem\ is\ unbounded\ below.$
- $f_0(x) = xlog(x);\ q^* = -\frac{1}{e},\ achived\ at\ the\ (unique)\ optimal\ point$

$$x^* = 1/e$$

Equivalent forms of optimization problems

Minimization Problem ⟺ Maximization Problem

$$\underset{x\in C}{\text{Max}}\ f_0(x) = -\underset{x\in C}{\text{Min}}\ \{-f_0(x)\} \equiv \underset{x\in C}{\text{Min}}\ \{-f_0(x)\}$$

$$\underset{x\in C}{\text{Max}}\ f_0(x) = \frac{1}{\underset{x\in C}{\text{Min}}\left(\frac{1}{f_0(x)}\right)} \equiv \underset{x\in C}{\text{Min}}\ \frac{1}{f_0(x)}$$

$$f_0(x) > 0$$

$$\underset{x\in C}{\text{Min}}\ f_0(x) = \frac{1}{\underset{x\in C}{\text{Max}}\left(\frac{1}{f_0(x)}\right)} \equiv \underset{x\in C}{\text{Max}}\ \frac{1}{f_0(x)}$$

Equivalent forms of optimization problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \Rightarrow \begin{array}{ll} \text{minimize} & \tilde{f}(x) = \alpha_0 f_0(x) \\ \text{subject to} & \tilde{f}_i(x) = \alpha_i f_i(x) \le 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, \quad i = 1, \dots, p, \end{array}$$

where $\alpha_i > 0$, $i = 1, \dots, m$, and $\beta_i \neq 0$, $i = 1, \dots, p$.

Slack variables

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \Rightarrow \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & s_i \ge 0, \quad i = 1, \dots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p, \end{array}$$

$$f_i(x) \ge 0 \Rightarrow f_i(x) - s_i = 0, \; s_i \ge 0$$

Surplus variable

x is free $\Rightarrow$ $x = x_2 - x_1, \quad x_2 \ge 0, \; x_1 \ge 0$

Equivalent forms of optimization problems

Eliminating equality constraints

Suppose the function $\phi : \mathbf{R}^k \to \mathbf{R}^n$ is such that x satisfies

$$h_i(x) = 0, \quad i = 1, \ldots, p,$$

if and only if there is some $z \in \mathbf{R}^k$ such that $x = \phi(z)$.

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

$$\Downarrow$$

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(z) = f_0(\phi(z)) \\ \text{subject to} & \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \quad i = 1, \ldots, m \end{array}$$

Equivalent forms of optimization problems

Eliminating linear equality constraints

$$Ax = b \quad \Rightarrow \quad x = Fz + x_0 \quad \Rightarrow \quad \begin{cases} x_0 = A^{\dagger} b \\ F = \mathcal{N}(A) \end{cases}$$

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ s.t. & f_i(\mathbf{x}) \le 0, , \quad i = 1, 2, \dots, m \\ & Ax = b, \end{array}$$

$$\Downarrow$$

$$\begin{array}{ll} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \le 0, \quad i = 1, \dots, m, \end{array}$$

Symmetric

$$\begin{array}{ll} \min & \mathbf{x}^T \mathbf{R} \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{array} \quad \Longrightarrow \quad \min_{z} \left(Fz + x_0\right)^T R \left(Fz + x_0\right)$$

Equivalent forms of optimization problems

Optimizing over some variables

$$\min_{x_1,x_2} f_0(x_1,x_2) = \min_{x_1} \tilde{f}_0(x_1)$$

where $\tilde{f}_0(x_1) = \min_{x_2} f_0(x_1,x_2)$

$$\begin{array}{ll} \text{minimize} & f_0(x_1,x_2) \\ \text{subject to} & f_i(x_1) \le 0, \quad i=1,\ldots,m_1 \\ & \tilde{f}_i(x_2) \le 0, \quad i=1,\ldots,m_2, \end{array}$$

$$\tilde{f}_0(x_1) = \min\left\{ f_0(x_1,x_2) \middle| \tilde{f}_i(x_2) \le 0, i=1,\ldots,m_2 \right\}$$

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \le 0, \quad i=1,\ldots,m_1. \end{array}$$

Equivalent forms of convex optimization problems

Introducing equality constraints

$$
\begin{array}{ll}
\text{minimize} & f_0(A_0x + b_0) \\
\text{subject to} & f_i(A_ix + b_i) \le 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p,
\end{array}
$$

$$\Big\downarrow \quad y_i = A_ix + b_i, \text{ for } i = 0, \ldots, m,$$

$$
\begin{array}{ll}
\text{minimize} & f_0(y_0) \\
\text{subject to} & f_i(y_i) \le 0, \quad i = 1, \ldots, m \\
& y_i = A_ix + b_i, \quad i = 0, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p.
\end{array}
$$

Equivalent forms of convex optimization problems

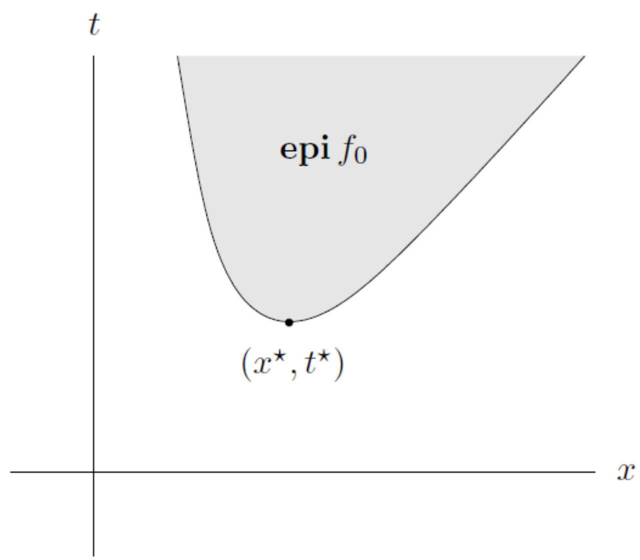
Epigraph problem form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

$\Rightarrow$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p, \end{array}$$

$$\mathbf{epi}\, f = \{(x, t) \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$$



Equivalent forms of convex optimization problems

Epigraph problem form

$$\min_{\mathbf{x}\in\mathbb{R}^n} \max_{i=1,\ldots,m} \{\mathbf{a}_i^T\mathbf{x} + b_i\} = \begin{cases} \min\ t \\ \text{s.t.}\ \ \max_{i=1,\ldots,m}\{\mathbf{a}_i^T\mathbf{x} + b_i\} \le t,\ \ \mathbf{x}\in\mathbb{R}^n \end{cases}$$

$$= \begin{cases} \min\ t \\ \text{s.t.}\ \ \mathbf{a}_i^T\mathbf{x} + b_i \le t,\ \ i = 1,\ldots,m,\ \ \mathbf{x}\in\mathbb{R}^n. \end{cases}$$

Equivalent forms of optimization problems

Change of variables

Suppose $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one, with image covering the problem domain $\mathcal{D}$, *i.e.*, $\phi(\mathbf{dom}\,\phi) \supseteq \mathcal{D}$. We define functions $\tilde{f}_i$ and $\tilde{h}_i$ as

$$\tilde{f}_i(z) = f_i(\phi(z)), \quad i = 0, \dots, m, \qquad \tilde{h}_i(z) = h_i(\phi(z)), \quad i = 1, \dots, p.$$

$$x = \phi(z) \quad \Longleftrightarrow \quad z = \phi^{-1}(x)$$

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \text{minimize} & \tilde{f}_0(z) \\ \text{subject to} & \tilde{f}_i(z) \le 0, \quad i = 1, \dots, m \\ & \tilde{h}_i(z) = 0, \quad i = 1, \dots, p, \end{array}$$

with variable z.

Equivalent forms of optimization problems

Transformation of objective and constraint functions

Suppose that $\psi_0 : \mathbf{R} \to \mathbf{R}$ is monotone increasing, $\psi_1, \ldots, \psi_m : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$, and $\psi_{m+1}, \ldots, \psi_{m+p} : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$. We define functions $\tilde{f}_i$ and $\tilde{h}_i$ as the compositions

$$\tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, \ldots, m, \qquad \tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \quad i = 1, \ldots, p.$$

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{minimize} & \tilde{f}_0(x) \\ \text{subject to} & \tilde{f}_i(x) \leq 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

Convex optimization problems in standard form

A *convex optimization problem* is one of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p, \end{array}$$

where $f_0, \ldots, f_m$ are convex functions.

- the objective function must be convex,
- the inequality constraint functions must be convex,
- the equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine.

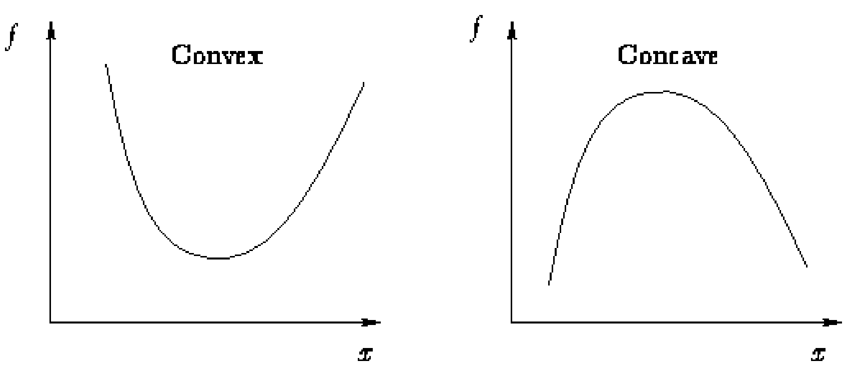
$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i$$

Concave maximization problems

Concave function

$$
\begin{array}{ll}
\text{maximize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & a_i^T x = b_i, \quad i = 1, \ldots, p,
\end{array}
$$

This *concave maximization problem* is readily solved by minimizing the convex objective function $-f_0$.



Abstract form convex optimization problem

A convex optimization problem is called convex optimization in abstract form if the objective function and associated feasible region are convex.

! Any convex problem in standard format is also convex in abstract form but the reverse is not true in general. !

Abstract form convex optimization problem

nonconvex function

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0, \end{array}$$

none-affine

Convex optimization in abstract form

! Nonconvex optimization in standard form !

It can be reduced to the standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & \tilde{f}_1(x) = x_1 \leq 0 \\ & \tilde{h}_1(x) = x_1 + x_2 = 0, \end{array}$$

Convex optimization in Standard form

Fundamental property of convex optimization problems

Any locally optimal point of a convex optimization is also (globally) optimal.

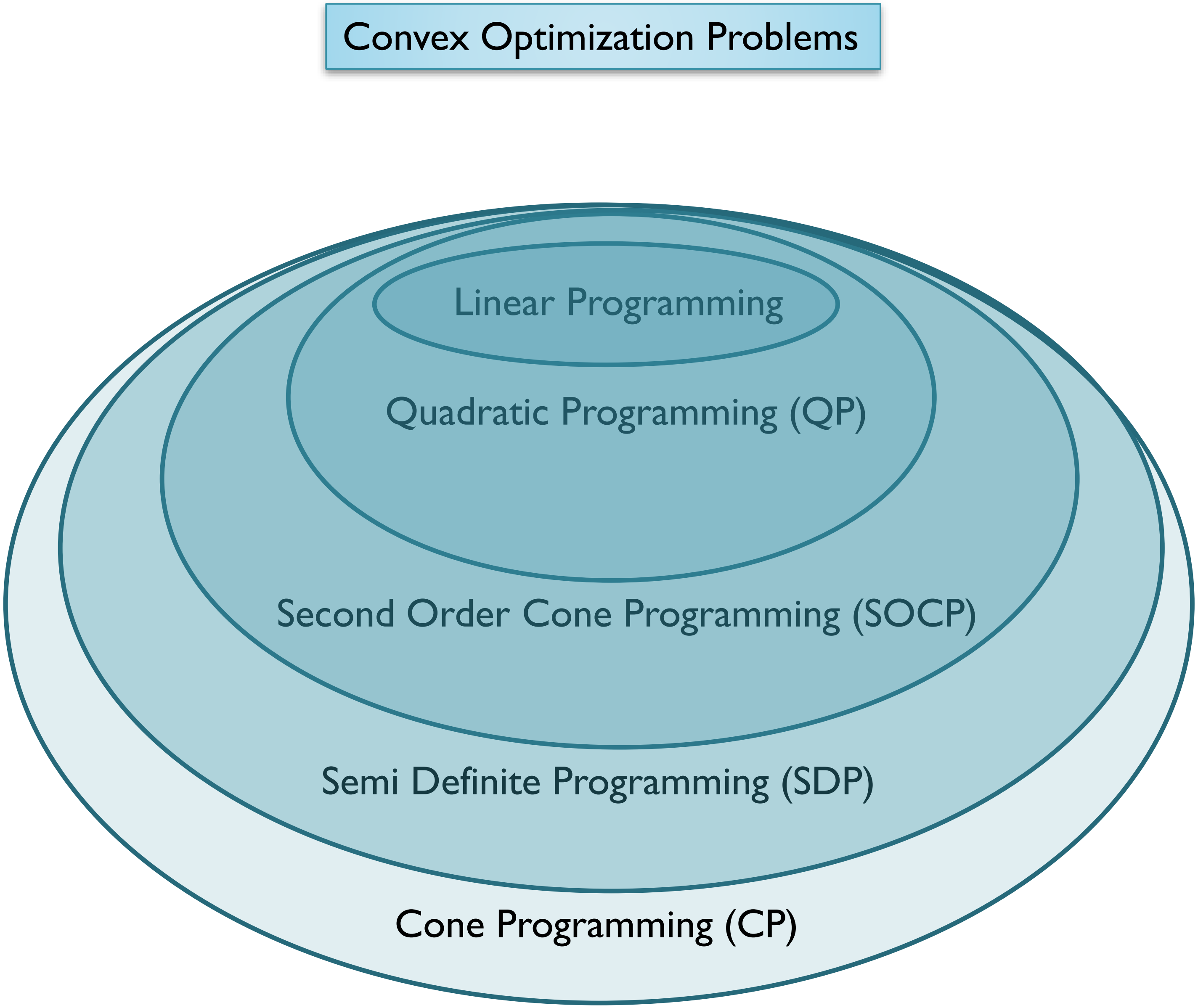
An optimality criterion for differentiable objective function

Optimality conditions

Constrained problem $\nabla f_0(x)^T(y-x) \geq 0 \text{ for all } y \in X$

Unconstrained problem $\nabla f_0(x) = 0$

Feasible set


Convex Optimization Problems
Linear Programming
Quadratic Programming (QP)
Second Order Cone Programming (SOCP)
Semi Definite Programming (SDP)
Cone Programming (CP)

Linear Programming (LP)

Standard form of LP

$$\begin{aligned} \min\ & c^T x \\ s.t\ \ & Ax = b \\ & x \geq 0 \end{aligned}$$

LP with inequality constraints

$$\begin{aligned} \min\ & c^T x \\ s.t\ \ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Linear Programming (LP)

A general linear program has the form

$$\begin{aligned} \min\ & c^T x + d \\ s.t.\ & Gx \le h \\ & Ax = b \end{aligned}$$

where $G \in \mathbf{R}^{m \times n}$ and $A \in \mathbf{R}^{p \times n}$.

Converting LPs to standard form

$$\begin{aligned} \min\ & c^T x + d \\ s.t.\ & Gx + s = h \\ & Ax = b \\ & s \ge 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & c^T x^+ - c^T x^- + d \\ s.t.\ & G x^+ - G x^- + s = h \\ & A x^+ - A x^- = b \\ & x^+ \ge 0,\ x^- \ge 0,\ s \ge 0 \end{aligned}$$

Quadratic Optimization Problems (QOP)

Linearly Constrained Quadratic Program (LCQP)

$$
\begin{aligned}
\min\ & (1/2)x^T P x + q^T x + r \\
s.t.\ & Gx \le h \\
& Ax = b
\end{aligned}
$$

Quadratically Constrained Quadratic Program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \le 0, \quad i = 1, \dots, m \\
& Ax = b,
\end{array}
$$

Least-squares and regression

$$
\|Ax - b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b
$$

Unconstrained QOP

Second Order Cone Programming (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \dots, m \\ & Fx = g, \end{array}$$

Second order cone constraint

! **Note.** This problem is not a quadratically constrained LP problem. !

If $c_i = 0, \ i = 1, 2, \dots, m$ SOCP → Quadratically Constrained LP Problem

If $A_i = 0, \ i = 1, 2, \dots, m$ SOCP → LP problem

Generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \dots, m \\ & Ax = b, \end{array}$$

Examples.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b. \end{array}$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \succeq_K 0 \\ & Ax = b \end{array}$$

Semi-Definite programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b, \end{array}$$

If $G,\ F_1, F_2, \ldots, F_n$ are diagonal matrices then SDP is reduced to LP problem

What is Linear Programming?

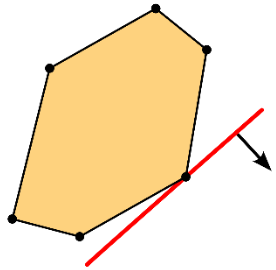
- A Linear Programming model seeks to maximize or minimize a linear function (usually **profit** or **cost** of production), subject to a set of linear constraints.

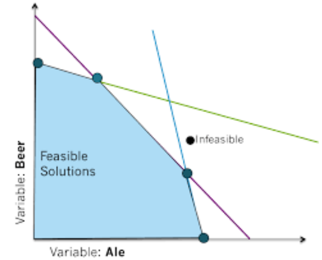
- The linear model consists of the following components:
  - **A set of decision variables.**
  - **An objective function.**
  - **A set of constraints.**

In 1939, **Leonid Kantorovich** presented a number of solutions to some problems related to production and transportation planning.

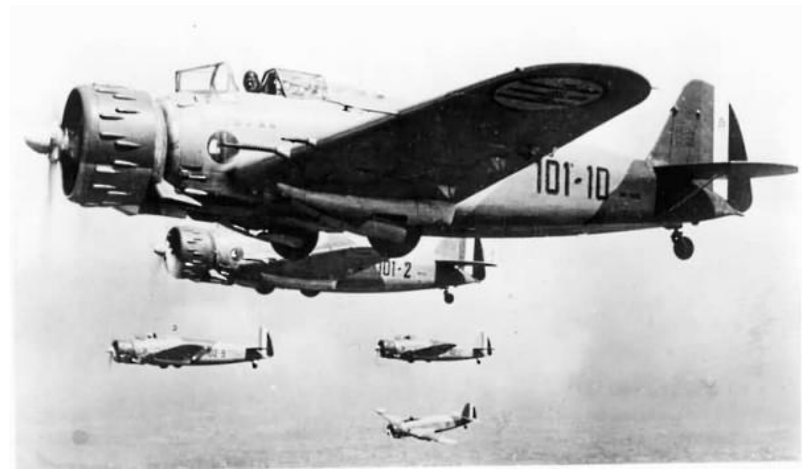
During World War II, **Tjalling Charles Koopmans** contributed significantly to the solution of transportation problems.

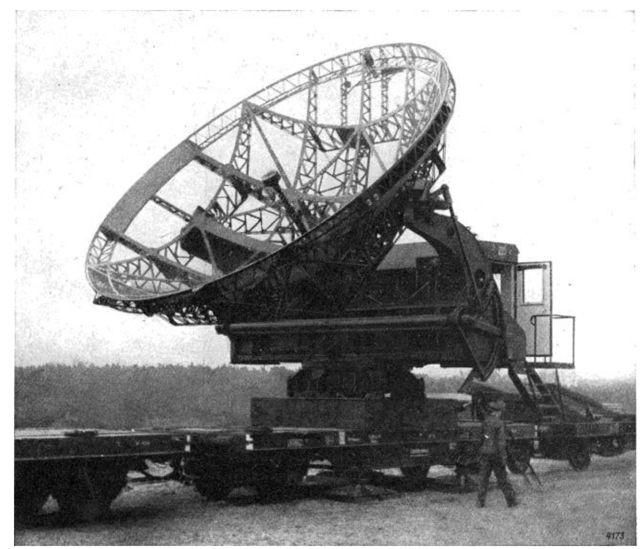
Kantorovich and Koopmans were awarded a Nobel Prize in economics in 1975 for their work on the theory of optimal allocation of resources.

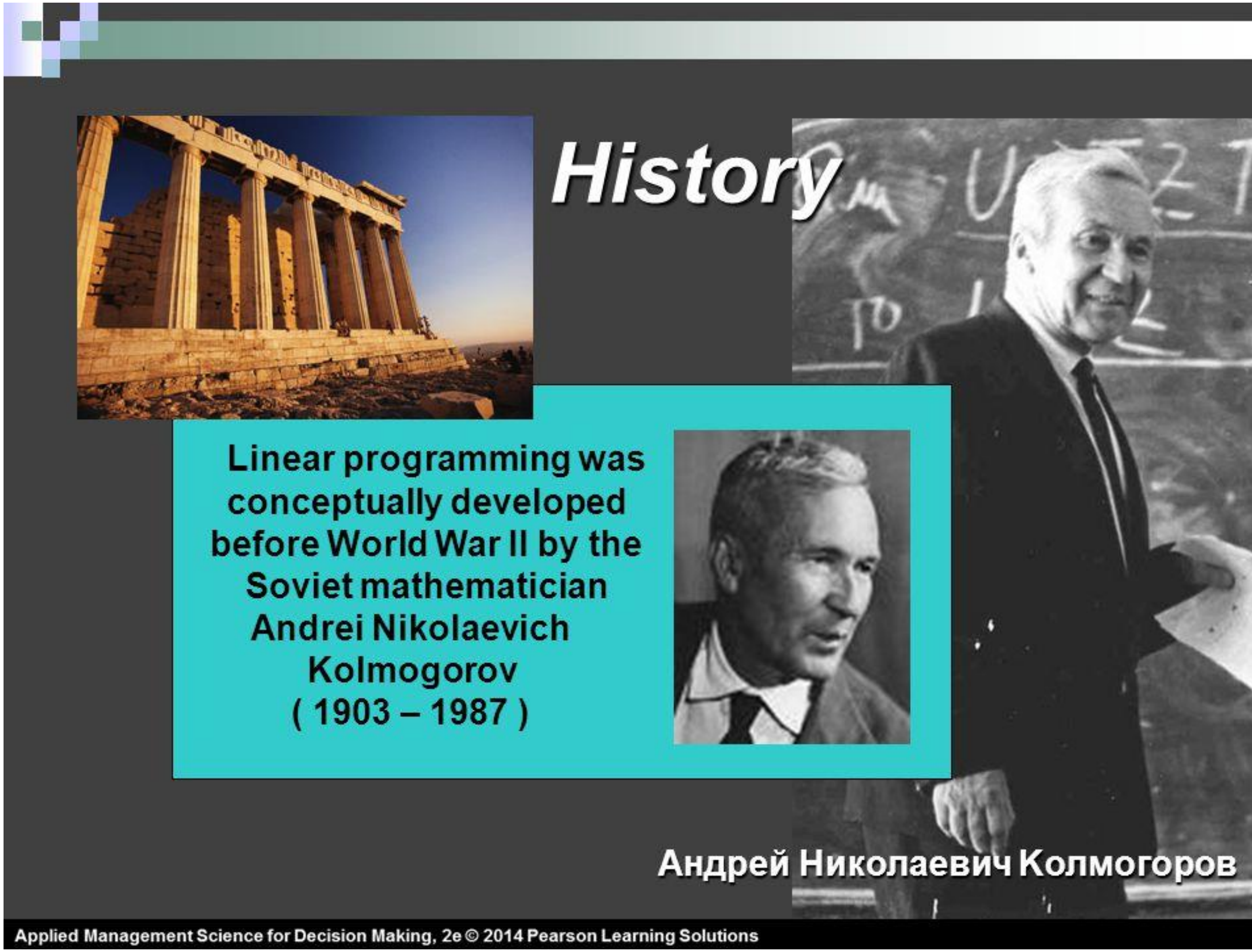



Infeasible
Feasible Solutions
Variable: Beer
Variable: Ale


X1
Maximum profit
feasible region
X2
Corner points


101-10
101-2




History
Linear programming was conceptually developed before World War II by the Soviet mathematician Andrei Nikolaevich Kolmogorov (1903 – 1987)
Андрей Николаевич Колмогоров
Applied Management Science for Decision Making, 2e © 2014 Pearson Learning Solutions

STANDARD FORM LINEAR PROGRAMS

Formally, a linear program is an optimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

$\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$, or $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. The vector inequality $\boldsymbol{x} \geq \boldsymbol{0}$ means that each component of $\boldsymbol{x}$ is nonnegative.

$$\text{minimize } c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Quadratic term $\boldsymbol{c}^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$

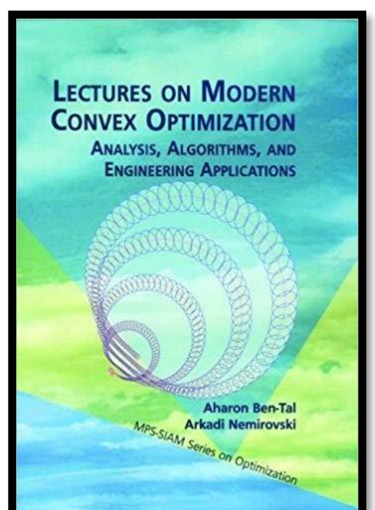
where $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

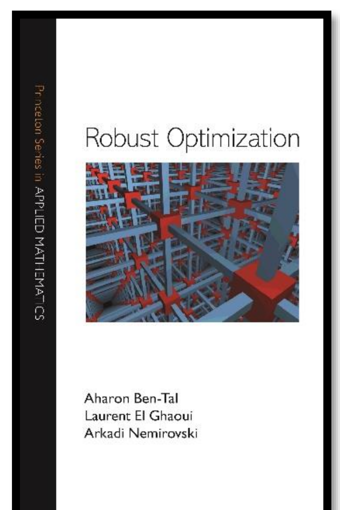
Positive definite matrix

Linear Programming Properties:

- ***<u>Linearity</u>*. The decision variables are linear in constraints and objective function.
- ***<u>Divisibility</u>*. Non-integer values of decision variables are acceptable.
- ***<u>Certainty.</u>*** Values of parameters are known and constant.
- ***<u>Nonnegativity.</u>*** Negative values of decision variables are unacceptable.

Arkadi Nemirovski





Aharon Ben-Tal

Linear Programming (LP) is a very special (the simplest) kind of convex optimization.

- The objective function is linear and as a result it is convex.
- The constrained set is convex**:** **intersection of halfspaces**

- **The local and global minimum-maximum of a LP are the same because of the convexity property of LP.**

Several applied problems in industry, **Manufacturing, Marketing, Finance (investment), Agriculture** can be modelled by LPs,

or

Some nonlinear models can be approximated by LPs

Linear approximation usually is not enough and quadratic term is required: Sequentially Quadratic programming

Transportation Problem

To / From	D	E	F	G	Supply
A	$7	$10	$14	$8	30
B	$7	$11	$12	$6	40
C	$5	$8	$15	$9	30
Demand	20	20	25	35	100

$$\text{minimize} \quad 7x_{11} + 10x_{12} + 14x_{13} + 8x_{14} + 7x_{21} + 11x_{22} + 12x_{23} + 6x_{24} + 5x_{31} + 8x_{32} + 15x_{33} + 9x_{34}$$

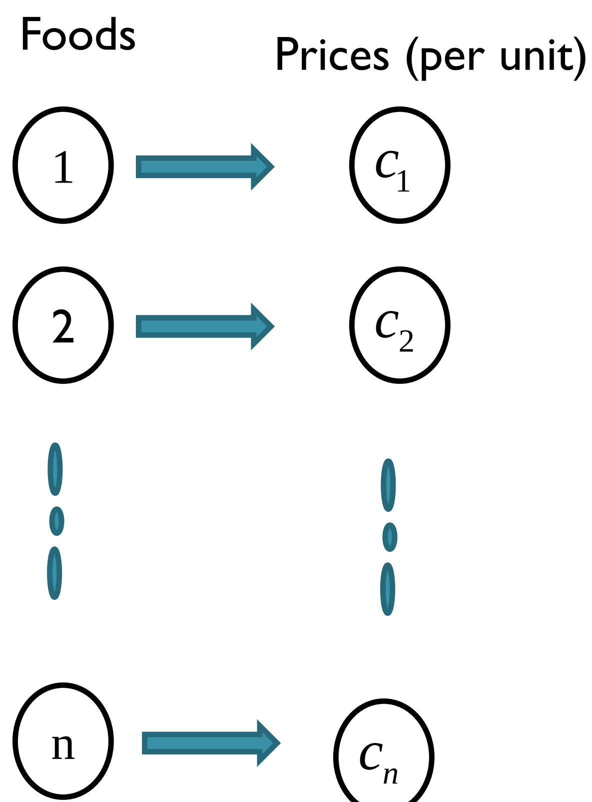
$$\begin{aligned}
\text{subject to} \quad x_{11} + x_{12} + x_{13} + x_{14} &= 30 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 40 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 30 \\
x_{11} + x_{21} + x_{31} &= 20 \\
x_{12} + x_{22} + x_{32} &= 20 \\
x_{13} + x_{23} + x_{33} &= 25 \\
x_{14} + x_{24} + x_{34} &= 35, \\
x_{11}, x_{12}, \ldots, x_{34} &\geq 0.
\end{aligned}$$

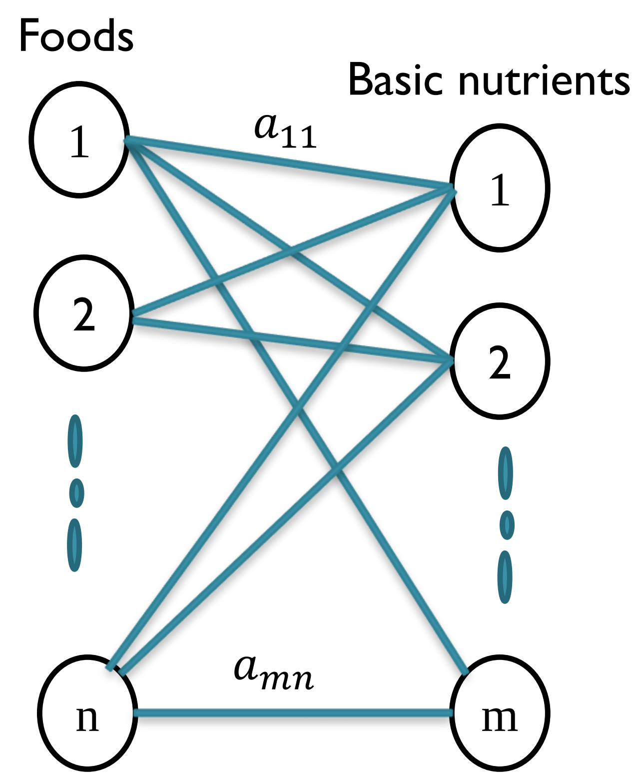
Diet Problem

x_j number of units of food j in the diet.

The jth food sells at a price c_j per unit.

Assume that each unit of food j contains a_{ij} units of the ith nutrient.





Diet Problem

$$\text{minimize } c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m,
\end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.$$

$$\begin{array}{ll}
\text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\
\text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0},
\end{array}$$

Production Problem

Example A manufacturer produces two different products X_1 and X_2 using three machines M_1, M_2, and M_3. Each machine can be used only for a limited amount of time. Production times of each product on each machine are given in

Machine	Production time (hours/unit) X_1	X_2	Available time (hours)
M_1	1	1	8
M_2	1	3	18
M_3	2	1	14
Total	4	5	

The objective is to maximize the combined time of utilization of all three machines.

$$\boxed{f(x_1, x_2) = 4x_1 + 5x_2.}$$

$$\boxed{\begin{aligned} x_1 + x_2 &\leq 8 \\ x_1 + 3x_2 &\leq 18 \\ 2x_1 + x_2 &\leq 14, \end{aligned}}$$

Production Problem

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0}, \end{array}$$

where

$$\begin{aligned} \boldsymbol{c}^T &= [4, 5], \\ \boldsymbol{x} &= [x_1, x_2]^T, \\ \boldsymbol{A} &= \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}, \\ \boldsymbol{b} &= [8, 18, 14]^T. \end{aligned}$$

minimize $\boldsymbol{c}^T\boldsymbol{x}$

Can be replaced by

maximize $-\boldsymbol{c}^T\boldsymbol{x}$

For decision variable which are free, we can substitute it by two new nonnegative variables

$$x_1 = \boldsymbol{u} - \boldsymbol{v} \text{ and } \boldsymbol{u}, \boldsymbol{v} \geq \mathbf{0}$$

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

<u>*Surplus variables*</u>

$$\text{minimize} \quad \boldsymbol{c}^T \boldsymbol{x}$$

$$\begin{array}{ll} \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - y_i = b_i, \;\; i = 1, \ldots, m \\ & x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \\ & y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \end{array}$$

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} - \boldsymbol{I}_m \boldsymbol{y} = [\boldsymbol{A}, -\boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \; \boldsymbol{y} \geq \boldsymbol{0}, \end{array}$$

$$
\boxed{\begin{aligned} Ax &\leq b \\ x &\geq 0, \end{aligned}}
$$

$$\Downarrow$$

<u>*Slack variables*</u>

$$
\boxed{\begin{aligned} &Ax + I_m y = [A, I_m] \begin{bmatrix} x \\ y \end{bmatrix} = b \\ &x \geq 0,\ y \geq 0, \end{aligned}}
$$

Consider the following optimization problem

$$\begin{array}{ll} \text{maximize} & x_2 - x_1 \\ \text{subject to} & 3x_1 = x_2 - 5 \\ & |x_2| \leq 2 \\ & x_1 \leq 0. \end{array}$$

How can we convert it to a LP?

$$\begin{aligned}
&\text{Min} && x_1 - x_2 \\
&\text{S.t.} && 3x_1 - x_2 = -5 \\
& && x_2 \le 2 \\
& && x_2 \ge -2 \\
& && x_1 \le 0
\end{aligned}$$

$$x_2 = u - v, \quad u \ge 0, v \ge 0.$$

$$x_1' = -x_1 \ge 0$$

$$\begin{aligned}
&\min && -x_1' - u + v \\
&\text{S.t.} && 3x' - v + u = 5 \\
& && u - v \le 2 \\
& && v - u \le 2 \\
& && x_1' \ge 0 \\
& && u \ge 0, v \ge 0.
\end{aligned}$$

$$\begin{aligned}
&\min && v - u - x_1' \\
&\text{S.t.} && 3x_1' + v - u = -5 \\
& && u - v + s_1 = 2 \\
& && u - v + s_2 = -2 \\
& && x_1' \ge 0,\ u \ge 0, v \ge 0, s_1 \ge 0,\ s_2 \ge 0.
\end{aligned}$$

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n| \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where $c_i \neq 0, i = 1, \ldots, n$. Convert the above problem into an equivalent standard form linear programming problem.

How can we convert it to a LP?

$$x^+ = \max(x,0), \quad x^- = -\min(x,0),$$

$$x^+ \ge 0, \; x^- \ge 0,$$

$$x = x^+ - x^-,$$

$$|x| = x^+ + x^-.$$

$$\min \quad c_1 x_1^+ + \ldots + c_1 x_n^+ + c_1 x_1^- + \ldots + c_1 x_n^-$$

$$Ax^+ - Ax^- = b,$$

$$x^+ \ge 0, \quad x^- \ge 0.$$

If we have 100 000 free variables then the mentioned method increases the variables.

!

Number of variables will be 200 000 (twice)

!

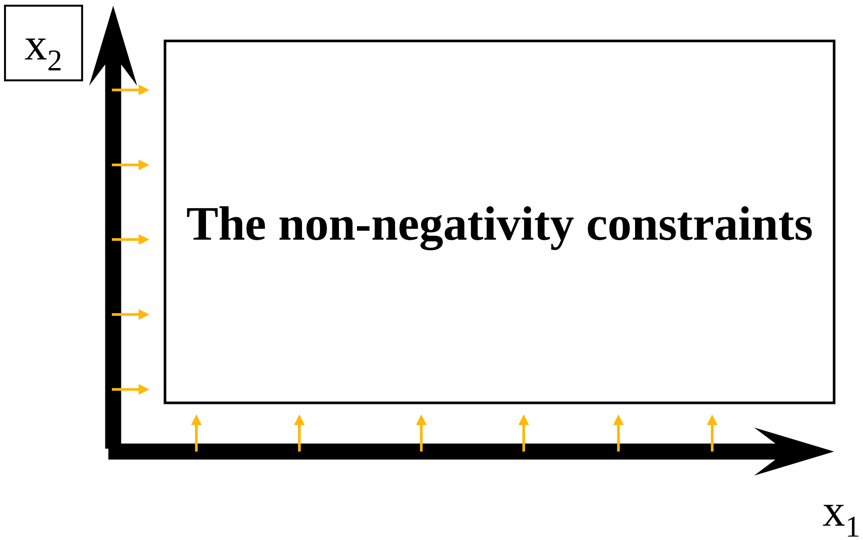
100 000 inequalities $(\leq)$ $\Rightarrow$ 100 000 new slack variables

10 000 free decision variable $\Rightarrow$ 20 000 new variables

---

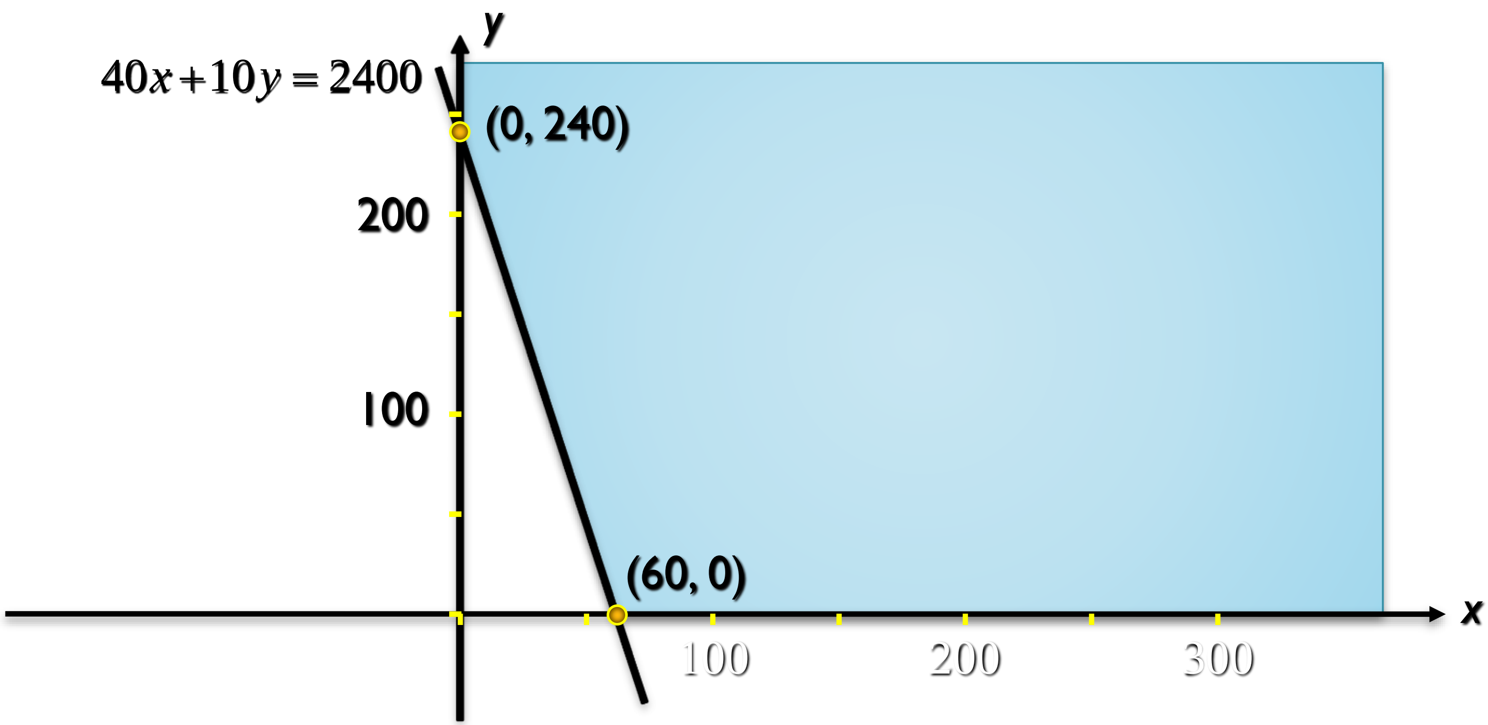
120 000 more variables

Graphical Analysis

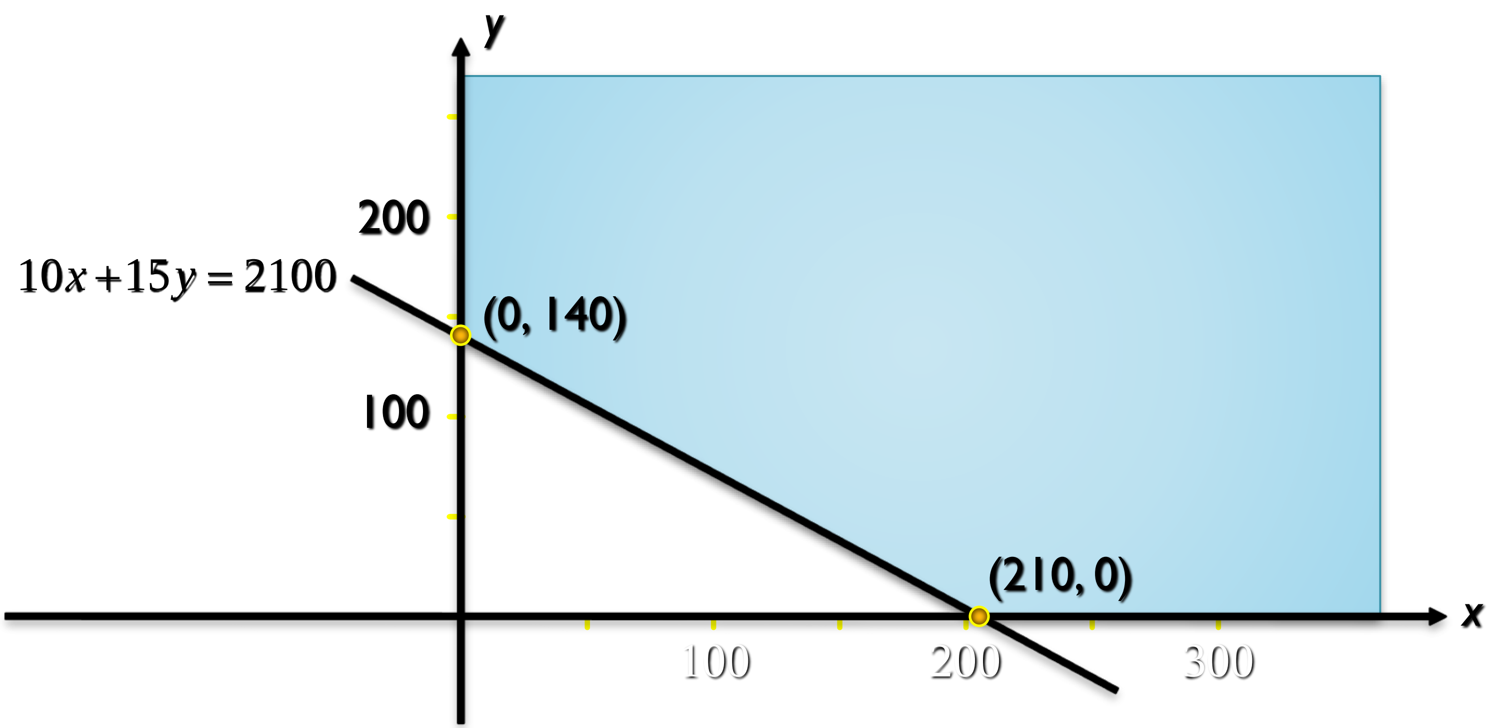


Graph the solution for the inequality

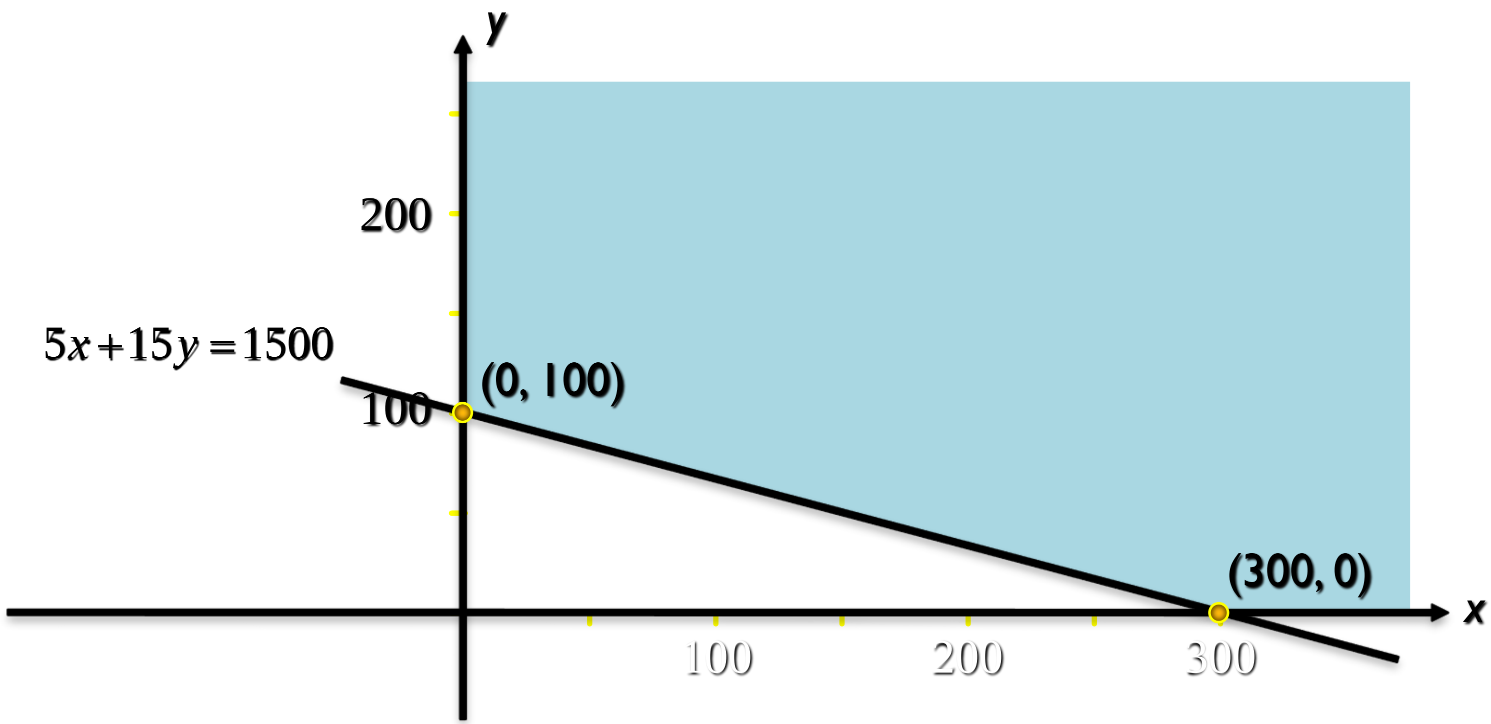
$$40x + 10y \geq 2400$$

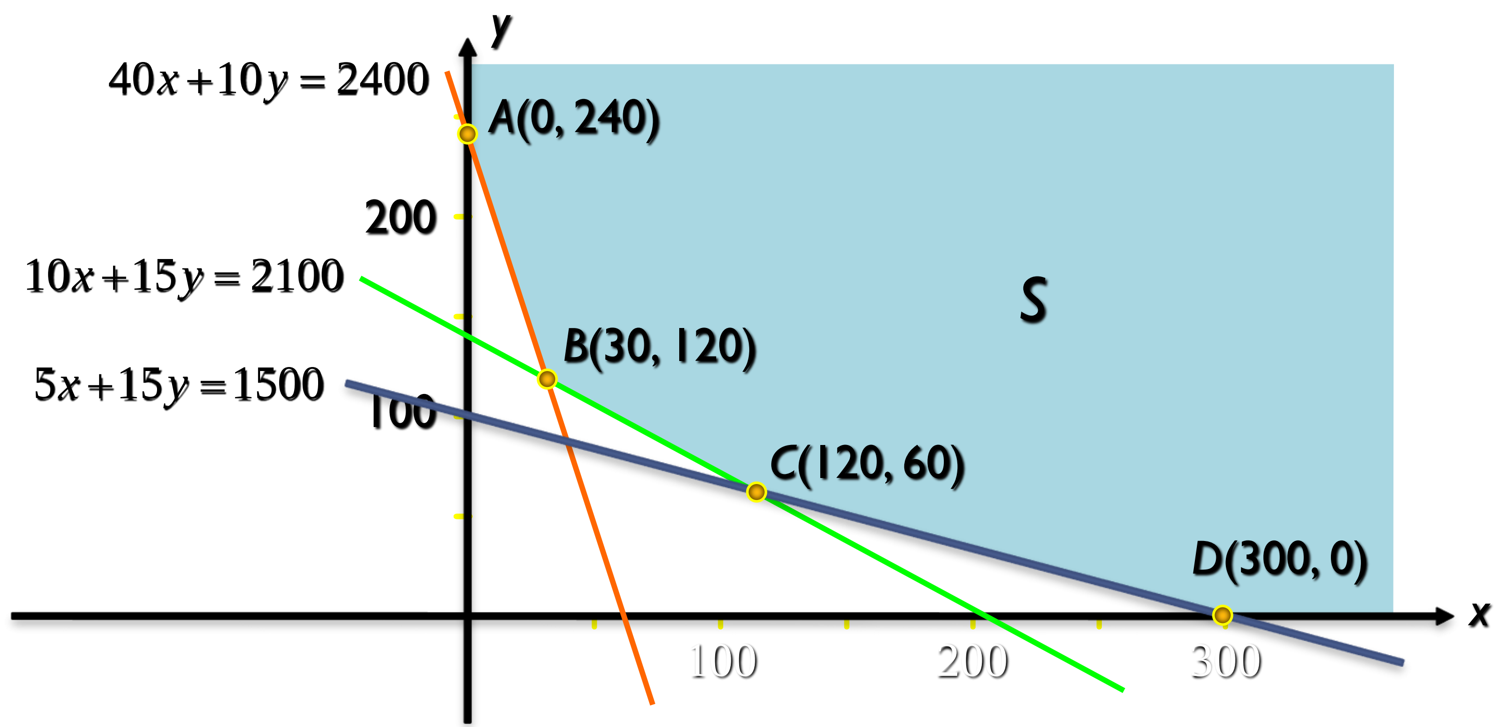


$$10x + 15y \geq 2100$$



$$5x + 15y \geq 1500$$





y
$40x+10y=2400$
A(0, 240)
200
$10x+15y=2100$
S
B(30, 120)
$5x+15y=1500$
100
C(120, 60)
D(300, 0)
x
100
200
300

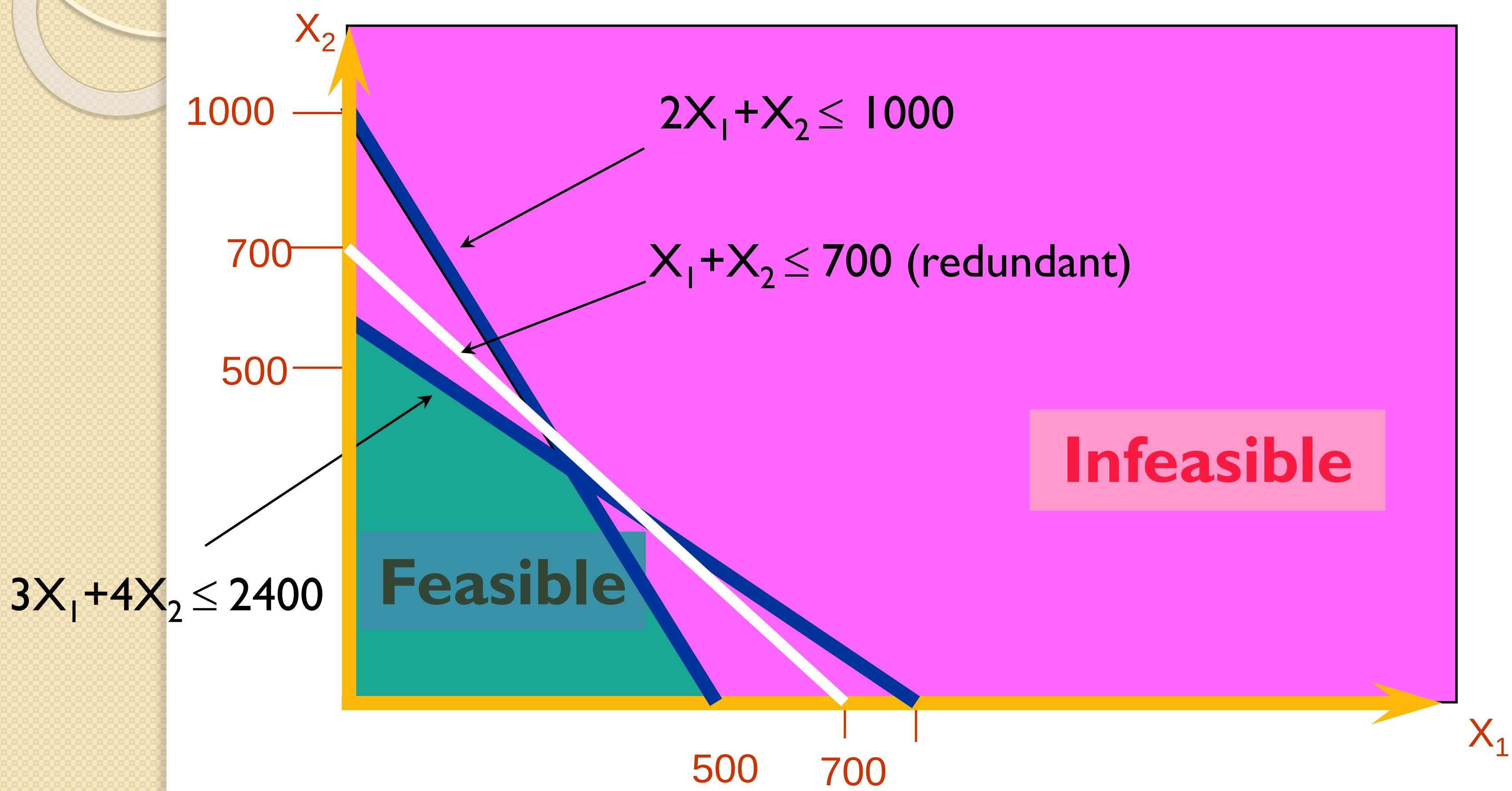
The Set of Feasible Solutions for Linear Programs

The set of all points that satisfy all the constraints of the model is called a

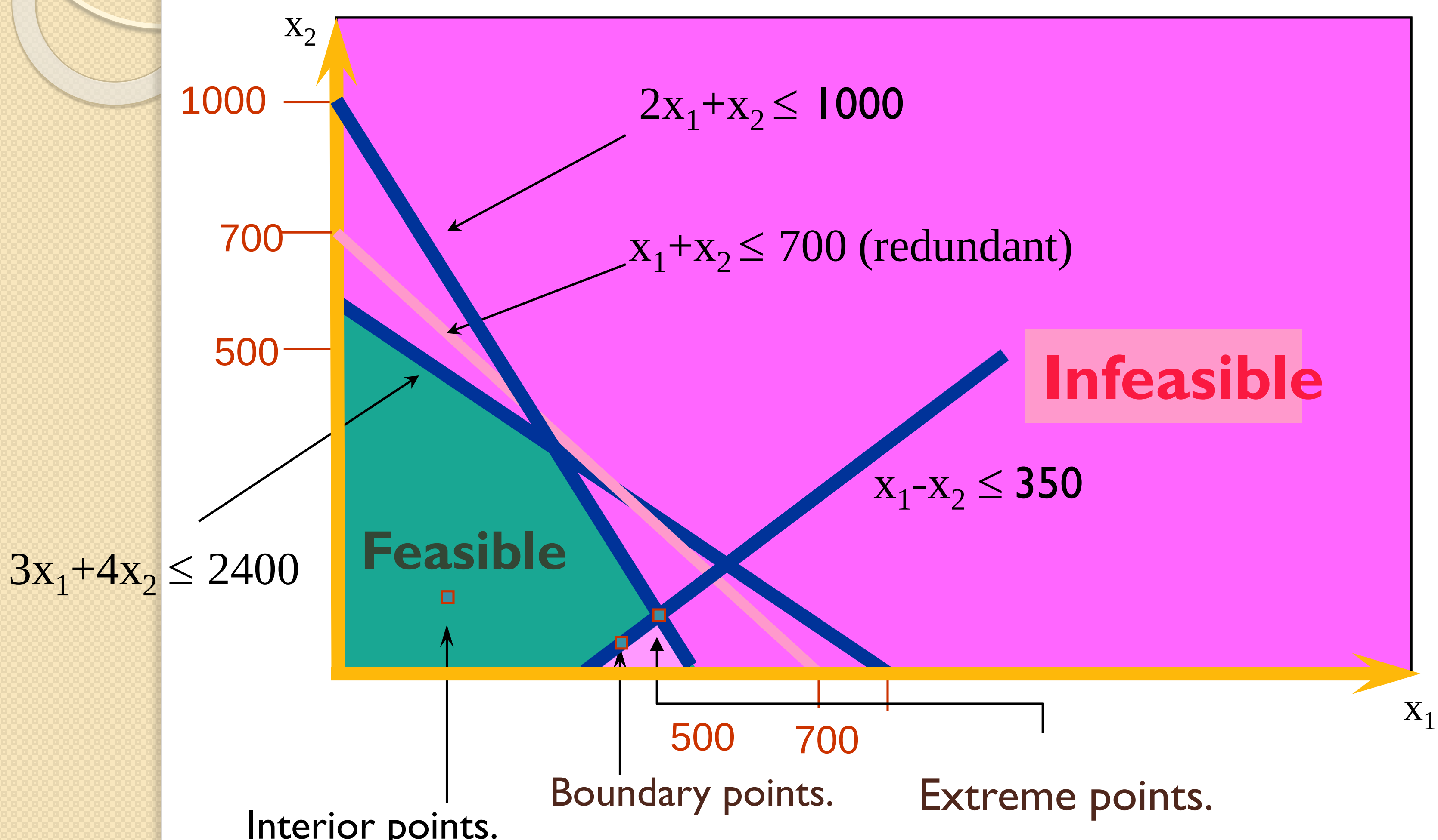
FEASIBLE REGION

Using a graphical presentation we can represent all the constraints, the objective function, and the three types of feasible points.

Graphical Analysis – the Feasible Region



Graphical Analysis – the Feasible Region



- **There are three types of feasible points**

Graphical Linear Programming

Graphical method for finding optimal solutions to two-variable problems

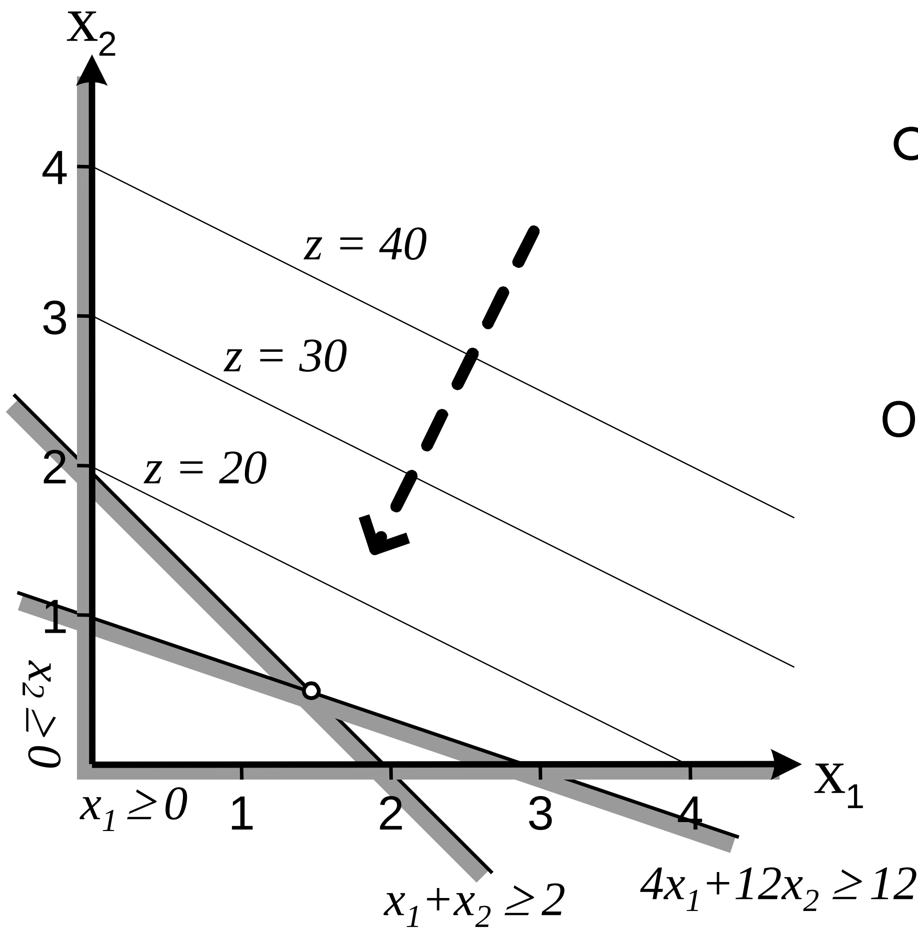
- ✓ Set up objective function and constraints in mathematical format.
- ✓ Plot the constraints.
- ✓ Identify the feasible solution space.
- ✓ Plot the objective function.
- ✓ Determine the optimum solution.

LP – Example

- The problem:

$$
\begin{aligned}
\min \quad & z = 5x_1 + 10x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

LP – Example



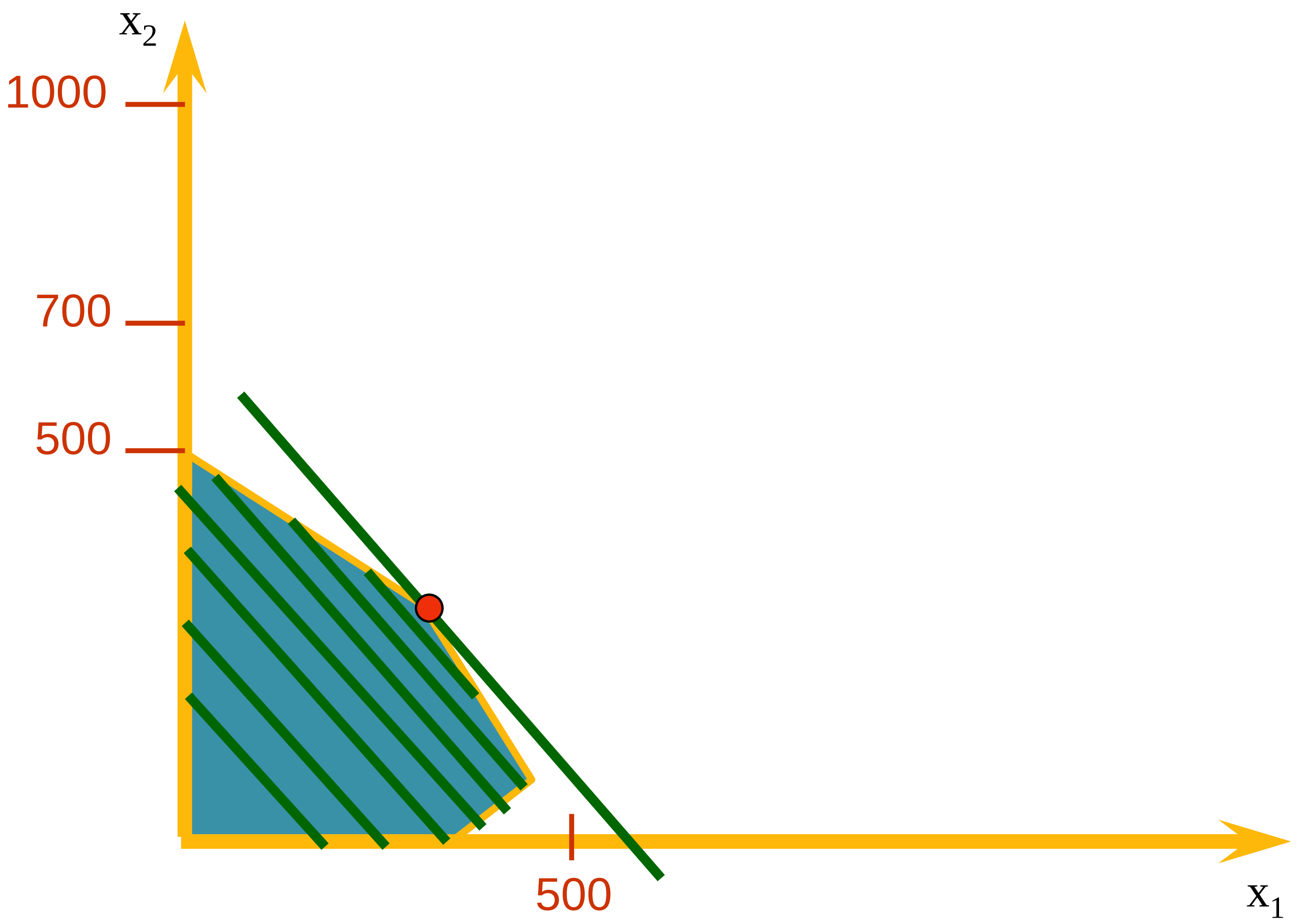
Optimum in:

$$x_1 = 1.5$$

$$x_2 = 0.5$$

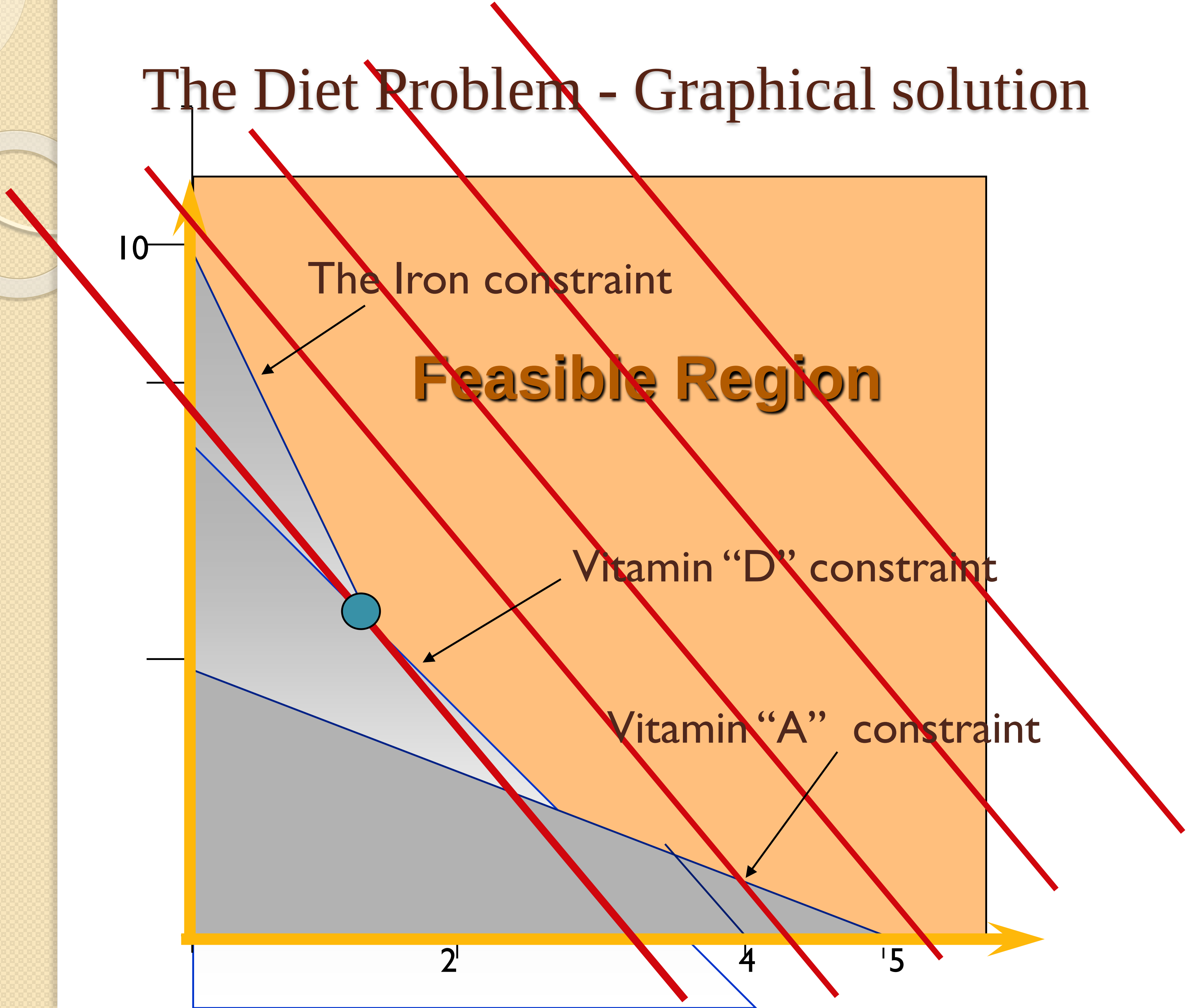
Objective:

$$z = 12.5$$



x2
1000
700
500
500
x1

The Diet Problem - Graphical solution



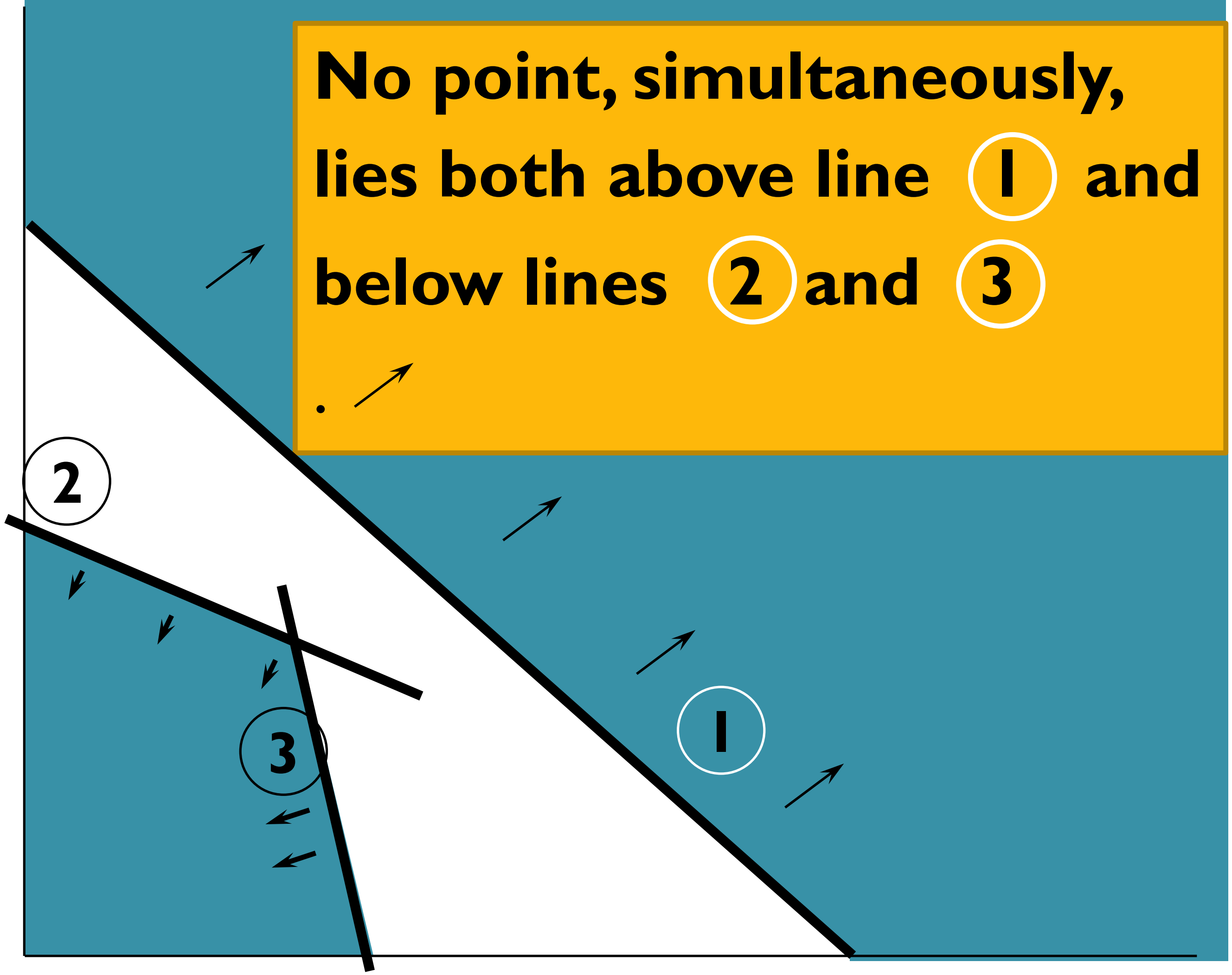
Models Without Unique Optimal Solutions

- **Infeasibility.** Occurs when a model has no feasible point.
- **Alternate solution.** Occurs when more than one point optimizes the objective function
- **Unboundness.** Occurs when the objective can become infinitely large (max), or infinitely small (min).

Infeasible Model

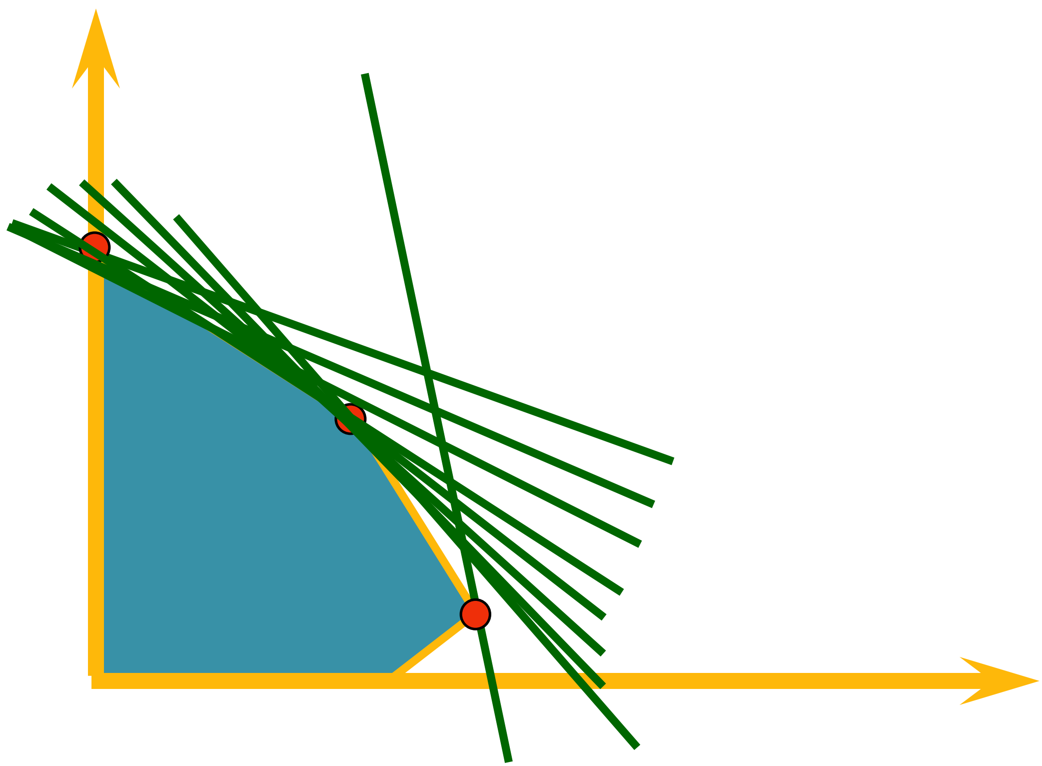
No point, simultaneously, lies both above line 1 and below lines 2 and 3

.



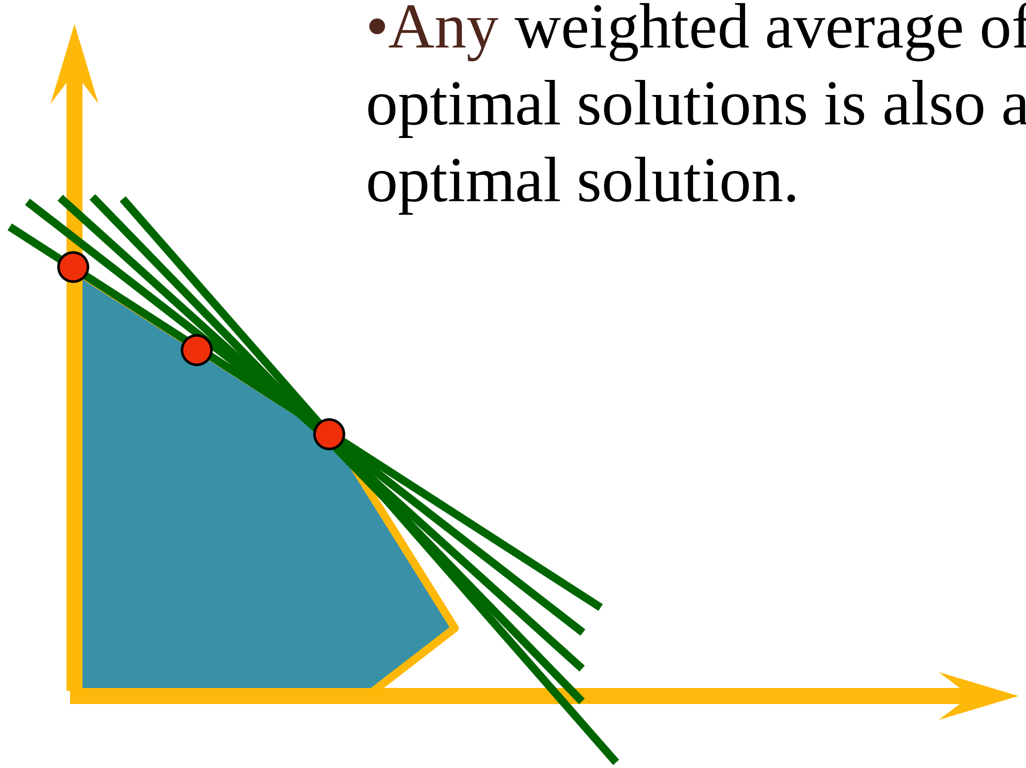
Extreme points and optimal solutions

- If a linear programming problem has an optimal solution, an extreme point is optimal.



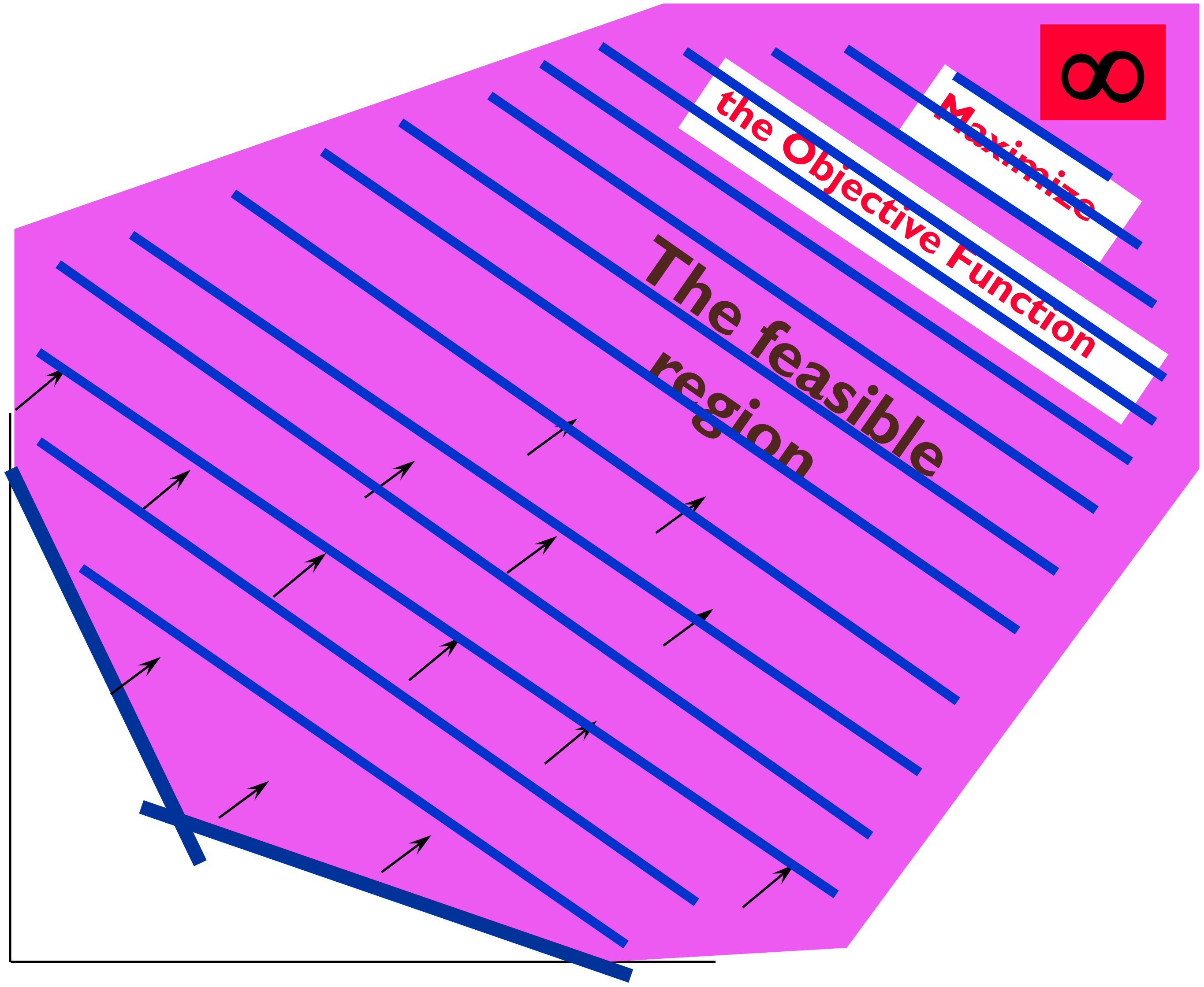
Multiple optimal solutions

- For multiple optimal solutions to exist, the objective function must be parallel to one of the constraints



- Any weighted average of optimal solutions is also an optimal solution.

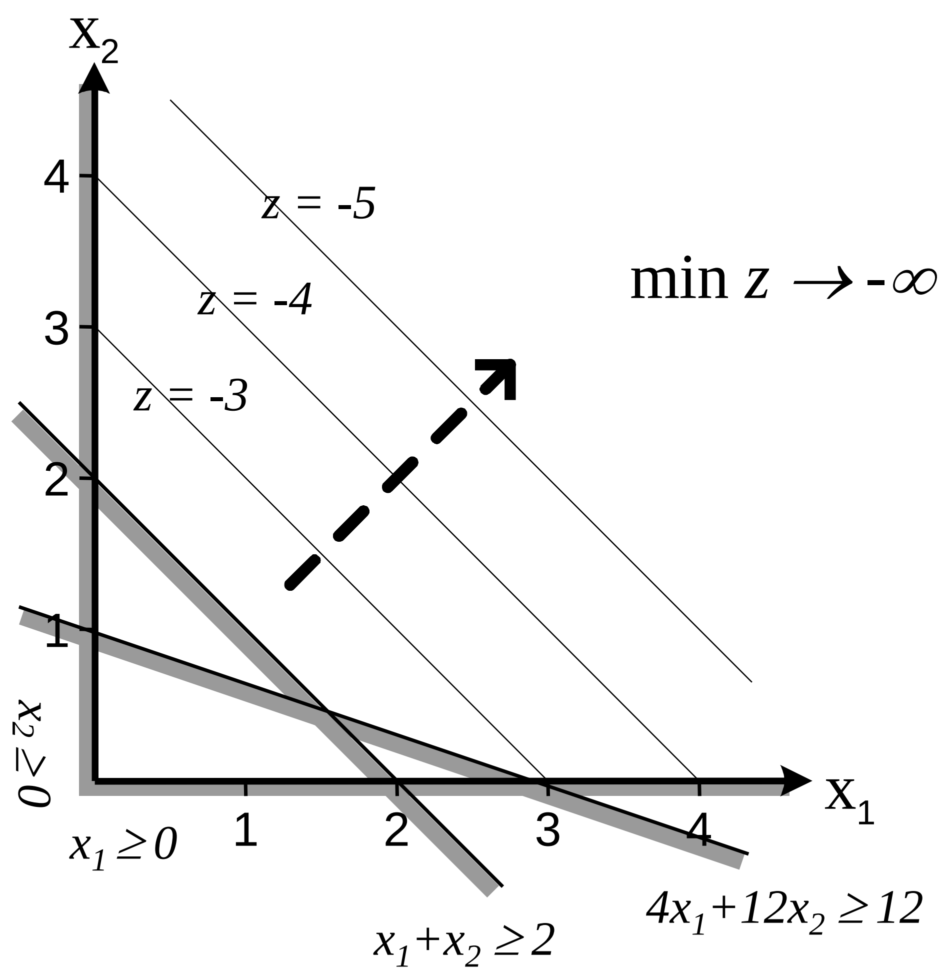
Unbounded solution



LP – No finite solution

$$
\begin{aligned}
\min \quad & z = -x_1 - x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 4x_1 + 12x_2 \geq 12 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

LP – No finite solution



BASIC SOLUTIONS

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

$$A = [B, N],$$

$$Ax = b, \quad \Rightarrow \quad [B, N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b, \quad Bx_B + Nx_N = b,$$

Basic variables

Basis columns

$$\text{where } B \text{ is nonsingular.}$$

$$x_N = 0, \quad x_B = B^{-1}b.$$

The number of basic variables is equal to the number of constrains.

If some of the basic variables of a basic solution are zero, then the basic solution is said to be a *degenerate basic solution*.

$$A = [a_1, a_2, a_3, a_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$\left.\begin{array}{ll} x = [6, 2, 0, 0]^T & B = [a_1, a_2] \\ x = [0, 0, 0, 2]^T & B = [a_3, a_4] \end{array}\right\}$$ Basic feasible solutions

$x = [0, 2, -6, 0]^T \qquad B = [a_2, a_3]$ Basic but infeasible solutions

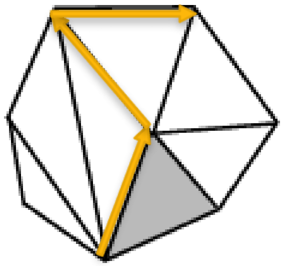
$x = [3, 1, 0, 1]^T$ Feasible but not basic solutions

Fundamental Theorem of LP. *Consider a linear program in standard form.*

1. *If there exists a feasible solution, then there exists a basic feasible solution;*
2. *If there exists an optimal feasible solution, then there exists an optimal basic feasible solution.*

So we need to just check the basic feasible solutions

Simplex algorithms looks for an optimal basic feasible solution.



Primal and Dual

Primal or original

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} \quad & \boldsymbol{\lambda}^T\mathbf{A} \leq \mathbf{c}^T \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

Symmetric Form of Duality

Note that the dual of the dual problem is the primal problem.

$$\boxed{\begin{aligned} \text{minimize} \quad & \boldsymbol{\lambda}^T(-\mathbf{b}) \\ \text{subject to} \quad & \boldsymbol{\lambda}^T(-\mathbf{A}) \geq -\mathbf{c}^T \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}} \Rightarrow \boxed{\begin{aligned} \text{maximize} \quad & (-\mathbf{c}^T)\mathbf{x} \\ \text{subject to} \quad & (-\mathbf{A})\mathbf{x} \leq -\mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}}$$

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

$$\boxed{\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\geq \boldsymbol{b} \\ -\boldsymbol{A}\boldsymbol{x} &\geq -\boldsymbol{b}. \end{aligned}}$$

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} \quad & \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \boldsymbol{x} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & [\boldsymbol{u}^T \ \boldsymbol{v}^T] \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \\ \text{subject to} \quad & [\boldsymbol{u}^T \ \boldsymbol{v}^T] \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \leq \boldsymbol{c}^T \\ & \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}. \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & (\boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{b} \\ \text{subject to} \quad & (\boldsymbol{u} - \boldsymbol{v})^T \boldsymbol{A} \leq \boldsymbol{c}^T \\ & \boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}. \end{aligned}$$

$\boldsymbol{\lambda} = \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0}$

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{\lambda}^T \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{\lambda}^T \boldsymbol{A} \leq \boldsymbol{c}^T. \end{aligned}$$

Asymmetric form of duality

Primal		Dual	
minimize	$\boldsymbol{c}^T\boldsymbol{x}$	maximize	$\boldsymbol{\lambda}^T\boldsymbol{b}$
subject to	$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$	subject to	$\boldsymbol{\lambda}^T\boldsymbol{A} \leq \boldsymbol{c}^T$
	$\boldsymbol{x} \geq \boldsymbol{0}$		

Please look at the following relations

Primal		Dual	
minimize	$\boldsymbol{c}^T\boldsymbol{x}$	maximize	$\boldsymbol{\lambda}^T\boldsymbol{b}$
subject to	$\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$	subject to	$\boldsymbol{\lambda}^T\boldsymbol{A} \leq \boldsymbol{c}^T$
	$\boldsymbol{x} \geq \boldsymbol{0}$		$\boldsymbol{\lambda} \geq \boldsymbol{0}$

Primal problem

Dual problem

$$objective\ function \qquad Min \quad \Leftrightarrow \quad Max$$

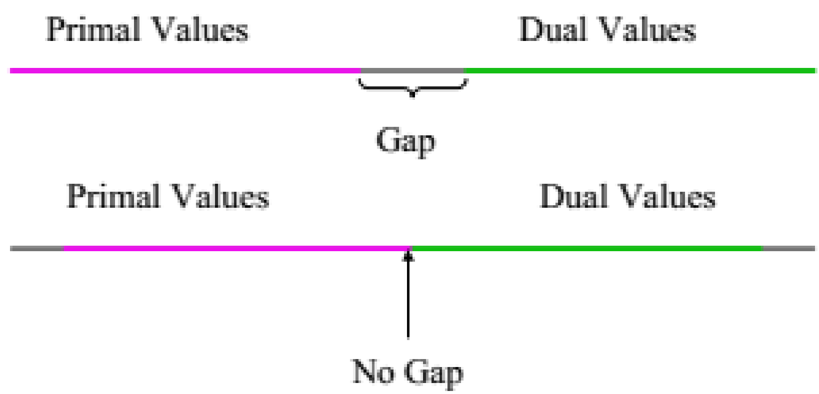
$$\text{for variables} \begin{Bmatrix} \geq 0 \\ \leq 0 \\ \text{free} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \leq \\ \geq \\ = \end{Bmatrix} \text{for constraints}$$

$$\text{for constraints} \begin{Bmatrix} \leq \\ \geq \\ = \end{Bmatrix} \quad \Leftrightarrow \quad \text{for variables} \begin{Bmatrix} \leq 0 \\ \geq 0 \\ \text{free} \end{Bmatrix}$$

Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?



Weak Duality Lemma. *Suppose that* $\boldsymbol{x}$ *and* $\boldsymbol{\lambda}$ *are feasible solutions to primal and dual LP problems, respectively (either in the symmetric or asymmetric form). Then,* $\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{\lambda}^T\boldsymbol{b}$. □

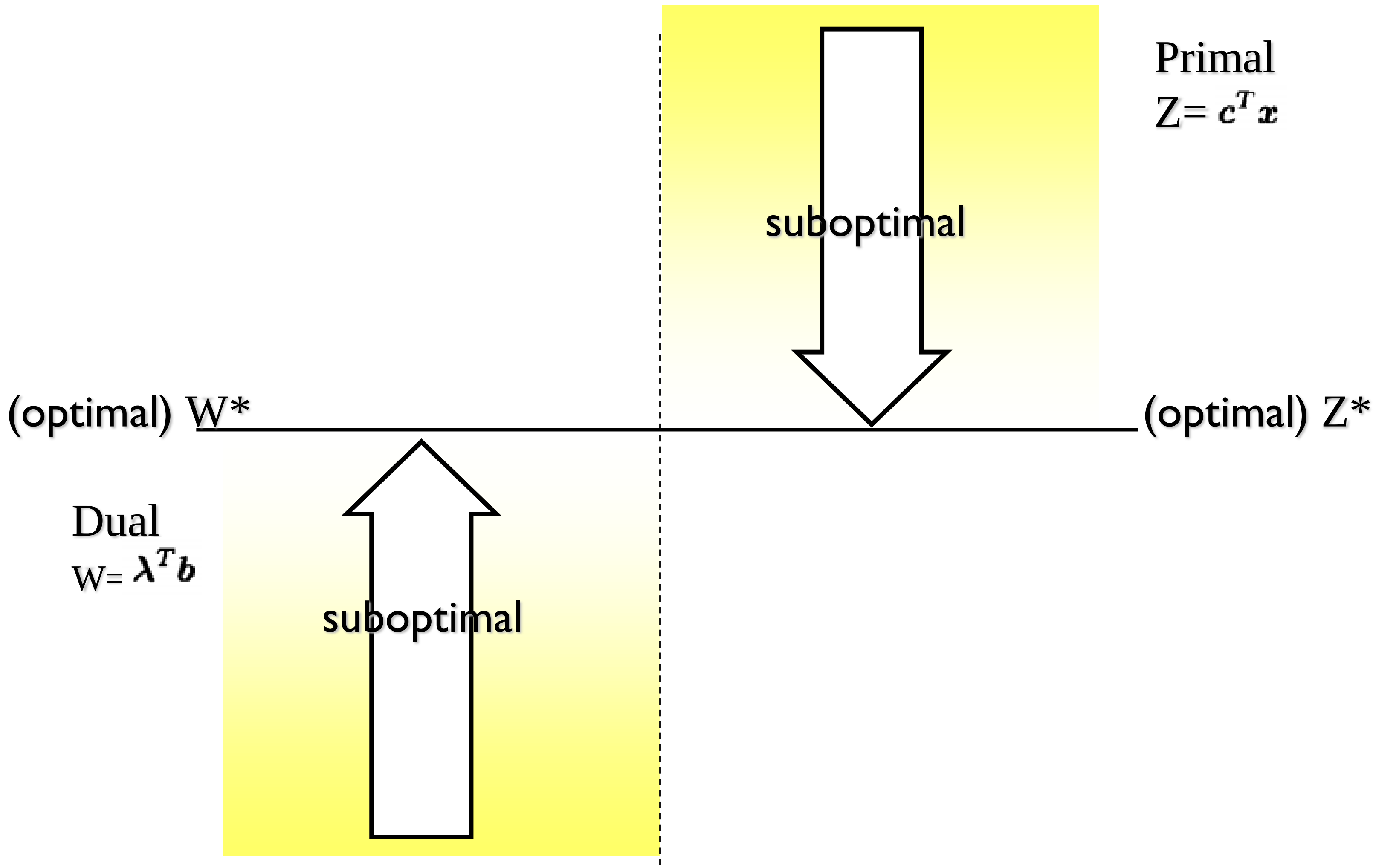
Primal optimal solution is –infinity $\Leftrightarrow$ Dual does not have feasible region

Dual optimal solution is +infinity $\Leftrightarrow$ Primal does not have feasible region

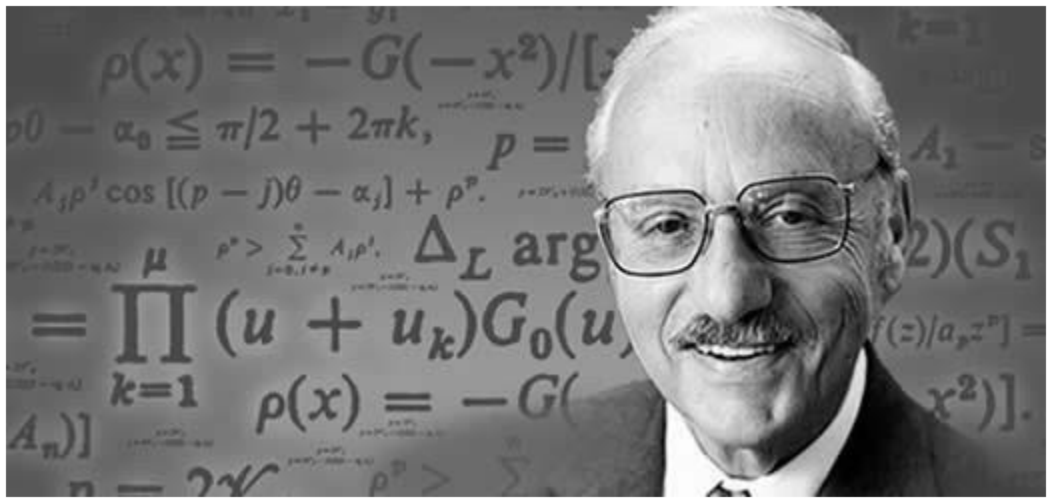
Theorem *Suppose that* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are feasible solutions to the primal and dual, respectively (either in symmetric or asymmetric form). If* $\boldsymbol{c}^T\boldsymbol{x}_0 = \boldsymbol{\lambda}_0^T\boldsymbol{b}$, *then* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are optimal solutions to their respective problems.* □

Duality Theorem. *If the primal problem (either in symmetric or asymmetric form) has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.* □

Primal and Dual



Simplex Algorithm



George Dantzig

Approach (Simplex Method):

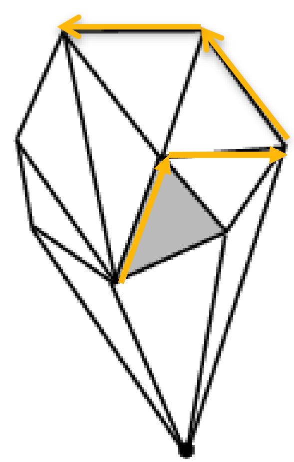
- Start with an initial basic feasible solution in standard form.
- Improve the solution by finding another basic feasible solution if possible.
- When a particular basic feasible solution is found, and cannot be improved by finding new basic feasible solutions, the optimality is reached.

Guaranteed to converge to the global optimal solution

Approach (Simplex Method):

Definition: An adjacent basic solution differs from a basic solution in exactly one basic variable.

Question: If one wants to find an adjacent feasible basic solution from one feasible basic solution (i.e., switch to another simplex), which adjacent basic solution gives lowest objective function?



Simplex Method

- Classical Simplex Method
- Big-M method
- Two phase method
- Modified Simplex method
- Primal Dual method

$$\max\ z = 70x_1 + 50x_2$$
$$4x_1 + 3x_2 \le 240$$
$$2x_1 + 1x_2 \le 100$$
$$x_1, x_2 \ge 0$$

$$\max\ z = 70x_1 + 50x_2$$
$$4x_1 + 3x_2 + s_1 = 240$$
$$2x_1 + x_2 + s_2 = 100$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

$$z - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$
$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + x_2 + 0s_1 + s_2 = 100$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Basic Variables	x_1	x_2	s_1	s_2	RHS
s_1	4	3	1	0	240
s_2	2	1	0	1	100
z	-70	-50	0	0	0

The tableau represents the initial solution

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 240, \quad s_2 = 100, \quad z = 0$$

Basic Variables	x_1	x_2	s_1	s_2	RHS
s_1	4	3	1	0	240
s_2	2	1	0	1	100
z	-70	-50	0	0	0

Pivot column

Import

Basic Variables	x_1	x_2	s_1	s_2	RHS	
s_1	4	3	1	0	240	$240/4=60$
s_2	2	1	0	1	100	$100/2=50$
z	-70	-50	0	0	0	

Take out

Pivot column

Pivot row

Pivot number

Basic Variables	x_1	x_2	s_1	s_2	RHS	
s_1	4	3	1	0	240	
x_1	1	1/2	0	1/2	50	$\frac{R_2}{2}$
z	-70	-50	0	0	0	

Basic Variables	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	**RHS**	
s_1	0	1	1	-2	40	$-4R_2 + R_1$
x_1	1	1/2	0	1/2	50	
z	0	-15	0	35	3500	$70R_2 + R_3$

Import

Basic Variables	x_1	x_2	s_1	s_2	RHS	
s_1	0	1	1	-2	40	$40/1 = 40$
x_1	1	1/2	0	1/2	50	$50/0.5 = 100$
z	0	-15	0	35	3500	

Take out

New pivot row

New pivot column

Basic Variables	x_1	x_2	s_1	s_2	RHS	
x_2	0	1	1	-2	40	
x_1	1	0	-1/2	3/2	30	$-\frac{1}{2}R_1 + R_2$
z	0	0	15	5	4100	$15R_1 + R_3$

In Simplex method, we need to start from basic feasible solutions but always we do not have such a case in our problems

Two Phase

BIG-M

Complexity of simplex algorithm

Klee and Minty in 1972

Consider the following LP problem:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

$$\boldsymbol{c}^T = [10^{n-1}, 10^{n-2}, \ldots, 10^1, 1], \boldsymbol{b} = [1, 10^2, 10^4, \ldots, 10^{2(n-1)}]^T$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2\times 10^1 & 1 & 0 & \cdots & 0 \\ 2\times 10^2 & 2\times 10^1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 2\times 10^{n-1} & 2\times 10^{n-2} & \cdots & 2\times 10^1 & 1 \end{bmatrix}$$

The simplex algorithm applied to the above LP problem requires $2^n - 1$ steps to find the solution.

$$O(2^n - 1)$$

Leonid Genrikhovich Khachiyan

Ellipsoid algorithm

1979

$$O(n^4 L)$$

L **represents the number of bits used in the computations.**

and

n is the variable dimension.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0. \end{array} \qquad \begin{array}{ll} \text{maximize} & \lambda^T b \\ \text{subject to} & \lambda^T A \leq c^T \\ & \lambda \geq 0. \end{array}$$

$$\begin{aligned} c^T x &= b^T \lambda \\ Ax &\geq b \\ A^T \lambda &\leq c \\ x &\geq 0 \\ \lambda &\geq 0. \end{aligned}$$

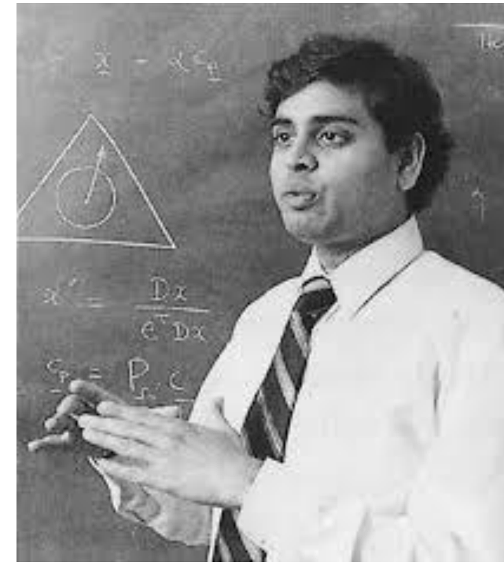
$$\begin{aligned} c^T x - b^T \lambda &\leq 0 \\ -c^T x + b^T \lambda &\leq 0. \end{aligned}$$

$$\begin{bmatrix} c^T & -b^T \\ -c^T & b^T \\ -A & 0 \\ -I_n & 0 \\ 0 & A^T \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -b \\ 0 \\ c \\ 0 \end{bmatrix}$$

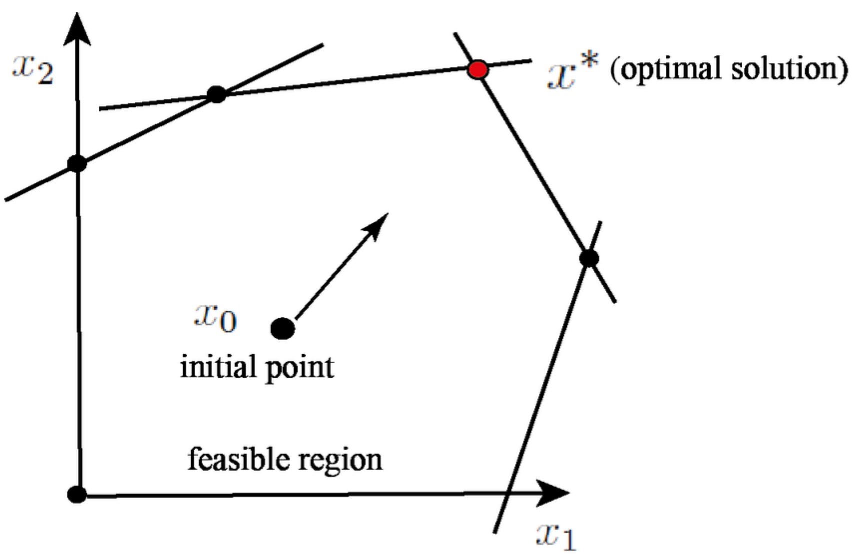
Interior point Algorithms for solving LPs

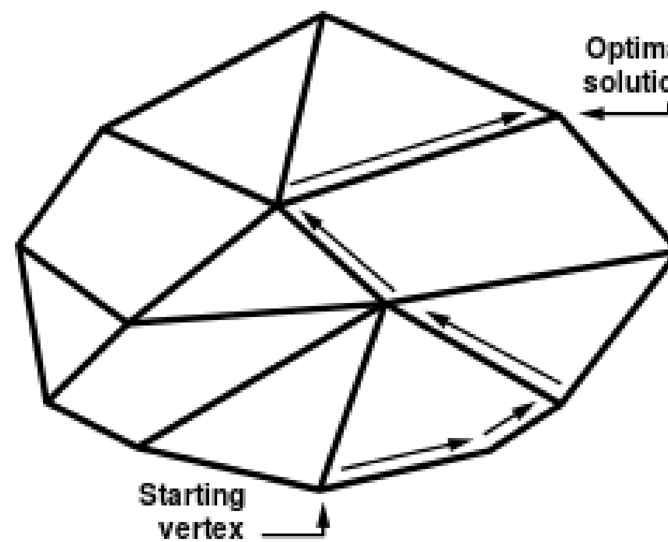
1984

$$O(n^{3.5}L)$$



Narendra Karmarkar





Norm minimization

Moore-Penrose Pseudoinverse

$$\operatorname*{Min}_{x\in \mathrm{R}^N} \|Ax-b\|_2$$

$$x^* = A^+ b + \tilde{x},\;\; \tilde{x} \in N(A)$$

$$\operatorname*{Min}_{Ax=b} \|x\|_2 \quad \Longrightarrow \quad x = A^+ b.$$

$$\|x^*\|_2^2 = \|A^+ b\|_2^2 + \|\bar{x}\|_2^2 + \langle A^+ b, \bar{x} \rangle = \|A^+ b\|_2^2 + \|\bar{x}\|_2^2,$$

Regularized Approximation

Classical least squares can have poor performance due to ill conditioning problem

$$\text{Min} \quad \|Ax-b\|_2^2+\lambda\|x\|_2^2, \quad \lambda>0 \text{ is tuning parameter}$$

$$\text{Min} \quad \left\| \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2$$

$$\operatorname*{Min}_{x\in\mathrm{R}^N} \|Ax-b\|_1$$

$$\|Ax-b\|_1 = \sum_{i=1}^{m} |a_i x - b_i|$$

$$|a_i x - b_i| \le t_i, \quad t_i \ge 0$$

$$i = 1, \ldots, m$$

$$\|Ax-b\|_1 = \sum_{i=1}^{m} |a_i x - b_i| \le \sum_{i=1}^{m} t_i$$

$$\text{Min } \sum_{i=1}^{m} t_i$$
$$|a_i x - b_i| \le t_i$$
$$t_1, t_2, \ldots, t_m \ge 0$$

$$\text{Min } \sum_{i=1}^{m} t_i$$
$$Ax - b \le t$$
$$-Ax + b \le t$$
$$t = [t_1, t_2, \ldots, t_m],$$
$$t_1, t_2, \ldots, t_m \ge 0.$$

$$\text{Min } \sum_{i=1}^{m} c_i t_i$$
$$Ax - t \le b$$
$$-Ax - t \le -b$$
$$t_1, t_2, \ldots, t_m \ge 0$$
$$c = [c_1, c_2, \ldots, c_m] = [1, 1, \ldots, 1]$$

$$\underset{x \in \mathrm{R}^N}{\mathrm{Min}} \; \|Ax - b\|_\infty$$

$$\|Ax - b\|_\infty = \underset{1 \le i \le m}{\mathrm{Max}} |a_i x - b_i|$$

$$|a_i x - b_i| \le t$$

$$1 \le i \le m$$

$$\|Ax - b\|_\infty \le t$$

$$\operatorname*{Min}_{x \in \mathbb{R}^N} \|Ax - b\|_\infty$$

$$\begin{aligned} \text{Min} \quad & t \\ & |a_i x - b_i| \le t \\ & t \ge 0 \end{aligned}$$

$$\Rightarrow$$

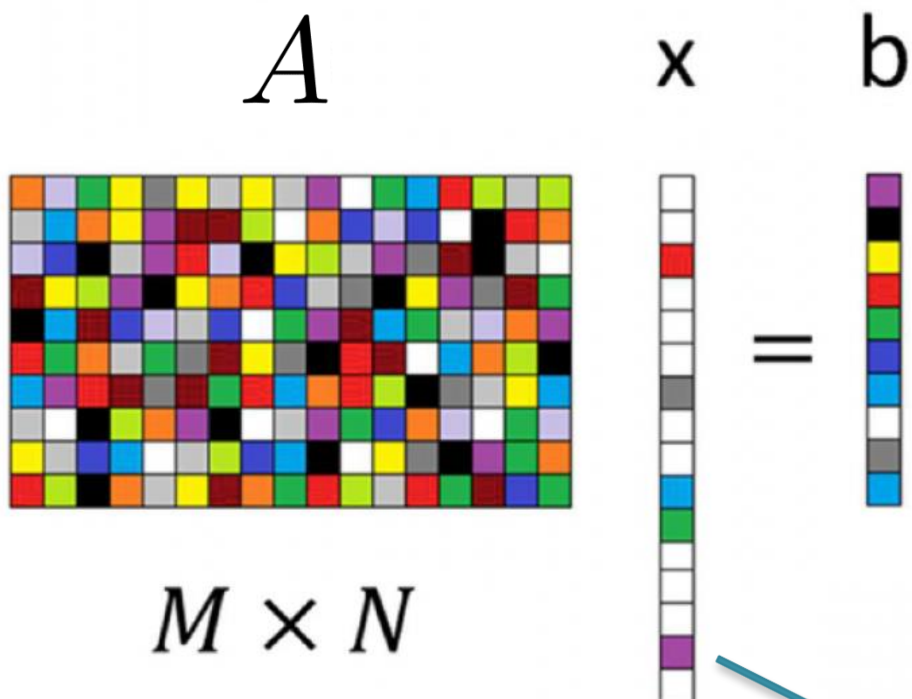
$$\begin{aligned} \text{Min} \quad & t \\ & Ax - b \le t.1 \\ & -Ax + b \le t.1 \\ & t \ge 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \text{Min} \quad & t \\ & Ax - t.1 \le b \\ & -Ax - t.1 \le -b \\ & t \ge 0 \end{aligned}$$

How about norm-l_1 ?

Compress sensing



Sparse solution

$$\boxed{\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \quad \text{subj. to } \boldsymbol{b} = A\boldsymbol{x}.}$$

$$\begin{aligned} \min \quad & 1^T u + 1^T v \\ & A(u - v) = b \\ & u \geq 0 \\ & v \geq 0 \end{aligned}$$

$$x = u - v, \quad u = \max(0, x)$$

$$|x| = u + v, \quad v = -\min(0, x)$$

How about the following problem?

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \quad \text{subj. to } \|\boldsymbol{b} - A\boldsymbol{x}\|_2 \leq \epsilon, \qquad \min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \quad \text{subj. to } \|\boldsymbol{b} - A\boldsymbol{x}\|_1 \leq \epsilon.$$

Thanks for your attention