A data matrix Y is made up of two submatrices, Y_1 and Y_2 of size $I \times J_n$. The task is to solve a distributed low-rank approximation by two agents, such that each agent has only one part of the data Y, and does not have access to the other parts of the data.

Derive the algorithms to update $\{U_n, V_n\}$ by the n-th agent for the following problem

$$\begin{aligned} \min \quad & \frac{1}{2}||Y_1 - U_1V_1^T||_F^2 + \frac{1}{2}||Y_2 - U_2V_2^T||_F^2 \\ s.t \quad & U_1 = U_2 \end{aligned}$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}||Y_1 - U_1V_1^T||_F^2 + \frac{1}{2}||Y_2 - U_2V_2^T||_F^2 + \frac{\alpha}{2}||U_1 - U_2 - T||_F^2 \\ \frac{\partial \mathcal{L}}{\partial V_1} &= U_1^T(Y_1 - U_1V_1^T) = 0 \quad => \quad V_1^* = ((U_1^TU_1)^{-1}U_1^TY_1)^T \\ ||Y_1 - U_1V_1^{*T}||_F^2 &= ||Y_1 - U_1(U_1^TU_1)^{-1}U_1^TY_1|| = [\text{SVD decomposition } U_1 = A\Sigma B^T] = \\ &= ||Y_1 - AA^TY_1||_F^2 = ||Y_1||_F^2 + ||AA^TY_1||_F^2 - 2tr(Y_1^TAA^TY_1) = \\ &= ||Y_1||_F^2 + ||A^TY_1||_F^2 - 2||A^TY_1||_F^2 = ||Y_1||_F^2 - ||A^TY_1||_F^2 \\ ||U_1 - U_2 - T||_F^2 &= ||U_1 - D||_F^2 = ||A\Sigma B^T - D||_F^2 \\ \mathcal{L} &= \frac{1}{2}||Y_1||_F^2 - \frac{1}{2}||A^TY_1||_F^2 + \frac{1}{2}||Y_2 - U_2V_2^T||_F^2 + \frac{\alpha}{2}||A\Sigma B^T - D||_F^2 \\ \frac{\partial \mathcal{L}}{\partial \Sigma} &= 0 => \Sigma^* = A^TDB \\ \mathcal{L} &= \frac{1}{2}||Y_1||_F^2 - \frac{1}{2}||A^TY_1||_F^2 + \frac{1}{2}||Y_2 - U_2V_2^T||_F^2 + \frac{\alpha}{2}||AA^TD - D||_F^2 \\ \min_A \mathcal{L} &= \min_A -\frac{1}{2}||A^TY_1||_F^2 + \frac{\alpha}{2}||AA^TD - D||_F^2 = \\ &= \min_A -\frac{1}{2}||A^TY_1||_F^2 + \frac{\alpha}{2}(||A^TD||_F^2 + ||D||_F^2 - 2||A^TD||_F^2) \\ &= \min_A -\frac{1}{2}||A^TY_1||_F^2 - \frac{\alpha}{2}||A^TD||_F^2 \\ &= \max_A tr(Y_1^TAA^TY_1 - \alpha D^TAA^TD) \\ &= \max_A tr(A^T(Y_1Y_1^T - \alpha DD^T)A) \end{aligned}$$

A = R -leading eigenvectors of matrix $Y_1Y_1^T - \alpha DD^T$
For V_2^* formula is similar to V_1^* because of simmetry.

$$||U_1 - U_2 - T||_F^2 = ||U_2 - U_1 + T||_F^2$$

$D_2 = U_1 - T$ and A_2 for U_2 should be similare to A for U_1 because of simmetry.

Final updates:

1. $$\begin{aligned} D &= U_2 + T \\ A &= R - \text{ leading eigenvectors of matrix } Y_1Y_1^T - \alpha DD^T \\ \Sigma^*B^T &= A^TD => U_1^* = AA^TD \\ V_1^* &= ((U_1^TU_1)^{-1}U_1^TY_1)^T \end{aligned}$$

2. $$\begin{aligned} D &= U_1 - T \\ A &= R - \text{ leading eigenvectors of matrix } Y_2Y_2^T - \alpha DD^T \\ U_2^* &= AA^TD \\ V_2^* &= ((U_2^TU_2)^{-1}U_2^TY_2)^T \end{aligned}$$

3. $$T = T_{prev} + U_2^* - U_1^*$$

In [ ]:
```python
import numpy as np
from numpy.linalg import svd, eig, inv
import matplotlib.pyplot as plt
from PIL import Image
import cv2

img = cv2.imread("airplane.jpg")
shape = (600, 600)
img = cv2.resize(img, (600, 600))
Y = cv2.cvtColor(img, cv2.COLOR_BGR2GRAY).astype(float)
```

In [ ]:
```python
J = Y.shape[1] // 2
Y1 = Y[:, :J]
Y2 = Y[:, J:]
```

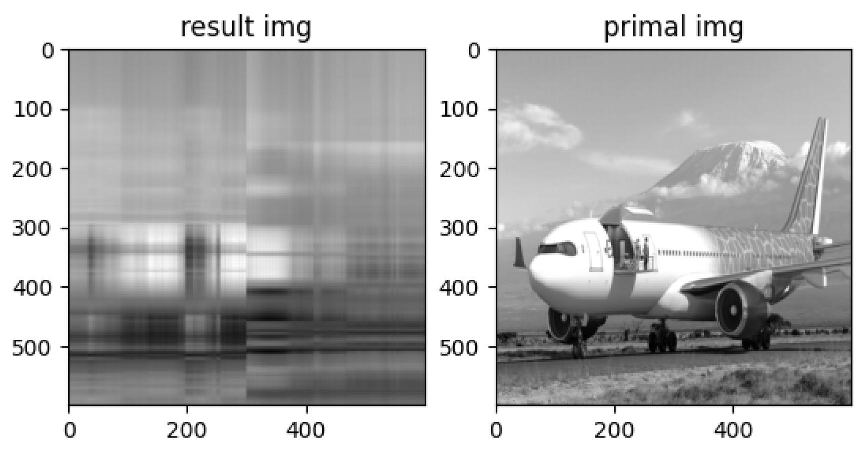
In [ ]:
```python
def get_U_V(Y, T, alpha, U1=None, U2=None):
    if U1 is None:
        D = U2 + T
    elif U2 is None:
        D = U1 - T
    R = D.shape[1]
    mat = Y @ Y.T - alpha * D @ D.T
    _, eigenvectors = eig(mat)
    A = eigenvectors[:, :R]
    U = A @ A.T @ D
    V = (inv(U.T @ U) @ U.T @ Y).T
    return U, V


def update_T(T_prev, U2, U1):
    return T_prev + U2 - U1


def get_score(Y, U, V):
    return np.linalg.norm(Y - U @ V.T, "fro") ** 2


def plot(rank, result_img, primal_img):
    fig, ax = plt.subplots(ncols=2)
    fig.suptitle(f" rank = {rank}")
    ax[0].set_title("result img")
    ax[1].set_title("primal img")
    ax[0].imshow(np.real(result_img), cmap="gray")
    ax[1].imshow(np.real(primal_img), cmap="gray")
    fig.show()
```

Give an example of two agents fitting an image Y.
Report the approximation error $err1$ and the difference $||U_1 - U_2||_F^2$ between $U1$ and $U2$.
Check the approximation error after exchanging U_1 and U_2 - $err2$.

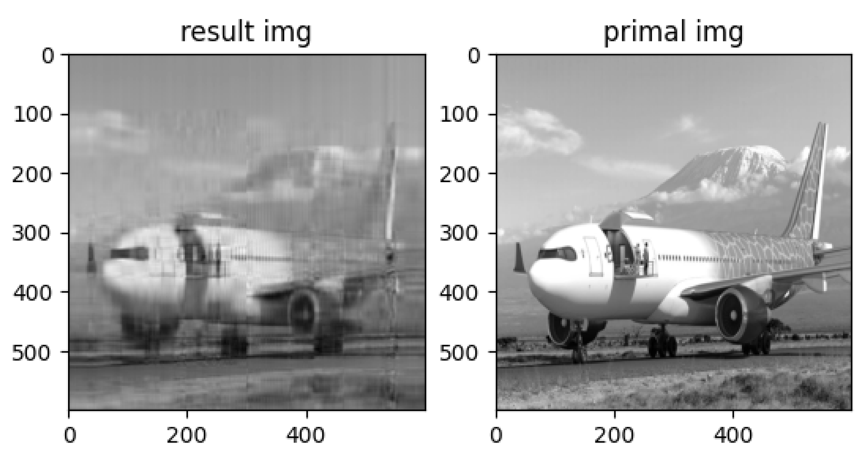
In [ ]:
```python
for R in [2, 10, 24]:
    V1 = np.random.random((J, R))
    U1 = np.random.random((Y.shape[0], R))

    V2 = np.random.random((J, R))
    U2 = np.random.random((Y.shape[0], R))

    T = np.random.random(U1.shape)
    alpha = 10
    for _ in range(3):
        U1, V1 = get_U_V(U2=U2, Y=Y1, T=T, alpha=alpha)
        U2, V2 = get_U_V(U1=U1, Y=Y2, T=T, alpha=alpha)
        T = update_T(T, U2=U2, U1=U1)
        # print(get_score(Y1, U1, V1))

    result_img = np.concatenate([U1 @ V1.T, U2 @ V2.T], axis=1)
    error1 = get_score(Y1, U1, V1) + get_score(Y2, U2, V2)
    error2 = get_score(Y1, U2, V1) + get_score(Y2, U1, V2)
    U_dist = np.linalg.norm(U1 - U2, "fro") ** 2
    print(
        f"rank = {R}, error1 = {error1}, error2 = {error2}, U1-U2 distance = {U_dist}"
    )
    plot(R, result_img, Y)
```

```
rank = 2, error1 = 229622028.5063237, error2 = 508835645.1671115, U1-U2 distance = 4.864087271041002
rank = 10, error1 = 69800916.99687403, error2 = 3268974718.5229964, U1-U2 distance = 13.087975037182579
rank = 24, error1 = 29840213.682771675, error2 = 5032585725.268271, U1-U2 distance = 21.095415568257792
```

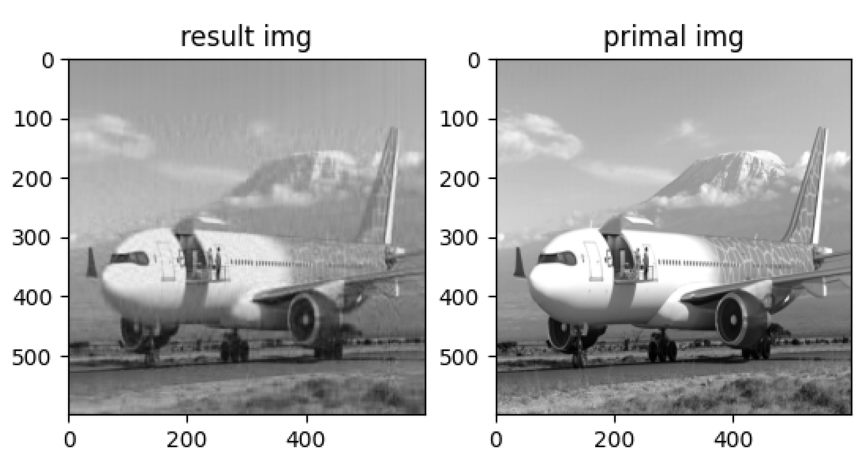
rank = 2



rank = 10



rank = 24



Are the two approximation errors are significantly different?
If so, explain why the algorithm fails and reformulate the problem if possible.

error1 and error2 are significantly different because U_1 and U_2 are different matrices with small scale factor. As far as we approximate Y_1 with $U_1V_1^T$ s.t. $U_1 = U_2$ the scale factor of $U_1V_1^T$ is contained in V_1 and U_1 remains relatively small in terms of norm. So then we compare U_1 and U_2 the distance error is small because of the small scale factor of U_1 and by the same logic U_2.

So if we force U to be orthogonal we could resolve that issue.

$$\begin{aligned} \min \quad & \frac{1}{2}||Y_1 - U_1V_1^T||_F^2 + \frac{1}{2}||Y_2 - U_2V_2^T||_F^2 \\ s.t \quad & U_1^TU_1 = I \\ & U_2^TU_2 = I \\ & U_1^TU_2 = I \end{aligned}$$