Midterm written test (20 points)

This assignment is worth 20 points. Students do not need to complete all the tasks.

1. (3 points) Show that the following set is convex $D = \{x : (\boldsymbol{x}^T \mathbf{Q} \boldsymbol{x} + 1)^2 + \ln \sum_i \exp(x_i)\}$, where $\mathbf{Q} \succ 0$.

2. (3 points) Find the optimal solution of the problem

$$\begin{aligned} \max_{\boldsymbol{x}=[x_1,x_2]} \quad & (x_1 - 2)^2 + 2(x_2 - 3)^2 && (1) \\ \text{s.t.} \quad & -2x_1 + x_2 \leq 0 && (2) \\ & x_1 + x_2 \leq 4 && (3) \\ & x_2 \geq 0 && (4) \end{aligned}$$

3. (3 points) Derive the solution for the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \|\boldsymbol{b} - \boldsymbol{x}\|_2^2 && (5) \\ \text{s.t} \quad & \boldsymbol{x} \geq 0 && (6) \end{aligned}$$

4. (3 points) Derive the optimality conditions and solution for the following optimization problem

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^n} \quad & \|\boldsymbol{b} - \boldsymbol{x}\|_2^2 && (7) \\ \text{s.t} \quad & \boldsymbol{x} \geq 0 && (8) \\ & \mathbf{1}^T \boldsymbol{x} \leq 1 && (9) \end{aligned}$$

   where $\boldsymbol{b} \in \mathbb{R}_+^n$.

5. (3 points) Derive the optimal solution for the following optimization problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad \|\boldsymbol{b} - \boldsymbol{x}\|_2^2 + \gamma\|\boldsymbol{x}\|_1 \qquad (10)$$

6. (10 points)

- (7 points)
  Derive optimal solution and implement algorithm for the following optimization problem

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times m}} \quad \|\mathbf{Y}-\mathbf{X}\|_F^2 \tag{11}$$

$$\text{s.t} \quad \|\mathbf{X}\|_* \leq \delta \tag{12}$$

  where $\mathbf{Y}$ is a matrix of size $n \times m$ and $\delta > 0$.

- (3 points) Derive optimal solution and implement algorithm for the following optimization problem

$$\min_{\mathbf{X}_1,\mathbf{X}_2} \quad \|\mathbf{Y}_1-\mathbf{X}_1\|_F^2 + \|\mathbf{Y}_2-\mathbf{X}_2\|_F^2 \tag{13}$$

$$\text{s.t} \quad \gamma_1\|\mathbf{X}_1\|_* + \gamma_2\|\mathbf{X}_2\|_* \leq \delta \tag{14}$$

  where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are matrices of size $n \times m$ and $\delta > 0$.

7. (10 points) Consider Support Matrix Machine problem which finds a hyperplane $\{\mathbf{X} \mid \mathrm{tr}(\mathbf{W}^T\mathbf{X}) + b = 0\}$ such that $\mathbf{W}$ has minimal rank

$$\begin{aligned} \min_{\mathbf{W},b} \quad & \frac{1}{2}\|\mathbf{W}\|_F^2 + \gamma\|\mathbf{W}\|_* \\ \text{s.t.} \quad & y_i(\mathrm{tr}(\mathbf{W}^T\mathbf{X}_i) + b) \geq 1, \quad i = 1, 2, \ldots, L. \end{aligned}$$

   where $\mathbf{X}_i$ are feature matrices of size $n \times m$ and the labels $y_i \in \{-1, 1\}$ are for the samples $i = 1, \ldots, L$, $\gamma > 0$. $\|\mathbf{W}\|_*$ represents the nuclear norm of $\mathbf{W}$.

   - Develop an algorithm based on the ADMM method to find $\mathbf{W}$ and b,
   - Show examples for training images, e.g., for the two digits 0 and 1 in the MNIST dataset
   - Report accuracy of the classifier and ranks of $\mathbf{W}$ for different values of the regularizer parameter γ