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1.1 矢量代数

▶ 高数笔记

1.1.1 矢量的运算

• 矢量的概念

这里只考虑三维的矢量(行向量).

$$\circ \ m{a} = (a_1, a_2, a_3) = a_1 m{i} + a_2 m{j} + a_3 m{k}.$$

。 模长:
$$|m{a}| = \sqrt{m{a}m{a}^{ ext{T}}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

$$\circ$$
 方向矢量: $e_a=rac{a}{|a|}=rac{a}{a}.$

• 矢量啪艇减 $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$

• 负数:
$$-\mathbf{a} = (-a_1, -a_2, -a_3)$$
.

• 减法:
$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

• 矢量的乘法

• 数乘:
$$k\mathbf{a} = (ka_1, ka_2, ka_3)$$
.

• 点乘:
$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
.

• 叉乘:
$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

○ 混合积:
$$(abc) = (a, b, c) = (a \times b) \cdot c$$
.

1.1.2 矢量的性质

基本性质(并不完整)

• 加法:零元、负元、结合律、交换律.

· 数乘:结合律、第一第二分配律、消去律.

。 点积:交换律、数乘结合律、加法分配律.

点乘

$$\circ \ \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \boldsymbol{b}^{\mathrm{T}} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cos \theta.$$

$$\circ |a| = \sqrt{aa^{\mathrm{T}}} = \sqrt{a \cdot a}.$$

$$\circ \ \mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0.$$

• 叉乘

。 叉乘性质

$$\bullet \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

$$|a \times b| = |a| \cdot |b| \cdot |\sin \theta|.$$

$$\bullet a // b \Leftrightarrow a \times b = 0.$$

。 其它表示

$$lack 行列式: oldsymbol{a} imesoldsymbol{b}=egin{array}{cccc} oldsymbol{i} & oldsymbol{j} & oldsymbol{k}\ a_1 & a_2 & a_3\ b_1 & b_2 & b_3 \ \end{pmatrix}.$$

■ 行向量:
$$oldsymbol{a} imesoldsymbol{b}=oldsymbol{a}\begin{pmatrix}0&-b_3&b_2\\b_3&0&-b_1\\-b_2&b_1&0\end{pmatrix}$$

■ 行列式:
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
.
■ 行向量: $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$.
■ 列向量: $\boldsymbol{a}^{\mathrm{T}} \times \boldsymbol{b}^{\mathrm{T}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \boldsymbol{b}^{\mathrm{T}}$.

混合积

$$egin{aligned} \circ & (oldsymbol{abc}) = (oldsymbol{a} imes oldsymbol{b}) oldsymbol{\cdot} oldsymbol{c} = egin{array}{ccc} a_1 & a_2 & a_3 \ b_1 & b_2 & b_3 \ c_1 & c_2 & c_3 \ \end{array}
ight]. \end{aligned}$$

$$\circ \ \ (\boldsymbol{abc}) = (\boldsymbol{bca}) = (\boldsymbol{cab}) = -(\boldsymbol{acb}) = -(\boldsymbol{cba}) = -(\boldsymbol{bac}).$$

• 向量共面
$$\Leftrightarrow$$
 $(abc) = 0 \Leftrightarrow \exists \alpha, \beta, \gamma : \alpha a + \beta b + \gamma c = 0.$

• 其它性质

。 对于非零矢量 a, 如果 $a \cdot b = a \cdot c$ 且 $a \times b = a \times c$, 则 b = c.

。 平行于 a 的 b 分量为 $b_{//}=rac{a\cdot b}{a\cdot a}a$,垂直分量为 $b_{\perp}=b-b_{//}$.

1.1.3 矢量恒等式

1. 拉格朗日恒等式

1.
$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}).$$

2.
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$
.

2. 雅可比恒等式: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

3. 四向量恒等式

1.
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

2.
$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{c} [\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{d})] - \boldsymbol{d} [\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})].$$

3.
$$\boldsymbol{a} \times [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})] = (\boldsymbol{a} \times \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \times \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

▶ 证明

1.1.4 矢量微积分

设 ${m a}={m a}(t), {m b}={m b}(t), f=f(t)$,此外 λ,μ 为常数, ${m k}$ 为常向量.

1. 求导

1.
$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda \boldsymbol{a} + \mu \boldsymbol{b}) = \lambda \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \mu \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}.$$

2.
$$\frac{\mathrm{d}}{\mathrm{d}t}(f\boldsymbol{a}) = \frac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{a} + f\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}.$$

3.
$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}.$$

4.
$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a} \times \boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$$
.

5.
$$\frac{\mathrm{d}}{\mathrm{d}t}a(f) = \frac{\mathrm{d}a}{\mathrm{d}f} \cdot \frac{\mathrm{d}f}{\mathrm{d}t}$$
.

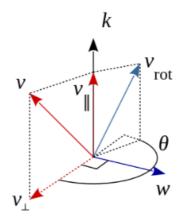
2. 积分

1.
$$\int (\lambda \boldsymbol{a} + \mu \boldsymbol{b}) dt = \lambda \int \boldsymbol{a} dt + \mu \int \boldsymbol{b} dt$$
.

2.
$$\int \mathbf{k} \cdot \mathbf{a} \, dt = \mathbf{k} \cdot \int \mathbf{a} \, dt$$
.

3.
$$\int \mathbf{k} \times \mathbf{a} \, dt = \mathbf{k} \times \int \mathbf{a} \, dt$$
.

1.1.5 矢量的旋转



设转轴的单位方向向量为 $m{k}$,旋转前的向量为 $m{v}$,逆时针旋转 heta 后的向量为 $m{v}'$ (即图中的 $m{v}_{
m rot}$),则

$$egin{aligned} oldsymbol{v}' &= \cos heta oldsymbol{v} + (1 - \cos heta) (oldsymbol{v} \cdot oldsymbol{k}) oldsymbol{k} + \sin heta oldsymbol{k} imes oldsymbol{v} \ &= oldsymbol{v} + (1 - \cos heta) oldsymbol{k} imes (oldsymbol{k} imes oldsymbol{v}) + \sin heta oldsymbol{k} imes oldsymbol{v}. \end{aligned}$$

▶ 证明

备注 有关空间旋转的更多内容可参考笔记平面旋转与空间旋转.

1.2 常用坐标系

1.2.1 正交坐标系

• 基本符号

坐标符号: (u₁, u₂, u₃).

 \circ 单位矢量: $a_{u_1}, a_{u_2}, a_{u_3}$.

有的教材又写为 $e_{u_1}, e_{u_2}, e_{u_3}$.

。 拉梅系数: $h_i=h_i(u_1,u_2,u_3)=rac{\mathrm{d} l_{u_i}}{\mathrm{d} u_i}.$

又称为度量系数或尺度因子.

• 空间度量

・ 长度元:
$$\begin{cases} \mathrm{d}l_{u_1} = h_1 \, \mathrm{d}u_1, \ \mathrm{d}l_{u_2} = h_2 \, \mathrm{d}u_2, \ \mathrm{d}l_{u_3} = h_3 \, \mathrm{d}u_3. \end{cases}$$
・ 面积元: $\begin{cases} \mathrm{d} \boldsymbol{S}_1 = \boldsymbol{a}_{u_1} (h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3), \ \mathrm{d} \boldsymbol{S}_2 = \boldsymbol{a}_{u_2} (h_3 h_1 \, \mathrm{d}u_3 \, \mathrm{d}u_1), \ \mathrm{d} \boldsymbol{S}_3 = \boldsymbol{a}_{u_3} (h_1 h_2 \, \mathrm{d}u_1 \, \mathrm{d}u_2). \end{cases}$
・ 体积元: $\mathrm{d}V = h_1 h_2 h_3 \, \mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}u_3.$

1.2.2 直角坐标系

• 基本符号

● 坐标变量: (x,y,z).

 \circ 单位矢量: $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$.

 \circ 位置矢量: $r = xa_x + ya_y + za_z$.

• 空间度量

$$\circ$$
 长度元: $egin{cases} \mathrm{d}l_x = \mathrm{d}x, \ \mathrm{d}l_y = \mathrm{d}y, \ \mathrm{d}l_z = \mathrm{d}z. \end{cases}$ \circ 面积元: $egin{cases} \mathrm{d}S_x = \mathrm{d}y\mathrm{d}z, \ \mathrm{d}S_y = \mathrm{d}z\mathrm{d}x, \ \mathrm{d}S_z = \mathrm{d}x\mathrm{d}y. \end{cases}$

 \circ 体积元: dV = dx dy dz.

• 坐标变量关系

• 单位矢量关系

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• 单位矢量的偏导、散度与旋度均为零.

1.2.3 圆柱坐标系

• 基本符号

• 坐标符号: (ρ, φ, z) .

。 単位矢量: $\boldsymbol{a}_{\rho}, \boldsymbol{a}_{\varphi}, \boldsymbol{a}_{z}$.

 \circ 位置矢量: $r = e_{\rho}\rho + e_{z}z$.

• 空间度量

・ 长度元:
$$\begin{cases} \mathrm{d}l_{\rho} = \mathrm{d}\rho, \\ \mathrm{d}l_{\varphi} = \rho\,\mathrm{d}\varphi, \\ \mathrm{d}l_{z} = \mathrm{d}z. \end{cases}$$
・ 面积元:
$$\begin{cases} \mathrm{d}S_{\rho} = \rho\,\mathrm{d}\varphi\mathrm{d}z, \\ \mathrm{d}S_{\varphi} = \mathrm{d}r\mathrm{d}z, \\ \mathrm{d}S_{z} = \rho\,\mathrm{d}\rho\mathrm{d}\varphi. \end{cases}$$

• 体积元: $dV = \rho d\rho d\varphi dz$.

• 坐标变量关系

$$\circ$$
 直角 o 圆柱: $egin{cases}
ho = \sqrt{x^2 + y^2}, \ arphi = \arctan rac{y}{x}, \ z = z. \ \end{cases}$ \circ 球极 o 圆柱: $egin{cases}
ho = r \sin heta, \ arphi = arphi, \ z = r \cos heta. \end{cases}$

• 单位矢量关系

$$egin{align*} ullet & ar{m{a}} eta & ar{m{a}} eta & ar{m{a}}_{arphi} \ m{a}_{arphi} \ m{a}_{z} \end{bmatrix} = egin{bmatrix} \cos arphi & \sin arphi & 0 \ -\sin arphi & \cos arphi & 0 \ 0 & 0 & 1 \end{bmatrix} egin{bmatrix} m{a}_{x} \ m{a}_{y} \ m{a}_{z} \end{bmatrix}. \ m{v} & ar{m{w}} m{w} & m{w} & m{w} & m{w} & m{w} \ m{a}_{arphi} \ m{a}_{arphi} \ m{a}_{arphi} \ m{w} & m{w} & m{w} & m{w} \end{bmatrix}. \ m{v} & m{w} \ m{w} & m{w} & m{w} & m{w} \ m{w} & m{w} & m{w} & m{w} \ m{w} & m{w} & m{w} \ m{w} & m{w} \ m{w} & m{w} \ m{w} \ m{w} \ m{w} \end{bmatrix}. \ m{w} & m{w} \ m{w} & m{w} & m{w} \ m{w} & m{w} \ m{w} \$$

• 单位矢量导数 (未标注的均为零)

。 対
$$\,arphi$$
求导: $egin{cases} rac{\mathrm{d}oldsymbol{a}_
ho}{\mathrm{d}arphi} = oldsymbol{a}_arphi, \ rac{\mathrm{d}oldsymbol{a}_arphi}{\mathrm{d}arphi} = -oldsymbol{a}_
ho. \end{cases}$

。 散度:
$$oldsymbol{
abla} \cdot oldsymbol{a}_{
ho} = rac{1}{
ho}.$$

。 旋度:
$$oldsymbol{
abla} imesoldsymbol{a}_{arphi}=rac{1}{
ho}oldsymbol{a}_{z}.$$

1.2.4 球极坐标系

• 基本符号

• 坐标符号: (r, θ, φ) .

 \circ 单位矢量: $\boldsymbol{a}_r, \boldsymbol{a}_{\theta}, \boldsymbol{a}_{\varphi}$.

○ 位置矢量: $r = ra_r$.

• 空间度量

・ 长度元:
$$\begin{cases} \mathrm{d}l_r = \mathrm{d}r, \\ \mathrm{d}l_{\theta} = r\,\mathrm{d}\theta, \\ \mathrm{d}l_{\varphi} = r\sin\theta\,\mathrm{d}\varphi. \end{cases}$$
・ 面积元: $\begin{cases} \mathrm{d}S_r = r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi, \\ \mathrm{d}S_{\theta} = r\sin\theta\,\mathrm{d}r\mathrm{d}\varphi, \\ \mathrm{d}S_{\varphi} = r\,\mathrm{d}r\mathrm{d}\theta. \end{cases}$

体积元: $dV = r^2 \sin \theta \, dr d\theta d\varphi$.

• 坐标变量关系

$$egin{aligned} &\circ \;\; ext{in} \ = \sqrt{x^2 + y^2 + z^2}, \ & heta = rctan rac{\sqrt{x^2 + y^2}}{z}, \ &arphi = rctan rac{y}{x}. \end{aligned}$$
 $egin{aligned} &\circ \;\; ext{in} \ arphi \to ext{if} \ arphi : \ & heta = rctan rac{
ho}{z}, \ & heta = rctan rac{
ho}{z}, \ &arphi = arphi . \end{aligned}$

• 单位矢量关系

$$\circ \ \, \bar{\mathbb{D}} \hat{\mathbb{D}} \to \bar{\mathbb{D}} \hat{\mathbb{D}} : \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}.$$

$$\circ \ \, \bar{\mathbb{D}} \hat{\mathbb{D}} \to \bar{\mathbb{D}} \hat{\mathbb{D}} : \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\varphi \\ \boldsymbol{a}_z \end{bmatrix}.$$

• 单位矢量导数 (未标注的均为零)

• 两个位置矢量 $m{R}_1=(r_1, heta_1,arphi_1)$ 和 $m{R}_2=(r_2, heta_2,arphi_2)$ 夹角的余弦为 $\cos\gamma=\cos heta_1\cos heta_2+\sin heta_1\sin heta_2\cos(arphi_1-arphi_2).$

1.3 哈密顿算子

1.3.1 正交坐标系

• 定义

$$egin{aligned} &oldsymbol{
abla} f = oldsymbol{a}_n rac{\Delta f}{\Delta n} = oldsymbol{a}_n rac{\mathrm{d} f}{\mathrm{d} n}. \ &oldsymbol{
abla} oldsymbol{
abla} \cdot oldsymbol{F} = \lim_{\Delta V o 0} rac{1}{\Delta V} \oint_S oldsymbol{F} \cdot \mathrm{d} oldsymbol{S}. \ &oldsymbol{
abla} oldsymbol{
abla} \times oldsymbol{F} = oldsymbol{a}_{\Delta S} \lim_{\Delta S o 0} rac{1}{\Delta S} \oint_I oldsymbol{F} \cdot \mathrm{d} oldsymbol{l}. \end{aligned}$$

推论

$$\bullet \quad \boldsymbol{\nabla} f = \frac{\boldsymbol{a}_{u_1}}{h_1} \frac{\partial f}{\partial u_1} + \frac{\boldsymbol{a}_{u_2}}{h_2} \frac{\partial f}{\partial u_2} + \frac{\boldsymbol{a}_{u_3}}{h_3} \frac{\partial f}{\partial u_3}.$$

$$\bullet \quad \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} F_1 h_2 h_3 + \frac{\partial}{\partial u_2} h_1 F_2 h_3 + \frac{\partial}{\partial u_3} h_1 h_2 F_3 \right).$$

$$oldsymbol{\circ} oldsymbol{
abla} imes oldsymbol{F} = rac{1}{h_1 h_2 h_3} egin{bmatrix} h_1 oldsymbol{a}_{u_1} & h_2 oldsymbol{a}_{u_2} & h_3 oldsymbol{a}_{u_3} \ rac{\partial}{\partial u_1} & rac{\partial}{\partial u_2} & rac{\partial}{\partial u_3} \ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{bmatrix}.$$

• 例子: 设
$$P=(x,y,z)$$
 到 $P'=(x',y',z')$ 的距离为 R , 则 $\mathbf{\nabla}\frac{1}{R}=-\frac{\mathbf{R}}{R^3}=-\mathbf{\nabla}'\frac{1}{R}$.

1.3.2 直角坐标系

$$ullet \quad oldsymbol{
abla} f = oldsymbol{a}_x rac{\partial f}{\partial x} + oldsymbol{a}_y rac{\partial f}{\partial y} + oldsymbol{a}_z rac{\partial f}{\partial z}.$$

$$\bullet \ \ \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$ullet oldsymbol{
abla} ullet oldsymbol{
abla} imes oldsymbol{F} = egin{bmatrix} oldsymbol{a}_x & oldsymbol{a}_y & oldsymbol{a}_z \ oldsymbol{\partial} x & oldsymbol{\partial} y & oldsymbol{\partial} z \ F_x & F_y & F_z \end{bmatrix} .$$

1.3.3 圆柱坐标系

$$ullet \quad oldsymbol{
abla} f = oldsymbol{a}_
ho rac{\partial f}{\partial
ho} + rac{oldsymbol{a}_arphi}{
ho} rac{\partial f}{\partial arphi} + oldsymbol{a}_z rac{\partial f}{\partial z}.$$

$$ullet \ oldsymbol{
abla} oldsymbol{\cdot} \ oldsymbol{F} = rac{1}{
ho} rac{\partial (
ho F_
ho)}{\partial
ho} + rac{1}{
ho} rac{\partial F_arphi}{\partial arphi} + rac{\partial F_z}{\partial z}.$$

$$ullet oldsymbol{
abla} imes oldsymbol{
abla} imes oldsymbol{F} = rac{1}{
ho} egin{bmatrix} oldsymbol{a}_{
ho} &
ho oldsymbol{a}_{arphi} & rac{\partial}{\partial arphi} & rac{\partial}{\partial z} \ F_{
ho} &
ho F_{arphi} & F_z \ \end{pmatrix}.$$

1.3.4 球极坐标系

$$ullet \quad oldsymbol{
abla} f = oldsymbol{a}_r rac{\partial f}{\partial r} + rac{oldsymbol{a}_ heta}{r} rac{\partial f}{\partial heta} + rac{oldsymbol{a}_arphi}{r \sin heta} rac{\partial f}{\partial arphi}.$$

$$ullet \; oldsymbol{
abla} oldsymbol{\cdot} \; oldsymbol{F} = rac{1}{r^2} rac{\partial (r^2 F_r)}{\partial r} + rac{1}{r \sin heta} rac{\partial \sin heta F_ heta}{\partial heta} + rac{1}{r \sin heta} rac{\partial F_arphi}{\partial arphi}.$$

$$ullet oldsymbol{
abla} imes oldsymbol{F} imes oldsymbol{F} = rac{1}{r^2 \sin heta} egin{bmatrix} oldsymbol{a}_r & r oldsymbol{a}_{ heta} & r \sin heta oldsymbol{a}_{arphi} \ rac{\partial}{\partial r} & rac{\partial}{\partial heta} & rac{\partial}{\partial arphi} \ F_r & r F_{ heta} & r \sin heta F_{arphi} \ \end{pmatrix}.$$

1.3.5 记号与拓展

可以将哈密顿算子写成向量的形式,但此时数乘与点乘均不可交换.以直角坐标系为例:

•
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

左乘

$$egin{array}{ll} \bullet & \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right). \\ \bullet & \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \\ \bullet & \boldsymbol{\nabla} \times \boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right). \end{array}$$

右乘

$$\circ f \nabla = \left(f \frac{\partial}{\partial x}, f \frac{\partial}{\partial y}, f \frac{\partial}{\partial z} \right).$$

$$\circ \mathbf{F} \cdot \nabla = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}.$$

$$\circ \mathbf{F} \times \nabla = \left(F_3 \frac{\partial}{\partial y} - F_2 \frac{\partial}{\partial z}, F_1 \frac{\partial}{\partial z} - F_3 \frac{\partial}{\partial x}, F_2 \frac{\partial}{\partial x} - F_1 \frac{\partial}{\partial y} \right).$$

- 备注
 - $\circ \ \nabla \cdot \boldsymbol{F} \neq \boldsymbol{F} \cdot \nabla.$
 - $\circ \nabla \cdot \nabla f := \nabla \cdot (\nabla f) = (\nabla \cdot \nabla) f.$
 - $\circ \nabla \times \nabla f := \nabla \times (\nabla f) = (\nabla \times \nabla) f \equiv 0.$
- 约定
 - · 梯度、散度、旋度的优先级高于数乘、点乘与叉乘,并且从右向左计算.
 - 例 1: $\nabla \times \nabla \times \mathbf{F} := \nabla \times (\nabla \times \mathbf{F}) \neq (\nabla \times \nabla) \times \mathbf{F} \equiv \mathbf{0}$.
 - 例 2: $\nabla \times \boldsymbol{a} \times \boldsymbol{b} := (\nabla \times \boldsymbol{a}) \times \boldsymbol{b} \neq \nabla \times (\boldsymbol{a} \times \boldsymbol{b})$.
- 向量的梯度

$$oldsymbol{
abla} oldsymbol{F} = rac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = egin{bmatrix} rac{\partial F_1}{\partial x} & rac{\partial F_1}{\partial y} & rac{\partial F_1}{\partial z} \ rac{\partial F_2}{\partial x} & rac{\partial F_2}{\partial y} & rac{\partial F_3}{\partial z} \ rac{\partial F_3}{\partial x} & rac{\partial F_3}{\partial y} & rac{\partial F_3}{\partial z} \end{bmatrix}.$$

1.4 哈密顿算子的性质

以下均假设混合偏导连续.

1.4.1 哈密顿算子与叉乘

1. Hamilton 算子在右边

1.
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{\nabla}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{\nabla}) - (\boldsymbol{a} \cdot \boldsymbol{b}) \boldsymbol{\nabla}.$$

2.
$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{\nabla} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{\nabla}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{\nabla}).$$

2. Hamilton 算子在中间

1.
$$oldsymbol{a} imes(oldsymbol{
abla} imesoldsymbol{b})=oldsymbol{a}oldsymbol{
abla}oldsymbol{b}-(oldsymbol{a}oldsymbol{\cdot}oldsymbol{b}$$

2.
$$(oldsymbol{a} imesoldsymbol{
abla}) imesoldsymbol{b}=oldsymbol{a}(oldsymbol{
abla}oldsymbol{\cdot}oldsymbol{b})-oldsymbol{a}oldsymbol{
bla}oldsymbol{b}.$$

3. Hamilton 算子在左边

1.
$$oldsymbol{
abla} imes(oldsymbol{a} imesoldsymbol{b})=oldsymbol{a}(oldsymbol{
abla}\cdotoldsymbol{b})-(oldsymbol{a}\cdotoldsymbol{
abla})oldsymbol{b}-oldsymbol{b}(oldsymbol{
abla}\cdotoldsymbol{a})+(oldsymbol{b}\cdotoldsymbol{
abla})oldsymbol{a}.$$

2.
$$(\nabla \times \boldsymbol{a}) \times \boldsymbol{b} = (\boldsymbol{b} \cdot \nabla) \boldsymbol{a} - \boldsymbol{b} \nabla \boldsymbol{a}$$
.

4. 上述公式的推论

1.
$$\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{\nabla}) \times \boldsymbol{b} = \boldsymbol{a}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{\nabla}) \boldsymbol{b}.$$

2.
$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \times \nabla + \nabla \times \boldsymbol{a}) \times \boldsymbol{b} - (\boldsymbol{b} \times \nabla + \nabla \times \boldsymbol{b}) \times \boldsymbol{a}$$
.

▶ 证明

1.4.2 哈密顿算子与梯度

1. 标量的梯度——加减乘除与复合

1.
$$\nabla(\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g$$
.

2.
$$\nabla(fg) = f\nabla g + g\nabla f$$
.

3.
$$oldsymbol{
abla} \left(rac{f}{g}
ight) = rac{g oldsymbol{
abla} f - f oldsymbol{
abla} g}{g^2}.$$

4.
$$\nabla(\phi(f)) = \phi'(f)\nabla f$$
.

2. 标量的梯度——向量点乘与散度

1.
$$\nabla (\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \nabla \boldsymbol{b} + \boldsymbol{b} \nabla \boldsymbol{a}$$
.

2.
$$\nabla (a \cdot b) = a \times (\nabla \times b) + b \times (\nabla \times a) + (a \cdot \nabla)b + (b \cdot \nabla)a$$
.

3.
$$\nabla (a \cdot b) = a(\nabla \cdot b) + b(\nabla \cdot a) - (a \times \nabla) \times b - (b \times \nabla) \times a$$
.

4.
$$\nabla(\nabla \cdot a) = \Delta a + \nabla \times (\nabla \times a)$$
.

3. 向量的梯度
$$\nabla(\lambda a + \mu b) = \lambda \nabla a + \mu \nabla b$$
.

▶ 证明

1.4.3 哈密顿算子与散度

1.
$$\nabla \cdot (\lambda \boldsymbol{a} + \mu \boldsymbol{b}) = \lambda (\nabla \cdot \boldsymbol{a}) + \mu (\nabla \cdot \boldsymbol{b}).$$

2.
$$\nabla \cdot (fa) = f \nabla \cdot a + \nabla f \cdot a$$
.

3.
$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b}).$$

4.
$$\nabla \cdot (g\nabla f) = \nabla g \cdot \nabla f + g\Delta f$$
.

$$5. \nabla \cdot (\nabla \times \boldsymbol{a}) = 0.$$
 (旋度场无源)

6.
$$\nabla \cdot (f \nabla \times \boldsymbol{a}) = (\nabla f) \cdot (\nabla \times \boldsymbol{a}).$$

▶ 证明

1.4.4 哈密顿算子与旋度

1.
$$\nabla \times (\lambda \boldsymbol{a} + \mu \boldsymbol{b}) = \lambda (\nabla \times \boldsymbol{a}) + \mu (\nabla \times \boldsymbol{b}).$$

2.
$$\nabla \times (f\boldsymbol{a}) = f\nabla \times \boldsymbol{a} + \nabla f \times \boldsymbol{a}$$
.

3.
$$\nabla imes (m{a} imes m{b}) = m{a} (
abla \cdot m{b}) - m{b} (
abla \cdot m{a}) - (m{a} \cdot
abla) m{b} + (m{b} \cdot
abla) m{a}.$$

$$4. \nabla \times (\nabla f) = \mathbf{0}.$$
 (梯度场无旋)

5.
$$\nabla \times (q \nabla f) = (\nabla q) \times (\nabla f)$$
.

6.
$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla (\nabla \cdot \boldsymbol{a}) - \Delta \boldsymbol{a}$$
.

▶ 证明

1.4.5 哈密顿算子与积分

1.5 拉普拉斯算子

1.5.1 一般定义

• 标量的定义: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$.

• 矢量的定义: $\Delta a = \nabla(\nabla \cdot a) - \nabla \times (\nabla \times a)$.

• 例子: $\Delta \frac{1}{\boldsymbol{r}-\boldsymbol{r}'} = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}').$

1.5.2 二维空间

• 正交坐标: $\Delta f = \frac{1}{h_1 h_2} \sum \frac{\partial}{\partial u_1} \left(\frac{h_2}{h_1} \frac{\partial f}{\partial u_1} \right)$.

• 直角坐标: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$

• 极坐标系: $\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2}$.

1.5.3 三维空间

• 正交坐标: $\Delta f = rac{1}{h_1 h_2 h_3} \sum rac{\partial}{\partial u_1} igg(rac{h_2 h_3}{h_1} rac{\partial f}{\partial u_1}igg).$

• 直角坐标: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$

• 圆柱坐标: $\Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}.$

• 球坐标系: $\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}.$

1.5.4 三维矢量

• 直角坐标: $\Delta oldsymbol{F} = oldsymbol{a}_x \Delta F_x + oldsymbol{a}_y \Delta F_y + oldsymbol{a}_z \Delta F_z.$

• 圆柱坐标: $\Delta \boldsymbol{F} = \left(\Delta F_{\rho} - \frac{2}{\rho^{2}} \frac{\partial F_{\varphi}}{\partial \varphi} - \frac{F_{\rho}}{\rho^{2}}\right) \boldsymbol{a}_{\rho} + \left(\Delta F_{\varphi} + \frac{2}{\rho^{2}} \frac{\partial F_{\rho}}{\partial \varphi} - \frac{F_{\varphi}}{\rho^{2}}\right) \boldsymbol{a}_{\varphi} + \Delta F_{z} \cdot \boldsymbol{a}_{z}.$

• 球坐标系:

$$egin{aligned} \Delta oldsymbol{F} &= \left[\Delta F_r - rac{2}{r^2} igg(F_r + \cot heta F_ heta + \csc heta rac{\partial F_arphi}{\partial arphi} + rac{\partial A_ heta}{\partial heta} igg)
ight] oldsymbol{a}_r + \ & \left[\Delta F_ heta - rac{1}{r^2} igg(\csc^2 heta F_ heta - 2 rac{\partial F_r}{\partial heta} + 2 \cot heta \csc heta rac{\partial F_arphi}{\partial arphi} igg)
ight] oldsymbol{a}_ heta + \ & \left[\Delta F_arphi - rac{1}{r^2} igg(\csc^2 heta F_arphi - 2 \csc heta rac{\partial F_r}{\partial arphi} - 2 \cot heta \csc heta rac{\partial F_ heta}{\partial arphi} igg)
ight]. \end{aligned}$$

证明

对于圆柱坐标系

$$\begin{split} & \boldsymbol{F} = F_{\rho}\boldsymbol{a}_{\rho} + F_{\varphi}\boldsymbol{a}_{\varphi} + F_{z}\boldsymbol{a}_{z}, \\ & \frac{\partial \boldsymbol{F}}{\partial \varphi} = \left(\frac{\partial F_{\rho}}{\partial \varphi} - F_{\varphi}\right)\boldsymbol{a}_{\rho} + \left(\frac{\partial F_{\varphi}}{\partial \varphi} + F_{\rho}\right)\boldsymbol{a}_{\varphi} + \frac{\partial F_{z}}{\partial \varphi}\boldsymbol{a}_{z}, \\ & \frac{\partial^{2}\boldsymbol{F}}{\partial \varphi^{2}} = \left(\frac{\partial^{2}F_{\rho}}{\partial \varphi^{2}} - 2\frac{\partial F_{\varphi}}{\partial \varphi} - F_{\rho}\right)\boldsymbol{a}_{\rho} + \left(\frac{\partial^{2}F_{\varphi}}{\partial \varphi^{2}} + 2\frac{\partial F_{\rho}}{\partial \varphi} - F_{\varphi}\right)\boldsymbol{a}_{\varphi} + \frac{\partial^{2}F_{z}}{\partial \varphi^{2}}\boldsymbol{a}_{z}, \\ & \Delta \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \boldsymbol{F}}{\partial \rho}\right) + \frac{1}{\rho^{2}}\frac{\partial^{2}\boldsymbol{F}}{\partial \varphi^{2}} + \frac{\partial^{2}\boldsymbol{F}}{\partial z^{2}} \\ & = \frac{1}{\rho}\frac{\partial \boldsymbol{F}}{\partial \rho} + \frac{\partial^{2}\boldsymbol{F}}{\partial \rho^{2}} + \frac{1}{\rho^{2}}\frac{\partial^{2}\boldsymbol{F}}{\partial \varphi^{2}} + \frac{\partial^{2}F_{\rho}}{\partial z^{2}} \\ & = \left(\frac{1}{\rho}\frac{\partial F_{\rho}}{\partial \rho} + \frac{\partial^{2}F_{\rho}}{\partial \rho^{2}} + \frac{1}{\rho^{2}}\frac{\partial^{2}F_{\rho}}{\partial \varphi^{2}} + \frac{\partial^{2}F_{\rho}}{\partial z^{2}} - \frac{2}{\rho^{2}}\frac{\partial F_{\varphi}}{\partial \varphi} - \frac{F_{\rho}}{\rho^{2}}\right)\boldsymbol{a}_{\rho} + \\ & \left(\frac{1}{\rho}\frac{\partial F_{\varphi}}{\partial \rho} + \frac{\partial^{2}F_{\varphi}}{\partial \rho^{2}} + \frac{1}{\rho^{2}}\frac{\partial^{2}F_{\varphi}}{\partial \varphi^{2}} + \frac{\partial^{2}F_{\varphi}}{\partial z^{2}} + \frac{2}{\rho^{2}}\frac{\partial F_{\rho}}{\partial \varphi} - \frac{F_{\varphi}}{\rho^{2}}\right)\boldsymbol{a}_{\varphi} + \\ & \left(\frac{1}{\rho}\frac{\partial F_{z}}{\partial \rho} + \frac{\partial^{2}F_{z}}{\partial \rho^{2}} + \frac{1}{\rho^{2}}\frac{\partial^{2}F_{z}}{\partial \varphi^{2}} + \frac{\partial^{2}F_{z}}{\partial z^{2}}\right)\boldsymbol{a}_{z} \\ & = \left(\Delta F_{\rho} - \frac{2}{\rho^{2}}\frac{\partial F_{\varphi}}{\partial \varphi} - \frac{F_{\rho}}{\rho^{2}}\right)\boldsymbol{a}_{\rho} + \left(\Delta F_{\varphi} + \frac{2}{\rho^{2}}\frac{\partial F_{\rho}}{\partial \varphi} - \frac{F_{\varphi}}{\rho^{2}}\right)\boldsymbol{a}_{\varphi} + \Delta F_{z} \cdot \boldsymbol{a}_{z}. \end{split}$$

对于球坐标系,这里采用另一种思路

$$egin{aligned} \Delta oldsymbol{F} &= oldsymbol{
abla} (oldsymbol{
abla} \cdot oldsymbol{F}) - oldsymbol{
abla} imes (oldsymbol{
abla} imes oldsymbol{F}_{arphi}) &= igg[\Delta F_r - rac{2}{r^2} igg(F_r + \cot heta F_{ heta} + \csc heta rac{\partial F_{arphi}}{\partial arphi} + rac{\partial A_{ heta}}{\partial heta} igg) igg] oldsymbol{a}_r + \ igg[\Delta F_{ heta} - rac{1}{r^2} igg(\csc^2 heta F_{arphi} - 2 \csc heta rac{\partial F_r}{\partial arphi} + 2 \cot heta \csc heta rac{\partial F_{arphi}}{\partial arphi} igg) igg] oldsymbol{a}_{ heta} + \ igg[\Delta F_{arphi} - rac{1}{r^2} igg(\csc^2 heta F_{arphi} - 2 \csc heta rac{\partial F_r}{\partial arphi} - 2 \cot heta \csc heta rac{\partial F_{ heta}}{\partial arphi} igg) igg]. \end{aligned}$$

证毕.

1.5.5 算子性质

1. 标量函数

1.
$$\Delta(\lambda f + \mu g) = \lambda \Delta f + \mu \Delta g$$

2.
$$\Delta(fg) = (\Delta f)g + 2(\boldsymbol{\nabla} f) \cdot (\boldsymbol{\nabla} g) + f(\Delta g).$$

3.
$$\Delta(\phi(f)) = \phi''(f)(oldsymbol{
abla} f)^2 + \phi'(f)\Delta f$$

2. 向量函数

1.
$$\Delta(\lambda \boldsymbol{a} + \mu \boldsymbol{b}) = \lambda \Delta \boldsymbol{a} + \mu \Delta \boldsymbol{b}$$
.

2.
$$\Delta a = \nabla (\nabla \cdot a) - \nabla \times (\nabla \times a)$$
.

3.
$$\Delta \boldsymbol{a} = \left[\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \boldsymbol{a}) \right]^{\mathrm{T}} = \left(\boldsymbol{\nabla}^2 \boldsymbol{a} \right)^{\mathrm{T}}$$
.

▶ 证明

1.6 场论定理

1.6.1 场论的三大定理

• Green 公式

。 切向向量
$$\int\limits_{\partial D} m{r} \cdot m{s} \, \mathrm{d}s = \int\limits_{\partial D} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint\limits_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \mathrm{d}y.$$

。 外法向量
$$\int\limits_{\partial D} m{r} \cdot m{n} \, \mathrm{d}s = \int\limits_{\partial D} P \, \mathrm{d}y - Q \, \mathrm{d}x = \iint\limits_{D} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x \mathrm{d}y.$$

• Gauss 公式
$$\iint_{\partial\Omega} oldsymbol{a} \cdot \mathrm{d} oldsymbol{S} = \iiint_{\Omega} oldsymbol{
abla} \cdot oldsymbol{a} \, \mathrm{d} V.$$

• Stokes 公式
$$\int\limits_{\partial \sum} m{a} \cdot \mathrm{d} m{s} = \iint\limits_{\sum} (m{
abla} m{x} \cdot \mathbf{d} m{s}.$$

1.6.2 两个格林恒等式

• 分部积分
$$\iint\limits_{D} \left(P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y}\right) \mathrm{d}x \mathrm{d}y = \int\limits_{\partial D} P u \, \mathrm{d}y - Q u \, \mathrm{d}x - \iint\limits_{D} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) u \, \mathrm{d}x \mathrm{d}y.$$

• Green 第一公式

$$\circ \iiint\limits_{\Omega} \left(oldsymbol{
abla} f \cdot oldsymbol{
abla} g + f \Delta g
ight) \mathrm{d}V = \iint\limits_{\partial\Omega} f rac{\partial g}{\partial oldsymbol{n}} \, \mathrm{d}S.$$

$$\circ \iiint (oldsymbol{
abla} g oldsymbol{\cdot} oldsymbol{
abla} f + g \Delta f) \, \mathrm{d}V = \iint g rac{\partial f}{\partial oldsymbol{n}} \, \mathrm{d}S.$$

• Green 第二公式
$$\iiint\limits_{\Omega} (f\Delta g - g\Delta f) \,\mathrm{d}V = \iint\limits_{\partial\Omega} \left(f \frac{\partial g}{\partial \boldsymbol{n}} - g \frac{\partial f}{\partial \boldsymbol{n}}\right) \mathrm{d}S.$$

1.6.3 Helmholtz 定理

• Helmholtz 定理

$$egin{aligned} oldsymbol{F} &= -oldsymbol{
abla} u + oldsymbol{
abla} imes oldsymbol{A} \ u(oldsymbol{r}) &= rac{1}{4\pi} \iiint\limits_{V'} rac{oldsymbol{
abla}' \cdot oldsymbol{F}(oldsymbol{r}')}{|oldsymbol{r} - oldsymbol{r}'|} \, \mathrm{d}V' - rac{1}{4\pi} \iint\limits_{S'} rac{oldsymbol{e}_n \cdot oldsymbol{F}(oldsymbol{r}')}{|oldsymbol{r} - oldsymbol{r}'|} \, \mathrm{d}S' \ oldsymbol{A}(oldsymbol{r}) &= rac{1}{4\pi} \iiint\limits_{V'} rac{oldsymbol{
abla}' imes oldsymbol{F}(oldsymbol{r}')}{|oldsymbol{r} - oldsymbol{r}'|} \, \mathrm{d}V' - rac{1}{4\pi} \iint\limits_{S'} rac{oldsymbol{e}_n imes oldsymbol{F}(oldsymbol{r}')}{|oldsymbol{r} - oldsymbol{r}'|} \, \mathrm{d}S' \end{aligned}$$

• Helmholtz 定理的解释

- 。 若矢量场 F 单值且导数连续有界,则其由散度、旋度和边界条件唯一确定.
- 梯度场 $\mathbf{F}_t = -\nabla u$ 由散度和在边界上的法向量唯一确定.
- 。 旋度场 $\mathbf{F}_c = \mathbf{\nabla} \times \mathbf{A}$ 由旋度和在边界上的切线分量唯一确定.

$$egin{cases} oldsymbol{
abla} \cdot oldsymbol{F}_t = oldsymbol{
abla} \cdot oldsymbol{F}, \ oldsymbol{
abla} imes oldsymbol{F}_c = 0, \ oldsymbol{
abla} imes oldsymbol{F}_c = oldsymbol{
abla} imes oldsymbol{F}_c. \end{cases}$$

。 对于无界空间,若 $|m{F}|=o\left(|m{r}-m{r}'|^{-1}
ight)\left(|m{r}| o\infty
ight)$,则上述面积分为零.