1 矢量分析与场论

1.1 矢量代数

▶ **高数笔记**

1.1.1 矢量的运算

- 矢量的概念

  > 这里只考虑三维的矢量（行向量）.

  - $\boldsymbol{a}=(a_1,a_2,a_3)=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$.
  - 模长：$|\boldsymbol{a}|=\sqrt{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}=\sqrt{a_1^2+a_2^2+a_3^2}$.
  - 方向矢量：$\boldsymbol{e}_{\boldsymbol{a}}=\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=\dfrac{\boldsymbol{a}}{a}$.

- 矢量的加减
    - 加法：$\boldsymbol{a}+\boldsymbol{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$.
    - 负数：$-\boldsymbol{a}=(-a_1,-a_2,-a_3)$.
    - 减法：$\boldsymbol{a}-\boldsymbol{b}=(a_1-b_1,a_2-b_2,a_3-b_3)$.
- 矢量的乘法
    - 数乘：$k\boldsymbol{a}=(ka_1,ka_2,ka_3)$.
    - 点乘：$\boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2+a_3b_3$.
    - 叉乘：$\boldsymbol{a}\times\boldsymbol{b}=(a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1)$.
    - 混合积：$(\boldsymbol{abc})=(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.

1.1.2 矢量的性质

- 基本性质（并不完整）
    - 加法：零元、负元、结合律、交换律.
    - 数乘：结合律、第一第二分配律、消去律.
    - 点积：交换律、数乘结合律、加法分配律.
- 点乘
    - $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}=|\boldsymbol{a}|\cdot|\boldsymbol{b}|\cos\theta$.
    - $|\boldsymbol{a}|=\sqrt{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}=\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$.
    - $\boldsymbol{a}\perp\boldsymbol{b}\Leftrightarrow\boldsymbol{a}\cdot\boldsymbol{b}=0$.
- 叉乘
    - 叉乘性质
        - $\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$.
        - $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}|\cdot|\boldsymbol{b}|\cdot|\sin\theta|$.
        - $\boldsymbol{a}\parallel\boldsymbol{b}\Leftrightarrow\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$.
    - 其它表示
        - 行列式：$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$.
        - 行向量：$\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\begin{pmatrix}0&-b_3&b_2\\b_3&0&-b_1\\-b_2&b_1&0\end{pmatrix}$.
        - 列向量：$\boldsymbol{a}^{\mathrm{T}}\times\boldsymbol{b}^{\mathrm{T}}=\begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}\boldsymbol{b}^{\mathrm{T}}$.
- 混合积
    - $(\boldsymbol{abc})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$.
    - $(\boldsymbol{abc})=(\boldsymbol{bca})=(\boldsymbol{cab})=-(\boldsymbol{acb})=-(\boldsymbol{cba})=-(\boldsymbol{bac})$.
    - 向量共面 $\Leftrightarrow(\boldsymbol{abc})=0\Leftrightarrow\exists\alpha,\beta,\gamma\colon\alpha\boldsymbol{a}+\beta\boldsymbol{b}+\gamma\boldsymbol{c}=\boldsymbol{0}$.
- 其它性质

- 对于非零矢量 $\boldsymbol{a}$，如果 $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}$ 且 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{c}$，则 $\boldsymbol{b}=\boldsymbol{c}$.
- 平行于 $\boldsymbol{a}$ 的 $\boldsymbol{b}$ 分量为 $\boldsymbol{b}_{/\!/}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\boldsymbol{a}$，垂直分量为 $\boldsymbol{b}_{\perp}=\boldsymbol{b}-\boldsymbol{b}_{/\!/}$.

1.1.3 矢量恒等式

1. 拉格朗日恒等式
    1. $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$.
    2. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$.
2. 雅可比恒等式：$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{0}$.
3. 四向量恒等式
    1. $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$.
    2. $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{c}[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{d})]-\boldsymbol{d}[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})]$.
    3. $\boldsymbol{a}\times[\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})]=(\boldsymbol{a}\times\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\times\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$.

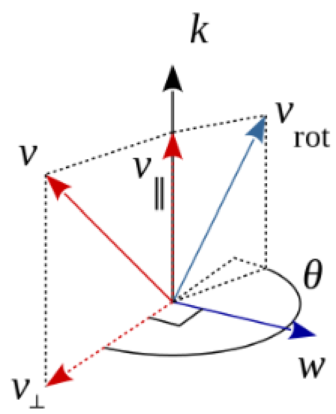
▶ **证明**

1.1.4 矢量微积分

设 $\boldsymbol{a}=\boldsymbol{a}(t),\boldsymbol{b}=\boldsymbol{b}(t),f=f(t)$，此外 λ,μ 为常数，$\boldsymbol{k}$ 为常向量.

1. 求导
    1. $\dfrac{\mathrm{d}}{\mathrm{d}t}(\lambda\boldsymbol{a}+\mu\boldsymbol{b})=\lambda\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}+\mu\dfrac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$.
    2. $\dfrac{\mathrm{d}}{\mathrm{d}t}(f\boldsymbol{a})=\dfrac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{a}+f\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}$.
    3. $\dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b})=\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\dfrac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$.
    4. $\dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\times\boldsymbol{b})=\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\times\boldsymbol{b}+\boldsymbol{a}\times\dfrac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$.
    5. $\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}(f)=\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}f}\cdot\dfrac{\mathrm{d}f}{\mathrm{d}t}$.
2. 积分
    1. $\displaystyle\int(\lambda\boldsymbol{a}+\mu\boldsymbol{b})\,\mathrm{d}t=\lambda\int\boldsymbol{a}\,\mathrm{d}t+\mu\int\boldsymbol{b}\,\mathrm{d}t$.
    2. $\displaystyle\int\boldsymbol{k}\cdot\boldsymbol{a}\,\mathrm{d}t=\boldsymbol{k}\cdot\int\boldsymbol{a}\,\mathrm{d}t$.
    3. $\displaystyle\int\boldsymbol{k}\times\boldsymbol{a}\,\mathrm{d}t=\boldsymbol{k}\times\int\boldsymbol{a}\,\mathrm{d}t$.

1.1.5 矢量的旋转



设转轴的单位方向向量为 $\boldsymbol{k}$，旋转前的向量为 $\boldsymbol{v}$，逆时针旋转 θ 后的向量为 $\boldsymbol{v}'$（即图中的 $\boldsymbol{v}_{\mathrm{rot}}$），则

$$\begin{aligned}\boldsymbol{v}' &= \cos\theta\boldsymbol{v} + (1-\cos\theta)(\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{k} + \sin\theta\boldsymbol{k}\times\boldsymbol{v}\\ &= \boldsymbol{v} + (1-\cos\theta)\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{v}) + \sin\theta\boldsymbol{k}\times\boldsymbol{v}.\end{aligned}$$

▶ **证明**

备注　有关空间旋转的更多内容可参考笔记平面旋转与空间旋转.

1.2 常用坐标系

1.2.1 正交坐标系

- 基本符号
  - 坐标符号：(u_1, u_2, u_3).
  - 单位矢量：$\boldsymbol{a}_{u_1}, \boldsymbol{a}_{u_2}, \boldsymbol{a}_{u_3}$.

    > 有的教材又写为 $\boldsymbol{e}_{u_1}, \boldsymbol{e}_{u_2}, \boldsymbol{e}_{u_3}$.

  - 拉梅系数：$h_i = h_i(u_1, u_2, u_3) = \dfrac{\mathrm{d}l_{u_i}}{\mathrm{d}u_i}$.

    > 又称为度量系数或尺度因子.

- 空间度量
  - 长度元：$\begin{cases}\mathrm{d}l_{u_1} = h_1\,\mathrm{d}u_1,\\ \mathrm{d}l_{u_2} = h_2\,\mathrm{d}u_2,\\ \mathrm{d}l_{u_3} = h_3\,\mathrm{d}u_3.\end{cases}$
  - 面积元：$\begin{cases}\mathrm{d}\boldsymbol{S}_1 = \boldsymbol{a}_{u_1}(h_2h_3\,\mathrm{d}u_2\,\mathrm{d}u_3),\\ \mathrm{d}\boldsymbol{S}_2 = \boldsymbol{a}_{u_2}(h_3h_1\,\mathrm{d}u_3\,\mathrm{d}u_1),\\ \mathrm{d}\boldsymbol{S}_3 = \boldsymbol{a}_{u_3}(h_1h_2\,\mathrm{d}u_1\,\mathrm{d}u_2).\end{cases}$
  - 体积元：$\mathrm{d}V = h_1h_2h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$.

1.2.2 直角坐标系

- 基本符号
  - 坐标变量：(x, y, z).
  - 单位矢量：$\boldsymbol{a}_x, \boldsymbol{a}_y, \boldsymbol{a}_z$.
  - 位置矢量：$\boldsymbol{r} = x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z$.
- 空间度量
  - 长度元：$\begin{cases} \mathrm{d}l_x = \mathrm{d}x, \\ \mathrm{d}l_y = \mathrm{d}y, \\ \mathrm{d}l_z = \mathrm{d}z. \end{cases}$
  - 面积元：$\begin{cases} \mathrm{d}S_x = \mathrm{d}y\mathrm{d}z, \\ \mathrm{d}S_y = \mathrm{d}z\mathrm{d}x, \\ \mathrm{d}S_z = \mathrm{d}x\mathrm{d}y. \end{cases}$
  - 体积元：$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.
- 坐标变量关系
  - 圆柱 $\to$ 直角：$\begin{cases} x = \rho\cos\varphi, \\ y = \rho\sin\varphi, \\ z = z. \end{cases}$
  - 球极 $\to$ 直角：$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta. \end{cases}$
- 单位矢量关系
  - 圆柱 $\to$ 直角：$\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\varphi \\ \boldsymbol{a}_z \end{bmatrix}$.
  - 球极 $\to$ 直角：$\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix}$.
- 单位矢量的偏导、散度与旋度均为零.

1.2.3 圆柱坐标系

- 基本符号
  - 坐标符号：(ρ, φ, z).
  - 单位矢量：$\boldsymbol{a}_\rho, \boldsymbol{a}_\varphi, \boldsymbol{a}_z$.
  - 位置矢量：$\boldsymbol{r} = \boldsymbol{e}_\rho\rho + \boldsymbol{e}_z z$.
- 空间度量
  - 长度元：$\begin{cases} \mathrm{d}l_\rho = \mathrm{d}\rho, \\ \mathrm{d}l_\varphi = \rho\,\mathrm{d}\varphi, \\ \mathrm{d}l_z = \mathrm{d}z. \end{cases}$
  - 面积元：$\begin{cases} \mathrm{d}S_\rho = \rho\,\mathrm{d}\varphi\mathrm{d}z, \\ \mathrm{d}S_\varphi = \mathrm{d}r\mathrm{d}z, \\ \mathrm{d}S_z = \rho\,\mathrm{d}\rho\mathrm{d}\varphi. \end{cases}$
  - 体积元：$\mathrm{d}V = \rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$.
- 坐标变量关系

- 直角 $\to$ 圆柱：$\begin{cases}\rho=\sqrt{x^2+y^2},\\ \varphi=\arctan\dfrac{y}{x},\\ z=z.\end{cases}$
- 球极 $\to$ 圆柱：$\begin{cases}\rho=r\sin\theta,\\ \varphi=\varphi,\\ z=r\cos\theta.\end{cases}$

- 单位矢量关系
  - 直角 $\to$ 圆柱：$\begin{bmatrix}\boldsymbol{a}_\rho\\ \boldsymbol{a}_\varphi\\ \boldsymbol{a}_z\end{bmatrix}=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_x\\ \boldsymbol{a}_y\\ \boldsymbol{a}_z\end{bmatrix}.$
  - 球极 $\to$ 圆柱：$\begin{bmatrix}\boldsymbol{a}_\rho\\ \boldsymbol{a}_\varphi\\ \boldsymbol{a}_z\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_r\\ \boldsymbol{a}_\theta\\ \boldsymbol{a}_\varphi\end{bmatrix}.$
- 单位矢量导数（未标注的均为零）
  - 对 φ 求导：$\begin{cases}\dfrac{\mathrm{d}\boldsymbol{a}_\rho}{\mathrm{d}\varphi}=\boldsymbol{a}_\varphi,\\ \dfrac{\mathrm{d}\boldsymbol{a}_\varphi}{\mathrm{d}\varphi}=-\boldsymbol{a}_\rho.\end{cases}$
  - 散度：$\boldsymbol{\nabla}\cdot\boldsymbol{a}_\rho=\dfrac{1}{\rho}.$
  - 旋度：$\boldsymbol{\nabla}\times\boldsymbol{a}_\varphi=\dfrac{1}{\rho}\boldsymbol{a}_z.$

1.2.4 球极坐标系

- 基本符号
  - 坐标符号：(r,θ,φ).
  - 单位矢量：$\boldsymbol{a}_r,\boldsymbol{a}_\theta,\boldsymbol{a}_\varphi$.
  - 位置矢量：$\boldsymbol{r}=r\boldsymbol{a}_r$.
- 空间度量
  - 长度元：$\begin{cases}\mathrm{d}l_r=\mathrm{d}r,\\ \mathrm{d}l_\theta=r\,\mathrm{d}\theta,\\ \mathrm{d}l_\varphi=r\sin\theta\,\mathrm{d}\varphi.\end{cases}$
  - 面积元：$\begin{cases}\mathrm{d}S_r=r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi,\\ \mathrm{d}S_\theta=r\sin\theta\,\mathrm{d}r\mathrm{d}\varphi,\\ \mathrm{d}S_\varphi=r\,\mathrm{d}r\mathrm{d}\theta.\end{cases}$
  - 体积元：$\mathrm{d}V=r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$.
- 坐标变量关系
  - 直角 $\to$ 球极：$\begin{cases}r=\sqrt{x^2+y^2+z^2},\\ \theta=\arctan\dfrac{\sqrt{x^2+y^2}}{z},\\ \varphi=\arctan\dfrac{y}{x}.\end{cases}$
  - 圆柱 $\to$ 球极：$\begin{cases}r=\sqrt{\rho^2+z^2},\\ \theta=\arctan\dfrac{\rho}{z},\\ \varphi=\varphi.\end{cases}$

- 单位矢量关系
  - 直角 $\to$ 球极: $\begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}.$
  - 圆柱 $\to$ 球极: $\begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\varphi \\ \boldsymbol{a}_z \end{bmatrix}.$
- 单位矢量导数（未标注的均为零）
  - 对 θ 求偏导: $\begin{cases} \dfrac{\partial \boldsymbol{a}_r}{\partial\theta} = \boldsymbol{a}_\theta, \\ \dfrac{\partial \boldsymbol{a}_\theta}{\partial\theta} = -\boldsymbol{a}_r, \\ \dfrac{\partial \boldsymbol{a}_\varphi}{\partial\theta} = \boldsymbol{0}. \end{cases}$
  - 对 φ 求偏导: $\begin{cases} \dfrac{\partial \boldsymbol{a}_r}{\partial\varphi} = \boldsymbol{a}_\varphi \sin\theta, \\ \dfrac{\partial \boldsymbol{a}_\theta}{\partial\varphi} = \boldsymbol{a}_\varphi \cos\theta, \\ \dfrac{\partial \boldsymbol{a}_\varphi}{\partial\varphi} = -\boldsymbol{a}_r \sin\theta - \boldsymbol{a}_\theta \cos\theta. \end{cases}$
  - 散度: $\begin{cases} \boldsymbol{\nabla}\cdot\boldsymbol{a}_r = \dfrac{2}{r}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{a}_\theta = \dfrac{\cot\theta}{r}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{a}_\varphi = 0. \end{cases}$
  - 旋度: $\begin{cases} \boldsymbol{\nabla}\times\boldsymbol{a}_r = \boldsymbol{0}, \\ \boldsymbol{\nabla}\times\boldsymbol{a}_\theta = \dfrac{1}{r}\boldsymbol{a}_\varphi, \\ \boldsymbol{\nabla}\times\boldsymbol{a}_\varphi = \dfrac{\cot\theta}{r}\boldsymbol{a}_r - \dfrac{1}{r}\boldsymbol{a}_\theta. \end{cases}$
- 两个位置矢量 $\boldsymbol{R}_1 = (r_1, \theta_1, \varphi_1)$ 和 $\boldsymbol{R}_2 = (r_2, \theta_2, \varphi_2)$ 夹角的余弦为

$$\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2).$$

1.3 哈密顿算子

1.3.1 正交坐标系

- 定义
  - $\boldsymbol{\nabla} f = \boldsymbol{a}_n \dfrac{\Delta f}{\Delta n} = \boldsymbol{a}_n \dfrac{\mathrm{d}f}{\mathrm{d}n}.$
  - $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \lim\limits_{\Delta V\to 0} \dfrac{1}{\Delta V} \oint_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}.$
  - $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{a}_{\Delta \boldsymbol{S}} \lim\limits_{\Delta S\to \boldsymbol{0}} \dfrac{1}{\Delta S} \oint_l \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}.$
- 推论
  - $\boldsymbol{\nabla} f = \dfrac{\boldsymbol{a}_{u_1}}{h_1}\dfrac{\partial f}{\partial u_1} + \dfrac{\boldsymbol{a}_{u_2}}{h_2}\dfrac{\partial f}{\partial u_2} + \dfrac{\boldsymbol{a}_{u_3}}{h_3}\dfrac{\partial f}{\partial u_3}.$
  - $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial}{\partial u_1}F_1h_2h_3 + \dfrac{\partial}{\partial u_2}h_1F_2h_3 + \dfrac{\partial}{\partial u_3}h_1h_2F_3\right).$

- $\nabla\times\boldsymbol{F}=\dfrac{1}{h_1h_2h_3}\begin{vmatrix}h_1\boldsymbol{a}_{u_1} & h_2\boldsymbol{a}_{u_2} & h_3\boldsymbol{a}_{u_3}\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$.

- 例子：设 $P=(x,y,z)$ 到 $P'=(x',y',z')$ 的距离为 R, 则 $\nabla\dfrac{1}{R}=-\dfrac{\boldsymbol{R}}{R^3}=-\nabla'\dfrac{1}{R}$.

1.3.2 直角坐标系

- $\nabla f=\boldsymbol{a}_x\dfrac{\partial f}{\partial x}+\boldsymbol{a}_y\dfrac{\partial f}{\partial y}+\boldsymbol{a}_z\dfrac{\partial f}{\partial z}$.
- $\nabla\cdot\boldsymbol{F}=\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}+\dfrac{\partial F_z}{\partial z}$.
- $\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$.

1.3.3 圆柱坐标系

- $\nabla f=\boldsymbol{a}_\rho\dfrac{\partial f}{\partial \rho}+\dfrac{\boldsymbol{a}_\varphi}{\rho}\dfrac{\partial f}{\partial \varphi}+\boldsymbol{a}_z\dfrac{\partial f}{\partial z}$.
- $\nabla\cdot\boldsymbol{F}=\dfrac{1}{\rho}\dfrac{\partial(\rho F_\rho)}{\partial \rho}+\dfrac{1}{\rho}\dfrac{\partial F_\varphi}{\partial \varphi}+\dfrac{\partial F_z}{\partial z}$.
- $\nabla\times\boldsymbol{F}=\dfrac{1}{\rho}\begin{vmatrix}\boldsymbol{a}_\rho & \rho\boldsymbol{a}_\varphi & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z}\\ F_\rho & \rho F_\varphi & F_z\end{vmatrix}$.

1.3.4 球极坐标系

- $\nabla f=\boldsymbol{a}_r\dfrac{\partial f}{\partial r}+\dfrac{\boldsymbol{a}_\theta}{r}\dfrac{\partial f}{\partial \theta}+\dfrac{\boldsymbol{a}_\varphi}{r\sin\theta}\dfrac{\partial f}{\partial \varphi}$.
- $\nabla\cdot\boldsymbol{F}=\dfrac{1}{r^2}\dfrac{\partial(r^2F_r)}{\partial r}+\dfrac{1}{r\sin\theta}\dfrac{\partial\sin\theta F_\theta}{\partial \theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial F_\varphi}{\partial \varphi}$.
- $\nabla\times\boldsymbol{F}=\dfrac{1}{r^2\sin\theta}\begin{vmatrix}\boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\sin\theta\boldsymbol{a}_\varphi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi}\\ F_r & rF_\theta & r\sin\theta F_\varphi\end{vmatrix}$.

1.3.5 记号与拓展

可以将哈密顿算子写成向量的形式，但此时数乘与点乘均不可交换. 以直角坐标系为例：

- $\nabla=\left(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial y},\dfrac{\partial}{\partial z}\right)$.
- 左乘

    - $\boldsymbol{\nabla} f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$.
    - $\boldsymbol{\nabla}\cdot\boldsymbol{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$.
    - $\boldsymbol{\nabla}\times\boldsymbol{F}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$.
- 右乘
    - $f\boldsymbol{\nabla}=\left(f\frac{\partial}{\partial x},f\frac{\partial}{\partial y},f\frac{\partial}{\partial z}\right)$.
    - $\boldsymbol{F}\cdot\boldsymbol{\nabla}=F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}$.
    - $\boldsymbol{F}\times\boldsymbol{\nabla}=\left(F_3\frac{\partial}{\partial y}-F_2\frac{\partial}{\partial z},F_1\frac{\partial}{\partial z}-F_3\frac{\partial}{\partial x},F_2\frac{\partial}{\partial x}-F_1\frac{\partial}{\partial y}\right)$.
- 备注
    - $\boldsymbol{\nabla}\cdot\boldsymbol{F}\neq\boldsymbol{F}\cdot\boldsymbol{\nabla}$.
    - $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f:=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f)=(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})f$.
    - $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f:=\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f)=(\boldsymbol{\nabla}\times\boldsymbol{\nabla})f\equiv\mathbf{0}$.
- 约定
    - 梯度、散度、旋度的优先级高于数乘、点乘与叉乘，并且从右向左计算.
    - 例 1：$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{F}:=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F})\neq(\boldsymbol{\nabla}\times\boldsymbol{\nabla})\times\boldsymbol{F}\equiv\mathbf{0}$.
    - 例 2：$\boldsymbol{\nabla}\times\boldsymbol{a}\times\boldsymbol{b}:=(\boldsymbol{\nabla}\times\boldsymbol{a})\times\boldsymbol{b}\neq\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b})$.
- 向量的梯度

$$\boldsymbol{\nabla}\boldsymbol{F}=\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}=\begin{bmatrix}\frac{\partial F_1}{\partial x}&\frac{\partial F_1}{\partial y}&\frac{\partial F_1}{\partial z}\\\frac{\partial F_2}{\partial x}&\frac{\partial F_2}{\partial y}&\frac{\partial F_2}{\partial z}\\\frac{\partial F_3}{\partial x}&\frac{\partial F_3}{\partial y}&\frac{\partial F_3}{\partial z}\end{bmatrix}.$$

1.4 哈密顿算子的性质

以下均假设混合偏导连续.

1.4.1 哈密顿算子与叉乘

1. Hamilton 算子在右边
    1. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{\nabla})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{\nabla})-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{\nabla}$.
    2. $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{\nabla}=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{\nabla})-\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{\nabla})$.
2. Hamilton 算子在中间
    1. $\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b})=\boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b}$. 🌙
    2. $(\boldsymbol{a}\times\boldsymbol{\nabla})\times\boldsymbol{b}=\boldsymbol{a}(\boldsymbol{\nabla}\cdot\boldsymbol{b})-\boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{b}$. 🌙
3. Hamilton 算子在左边
    1. $\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{a}(\boldsymbol{\nabla}\cdot\boldsymbol{b})-(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b}-\boldsymbol{b}(\boldsymbol{\nabla}\cdot\boldsymbol{a})+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a}$. 🌙
    2. $(\boldsymbol{\nabla}\times\boldsymbol{a})\times\boldsymbol{b}=(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a}-\boldsymbol{b}\boldsymbol{\nabla}\boldsymbol{a}$.
4. 上述公式的推论

1. $\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{\nabla}) \times \boldsymbol{b} = \boldsymbol{a}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b}$.
2. $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \times \boldsymbol{\nabla} + \boldsymbol{\nabla} \times \boldsymbol{a}) \times \boldsymbol{b} - (\boldsymbol{b} \times \boldsymbol{\nabla} + \boldsymbol{\nabla} \times \boldsymbol{b}) \times \boldsymbol{a}$.

▶ **证明**

1.4.2 哈密顿算子与梯度

1. 标量的梯度——加减乘除与复合
   1. $\boldsymbol{\nabla}(\lambda f + \mu g) = \lambda \boldsymbol{\nabla} f + \mu \boldsymbol{\nabla} g$.
   2. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$.
   3. $\boldsymbol{\nabla}\left(\dfrac{f}{g}\right) = \dfrac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$.
   4. $\boldsymbol{\nabla}(\phi(f)) = \phi'(f)\boldsymbol{\nabla} f$.
2. 标量的梯度——向量点乘与散度
   1. $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\boldsymbol{\nabla}\boldsymbol{a}$. 🌙
   2. $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) + (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a}$.
   3. $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) + \boldsymbol{b}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - (\boldsymbol{a} \times \boldsymbol{\nabla}) \times \boldsymbol{b} - (\boldsymbol{b} \times \boldsymbol{\nabla}) \times \boldsymbol{a}$.
   4. $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) = \Delta\boldsymbol{a} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$.
3. 向量的梯度 $\boldsymbol{\nabla}(\lambda\boldsymbol{a} + \mu\boldsymbol{b}) = \lambda\boldsymbol{\nabla}\boldsymbol{a} + \mu\boldsymbol{\nabla}\boldsymbol{b}$.

▶ **证明**

1.4.3 哈密顿算子与散度

1. $\boldsymbol{\nabla} \cdot (\lambda\boldsymbol{a} + \mu\boldsymbol{b}) = \lambda(\boldsymbol{\nabla} \cdot \boldsymbol{a}) + \mu(\boldsymbol{\nabla} \cdot \boldsymbol{b})$.
2. $\boldsymbol{\nabla} \cdot (f\boldsymbol{a}) = f\boldsymbol{\nabla} \cdot \boldsymbol{a} + \boldsymbol{\nabla} f \cdot \boldsymbol{a}$.
3. $\boldsymbol{\nabla} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})$.
4. $\boldsymbol{\nabla} \cdot (g\boldsymbol{\nabla} f) = \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f + g\Delta f$.
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) = 0$.（旋度场无源）
6. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} \times \boldsymbol{a}) = (\boldsymbol{\nabla} f) \cdot (\boldsymbol{\nabla} \times \boldsymbol{a})$.

▶ **证明**

1.4.4 哈密顿算子与旋度

1. $\boldsymbol{\nabla} \times (\lambda\boldsymbol{a} + \mu\boldsymbol{b}) = \lambda(\boldsymbol{\nabla} \times \boldsymbol{a}) + \mu(\boldsymbol{\nabla} \times \boldsymbol{b})$.
2. $\boldsymbol{\nabla} \times (f\boldsymbol{a}) = f\boldsymbol{\nabla} \times \boldsymbol{a} + \boldsymbol{\nabla} f \times \boldsymbol{a}$.
3. $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a}$.
4. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$.（梯度场无旋）
5. $\boldsymbol{\nabla} \times (g\boldsymbol{\nabla} f) = (\boldsymbol{\nabla} g) \times (\boldsymbol{\nabla} f)$.
6. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \Delta\boldsymbol{a}$.

▶ **证明**

1.4.5 哈密顿算子与积分

1.5 拉普拉斯算子

1.5.1 一般定义

- 标量的定义：$\Delta f = \boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f.$
- 矢量的定义：$\Delta \boldsymbol{a} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{a}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a}).$
- 例子：$\Delta\dfrac{1}{\boldsymbol{r}-\boldsymbol{r}'} = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}').$

1.5.2 二维空间

- 正交坐标：$\Delta f = \dfrac{1}{h_1 h_2}\sum\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2}{h_1}\dfrac{\partial f}{\partial u_1}\right).$
- 直角坐标：$\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}.$
- 极坐标系：$\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial f}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial\varphi^2}.$

1.5.3 三维空间

- 正交坐标：$\Delta f = \dfrac{1}{h_1 h_2 h_3}\sum\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right).$
- 直角坐标：$\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}.$
- 圆柱坐标：$\Delta f = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial f}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial\varphi^2} + \dfrac{\partial^2 f}{\partial z^2}.$
- 球坐标系：$\Delta f = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial f}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial\varphi^2}.$

1.5.4 三维矢量

- 直角坐标：$\Delta\boldsymbol{F} = \boldsymbol{a}_x\Delta F_x + \boldsymbol{a}_y\Delta F_y + \boldsymbol{a}_z\Delta F_z.$
- 圆柱坐标：
$$\Delta\boldsymbol{F} = \left(\Delta F_\rho - \frac{2}{\rho^2}\frac{\partial F_\varphi}{\partial\varphi} - \frac{F_\rho}{\rho^2}\right)\boldsymbol{a}_\rho + \left(\Delta F_\varphi + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial\varphi} - \frac{F_\varphi}{\rho^2}\right)\boldsymbol{a}_\varphi + \Delta F_z\cdot\boldsymbol{a}_z.$$
- 球坐标系：

$$
\begin{aligned}
\Delta \boldsymbol{F} = &\left[\Delta F_r - \frac{2}{r^2}\left(F_r + \cot\theta F_\theta + \csc\theta\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial A_\theta}{\partial\theta}\right)\right]\boldsymbol{a}_r + \\
&\left[\Delta F_\theta - \frac{1}{r^2}\left(\csc^2\theta F_\theta - 2\frac{\partial F_r}{\partial\theta} + 2\cot\theta\csc\theta\frac{\partial F_\varphi}{\partial\varphi}\right)\right]\boldsymbol{a}_\theta + \\
&\left[\Delta F_\varphi - \frac{1}{r^2}\left(\csc^2\theta F_\varphi - 2\csc\theta\frac{\partial F_r}{\partial\varphi} - 2\cot\theta\csc\theta\frac{\partial F_\theta}{\partial\varphi}\right)\right].
\end{aligned}
$$

证明

对于圆柱坐标系，

$$
\begin{aligned}
\boldsymbol{F} &= F_\rho\boldsymbol{a}_\rho + F_\varphi\boldsymbol{a}_\varphi + F_z\boldsymbol{a}_z, \\
\frac{\partial\boldsymbol{F}}{\partial\varphi} &= \left(\frac{\partial F_\rho}{\partial\varphi} - F_\varphi\right)\boldsymbol{a}_\rho + \left(\frac{\partial F_\varphi}{\partial\varphi} + F_\rho\right)\boldsymbol{a}_\varphi + \frac{\partial F_z}{\partial\varphi}\boldsymbol{a}_z, \\
\frac{\partial^2\boldsymbol{F}}{\partial\varphi^2} &= \left(\frac{\partial^2 F_\rho}{\partial\varphi^2} - 2\frac{\partial F_\varphi}{\partial\varphi} - F_\rho\right)\boldsymbol{a}_\rho + \left(\frac{\partial^2 F_\varphi}{\partial\varphi^2} + 2\frac{\partial F_\rho}{\partial\varphi} - F_\varphi\right)\boldsymbol{a}_\varphi + \frac{\partial^2 F_z}{\partial\varphi^2}\boldsymbol{a}_z, \\
\Delta\boldsymbol{F} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\boldsymbol{F}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\boldsymbol{F}}{\partial\varphi^2} + \frac{\partial^2\boldsymbol{F}}{\partial z^2} \\
&= \frac{1}{\rho}\frac{\partial\boldsymbol{F}}{\partial\rho} + \frac{\partial^2\boldsymbol{F}}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\boldsymbol{F}}{\partial\varphi^2} + \frac{\partial^2\boldsymbol{F}}{\partial z^2} \\
&= \left(\frac{1}{\rho}\frac{\partial F_\rho}{\partial\rho} + \frac{\partial^2 F_\rho}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 F_\rho}{\partial\varphi^2} + \frac{\partial^2 F_\rho}{\partial z^2} - \frac{2}{\rho^2}\frac{\partial F_\varphi}{\partial\varphi} - \frac{F_\rho}{\rho^2}\right)\boldsymbol{a}_\rho + \\
&\quad \left(\frac{1}{\rho}\frac{\partial F_\varphi}{\partial\rho} + \frac{\partial^2 F_\varphi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 F_\varphi}{\partial\varphi^2} + \frac{\partial^2 F_\varphi}{\partial z^2} + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial\varphi} - \frac{F_\varphi}{\rho^2}\right)\boldsymbol{a}_\varphi + \\
&\quad \left(\frac{1}{\rho}\frac{\partial F_z}{\partial\rho} + \frac{\partial^2 F_z}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 F_z}{\partial\varphi^2} + \frac{\partial^2 F_z}{\partial z^2}\right)\boldsymbol{a}_z \\
&= \left(\Delta F_\rho - \frac{2}{\rho^2}\frac{\partial F_\varphi}{\partial\varphi} - \frac{F_\rho}{\rho^2}\right)\boldsymbol{a}_\rho + \left(\Delta F_\varphi + \frac{2}{\rho^2}\frac{\partial F_\rho}{\partial\varphi} - \frac{F_\varphi}{\rho^2}\right)\boldsymbol{a}_\varphi + \Delta F_z \cdot \boldsymbol{a}_z.
\end{aligned}
$$

对于球坐标系，这里采用另一种思路，

$$
\begin{aligned}
\Delta\boldsymbol{F} &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) \\
&= \left[\Delta F_r - \frac{2}{r^2}\left(F_r + \cot\theta F_\theta + \csc\theta\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial A_\theta}{\partial\theta}\right)\right]\boldsymbol{a}_r + \\
&\quad \left[\Delta F_\theta - \frac{1}{r^2}\left(\csc^2\theta F_\theta - 2\frac{\partial F_r}{\partial\theta} + 2\cot\theta\csc\theta\frac{\partial F_\varphi}{\partial\varphi}\right)\right]\boldsymbol{a}_\theta + \\
&\quad \left[\Delta F_\varphi - \frac{1}{r^2}\left(\csc^2\theta F_\varphi - 2\csc\theta\frac{\partial F_r}{\partial\varphi} - 2\cot\theta\csc\theta\frac{\partial F_\theta}{\partial\varphi}\right)\right].
\end{aligned}
$$

证毕.

1.5.5 算子性质

1. 标量函数
   1. $\Delta(\lambda f + \mu g) = \lambda\Delta f + \mu\Delta g$.
   2. $\Delta(fg) = (\Delta f)g + 2(\boldsymbol{\nabla} f)\cdot(\boldsymbol{\nabla} g) + f(\Delta g)$.
   3. $\Delta(\phi(f)) = \phi''(f)(\boldsymbol{\nabla} f)^2 + \phi'(f)\Delta f$.
2. 向量函数

1. $\Delta(\lambda \boldsymbol{a}+\mu \boldsymbol{b})=\lambda \Delta \boldsymbol{a}+\mu \Delta \boldsymbol{b}$.
2. $\Delta \boldsymbol{a}=\nabla(\nabla \cdot \boldsymbol{a})-\nabla \times(\nabla \times \boldsymbol{a})$.
3. $\Delta \boldsymbol{a}=[\nabla \cdot(\nabla \boldsymbol{a})]^{\mathrm{T}}=\left(\nabla^2 \boldsymbol{a}\right)^{\mathrm{T}}$.

▶ **证明**

1.6 场论定理

1.6.1 场论的三大定理

- Green 公式
  - 切向向量 $\displaystyle\int\limits_{\partial D} \boldsymbol{r} \cdot \boldsymbol{s} \,\mathrm{d} s=\int\limits_{\partial D} P \,\mathrm{d} x+Q \,\mathrm{d} y=\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) \mathrm{d} x \mathrm{d} y$.
  - 外法向量 $\displaystyle\int\limits_{\partial D} \boldsymbol{r} \cdot \boldsymbol{n} \,\mathrm{d} s=\int\limits_{\partial D} P \,\mathrm{d} y-Q \,\mathrm{d} x=\iint\limits_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) \mathrm{d} x \mathrm{d} y$.
- Gauss 公式 $\displaystyle\iint\limits_{\partial \Omega} \boldsymbol{a} \cdot \mathrm{d} \boldsymbol{S}=\iiint\limits_{\Omega} \nabla \cdot \boldsymbol{a} \,\mathrm{d} V$.
- Stokes 公式 $\displaystyle\int\limits_{\partial \sum} \boldsymbol{a} \cdot \mathrm{d} \boldsymbol{s}=\iint\limits_{\sum}(\nabla \times \boldsymbol{a}) \cdot \mathrm{d} \boldsymbol{S}$.

1.6.2 两个格林恒等式

- 分部积分 $\displaystyle\iint\limits_D\left(P \frac{\partial u}{\partial x}+Q \frac{\partial u}{\partial y}\right) \mathrm{d} x \mathrm{d} y=\int\limits_{\partial D} P u \,\mathrm{d} y-Q u \,\mathrm{d} x-\iint\limits_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) u \,\mathrm{d} x \mathrm{d} y$.
- Green 第一公式
  - $\displaystyle\iiint\limits_{\Omega}(\nabla f \cdot \nabla g+f \Delta g) \,\mathrm{d} V=\iint\limits_{\partial \Omega} f \frac{\partial g}{\partial \boldsymbol{n}} \,\mathrm{d} S$.
  - $\displaystyle\iiint\limits_{\Omega}(\nabla g \cdot \nabla f+g \Delta f) \,\mathrm{d} V=\iint\limits_{\partial \Omega} g \frac{\partial f}{\partial \boldsymbol{n}} \,\mathrm{d} S$.
- Green 第二公式 $\displaystyle\iiint\limits_{\Omega}(f \Delta g-g \Delta f) \,\mathrm{d} V=\iint\limits_{\partial \Omega}\left(f \frac{\partial g}{\partial \boldsymbol{n}}-g \frac{\partial f}{\partial \boldsymbol{n}}\right) \mathrm{d} S$.

1.6.3 Helmholtz 定理

- Helmholtz 定理

$$
\begin{aligned}
\boldsymbol{F} &= -\boldsymbol{\nabla} u + \boldsymbol{\nabla} \times \boldsymbol{A} \\
u(\boldsymbol{r}) &= \frac{1}{4\pi} \iiint\limits_{V'} \frac{\boldsymbol{\nabla}' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V' - \frac{1}{4\pi} \iint\limits_{S'} \frac{\boldsymbol{e}_n \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}S' \\
\boldsymbol{A}(\boldsymbol{r}) &= \frac{1}{4\pi} \iiint\limits_{V'} \frac{\boldsymbol{\nabla}' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V' - \frac{1}{4\pi} \iint\limits_{S'} \frac{\boldsymbol{e}_n \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}S'
\end{aligned}
$$

- Helmholtz 定理的解释
  - 若矢量场 $\boldsymbol{F}$ 单值且导数连续有界，则其由散度、旋度和边界条件唯一确定.
  - 梯度场 $\boldsymbol{F}_t = -\boldsymbol{\nabla} u$ 由散度和在边界上的法向量唯一确定.
  - 旋度场 $\boldsymbol{F}_c = \boldsymbol{\nabla} \times \boldsymbol{A}$ 由旋度和在边界上的切线分量唯一确定.
  - $$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{F}_t = \boldsymbol{\nabla} \cdot \boldsymbol{F}, \\ \boldsymbol{\nabla} \times \boldsymbol{F}_t = 0. \end{cases} \qquad \begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{F}_c = 0, \\ \boldsymbol{\nabla} \times \boldsymbol{F}_c = \boldsymbol{\nabla} \times \boldsymbol{F}. \end{cases}$$
  - 对于无界空间, 若 $|\boldsymbol{F}| = o\left(|\boldsymbol{r} - \boldsymbol{r}'|^{-1}\right)$ $(|\boldsymbol{r}| \to \infty)$，则上述面积分为零.