Hidden Markov Model Cheatsheet

Kanru Hua*

University of Illinois at Urbana-Champaign

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1 Preliminaries

Conditional (Posterior) Probability

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

where $P(A,B)$ is called **joint probability** and $P(B)$ is called **prior probability**.

Statistical Independence

If A and B are independent events,

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$
$$\text{or } P(A,B) = P(A)P(B)$$

Marginal Probability

$$P(A) = \sum_b P(A, B = b)$$

Expectation

$$E[X] = \sum_j P(X = x_j) x_j$$

Jensen's Inequality

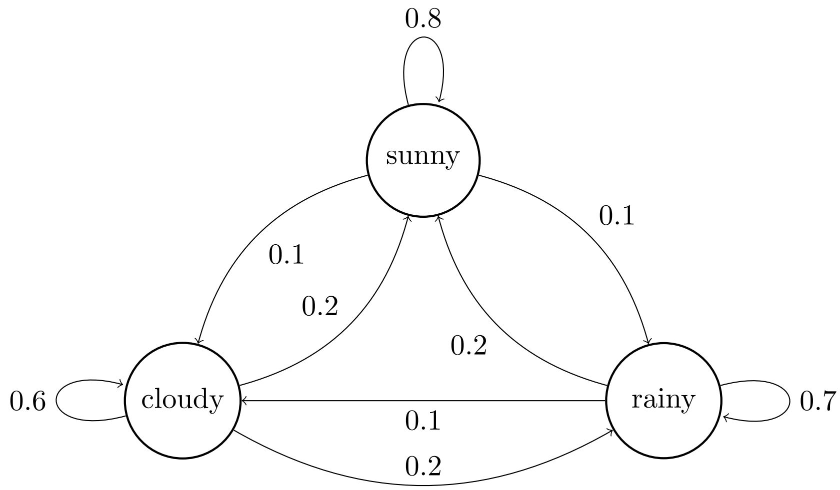
(You are **not** required to know this beforehand)

$$E[f(X)] \leq f(E[X]) \text{ for any concave function } f$$
$$E[f(X)] = f(E[X]) \text{ if X is constant}$$

*Email address: `khua5@illinois.edu`

2 Markov Chain

2.1 Example



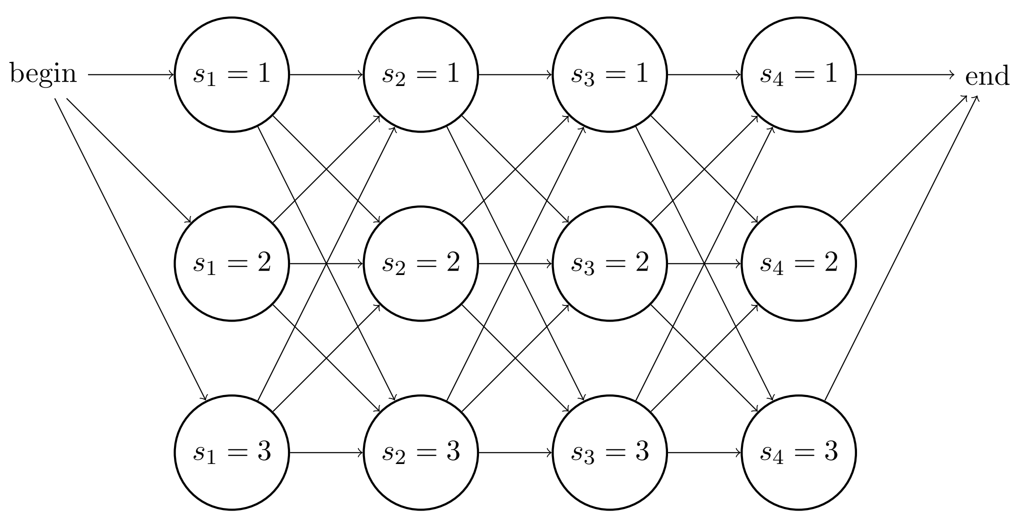
2.2 Definition

$$
\begin{aligned}
\text{parameter set:} \quad & \lambda = \{\pi, \mathbf{A}\} \\
\text{initial probability:} \quad & \pi_j = P(s_1 = j|\lambda) \\
\text{transition probability:} \quad & a_{ij} = P(s_t = j|s_{t-1} = i, \lambda)
\end{aligned}
$$

Markov Property

$$P(s_t = j|s_1, s_2, ..., s_{t-2}, s_{t-1} = i, \lambda) = P(s_t = j|s_{t-1} = i, \lambda) = a_{ij}$$

A lateral view where each edge has a transition probability, or an initial probability for $t = 1$:

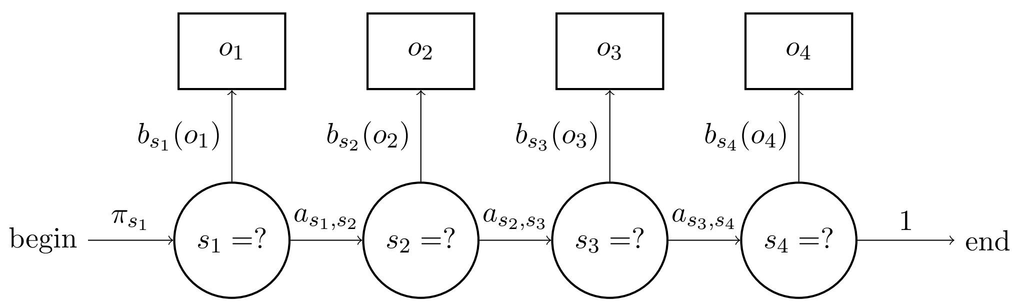


3 Hidden Markov Model

3.1 Definition

$$
\begin{aligned}
\text{parameter set:} \quad & \lambda = \{\pi, \mathbf{A}, \mathbf{B}\} \\
\text{initial probability:} \quad & \pi_j = P(s_1 = j|\lambda) \\
\text{transition probability:} \quad & a_{ij} = P(s_t = j|s_{t-1} = i, \lambda) \\
\\
\text{output probability (discrete case):} \quad & b_{jk} = P(o_t = k|s_t = j, \lambda) \\
\text{output probability density (continuous case):} \quad & b_j(o_t) = p(o_t|s_t = j, \lambda)
\end{aligned}
$$

3.2 Graph



3.3 List of Operations & Algorithms

What can we do with HMM?

- Generation
  - Random walk
- Inference
  - **Forward/Backward algorithm**
  - **Viterbi algorithm**
  - Total probability: $P(O|\lambda)$
  - State occupancy probability: $P(s_t = j|O, \lambda)$
  - Optimal probability & state sequence: $\arg\max_s P(O, s|\lambda)$
- Parameter estimation (training)
  - Initialization
  - Viterbi training
  - **Baum-Welch algorithm** (expectation-maximization)

4 HMM Inference

4.1 Building Blocks

What we already know:

$$
\begin{aligned}
P(s_1 = i|\lambda) &= \pi_i \\
P(s_t = j|s_{t-1} = i, \lambda) &= a_{ij} \\
P(o_t|s_t = j, \lambda) &= b_j(o_t)
\end{aligned}
$$

Trivial extensions using Markov property:

$$
\begin{aligned}
P(s_1, s_2, ..., s_t|\lambda) &= \pi_{s_1} \prod_{\tau=2}^{t} a_{\tau-1,\tau} \\
P(o_1, o_2, ..., o_t|s_1, s_2, ..., s_t, \lambda) &= \prod_{\tau=1}^{t} b_{s_\tau}(o_\tau)
\end{aligned}
$$

By combining above equations, we get

$$
P(o_1, o_2, ..., o_t, s_1, s_2, ..., s_t|\lambda) = \pi_{s_1} b_{s_1}(o_1) \prod_{\tau=2}^{t} a_{\tau-1,\tau} b_{s_\tau}(o_\tau)
$$

4.2 Inference Algorithms

Forward Algorithm

$$
\begin{aligned}
\alpha_t(i) &= P(o_1, o_2, ..., o_t, s_t = i|\lambda) \\
&= P(o_t|s_t = i, \lambda) \sum_j P(o_1, ..., o_{t-1}, s_{t-1} = j|\lambda) P(s_t = i|s_{t-1} = j, \lambda) \\
&= b_i(o_t) \sum_j a_{ji} \alpha_{t-1}(j) \\
\alpha_1(i) &= P(o_1|s_1 = i, \lambda) P(s_1 = i|\lambda) = b_i(o_1)\pi_i
\end{aligned}
$$

Backward Algorithm

$$
\begin{aligned}
\beta_t(i) &= P(o_{t+1}, o_{t+2}, ..., o_t|s_t = i, \lambda) \\
&= \sum_j P(o_{t+1}|s_{t+1} = j, \lambda) P(o_{t+2}, ..., o_T|s_{t+1} = j, \lambda) P(s_{t+1} = j|s_t = i, \lambda) \\
&= \sum_j b_j(o_{t+1}) \beta_{t+1}(j) a_{ij} \\
\beta_T(i) &= 1
\end{aligned}
$$

Viterbi Algorithm

$$\begin{aligned}
\alpha_t^*(i) &= \max_{s_1,\ldots,s_{t-1}} P(o_1,...,o_t,s_1,...,s_{t-1},s_t=i|\lambda) \\
&= P(o_t|s_t=i,\lambda)\max_j \left(P(s_t=i|s_{t-1}=j,\lambda) \max_{s_1,\ldots,s_{t-2}} P(o_1,...,o_{t-1},s_1,...,s_{t-2},s_{t-1}=i|\lambda) \right) \\
&= b_i(o_t)\max_j a_{ji}\alpha_{t-1}^*(j) \\
p_t^*(i) &= \arg\max_j \ a_{ji}\alpha_{t-1}^*(j) \\
\alpha_1^*(i) &= b_i(o_1)\pi_i \\
p_1^*(i) &= 0
\end{aligned}$$

Total Probability

$$\begin{aligned}
P(O|\lambda) &= \sum_j P(o_1,...,o_T,s_t=j|\lambda) = \sum_j \alpha_T(j) \quad \text{(from forward probability)} \\
P(O|\lambda) &= \sum_j P(o_1|s_1=j,\lambda)P(o_2,...,o_T|s_1=j,\lambda)P(s_1=j|\lambda) \\
&= \sum_j b_j(o_1)\beta_1(j)\pi_j \quad \text{(from backward probability)} \\
P(O|\lambda) &= \sum_j P(o_1,...,o_t,o_{t+1},...,o_T,s_t=j|\lambda) \\
&= \sum_j P(o_1,...,o_t,s_t=j|\lambda)P(o_{t+1},...,o_T|s_t=j,\lambda) \\
&= \sum_j \alpha_t(j)\beta_t(j) \quad \text{(from both forward and backward probability, for arbitrary } t\text{)}
\end{aligned}$$

State Occupancy Probability

$$\begin{aligned}
\gamma_t(j) &= P(s_t=j|O,\lambda) \\
&= \frac{P(o_1,...,o_t,s_t=j|\lambda)P(o_{t+1},...,o_T|s_t=j,\lambda)}{P(O|\lambda)} \\
&= \frac{\alpha_t(j)\beta_t(j)}{P(O|\lambda)}
\end{aligned}$$

note that $\sum_j \gamma_t(j) = 1$

State Transition Probability (not to be confused with a_{ij})

$$\begin{aligned}
\gamma_t(i,j) &= P(s_{t-1}=i,s_t=j|O,\lambda) \\
&= \alpha_{t-1}(j)\beta_t(j)b_j(o_t)a_{ij} \quad \text{whose derivation is similar to } \gamma_t(j)
\end{aligned}$$

5 HMM Parameter Estimation

5.1 Expectation-Maximization Algorithm

Goal: obtain maximum likelihood estimation of λ:

$$\lambda^* = \arg\max_{\lambda} l(\lambda) = \arg\max_{\lambda} \ \log P(O|\lambda)$$
$$l(\lambda) = \log P(O|\lambda) = \log \sum_{\mathbf{s}} P(O, \mathbf{s}|\lambda)$$

Motivation: find an alternative likelihood function whose derivatives are easier to calculate.

Assume we have a function $Q(\mathbf{s})$ such that $\sum_{\mathbf{s}} Q(\mathbf{s}) = 1$ and $Q(\mathbf{s}) > 0 \ \forall \mathbf{s}$,

$$\begin{aligned} l(\lambda) &= \log \sum_{\mathbf{s}} Q(\mathbf{s}) \frac{P(O, \mathbf{s}|\lambda)}{Q(\mathbf{s})} \\ &= \log E_{\mathbf{s}}[\frac{P(O, \mathbf{s}|\lambda)}{Q(\mathbf{s})}] \\ &\geq E_{\mathbf{s}}[\log \frac{P(O, \mathbf{s}|\lambda)}{Q(\mathbf{s})}] \quad \text{(whose partial derivatives have closed form)} \end{aligned}$$

To make the lower bound more "effective", i.e., we want $\log E_s[...] = E_s[\log(...)]$,

$$\begin{cases} \frac{P(O,\mathbf{s}|\lambda}{Q(\mathbf{s})} & = c \\ \sum_{\mathbf{s}} Q(\mathbf{s}) & = 1 \end{cases} \Rightarrow Q(\mathbf{s}) = P(\mathbf{s}|O, \lambda)$$

EM Algorithm

Repeat until convergence {
  Expectation: $Q(\mathbf{s}) = P(\mathbf{s}|O, \lambda)$
  Maximization: $\lambda^* = \arg\max_{\lambda} \ \sum_{\mathbf{s}} Q(\mathbf{s}) \log \frac{P(O,\mathbf{s}|\lambda)}{Q(\mathbf{s})}$
}

5.2 Baum-Welch Algorithm

Error function (in maximization step):

$$\begin{aligned} J(\lambda) &= \sum_{\mathbf{s}} Q(\mathbf{s}) \log \frac{P(O, \mathbf{s}|\lambda)}{Q(\mathbf{s})} \\ &= \sum_{\mathbf{s}} Q(\mathbf{s}) \left(\log \pi_{s_1} + \sum_{t=2}^{T} \log a_{s_{t-1}, s_t} + \sum_{t=1}^{T} \log b_{s_t}(o_t) - \log(Q(\mathbf{s})) \right) \end{aligned}$$

Take partial derivative with respect to, for example, a_{ij},

$$\begin{aligned}
\frac{\partial J}{\partial a_{ij}} &= \frac{\partial}{\partial a_{ij}} \sum_{\mathbf{s}} Q(\mathbf{s}) \sum_{t=2}^{T} \log a_{s_{t-1},s_t} \\
&= \frac{\partial}{\partial a_{ij}} \sum_{t=2}^{T} \sum_{m} \sum_{n} \log a_{mn} \sum_{\substack{s_1,\ldots,s_{t-2} \\ s_{t+1},\ldots,s_T}} Q(\mathbf{s}) \\
&= \frac{\partial}{\partial a_{ij}} \sum_{t=2}^{T} \sum_{m} \sum_{n} \log a_{mn} \underbrace{P(s_{t-1} = m, s_t = n | O, \lambda')}_{\gamma_t(m,n)} \\
&= \frac{1}{a_{ij}} \sum_{t=2}^{T} \gamma_t(i,j)
\end{aligned}$$

To make sure $\sum_j a_{ij} = 1 \ \forall i$, introduce Lagrange multiplier l,

$$\begin{cases} \sum_{t=2}^{T} \gamma_t(i,j) & = l a_{ij} \\ \sum_j a_{ij} & = 1 \end{cases}$$

Solve the equations,

$$l = \sum_{t=2}^{T} \sum_{n} \gamma_t(i,n), \quad a_{ij} = \frac{\sum_{t=2}^{T} \gamma_t(i,j)}{\sum_{t=2}^{T} \sum_n \gamma_t(i,n)} = \frac{\sum_{t=2}^{T} \gamma_t(i,j)}{\sum_{t=2}^{T} \gamma_{t-1}(i)}$$

Similarly for π_i and b_{ik} (discrete case) we get,

$$\begin{aligned}
\pi_i &= \gamma_1(i) \\
b_{ik} &= \frac{\sum_{t:o_t=k} \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}
\end{aligned}$$

Multiple Observation Sequences

$$\begin{aligned}
\pi_i &= \frac{1}{L} \sum_{l=0}^{L} \gamma_1^l(i) \\
a_{ij} &= \frac{\sum_{l=0}^{L} \sum_{t=2}^{T} \gamma_t^l(i,j)}{\sum_{l=0}^{L} \sum_{t=2}^{T} \gamma_{t-1}^l(i)} \quad b_{ik} = \frac{\sum_{l=0}^{L} \sum_{t:o_t=k} \gamma_t^l(i)}{\sum_{l=0}^{L} \sum_{t=1}^{T} \gamma_t^l(i)}
\end{aligned}$$

5.3 Baum-Welch Algorithm for Continuous Output Distributions

Multivariate Normal Distribution

$$\mu_j = \frac{\sum_{t=1}^{T} \gamma_t(j) o_t}{\sum_{t=1}^{T} \gamma_t(j)} \quad \Sigma_j = \frac{\sum_{t=1}^{T} \gamma_t(j) o_t o_t^T}{\sum_{t=1}^{T} \gamma_t(j)} - \mu_j \mu_j^T$$

Multivariate Gaussian Mixture Model

Gaussian mixture model:

$$p(x \in \mathbf{R}^k|\boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{k}{2}} \sum_j c_j |\Sigma_j|^{-\frac{k}{2}} e^{-\frac{k}{2}(x-\mu_j)^T \Sigma_j^{-1}(x-\mu_j)}$$

HMM-GMM:

$$\begin{aligned} g_{jk}(o_t) &= p(o_t|s_t = j, m_t = k, \lambda) = p(o_t|\boldsymbol{\mu}_{jk}, \boldsymbol{\Sigma}_{jk}) \\ b_j(o_t) &= p(o_t|s_t = j, \lambda) = \sum_k c_{jk} g_{jk}(o_t) \end{aligned}$$

HMM-GMM inference:

$$\xi_t(j,k) = p(s_t = j, m_t = k|O, \lambda) = \frac{\sum_i \alpha_{t-1}(i)\beta_t(j) a_{ij} c_{jk} g_{jk}(o_t)}{P(O|\lambda)}$$

HMM-GMM parameter estimation:

$$\begin{aligned} c_{jk} &= \frac{\sum_{t=1}^T \xi_t(j,k)}{\sum_{t=1}^T \gamma_t(j)} \\ \mu_{jk} &= \frac{\sum_{t=1}^T \xi_t(j,k) o_t}{\sum_{t=1}^T \xi_t(j,k)} \quad \Sigma_{jk} = \frac{\sum_{t=1}^T \xi_t(j,k) o_t o_t^T}{\sum_{t=1}^T \xi_t(j,k)} - \mu_{jk}\mu_{jk}^T \end{aligned}$$

References

[1] Fink, Gernot A. “Markov models for pattern recognition: from theory to applications”. Springer Science & Business Media, 2014.

[2] Ng, Andrew. “Mixtures of Gaussians and the EM algorithm.” Stanford University. CS229 Lecture Notes (2014).

[3] Bilmes, Jeff A. “A gentle tutorial of the EM algorithm and its application to parameter estimation for Gaussian mixture and hidden Markov models.” International Computer Science Institute 4.510 (1998): 126.