2D Spline Curves

CS 4620 Lecture 13

Motivation: smoothness

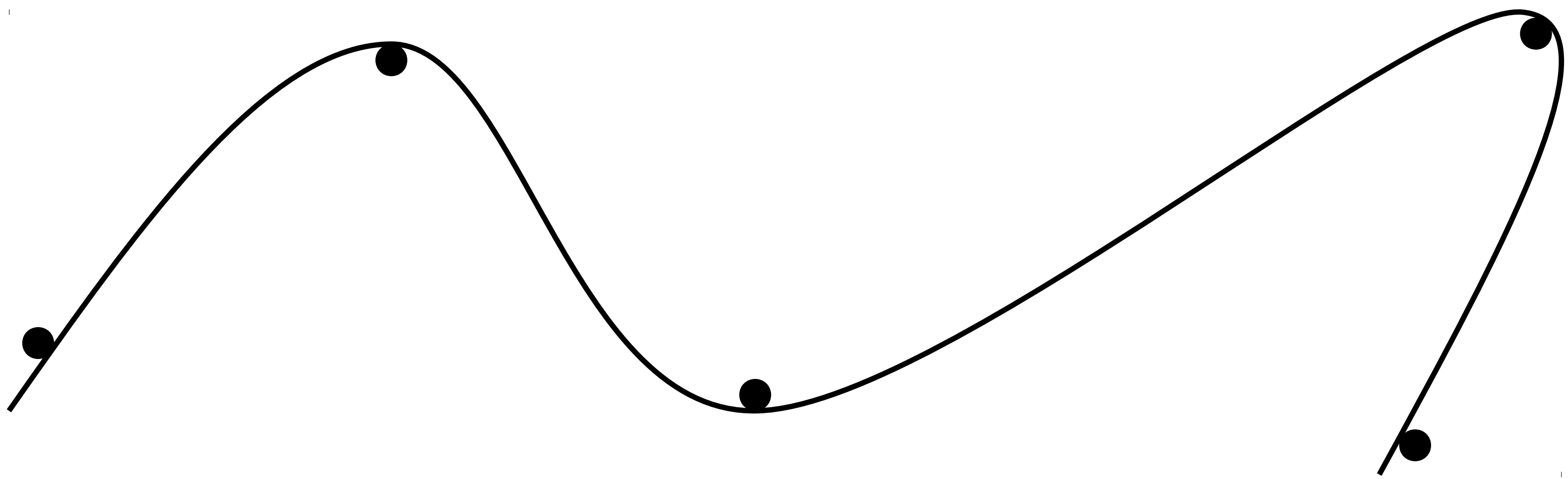
- In many applications we need smooth shapes
  - that is, without discontinuities

[Boeing]

- So far we can make
  - things with corners (lines, squares, rectangles, ...)
  - circles and ellipses (only get you so far!)

Classical approach

- Pencil-and-paper draftsmen also needed smooth curves
- Origin of “spline:” strip of flexible metal
  - held in place by pegs or weights to constrain shape
  - traced to produce smooth contour



Translating into usable math

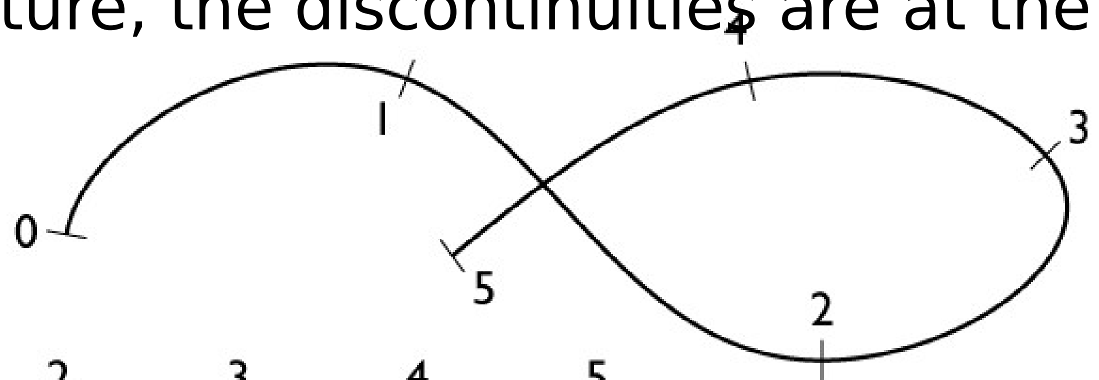
- Smoothness
  - in drafting spline, comes from physical curvature minimization
  - in CG spline, comes from choosing smooth functions
    - usually low-order polynomials 3차 다항식
- Control
  - in drafting spline, comes from fixed pegs
  - in CG spline, comes from user-specified *control points*

Defining spline curves

- At the most general they are parametric curves

$$S = \{\mathbf{p}(t) \mid t \in [0, N]\}$$

- Generally *f*(*t*) is a piecewise polynomial
  – for this lecture, the discontinuities are at the integers



시간개념

Defining spline curves

- Generally *f*(*t*) is a piecewise polynomial
  - for this lecture, the discontinuities are at the integers
  - e.g., a cubic spline has the following form over [k, k + 1]:

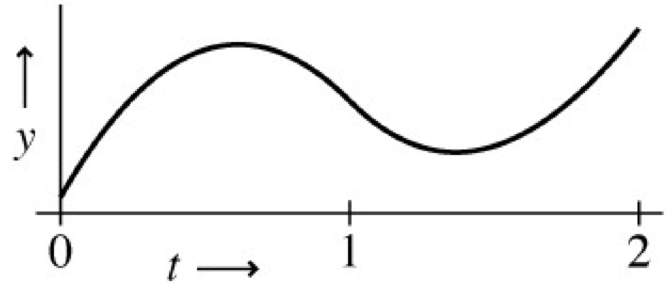
$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

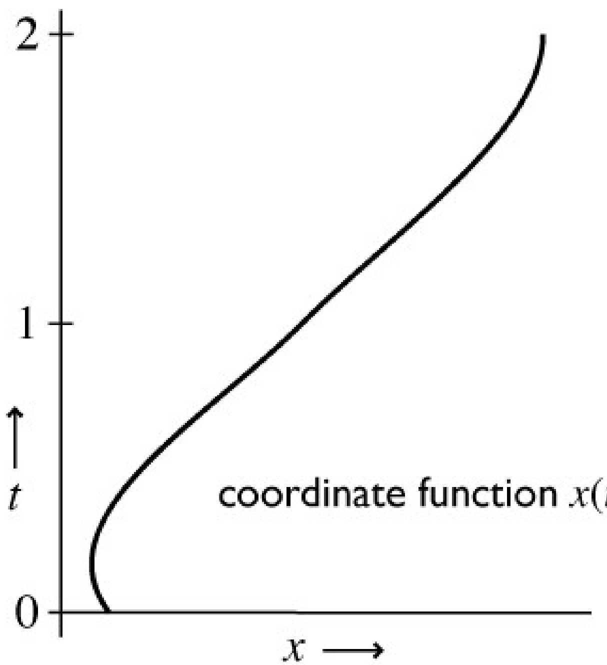
  - Coefficients are different for every interval

Coordinate functions

2D spline

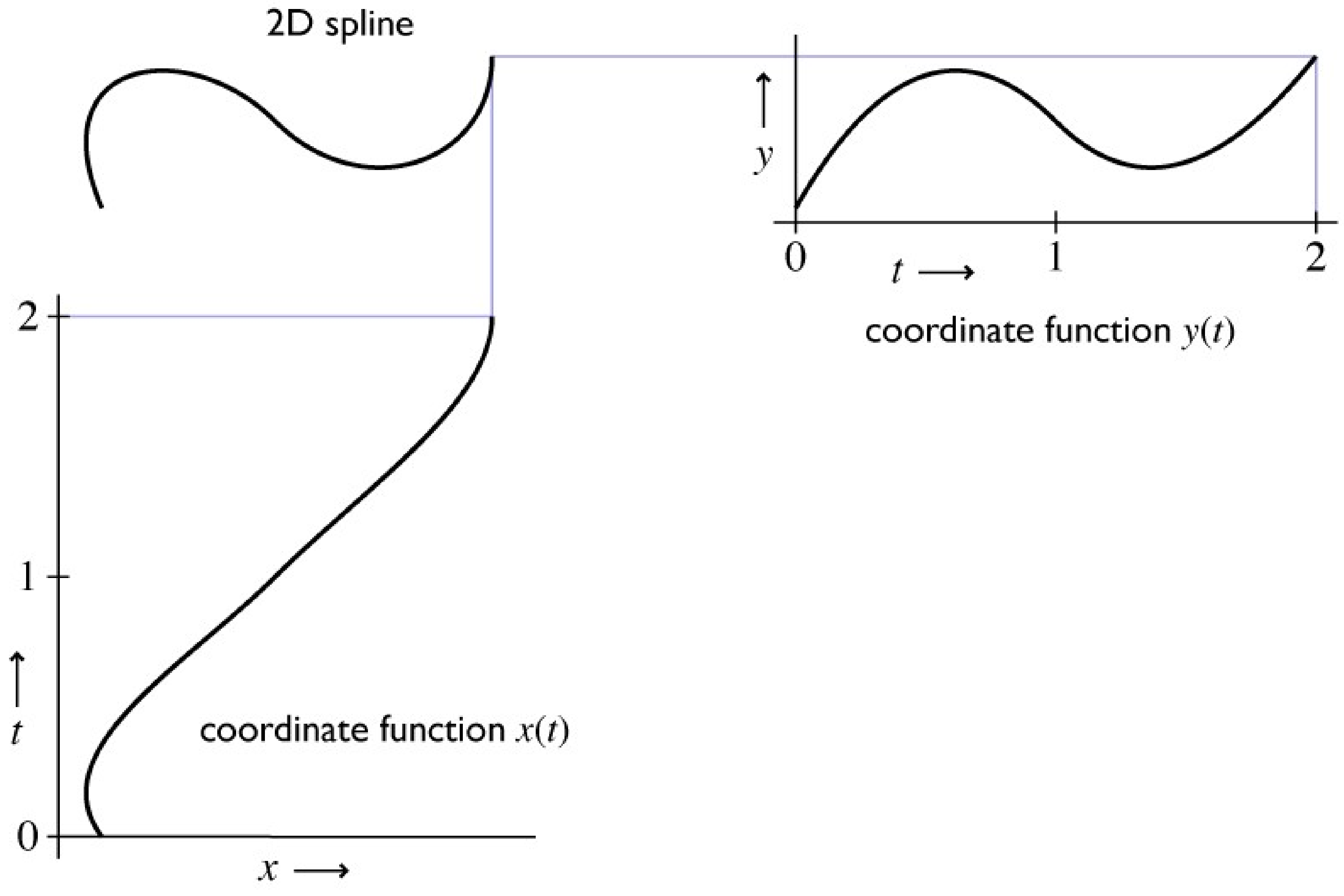


coordinate function $y(t)$



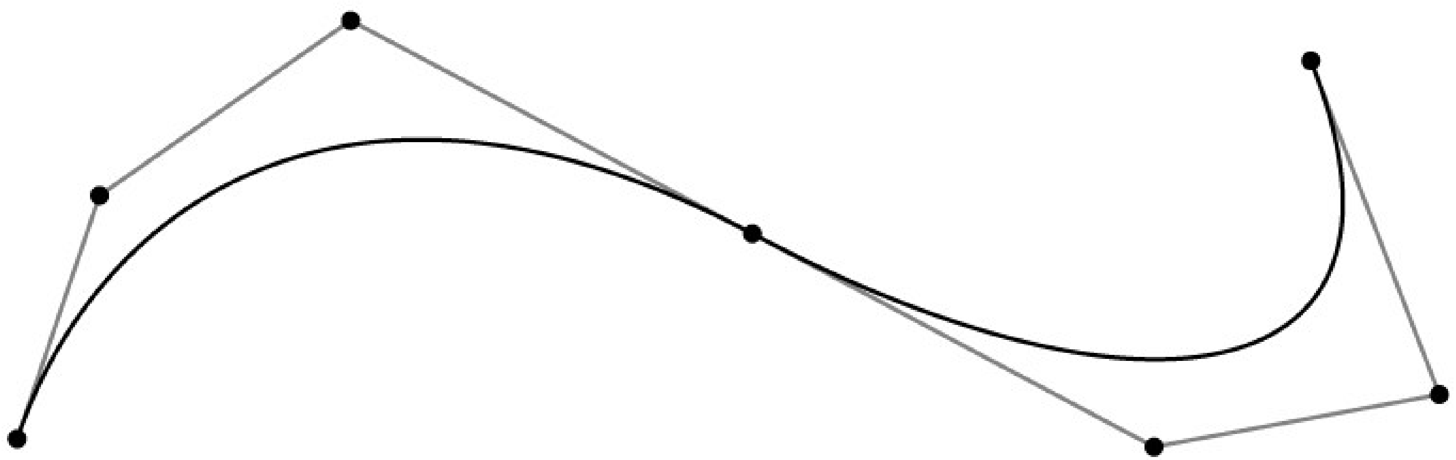
coordinate function $x(t)$

Coordinate functions



Control of spline curves

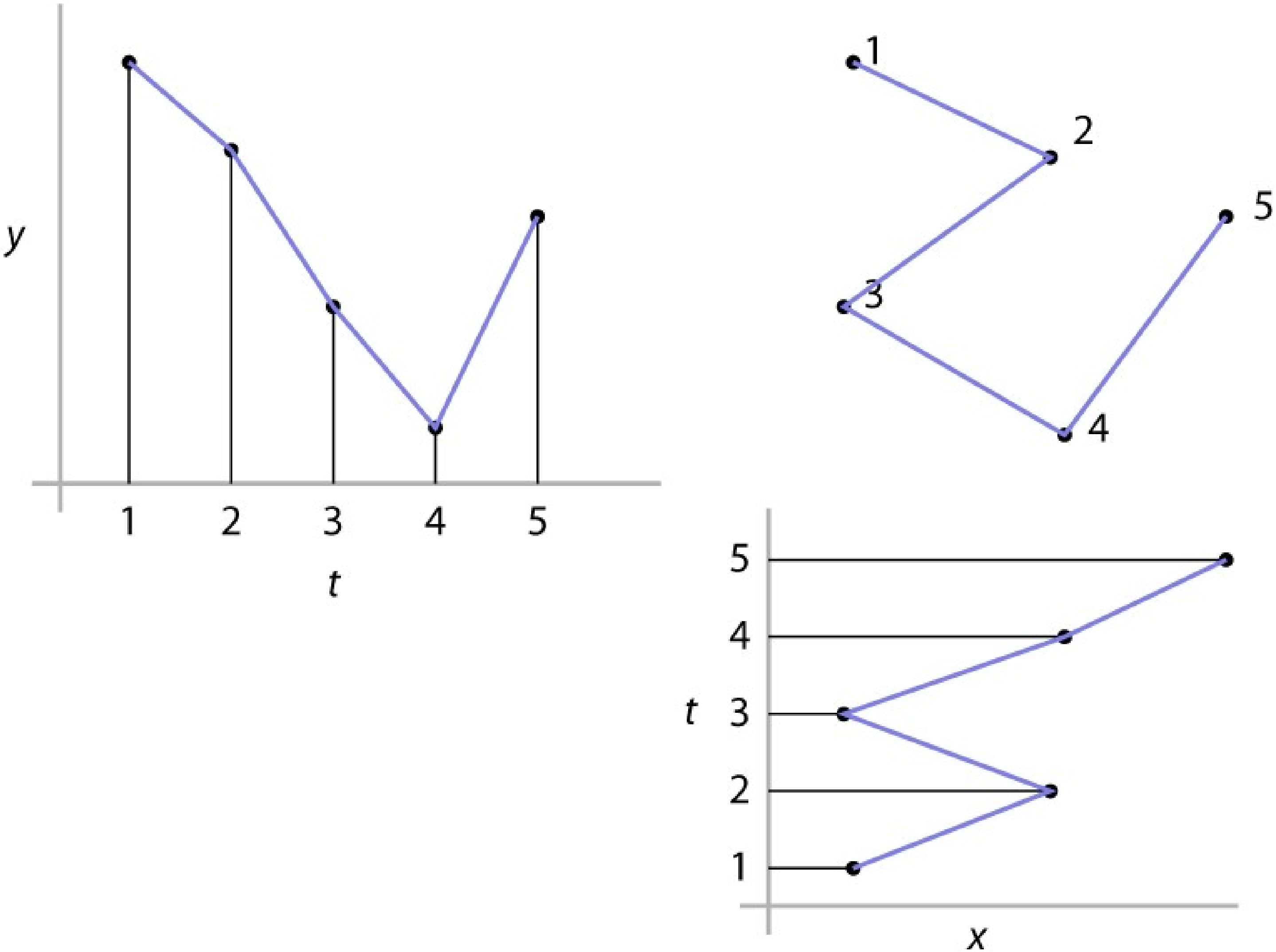
- Specified by a sequence of control points
- Shape is guided by control points (aka control polygon)
  - interpolating: passes through points
  - approximating: merely guided by points



How splines depend on their controls

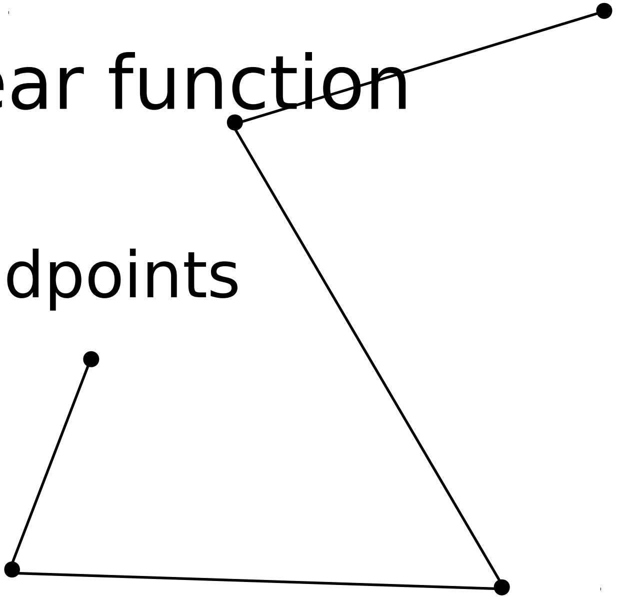
- Each coordinate is separate
  - the function $x(t)$ is determined solely by the x coordinates of the control points
  - this means 1D, 2D, 3D, … curves are all really the same
- Spline curves are **linear** functions of their controls
  - moving a control point two inches to the right moves $x(t)$ twice as far as moving it by one inch
  - $x(t)$, for fixed t, is a linear combination (weighted sum) of the control points' x coordinates
  - $\mathbf{p}(t)$, for fixed t, is a linear combination (weighted sum) of the control points

Splines as reconstruction



Trivial example: piecewise linear

- This spline is just a polygon
  - control points are the vertices
- But we can derive it anyway as an illustration
- Each interval will be a linear function
  - $x(t) = at + b$
  - constraints are values at endpoints
  - $b = x_0$; $a = x_1 - x_0$
  - this is linear interpolation



Trivial example: piecewise linear

- Vector formulation

$$x(t) = (x_1 - x_0)t + x_0$$

$$y(t) = (y_1 - y_0)t + y_0$$

$$\mathbf{p}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

- Matrix formulation

t // coef // control point

$$\mathbf{p}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

- Basis function formulation
  - regroup expression by **p** rather than *t*

$$\begin{aligned}\mathbf{p}(t) &= (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 \\ &= (1-t)\mathbf{p}_0 + t\mathbf{p}_1\end{aligned}$$

  - interpretation in matrix viewpoint

$$\mathbf{p}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

- Basis function formulation
  - regroup expression by **p** rather than *t*

$$\mathbf{p}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$
$$= (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\boldsymbol{p}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}$$

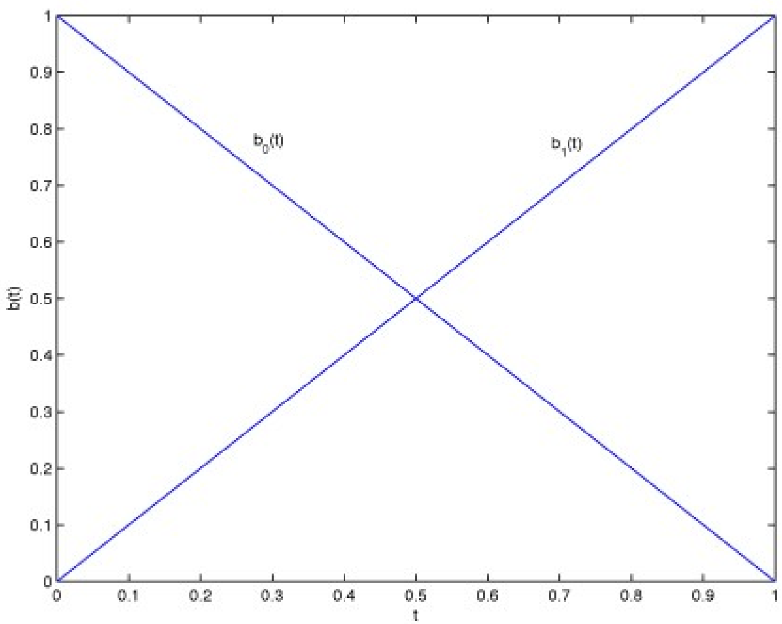
  - interpretation in matrix viewpoint $\begin{bmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \end{bmatrix} = \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix}$

$$\mathbf{p}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

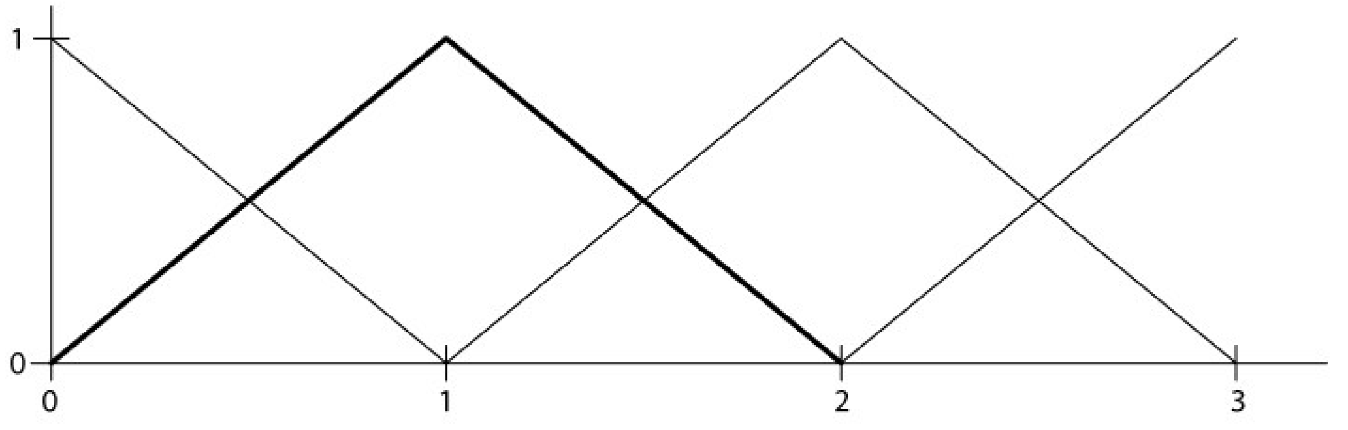
- Vector blending formulation: “average of points”
  - blending functions: contribution of each point as *t* changes

control point 의 가중치 (weight)



Trivial example: piecewise linear

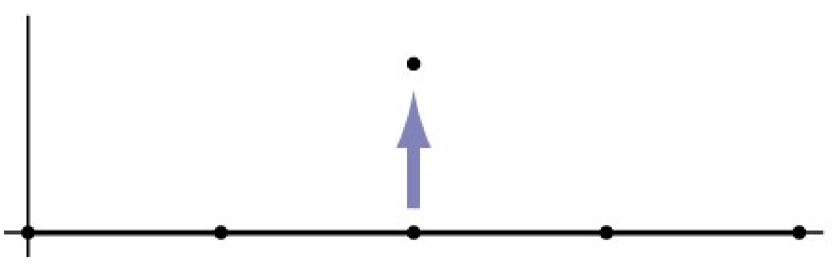
- Basis function formulation: “function times point”
  - basis functions: contribution of each point as *t* changes

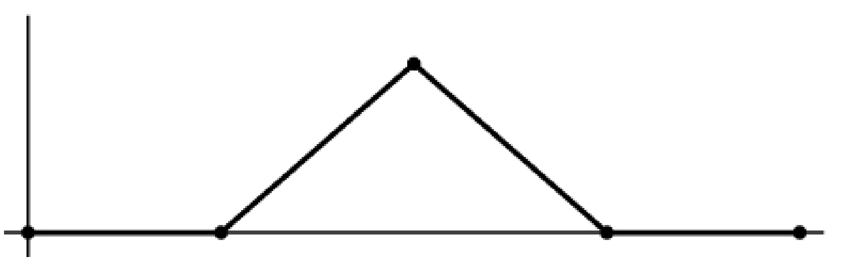


  - can think of them as blending functions glued together
  - this is just like a reconstruction filter!

Seeing the basis functions

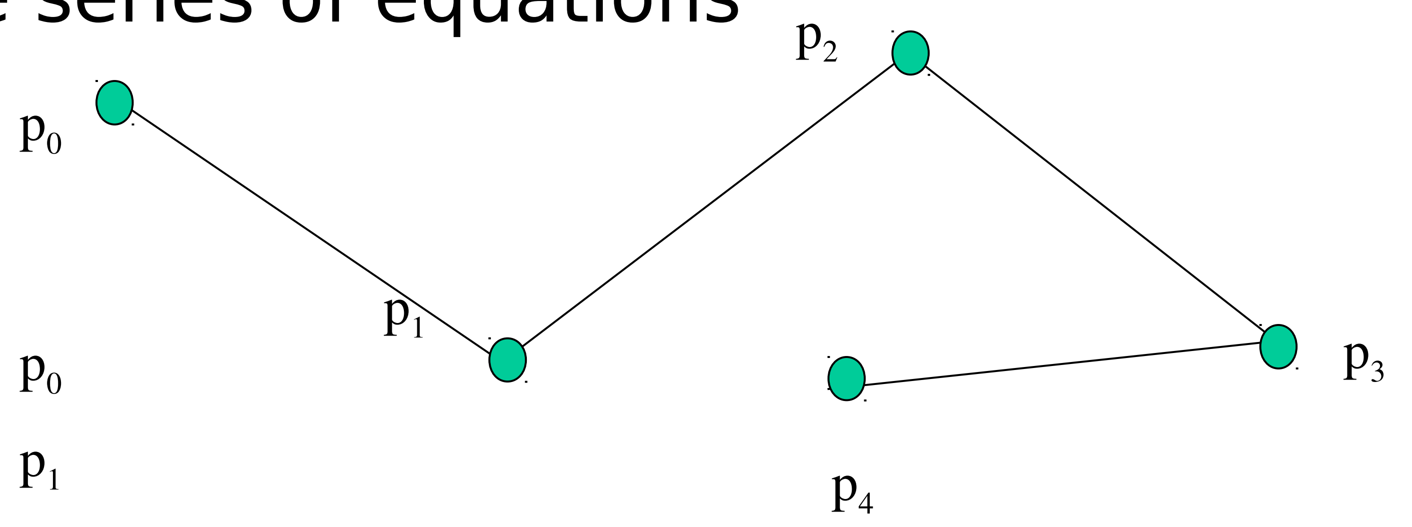
- Basis functions of a spline are revealed by how the curve changes in response to a change in one control
  - to get a graph of the basis function, start with the curve laid out in a straight, constant-speed line
    - what are $x(t)$ and $y(t)$?
  - then move one control straight up

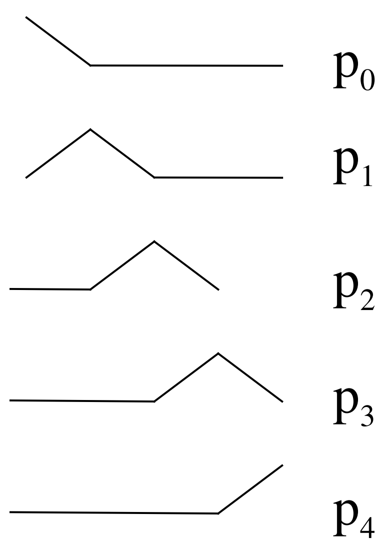




Piece these together

Create series of equations



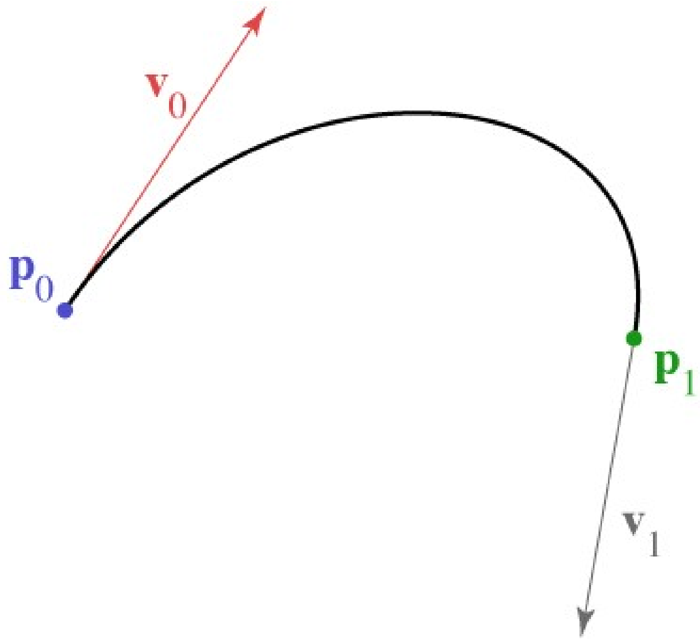


Blending Functions

$$(x,y)=\begin{cases} 0\le t<1, & (1-t)\,p_0+tp_1 \\ 1\le t<2, & (2-t)\,p_1+(t-1)\,p_2 \\ 2\le t<3, & (3-t)\,p_2+(t-2)\,p_3 \\ 3\le t<4, & (4-t)\,p_3+(t-3)\,p_4 \\ 4\le t<5, & (5-t)\,p_4+(t-4)\,p_5 \end{cases}$$

Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
- Constraints: endpoints and tangents (derivatives)



Hermite splines

- Solve constraints to find coefficients

$$x(t) = at^3 + bt^2 + ct + d$$
$$x'(t) = 3at^2 + 2bt + c$$
$$x(0) = x_0 = d$$
$$x(1) = x_1 = a + b + c + d$$
$$x'(0) = x'_0 = c$$
$$x'(1) = x'_1 = 3a + 2b + c$$

미지수 // 입력값(아는값)

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix}$$

Hermite splines

- Solve constraints to find coefficients

$$x(t) = at^3 + bt^2 + ct + d$$

$$x'(t) = 3at^2 + 2bt + c$$

$$x(0) = x_0 = d$$

$$x(1) = x_1 = a + b + c + d$$

$$x'(0) = x'_0 = c$$

$$x'(1) = x'_1 = 3a + 2b + c$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix}$$

calculate A^{-1}

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix}$$

$$d = x_0$$

$$c = x'_0$$

$$a = 2x_0 - 2x_1 + x'_0 + x'_1$$

$$b = -3x_0 + 3x_1 - 2x'_0 - x'_1$$

Hermite splines

- Matrix form is much simpler

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix}$$

  – coefficients = rows
  – basis functions = columns
    - note **p** columns sum to $[0\ 0\ 0\ 1]^T$

Hermite splines

$$\begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \end{bmatrix} = \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x'_0 & y'_0 \\ x'_1 & y'_1 \end{bmatrix}$$

- Matrix form is much simpler

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  - coefficients = rows
  - basis functions = columns
    - note **p** columns sum to [0 0 0 1]^T

A^{-1}

Coefficients = rows

$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

Basis functions=columns

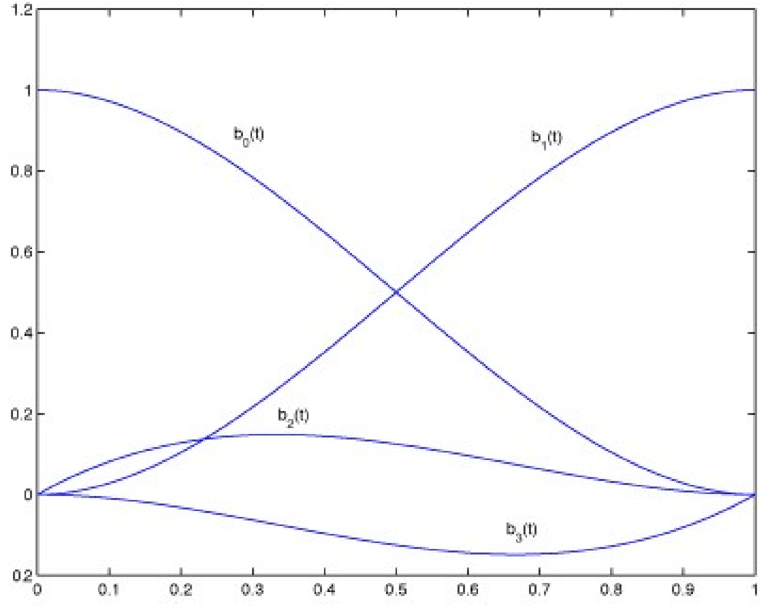
$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

Hermite splines

- Hermite blending functions



Longer Hermite splines

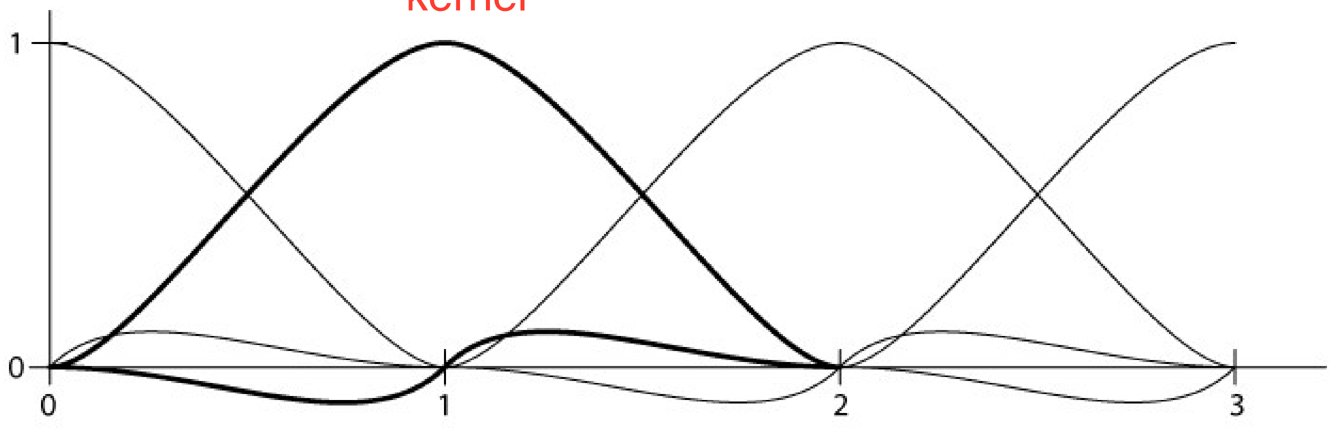
- Can only do so much with one Hermite spline
- Can use these splines as segments of a longer curve
  - curve from t = 0 to t = 1 defined by first segment
  - curve from t = 1 to t = 2 defined by second segment
- To avoid discontinuity, match derivatives at junctions
  - this produces a C^1 curve

Hermite splines

- Hermite basis functions

weight가 1이고 나머지 0이면 그 point 를지남

hermit
kernel

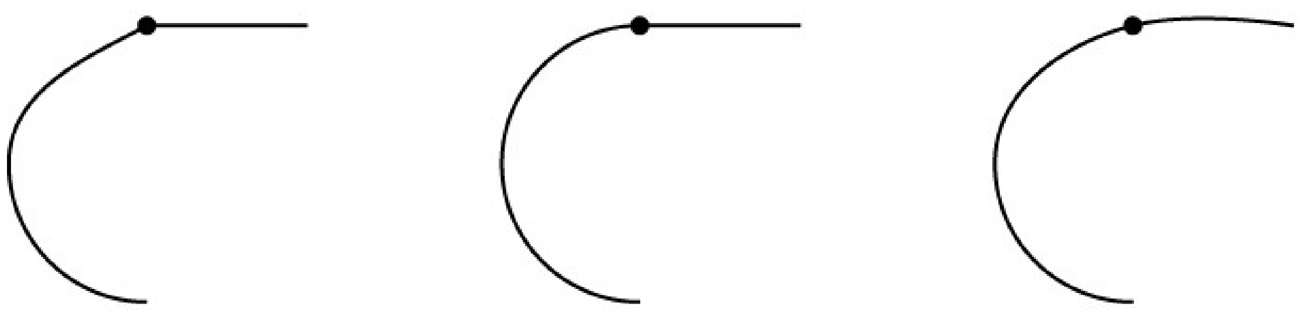


Continuity

- Smoothness can be described by degree of continuity
  - zero-order (C^0): position matches from both sides
  - first-order (C^1): tangent matches from both sides
  - second-order (C^2): curvature matches from both sides
  - G^n vs. C^n

변화율까지 동일(좌, 우 미분)

zero order first order second order

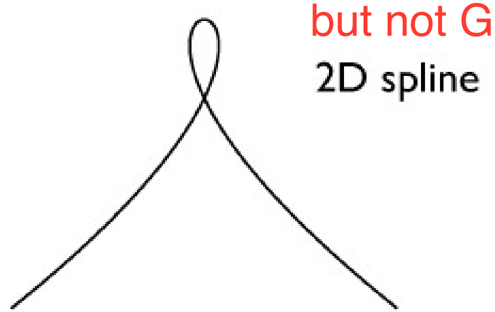


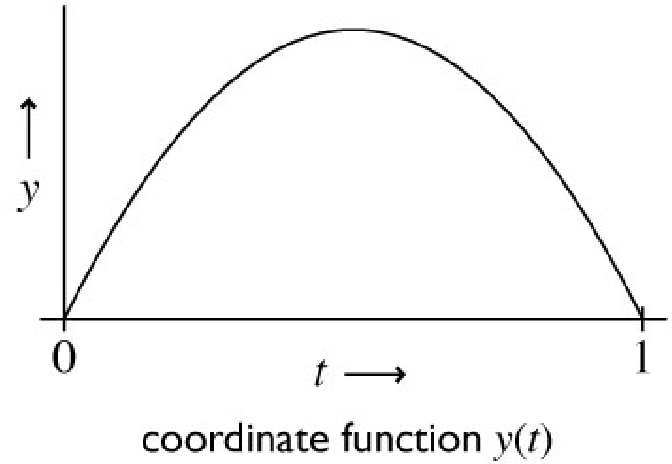
Continuity

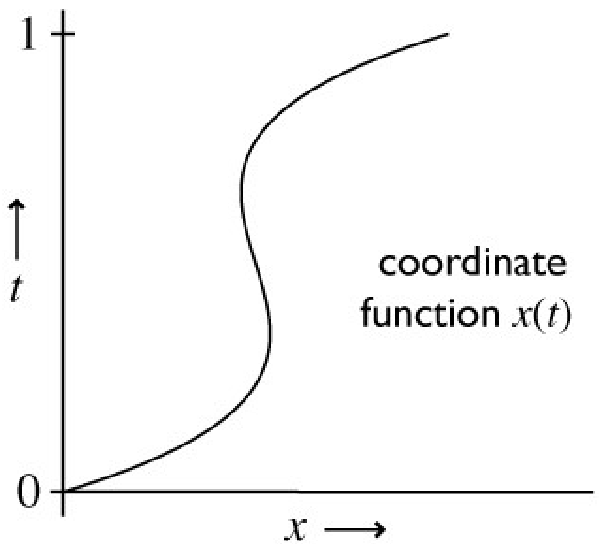
- Parametric continuity (C) of spline is continuity of coordinate functions
- Geometric continuity (G) is continuity of the curve itself
- Neither form of continuity is guaranteed by the other
  - Can be C^1 but not G^1 when $\mathbf{p}(t)$ comes to a halt (next slide)
  - Can be G^1 but not C^1 when the tangent vector changes length abruptly

Geometric vs. parametric continuity

C1
but not G1



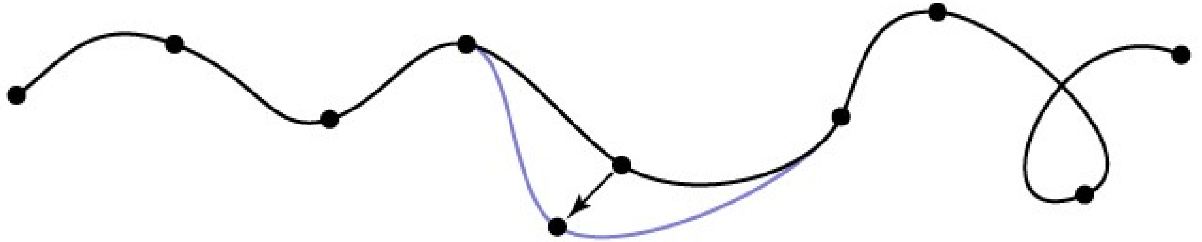




그림이 다르다, 위 쪽 동그라미 x

Control

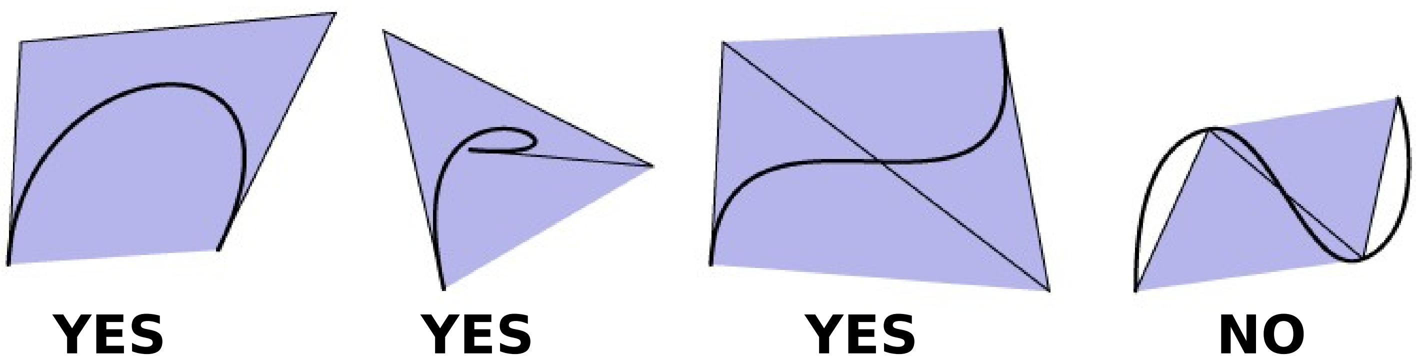
- Local control
  - changing control point only affects a limited part of spline
  - without this, splines are very difficult to use
  - many likely formulations lack this
    - natural spline
    - polynomial fits



Control

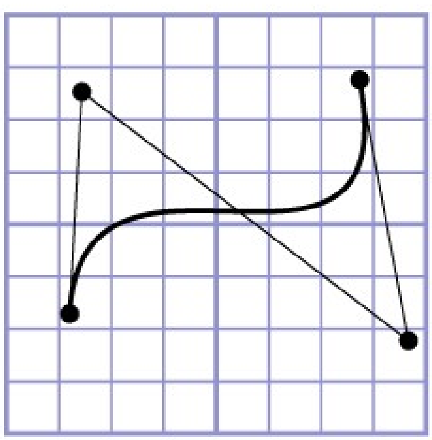
- Convex hull property
  - convex hull = smallest convex region containing points
    - think of a rubber band around some pins
  - some splines stay inside convex hull of control points

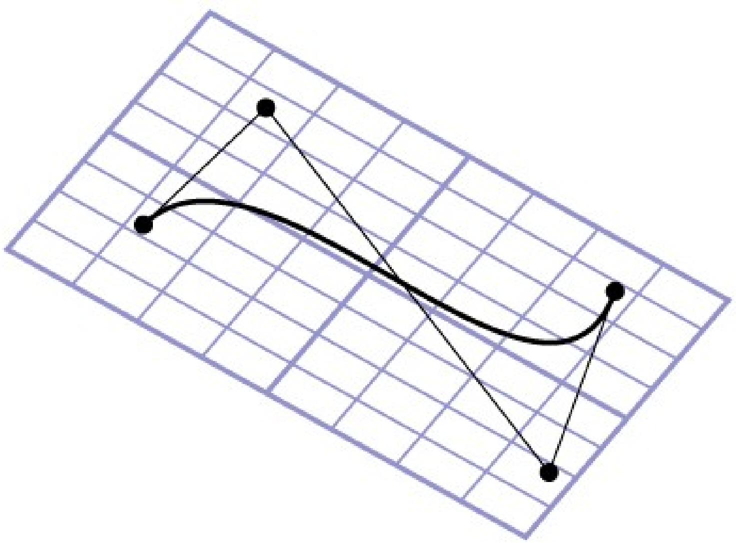
er



Affine invariance

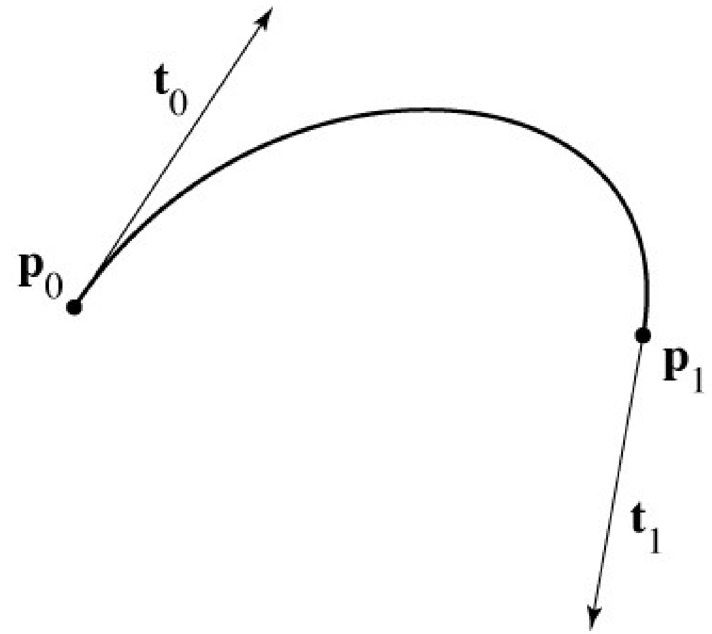
- Transforming the control points is the same as transforming the curve
  - true for all commonly used splines
  - extremely convenient in practice...





Hermite splines

- Constraints are endpoints and endpoint tangents



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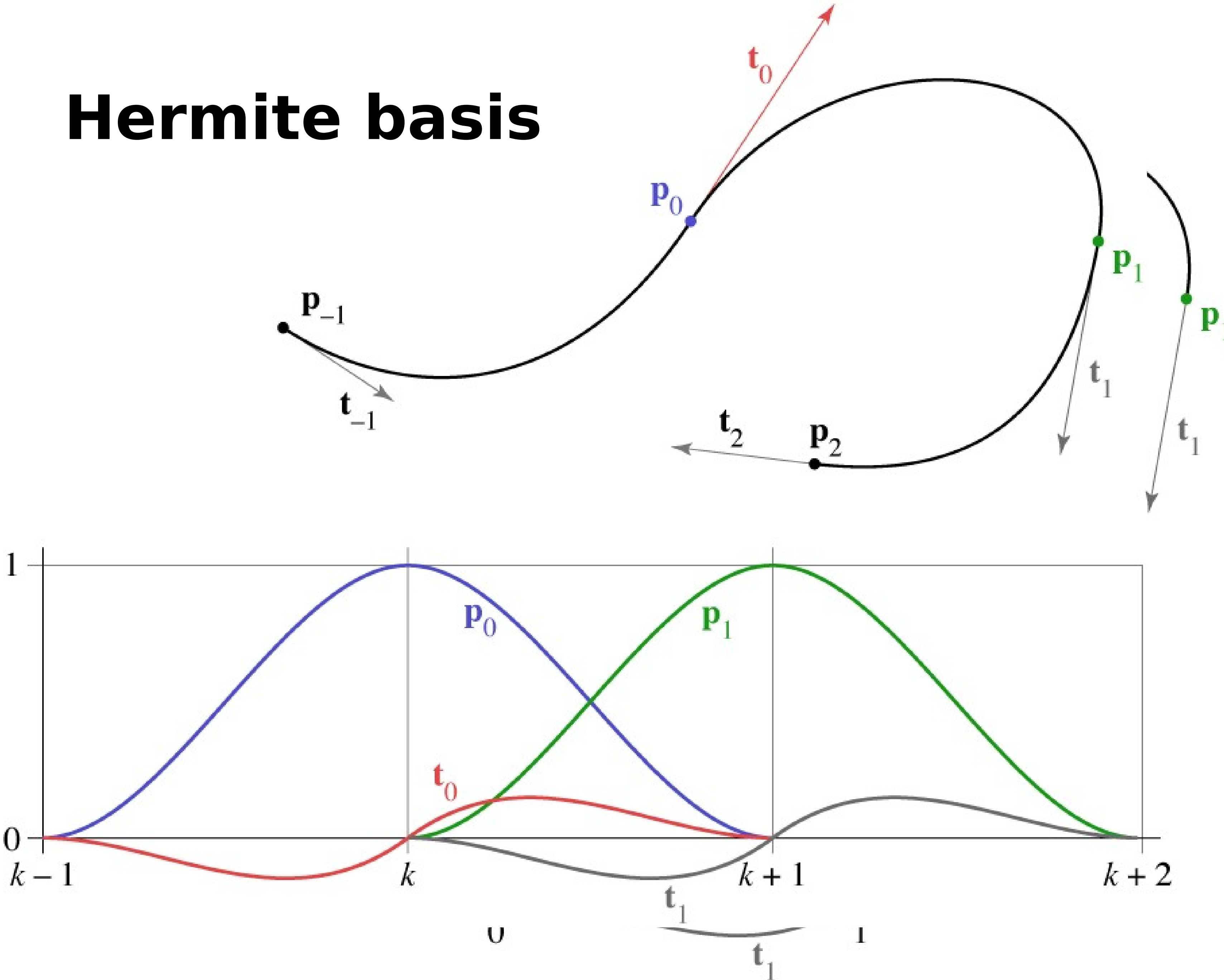
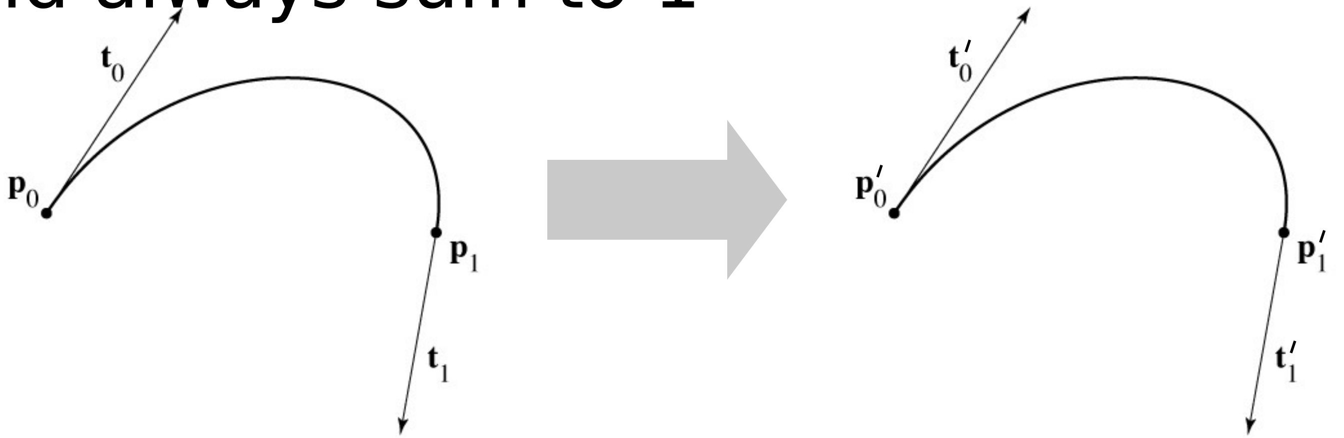
Hermite basis



Affine invariance

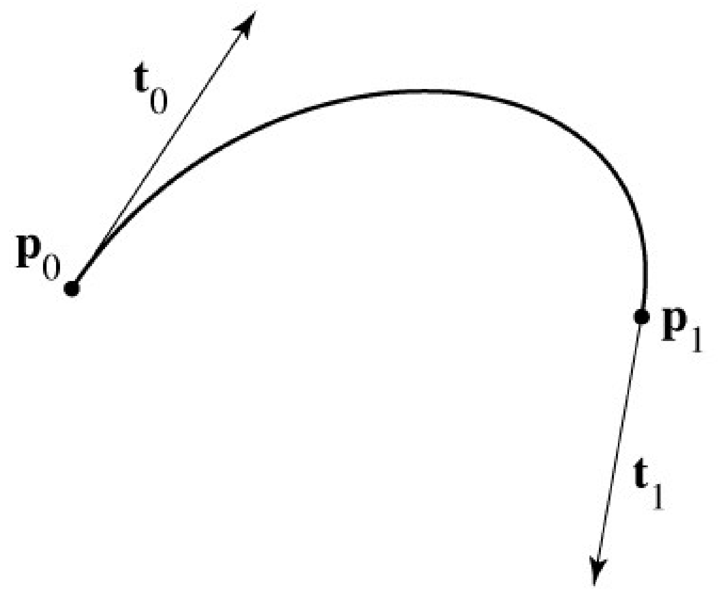
- Basis functions associated with points should always sum to 1



$$
\begin{aligned}
\mathbf{p}(t) &= b_0\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 \\
\mathbf{p}'(t) &= b_0(\mathbf{p}_0 + \mathbf{u}) + b_1(\mathbf{p}_1 + \mathbf{u}) + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 \\
&= b_0\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{v}_0 + b_3\mathbf{v}_1 + (b_0 + b_1)\mathbf{u} \\
&= \mathbf{p}(t) + \mathbf{u}
\end{aligned}
$$

Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points



Hermite to Bézier

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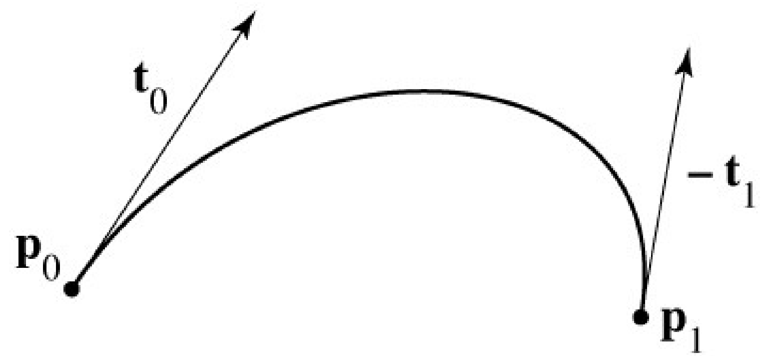

Hermite to Bézier

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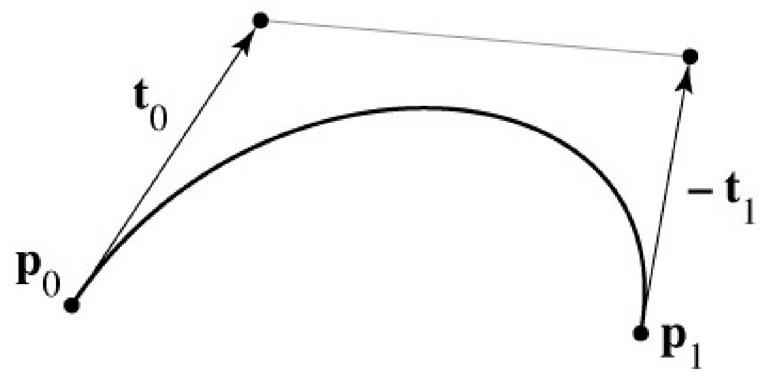

Hermite to Bézier

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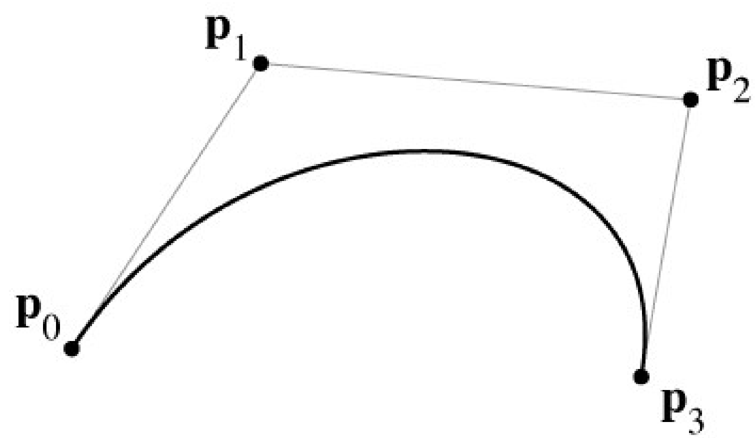

Hermite to Bézier

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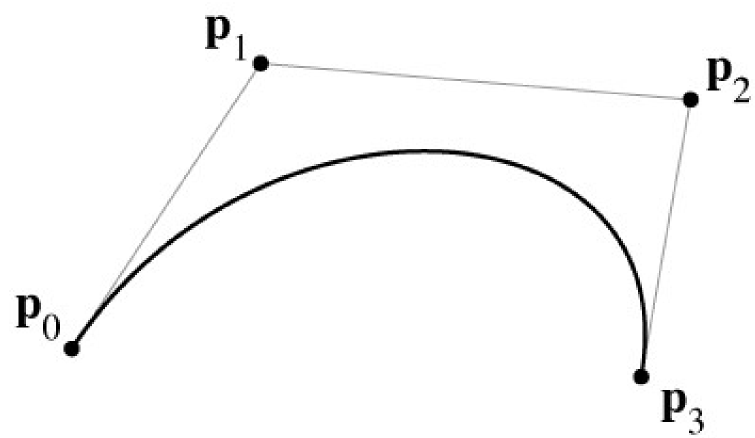
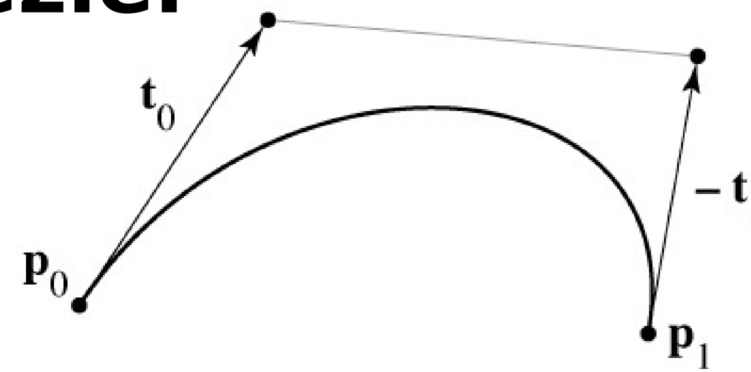


– note derivative is defined as 3 times offset t

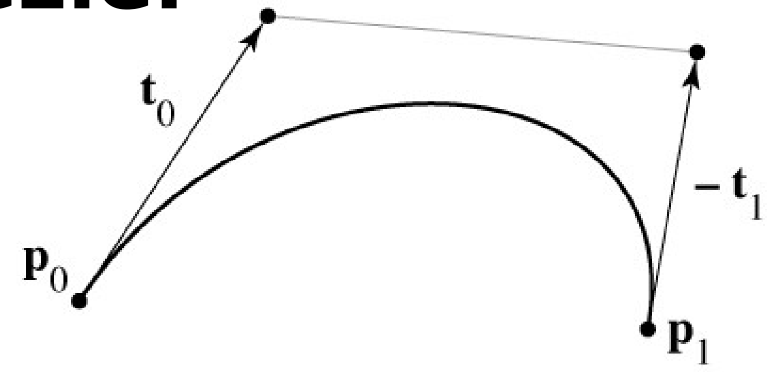
- reason is illustrated by linear case

Hermite to Bézier



$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$

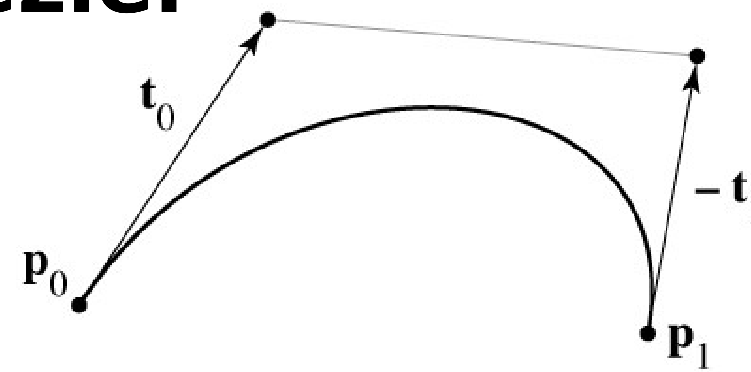
Hermite to Bézier



$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{v}_0 \\ \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Hermite to Bézier



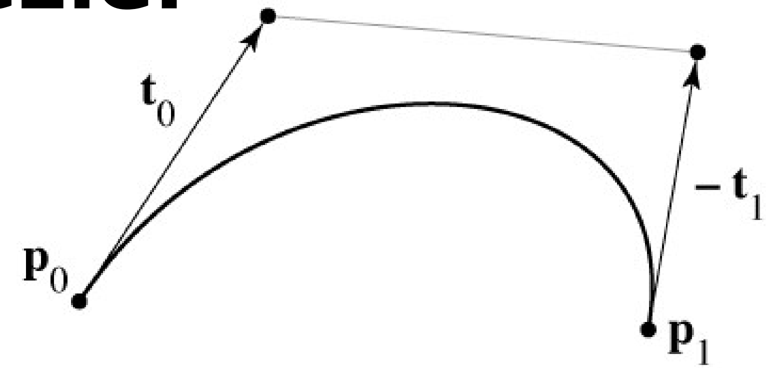
$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Hermite matrix

Hermite to Bézier

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$



$$\mathbf{p}_0 = \mathbf{q}_0$$

$$\mathbf{p}_1 = \mathbf{q}_3$$

$$\mathbf{v}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$

$$\mathbf{v}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Bézier matrix

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

– note that these are the Bernstein polynomials

$$C(n,k)\ t^k\ (1 - t)^{n-k}$$

and that defines Bézier curves for any degree

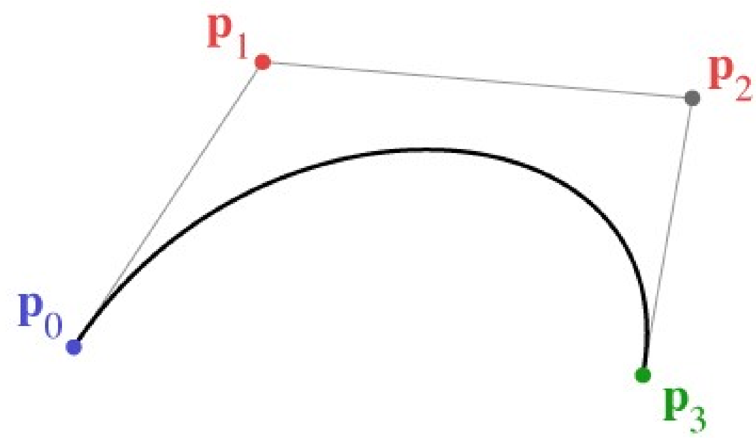
Apply Constraint Matrix

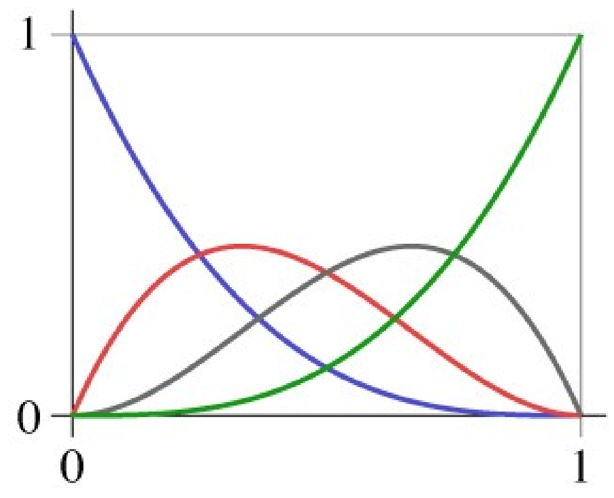
$$\begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\begin{bmatrix} (1-u)^3 & 3u(u-1)^2 & 3u^2(u-1) & u^3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

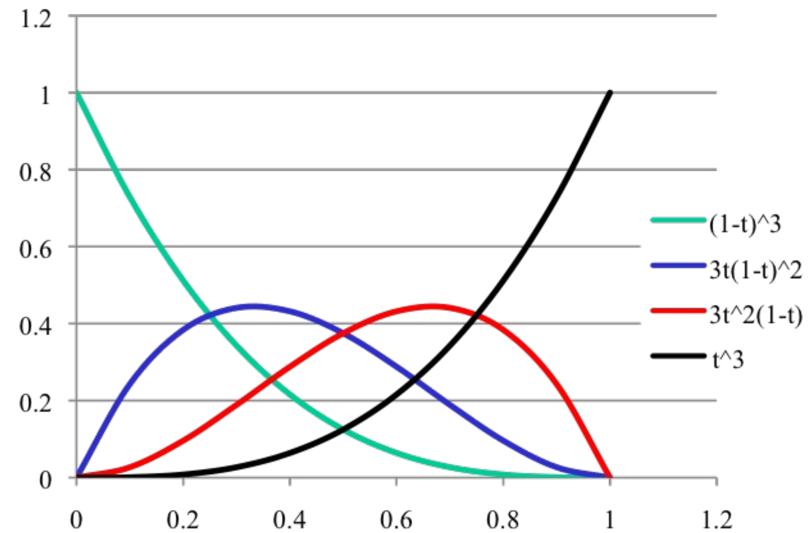
$$(1-u)^3 p_0 + 3u(1-u)^2 p_1 + 3u^2(1-u) p_2 + u^3 p_3$$

Bézier basis





Bezier Polynomials sum to one



$$(1-u)+u=1$$

$$((1-u)+u)=1$$

$$((1-u)+u)^3=1$$

$$(1-u)^3+3u(1-u)^2+3u^2(1-u)+u^3=1$$

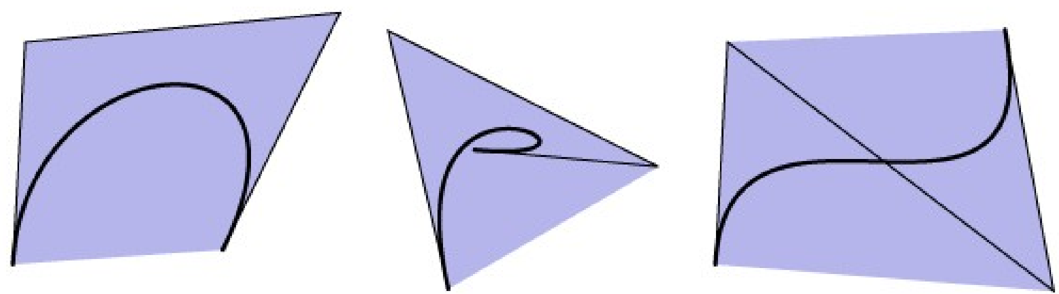
So each point on the curve is a convex sum of the control points

Thus the curve lies inside the convex hull of the control points

Convex hull

- If basis functions are all positive, the spline has the convex hull property
  - we’re still requiring them to sum to 1

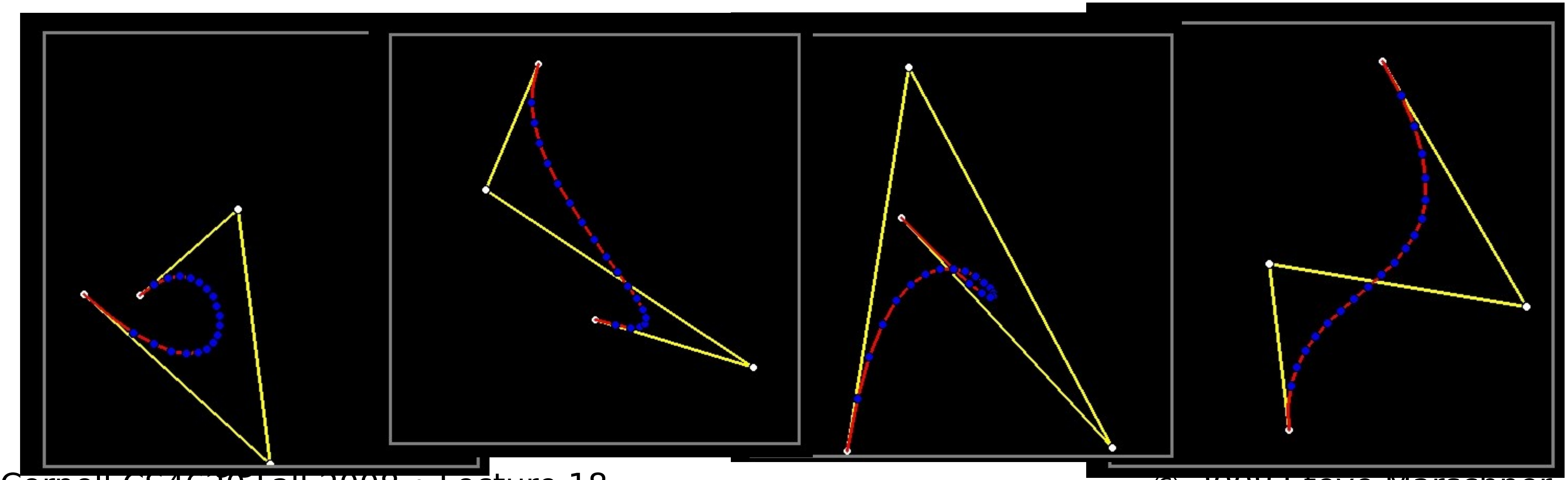
Client click check



  - if any basis function is ever negative, no convex hull prop.
    - proof: take the other three points at the same place

Convex Hull

Check that the curve remains inside the convex hull of the control points in our examples

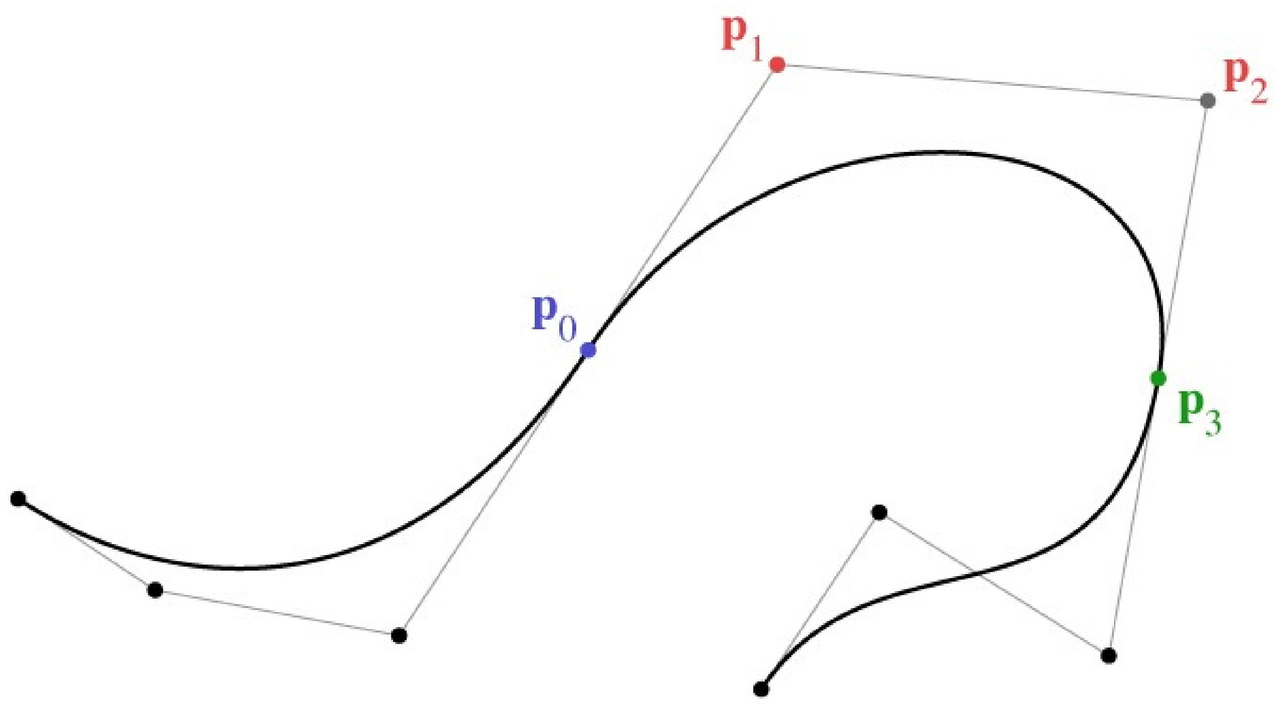


Chaining spline segments

- Hermite curves are convenient because they can be made long easily
- Bézier curves are convenient because their controls are all points and they have nice properties
  - and they interpolate every 4th point, which is a little odd
- We derived Bézier from Hermite by defining tangents from control points
  - a similar construction leads to the interpolating *Catmull-Rom* spline

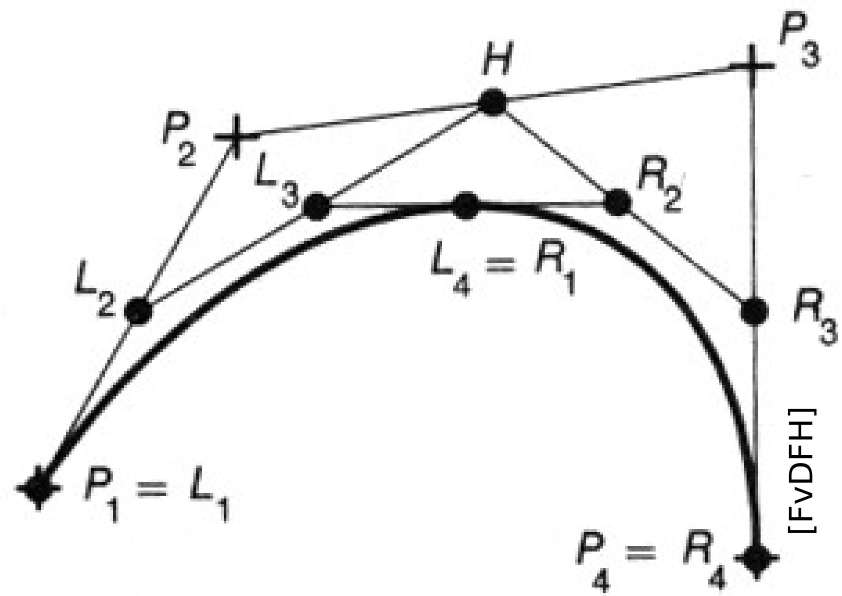
Chaining Bézier splines

- No continuity built in
- Achieve C^1 using collinear control points



Subdivision

- A Bézier spline segment can be split into a two-segment curve:



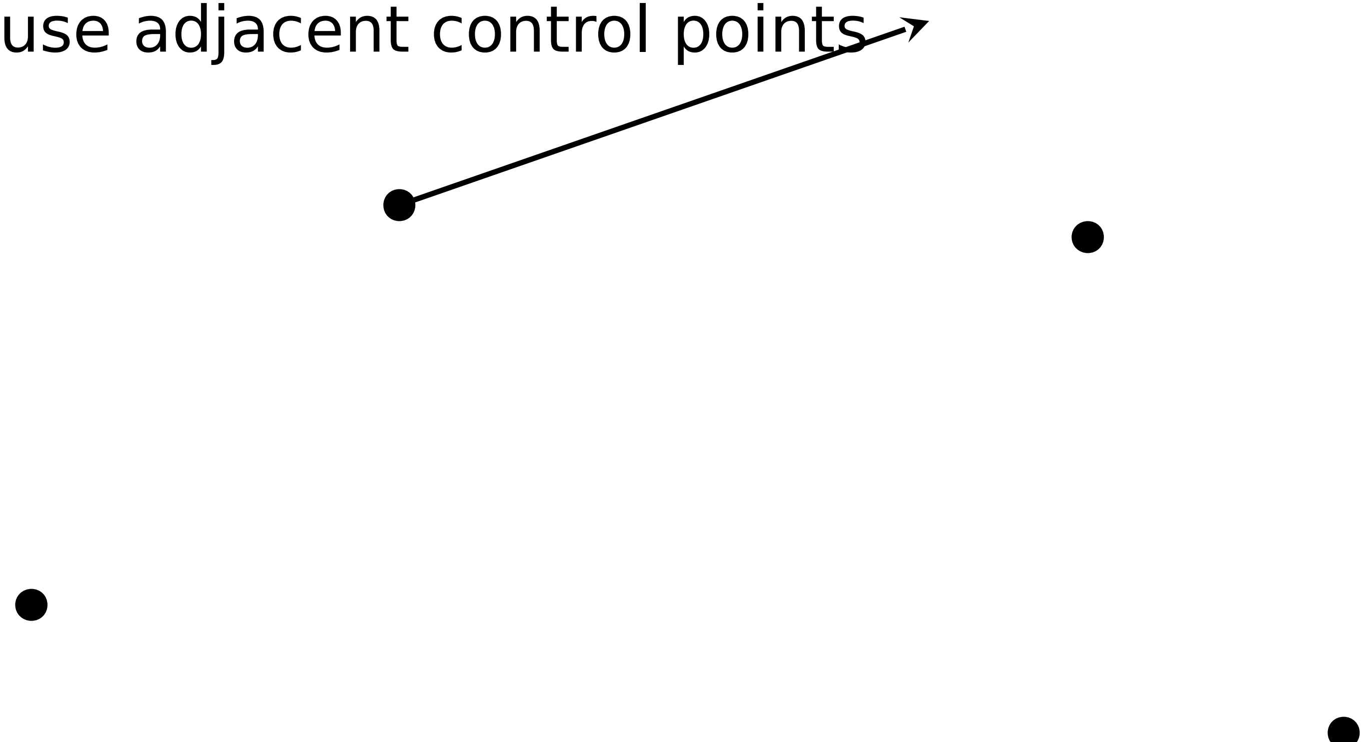
  – de Casteljau’s algorithm
  – also works for arbitrary *t*

Cubic Bézier splines

- Very widely used type, especially in 2D
  - e.g. it is a primitive in PostScript/PDF
- Can represent C^1 and/or G^1 curves with corners
- Can easily add points at any position
- Illustrator demo

Hermite to Catmull-Rom

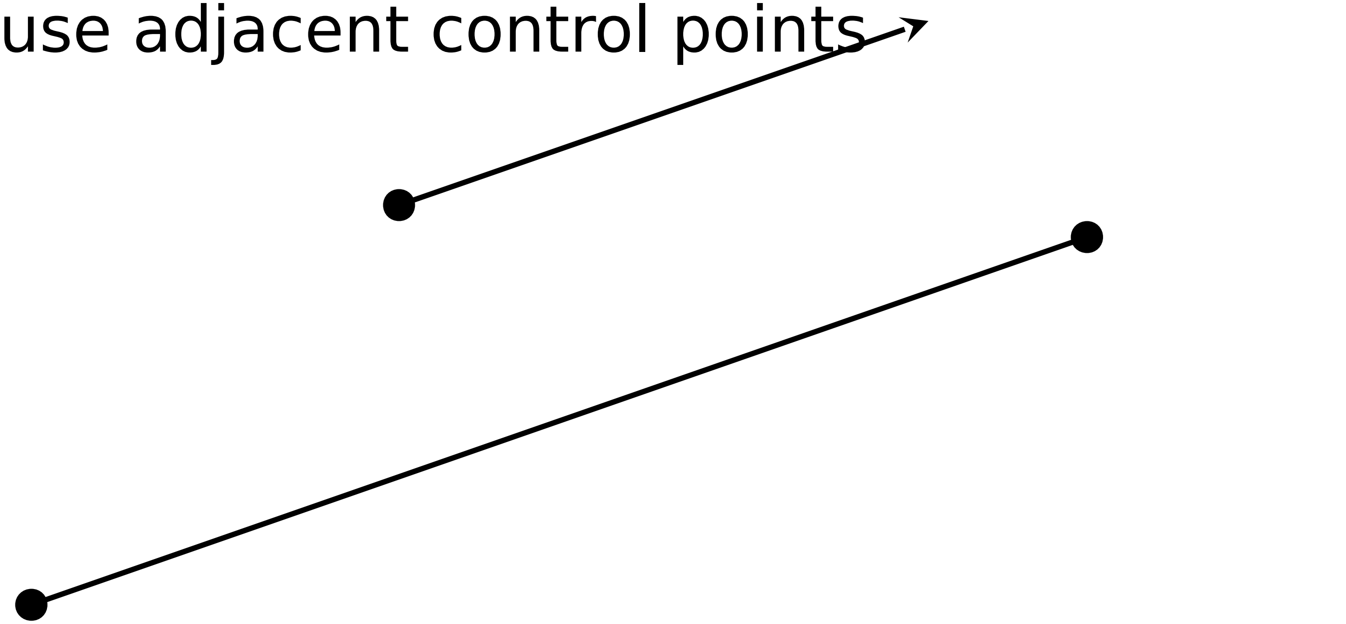
- Have not yet seen any interpolating splines
- Would like to define tangents automatically
  - use adjacent control points



  - end tangents: extra points or zero

Hermite to Catmull-Rom

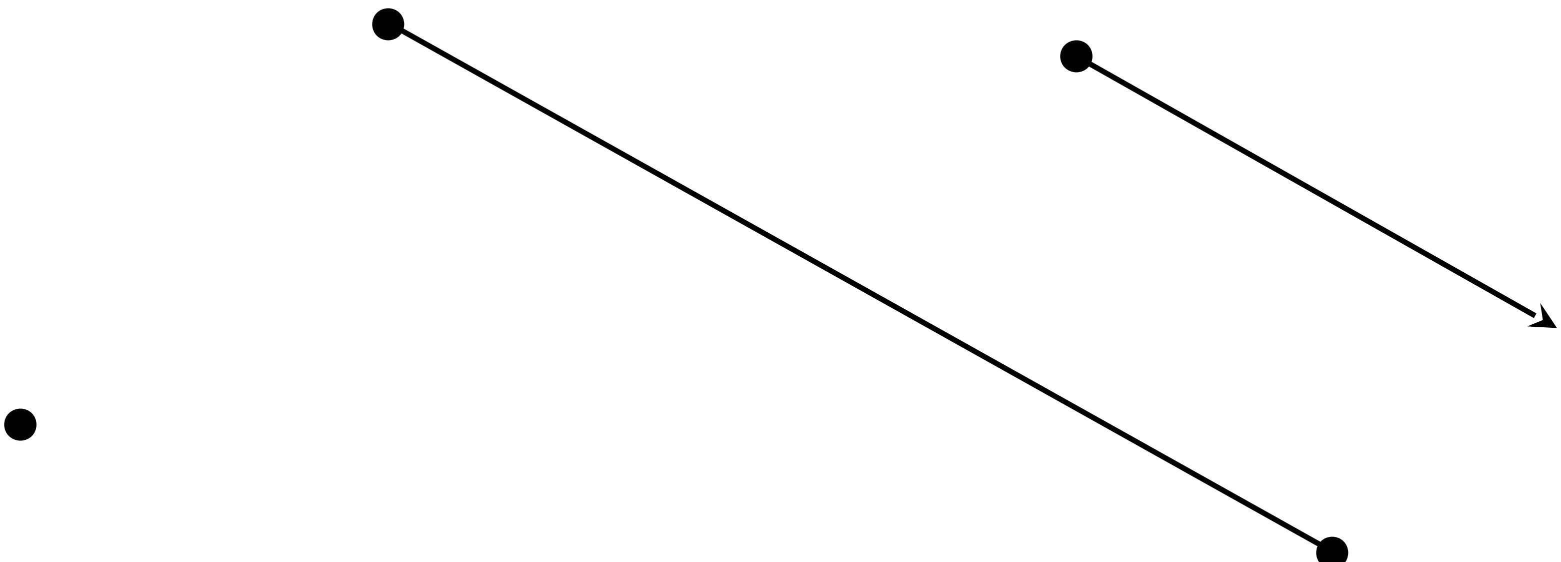
- Have not yet seen any interpolating splines
- Would like to define tangents automatically
  - use adjacent control points



  - end tangents: extra points or zero

Hermite to Catmull-Rom

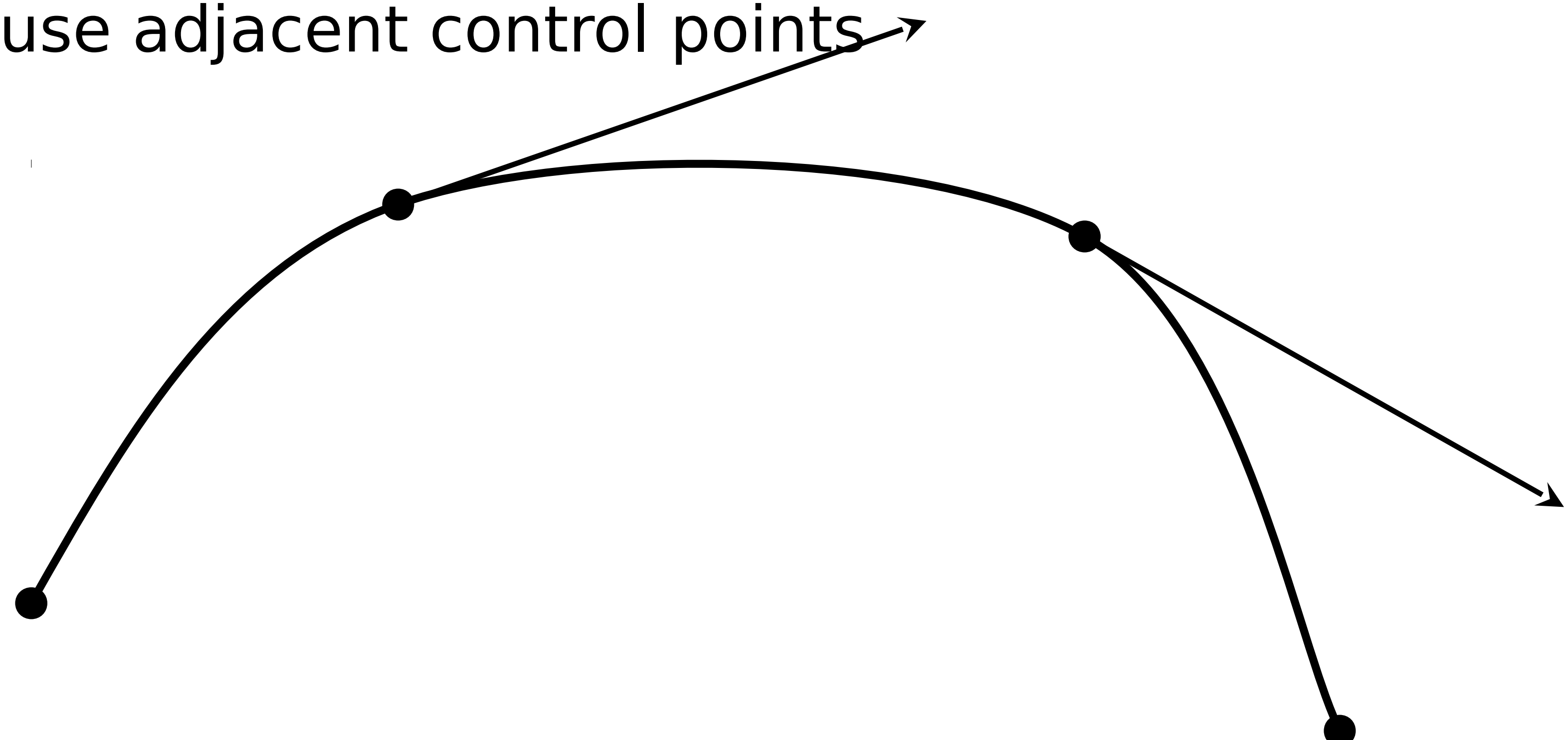
- Have not yet seen any interpolating splines
- Would like to define tangents automatically
  - use adjacent control points



  - end tangents: extra points or zero

Hermite to Catmull-Rom

- Have not yet seen any interpolating splines
- Would like to define tangents automatically
  - use adjacent control points



  - end tangents: extra points or zero

Hermite to Catmull-Rom

- Tangents are $(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}) / 2$
  - scaling based on same argument about collinear case

$$\mathbf{p}_0 = \mathbf{q}_k$$

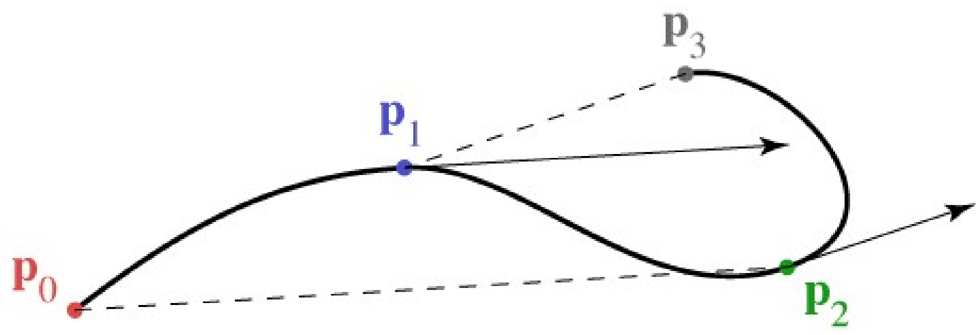
$$\mathbf{p}_1 = \mathbf{q}_k + 1$$

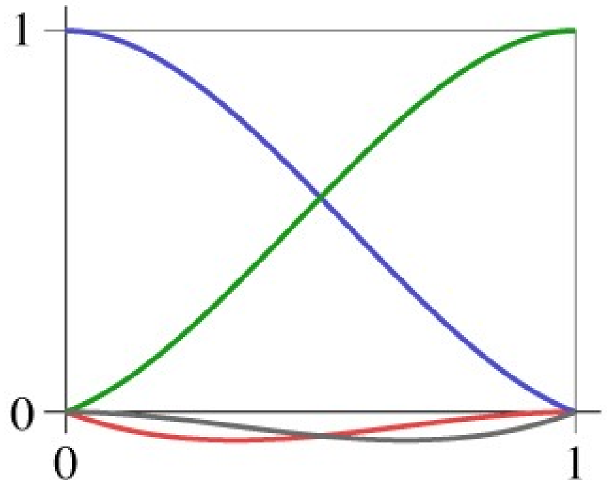
$$\mathbf{v}_0 = 0.5(\mathbf{q}_{k+1} - \mathbf{q}_{k-1})$$

$$\mathbf{v}_1 = 0.5(\mathbf{q}_{k+2} - \mathbf{q}_K)$$

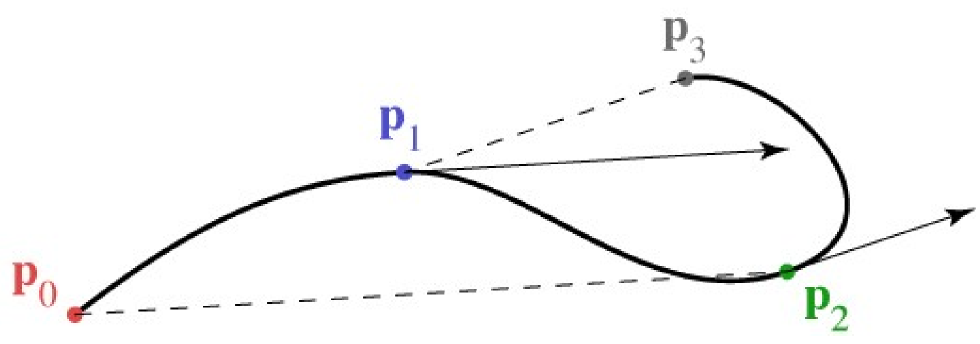
$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{d}\end{bmatrix} = \begin{bmatrix}2 & -2 & 1 & 1\\ -3 & 3 & -2 & -1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -.5 & 0 & .5 & 0\\ 0 & -.5 & 0 & .5\end{bmatrix} \begin{bmatrix}\mathbf{q}_{k-1}\\ \mathbf{q}_k\\ \mathbf{q}_{k+1}\\ \mathbf{q}_{k+2}\end{bmatrix}$$

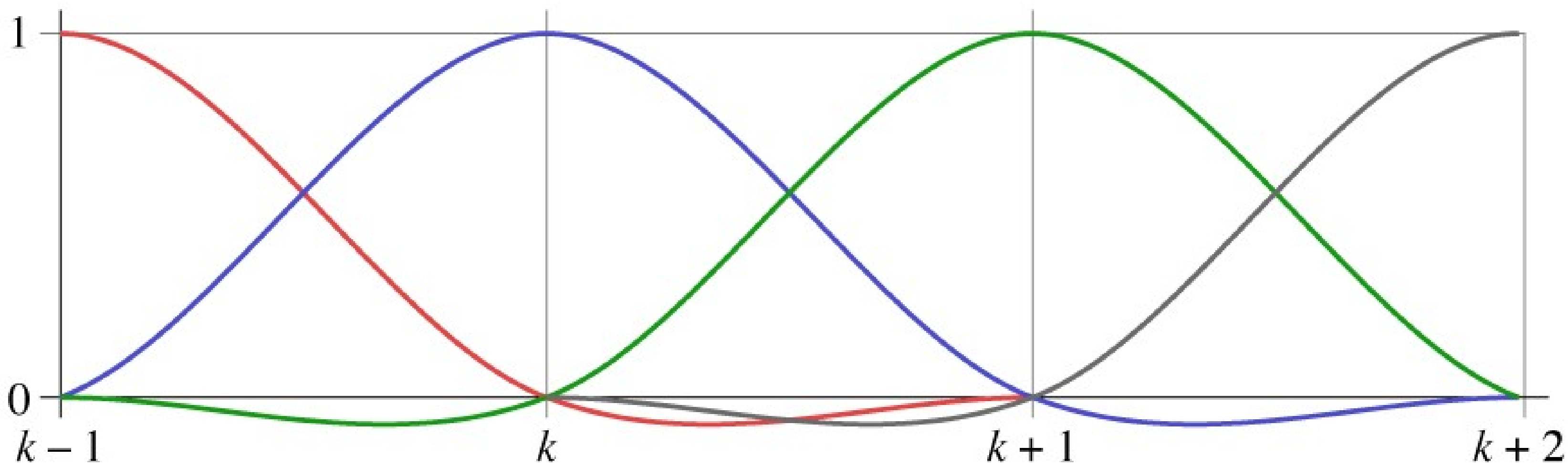
Catmull-Rom basis





Catmull-Rom basis



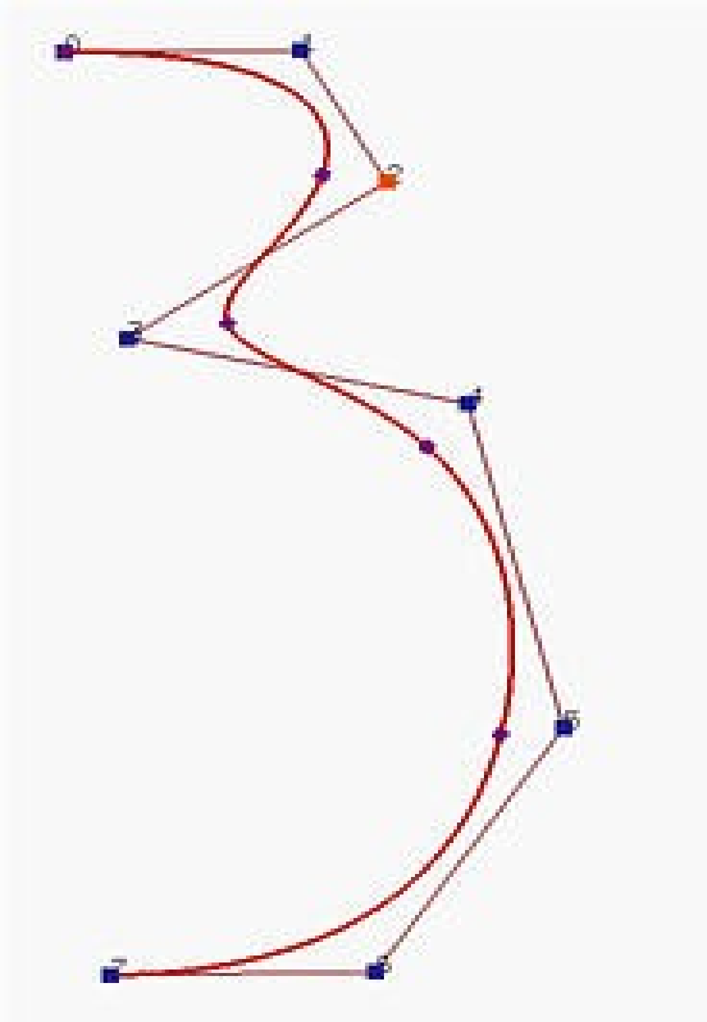


Catmull-Rom splines

- Our first example of an interpolating spline
- Like Bézier, equivalent to Hermite
  - in fact, all splines of this form are equivalent
- First example of a spline based on just a control point sequence
- Does not have convex hull property

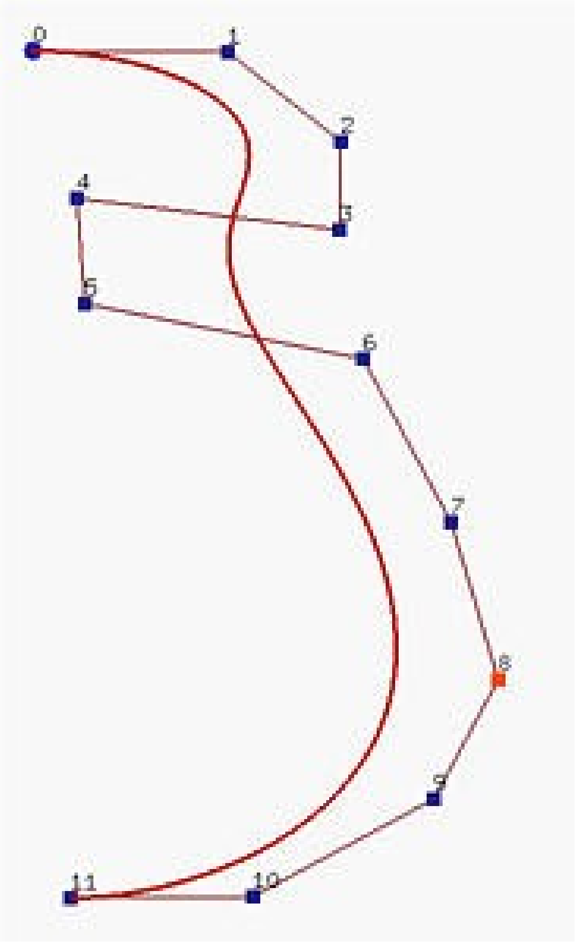
B-Spline

- We may want more continuity than C^1
- We may not need an interpolating spline
- B-splines are a clean, flexible way of making long splines with arbitrary order of continuity



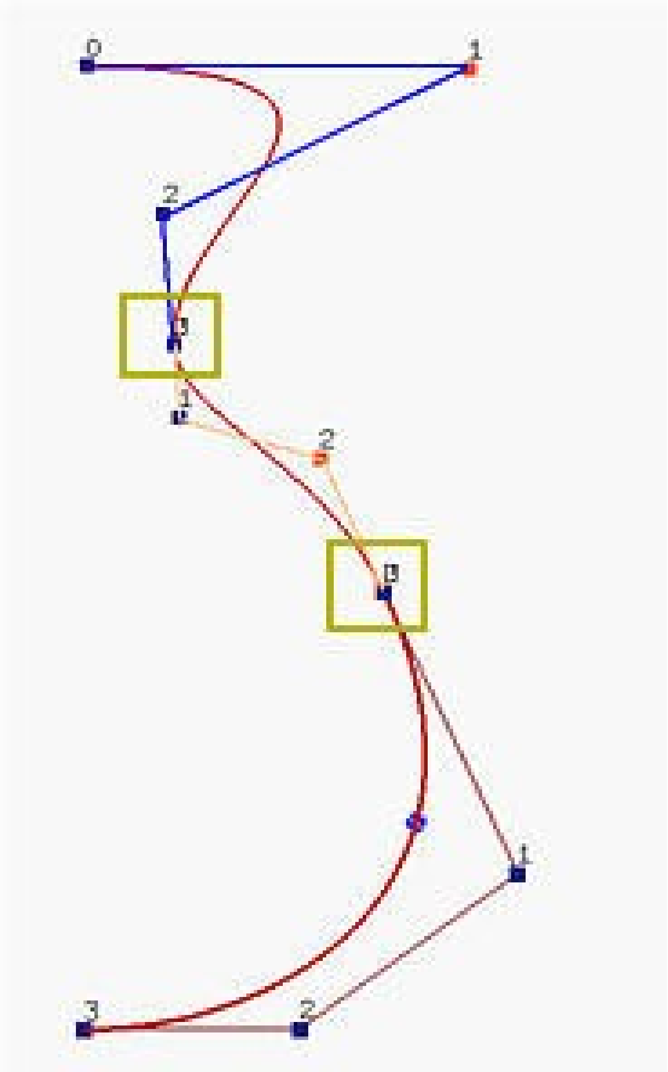
Why B-spline. 1. High-order bezier curve instead?

- Recall bezier curve
  - The degree of a Bezier Curve is determined by the number of control points
  - E. g. bezier curve degree 11 – difficult to bend the "neck" toward the line segment $\mathbf{P}_4\mathbf{P}_5$.
  - We can add more control points, BUT this will increase the degree of the curve → increase computational burden and smoothness



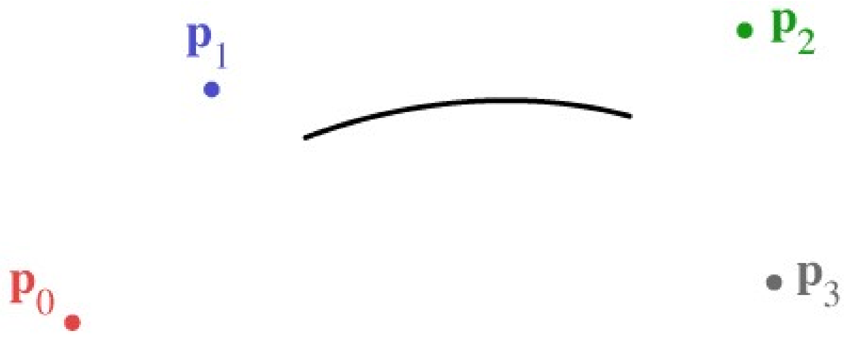
Why B-Spline.
2. Chaining cubic Bezier curves instead?

- Joint many bezier curves of lower degree together (right figure)
  - You can chain Hermite or Bezier curves
  - Catmull-Rom spline is also in this form
  - Unintuitive to control and sometimes not smooth enough



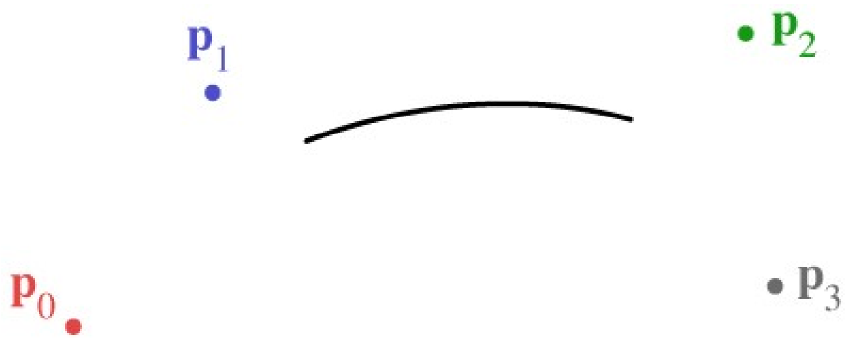
B-splines

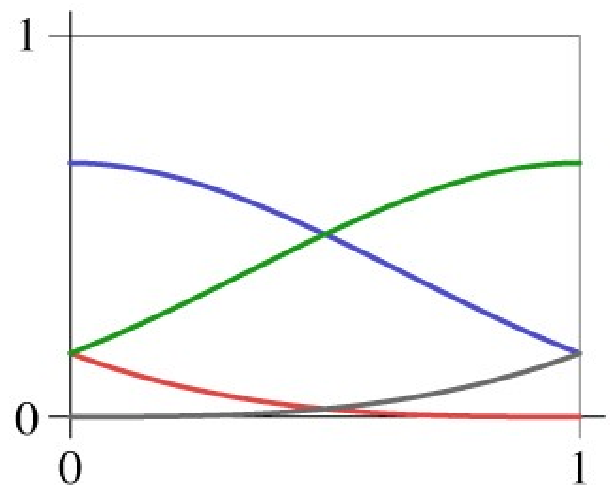
- Use 4 points, but approximate only middle two



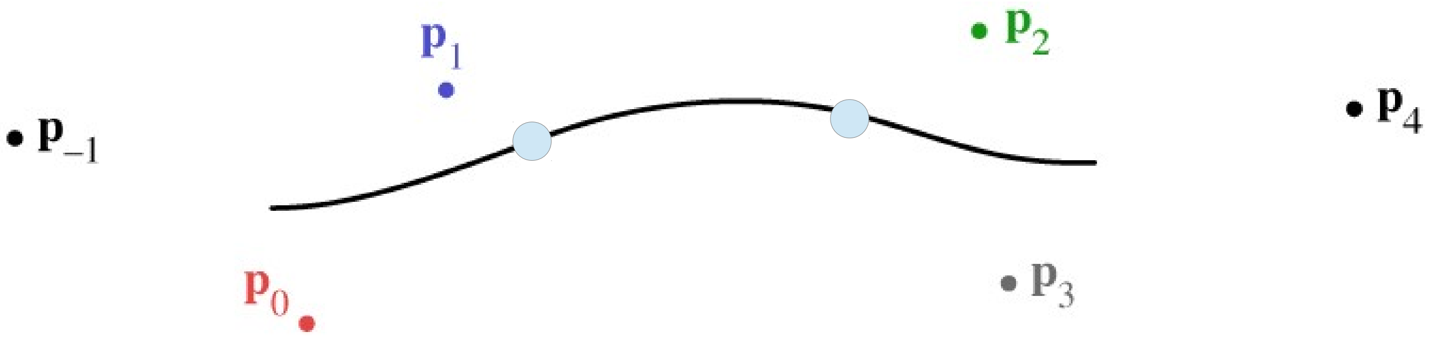
- Draw curve with overlapping segments
  - 0-1-2-3, 1-2-3-4, 2-3-4-5, 3-4-5-6, etc
- Curve may miss all control points
- Smoother at joint points (why? - later)

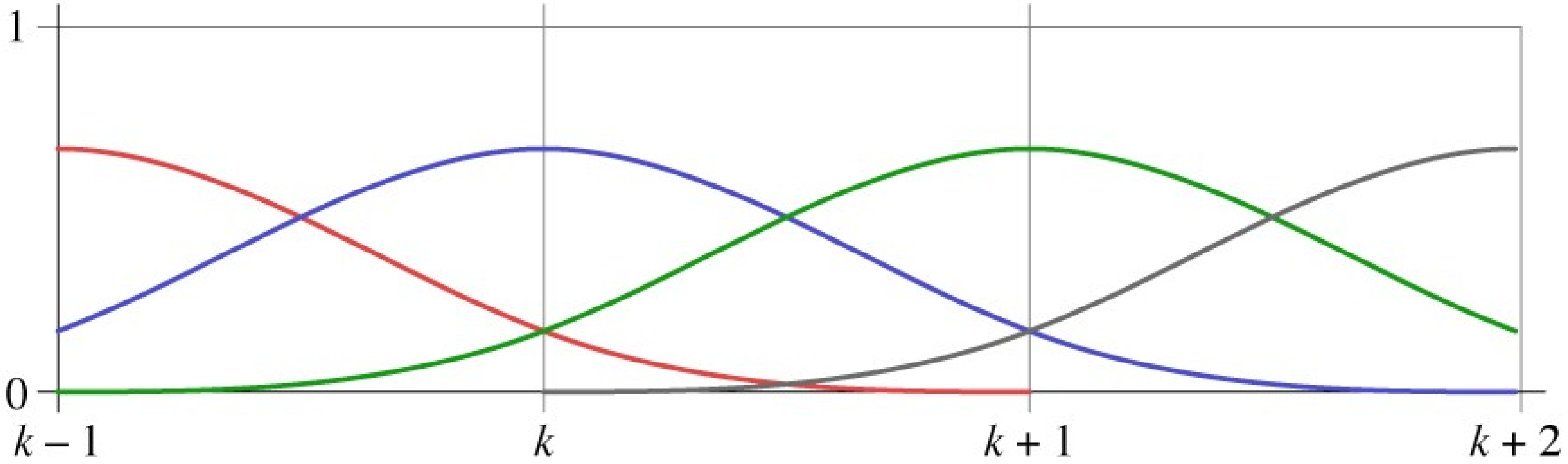
Cubic B-spline basis





Cubic B-spline basis





Deriving the B-Spline

- Want a cubic spline; therefore 4 active control points
- Want C2 continuity
  - Turns out that is enough to determine everything

Efficient construction of any B-spline

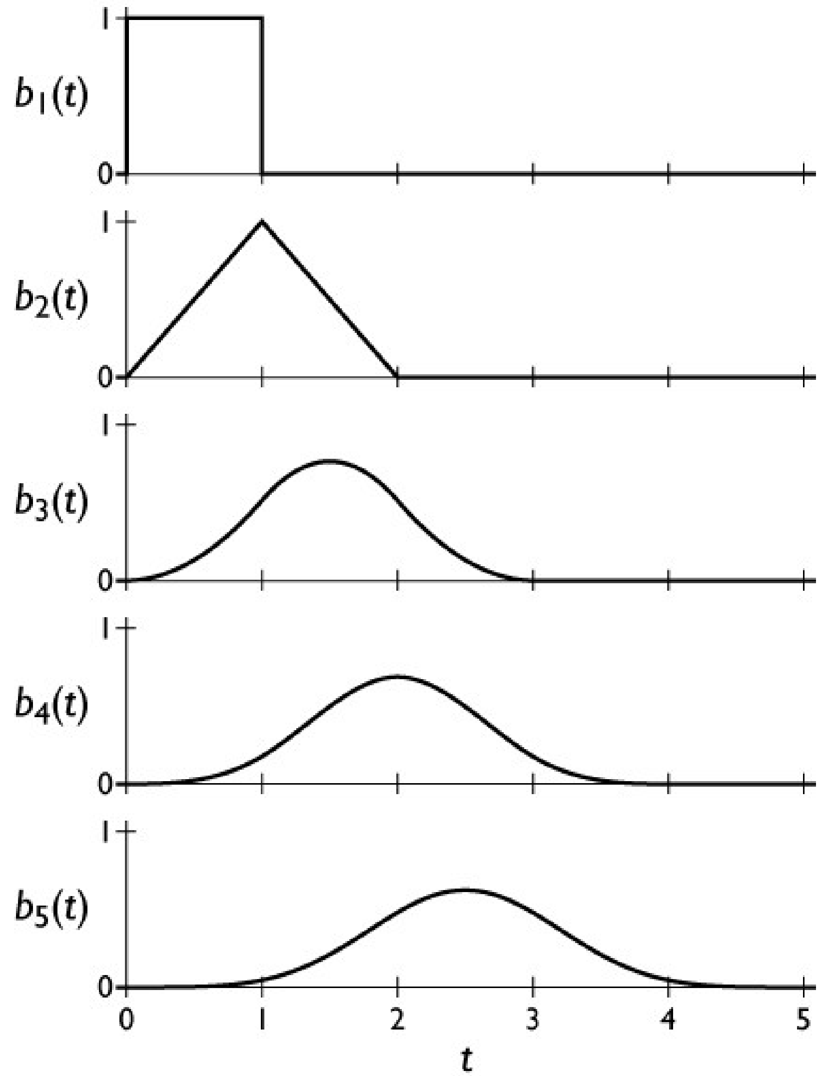
- B-splines defined for all orders
  - order *d*: degree *d* – 1
  - order *d*: *d* points contribute to value
- One definition: Cox-deBoor recurrence

$$b_1 = \begin{cases} 1 & 0 \le u < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_d = \frac{t}{d-1} b_{d-1}(t) + \frac{d-t}{d-1} b_{d-1}(t-1)$$

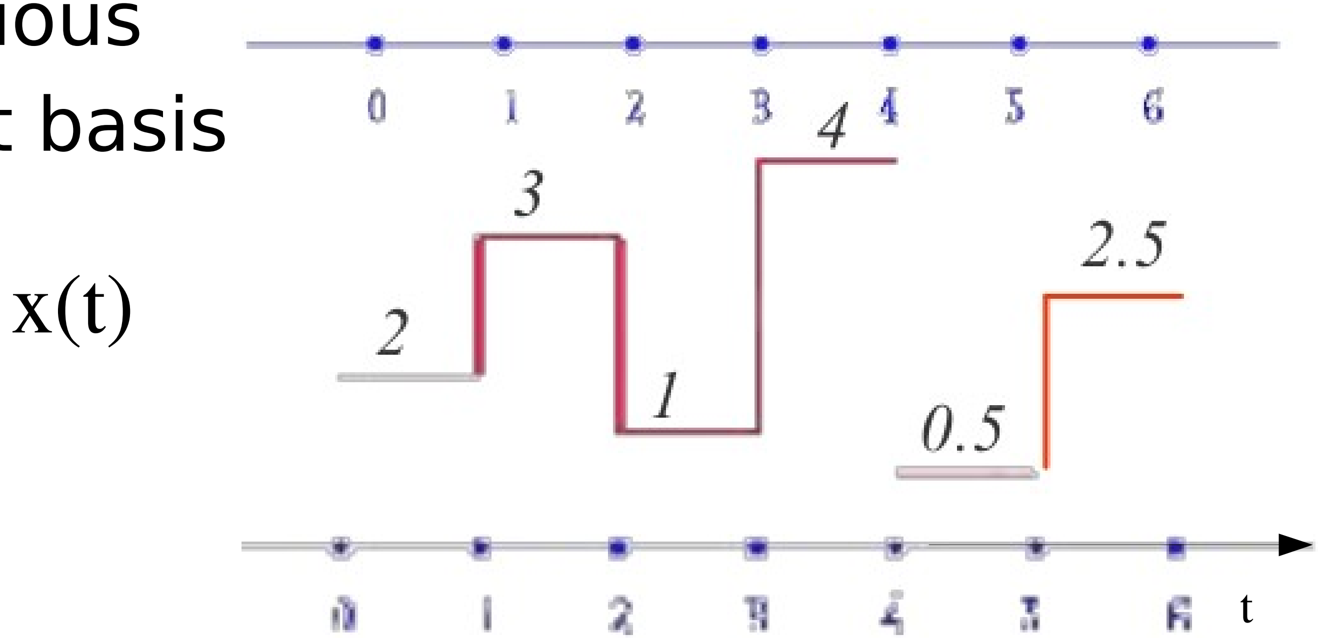
B-spline construction, alternate view

- Recurrence
  - ramp up/down
- Convolution
  - smoothing of basis fn
  - smoothing of curve



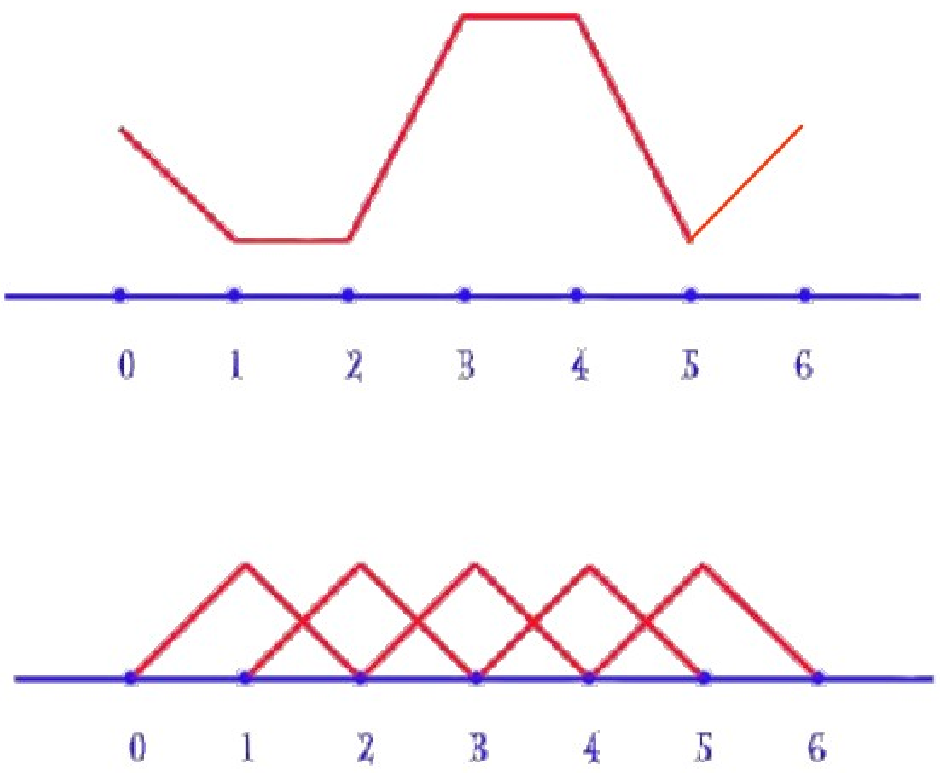
B-spline of order 1 using b1(t)

- Order =1
- Degree =0
- Discontinuous
- 1 segment basis function



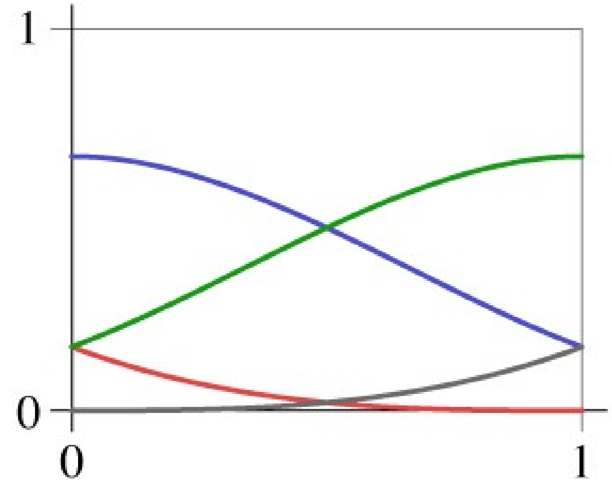
B-spline of order 2 (Linear B-Splines)

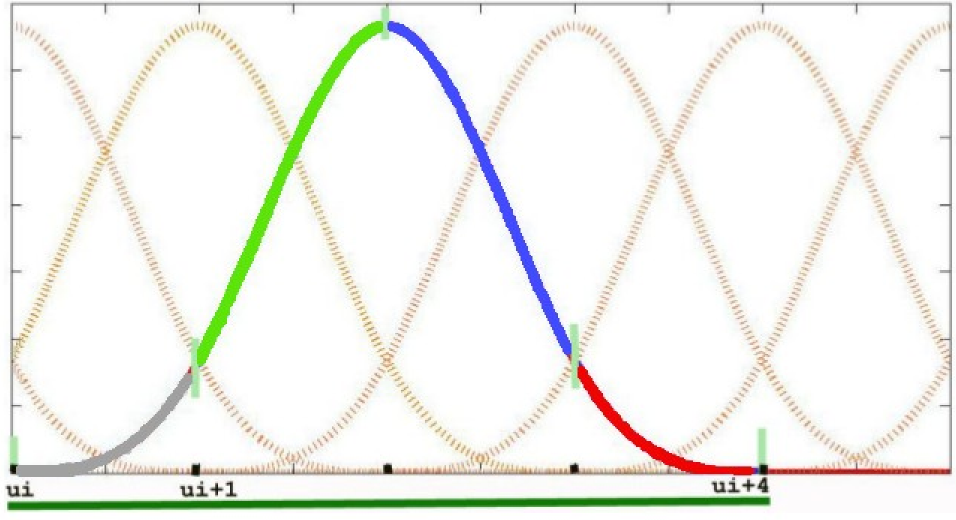
- Order =2
- Degree =1
- C0 continuous
- 2 segments



B-spline or order 4 (cubic B-spline)

- Order =4
- Degree =3
- C2 continuous
- 4 segments



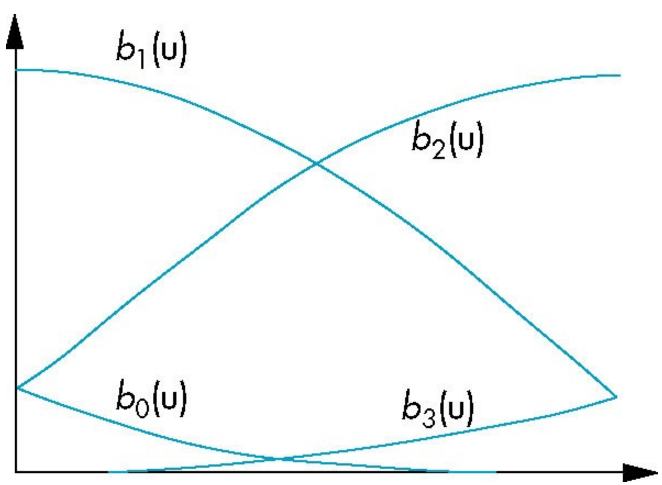


Cubic B-spline matrix

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{bmatrix}$$

$$\mathrm{p(t)} = \mathbf{t}^{\mathrm{T}} \mathbf{M}_{S} \mathbf{p} = \mathbf{b(t)}^{\mathrm{T}} \mathbf{p}$$

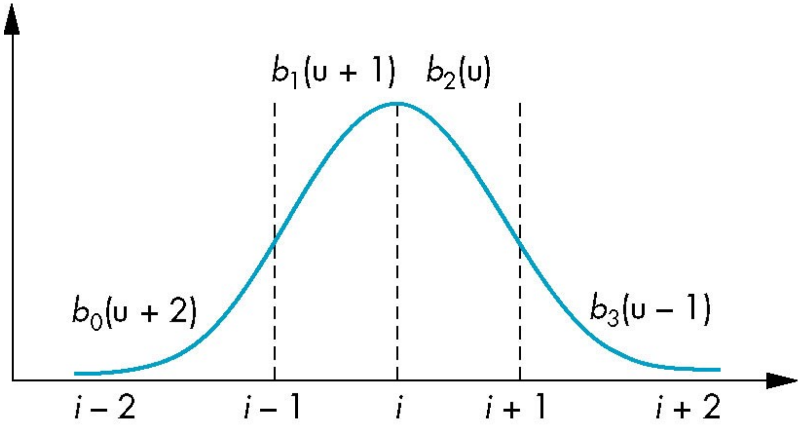
$$b(t) = \frac{1}{6} \begin{bmatrix} (1-t)^3 \\ 3t^3 - 6t^2 + 4 \\ -3t^3 + 3t^2 + 3t + 1 \\ t^3 \end{bmatrix}$$



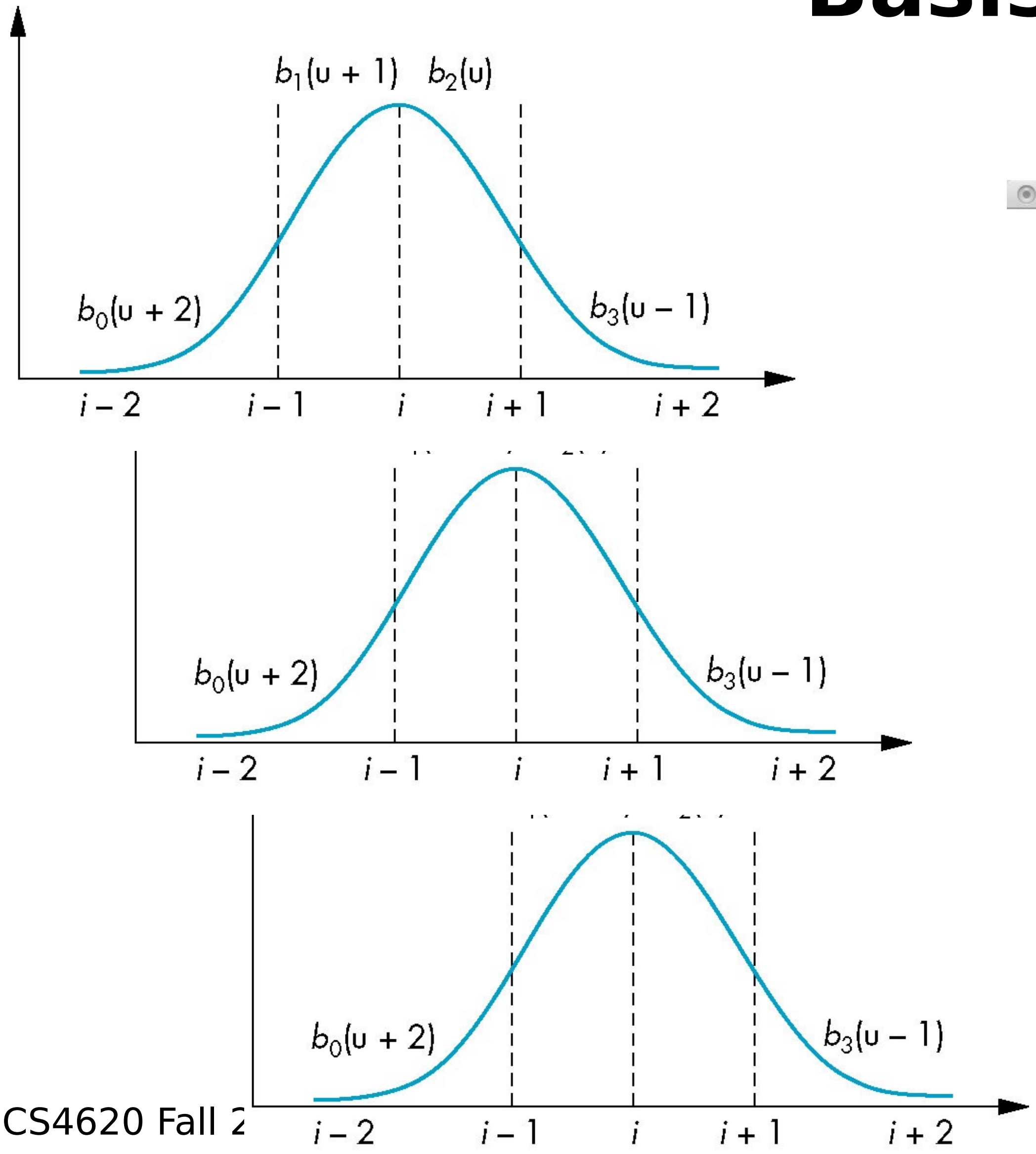
Basis Functions

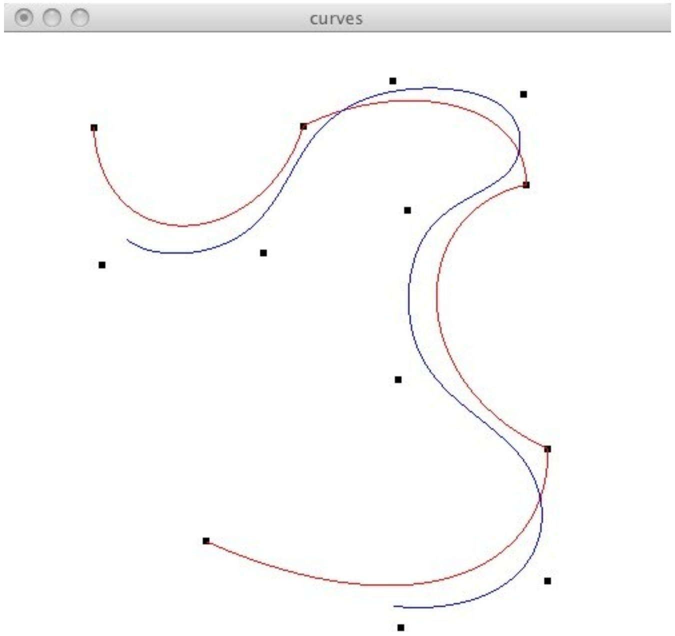
In terms of the blending polynomials

$$B_i(u) = \begin{cases} 0 & u < i-2 \\ b_0(u+2) & i-2 \le u < i-1 \\ b_1(u+1) & i-1 \le u < i \\ b_2(u) & i \le u < i+1 \\ b_3(u-1) & i+1 \le u < i+2 \\ 0 & u \ge i+2 \end{cases}$$



Basis Functions





Other types of B-splines

- Nonuniform B-splines
  - discontinuities not evenly spaced
  - allows control over continuity or interpolation at certain points
  - e.g. interpolate endpoints (commonly used case)
- Nonuniform Rational B-splines (NURBS)
  - ratios of nonuniform B-splines: $x(t)$ / $w(t)$; $y(t)$ / $w(t)$
  - key properties:
    - invariance under perspective as well as affine
    - ability to represent conic sections exactly

Non-uniform B-Spline basis function

$$N_{i,k}(u) = (u - u_i)\frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u)\frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}} \qquad (1.1)$$

$$N_{i,1} = \begin{cases} 1 & u_i \le u \le u_{i+1} \\ 0 & \text{Otherwise} \end{cases} \qquad (1.2)$$

→In equation (1.1), the denominators can have a value of zero, 0/0 is presumed to be zero.

Type of B-Spline knot vector (the set of parameters t)

Non-periodic knots
(open knots)

Periodic knots
(non-open knots)

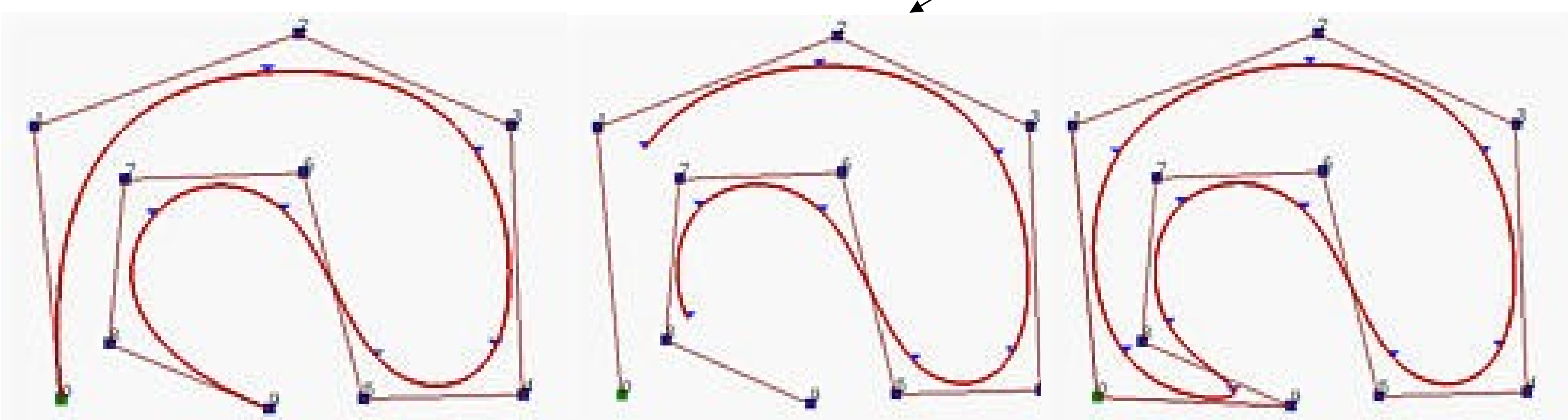
-First and last knots are duplicated k times.
-E.g (0,0,0,1,2,2,2)
-Curve pass through the first and last control points

-First and last knots are not duplicated – same contribution.
-E.g (0, 1, 2, 3)
-Curve doesn’t pass through end points.
- can used to generate closed curves (when first = last control points)

Type of B-Spline knot vector

Non-periodic knots (open knots)

Periodic knots (non-open knots)



(0,0,0,1,2,2,2)

(0, 1, 2, 3...)

(Closed knots)

Converting spline representations

- All the splines we have seen so far are equivalent
  - all represented by geometry matrices

$$\mathbf{p}_S(t) = T(t)M_S P_S$$

    - where *S* represents the type of spline
  - therefore the control points may be transformed from one type to another using matrix multiplication

$$P_1 = M_1^{-1} M_2 P_2$$

베지에spline 으로 컨버전 후,, 사용

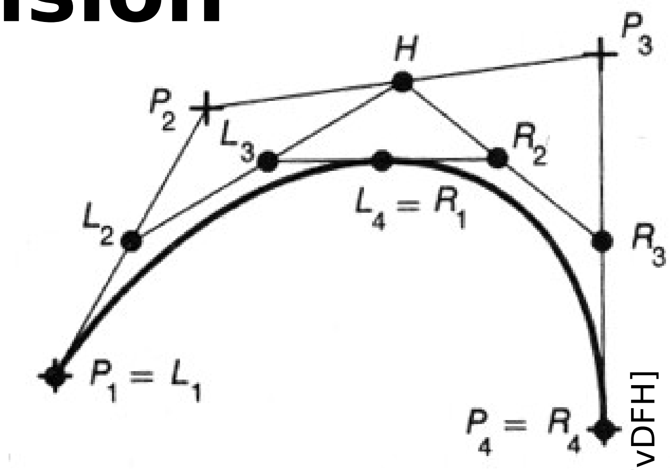
$$\mathbf{p}_1(t) = T(t)M_1(M_1^{-1}M_2P_2)$$
$$= T(t)M_2P_2 = \mathbf{p}_2(t)$$

Evaluating splines for display

- Need to generate a list of line segments to draw
  - generate efficiently
  - use as few as possible
  - guarantee approximation accuracy
- Approaches
  - reccursive subdivision (easy to do adaptively)
  - uniform sampling (easy to do efficiently)

Evaluating by subdivision

- Recursively split spline
  - stop when polygon is within epsilon of curve
- Termination criteria
  - distance between control points
  - distance of control points from line



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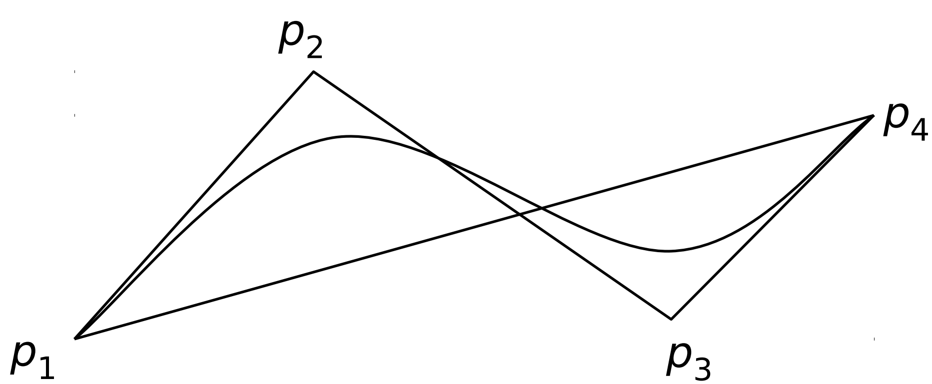
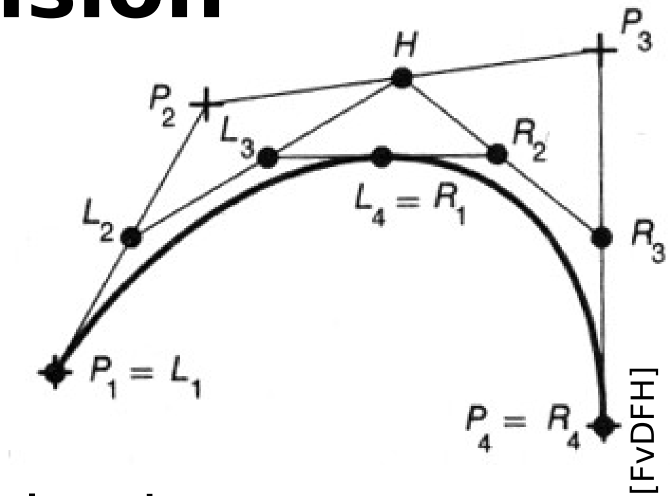
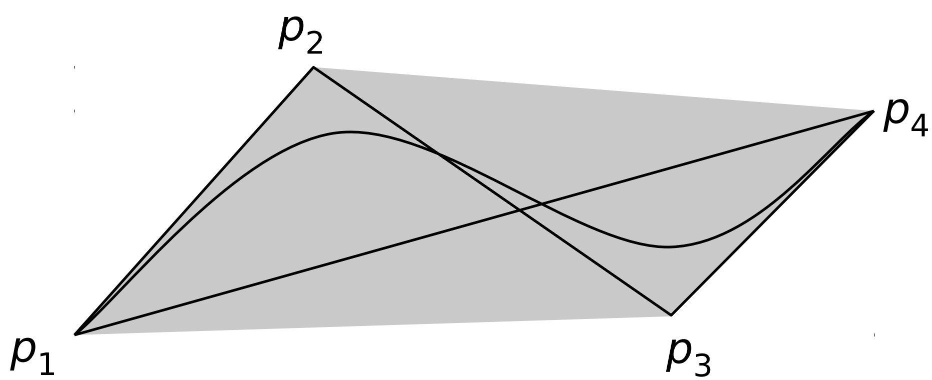


Evaluating by subdivision



- Recursively split spline
  - stop when polygon is within epsilon of curve
- Termination criteria
  - distance between control points
  - distance of control points from line



Evaluating by subdivision

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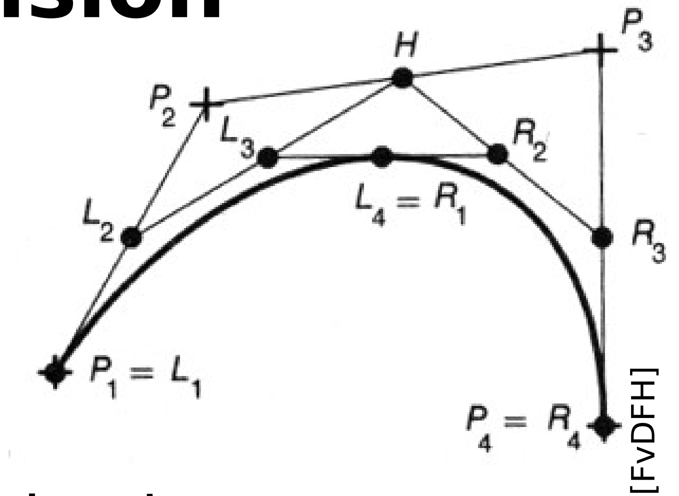
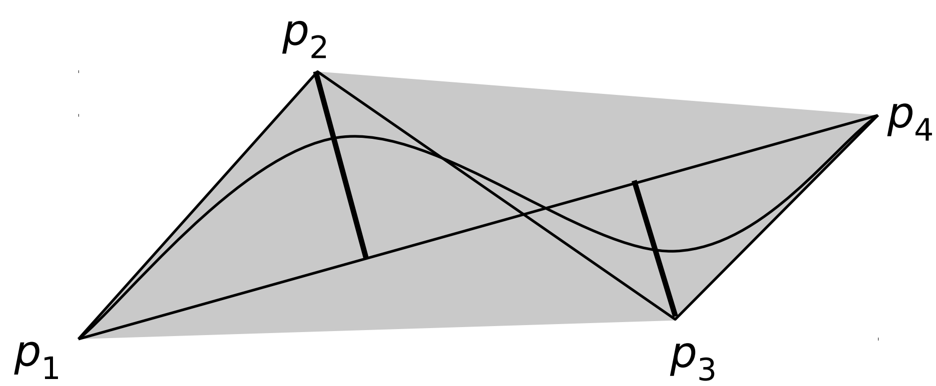




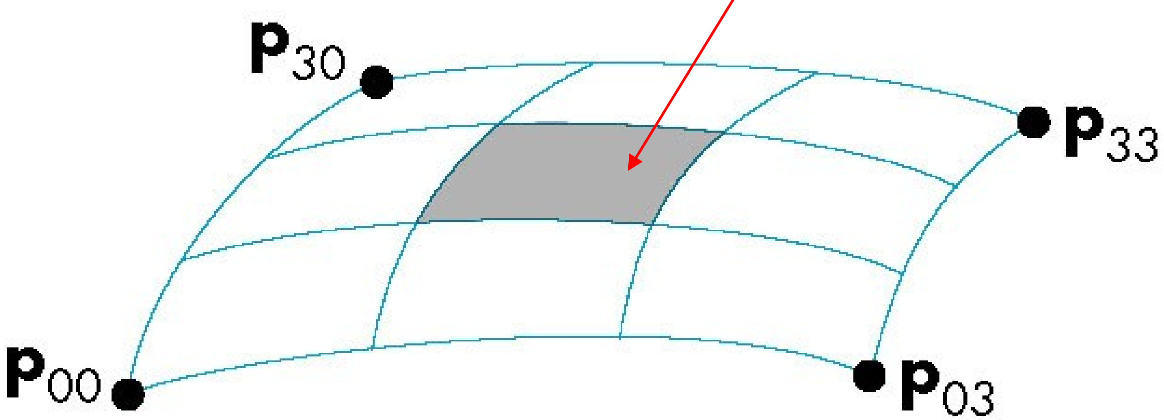
Evaluating with uniform spacing

- Forward differencing
  - efficiently generate points for uniformly spaced *t* values
  - evaluate polynomials using repeated differences

B-Spline Patches

$$p(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_i(u)\, b_j(v)\, p_{ij} = u^T \mathbf{M}_S \mathbf{P} \mathbf{M}_S^T v$$

defined over only 1/9 of region



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