Floating Point

Lecture 3

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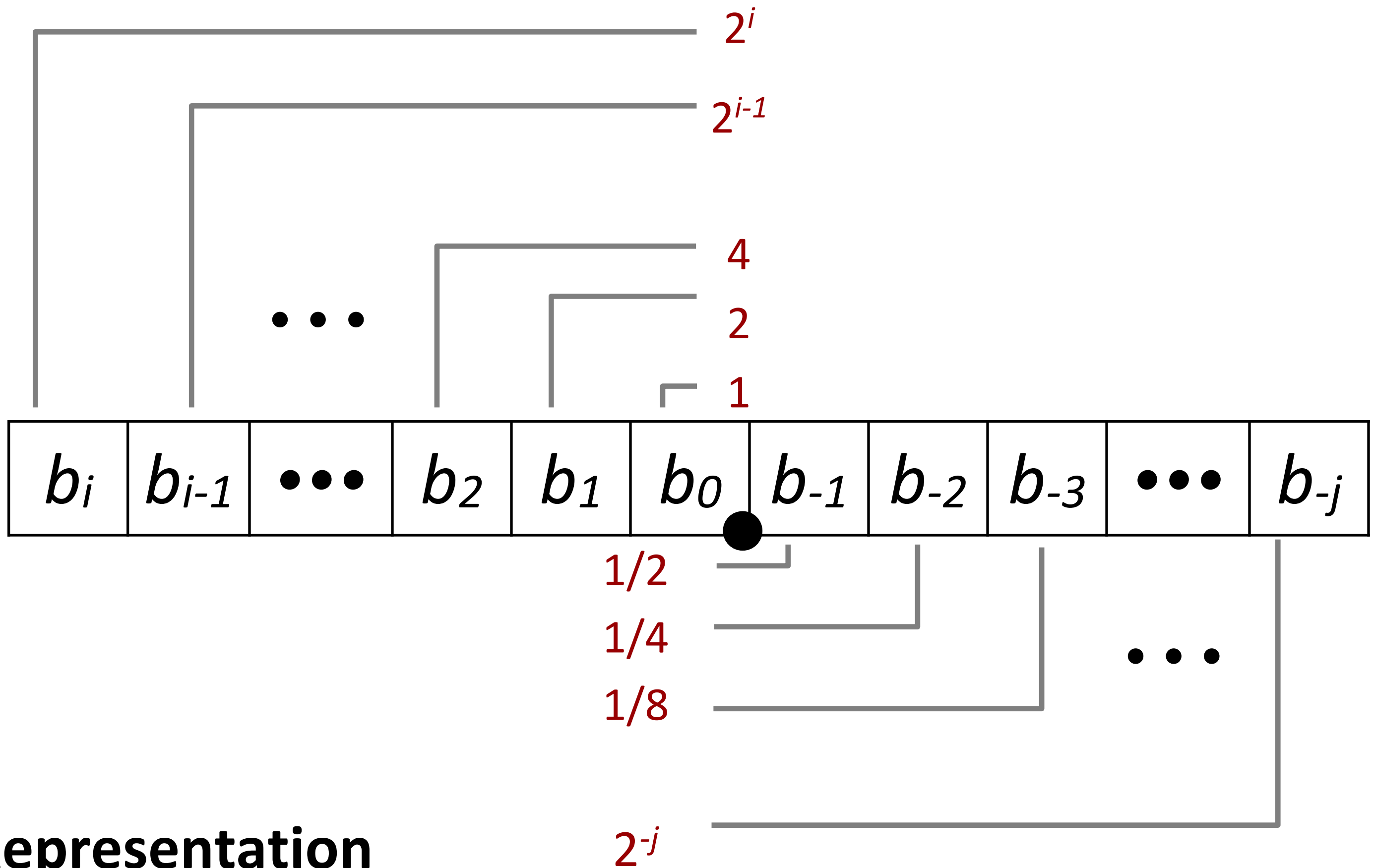
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Creating Floating Point Number
- Floating point in C
- Summary

Fractional binary numbers

- **What is 1011.101_2?**
  - 8 + 0 + 2 + 1 + 1/2 + 0 + 1/8 = 11 + 5/8

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation #1**
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 $\mathbf{0.0101010101[01]...}_2$
    - 1/5 $\mathbf{0.001100110011[0011]...}_2$
    - 1/10 $\mathbf{0.0001100110011[0011]...}_2$

- **Limitation #2**
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers
      - both 2^{100} and 2^{-100} need at least 100 bits

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

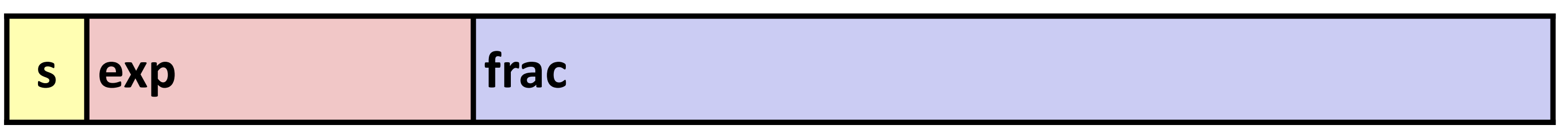
Floating Point Representation

- **Numerical Form:**

$$(-1)^s\, M\; 2^E$$

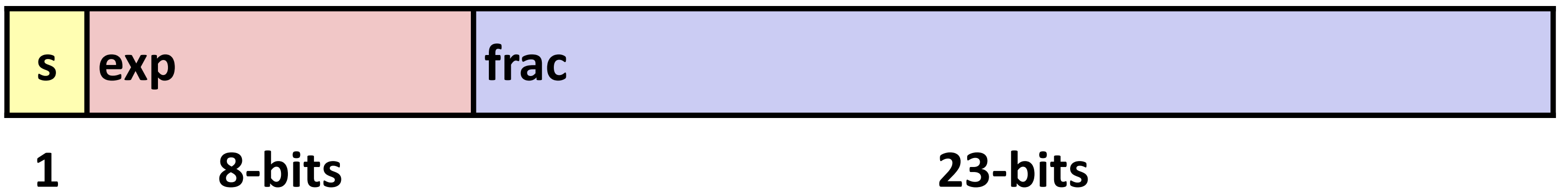
  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand or Mantissa *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two
    - positive and negative both are possible

- **Encoding**
  - MSB s is sign bit ***s***
  - **exp** field encodes ***E*** (but is not equal to E)
  - **frac** field encodes ***M*** (but is not equal to M)



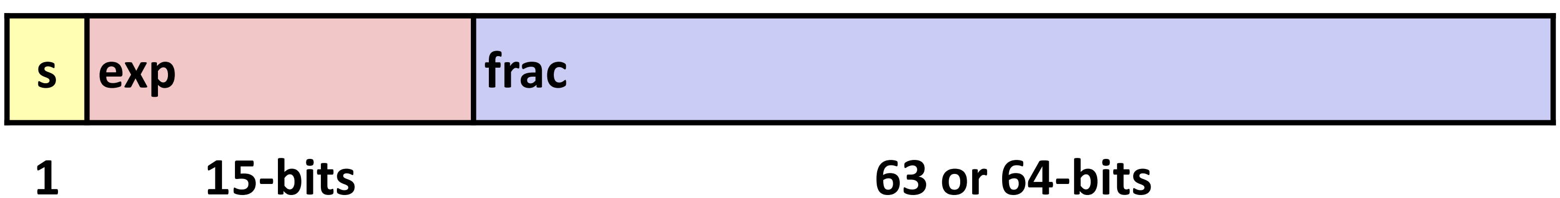
Precision options

- Single precision: 32 bits (float in C)



- Double precision: 64 bits (double in C)

- Extended precision: 80 bits (Intel only, long double in C)



"Normalized" Values

$v = (-1)^s \, M \, 2^E$

- **When: exp ≠ 000…0 and exp ≠ 111…1**
- **Exponent coded as a *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value of exp field
  - *Bias* = $2^{k-1} - 1$, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)
    - ex) Exp = 127 with E = 0, Exp = 119 with E = -8
    - Why use the bias?
      - Making Exp to be proportional to the size of the floating point number
      - Speeding up the comparison between floating point numbers
- **Significand coded with implied leading 1: *M* =** $1.xxx…x_2$
  - xxx…x: bits of frac field
    - Get extra leading bit for "free"
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)

Normalized Encoding Example

$v = (-1)^s \, M \, 2^E$
$E = Exp - Bias$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**
  M = $1.\underline{1101101101101}_2$
  `frac`= $\underline{1101101101101}0000000000_2$

- **Exponent**
  E = 13
  $Bias$ = 127
  Exp = 140 = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s \, M \, 2^E$

$E = 1 - Bias$

- **Condition:** exp = 000…0
- **Exponent value: E = 1 – Bias (instead of E = 0 – *Bias*)**
- **Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$**
  - **`xxx…x`**: bits of **`frac`**
  - Adding 1 to E is needed because the implicit leading 1 of M is removed unlike the Normalized case
    - allows normalized and denormalized values to be linked
    - 1×2^{-125} (exp = 2) → 1×2^{-126} (exp = 1) → 0.1×2^{-126} (exp = 0)
- **Cases**
  - **`exp`** = `000…0`, **`frac`** = `000…0`
    - Represents zero value
      - impossible in the normalized value due to its implicit leading 1
    - Note distinct values: +0 and –0
      - but these are considered identical
  - **`exp`** = `000…0`, **`frac`** ≠ `000…0`
    - Numbers closest to 0.0
      - The implicit leading 1 makes the size of a number bound to the exp
      - ex) 1.0001×2^{-127} (exp = 0) vs 0.0001×2^{-126} (exp = 0)

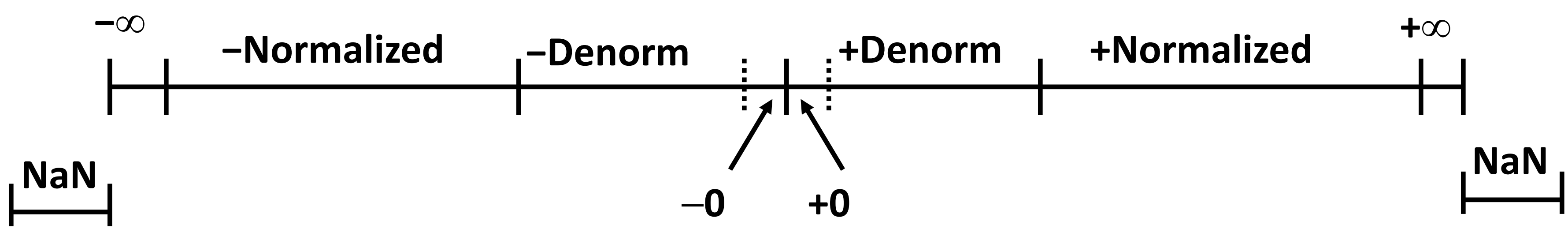
Special Values

- **Condition: `exp` = `111`…`1`**

- **Case: `exp` = `111`…`1`, `frac` = `000`…`0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

- **Case: `exp` = `111`…`1`, `frac` ≠ `000`…`0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty \times 0$, uninitialized value (possibly)

Visualization: Floating Point Encodings



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Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$

n: E = Exp – Bias

d: E = 1 – Bias

	s	exp	frac	E	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

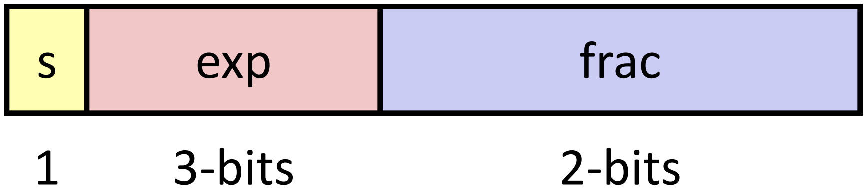
- We can sort float pointing numbers using integer sorting algorithms.

Dynamic Range

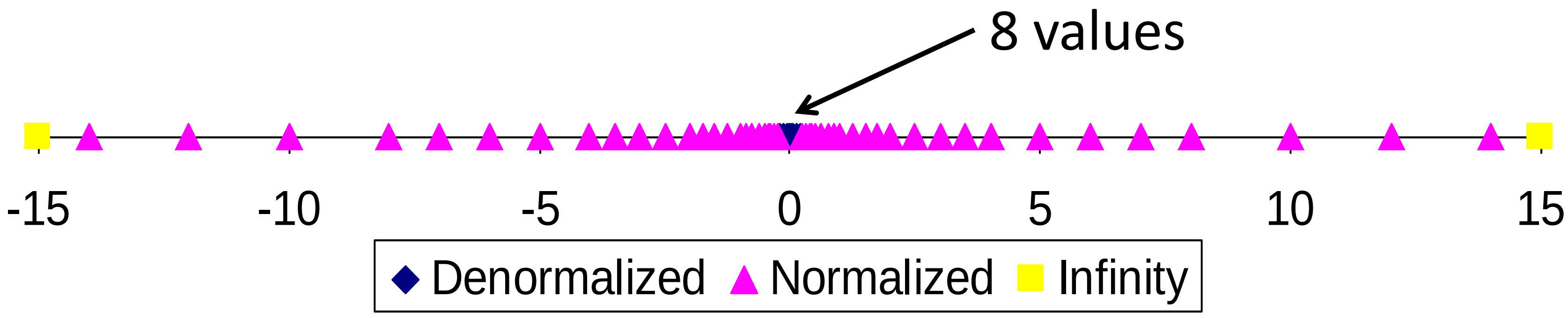
			Single precision		Double precision	
Description	**exp**	**frac**	**Value**	**Decimal**	**Value**	**Decimal**
Zero	00…00	0…00	0	0.0	0	0.0
MIN denormalized	00…00	0…01	$2^{-23}\times2^{-126}$	1.4×10^{-45}	$2^{-52}\times2^{-1022}$	4.9×10^{-324}
MAX denormalized	00…00	1…11	$(1-\varepsilon)\times2^{-126}$	1.2×10^{-38}	$(1-\varepsilon)\times2^{-1022}$	2.2×10^{-308}
MIN normalized	00…01	0…00	1×2^{-126}	1.2×10^{-38}	1×2^{-1022}	2.2×10^{-308}
One	01…11	0…00	1×2^{0}	1.0	1×2^{0}	1.0
MAX normalized	11…10	1…11	$(2-\varepsilon)\times2^{127}$	3.4×10^{38}	$(2-\varepsilon)\times2^{1023}$	1.8×10^{308}

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 2^{3-1}-1 = 3

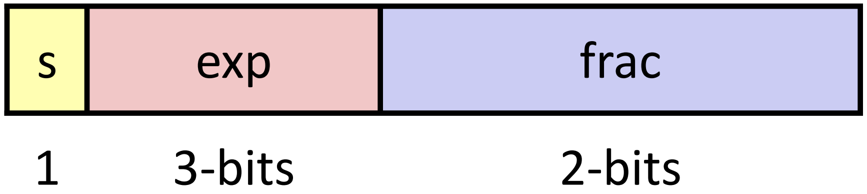


- **Notice how the distribution gets denser toward zero.**

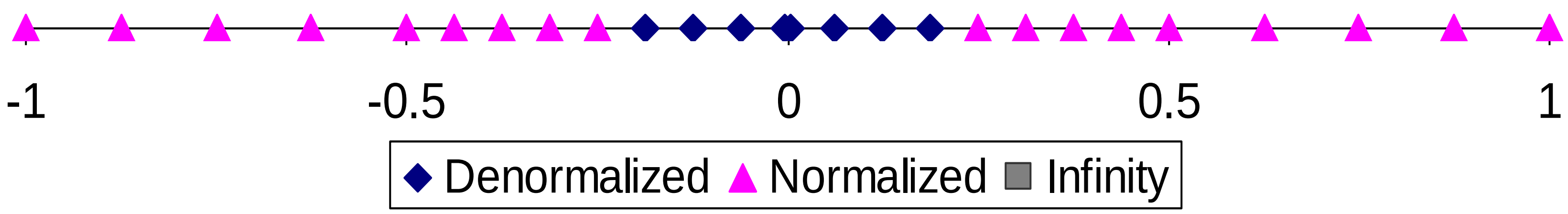


Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3



- **Denormalized value is distributed uniformly**



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Floating Point Operations: Basic Idea

- **Floating point computations are approximated due to limited range and precision.**
  - $\mathbf{x} +_f \mathbf{y} = \mathrm{Round}(\mathbf{x} + \mathbf{y})$
  - $\mathbf{x} \times_f \mathbf{y} = \mathrm{Round}(\mathbf{x} \times \mathbf{y})$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Approximating all values in one direction (either upwards or downwards) results in over-estimation or under-estimation.
  - Only the round-to-even method avoids this bias by approximating values upward and downward by half.
- **Applying to Other Decimal Places / Bit Positions**
  - Round to the nearest significant figures
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is `0`
  - "Half way" when bits to right of rounding position = $\texttt{100}\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

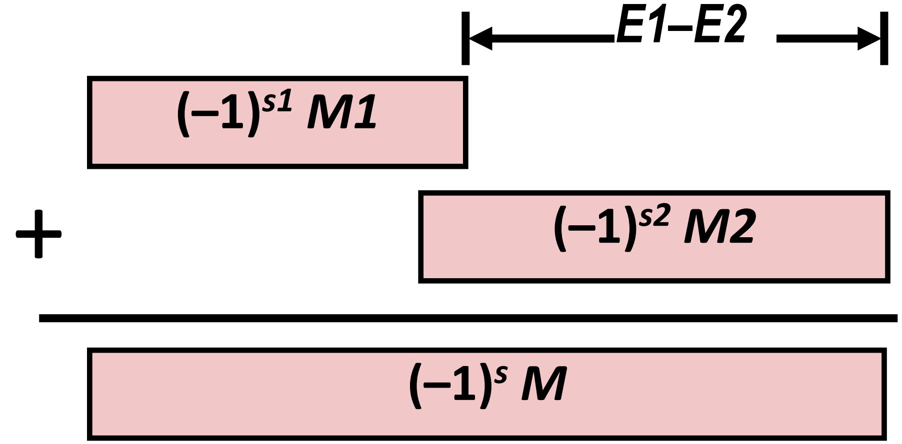
- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*: $s1$ ^ $s2$
  - Significand *M*: $M1 \times M2$
  - Exponent *E*: $E1 + E2$

- **Fixing**
  - If $M \geq 2$, shift *M* right, increment *E*
    - Decrease M to [1.0, 2.0)
  - If *E* out of range, overflow
  - Round *M* to fit **`frac`** precision

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1
- **Fixing**
  - If M ≥ 2, shift M right, increment E
    - Decrease M to [1.0, 2.0)
  - if M < 1, shift M left k positions, decrement E by k
    - Increase M to [1.0, 2.0)
  - Overflow if E out of range
  - Round M to fit frac precision

Get binary points lined up



Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? ***Yes***
    - a + 0 = a
  - Every element has additive inverse? ***Almost***
    - Yes, except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? ***Yes***
    - But may generate infinity or NaN
  - Multiplication Commutative? ***Yes***
  - Multiplication is Associative? ***No***
    - Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20`= `inf`, `1e20*(1e20*1e-20)`= `1e20`
  - 1 is multiplicative identity? ***Yes***
  - Multiplication distributes over addition? ***No***
    - Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)`= `0.0`, `1e20*1e20 - 1e20*1e20` = `NaN`
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
    - Except for infinities & NaNs

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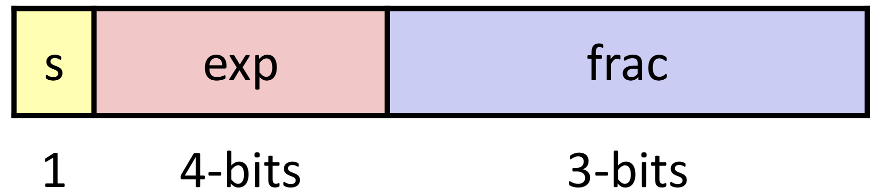
Creating Floating Point Number

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding



- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

`128`	`10000000`
`15`	`00001101`
`33`	`00010001`
`35`	`00010011`
`138`	`10001010`
`63`	`00111111`

Normalize

s	exp	frac
1	4-bits	3-bits

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
`128`	`10000000`	`1.0000000`	`7`
`15`	`00001101`	`1.1010000`	`3`
`17`	`00010001`	`1.0001000`	`4`
`19`	`00010011`	`1.0011000`	`4`
`138`	`10001010`	`1.0001010`	`7`
`63`	`00111111`	`1.1111100`	`5`

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
`128`	`1.000`	`7`		`128`
`15`	`1.101`	`3`		`15`
`17`	`1.000`	`4`		`16`
`19`	`1.010`	`4`		`20`
`138`	`1.001`	`7`		`134`
`63`	`10.000`	`5`	`1.000/6`	`64`

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Floating Point in C

- **C Guarantees Two Levels**
  - **`float`** single precision
  - **`double`** double precision
- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
      - 1.999 → 1, -1.999 → -1
    - Not defined when out of range or NaN
      - Generally sets to TMin (Positive float may become negative int)
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
      - double's faction is 52-bit
  - **`int`** → **`float`**
    - Will round according to rounding mode

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Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form $M \times 2^E$**
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers