Dynamic Programming

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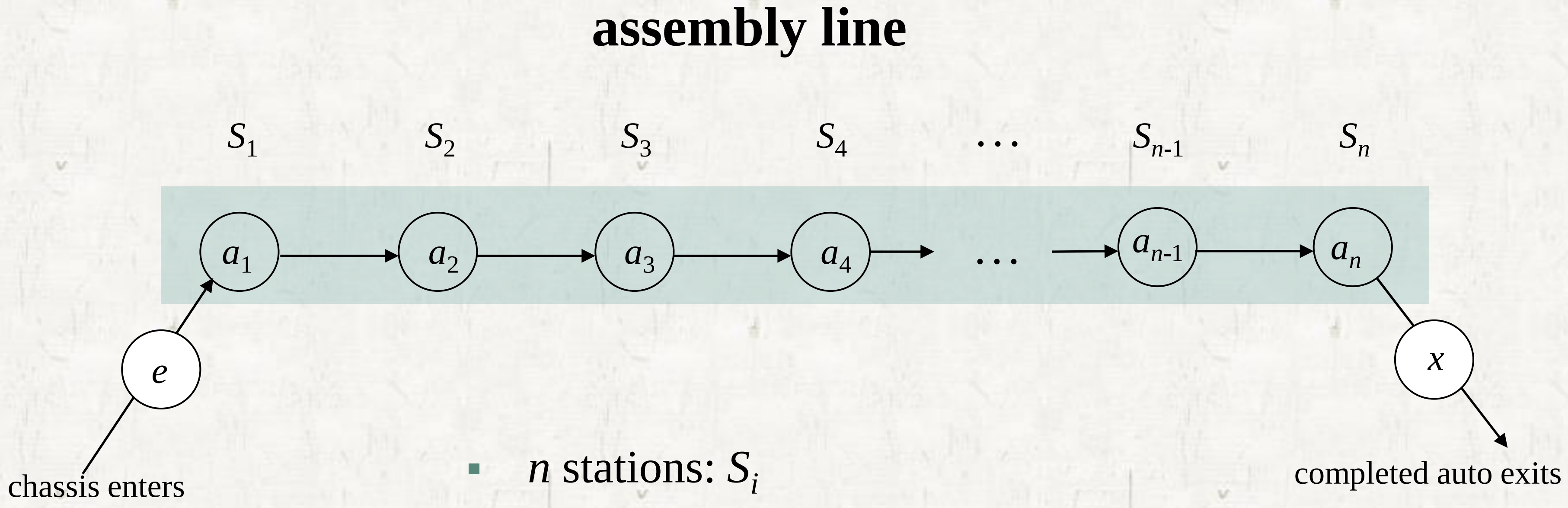
Contents

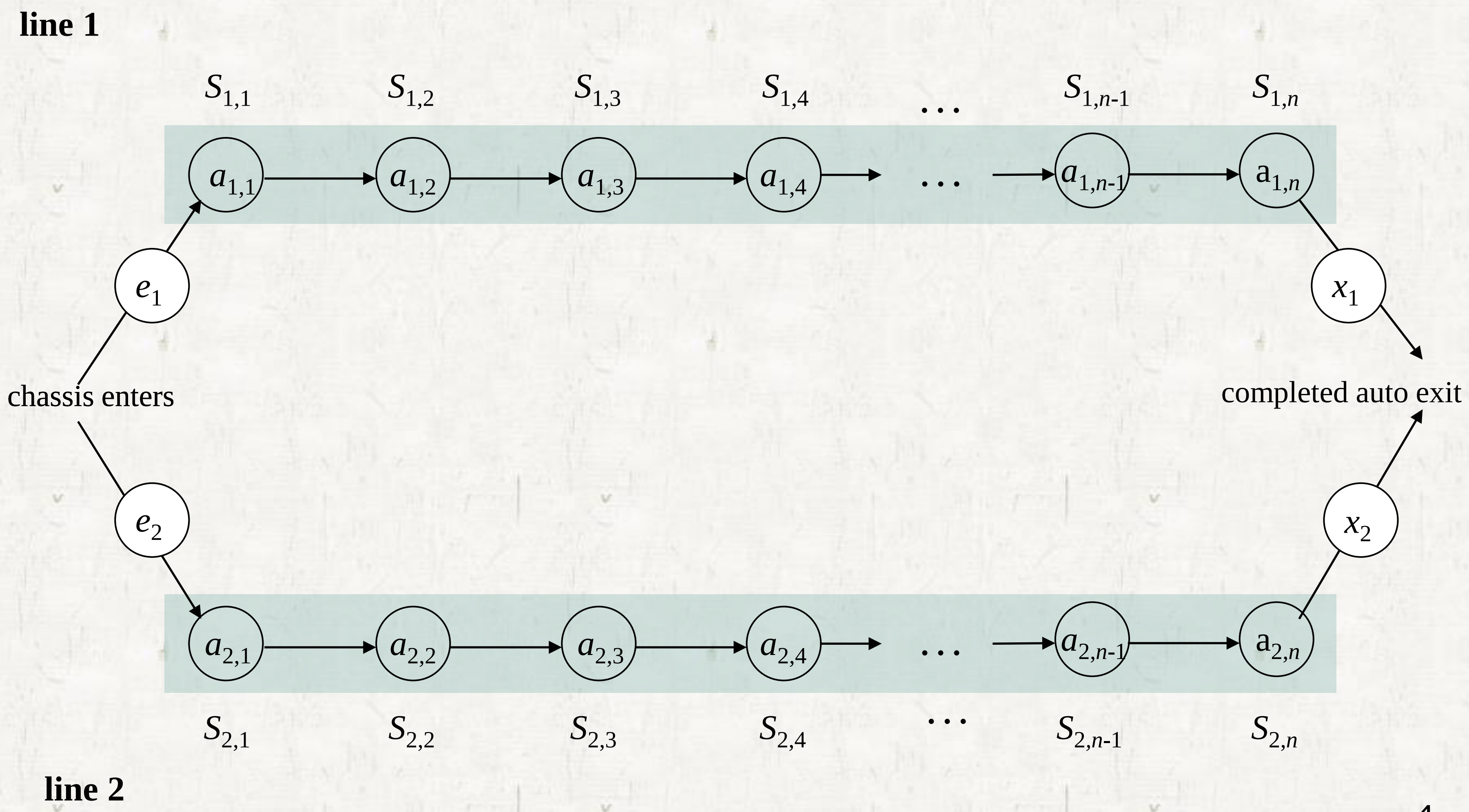
Assembly-line scheduling



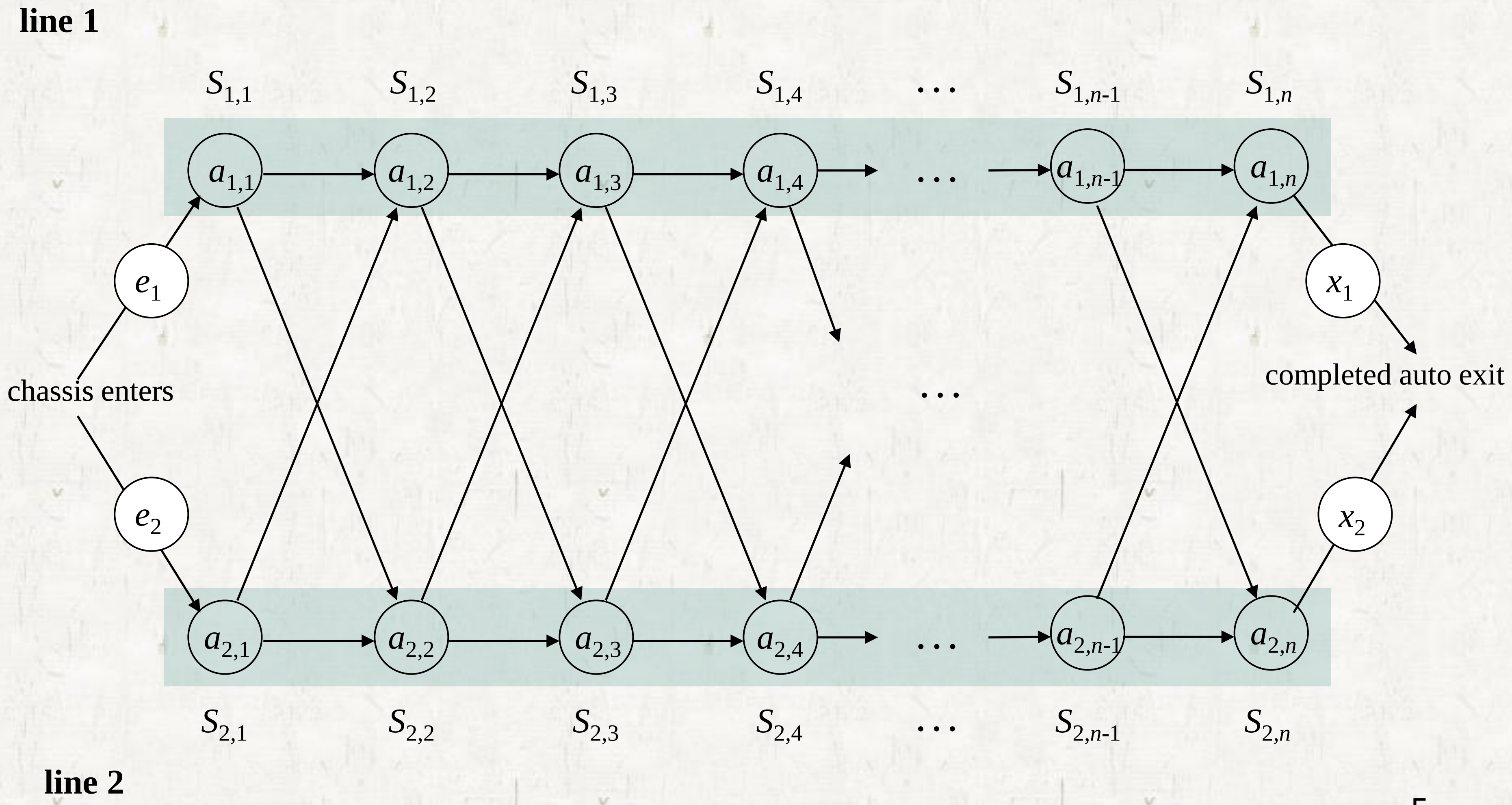
- n stations: S_i
- Assembly time in the ith station: a_i
- Entry time : e
- Exit time : x

- **The problem:** Determine the fastest assembly time

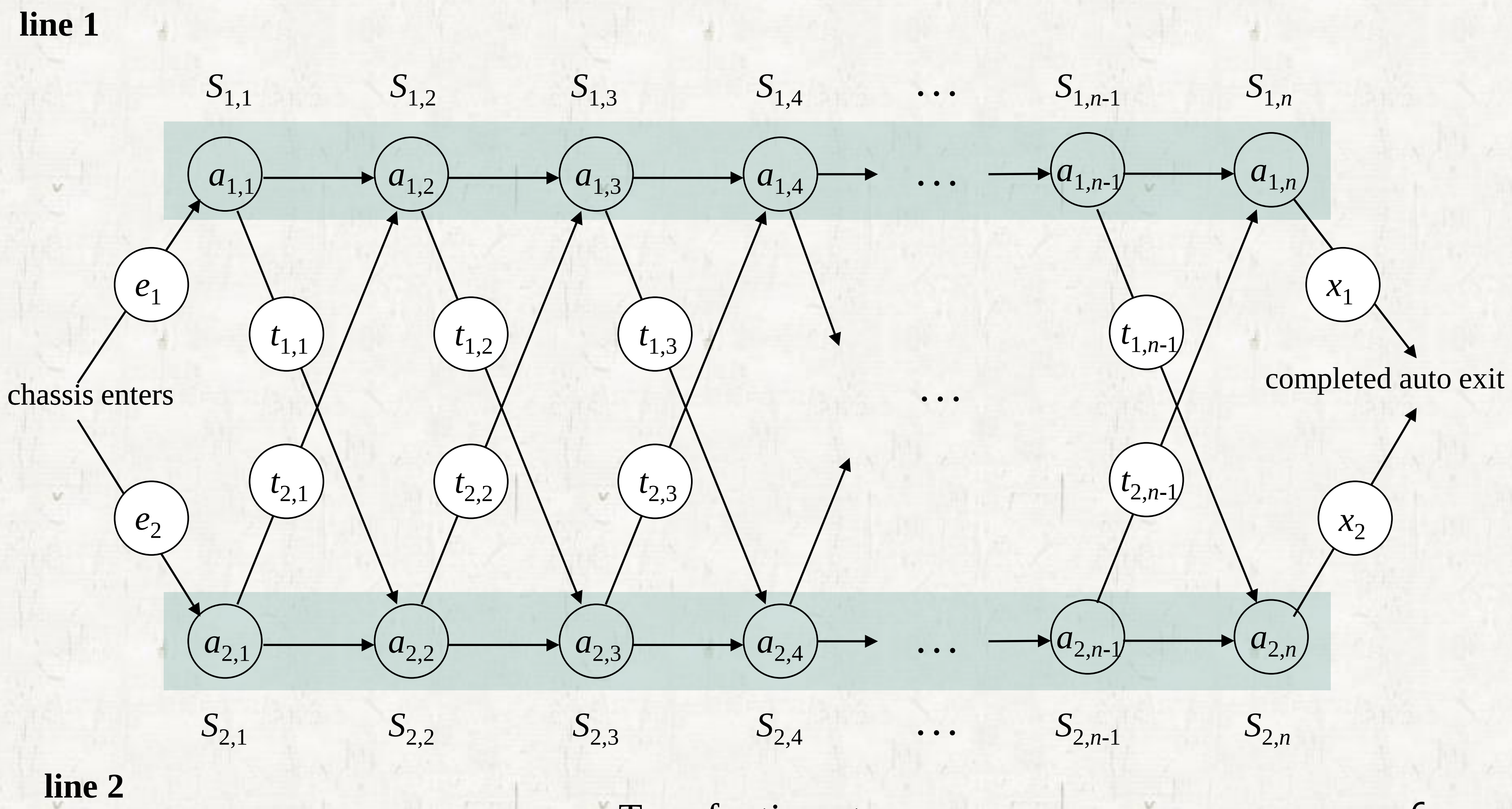
Assembly-line scheduling



Assembly-line scheduling



Assembly-line scheduling



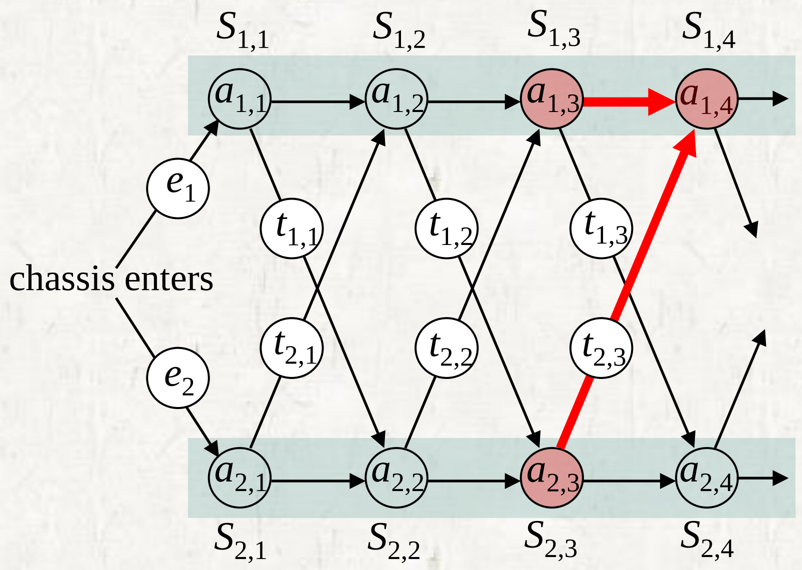
- Transfer time: $t_{i,j}$

Assembly-line scheduling

- **Brute-force approach**
  - Enumerate all possible ways and find a fastest way.
  - There are 2^n possible ways: Too many.

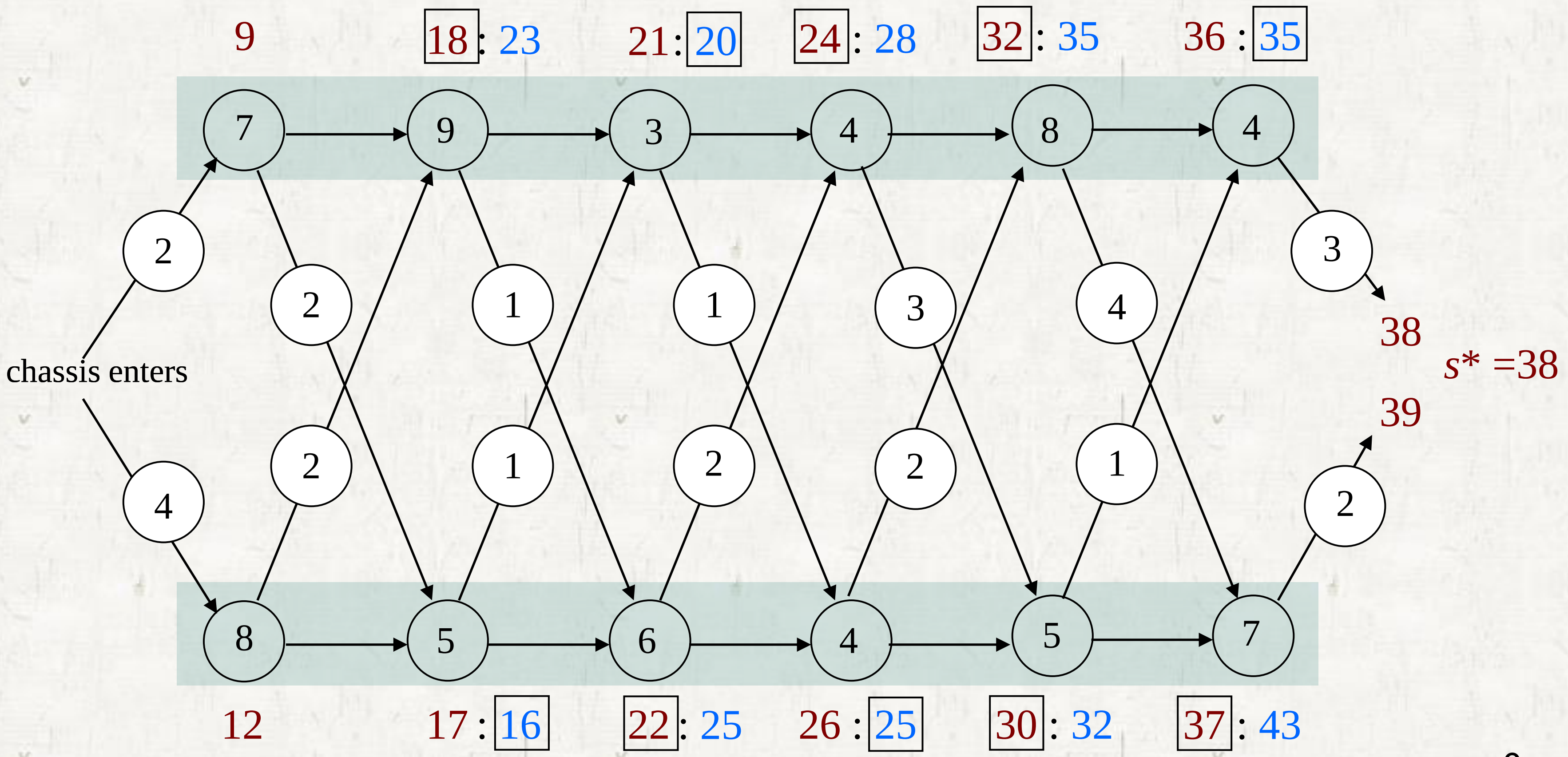
Assembly-line scheduling

The fastest way to $S_{i,j}$ goes through $S_{1,j\text{-}1}$ or $S_{2,j\text{-}1}$.

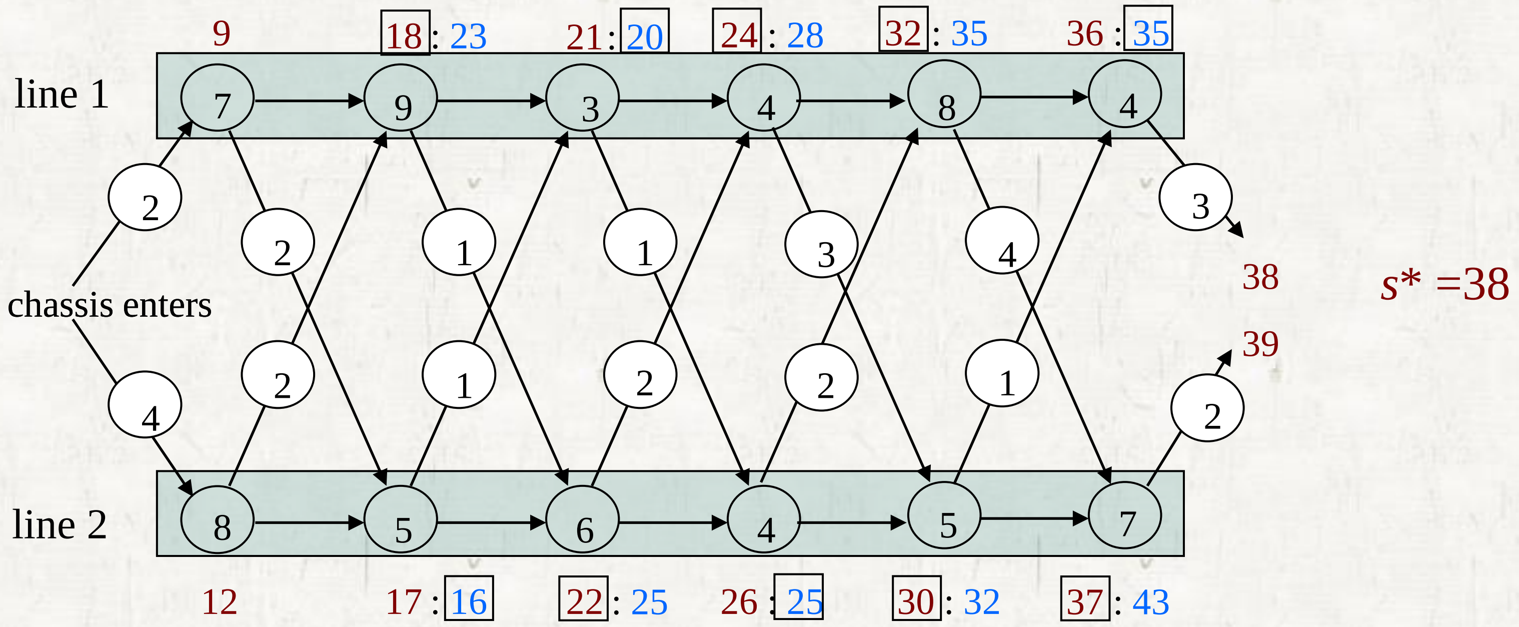


Assembly-line scheduling

- ***Dynamic programming***



Assembly-line scheduling



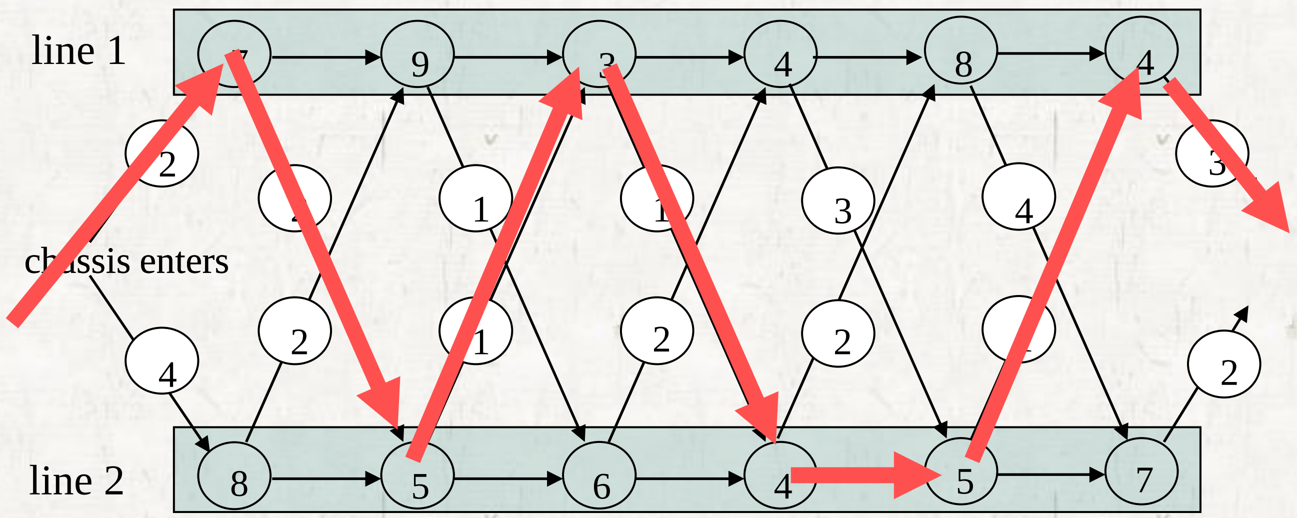
$s^* = 38$

S	1	2	3	4	5	6
1	9	18	20	24	32	35
2	12	16	22	25	30	37

$s^* = 38$

- ***Fastest time***

Assembly-line scheduling



S	1	2	3	4	5	6
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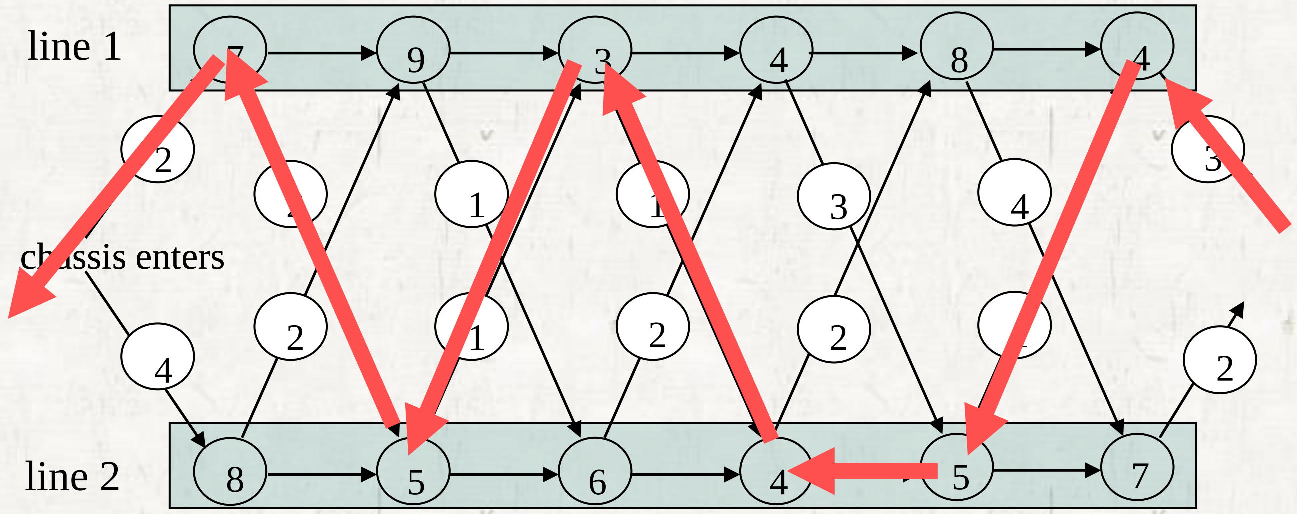
2	12	16	22	25	30	37

$s^* = 38$

- ***Fastest way***

Assembly-line scheduling



S	1	2	3	4	5	6
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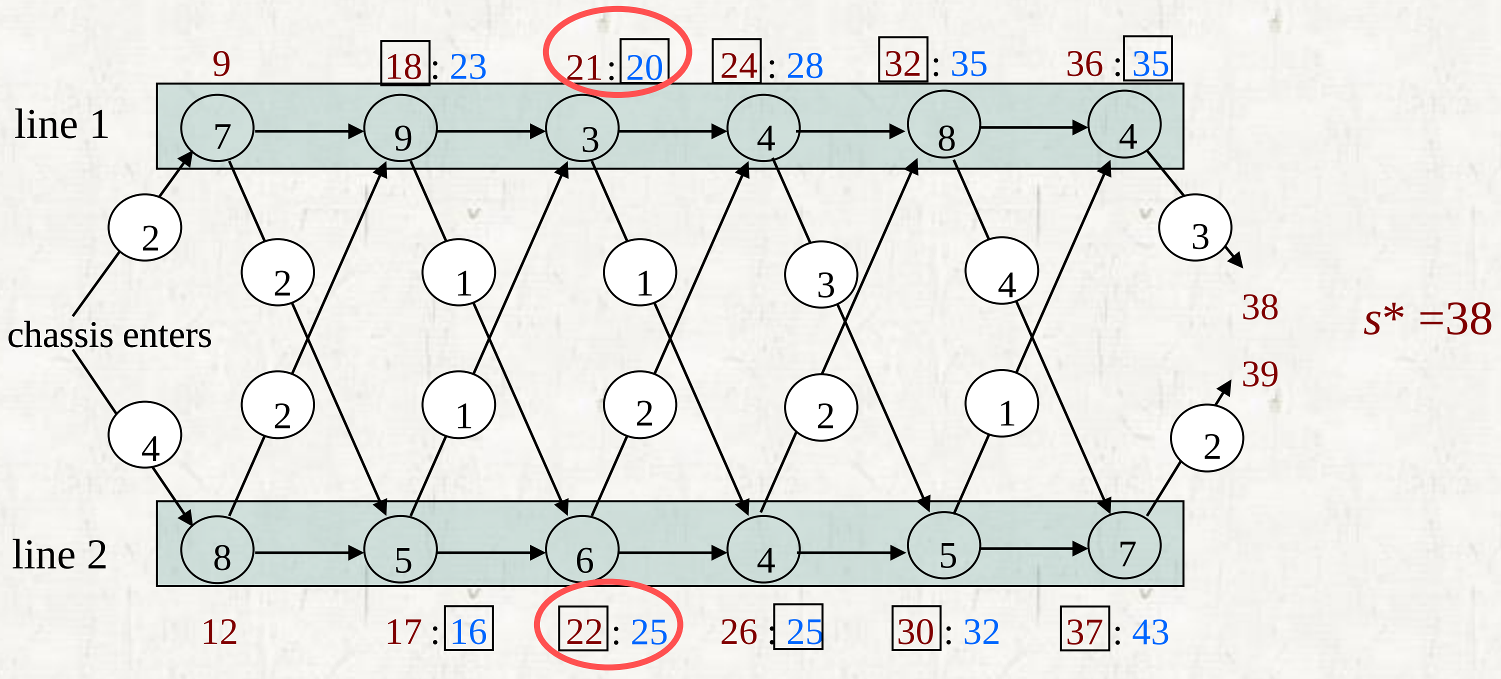
$s^* = 38$

- ***Fastest way***

Assembly-line scheduling



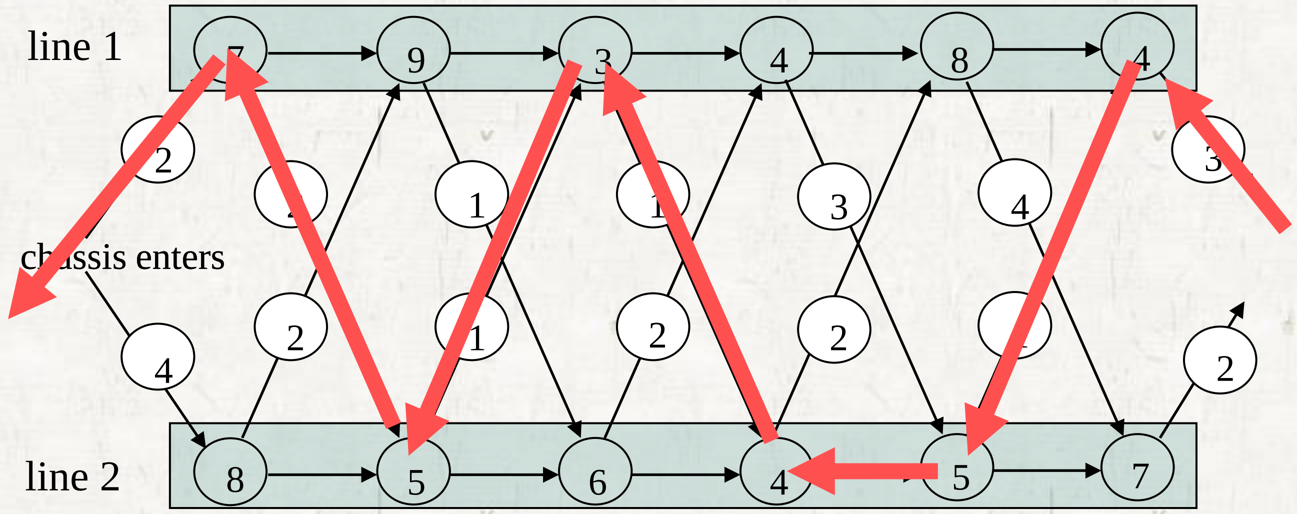
S	1	2	3	4	5	6
1	9	18	20	24	32	35
2	12	16	22	25	30	37

s^* =38

L	1	2	3	4	5	6
1	-	1	2	1	1	2
2	-	1	2	1	2	2

$l^* = 1$

Assembly-line scheduling



S	1	2	3	4	5	6
1	9	18	20	24	32	35
2	12	16	22	25	30	37

*s** =38

L	1	2	3	4	5	6
1	-	1	2	1	1	2
2	-	1	2	1	2	2

*l** = 1

Assembly-line scheduling

```
FASTEST-WAY ($a, t, e, x, n$)
    $s[1][1] = e[1] + a[1][1]$
    $s[2][1] = e[2] + a[2][1]$

    **for** $j = 2$ **to** $n$
        **if** $s[1][j-1] \leq s[2][j-1] + t[2][j-1]$
            $s[1][j] = s[1][j-1] + a[1][j]$
            $l[1][j] = 1$
        **else** $s[1][j] = s[2][j-1] + t[2][j-1] + a[1][j]$
            $l[1][j] = 2$

        **if** $s[2][j-1] \leq s[1][j-1] + t[1][j-1]$
            $s[2][j] = s[2][j-1] + a[2][j]$
            $l[2][j] = 2$
        **else** $s[2][j] = s[1][j-1] + t[1][j-1] + a[2][j]$
            $l[2][j] = 1$

    **if** $s[1][n] + x[1] \leq s[2][n] + x[2]$
        $s^* = s[1][n] + x[1]$
        $l^* = 1$
    **else** $s^* = s[2][n] + x[2]$
        $l^* = 2$
```

Assembly-line scheduling

```
PRINT-STATIONS ($l$, $l^*$, $n$)
1   $i = l^*$
2   print "line " $i$ ", station" $n$
3   for $j = n$ downto 2
4       $i = l[i][j]$
5       print "line " $i$ ", station" $j - 1$
```

- **Result of PRINT-STATIONS**

line 1, station 6
line 2, station 5
line 2, station 4
line 1, station 3
line 2, station 2
line 1, station 1

L	1	2	3	4	5	6
1	-	1	2	1	1	2
2	-	1	2	1	2	2

$l^* = 1$

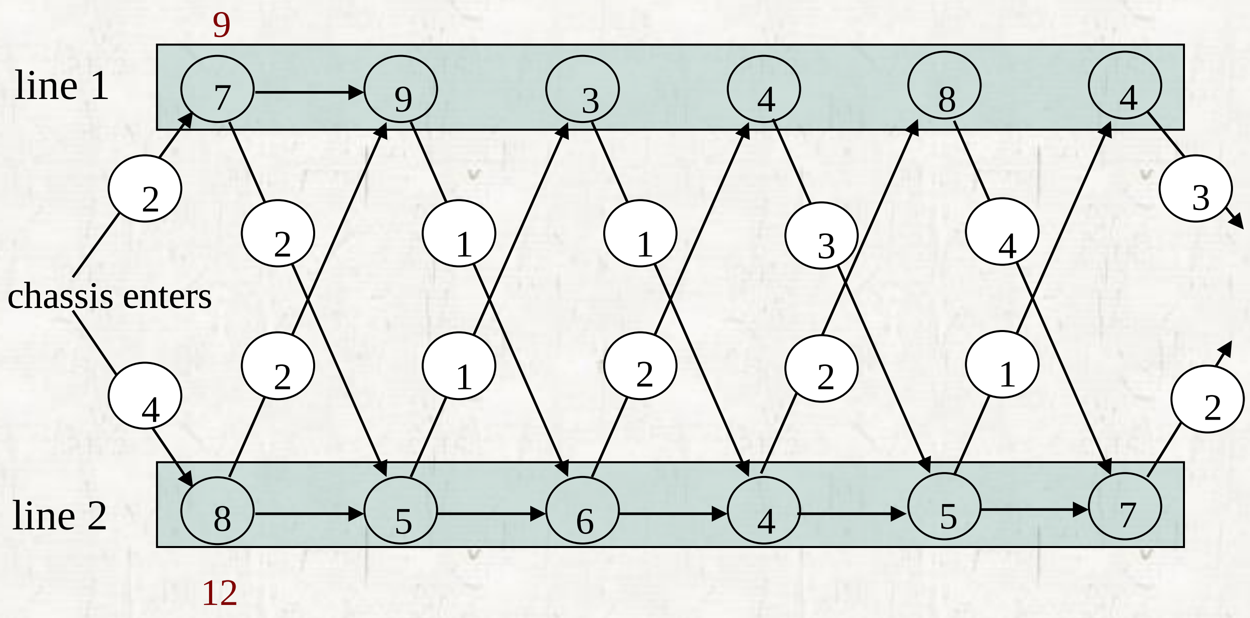
Assembly-line scheduling

- **Space consumption**
  - Table ***s*****:** $2n$
  - Table ***l*****:** $2n$
  - $\Theta(n)$ elements in total.
- **Running time**
  - Computing each element requires $\Theta(1)$ time.
  - $\Theta(n)$ time in total

Assembly-line scheduling

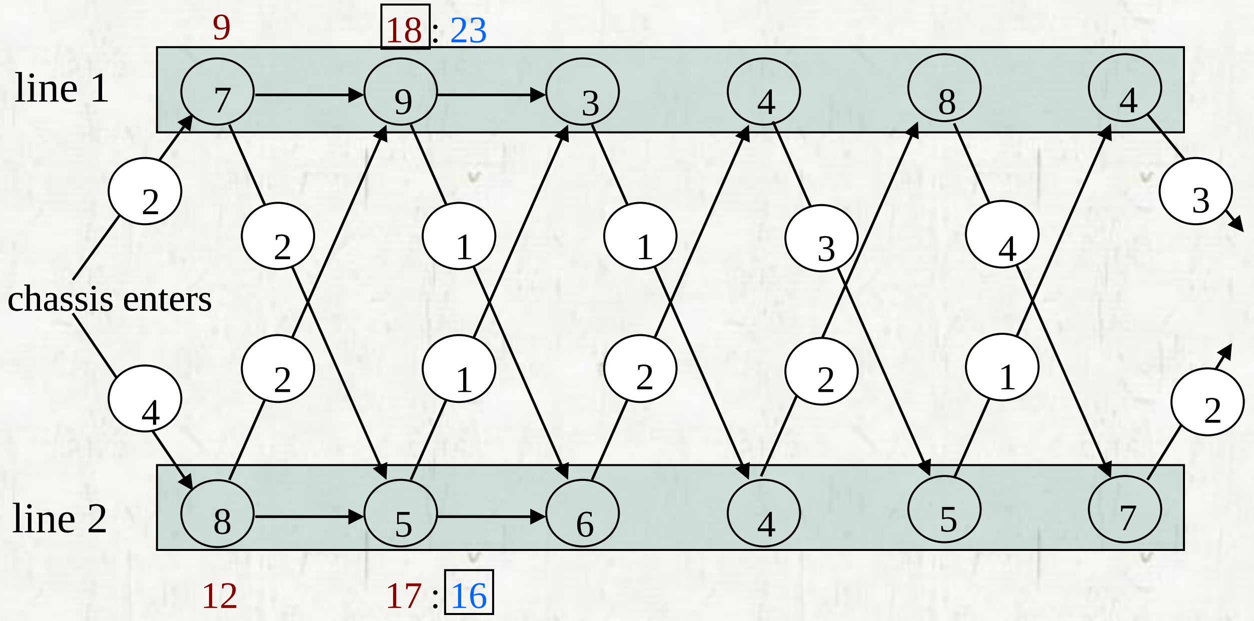
- **Fastest time only**
  - $\Theta(1)$ space
    - Table ***l*** is not necessary.
    - Table ***s***
      - $2n$ elements → 4 elements

Assembly-line scheduling



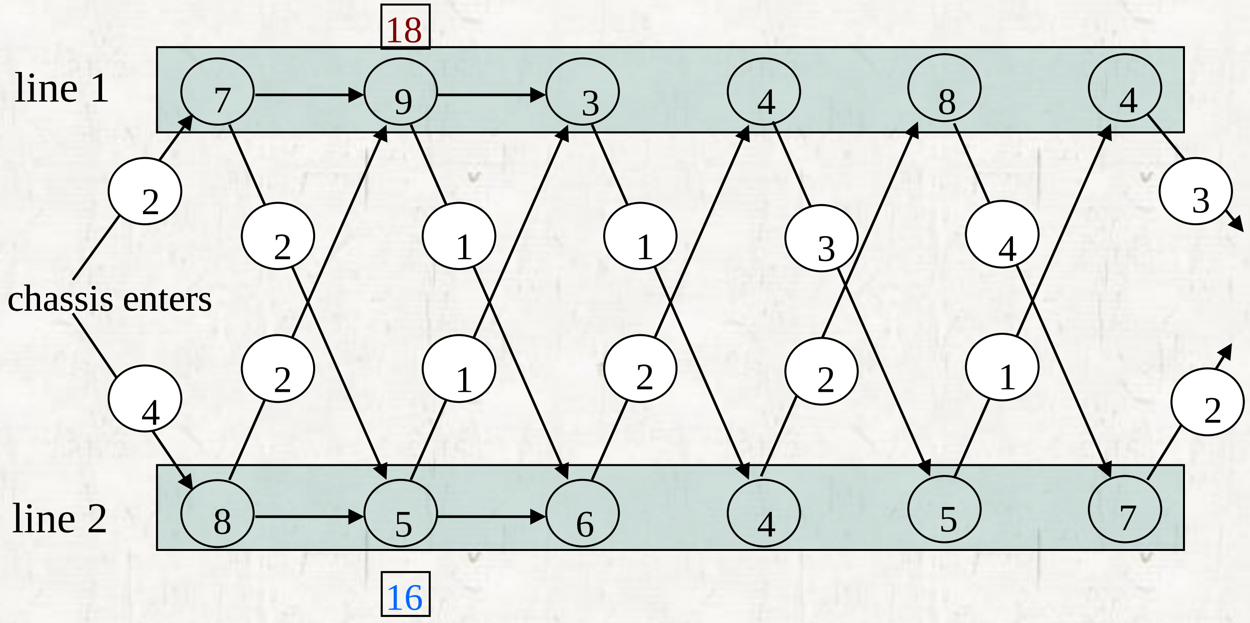
S	1	2	3	4	5	6
1	9					
2	12					

Assembly-line scheduling



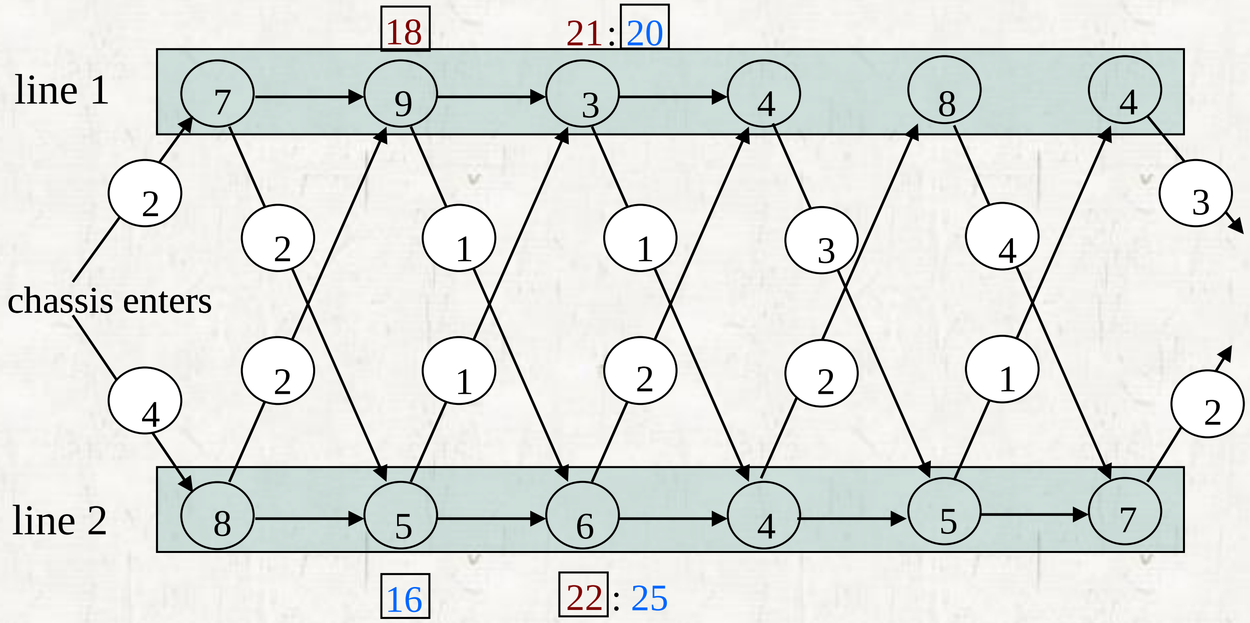
S	1	2	3	4	5	6
1	9	18				
2	12	16				

Assembly-line scheduling



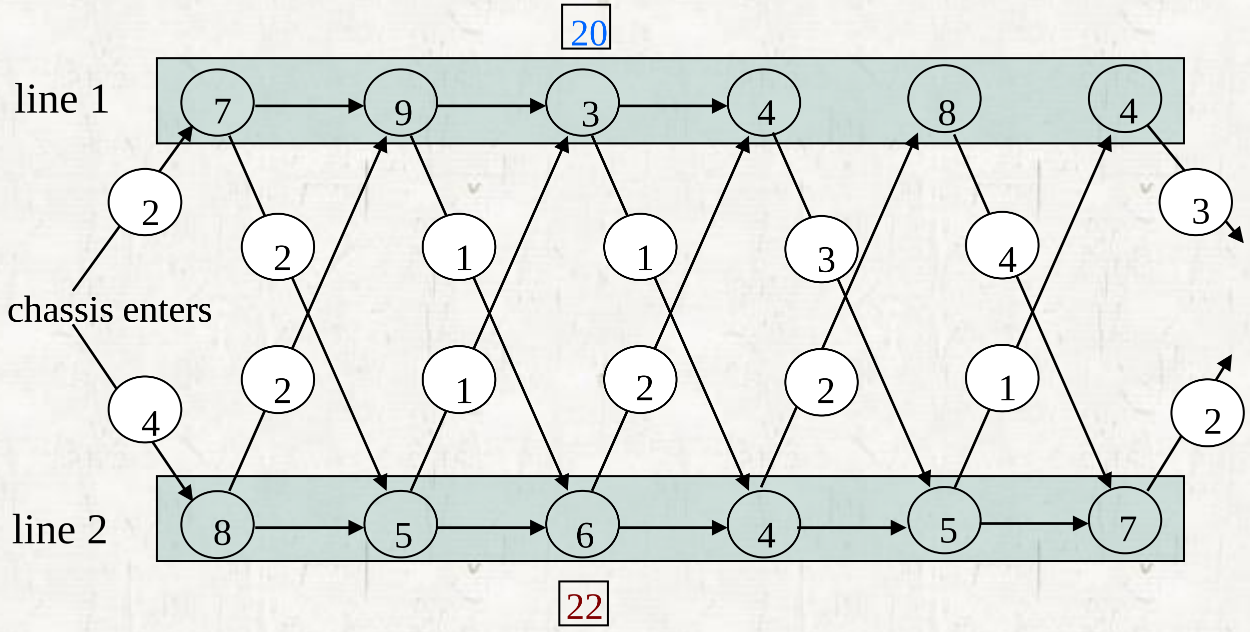
S	1	2	3	4	5	6
1		18				
2		16				

Assembly-line scheduling



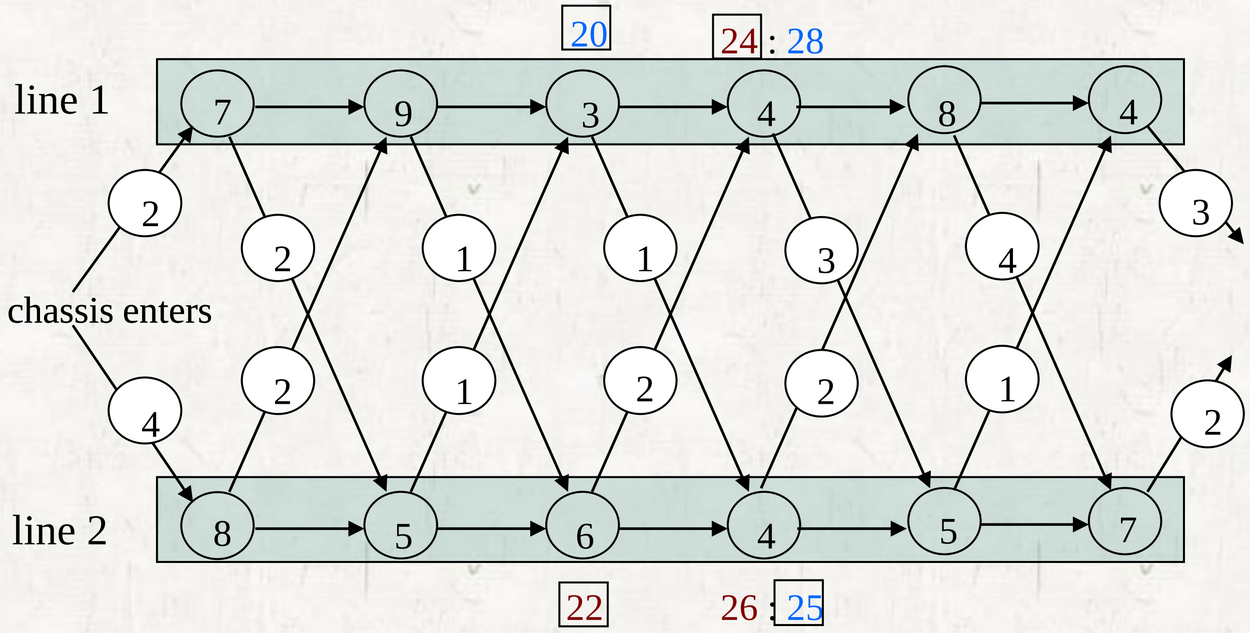
S	1	2	3	4	5	6
1		18	20			
2		16	22			

Assembly-line scheduling



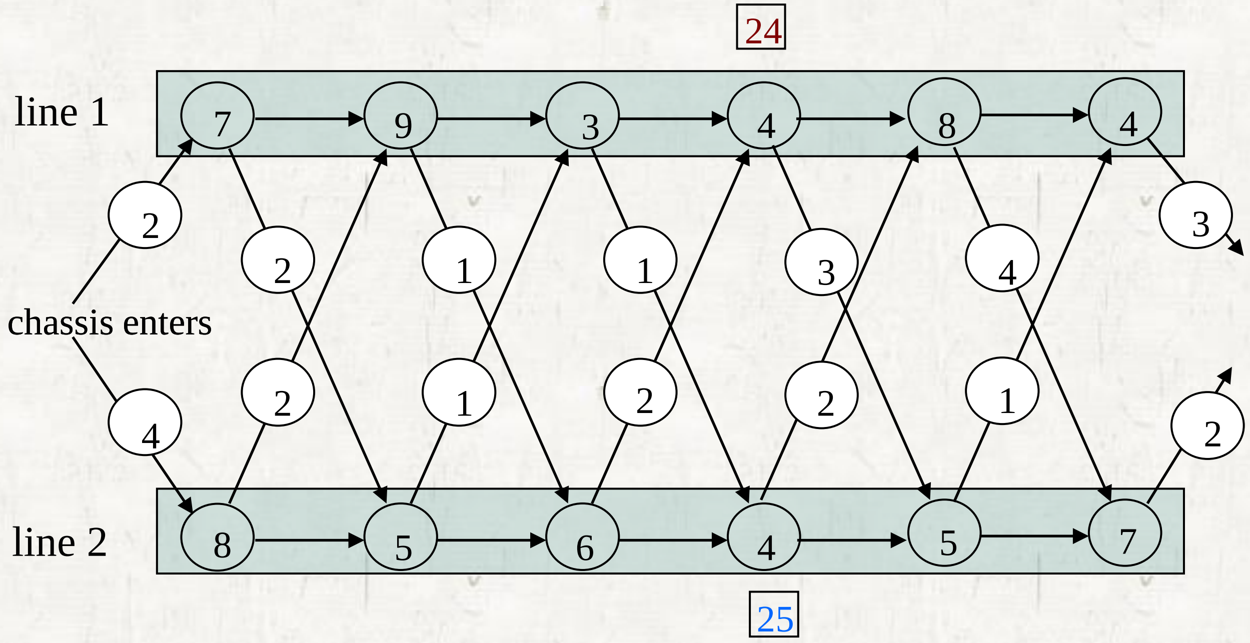
S	1	2	3	4	5	6
1	▨	▨	20	▨	▨	▨
2	▨	▨	22	▨	▨	▨

Assembly-line scheduling



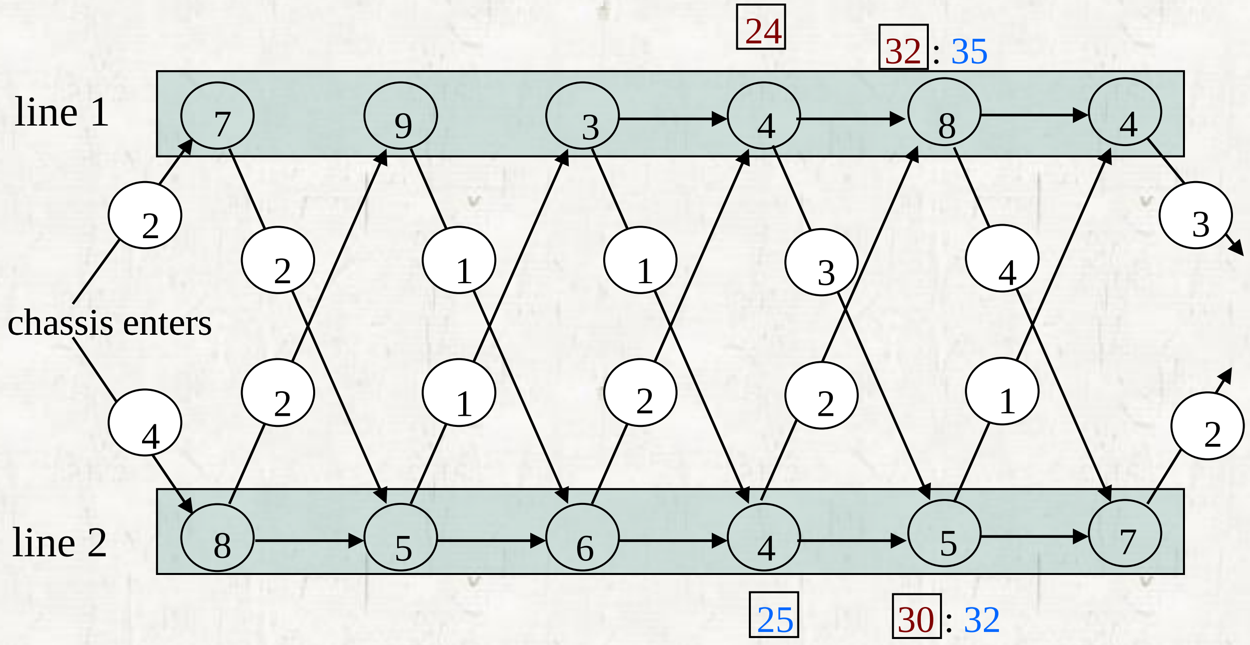
S	1	2	3	4	5	6
1			20	24		
2			22	25		

Assembly-line scheduling



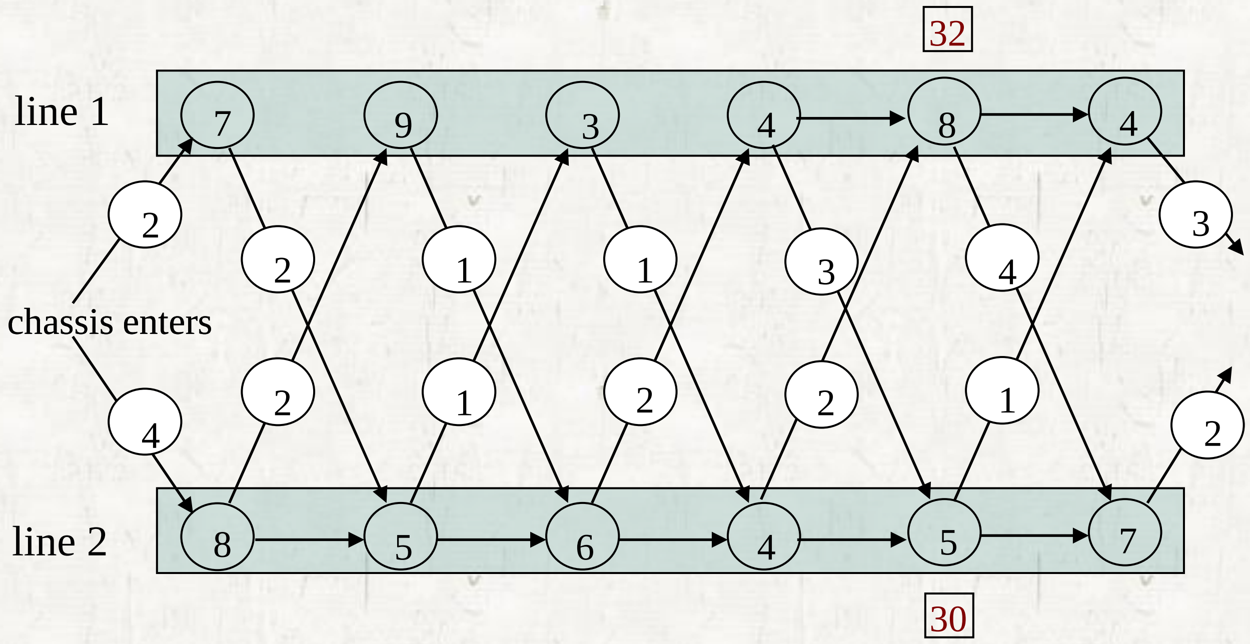
S	1	2	3	4	5	6
1				24		
2				25		

Assembly-line scheduling



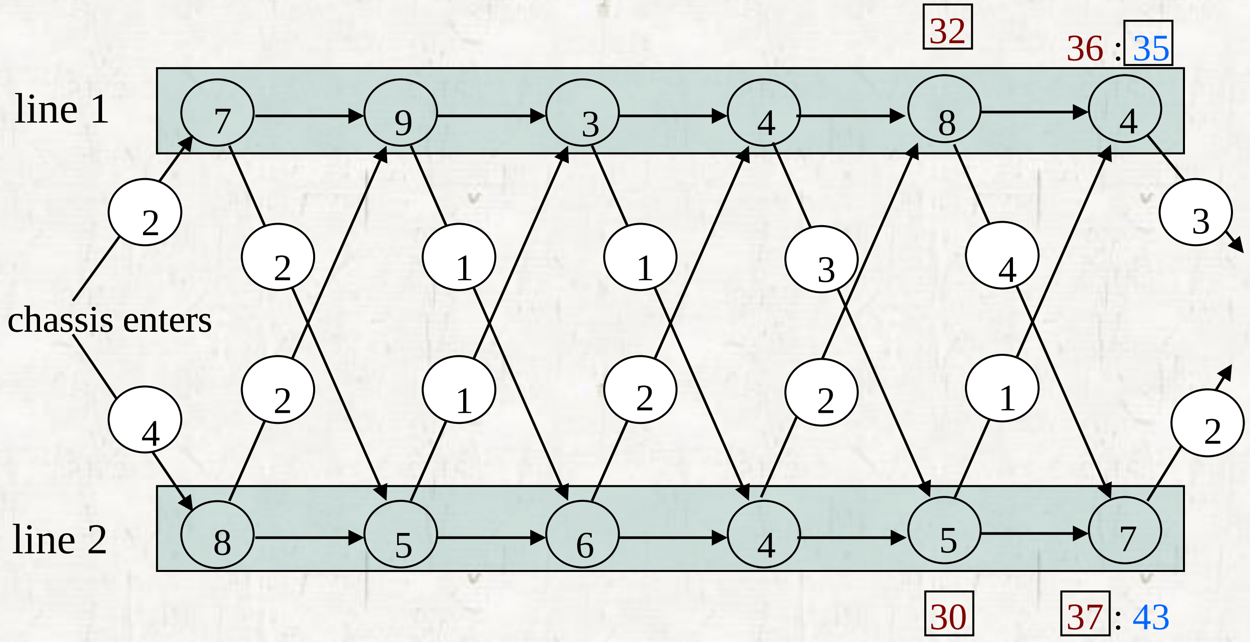
S	1	2	3	4	5	6
1				24	32	
2				25	30	

Assembly-line scheduling



S	1	2	3	4	5	6
1					32	
2					30	

Assembly-line scheduling



S	1	2	3	4	5	6
1					32	35
2					30	37

Assembly-line scheduling

- **Fastest time only**
  - $\Theta(1)$ space
    - Table ***s***
      - $2n$ elements → 4 elements

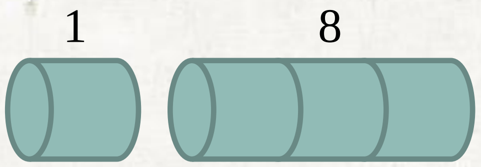
Contents

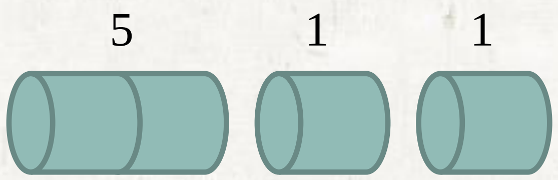
Problems	Space	Time
Assembly-line scheduling	$\Theta(n)$	$\Theta(n)$
Rod cutting		
Longest common subsequence		
Matrix-chain multiplication		

Rod cutting

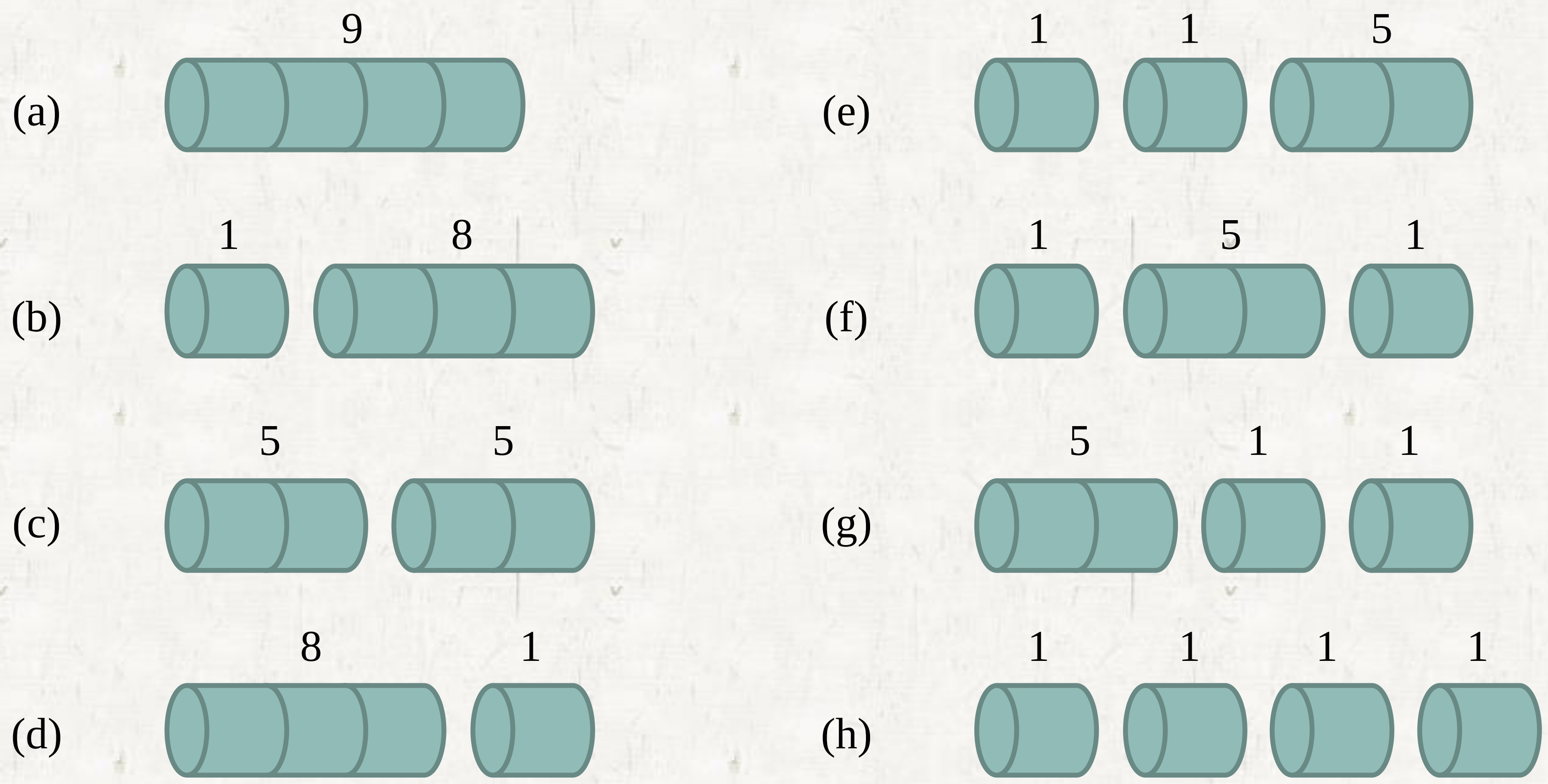
- The ***rod-cutting problem*****:** Given a rod of length n inches and a table of prices p_i for $i = 1, 2, \ldots n$, determine the maximum revenue r_n obtainable by cutting up the rod and selling the pieces.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30





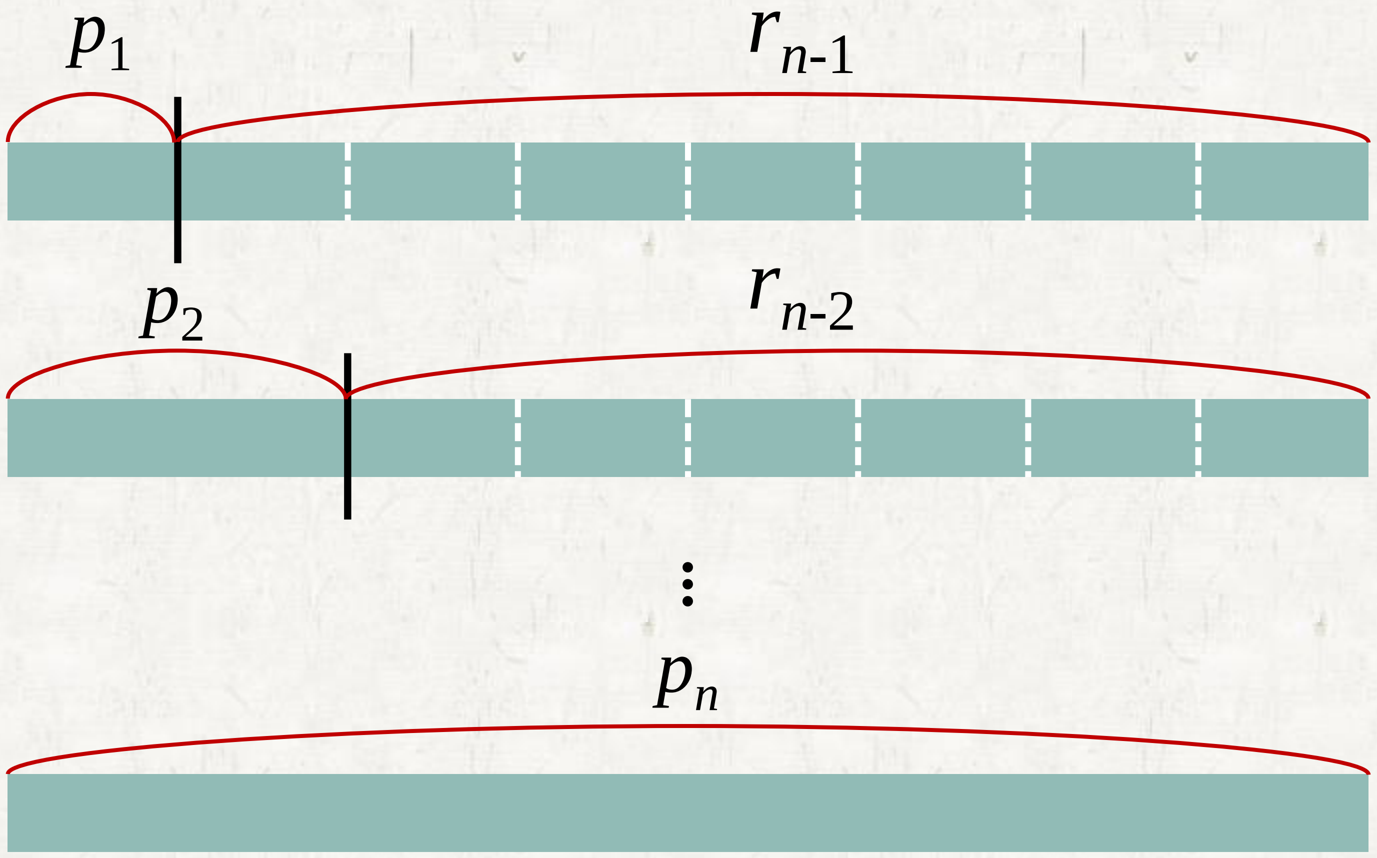
Rod cutting



length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Rod cutting

$$r_n = \max_{1 \le i \le n}(p_i + r_{n-i})$$

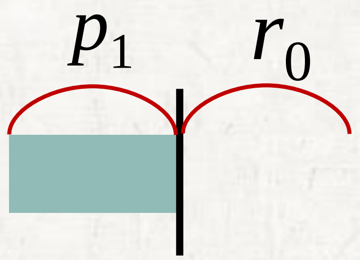


Rod cutting

i	0	1	2	3	4	5	6	7	8	9	10
$p[i]$	0	1	5	8	9	10	17	17	20	24	30
$r[i]$	0	**1**									

$r[0] = 0$

$r[1] = p[1] + r[0] = 1 + 0 = \mathbf{1}$



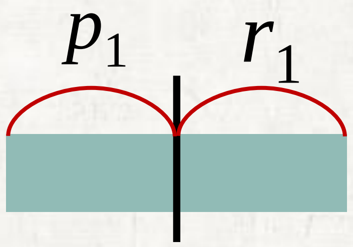
Rod cutting

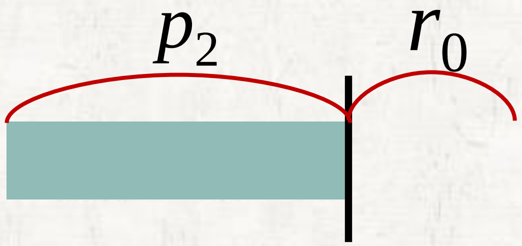
i	0	1	2	3	4	5	6	7	8	9	10
$p[i]$	0	1	5	8	9	10	17	17	20	24	30
$r[i]$	0	1	**5**								

$r[0] = 0$

$r[1] = p[1] + r[0] = 1 + 0 = 1$

$$r[2] = \begin{cases} p[1] + r[1] = 1 + 1 = 2 \\ p[2] + r[0] = 5 + 0 = 5 \end{cases}$$





Rod cutting

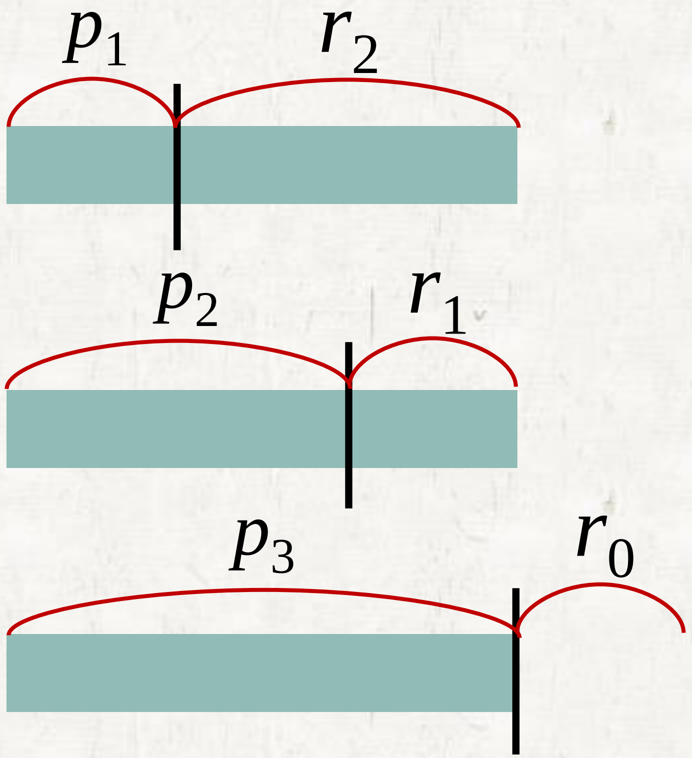
i	0	1	2	3	4	5	6	7	8	9	10
$p[i]$	0	1	5	8	9	10	17	17	20	24	30
$r[i]$	0	1	5	**8**							

$r[0] = 0$

$r[1] = p[1] + r[0] = 1 + 0 = 1$

$$r[2] = \begin{cases} p[1] + r[1] = 1 + 1 = 2 \\ p[2] + r[0] = 5 + 0 = \mathbf{5} \end{cases}$$

$$r[3] = \begin{cases} p[1] + r[2] = 1 + 5 = 6 \\ p[2] + r[1] = 5 + 1 = 6 \\ p[3] + r[0] = 8 + 0 = \mathbf{8} \end{cases}$$



Rod cutting

i	0	1	2	3	4	5	6	7	8	9	10
$p[i]$	0	1	5	8	9	10	17	17	20	24	30
$r[i]$	0	1	5	8	10	13	17	18	22	25	30

$r[0] = 0$

$r[1] = p[1] + r[0] = 1 + 0 = \mathbf{1}$

$$r[2] = \begin{cases} p[1] + r[1] = 1 + 1 = 2 \\ p[2] + r[0] = 5 + 0 = \mathbf{5} \end{cases}$$

$$r[3] = \begin{cases} p[1] + r[2] = 1 + 5 = 6 \\ p[2] + r[1] = 5 + 1 = 6 \\ p[3] + r[0] = 8 + 0 = \mathbf{8} \end{cases}$$

$\vdots$

$$r[10] = \begin{cases} p[1] + r[9] = 1 + 25 = 26 \\ p[2] + r[8] = 5 + 22 = 27 \\ \vdots \\ p[10] + r[0] = 30 + 0 = 30 \end{cases}$$

Rod cutting

i	0	1	2	3	4	5	6	7	8	9	10
$p[i]$	0	1	5	8	9	10	17	17	20	24	30
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

Rod cutting

```
EXTENDED-BOTTOM-UP-CUT-ROD ($p$, $n$)
 1  let $r[0 .. n]$ and $s[0 .. n]$ be new arrays
 2  $r[0] = 0$
 3  for $j = 1$ to $n$
 4      $q = -\infty$
 5      for $i = 1$ to $j$
 6          if $q < p[i] + r[j - i]$
 7              $q = p[i] + r[j - i]$
 8              $s[j] = i$
 9      $r[j] = q$
10  return $r$ and $s$
```

Rod cutting

```
PRINT-CUT-ROD-SOLUTION($p$, $n$)
1  ($r$, $s$) = EXTENDED-BOTTOM-UP-CUT-ROD ($p$, $n$)
2  **while** $n > 0$
3      print $s[n]$
4      $n = n - s[n]$
```

Rod cutting

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

Rod cutting

- Space consumption
  - $\Theta(n)$
- Running time
  - $\Theta(n^2)$
  - $1 + 2 + 3 + 4 + \cdots + n = n(n+1)/2$

Rod cutting

- Table ***r*** can be reduced like table ***s*** in assembly-line scheduling?

Contents

Problems	Space	Time
Assembly-line scheduling	$\Theta(n)$	$\Theta(n)$
Rod cutting	$\Theta(n)$	$\Theta(n^2)$
Longest common subsequence		
Matrix-chain multiplication		

Longest common subsequence

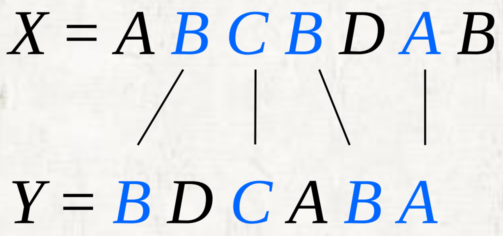
- **Definition**
  - ***Character***
  - ***String*** (or ***sequence***): A list of characters
    - ex> strings over {0,1}: Binary strings
    - ex> strings over {A,C,G,T}: DNA sequences

Longest common subsequence

- ***Substring***
  - *CBD* is a substring of *ABCBDAB*
- ***Subsequence***
  - *BCDB* is a subsequence of *ABCBDAB*
- ***Common subsequence***
  - *BCA* is a common subsequence of *X*=*ABCBDAB* and *Y*=*BDCABA*

Longest common subsequence

- ***Longest common subsequence* (LCS)**
  - *BCBA* is the longest common subsequence of *X* and *Y*



Longest common subsequence

- **Brute force approach**
  - Enumerate all subsequences of X and check each subsequence if it is also a subsequence of Y and find the longest one.

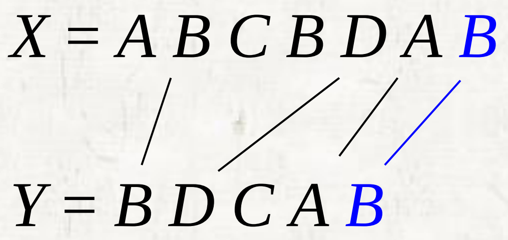
  - Infeasible!
    - The number of subsequences of X is 2^m.

Longest common subsequence

- **Dynamic programming**
  - **The *i*th *prefix* $\boldsymbol{X_i}$ of $\boldsymbol{X}$** is $X_i = x_1 x_2 \ldots x_i$
  - If $X = ABCBDAB$
    - $X_4 = ABCB$
    - $X_0 =$

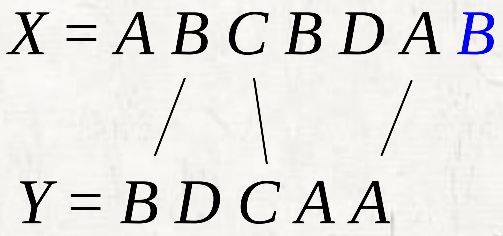
Longest common subsequence

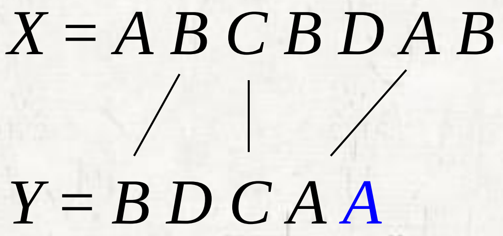
1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.



Longest common subsequence

2. If $x_m \neq y_n$, Z is an LCS of X_{m-1} and Y or an LCS of X and Y_{n-1}.





Longest common subsequence

- $c[i][j]$: The length of an LCS of the sequences X_i and Y_j .

$$c[i][j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0, \\ c[i-1][j-1]+1 & \text{if } i, j > 0 \text{ and } x_i = y_j , \\ \max(c[i][j-1], c[i-1][j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j . \end{cases}$$

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	$\textcircled{B}$	D	$\textcircled{C}$	A	$\textcircled{B}$	$\textcircled{A}$
0	x_i	0	0	0	0	0	0	0
1	A	$\boxed{0}$	$\uparrow 0$	$\uparrow 0$	$\uparrow 0$	$\nwarrow 1$	$\leftarrow 1$	$\nwarrow 1$
2	$\textcircled{B}$	0	$\boxed{\nwarrow 1}$	$\boxed{\leftarrow 1}$	$\leftarrow 1$	$\uparrow 1$	$\nwarrow 2$	$\leftarrow 2$
3	$\textcircled{C}$	0	$\uparrow 1$	$\uparrow 1$	$\boxed{\nwarrow 2}$	$\boxed{\leftarrow 2}$	$\uparrow 2$	$\uparrow 2$
4	$\textcircled{B}$	0	$\nwarrow 1$	$\uparrow 1$	$\uparrow 2$	$\uparrow 2$	$\boxed{\nwarrow 3}$	$\leftarrow 3$
5	D	0	$\uparrow 1$	$\nwarrow 2$	$\uparrow 2$	$\uparrow 2$	$\boxed{\uparrow 3}$	$\uparrow 3$
6	$\textcircled{A}$	0	$\uparrow 1$	$\uparrow 2$	$\uparrow 2$	$\nwarrow 3$	$\uparrow 3$	$\boxed{\nwarrow 4}$
7	B	0	$\nwarrow 1$	$\uparrow 2$	$\uparrow 2$	$\uparrow 3$	$\nwarrow 4$	$\boxed{\uparrow 4}$

Longest common subsequence

```
LCS-LENGTH ($X$, $Y$)
    $m = X.length$
    $n = Y.length$
    let $b[1\,..\,m][1\,..\,n]$ and $c[0\,..\,m][0\,..\,n]$ be new tables
    **for** $i = 1$ **to** $m$
        $c[i][0] = 0$
    **for** $j = 0$ **to** $n$
        $c[0][j] = 0$
    **for** $i = 1$ **to** $m$
        **for** $j = 1$ **to** $n$
            **if** $x_i == y_j$
                $c[i][j] = c[i-1][j-1] + 1$
                $b[i][j] = $ "$\boxed{\nwarrow}$"
            **elseif** $c[i-1][j] \geq c[i][j-1]$
                $c[i][j] = c[i-1][j]$
                $b[i][j] = $ "$\uparrow$"
            **else** $c[i][j] = c[i][j-1]$
                $b[i][j] = $ "$\leftarrow$"
    **return** $c$ and $b$
```

Longest common subsequence

```
PRINT-LCS ($b$, $X$, $i$, $j$)
    **if** $i == 0$ or $j == 0$
        **return**
    **if** $b[i][j] ==$ "$\boxed{\nwarrow}$"
        PRINT-LCS ($b$, $X$, $i - 1$, $j - 1$)
        print $x_i$
    **elseif** $b[i][j] ==$ "$\uparrow$"
        PRINT-LCS ($b$, $X$, $i - 1$, $j$)
    **else** PRINT-LCS ($b$, $X$, $i$, $j - 1$)
```

Longest common subsequence

- Multiple LCSs

i	j	0	1	2	3	4	5	6
	y_j		Ⓑ	D	Ⓒ	A	Ⓑ	Ⓐ
0	x_i	0	0	0	0	0	0	0
1	A	$\boxed{0}$	← ↑ 0	← ↑ 0	← ↑ 0	↖ 1	← 1	↖ 1
2	Ⓑ	0	$\boxed{\nwarrow 1}$	$\boxed{\leftarrow 1}$	← 1	← ↑ 1	↖ 2	← 2
3	Ⓒ	$\boxed{0}$	↑ 1	← ↑ 1	$\boxed{\nwarrow 2}$	$\boxed{\leftarrow 2}$	← ↑ 2	← ↑ 2
4	Ⓑ	0	$\boxed{\nwarrow 1}$	← ↑ 1	$\boxed{\uparrow 2}$	← ↑ 2	$\boxed{\nwarrow 3}$	← 3
5	D	0	↑ 1	$\boxed{\nwarrow 2}$	$\boxed{\leftarrow \uparrow 2}$	← ↑ 2	$\boxed{\uparrow 3}$	← ↑ 3
6	Ⓐ	0	↑ 1	↑ 2	← ↑ 2	$\boxed{\nwarrow 3}$	← ↑ 3	$\boxed{\nwarrow 4}$
7	B	0	↖ 1	↑ 2	← ↑ 2	↑ 3	$\boxed{\nwarrow 4}$	$\boxed{\leftarrow \uparrow 4}$

Longest common subsequence

- Multiple LCSs

	j	0	1	2	3	4	5	6
i		y_j	$\textcircled{B}$	D	$\textcircled{C}$	A	$\textcircled{B}$	$\textcircled{A}$
0	x_i	0	0	0	0	0	0	0
1	A	0	← ↑ 0	← ↑ 0	← ↑ 0	↖ 1	← 1	↖ 1
2	$\textcircled{B}$	0	↖ 1	← 1	← 1	← ↑ 1	↖ 2	← 2
3	$\textcircled{C}$	0	↑ 1	← ↑ 1	↖ 2	← 2	← ↑ 2	← ↑ 2
4	$\textcircled{B}$	0	↖ 1	← ↑ 1	↑ 2	← ↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	← ↑ 2	← ↑ 2	↑ 3	← ↑ 3
6	$\textcircled{A}$	0	↑ 1	↑ 2	← ↑ 2	↖ 3	← ↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	← ↑ 2	↑ 3	↖ 4	← ↑ 4

Longest common subsequence

- Multiple LCSs

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	← ↑ 0	← ↑ 0	← ↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	← ↑ 1	↖ 2	← 2
3	C	0	↑ 1	← ↑ 1	↖ 2	← 2	← ↑ 2	← ↑ 2
4	B	0	↖ 1	← ↑ 1	↑ 2	← ↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	← ↑ 2	← ↑ 2	↑ 3	← ↑ 3
6	A	0	↑ 1	↑ 2	← ↑ 2	↖ 3	← ↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	← ↑ 2	↑ 3	↖ 4	← ↑ 4

Longest common subsequence

- Multiple LCSs

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	← ↑ 0	← ↑ 0	← ↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	← ↑ 1	↖ 2	← 2
3	C	0	↑ 1	← ↑ 1	↖ 2	← 2	← ↑ 2	← ↑ 2
4	B	0	↖ 1	← ↑ 1	↑ 2	← ↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	← ↑ 2	← ↑ 2	↑ 3	← ↑ 3
6	A	0	↑ 1	↑ 2	← ↑ 2	↖ 3	← ↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	← ↑ 2	↑ 3	↖ 4	← ↑ 4

Longest common subsequence

- Space: $\Theta(mn)$
- Time: $\Theta(mn)$
- Space reduction: Θ (min(m,n)) (LCS length only)

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A							
2	B							
3	C							
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B							
3	C							
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A	0	0	0	0	1	1	1
2	B							
3	C							
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C							
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B	0	1	1	1	1	2	2
3	C							
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C	0	1	1	2	2	2	2
4	B							
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B	0	1	1	2	2	3	3
5	D							
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D	0	1	2	2	2	3	3
6	A							
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D							
6	A	0	1	2	2	3	3	4
7	B							

Longest common subsequence

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D							
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

Longest common subsequence

	j	0	1	2	3	4	5	6	…	100,000
i		y_j	B	D	A	C	B	A	…	A
0	x_i	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥 …	🟥
1	A	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥 …	🟥
2	B								…	
3	C								…	
4	B								…	
5	A								…	

Longest common subsequence

	j	0	1	2	3	4	5
i		y_j	A	B	C	B	A
0	x_i	🟥	🟥	🟥	🟥	🟥	🟥
1	B	🟥	🟥	🟥	🟥	🟥	🟥
2	D						
3	A						
4	C						
5	B						
6	A						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100,000	A						

Longest common subsequence

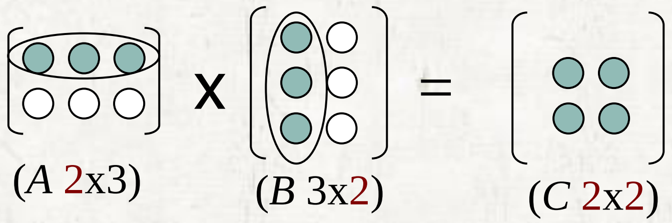
- Space reduction: Θ (min(m,n)) (LCS length only)

Contents

Problems	Space	Time
Assembly-line scheduling	$\Theta(n)$	$\Theta(n)$
Rod cutting	$\Theta(n)$	$\Theta(n^2)$
Longest common subsequence	$\Theta(mn)$ $\Theta(n^2)$	$\Theta(mn)$ $\Theta(n^2)$
Matrix-chain multiplication		

Matrix-chain multiplication

- **Multiplying two matrices *A* and *B***
  - ***A* ($p \times q$) and *B* ($r \times s$) can be multiplied only if $q = r$.**



  - If *A* is a $p \times q$ matrix and *B* is a $q \times r$ matrix,
    the resulting matrix is a $p \times r$ matrix.
- **The number of scalar multiplications is *pqr* .**
  - ***pr*** elements are computed and each element needs ***q*** scalar multiplications.

Matrix-chain multiplication

- **The order of multiplications**

$$(A_1 \cdot A_2) \cdot A_3 = A_1 \cdot (A_2 \cdot A_3)$$

- It does not change ***the value of computation.***
- However, it changes ***the amount of computation.***

Matrix-chain multiplication

- Computing $\boldsymbol{A_1A_2A_3}$ where $\boldsymbol{A_1}$: 10×100 $\boldsymbol{A_2}$: 100×5 $\boldsymbol{A_3}$: 5×50

- **$(A_1\ A_2)\ A_3$**
  - $(A_1\ A_2)$ = 10*100*5 = 5000, $(10\times5)\ A_3$ = 10*5*50 = 2500
    =>5000 + 2500 = **7,500**

- **$A_1\ (A_2\ A_3)$**
  - $(A_2\ A_3)$ = 100*5*50 =25000, $A_1(100\times50)$ = 10 *100*50 = 50000
    =>25000 + 50000 = **75,000**

- **Computing $(A_1\ A_2)\ A_3$ is 10 times faster.**

Matrix-chain multiplication

- **Matrix-chain multiplication problem**
  - Given a chain $A_1, A_2, \ldots, A_n$ of n matrices, where matrix A_i has dimension $p_{i\text{-}1} \times p_i$, find the order of matrix multiplications minimizing the scalar multiplications to compute the product.
  - That is, to fully parenthesize the product of matrices minimizing scalar
  - A_1: $p_0 \times p_1$, A_2: $p_1 \times p_2$, A_3: $p_2 \times p_3$ …

Matrix-chain multiplication

- The product $A_1\ A_2\ A_3\ A_4$ can be fully parenthesized in five distinct ways.

  $A_1(A_2(A_3\ A_4)),\quad A_1((A_2\ A_3)\ A_4),$

  $(A_1\ A_2)(A_3\ A_4),$

  $(A_1(A_2\ A_3))A_4,\quad ((A_1\ A_2)\ A_3)\ A_4.$

Matrix-chain multiplication

- **The Brute-force approach is inefficient.**
  - The number of parenthesizations of a product of *n* matrices, denoted by *P*(*n*), is as follows.

$$P(n) = \begin{cases} 1 & \text{if } n=1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n\geq 2 \end{cases}$$

  - The number of enumerated parenthesizations is $\Omega(4^n/n^{3/2})$.

Matrix-chain multiplication

- **Dynamic programming**
- **Optimal substructure**
  - $m[i][j]$: The minimum number of scalar multiplications for computing $A_i A_{i+1} \ldots A_{j.}$

$$m[i][j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i][k] + m[k+1][j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

  - matrix $A_i : p_{i\text{-}1} \times p_i$
  - computing $A_{i..k} A_{k+1..j}$ takes $p_{i\text{-}1}\, p_k\, p_j$ scalar multiplications.
  - $s[i][j]$ stores the optimal k for tracing the optimal solution.

Matrix-chain multiplication

i \ j	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

m

i \ j	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

s

0	30
1	35
2	15
3	5
4	10
5	20
6	25

p

$$m[2][5] = \min\begin{cases} m[2][2] + m[3][5]+p[1]p[2]p[5] = 0 +2500+35\cdot15\cdot20 = 13000, \\ m[2][3] + m[4][5]+p[1]p[3]p[5] = 2625+1000+35\cdot5\cdot20 = 7125, \\ m[2][4] + m[5][5]+p[1]p[4]p[5] = 4375+0 +35\cdot10\cdot20 = 11375 \end{cases}$$

i=2, j=5, $i \le k < j$

$(A_2)(A_3 A_4 A_5)$

$(A_2 A_3)(A_4 A_5)$

$(A_2 A_3 A_4)(A_5)$

Matrix-chain multiplication

```
MATRIX-CHAIN-ORDER ($p$)
1   $n = p.length - 1$
2   let $m[1 .. n][1 .. n]$ and $s[1 .. n - 1][2 .. n]$ be new tables
3   **for** $i = 1$ **to** $n$
4       $m[i][i] = 0$
5   **for** $l = 2$ **to** $n$                    // $l$ is the chain length
6       **for** $i = 1$ **to** $n - l + 1$
7           $j = i + l - 1$
8           $m[i][j] = \infty$
9           **for** $k = i$ **to** $j - 1$
10              $q = m[i][k] + m[k + 1][j] + p[i-1]p[k]p[j]$
11              **if** $q < m[i][j]$
12                  $m[i][j] = q$
13                  $s[i][j] = k$
14  **return** $m$ and $s$
```

Matrix-chain multiplication

```
PRINT-OPTIMAL-PARENS ($s$, $i$, $j$)
1   if $i == j$
2       print "$A_i$"
3   else print "("
4       PRINT-OPTIMAL-PARENS($s$, $i$, $s[i][j]$)
5       PRINT-OPTIMAL-PARENS($s$, $s[i][j] + 1$, $j$)
6       print ")"
```

Matrix-chain multiplication

- **Space consumption**
  - $\Theta(n^2)$ space to store *m* and *s* tables.

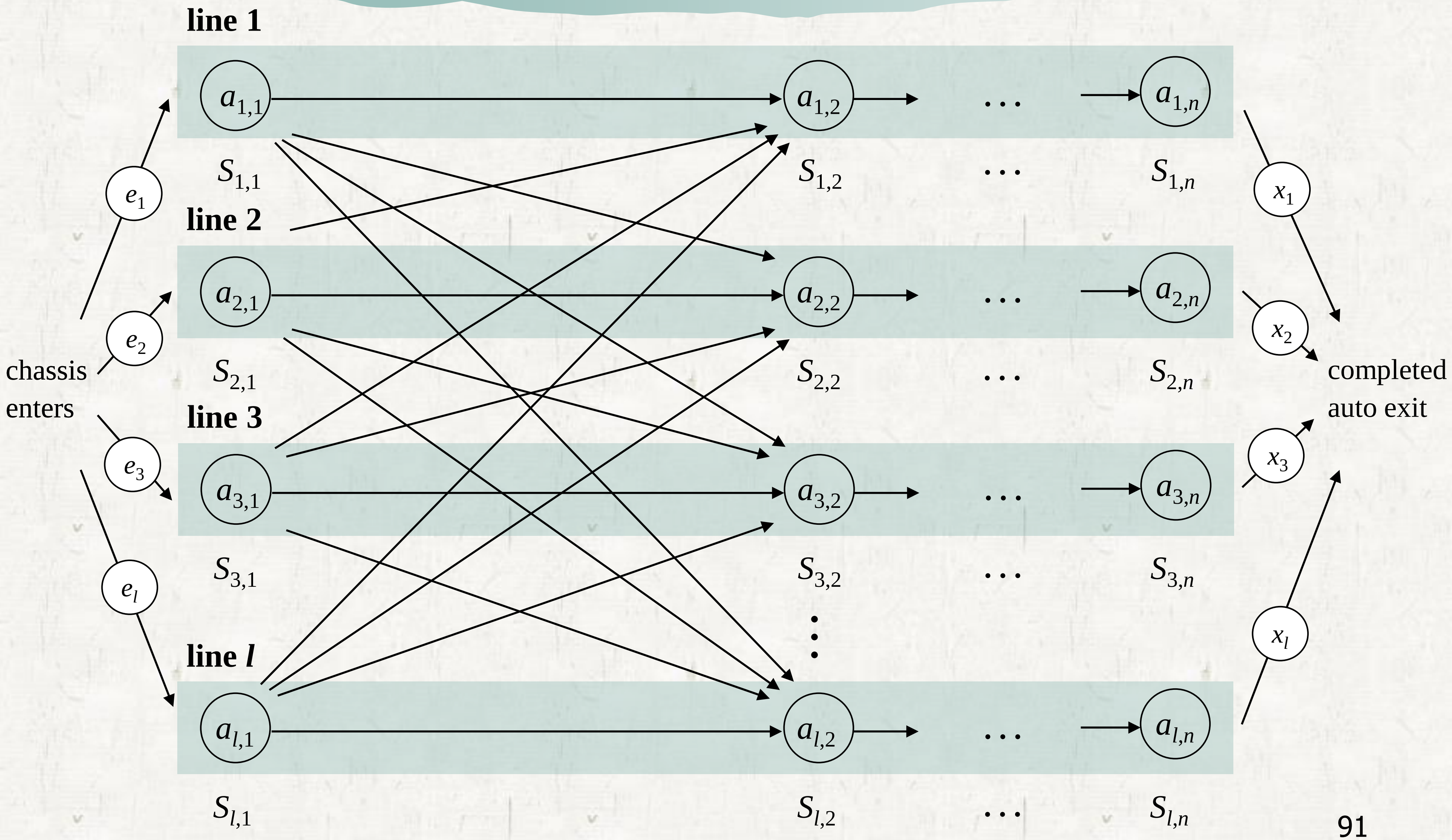
- **Running time**
  - $\Theta(n^3)$

$$1\cdot n + 1\cdot n\text{-}1 + 2\cdot n\text{-}2 + \cdots + (n-1)\cdot 1$$
$$= 1\cdot n + \sum_{k=1}^{n-1} k(n-k)$$
$$= 1\cdot n + \sum_{k=1}^{n-1} kn - \sum_{k=1}^{n-1} k^2$$
$$= n + n^2(n-1)/2 - n(n-1)(2n-1)/6$$
$$= (n^3 + 5n)/6$$
$$= \Theta(n^3)$$

Conclusion

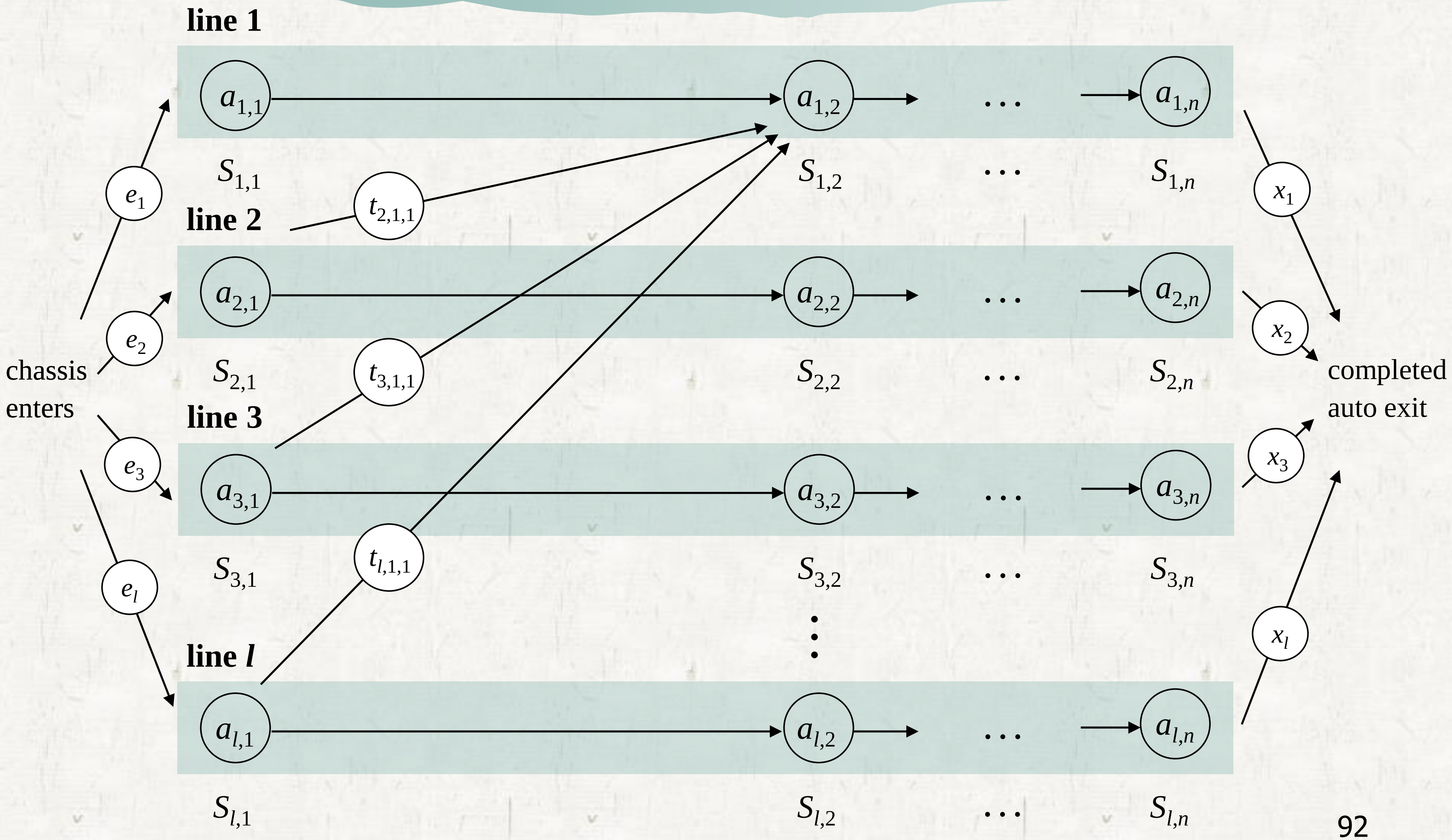
Problems	Space	Time
Assembly-line scheduling	$\Theta(n)$	$\Theta(n)$
Rod cutting	$\Theta(n)$	$\Theta(n^2)$
Longest common subsequence	$\Theta(n^2)$	$\Theta(n^2)$
Matrix-chain multiplication	$\Theta(n^2)$	$\Theta(n^3)$

Assembly-line scheduling (L lines)



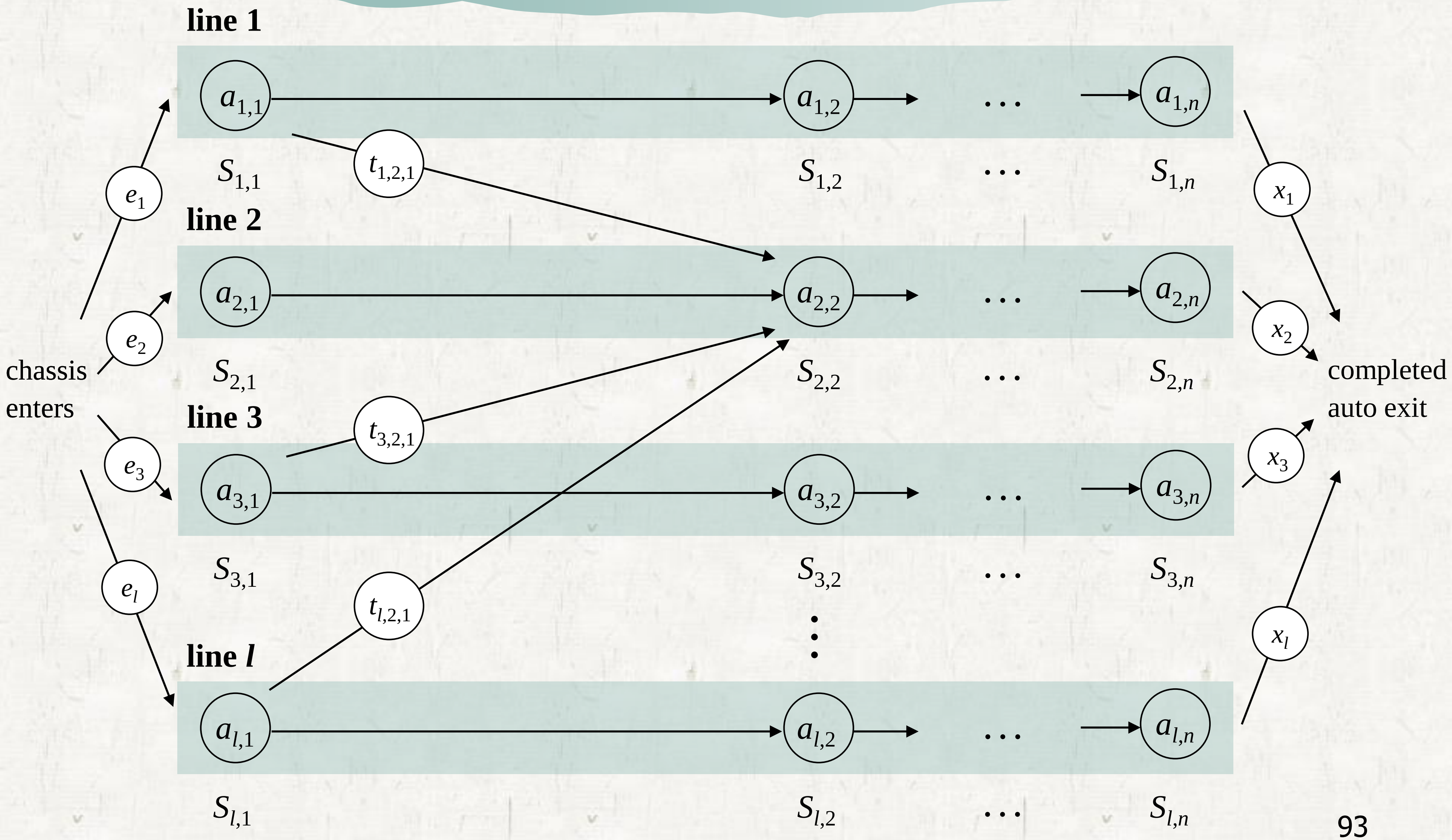
- Transfer time: $t_{i,j,k}$

Assembly-line scheduling (L lines)



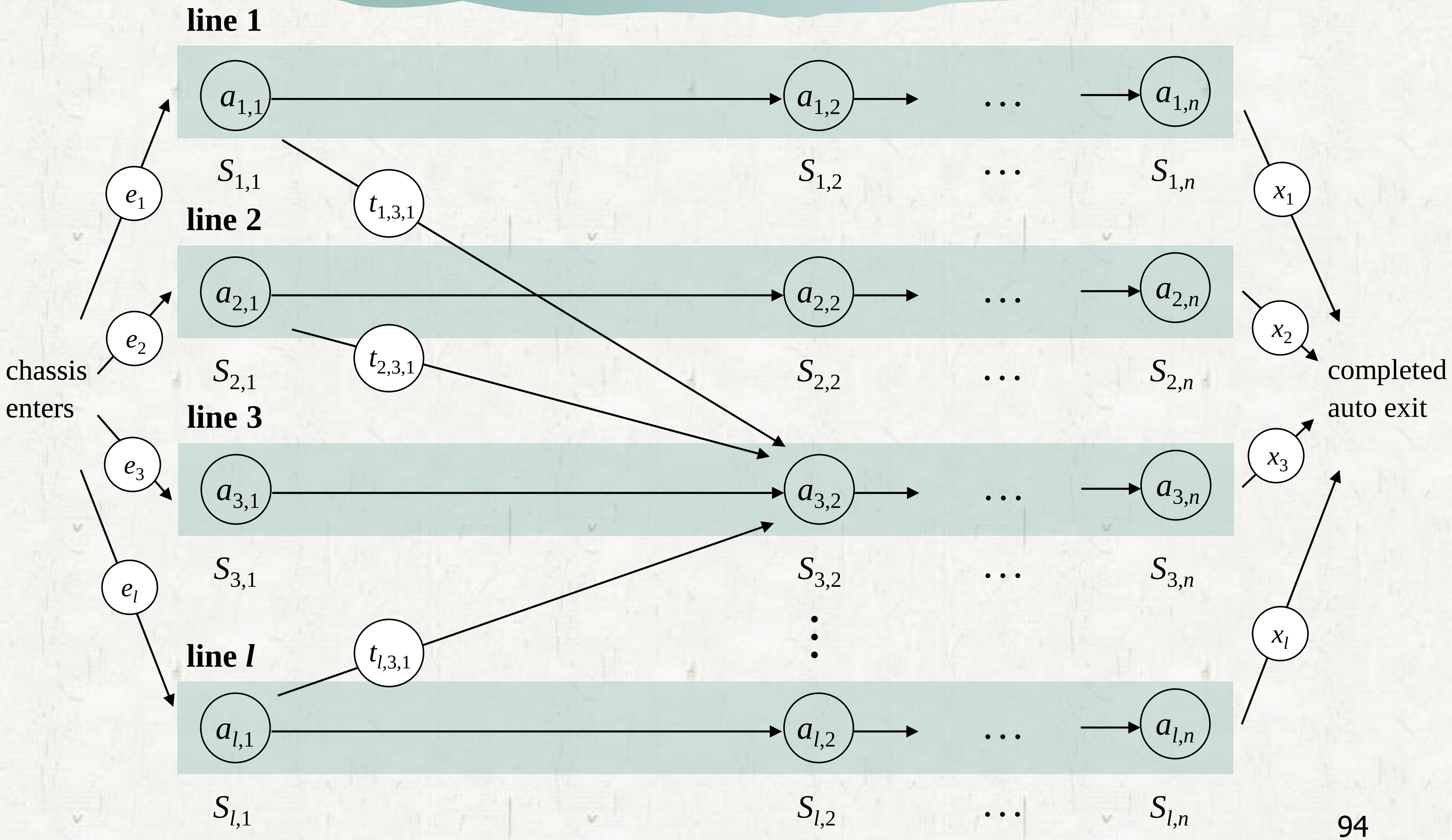
- Transfer time: $t_{i,j,k}$

Assembly-line scheduling (L lines)



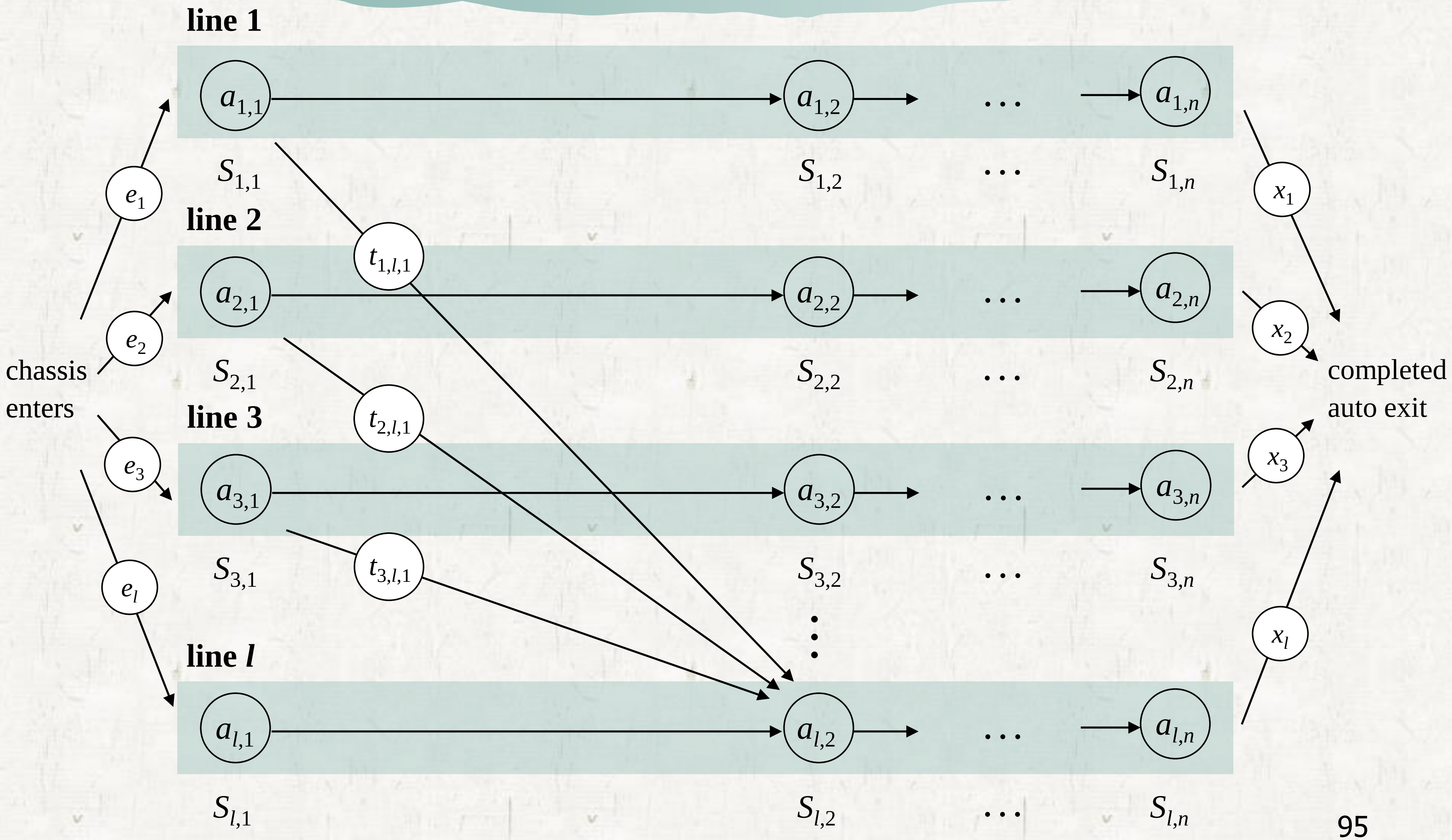
- Transfer time: $t_{i,j,k}$

Assembly-line scheduling (L lines)



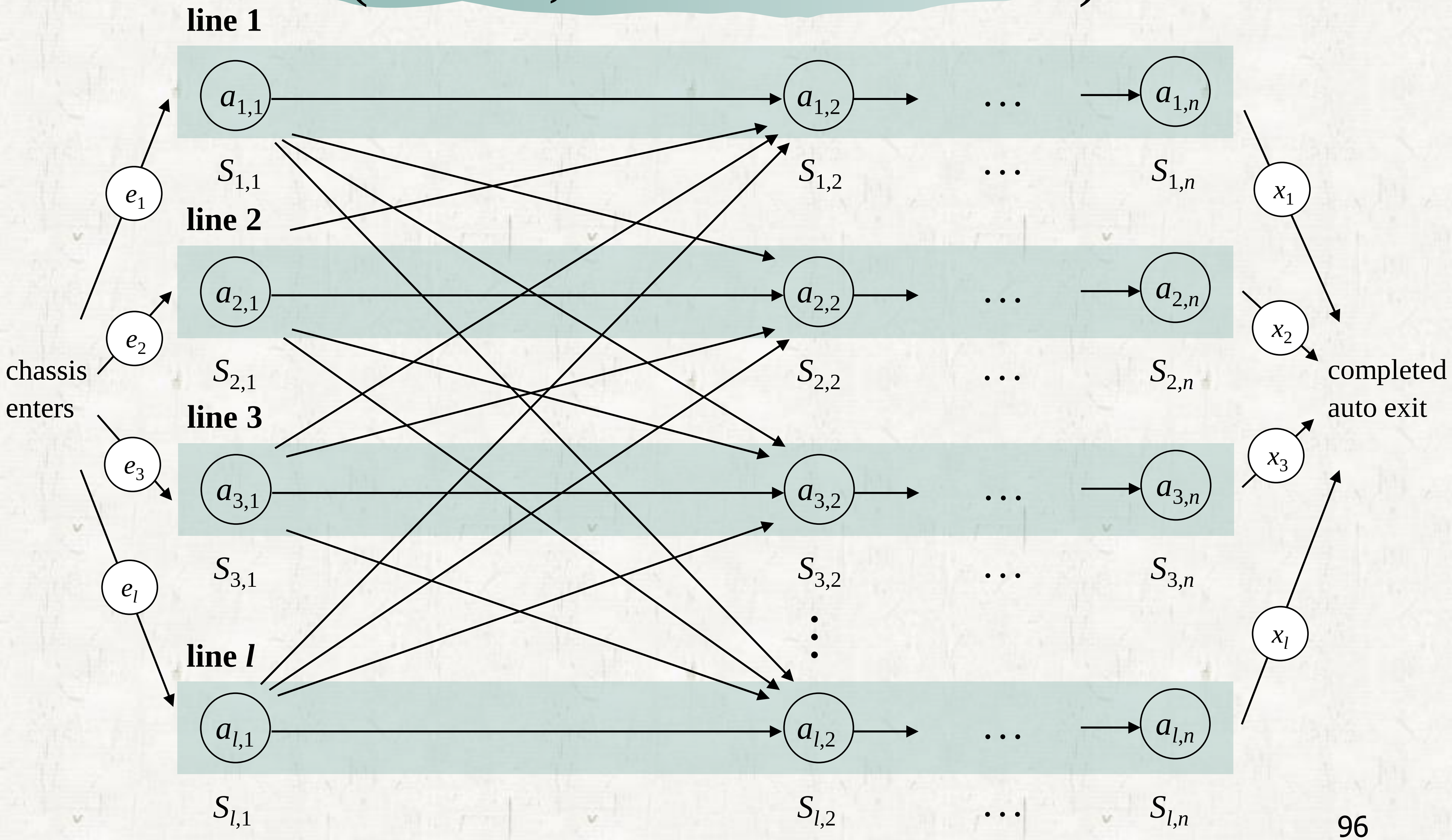
- Transfer time: $t_{i,j,k}$

Assembly-line scheduling (L lines)



- Transfer time: $t_{i,j,k}$

Assembly-line scheduling (L lines, same transfer costs)



- Transfer time: $t_{i,j,k}$