Data Structures for Disjoint Sets

Heejin Park

Hanyang University

Contents

- Disjoint-sets
- Disjoint-set operations
- An application of disjoint-set data structures
- Disjoint-set data structures

Disjoint sets

- ***Disjoint sets***
  - Two sets A and B are disjoint if $A \cap B = \{\}$.
    Ex> $A = \{1, 2\}$, $B = \{3, 4\}$
  - Sets $S_1, S_2, \ldots, S_k$ are disjoint
    if every two distinct sets S_i and S_j are disjoint.
    Ex> $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 8\}$, $S_3=\{5,7\}$

Disjoint sets

- A ***collection*** of disjoint sets
  - A set of disjoint sets is called a collection of disjoint sets.

    Ex> {{1, 2, 3}, {4, 8}, {5,7}}

  - Each set in a collection has a ***representative member*** and the set is identified by the member.

    Ex> {{1, 2, 3}, {4, 8}, {5,7}}

Disjoint sets

- A collection of ***dynamic disjoint sets***
  - **Dynamic**: Sets are changing.
    - New sets are created.
      - {{1, 2, 3}, {4, 8}, {5,7}} ➔ {{1, 2, 3}, {4, 8}, {5,7}, {9}}
    - Two sets are united.
      - {{1, 2, 3}, {4, 8}, {5,7} } ➔ {{1, 2, 3}, {4, 8, 5, 7}}

Disjoint-set operations

- **Disjoint-set operations**
  - **MAKE-SET(x)**
  - **UNION(x, y)**
  - **FIND-SET(x)**

Disjoint-set operations

- **MAKE-SET(*x*)**
  - Given a member *x*, generate a set for *x*.
  - MAKE-SET(9)

    {{1, 2, 3}, {4, 8}, {5,7}} ➔ {{1, 2, 3}, {4, 8}, {5,7}, {9} }

Disjoint-set operations

- **UNION(*x*, *y*)**
  - Given two members *x* and *y*, unite the set containing *x* and another set containing *y*.
  - UNION(1,4)
  - {{1, 2, 3}, {4, 8}, {5,7}} → {{1, 2, 3, 4, 8}, {5,7}}
- **FIND-SET(*x*)**
  - Find the representative of the set containing *x*.
  - FIND-SET(5): 7

Disjoint-set data structures

- **Problem**
  - ***Developing data structures*** to maintain a collection of dynamic disjoint sets supporting disjoint-set operations, which are MAKE-SET(x), UNION(x,y), FIND-SET(x).

Disjoint-set data structures

- **Parameters for running time analysis**
  - #Total operations: m
  - #MAKE-SET ops: n
  - #UNION ops: u
  - #FIND-SET ops: f
  - $m = n + u + f$

Disjoint-set data structures

- $\boldsymbol{u \leq n - 1}$
  - n is the number of sets are generated by MAKE-SET ops.
  - Each UNION op reduces the number of sets by 1.
  - So, after n-1 UNION ops, we have only 1 set and then we cannot do UNION op more.

Assumption

- The first n operations are MAKE-SET operations.

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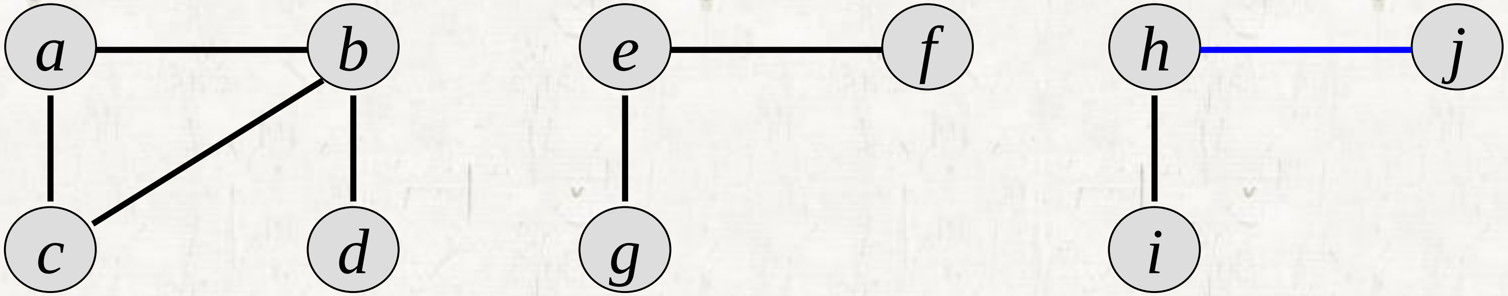
Application

- **Computing connected components (CC)**
  - Static graph
    - Depth-first search: $\Theta(V+E)$
  - Dynamic graph
    - Depth-first search is inefficient.
    - Maintaining a disjoint-set data structure is more efficient.

Connected component computation



{{a,b,c,d}, {e,f,g}, {h,i}, {j}}

➔ {{a,b,c,d}, {e,f,g}, {h, i, j}}

Depth first search: $\Theta(V + E)$

Disjoint-set data structures: UNION(h, j)

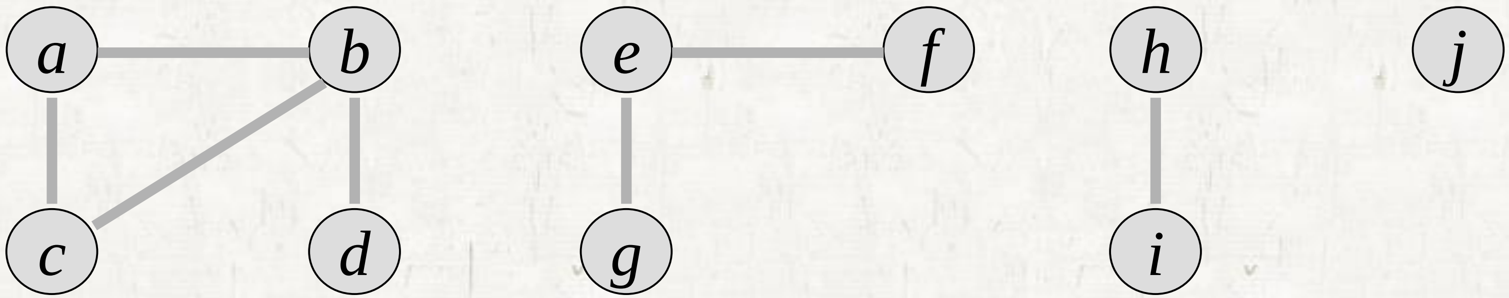
Connected component computation

- **Computing CC using disjoint set operations**

CONNECTED-COMPONENTS(*G*)

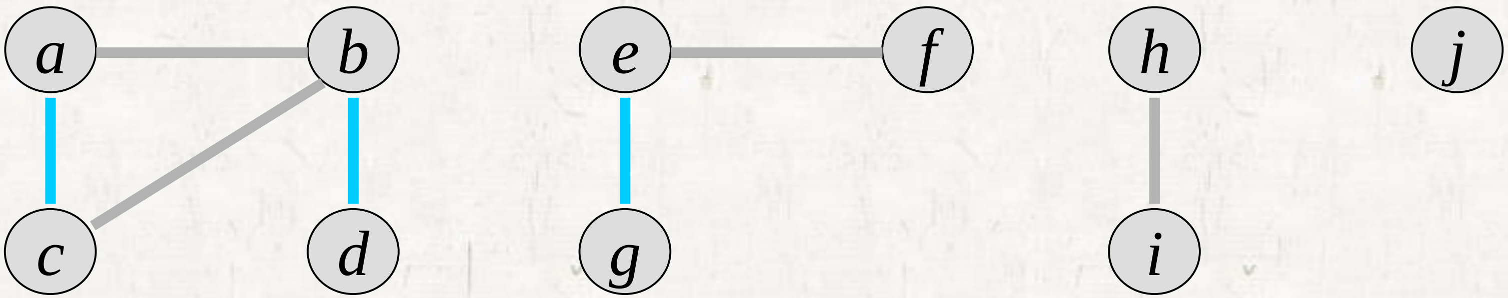
for each vertex $v \in G.V$
    MAKE-SET(v)
for each edge $(u, v) \in G.E$
    if FIND-SET(u) $\neq$ FIND-SET(v)
        UNION(u, v)

Connected component computation



Initial sets	{*a*}	{*b*}	{*c*}	{*d*}	{*e*}	{*f*}	{*g*}	{*h*}	{*i*}	{*j*}
(*b*,*d*)	{*a*}	**{*b*,*d*}**	{*c*}	{*e*}	{*f*}	{*g*}	{*h*}	{*i*}	{*j*}	
(*e*,*g*)	{*a*}	{*b*,*d*}	{*c*}	**{*e*,*g*}**	{*f*}	{*h*}	{*i*}	{*j*}		
(*a*,*c*)	**{*a*,*c*}**	{*b*,*d*}	{*e*,*g*}	{*f*}	{*h*}	{*i*}	{*j*}			

Connected component computation



(*a*,*c*)	{*a*,*c*}	{*b*,*d*}	{*e*,*g*}	{*f*}	{*h*}	{*i*}	{*j*}
(*h*,*i*)	{*a*,*c*}	{*b*,*d*}	{*e*,*g*}	{*f*}	{*h*,*i*}		{*j*}
(*a*,*b*)	{*a*,*b*,*c*,*d*}		{*e*,*g*}	{*f*}	{*h*,*i*}		{*j*}
(*e*,*f*)	{*a*,*b*,*c*,*d*}		{*e*,*f*,*g*}		{*h*,*i*}		{*j*}
(*b*,*c*)	{*a*,*b*,*c*,*d*}		{*e*,*f*,*g*}		{*h*,*i*}		{*j*}

Connected component computation

```
SAME-COMPONENT(u, v)
if FIND-SET(u) == FIND-SET(v)
    return TRUE
else return FALSE
```

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Disjoint-set data structures

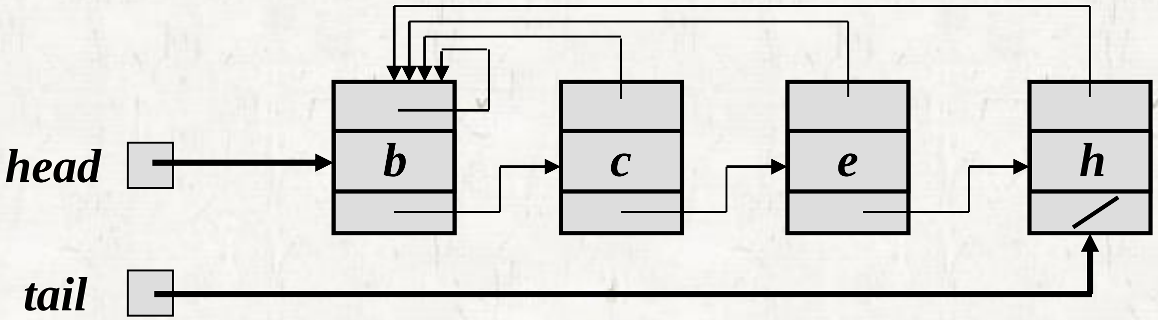
- **Disjoint-set data structures**
  - Linked-list representation
  - Forest representation

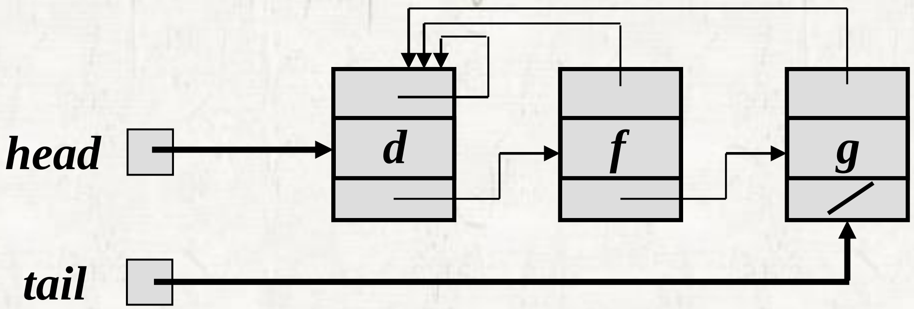
Linked-list representation

- **Linked-list representation**
  - Each set is represented by a linked list.
  - Members of a disjoint set are objects in a linked list.
  - The first object in the linked list is the representative.
  - All objects have pointers to the representative.

Linked-list representation

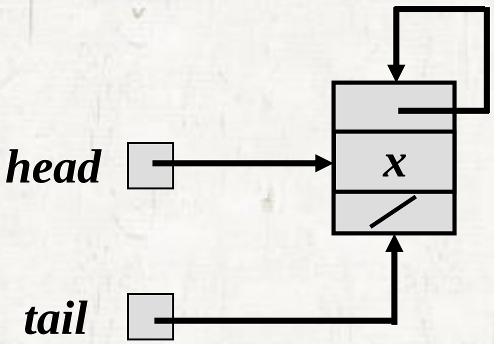
$\{\{b,c,e,h\}, \{d,f,g\}\}$: Two linked lists are needed.





Linked-list representation

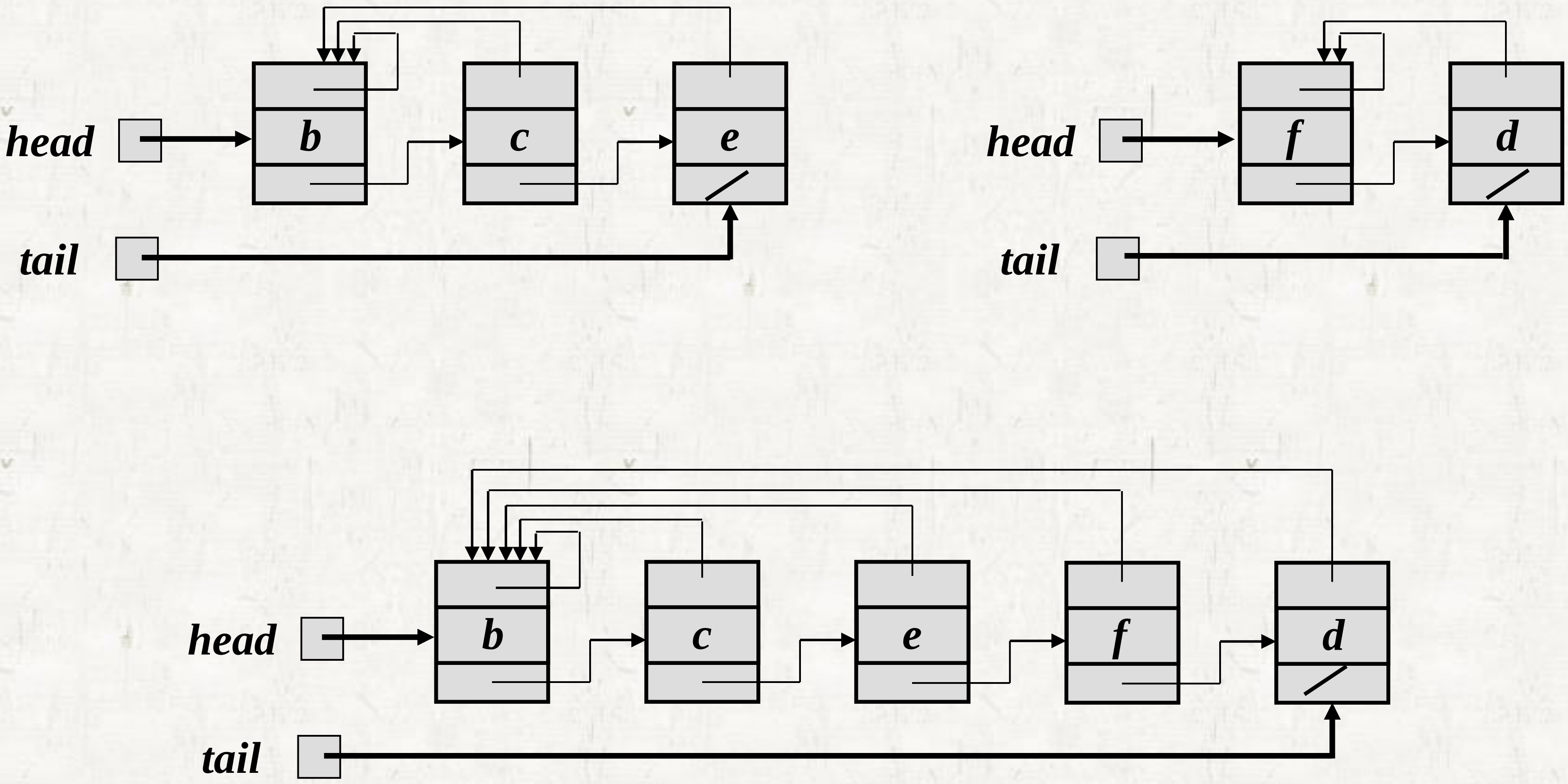
- **MAKE-SET(*x*)**
  - $\Theta(1)$



- **FIND-SET(*x*)**
  - $\Theta(1)$

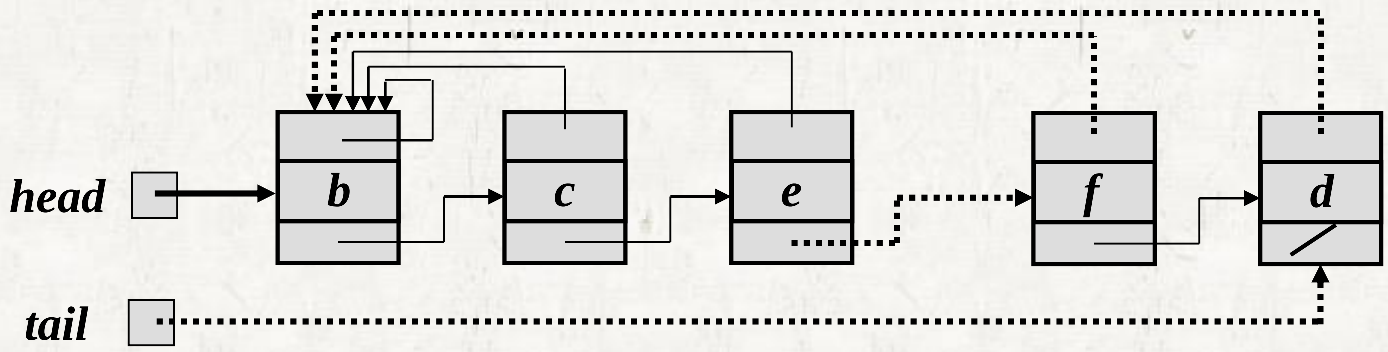
Linked-list representation

- **UNION(*x*,*y*):** Attaching a linked list to the other



Linked-list representation

- **UNION(*x*,*y*):** Attaching a linked list to the other



- $\Theta(m_2)$ time where m_2 is the number of objects in the linked list being attached.
  - Changing tail pointer & linking two linked lists: $\Theta(1)$
  - Changing pointers to the representative: $\Theta(m_2)$

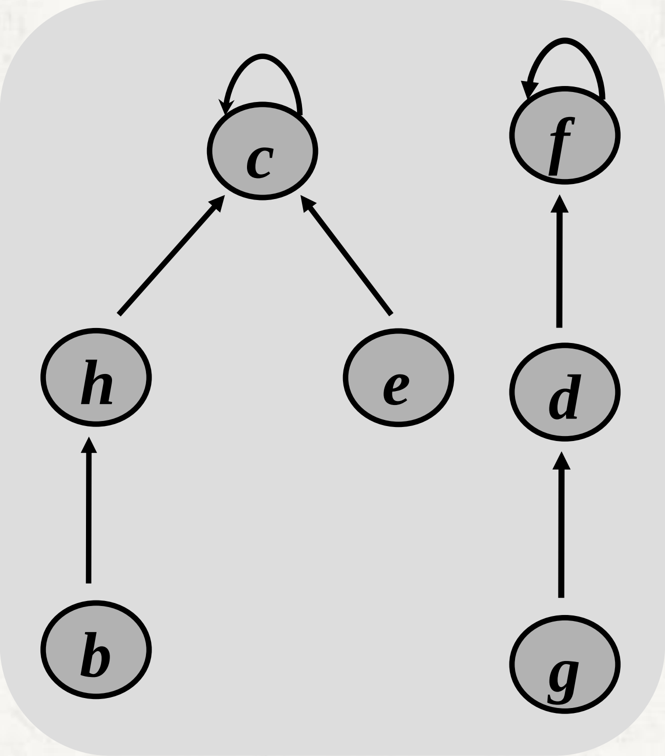
Linked-list representation

- **Running time for m (= n + f + u) operations**
  - Simple implementation of union
    - $O(n+f+un)$ time ➔ $O(m+n^2)$ time
      - Because $u < n$
  - A weighted-union heuristic
    - $O(n+f+u\lg n)$ time ➔ $O(m+n\lg n)$ time

Forest representation

$\{\{b,c,e,h\},\ \{f,d,g\}\}$

- **Forest representation**
  - Each set is represented by a tree.
  - Each member points to its parent.
  - The root of each tree is the rep.



Forest representation

MAKE-SET(x)

$x.p = x$

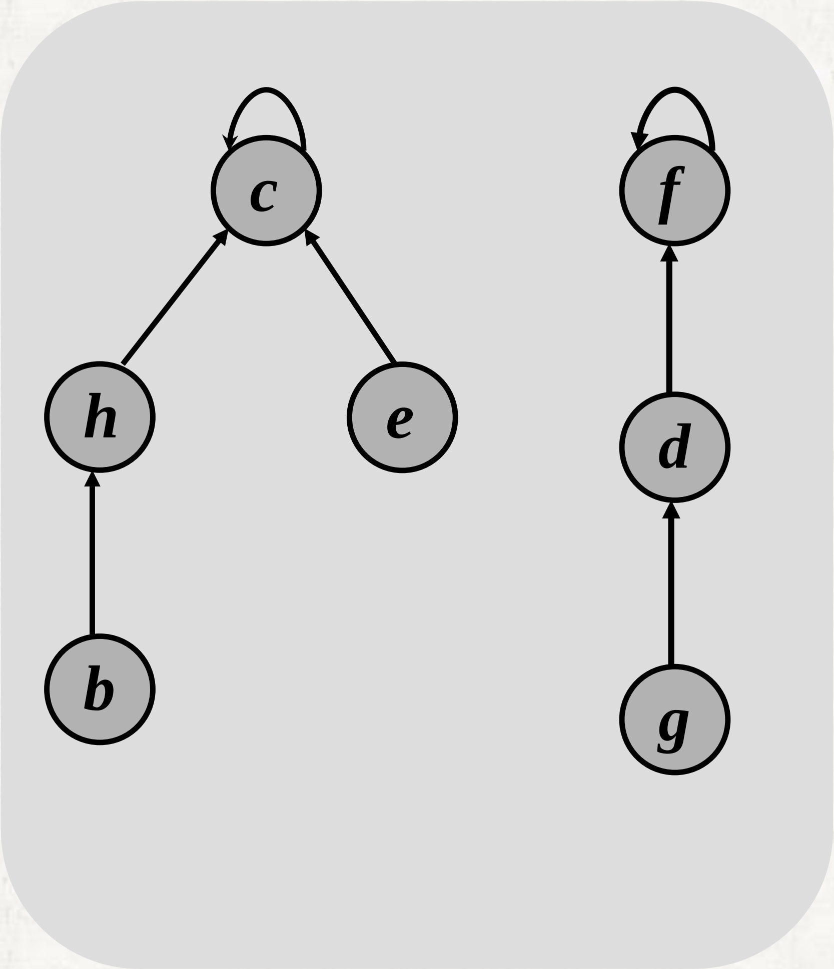
FIND-SET(x)

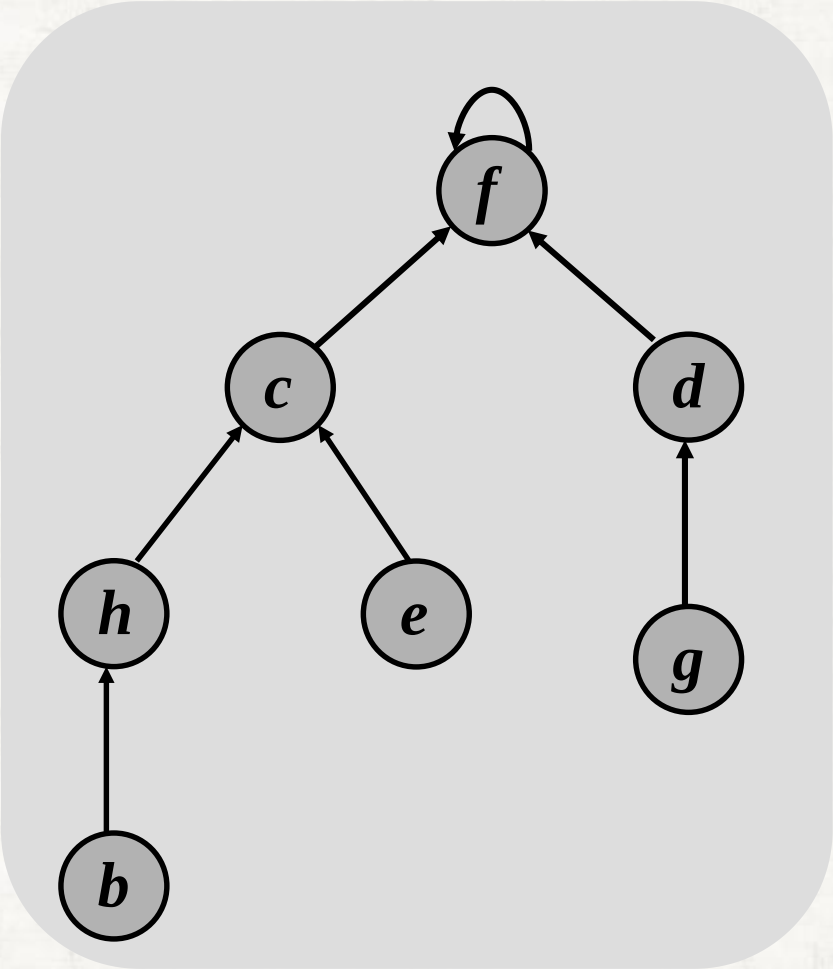
if $x == x.p$
  return x
else return FIND-SET($x.p$)

Forest representation

- **Union by rank**
  - ***Idea*: ** Attach the shorter tree to the higher tree.
  - Each node maintains a ***rank***,which is an upper bound on the height of the node.
  - Compare the ranks of the two roots and attach the tree whose root's rank is smaller to the other.

Forest representation





Forest representation

MAKE-SET(x)

```
x.p = x
x.rank = 0
```

UNION(x, y)

```
LINK(FIND-SET(x), FIND-SET(y))
```

LINK(x, y)

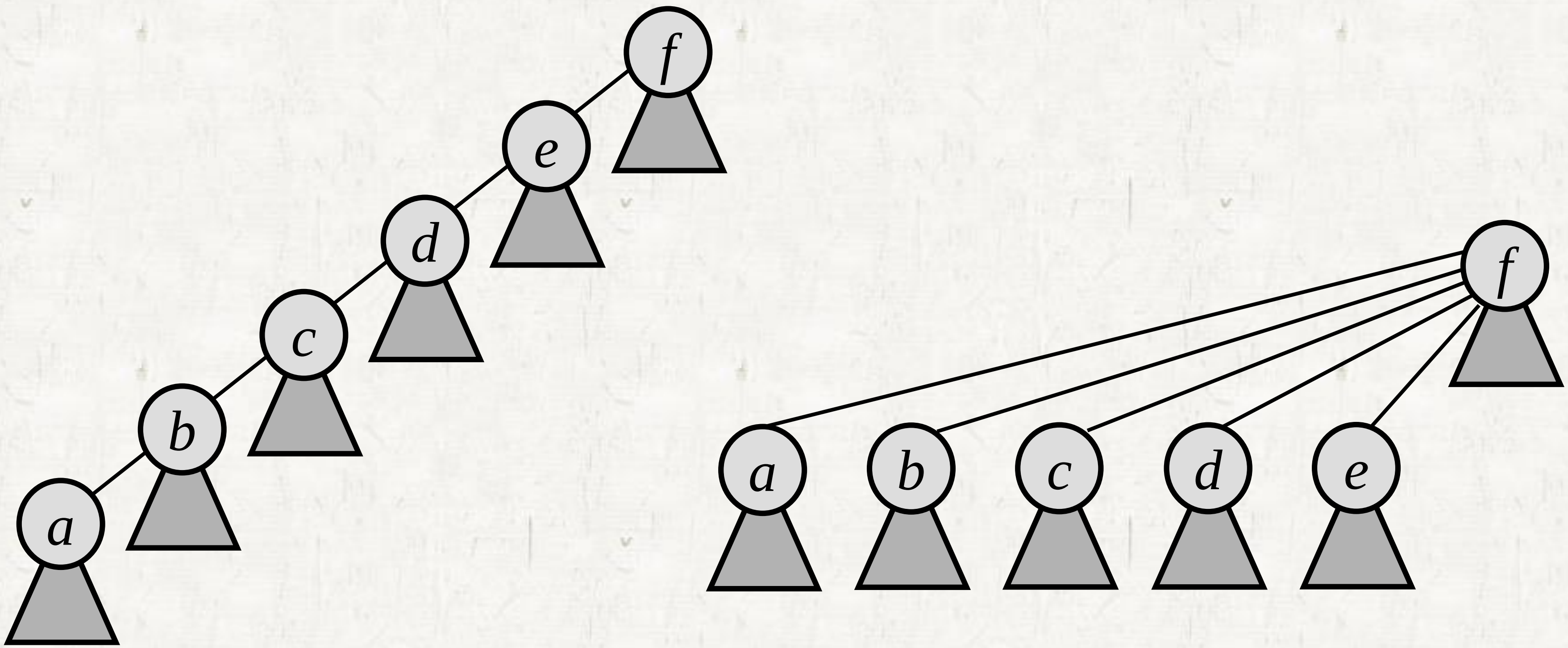
```
if x.rank > y.rank
    y.p = x
else x.p = y
    if x.rank == y.rank
        y.rank = y.rank + 1
```

Forest representation

- **Path compression**
  - Change the parent to the root during FIND-SET(x).

```
FIND-SET(x)
1  if x ≠ x.p
2      x.p = FIND-SET(x.p)
3  return x.p
```

Forest representation



Forest representation

- Worst case running time : $O(m\,\alpha(n))$
- $\alpha(n) \leq 4$:for all practical situations.