Growth of Functions

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Contents

- **Asymptotic notation**
  - Θ-notation
  - O-notation
  - Ω-notation

Simple examples

- $\Theta(n^2) = 3n^2 + 2n - 1$
- $\Theta(n) = 3n - 1$
- $\Theta(n^2) \neq 3n - 1$

- $O(n^2) = 3n^2 + 2n - 1$
- $O(n) = 3n - 1$
- $O(n^2) = 3n - 1$

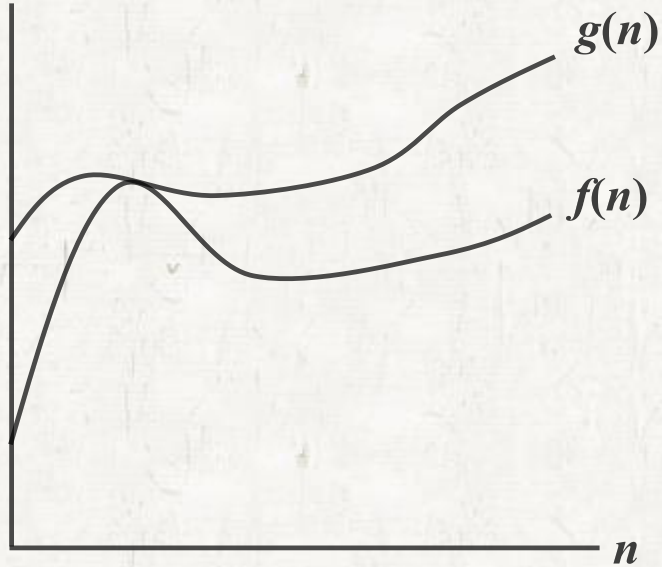
- $\Omega(n) = 3n - 1$
- $\Omega(n) = 3n^2 - 1$

Analogy

- **Analogy**
  - $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
  - $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
  - $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.

O-notation

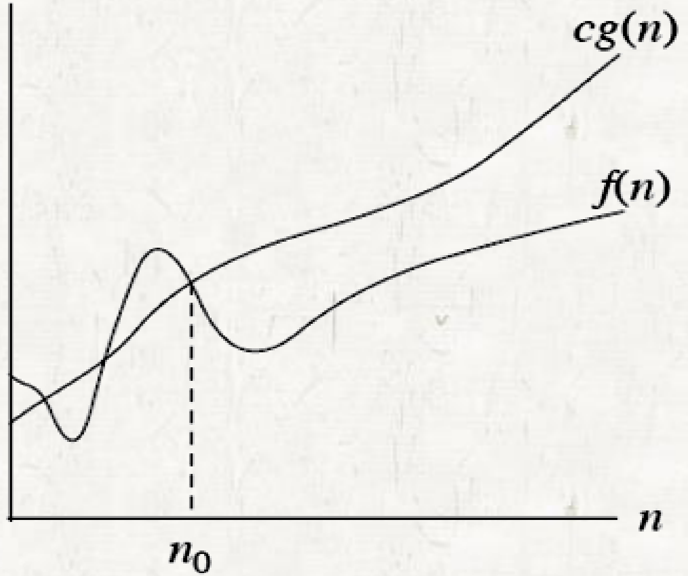
- ***Upper bound***
  - $g(n)$ is an *upper bound* of $f(n)$.



O-notation

- $g(n)$ is an ***asymptotic upper bound*** of $f(n)$.
  - $f(n) = O(g(n))$

There exist positive constants c and n_0 such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0$.



O-notation

- Example

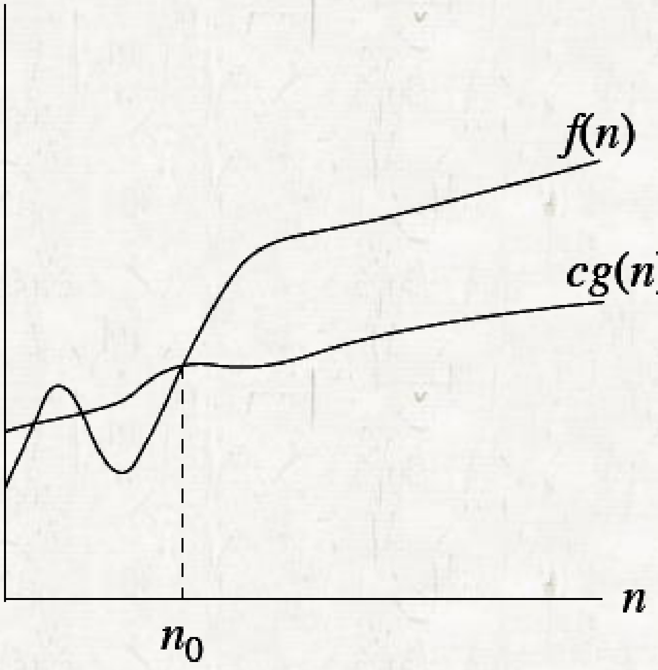
$$3n + 1 = \mathbf{O}(n^2)$$

- Show there are c and n_o such that $3n + 1 \le cn^2$ for all $n \ \geq \ n_0$.
- Dividing by n^2 yields $\dfrac{3}{n} + \dfrac{1}{n^2} \le c.$
- The inequality holds for any $n \geq 1$ ($n_0 = 1$) and $c = 4$.

Ω-notation

- **Asymptotic lower bound**
  - $f(n) = \Omega(g(n))$

There exist positive constants c and n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0$.



Ω-notation

- Example

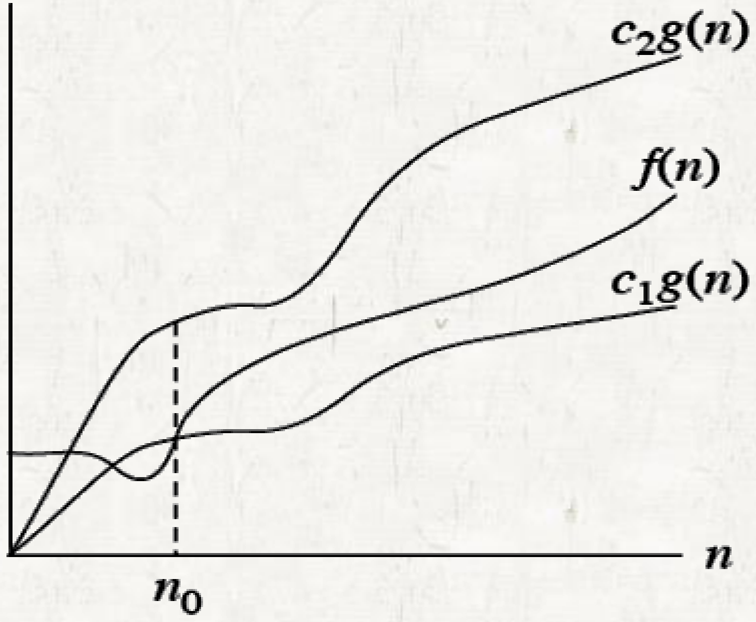
$$3n^2 - 4n + 1 = \Omega(n)$$

- Show there are c and n_o such that $3n^2 - 4n + 1 \geq cn$ for all $n \geq n_0$.
- Dividing by n yields $3n - 4 + \frac{1}{n} \geq c$.
- The inequality holds for any $n \geq 2$ ($n_0 = 2$) and $c = 2$.

Θ-notation

- **Asymptotically tight bound**
  - $f(n) = \Theta(g(n))$

there exist positive constants c_1, c_2 ,and n_0 such that
$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
for all $n \geq n_0$.



Θ-notation

- Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

To show there exist positive constants c_1, c_2 and n_o such that

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Dividing by n^2 yields $c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$

Θ-notation

- Example

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$$

- The right-hand inequality holds for $n \ge 1$ by choosing $c_2 \ge 1/2$.
- The left-hand inequality holds for $n \ge 7$ by choosing $c_1 \le 1/14$.
- Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$,

we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Θ-notation

- Example
  - Consider any quadratic function $f(n) = an^2 + bn + c$, where a, b, and c are constants and $a > 0$.
  - Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$.
  - The reader may verify that $0 \le c_1 n^2 \le an^2 + bn + c \le c_2 n^2$ for all $n \ge n_0$. (Self-study)
  - In general, for any polynomial $p(n) = \sum_{i=0}^{d} a_i n^i$ where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$.

Examples

- **Insertion sort**
  - $O(n^2)$, $\Omega(n)$
- **Selection sort**
  - $\Theta(n^2)$
- **Merge sort**
  - $\Theta(n \lg n)$
- **Binary search**
  - $O(\lg n)$, $\Omega(1)$

Analogy

- **Analogy**
  - $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
  - $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
  - $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
  - $f(n) = o(g(n)) \approx f(n) < g(n)$ in degree.
  - $f(n) = \omega(g(n)) \approx f(n) > g(n)$ in degree.

Comparison of functions

- **Comparison of functions**
  - Transitivity
  - Reflexivity
  - Symmetry
  - Transpose symmetry

Comparison of functions

- **Comparison of functions**
  - Transitivity ($=, \leq, \geq, <, >$)
  - Reflexivity ($=, \leq, \geq$)
  - Symmetry ($=$)
  - Transpose symmetry ($\leq \leftrightarrow \geq, < \leftrightarrow >$)

Transitivity

- **Transitivity** ($=, \leq, \geq, <, >$)
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$,
  - $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$,
  - $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$,
  - $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$,
  - $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$.

Reflexivity

- **Reflexivity** ($=, \leq, \geq$)
  - $f(n) = \Theta(f(n))$
  - $f(n) = O(f(n))$
  - $f(n) = \Omega(f(n))$

Symmetry and transpose symmetry

- **Symmetry** (=)
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- **Transpose symmetry** ($\leq \leftrightarrow \geq$, $< \leftrightarrow >$)
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$,
  - $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Comparison of functions

- **Trichotomy**
  - For any two real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, $a > b$.
  - That is, any two numbers are comparable.
  - Are any two functions asymptotically comparable?
    - Is it possible $f(n) \neq O(g(n))$ and $f(n) \neq \Omega(g(n))$?
    - n and $n^{1+\sin n}$

Self-study

- **Exercise 3.1-1**
  - Show $\max(f(n), g(n)) = \Theta (f(n) + g(n))$
- **Exercise 3.1-4**
  - Is $2^{n+1} = O(2^n)$?
  - Is $2^{2n} = O(2^n)$?
- **Problem 3-2 for O, Θ, and Ω.**
  - Use $\lg(n!) = \Theta (n\lg n)$