Elementary Graph Algorithms

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Contents

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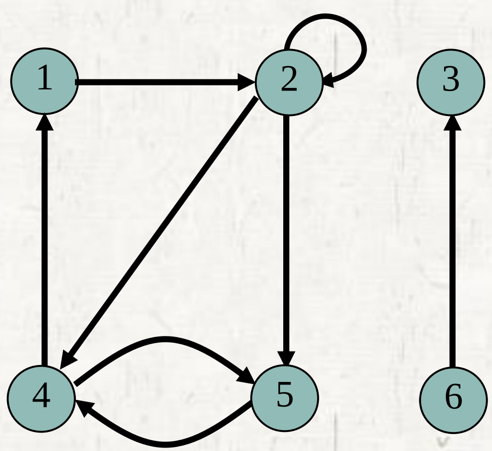
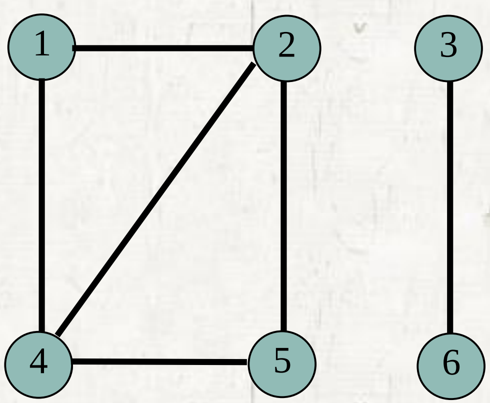
  - Graph representation
- **Searching a graph**
  - Breadth-first search
  - Depth-first search
- **Applications of depth-first search**
  - Topological sort

Graph basics

- A ***graph G*** is a pair (***V***, ***E***)
  where ***V*** is a ***vertex*** set and ***E*** is an ***edge*** set.
- A vertex (node)
  - a circle.
- An edge (link).
  - an arrow or a line.



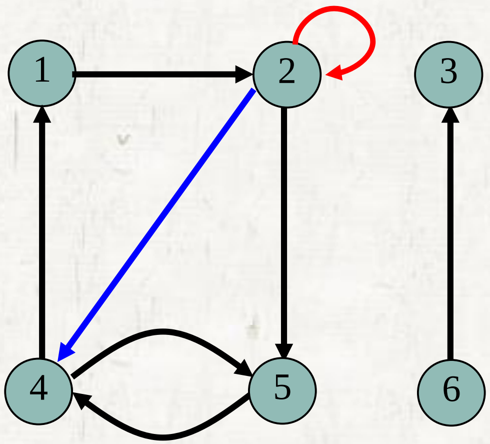
A directed graph



An undirected graph

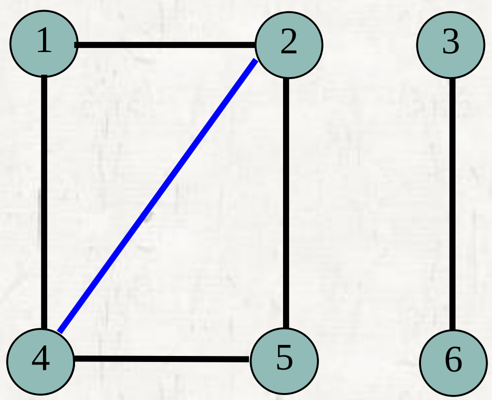
Graph basics

- ***Directed graph*** (or ***digraph***)
  - The blue edge ***leaves*** vertex 2 and *enters* vertex 4.
  - The blue edge is ***incident from*** vertex 2 and ***incident to*** vertex 4.
  - The red edge is a ***self-loop.*** (an edge from a vertex to itself)



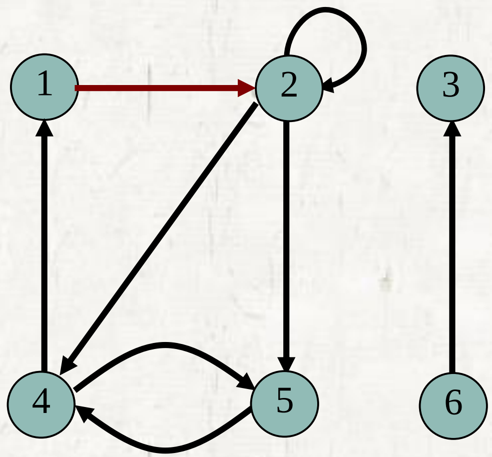
Graph basics

- ***Undirected graph***.
  - Edges have no directions
  - No Self-loops



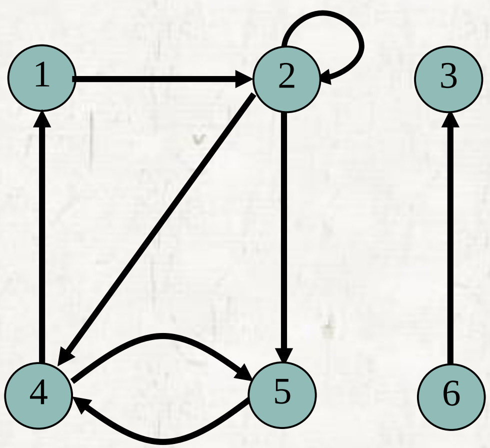
Graph basics

- ***Adjacency***
  - If (u,v) is an edge, vertex v is ***adjacent*** to vertex u.
  - In an undirected graph, adjacency relation is symmetric.
    - If u is adjacent to v, v is adjacent to u.
  - In a directed graph, it is not symmetric.
    - Vertex 2 is adjacent to 1.
    - But vertex 1 is not adjacent to 2.



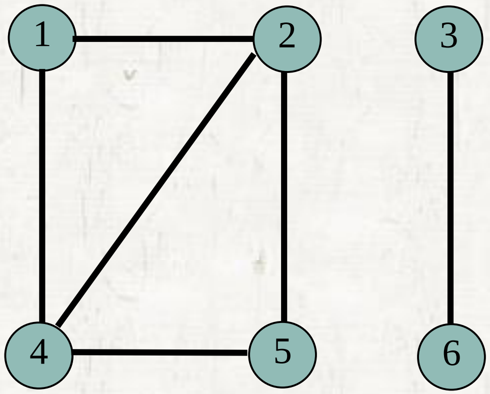
Graph basics

- ***Degree***
  - The ***out-degree*** of vertex 2 is 3.
  - The ***in-degree*** of vertex 2 is 2.
  - ***degree* = out-degree + in-degree.**



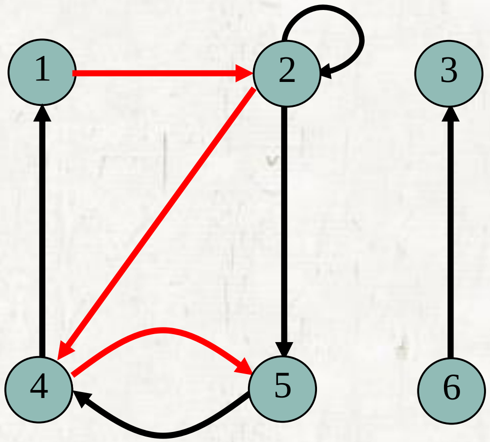
Graph basics

- ***Degree***
  - In an undirected graph,
    - The **degree** of vertex 2 is 3.



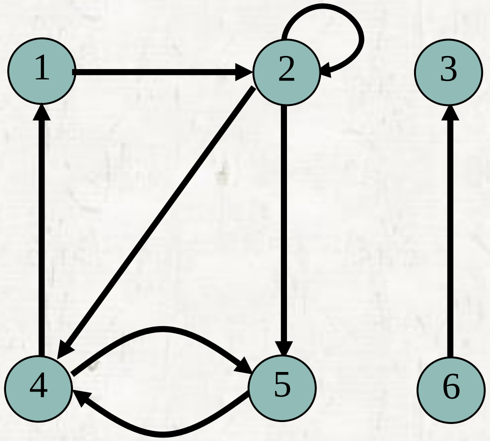
Graph basics

- ***Path***
  - A sequence of consecutive edges
    - <1, 2>, <2, 4>, <4, 5> is a path.
    - <1, 2, 4, 5> for short
    - <1, 2, 4, 1, 2> is a path.
    - <1, 2, 4, 2> is not a path.



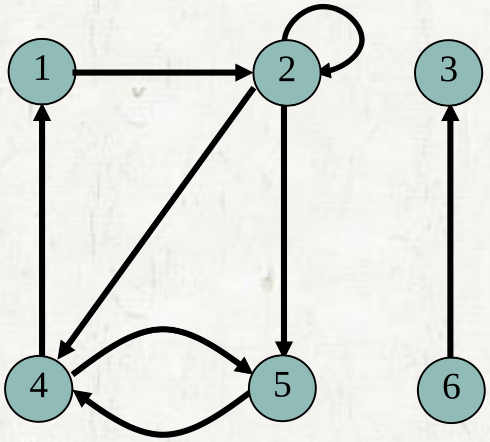
Graph basics

- ***Path***
  - The ***length*** of a path is the number of edges in the path.
    - The length of a path <1, 2, 4, 5> is 3.
    - If there is a path from vertex *u* to vertex *v*, *v* is called ***reachable*** from *u*.
      - Vertex 5 is reachable from vertex 1.
      - Vertex 3 is not reachable from vertex 1.



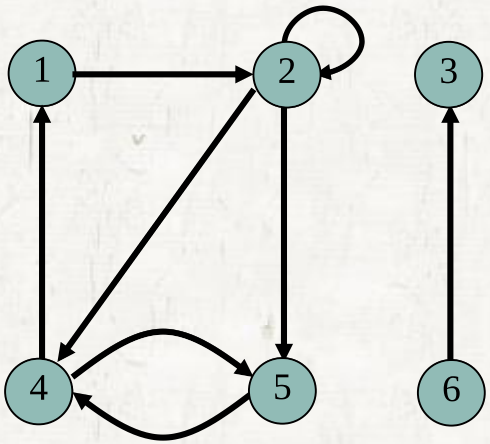
Graph basics

- ***Simple path***
  - A path is simple if all vertices in the path are distinct.
  - A path <1, 2, 4, 5> is a simple path.
  - A path <1, 2, 4, 1, 2> is not a simple path.



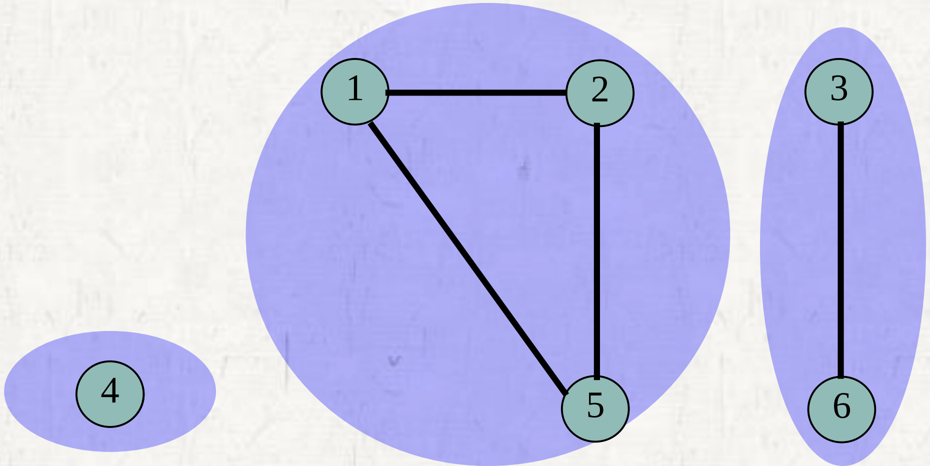
Graph basics

- ***Cycle* and *simple cycle***
  - A path <v_0, v_1, v_2,…, v_k> is a cycle if $v_0 = v_k$
  - A cycle <v_0, v_1, v_2,…, v_k> is simple if v_1, v_2,…, v_k are distinct.
  - A path <1, 2, 4, 5, 4, 1> is a cycle but it is not a simple cycle.
  - A path <1, 2, 4, 1> is a simple cycle.



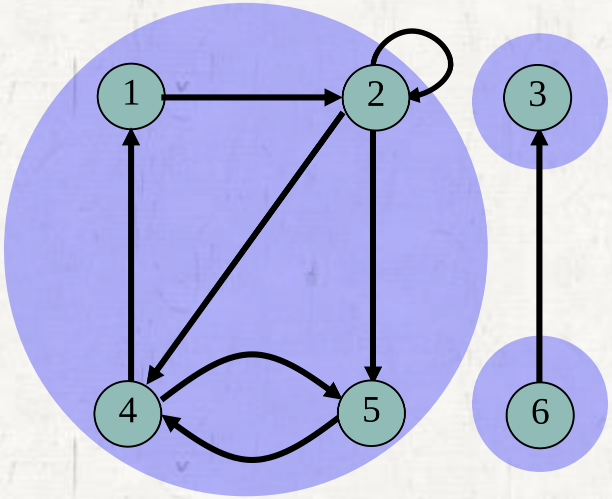
Graph basics

- ***An acyclic graph***
  - A graph without cycles
- ***A connected graph***
  - An undirected graph is ***connected***
    if every pair of vertices is connected by a path.
- ***Connected components***
  - Maximally connected subsets of vertices of an undirected graph.



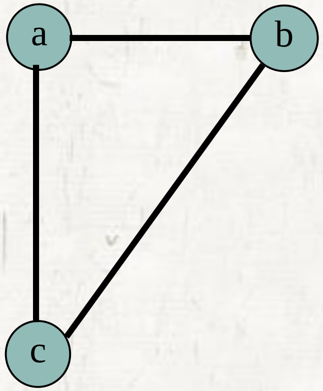
Graph basics

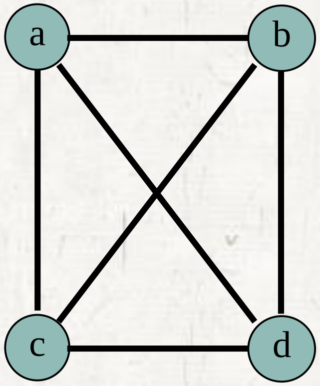
- ***Strongly connected***
  - A directed graph is ***strongly connected*** if every pair of vertices is reachable from each other.
- ***Strongly connected components***
  - Maximally strongly connected subsets of vertices in a directed graph.



Graph basics

- ***A complete graph***
  - An undirected graph in which every pair of vertices is adjacent.

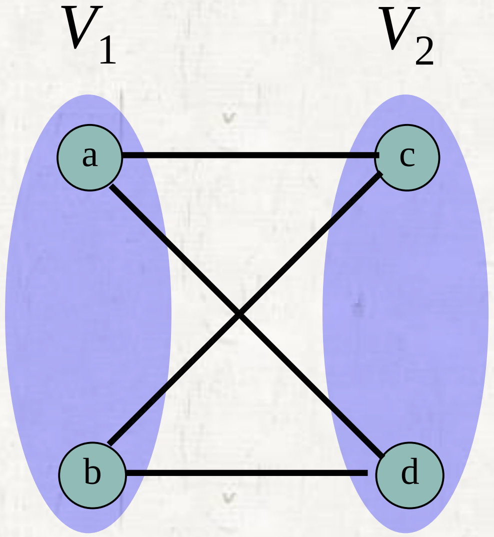




  - The number of edges with *n* vertices?

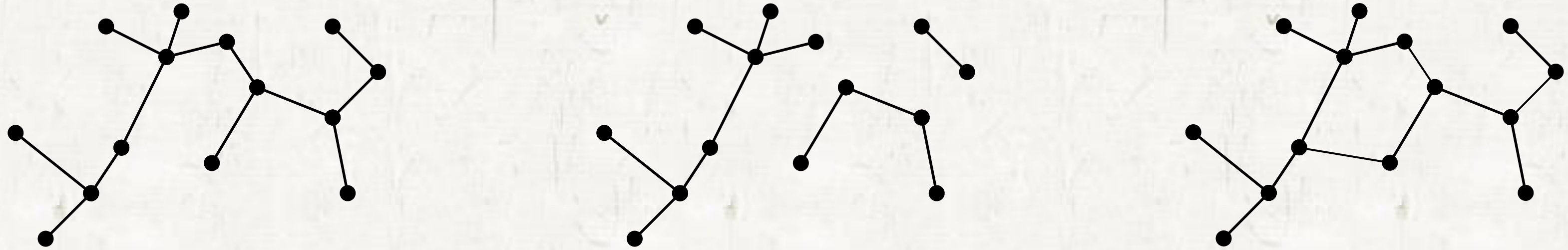
Graph basics

- ***A bipartite graph***
  - An undirected graph $G = (V,E)$ in which V can be partitioned into two sets V_1 and V_2 such that for each edge (u,v), either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.



Graph basics

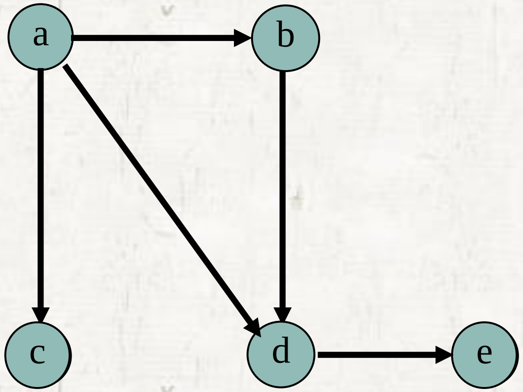
- ***Forest***
  - An acyclic, undirected graph
- ***Tree***
  - A connected forest
  - A connected, acyclic, undirected graph



- Is a connected component of a forest a tree?

Graph basics

- ***Dag***
  - A directed acyclic graph

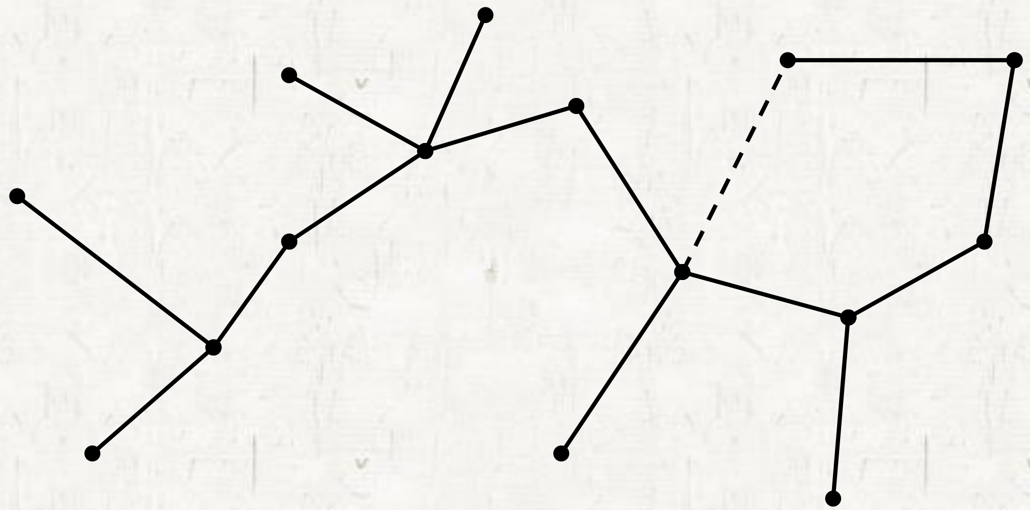


- ***Handshaking lemma***
  - If $G = (V, E)$ is an undirected graph

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

Graph basics

- **Tree: connected, acyclic, and undirected graph**
  - Any two vertices are connected by a unique simple path.
  - If any edge is removed, the resulting graph is disconnected.
  - If any edge is added, the resulting graph contains a cycle.
  - |E| = |V| - 1



Graph basics

- ***G* is a tree.**

= G is a connected, acyclic, and undirected graph

= In G, any two vertices are connected by a unique simple path.

= G is connected, and if any edge is removed, the resulting graph is disconnected.

= G is connected, $|E| = |V| - 1$.

= G is acyclic, $|E| = |V| - 1$.

= G is acyclic, but if any edge is added, the resulting graph contains a cycle.

Graph basics

- **The number of edges**
  - Directed graph
    - $|E| \le |V|^2$
  - Undirected graph
    - $|E| \le |V| (|V|-1) / 2$

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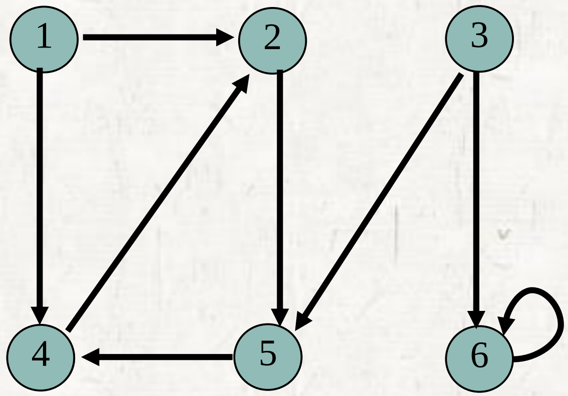
  - Topological sort

Graph representation

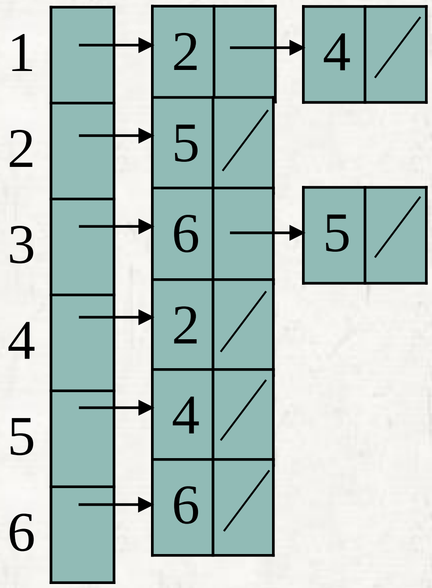
- **Representations of graphs**
  - Adjacency-list representation
  - Adjacency-matrix representation

Graph representation

- **Adjacency-list representation**
  - An array of |V| lists, one for each vertex.
  - For vertex *u*, its adjacency list contains all vertices adjacent to *u*.

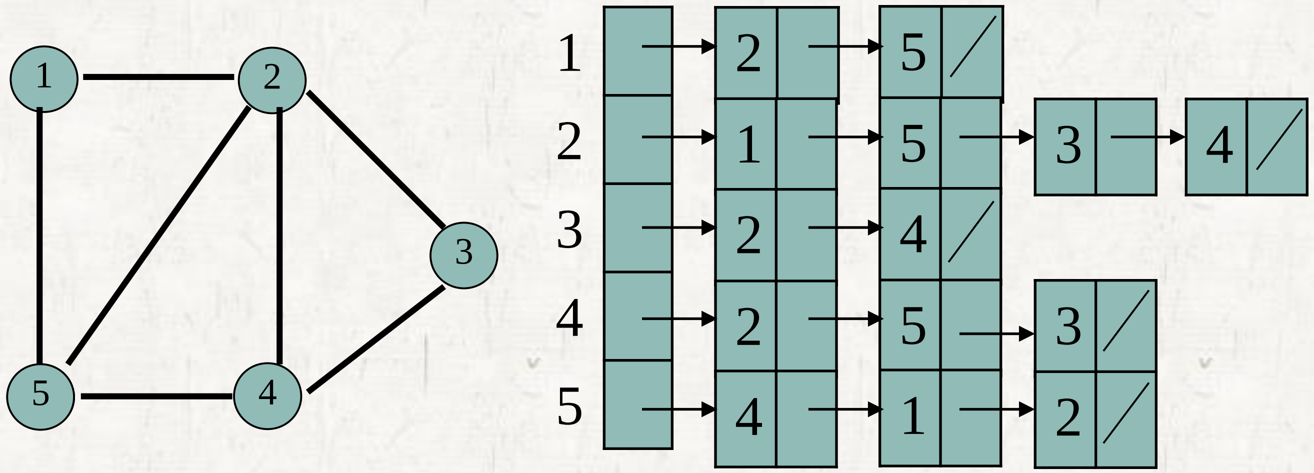


A directed graph



Graph representation

- **Adjacency-list representation**
  - For an undirected graph, its directed version is stored.

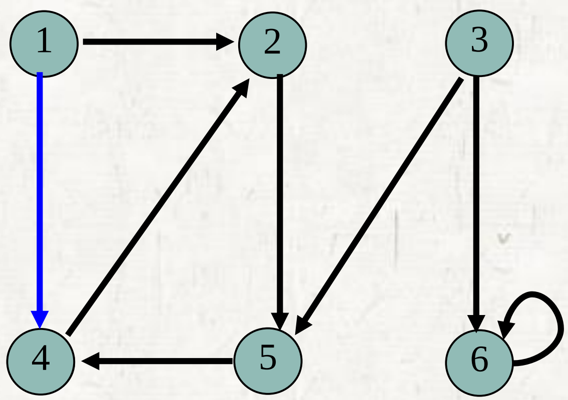


An undirected graph

  - $\Theta(V + E)$ space

Graph representation

- **Adjacency-matrix representation**
  - $|V| \times |V|$ matrix: $\Theta(V^2)$ space
  - Entry (i,j) is 1 if there is an edge and 0 otherwise.

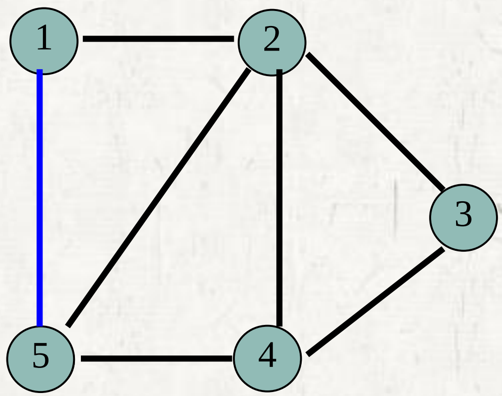


A directed graph

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Graph representation

- **Adjacency-matrix representation**
  - $|V| \times |V|$ matrix
  - Entry (i,j) is 1 if there is an edge and 0 otherwise.

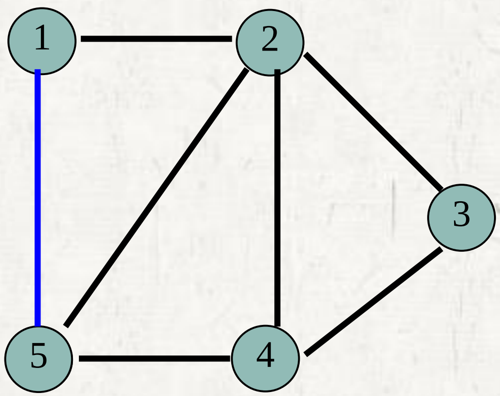


An undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph representation

- **Adjacency-matrix representation**
  - For an undirected graph, there is a symmetry along the main diagonal of its adjacency matrix.
  - Storing the lower matrix is enough.



An undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph representation

- **Comparison of adjacency list an adjacency matrix**
  - Storage
    - If G is sparse, adjacency list is better.
      - because $|E| < |V|^2$.
    - If G is dense, adjacency matrix is better.
      - because adjacency matrix uses only one bit for an entry.

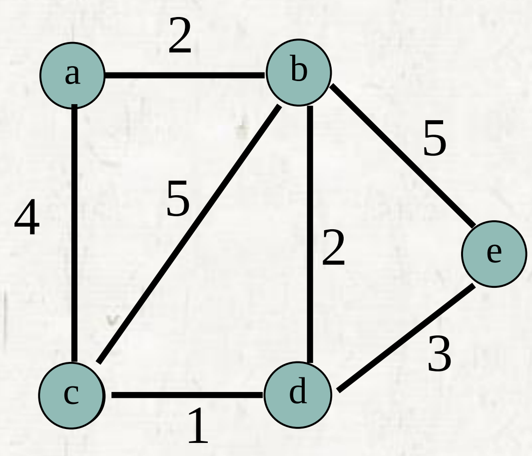
  - Edge present test: does an edge (i,j) exist?
    - Adjacency matrix: $\Theta(1)$ time.
    - Adjacency list: $O(V)$ time.

Graph representation

- **Comparison of adjacency list and adjacency matrix**
  - Listing or visiting all edges
    - Adjacency matrix: $\Theta(V^2)$ time.
    - Adjacency list: $\Theta(V + E)$ time.

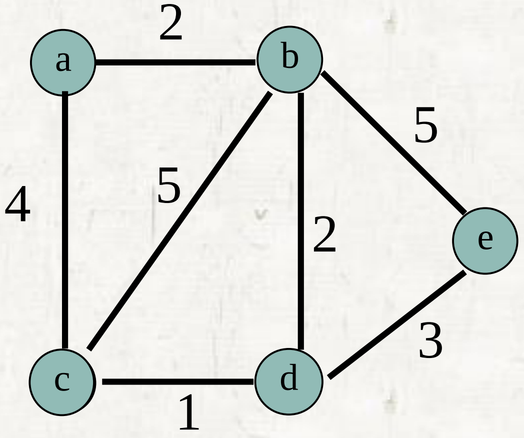
Graph representation

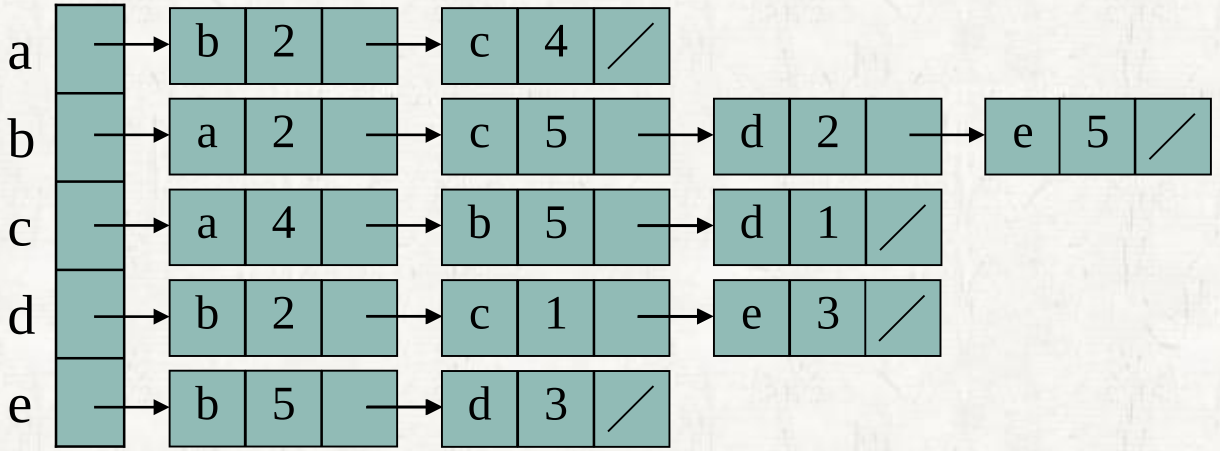
- **Weighted graph**
  - Edges have weights.



Graph representation

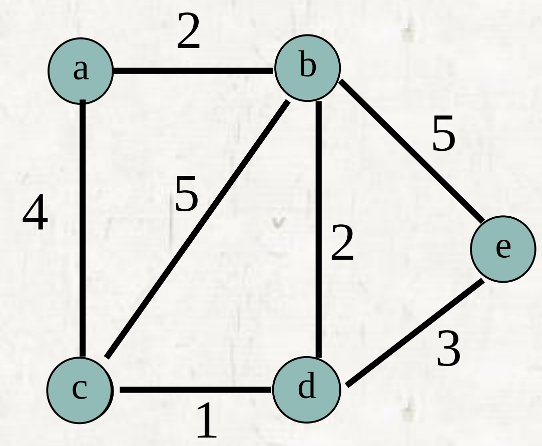
- **Weighted graph representation**
  - adjacency list





Graph representation

- **Weighted graph representation**
  - adjacency matrix
    - $\Theta(V^2)$ space



	a	b	c	d	e
a	0	2	4	∞	∞
b	2	0	5	2	5
c	4	5	0	1	∞
d	∞	2	1	0	3
e	∞	5	∞	3	0

Graph representation

- **Transpose of a matrix**
  - The ***transpose*** of a matrix $A = (a_{ij})$ is
  - $A^T = (a_{ij}^T)$ where $a_{ij}^T = a_{ji}$
  - An undirected graph is its own transpose: $A = A^{\mathrm{T}}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Self-study

- **Exercise 22.1-3**
  - The transpose of a directed graph
- **Exercise 22.1-4**
  - Removing duplicate edges in a multigraph in $O(V + E)$ time.
- **Exercise 22.1-6**
  - Universal sink detection in $O(V)$ time.

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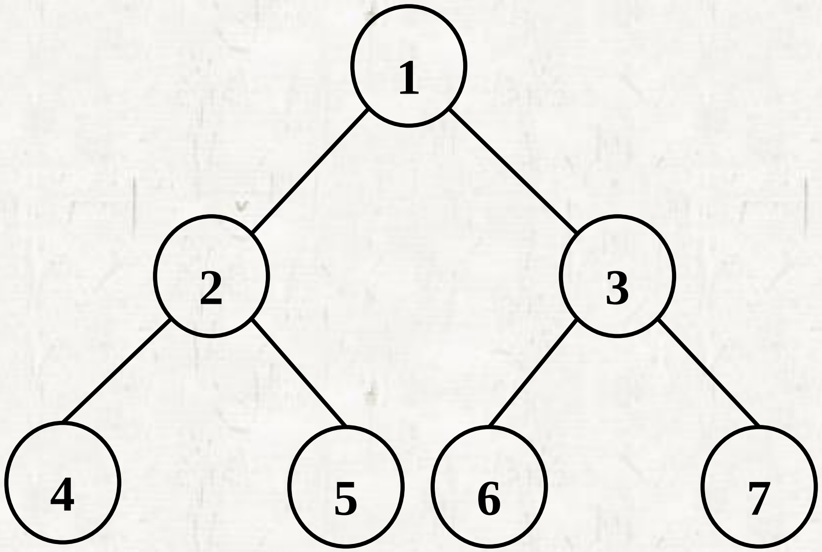
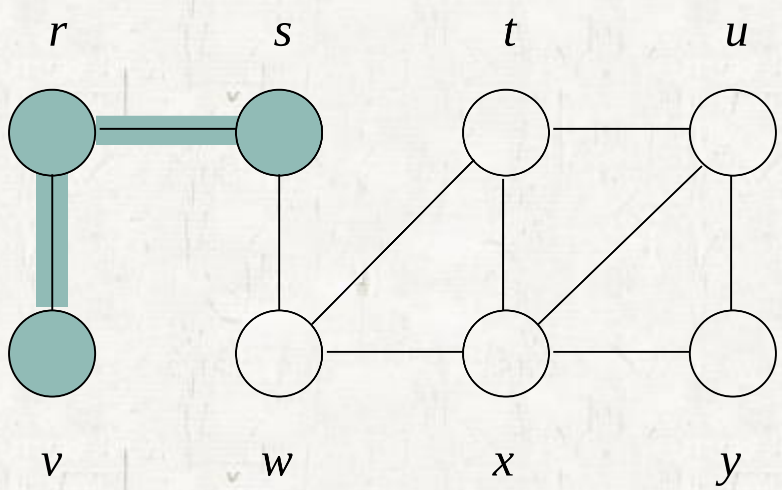
Searching a tree



- Breadth-first search
- Depth-first search

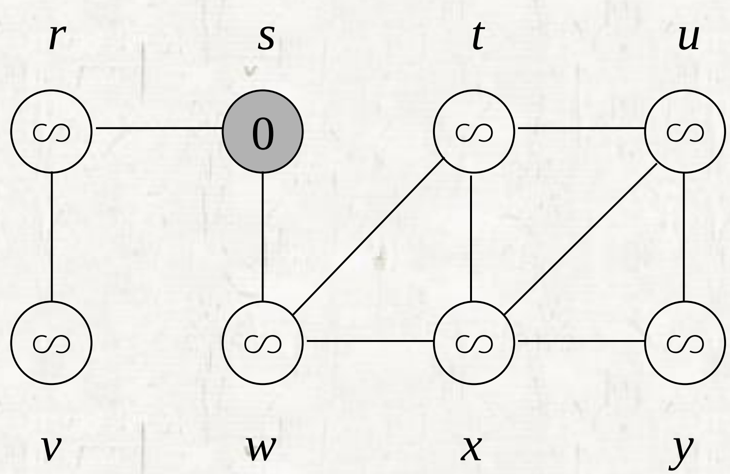
Breadth-first search

- ***Distance***
  - Distance from *u* to *v*
    - The number of edges in the shortest path from *u* to *v*.
    - The distance from *s* to *v* is 2.



Breadth-first search

- **Breadth-first search**
  - Given a graph $G = (V, E)$ and a ***source*** vertex s, it explores the edges of G to "discover" every reachable vertex from s.
  - It discovers vertices in the increasing order of distance from the source. It first discovers all vertices at distance 1, then 2, and etc.



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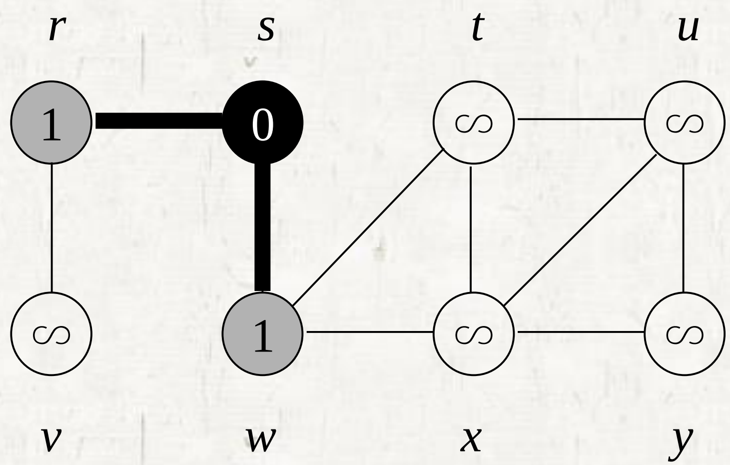

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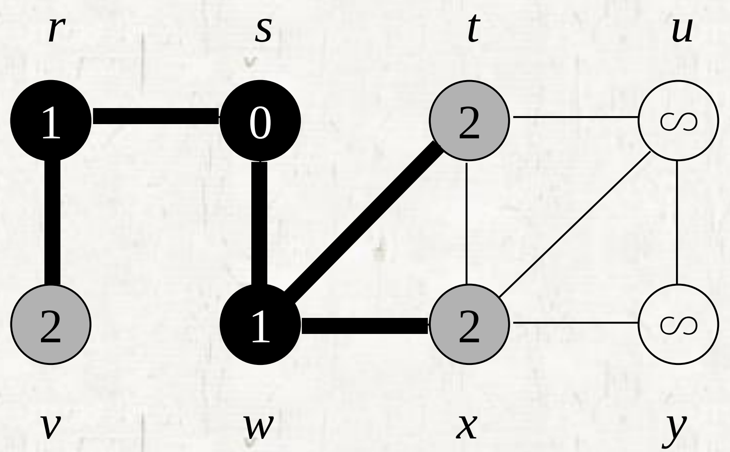

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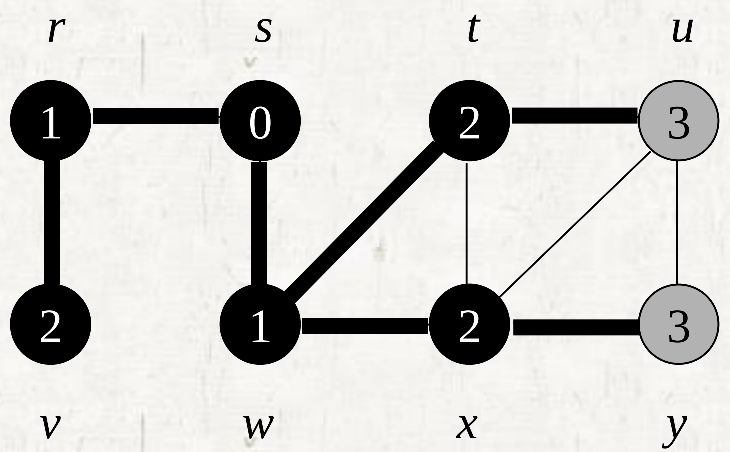
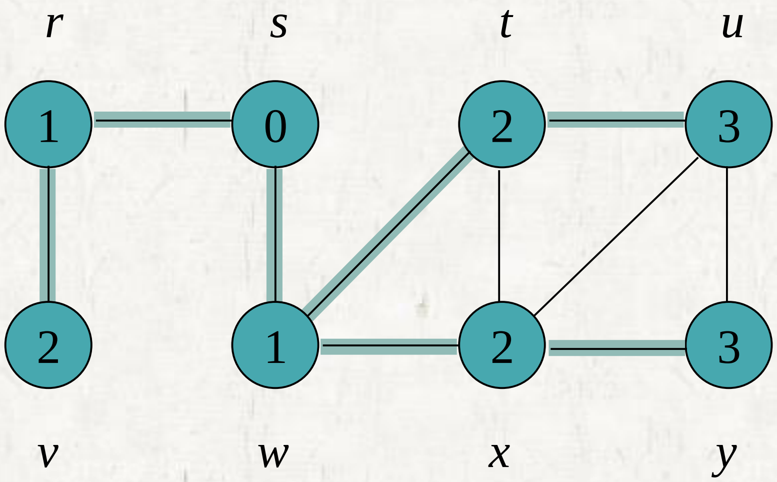


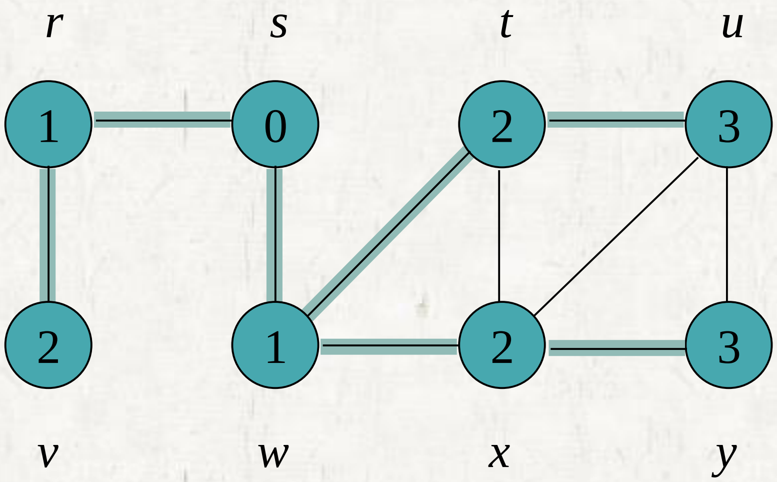
Breadth-first search

- **Breadth-first search**
  - It also computes
    - the distance of vertices from the source: $u.d = 3$
    - the predecessor of vertices: $u.\pi = t$



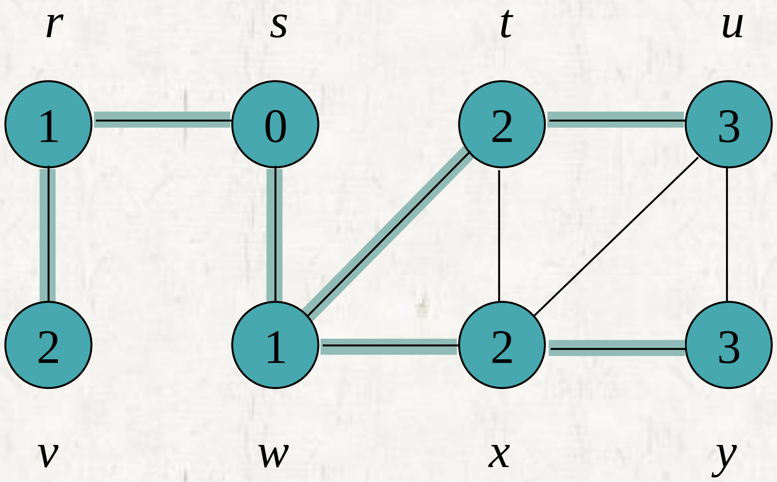
Breadth-first search

- The ***predecessor subgraph*** of G as $G_\pi = (V_\pi, E_\pi)$,
  - $V_\pi = \{v \in V: v.\pi \neq \text{NIL}\} \cup \{ s\}$
  - $E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}$.



Breadth-first search

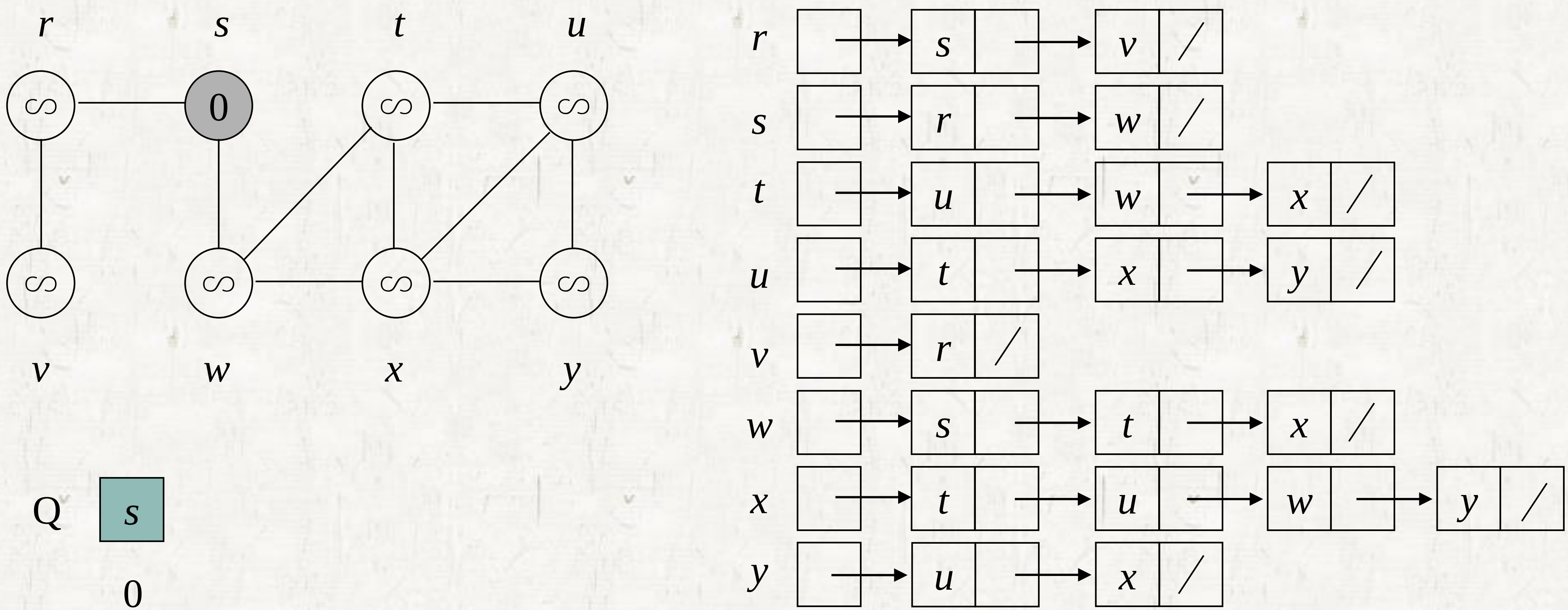
- The predecessor subgraph G_π is a ***breadth-first tree.***
  - since it is connected and $|E_\pi| = |V_\pi| -1$.
  - The edges in E_π are called ***tree edges***.



Breadth-first search

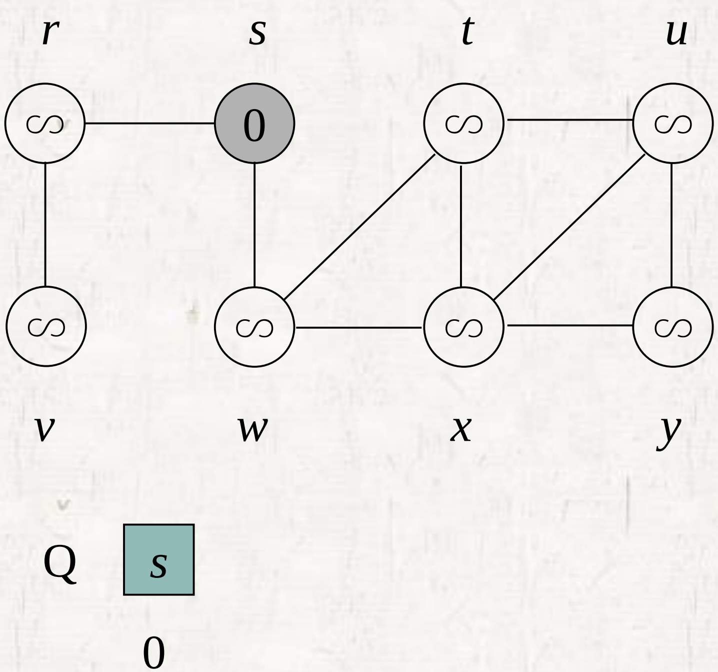
```
BFS(G, s)
for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
        u = DEQUEUE(Q)
        for each v ∈ G.Adj[u]
                if v.color == WHITE
                        v.color = GRAY
                        v.d = u.d + 1
                        v.π = u
                        ENQUEUE(Q, v)
        u.color = BLACK
```

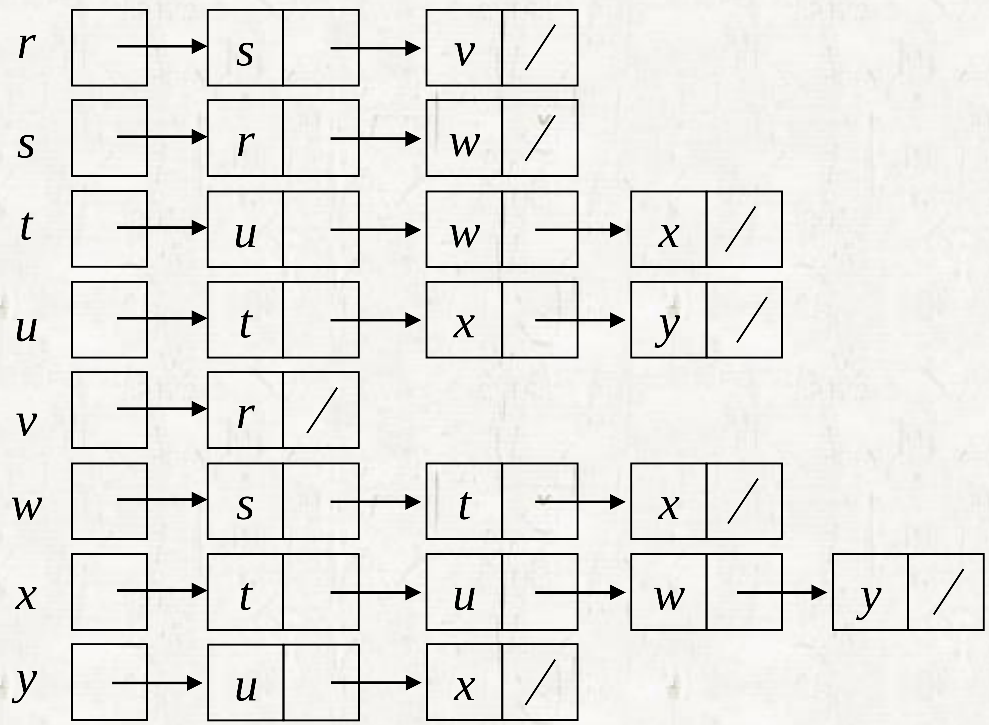
Breadth-first search



- white: not discovered (not entered the Q)
- gray: discovered (in the Q)
- black: finished (out of the Q)

Breadth-first search





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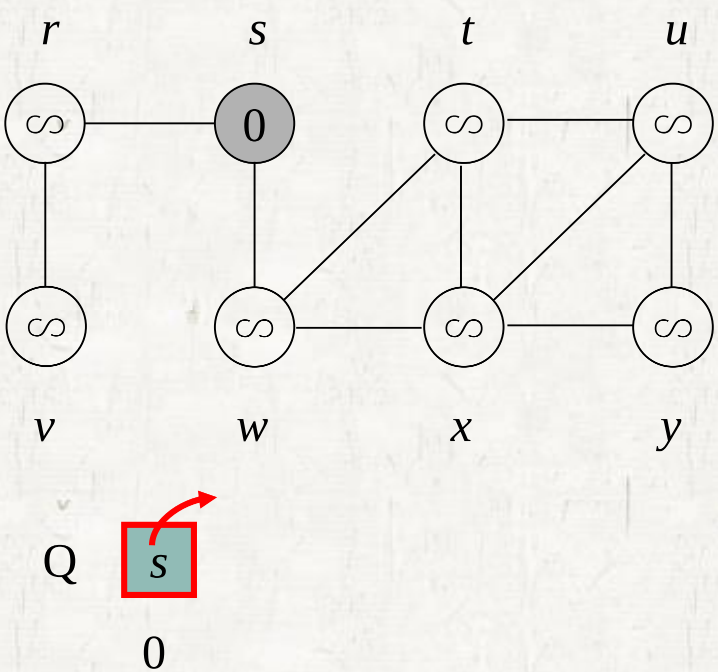

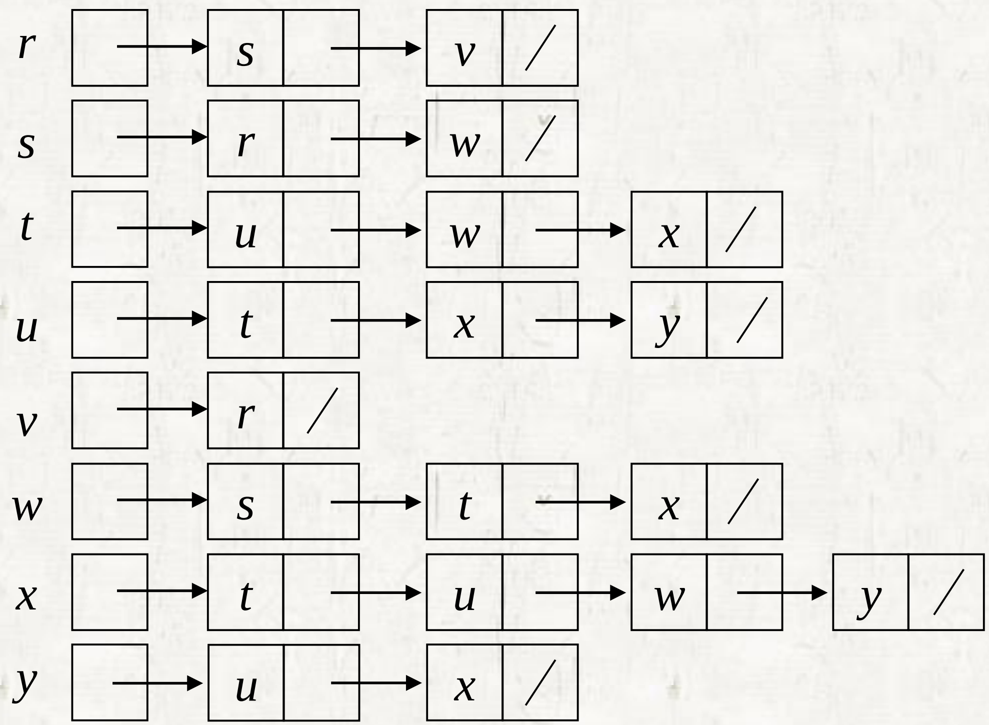

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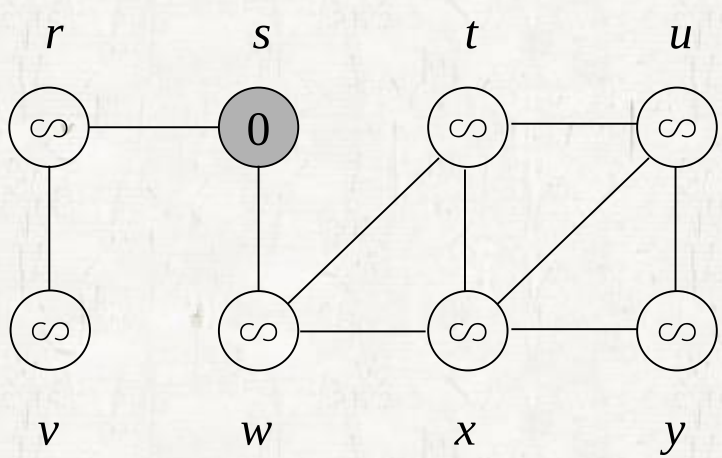

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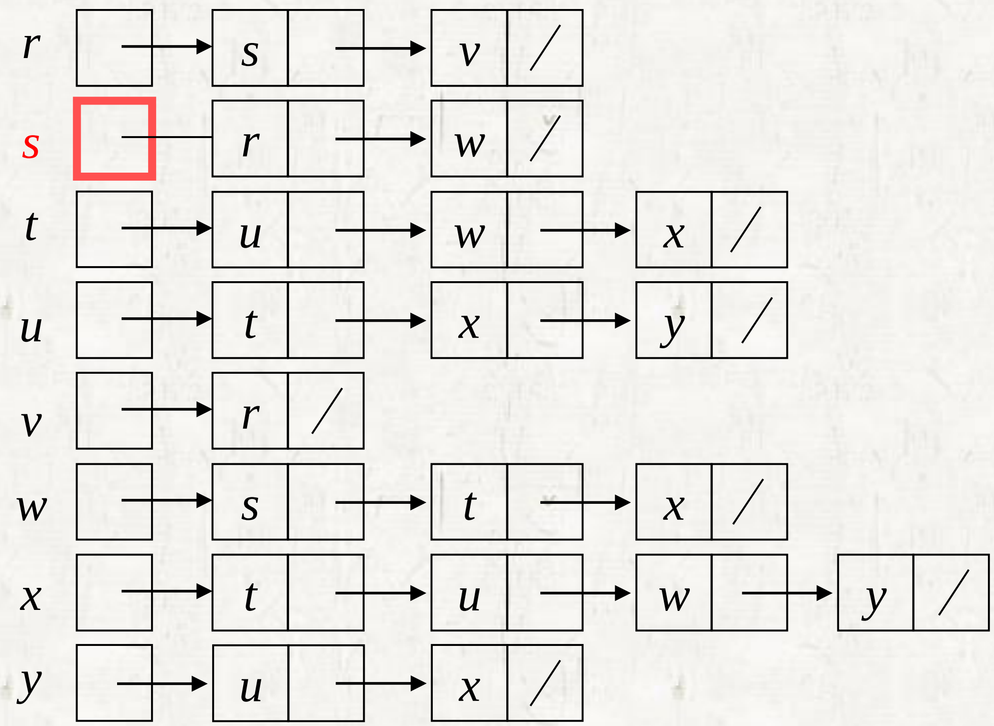

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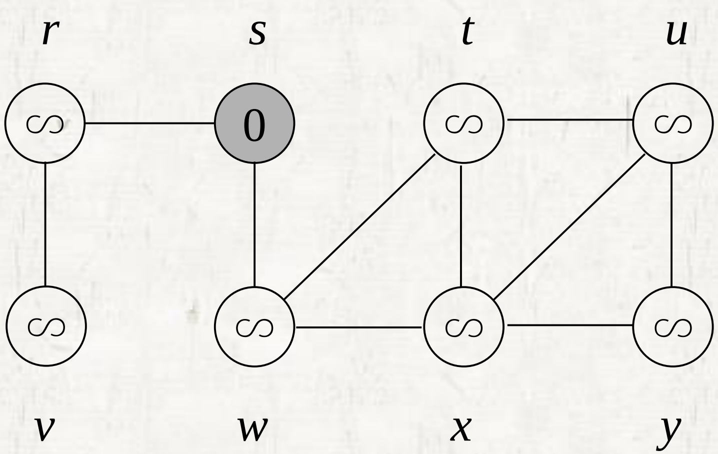

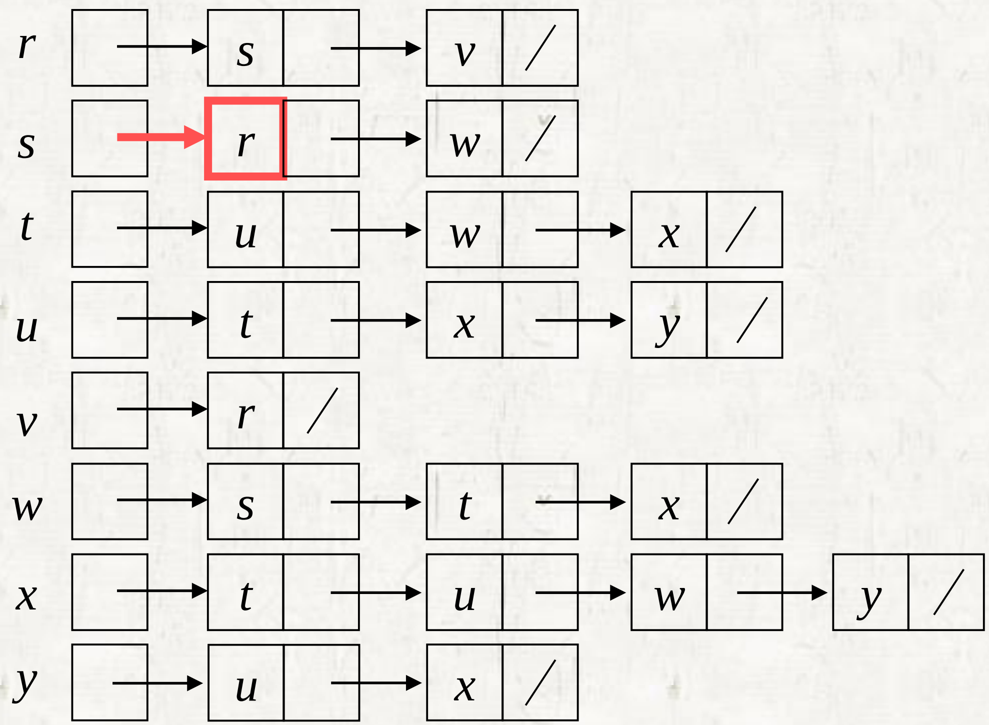

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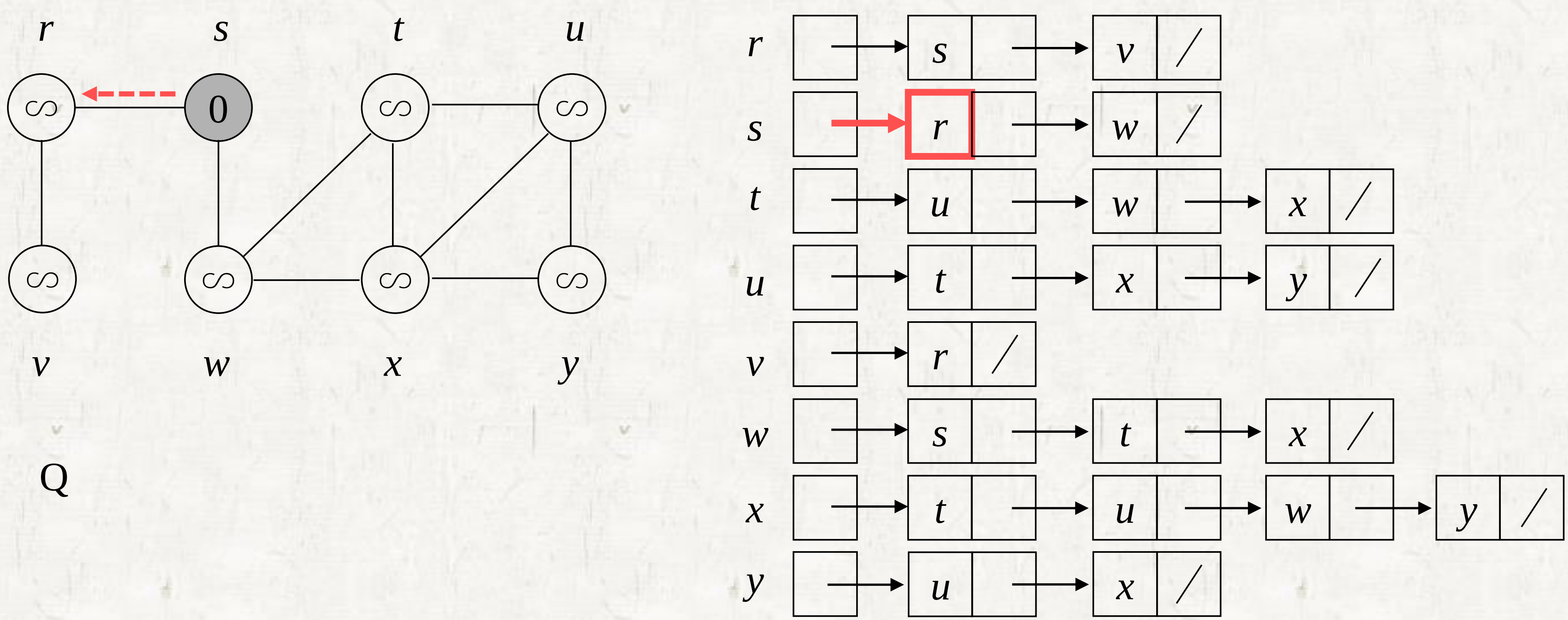

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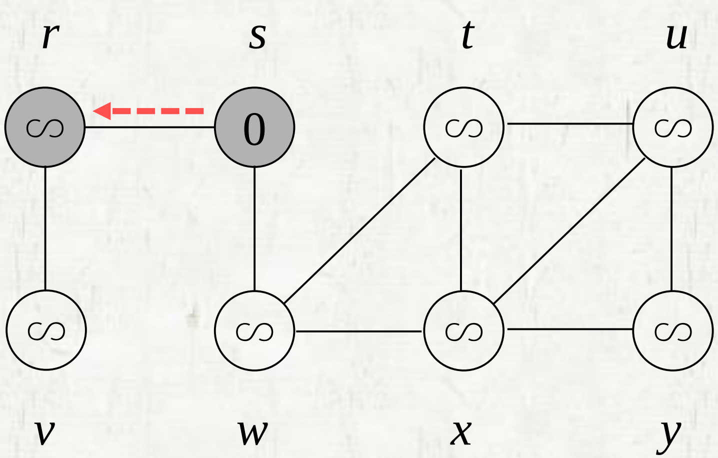

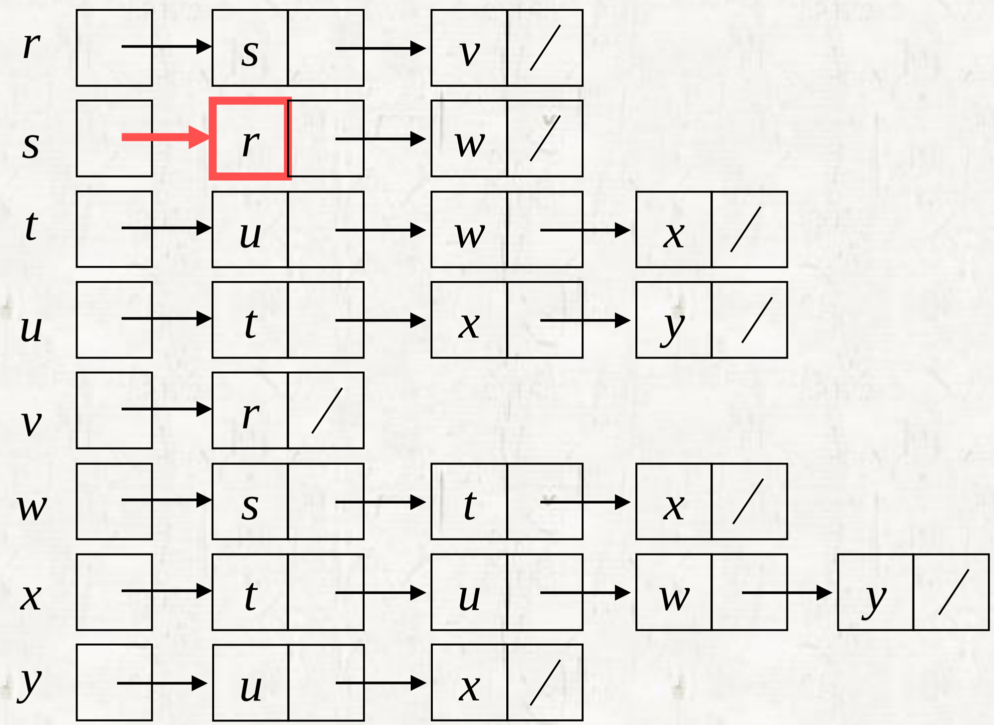

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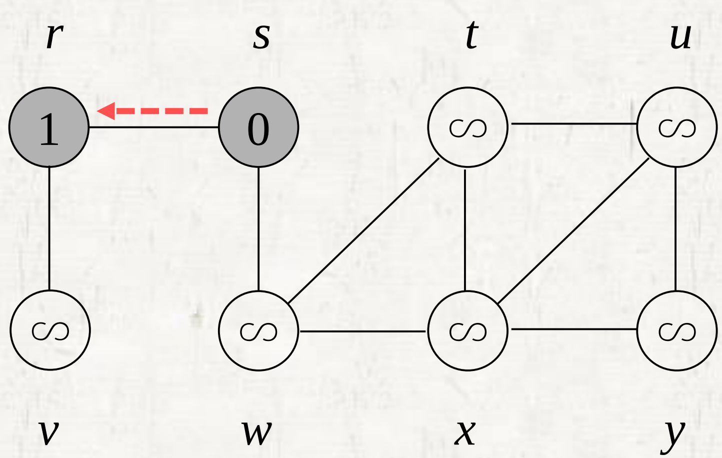

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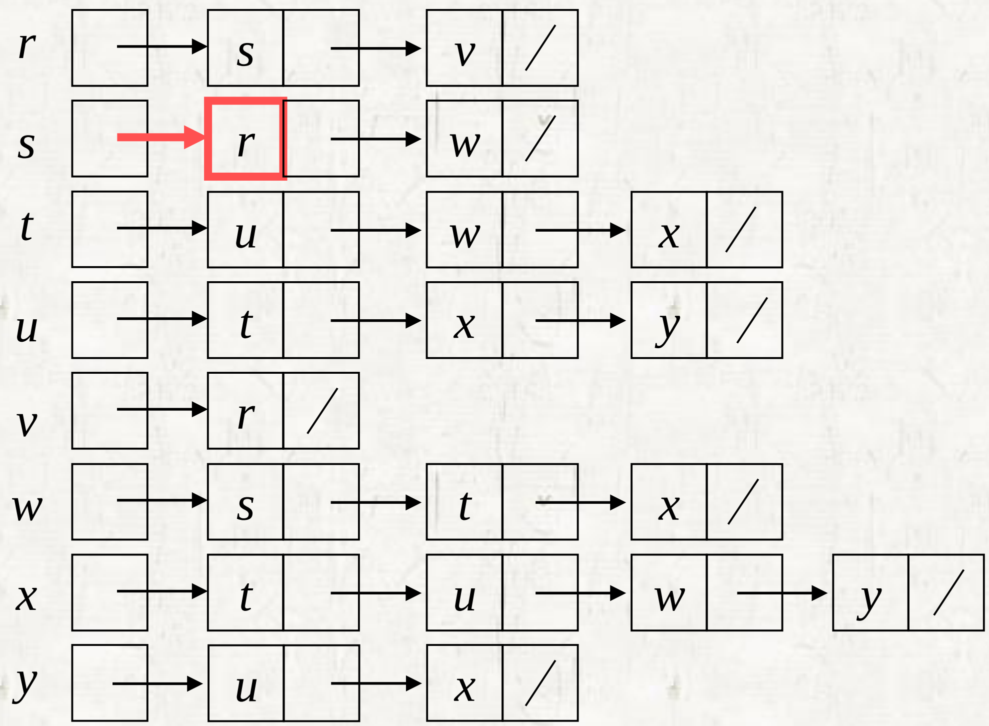

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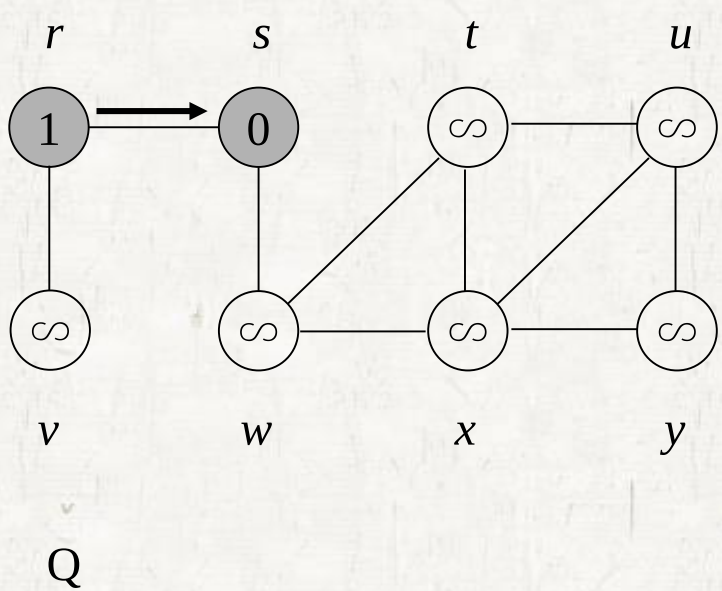

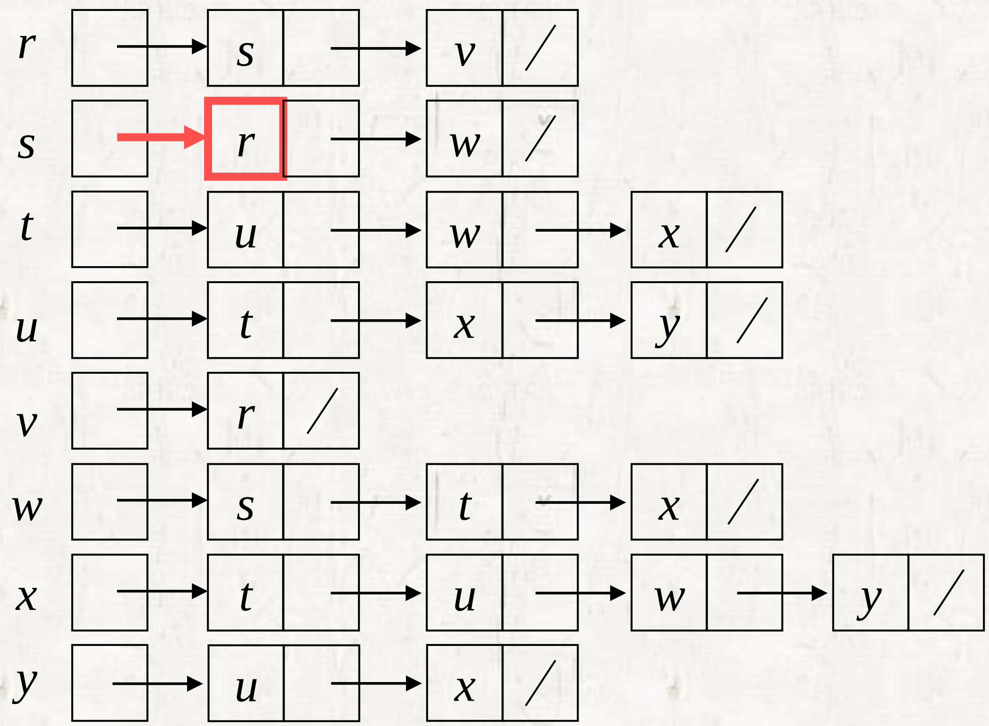

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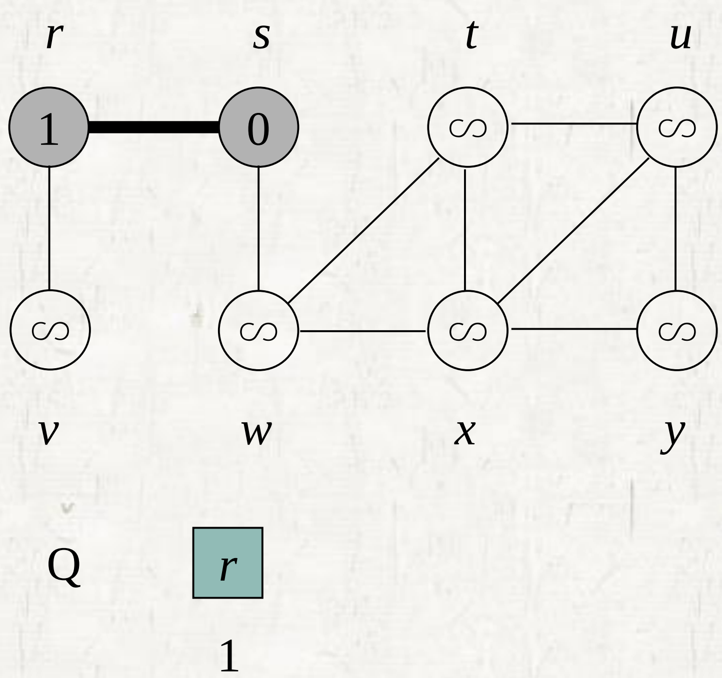

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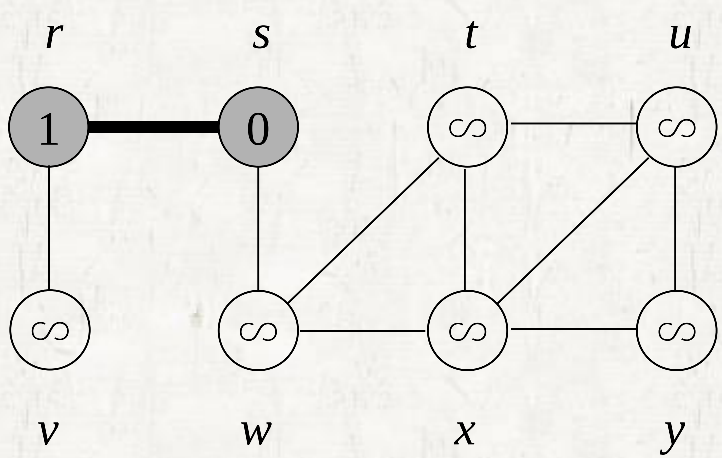

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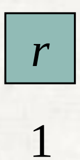

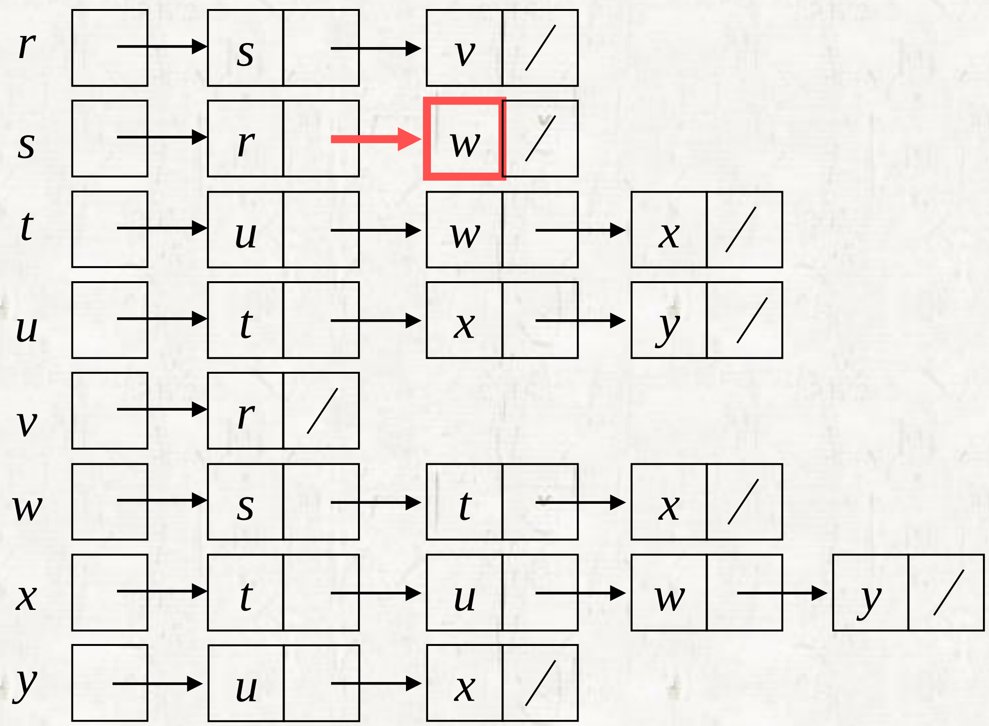

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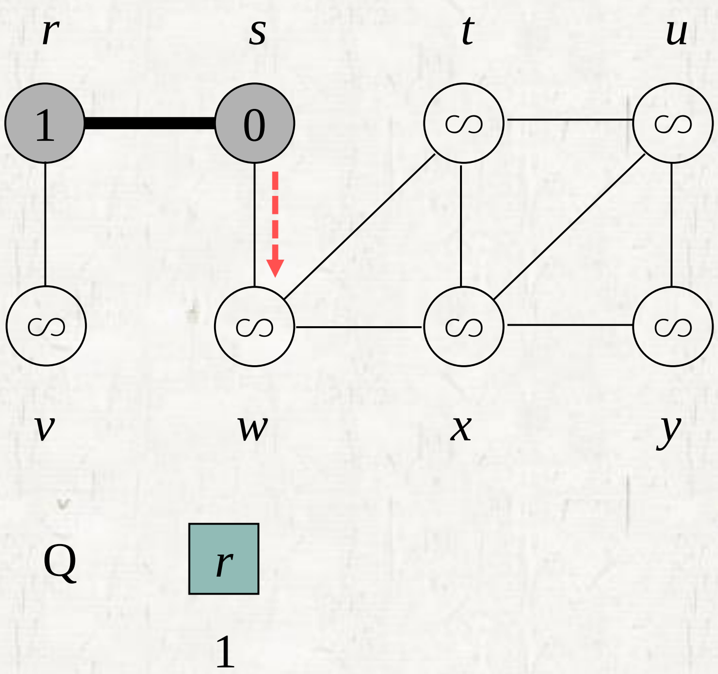

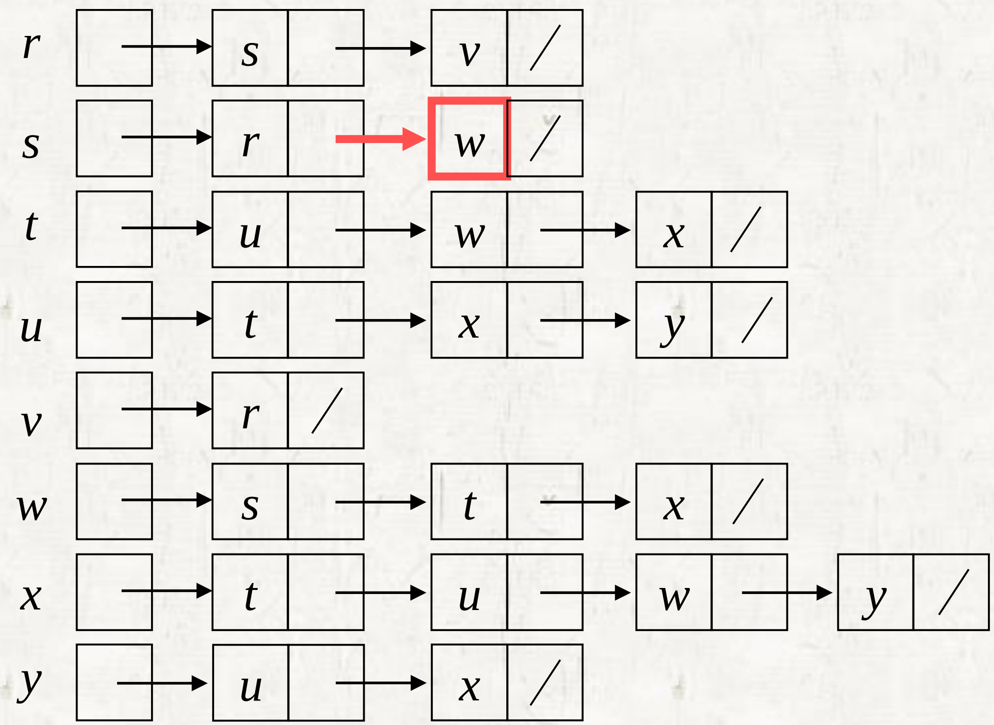

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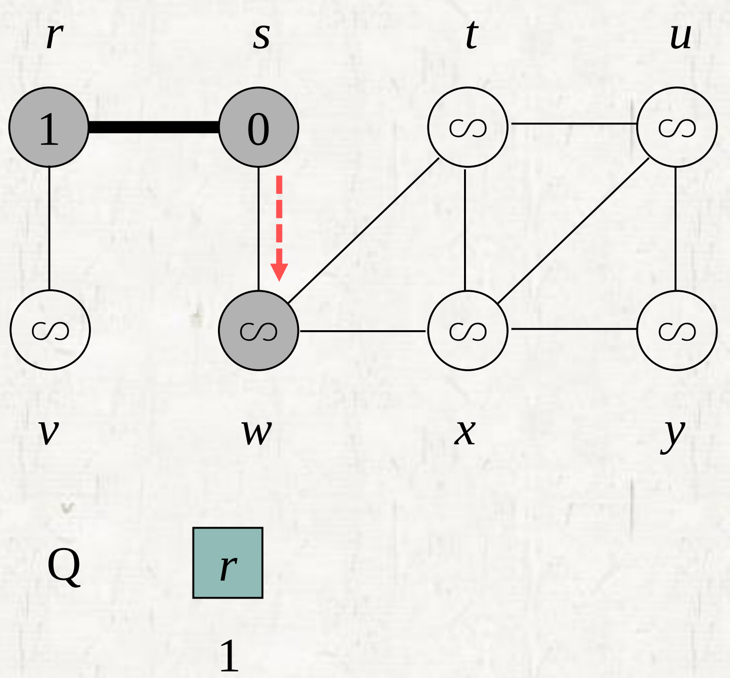

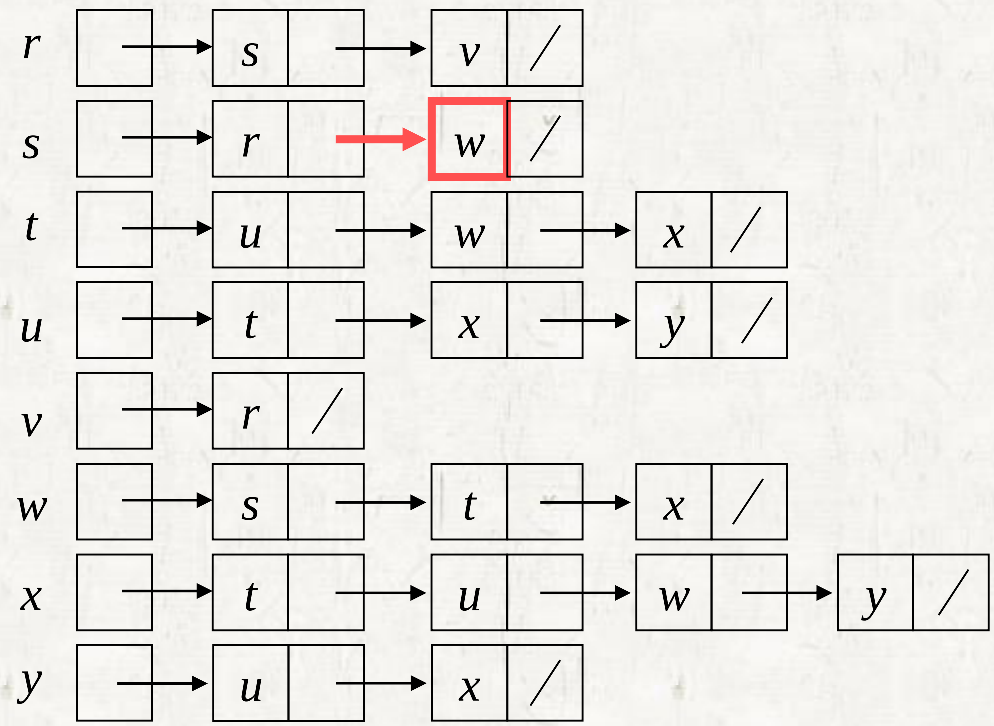

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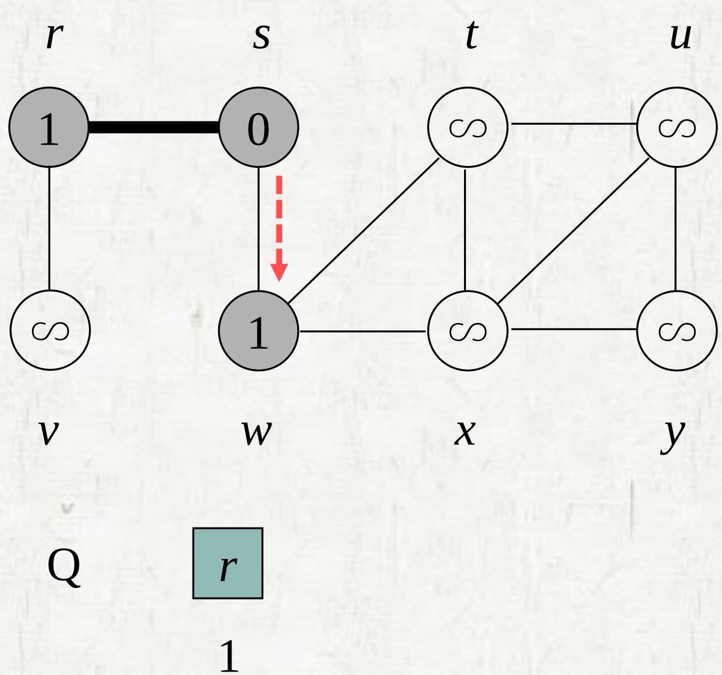

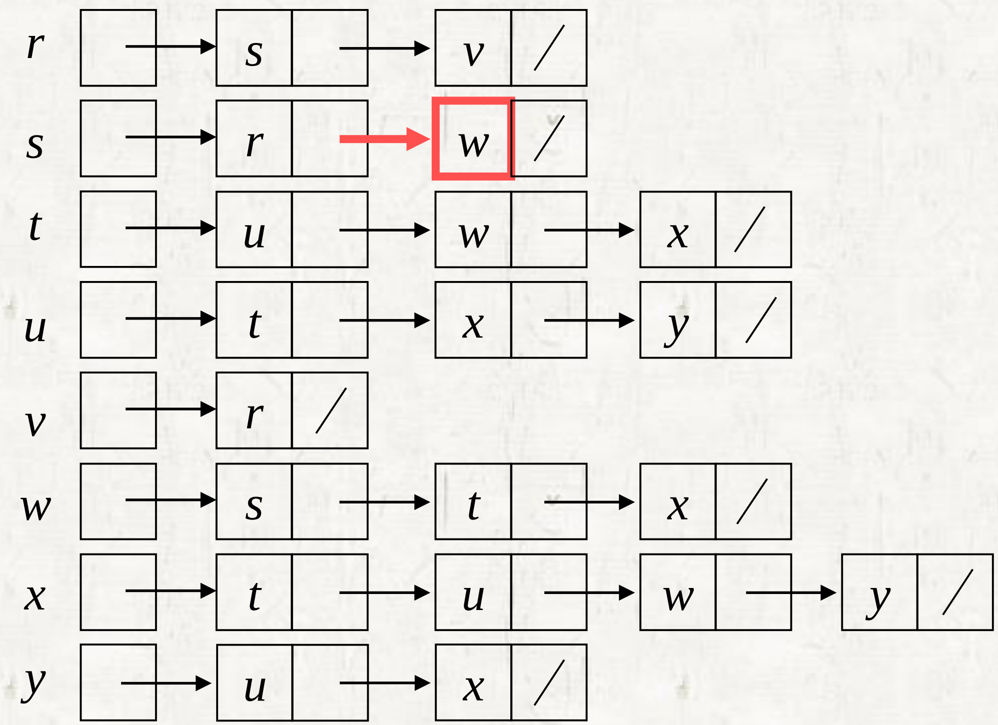

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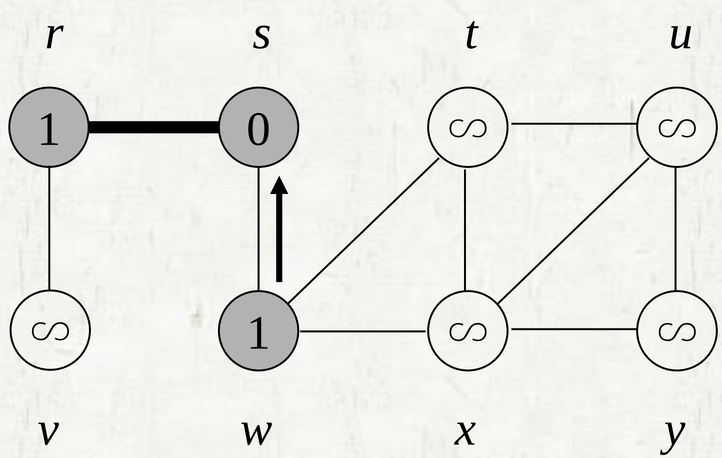

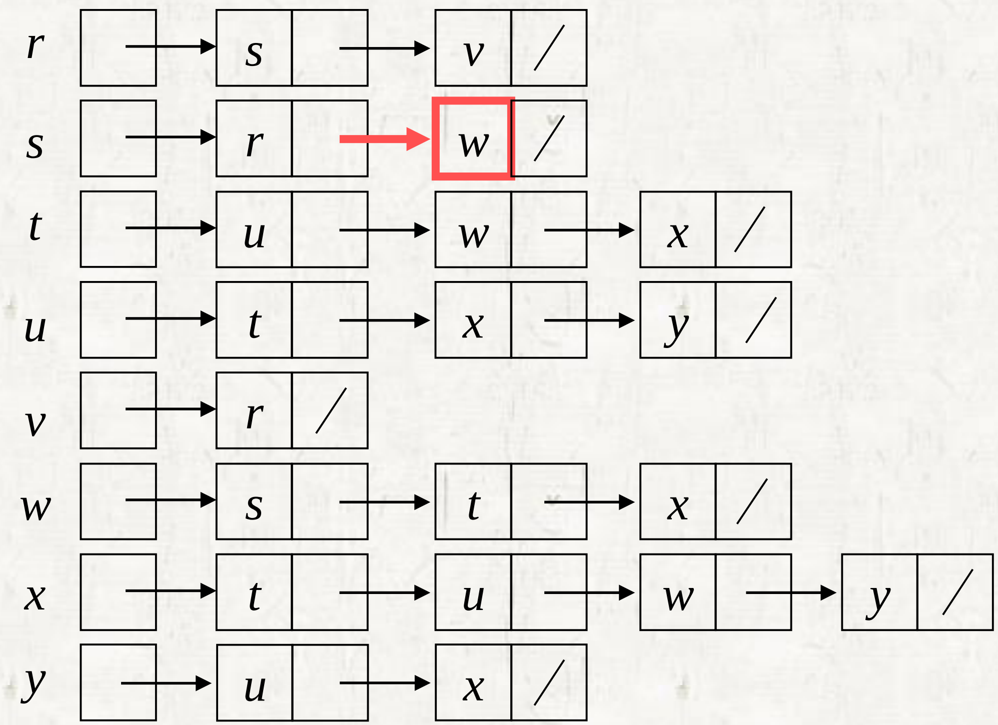

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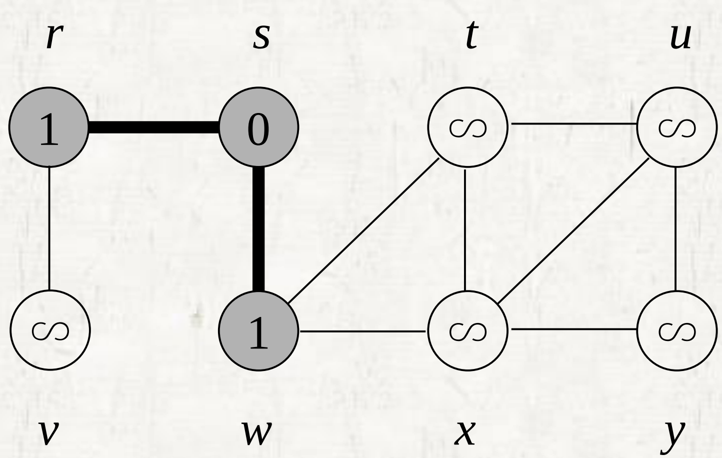

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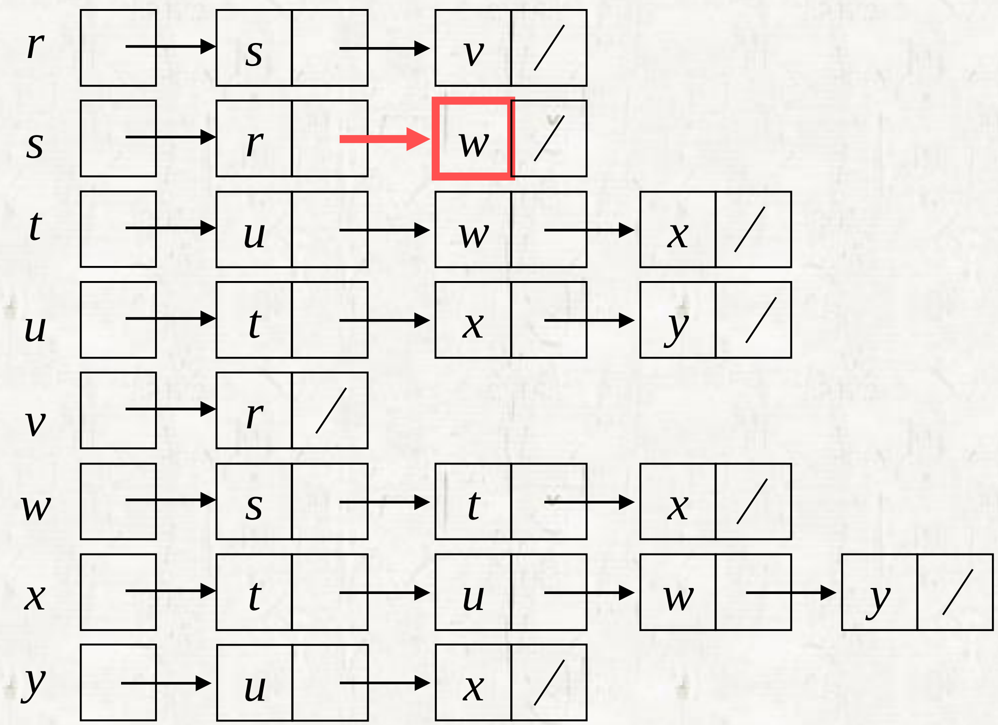

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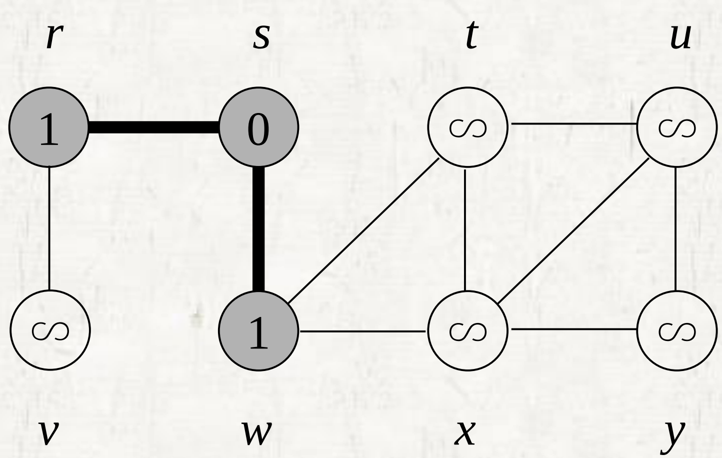

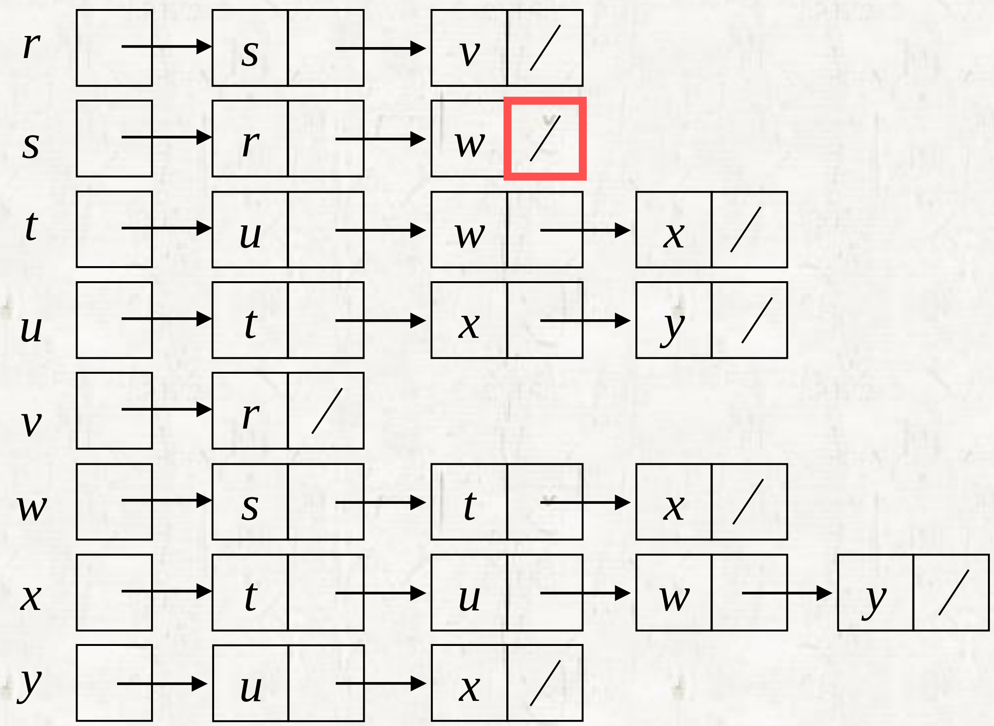

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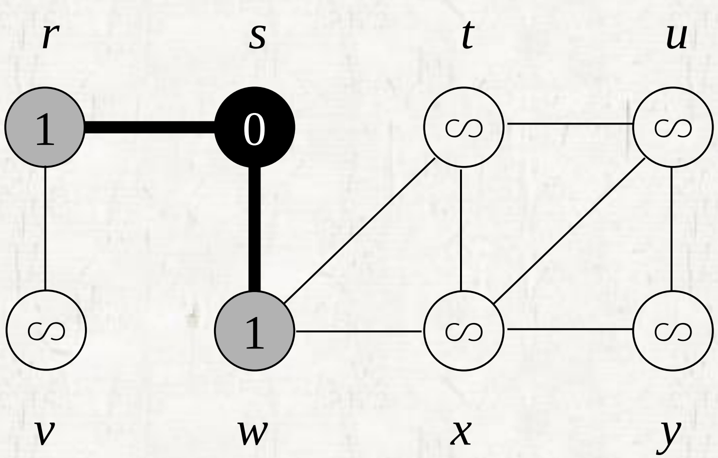

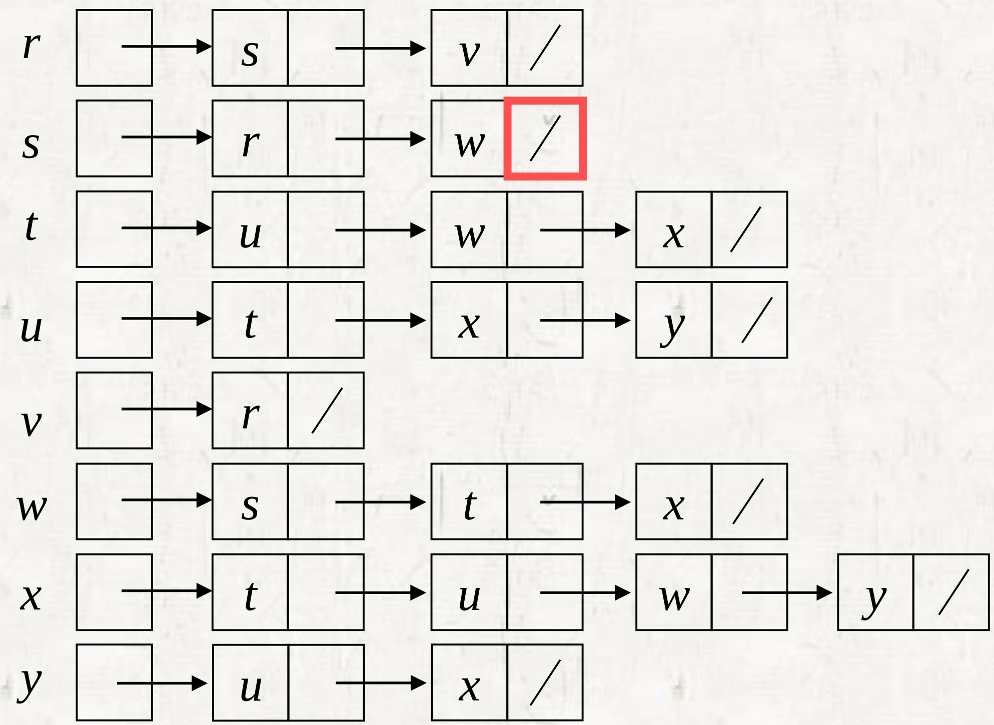

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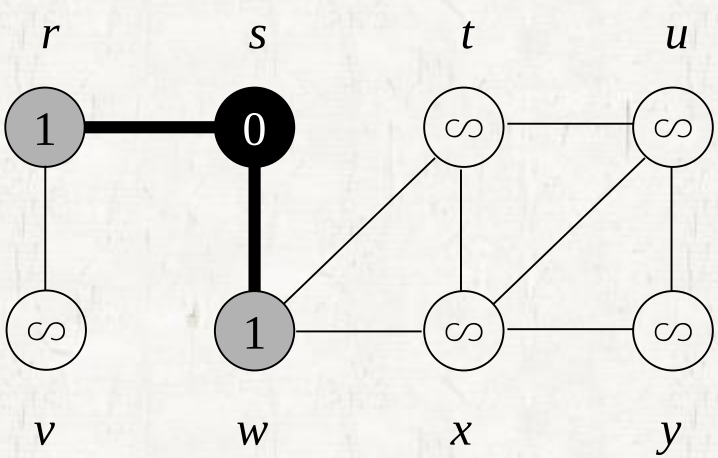

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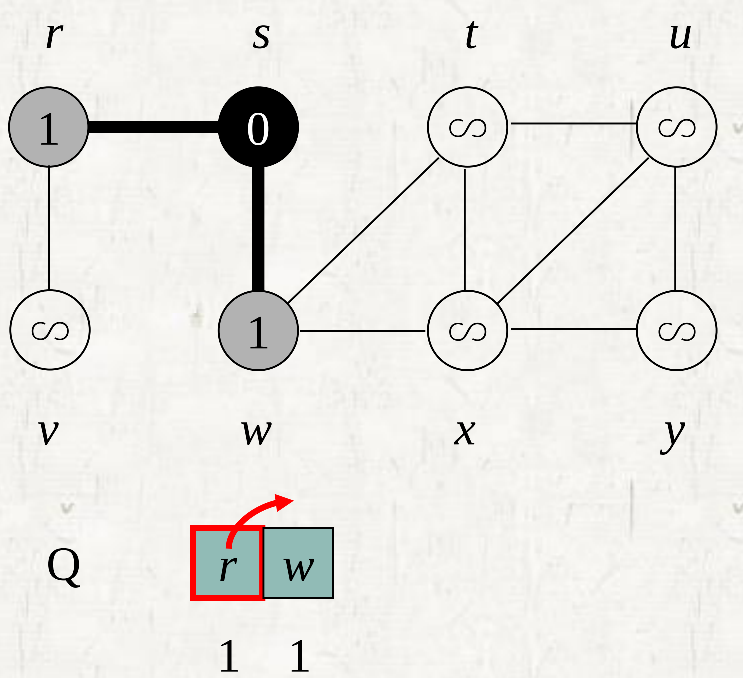

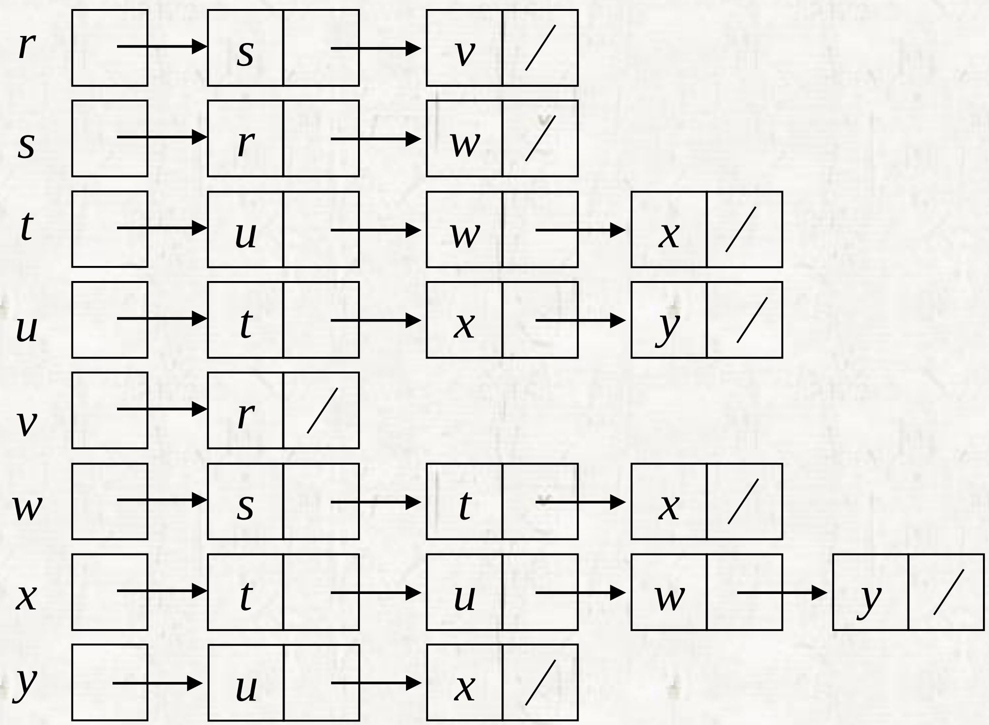

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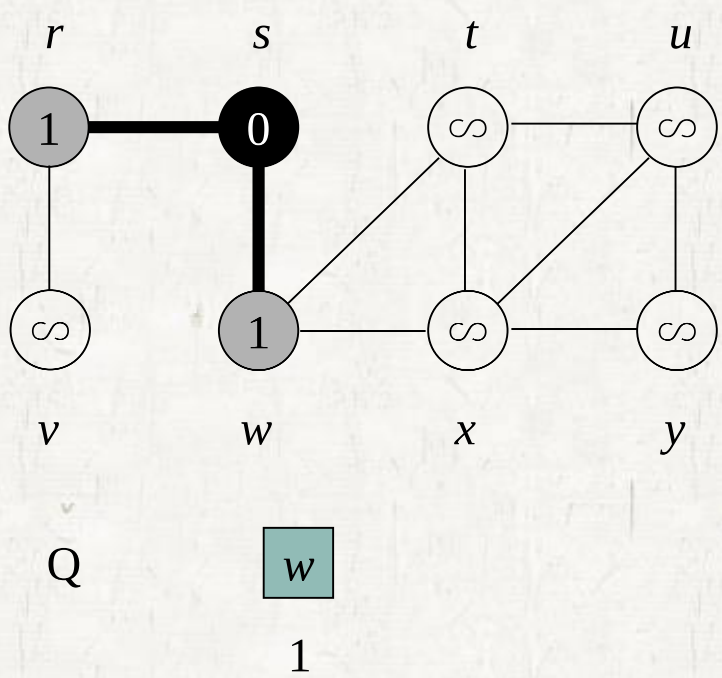

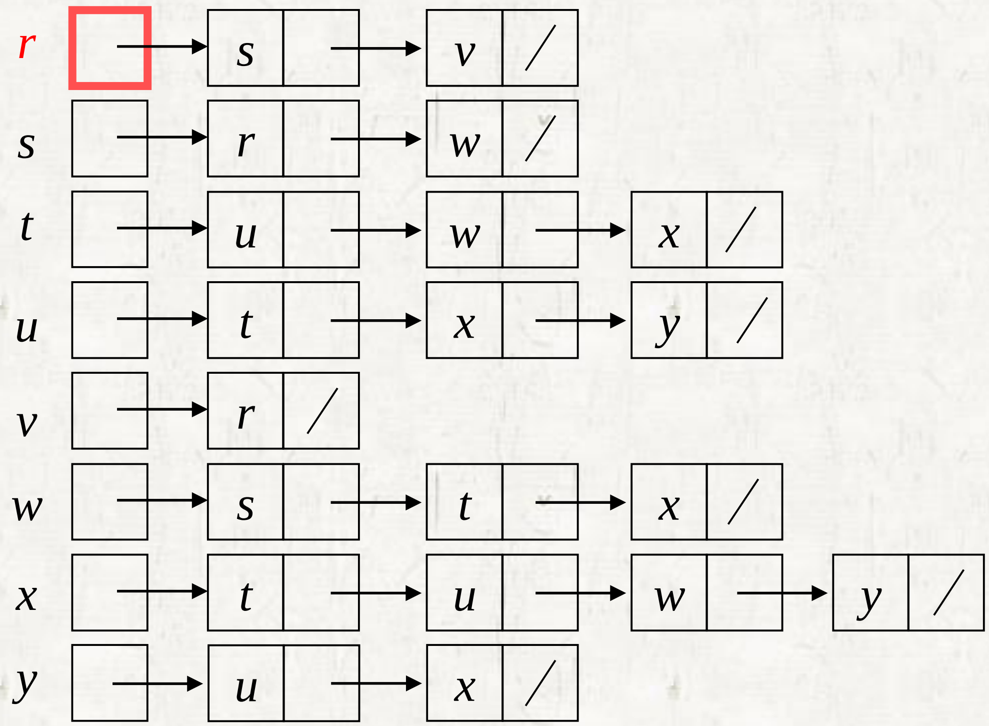

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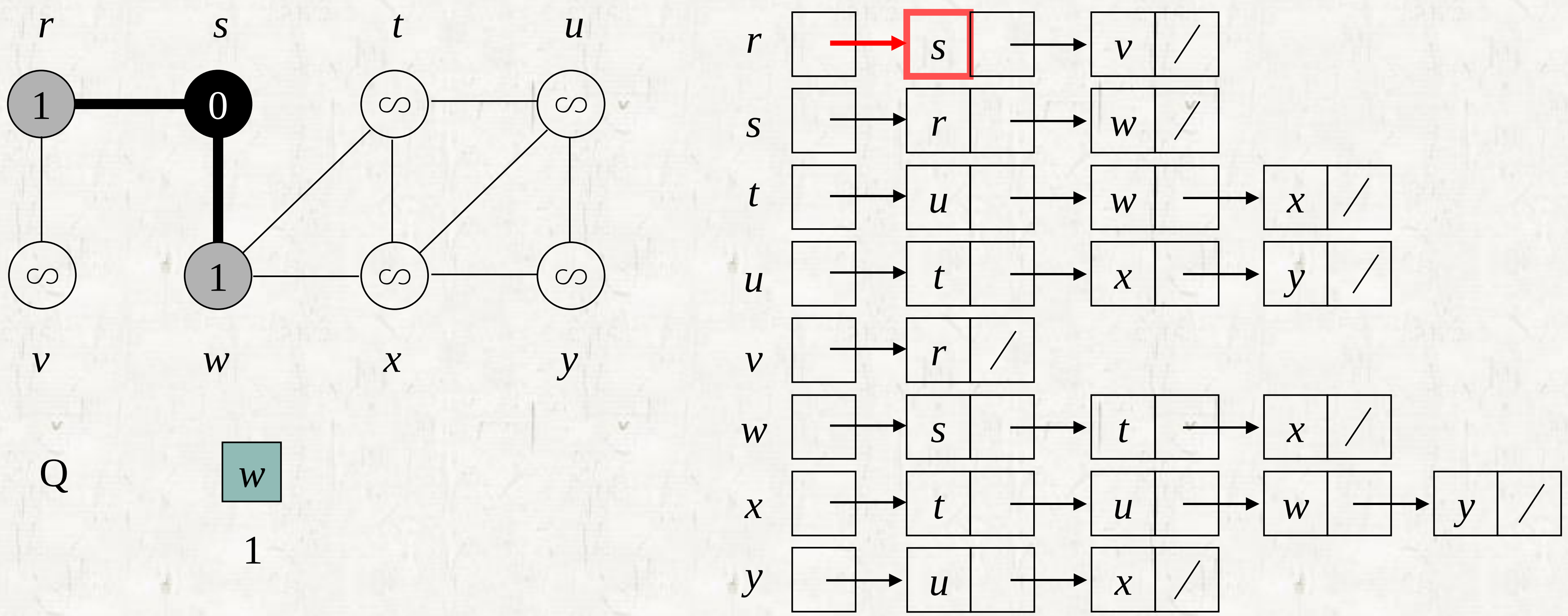

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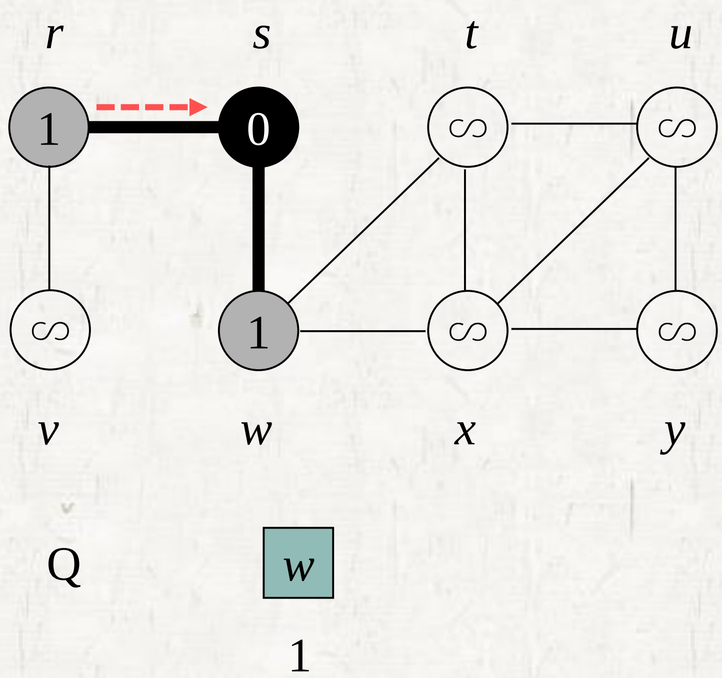

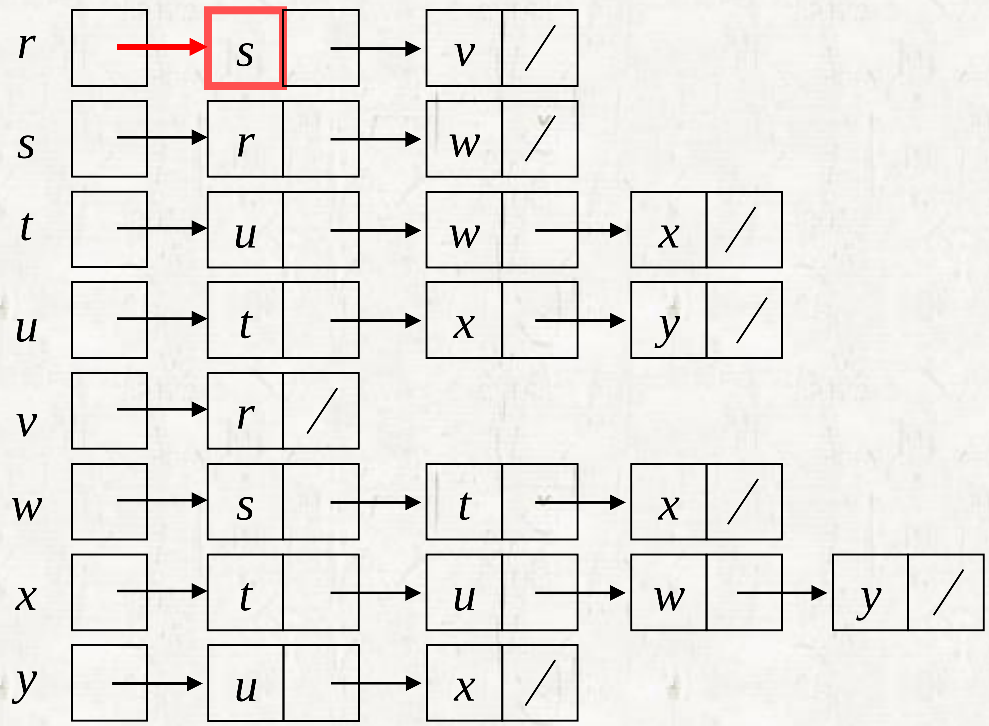

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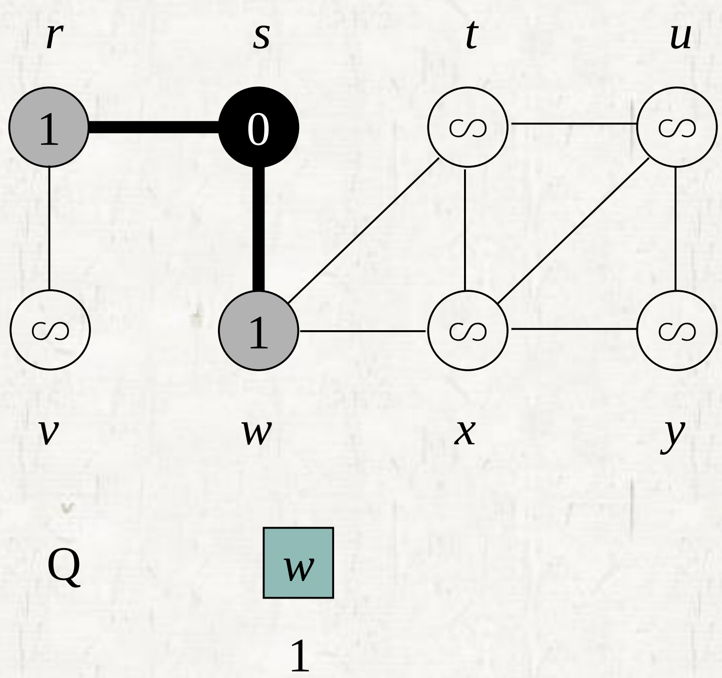

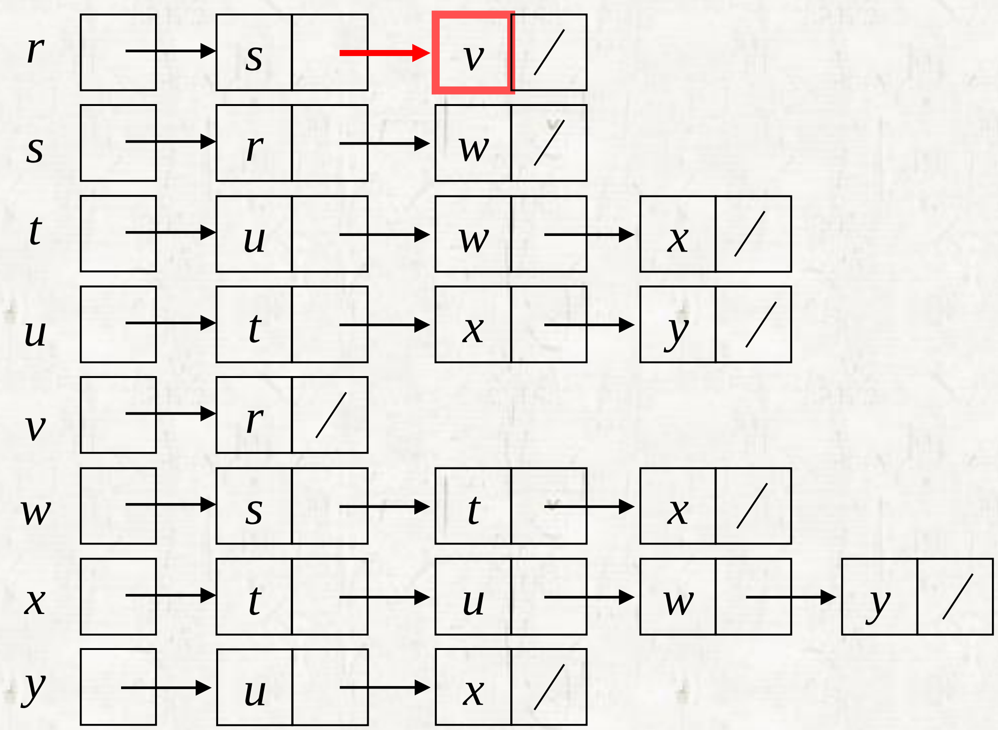

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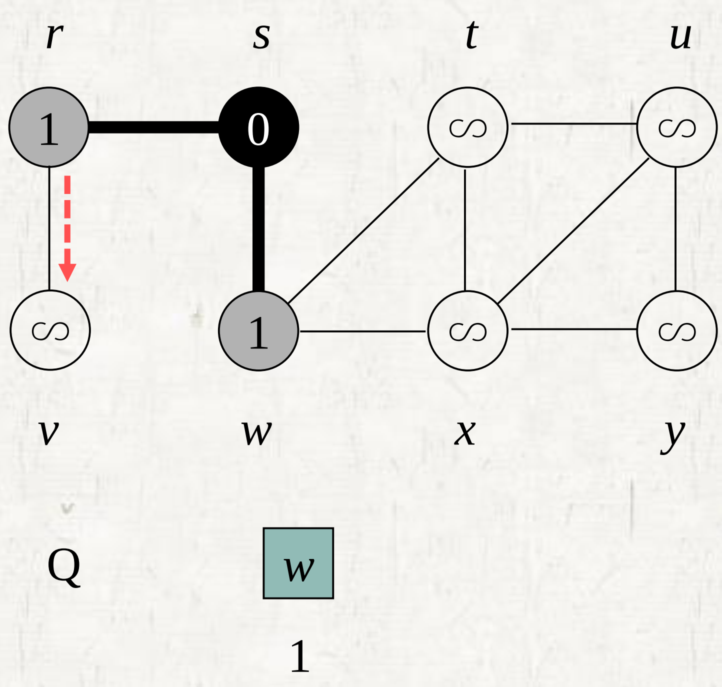

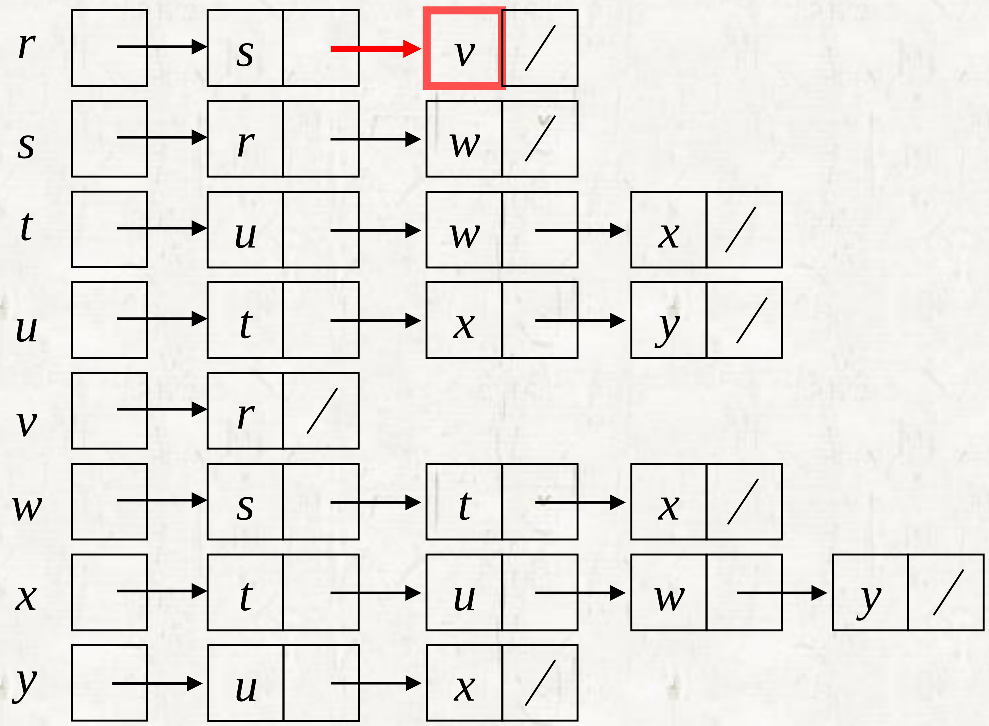

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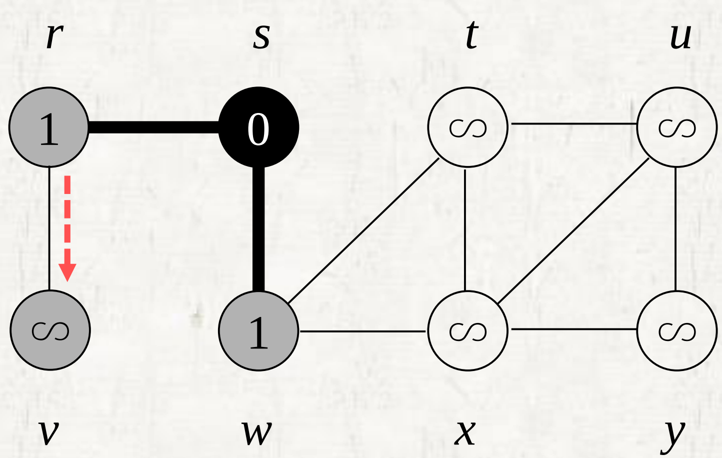

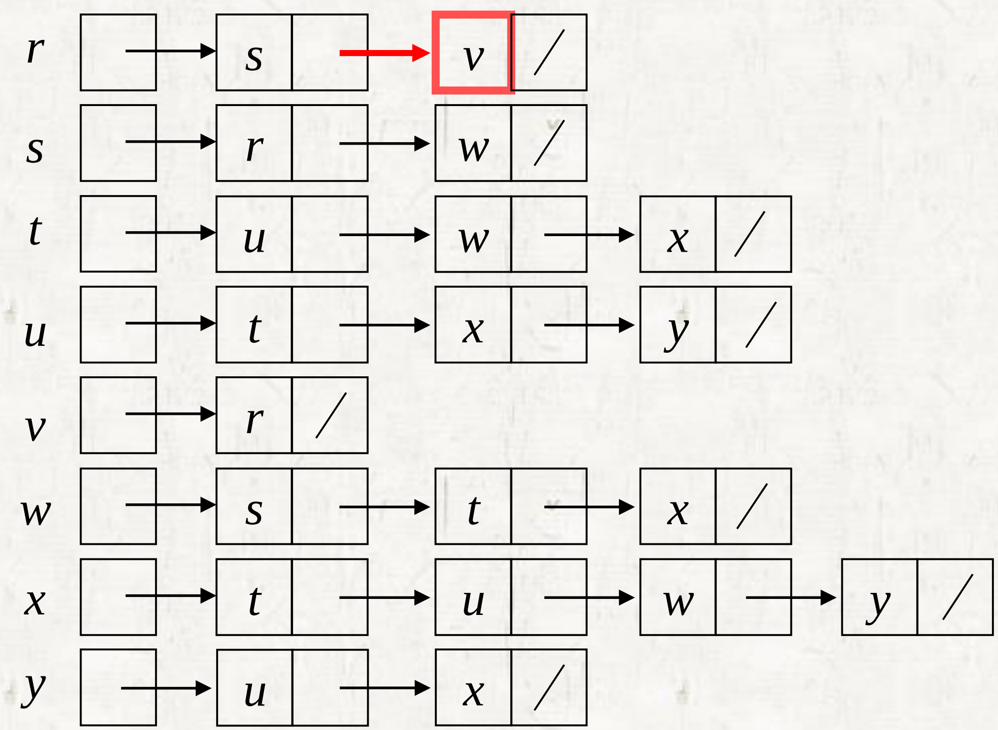

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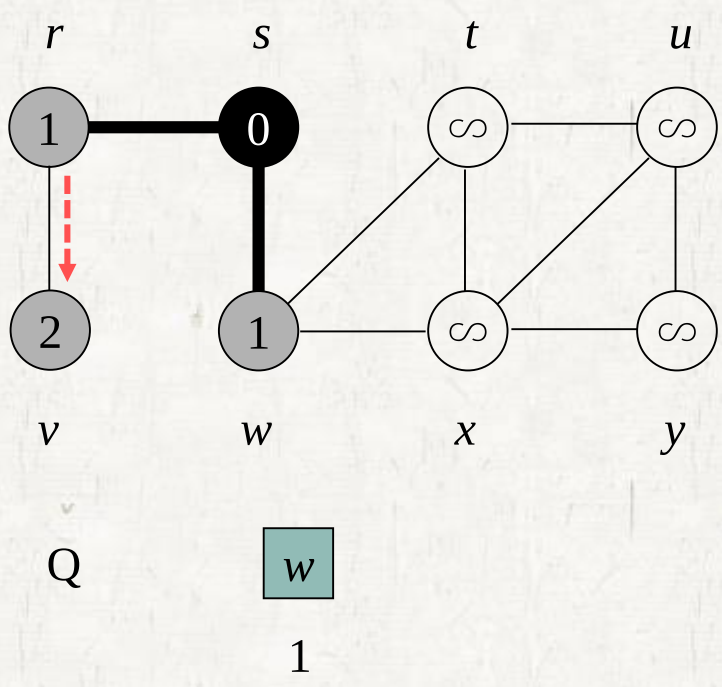

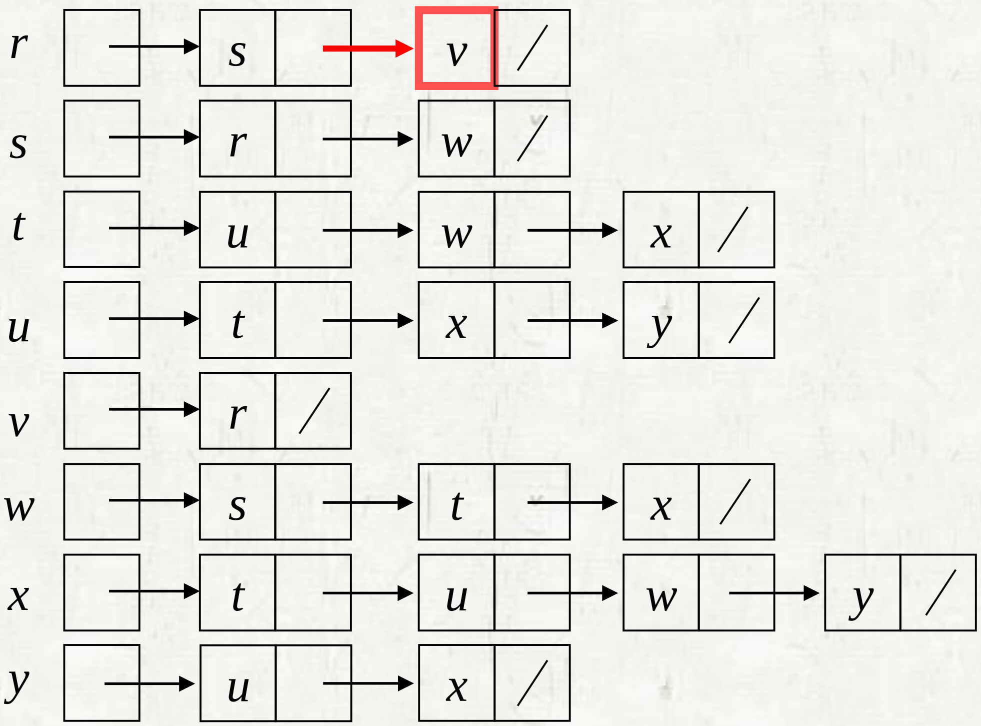

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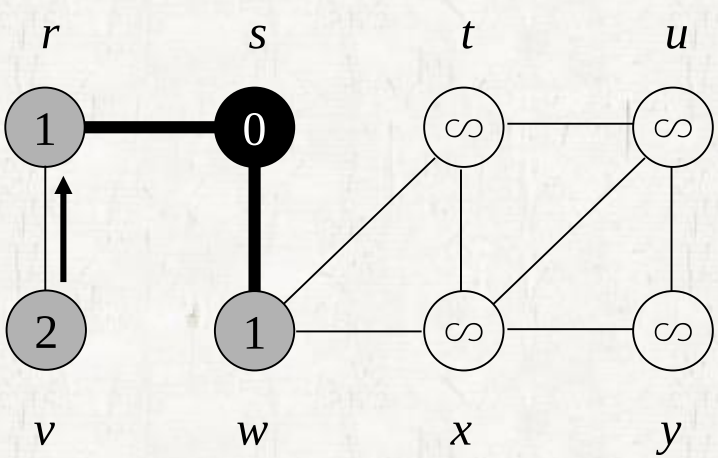

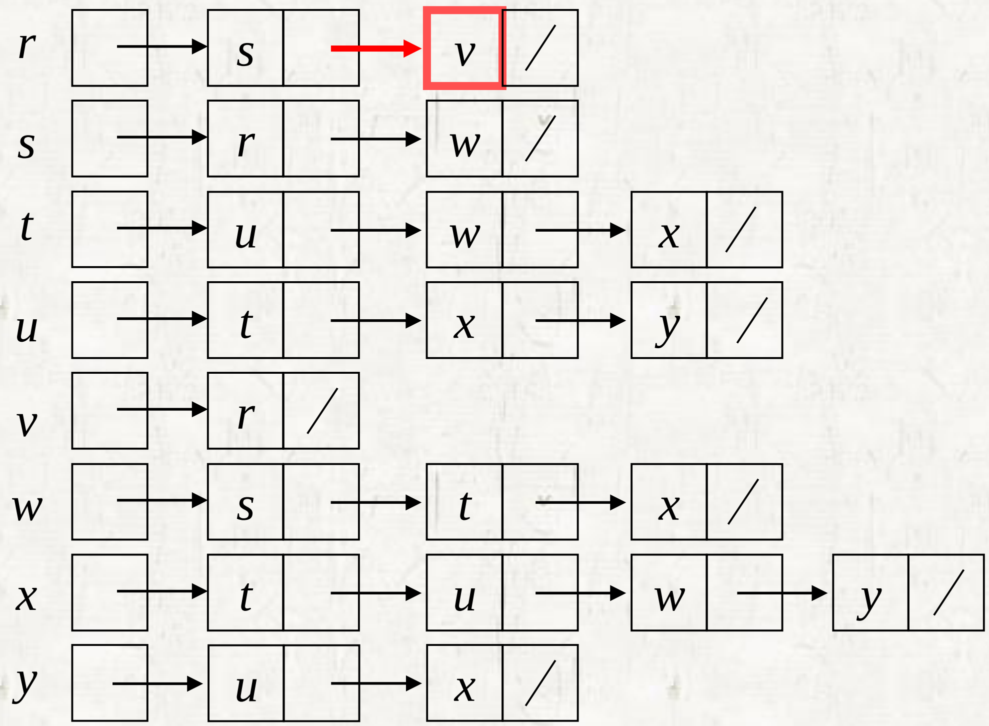

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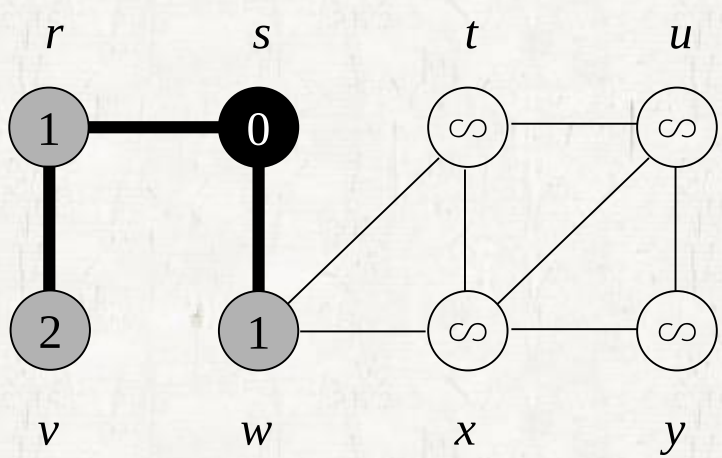

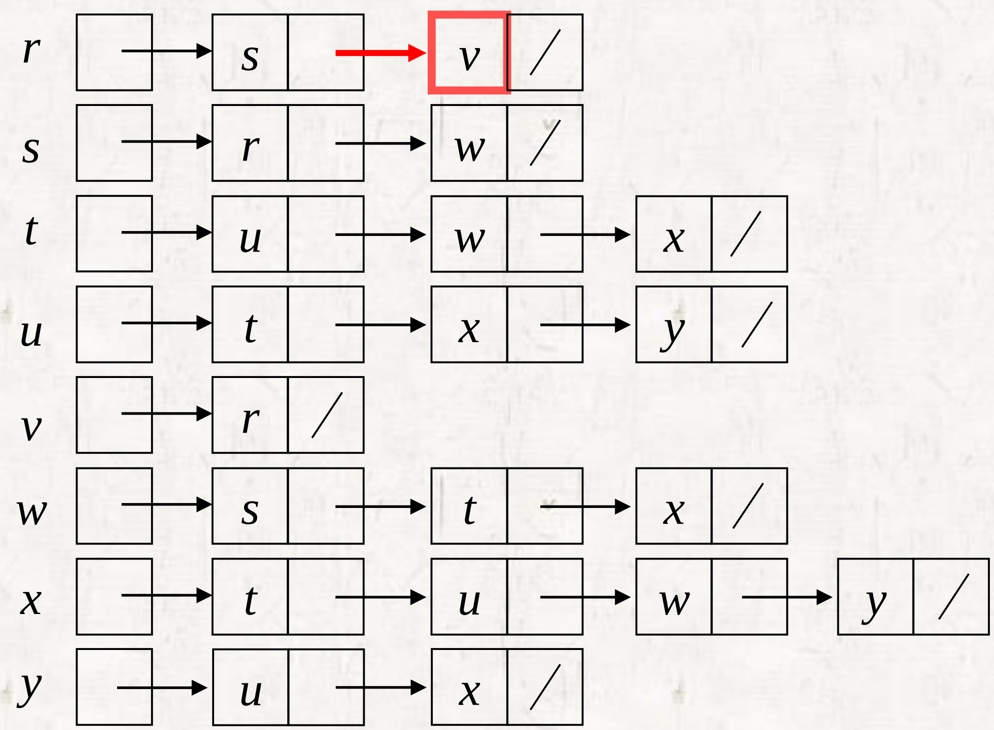

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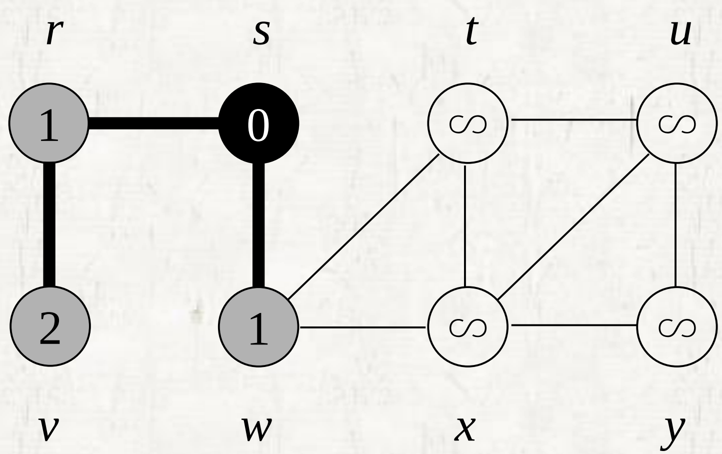

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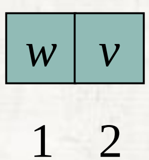

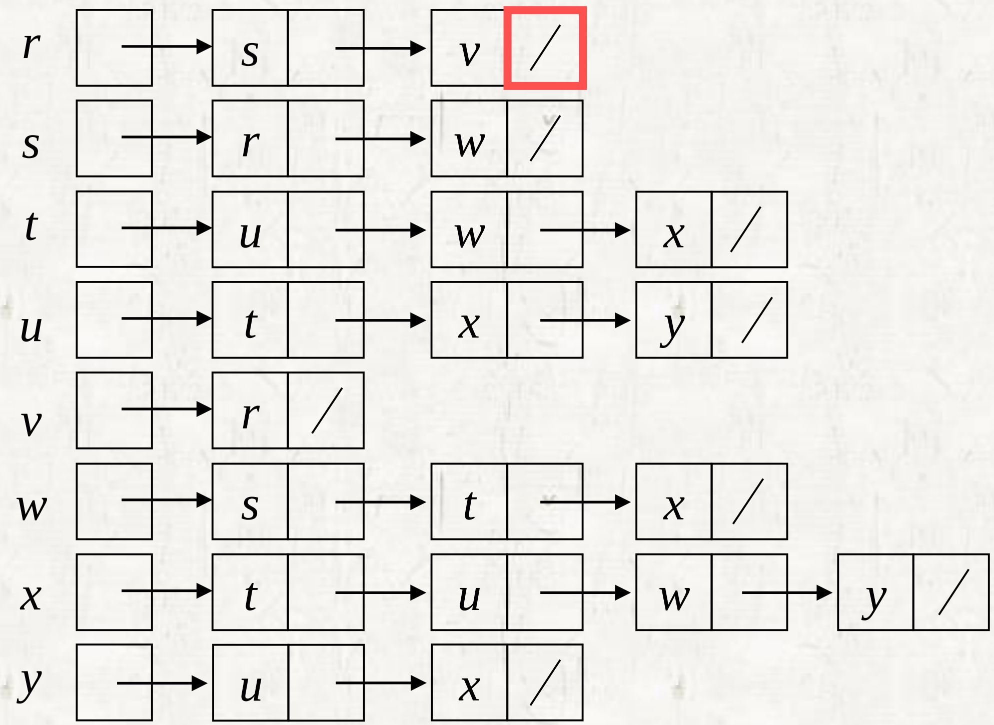

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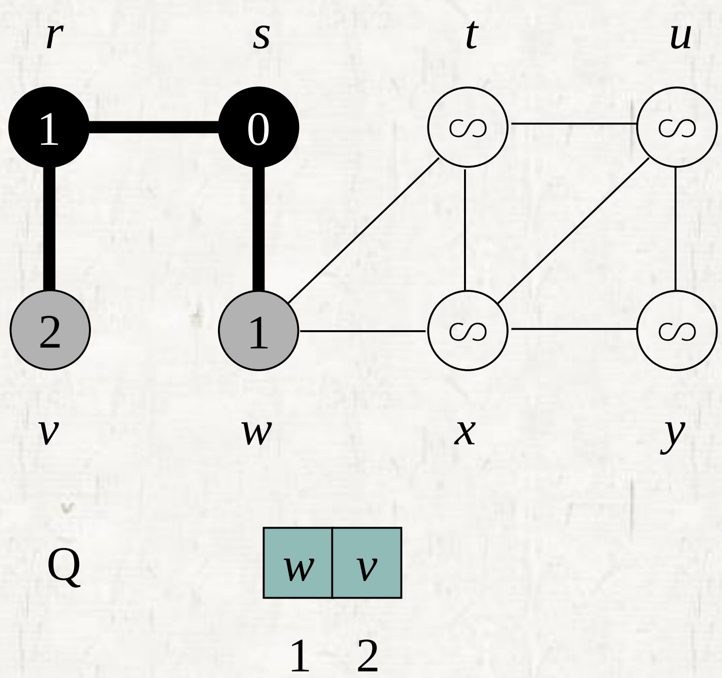

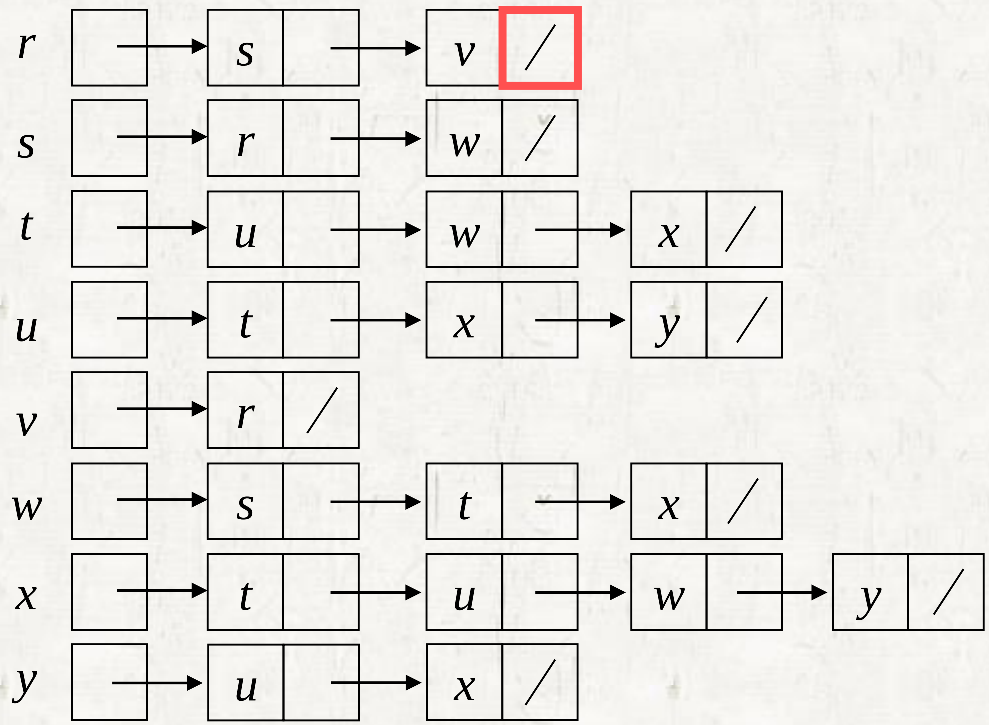

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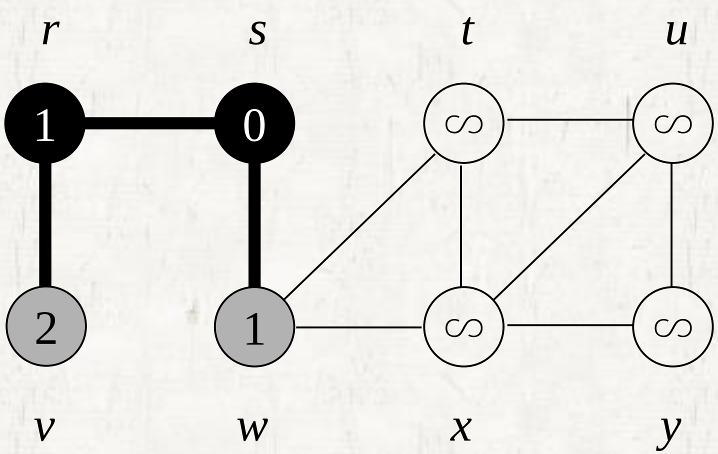

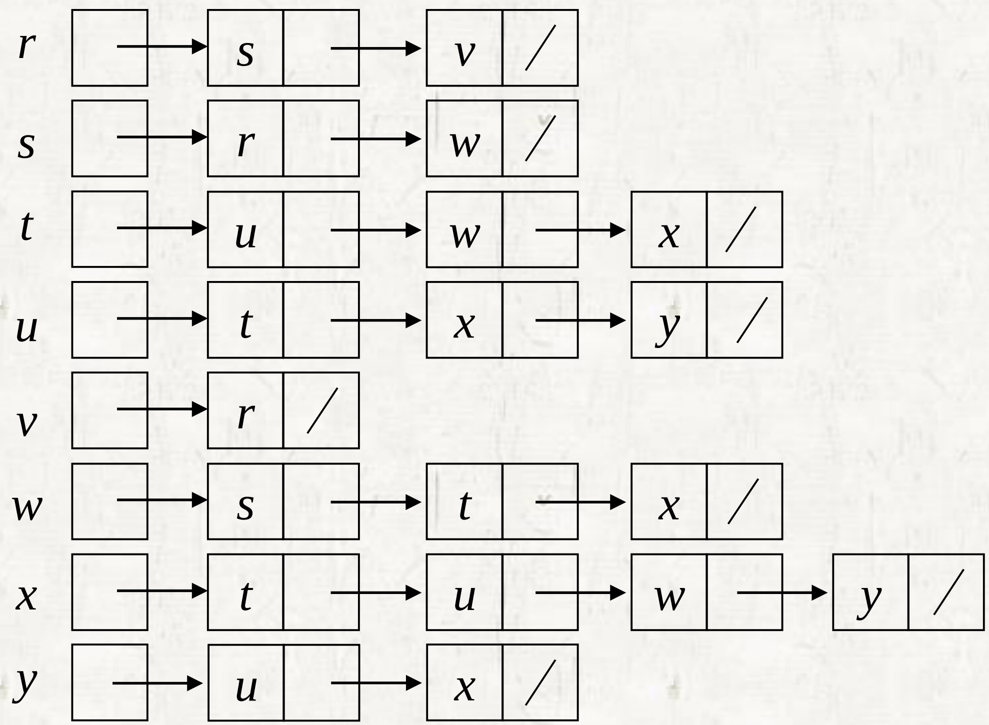

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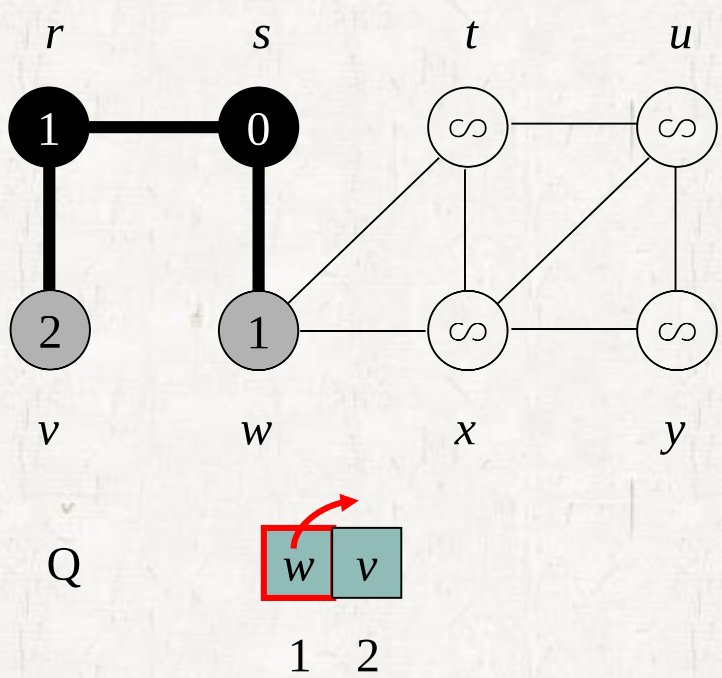

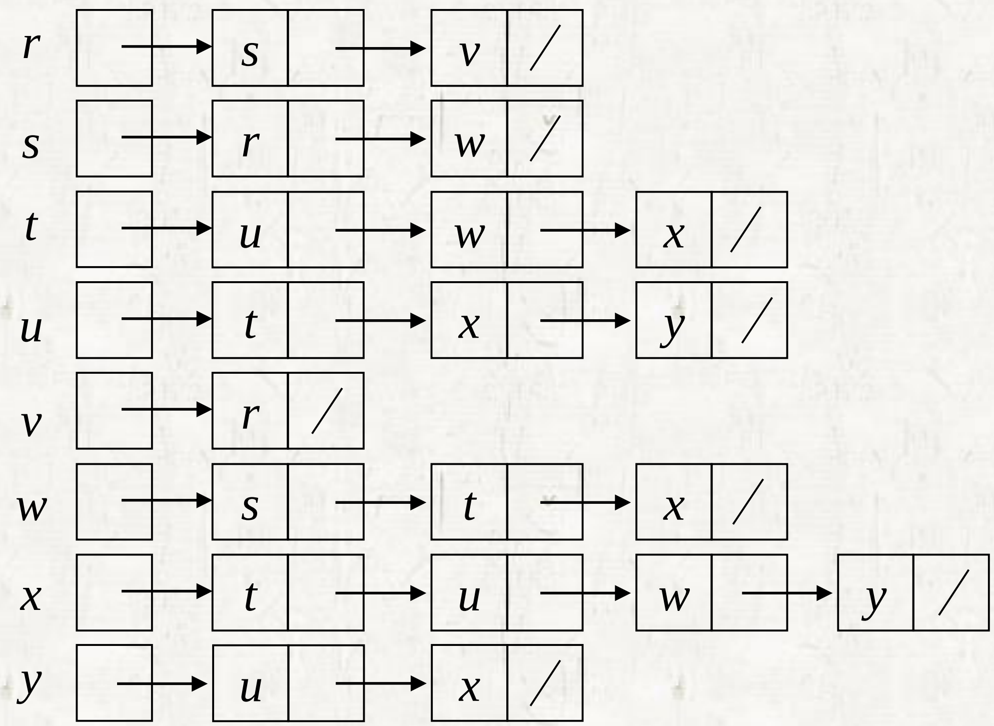

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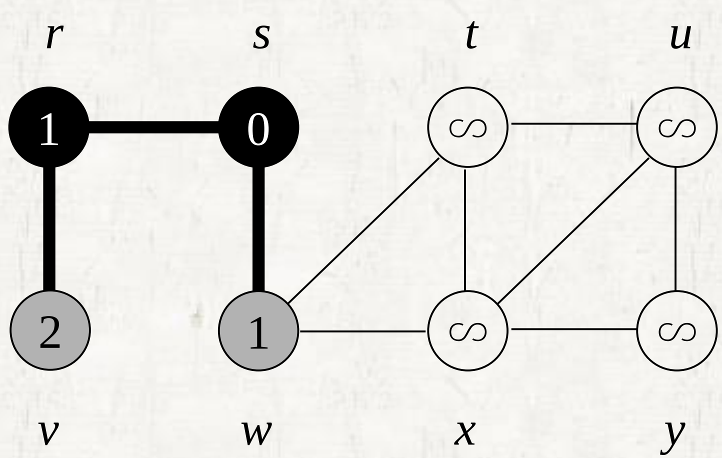

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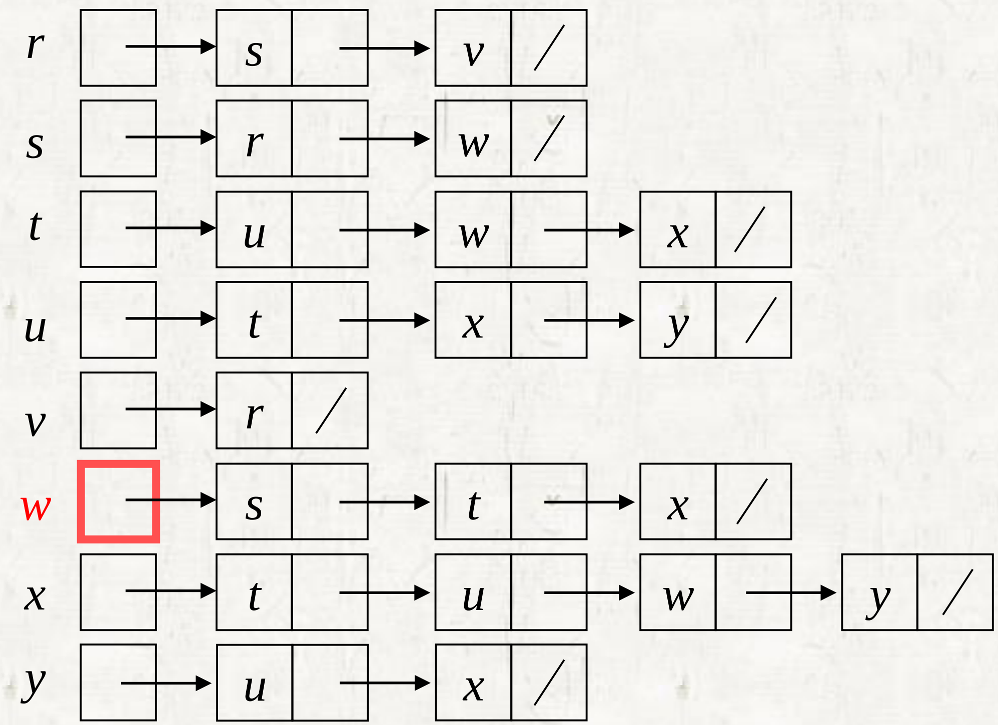

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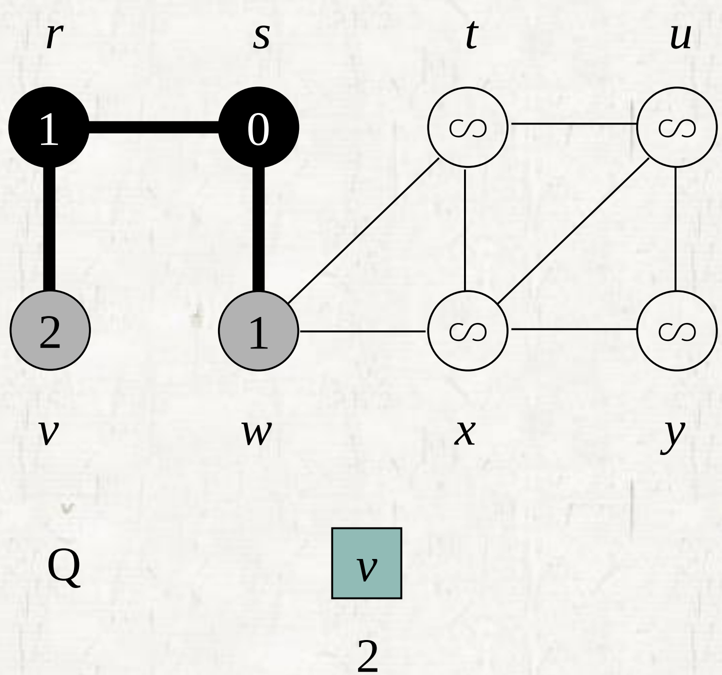

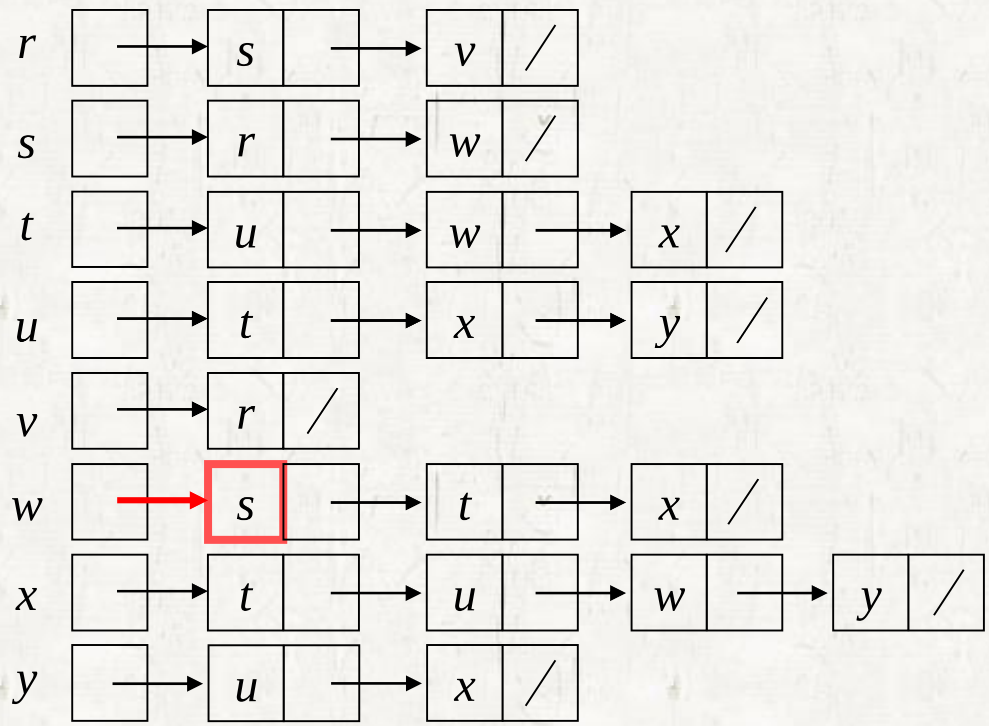

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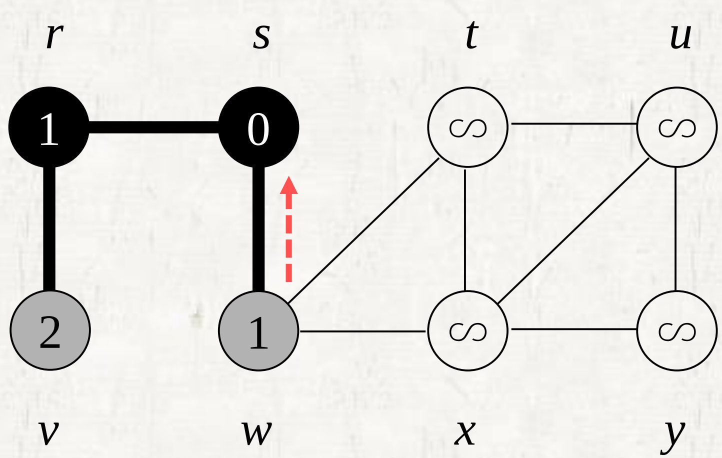

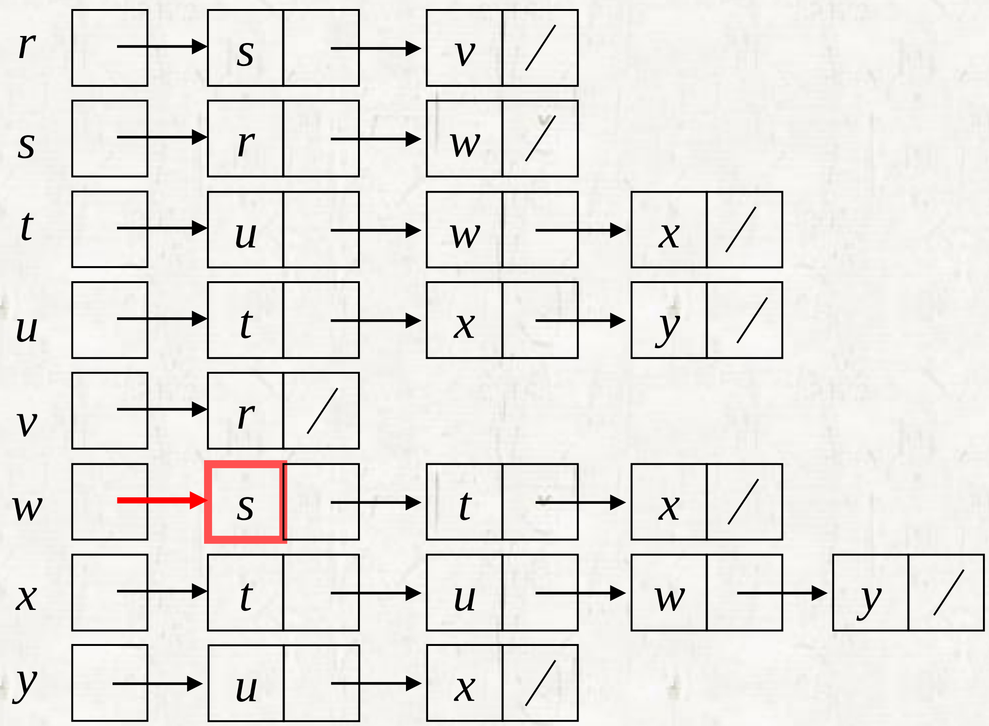

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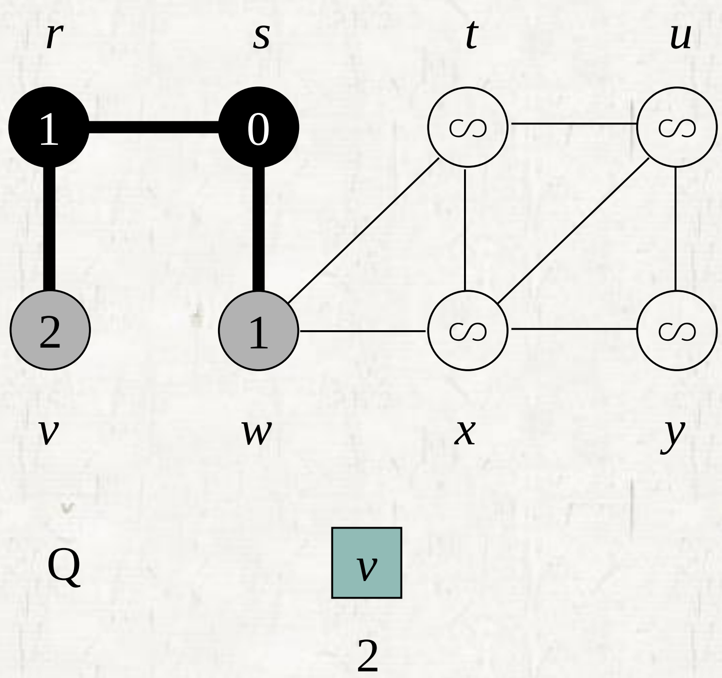

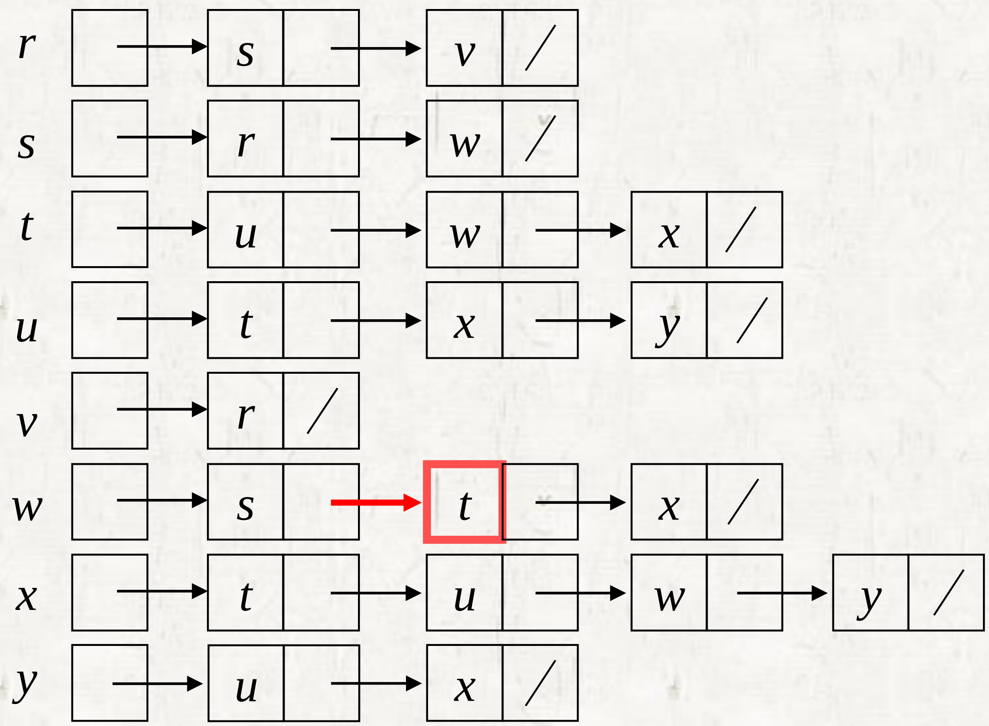

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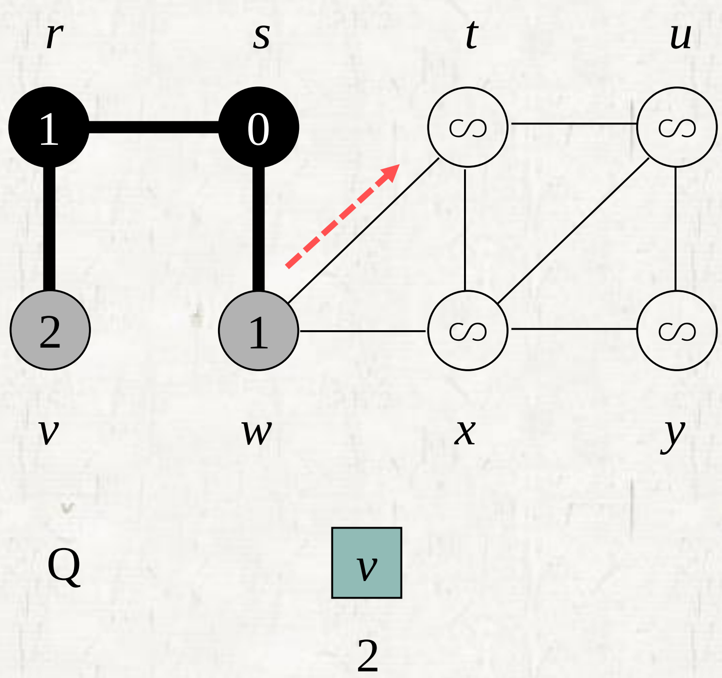

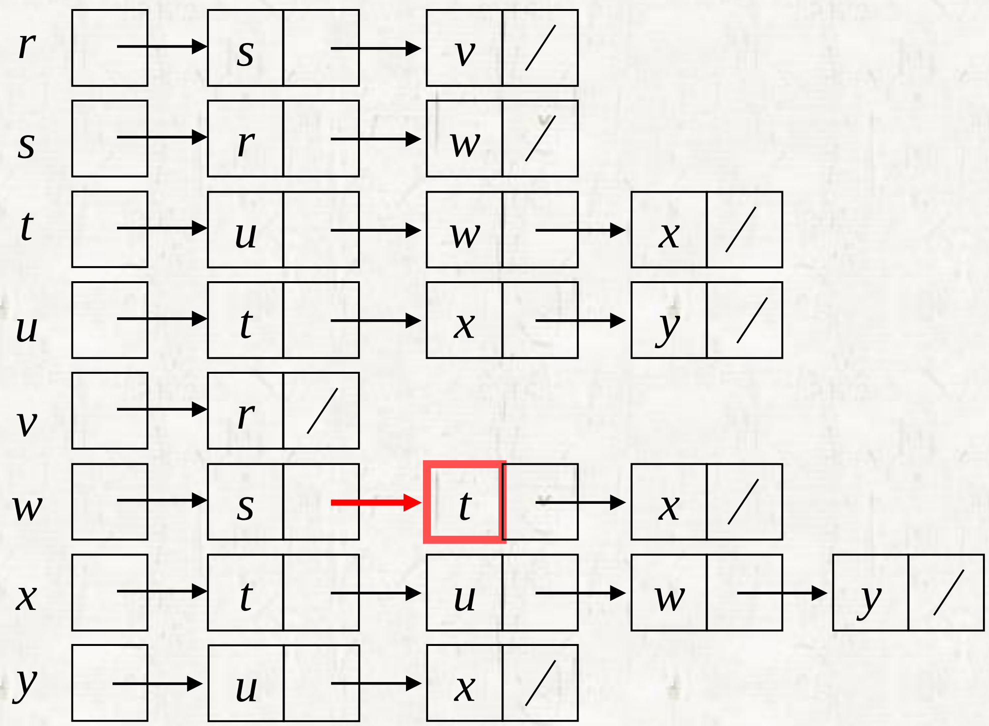

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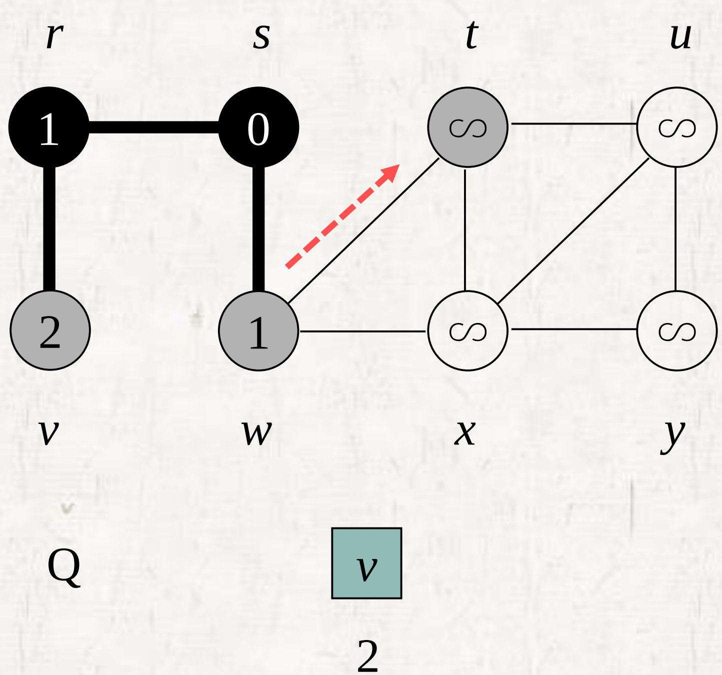
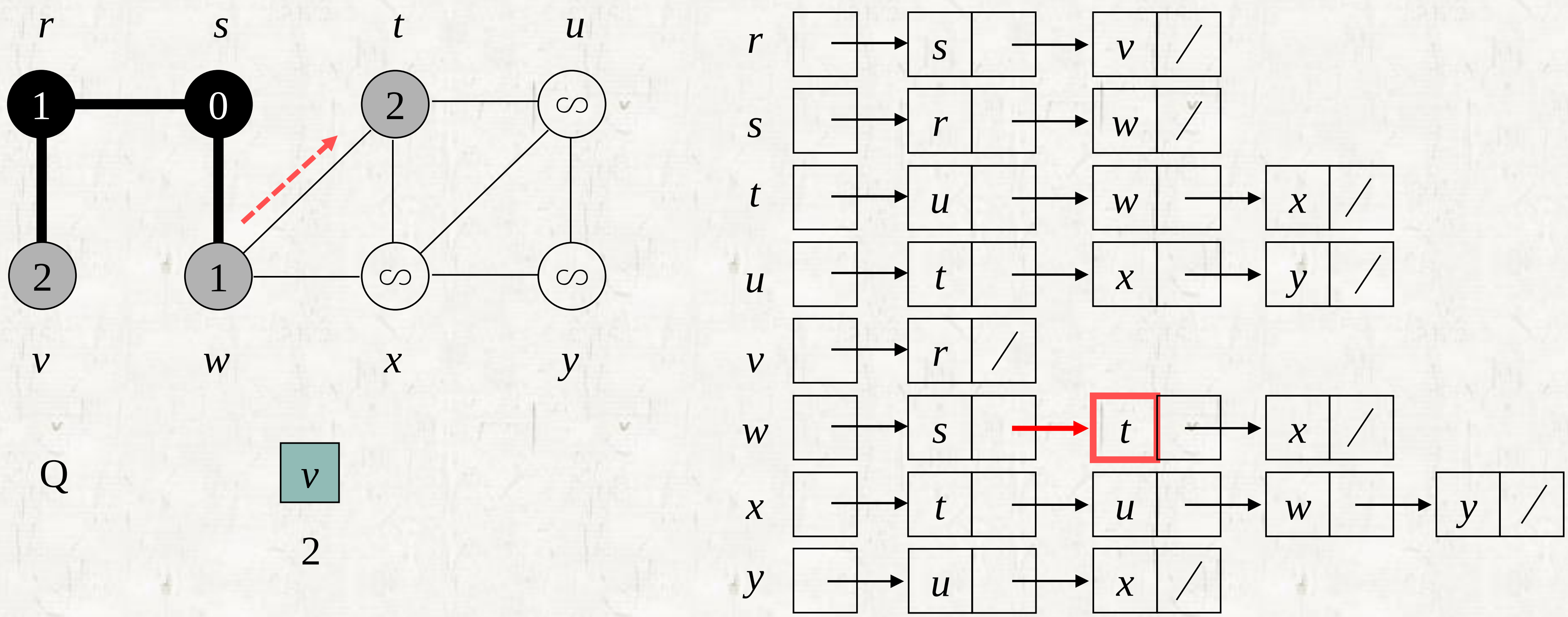


r		s	v /		
s		r	w /		
t		u	w	x /	
u		t	x	y /	
v		r /			
w		s	$\boxed{t}$	x /	
x		t	u	w	y /
y		u	x /		

Breadth-first search



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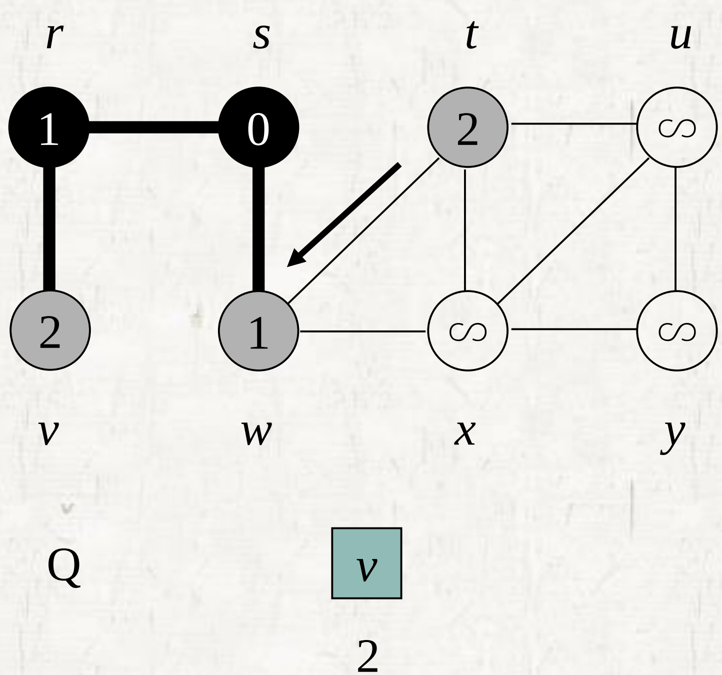

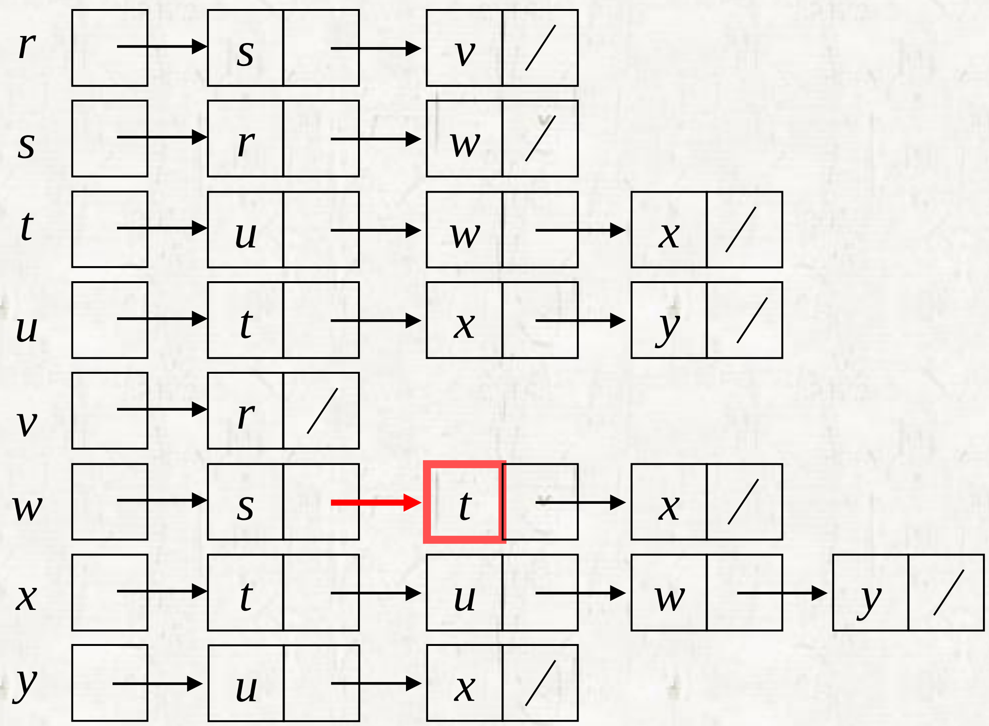

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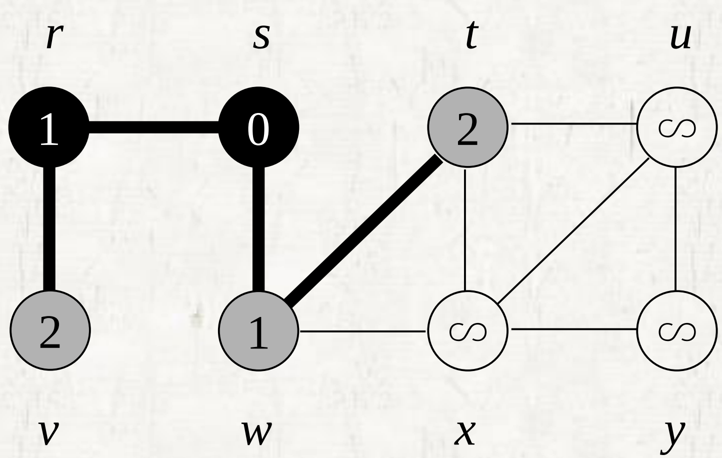
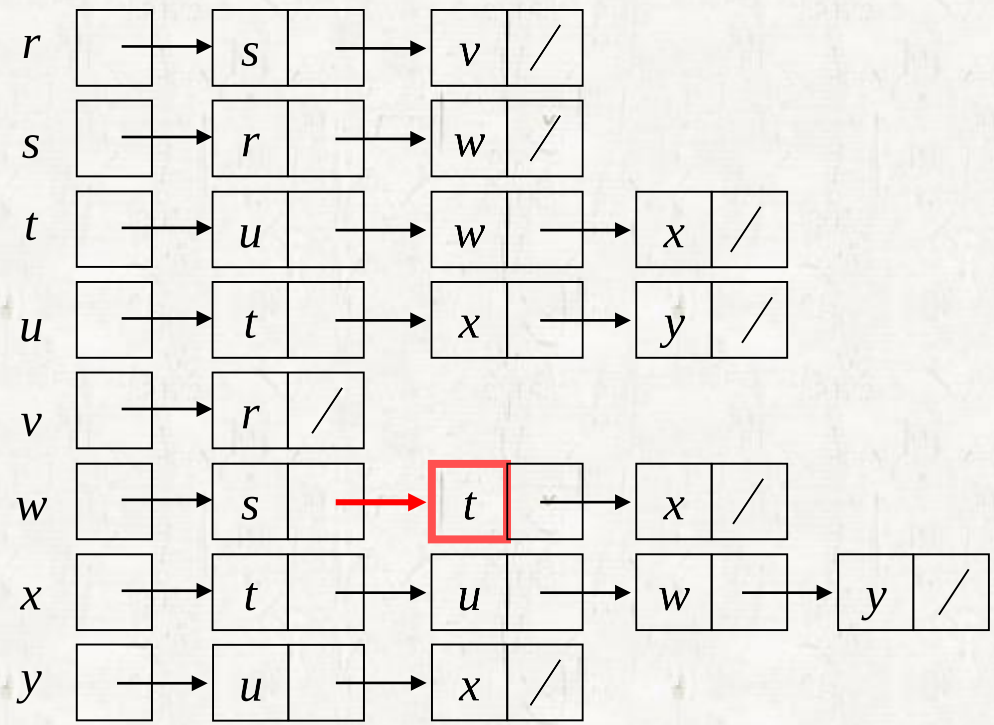
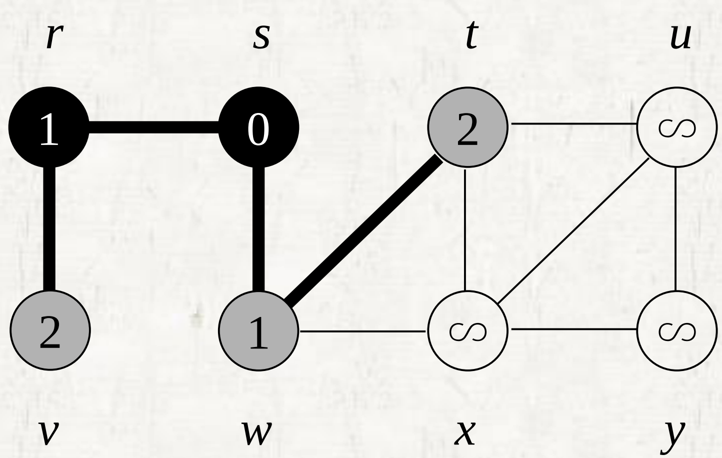




Q		
	v	*t*
	2	2

Breadth-first search

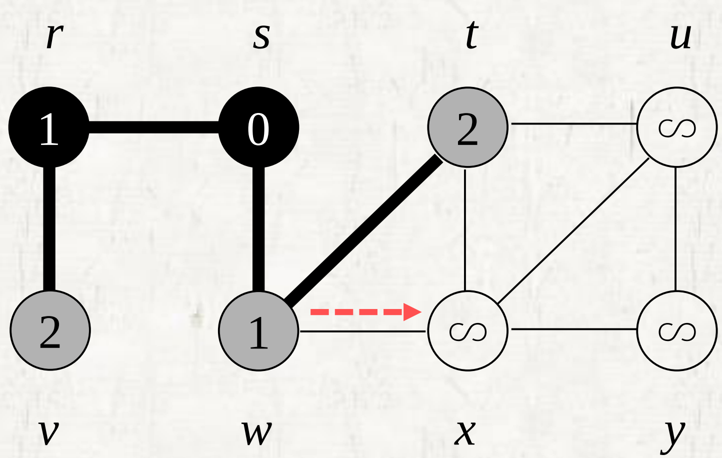


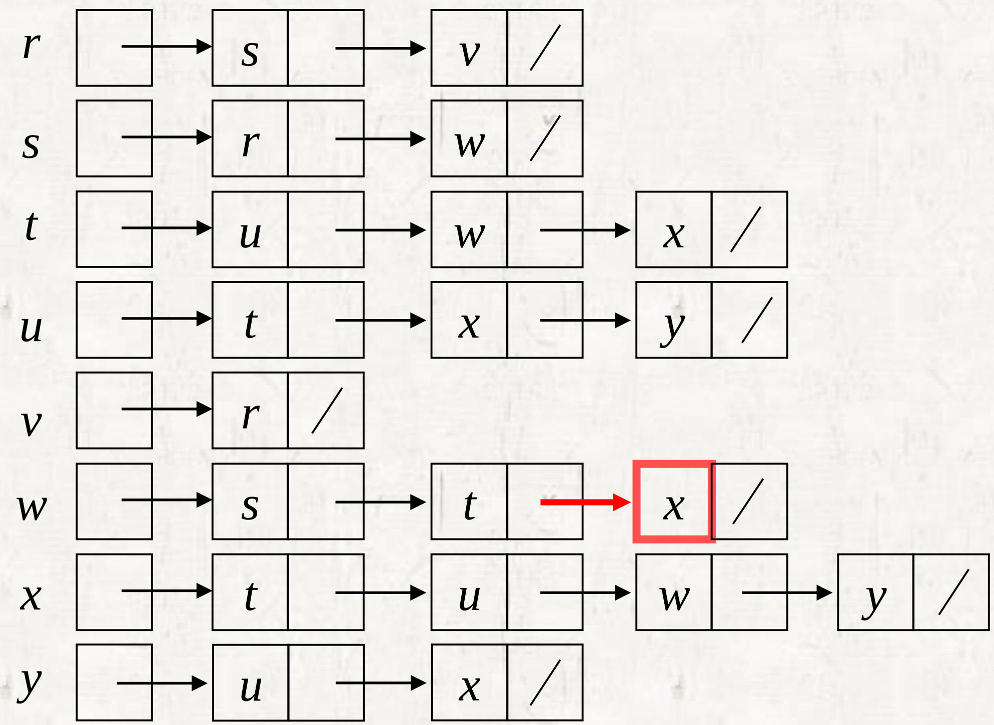
r	→ *s*	→ *v* /		
s	→ *r*	→ *w* /		
t	→ *u*	→ *w*	→ *x* /	
u	→ *t*	→ *x*	→ *y* /	
v	→ *r* /			
w	→ *s*	→ *t*	→ $\boxed{x}$ /	
x	→ *t*	→ *u*	→ *w*	→ *y* /
y	→ *u*	→ *x* /		

Q

v	*t*
2	2

Breadth-first search





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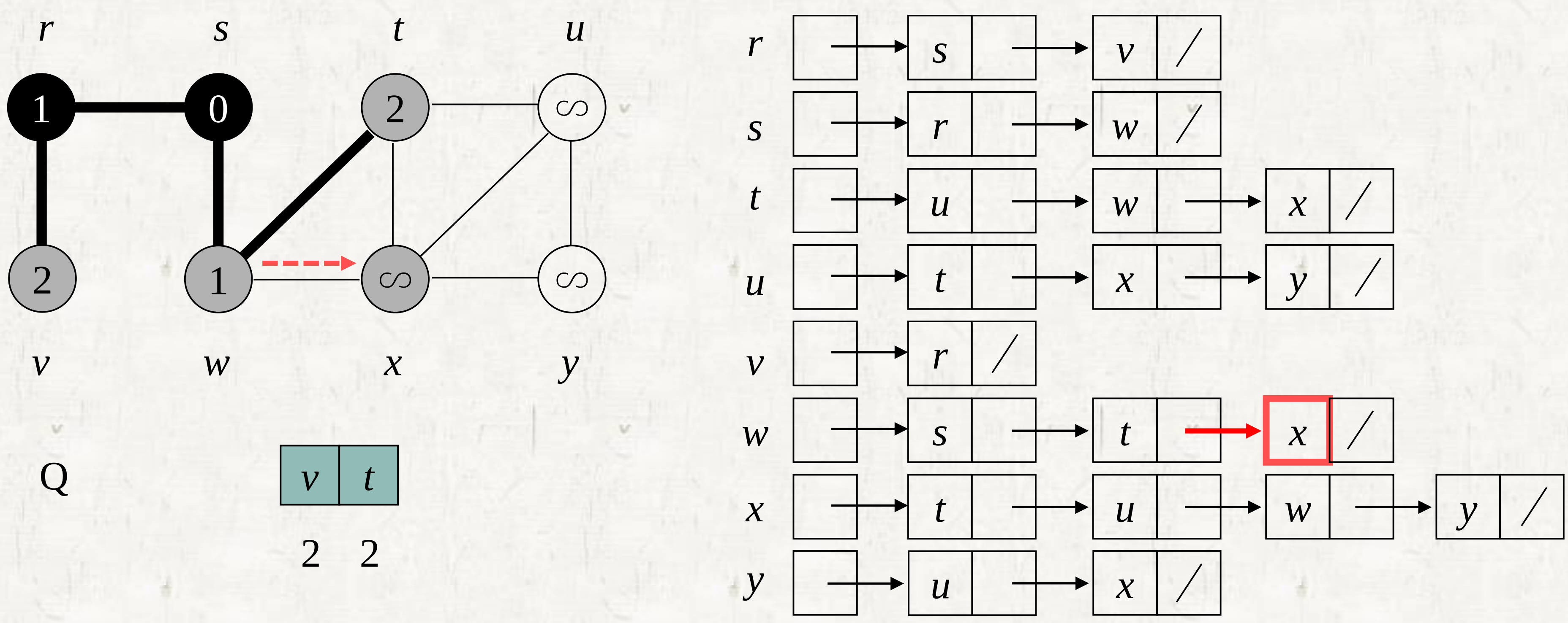

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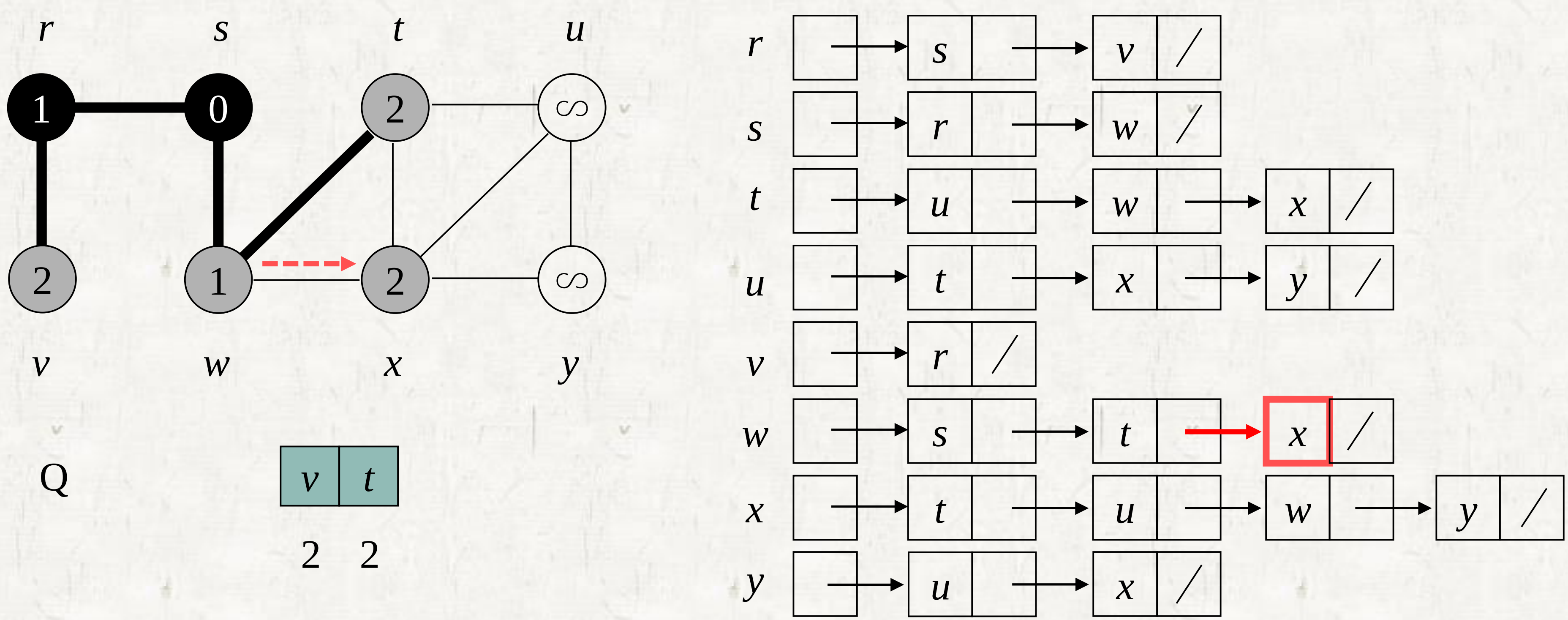

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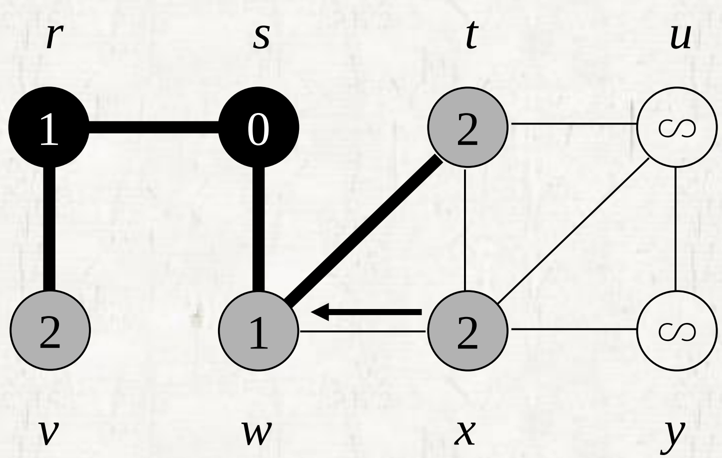
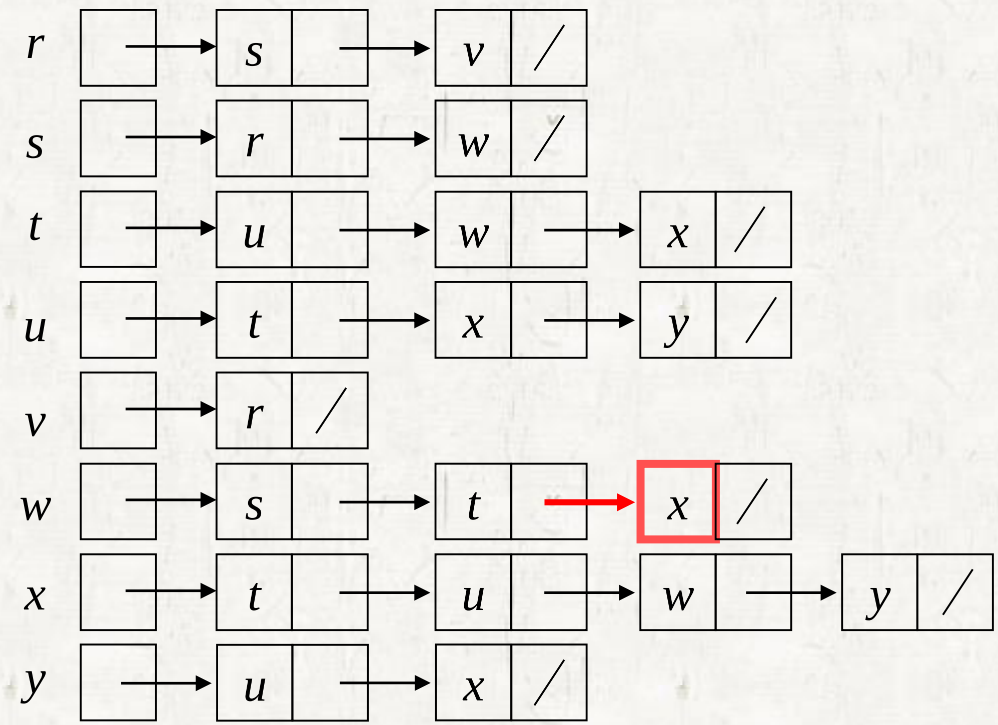
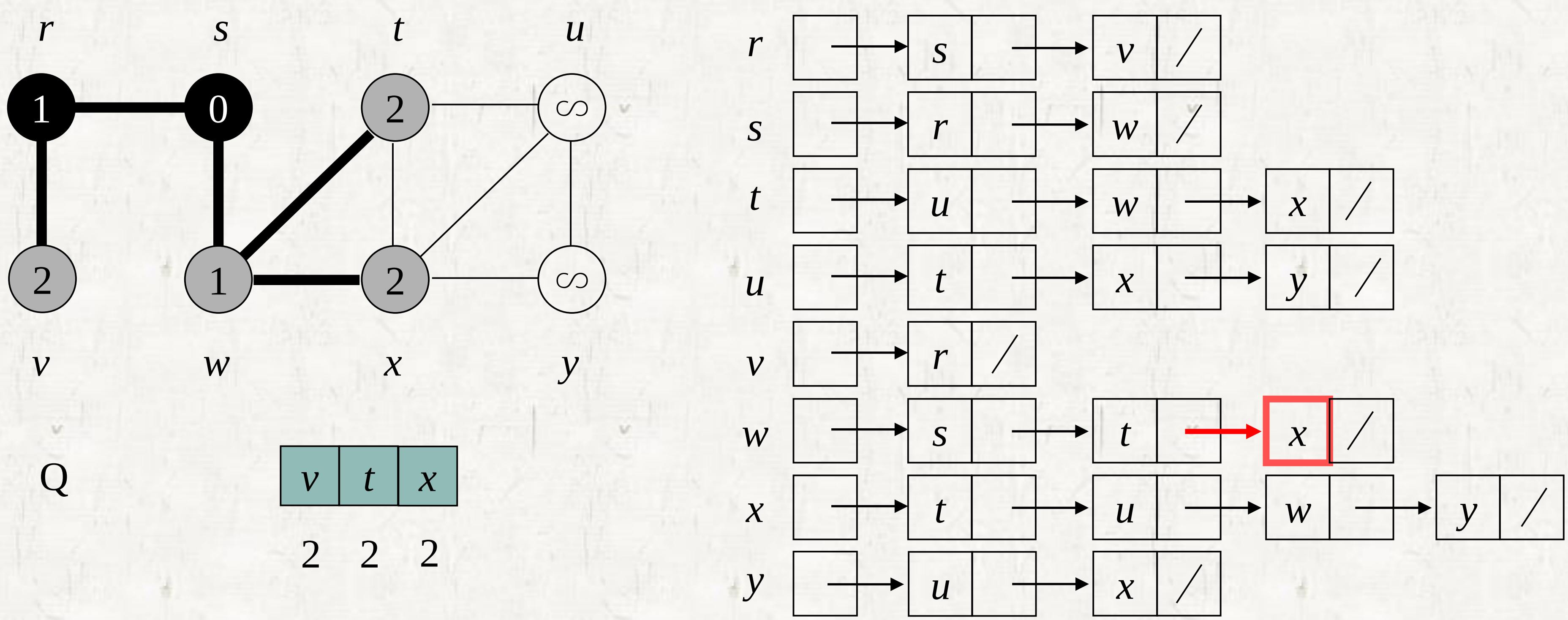




Q

v	t
2	2

Breadth-first search



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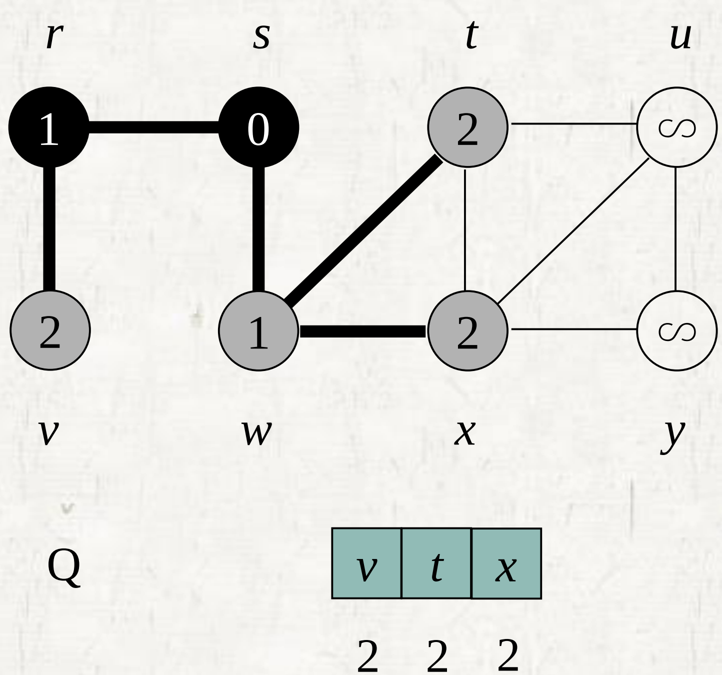

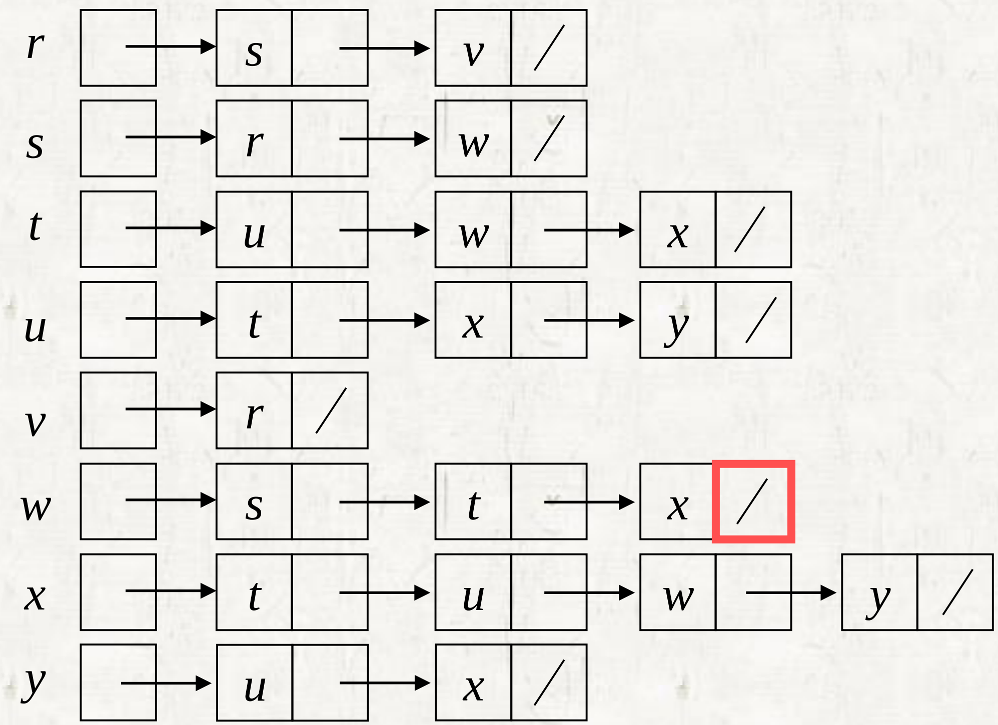

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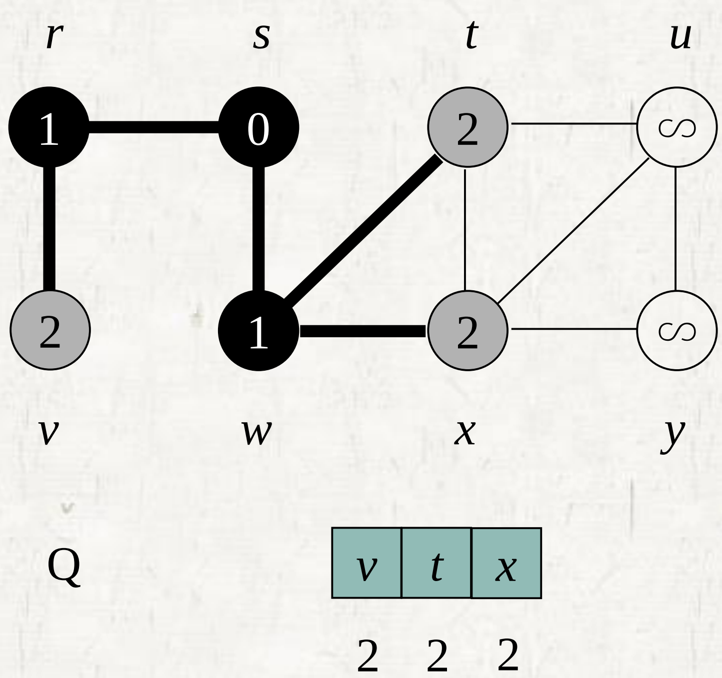

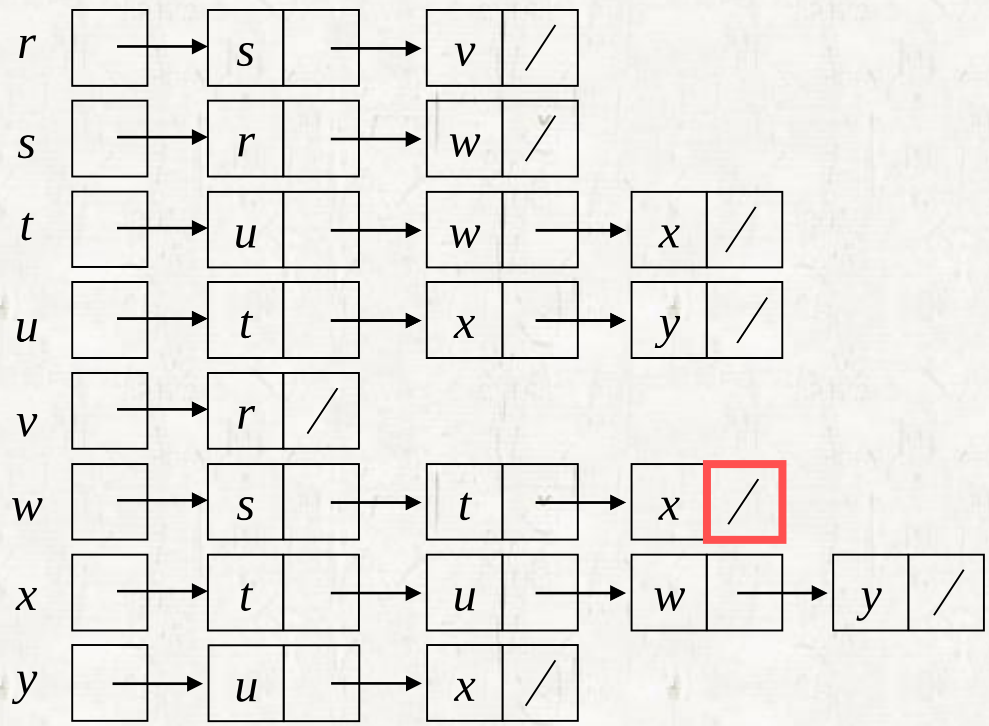

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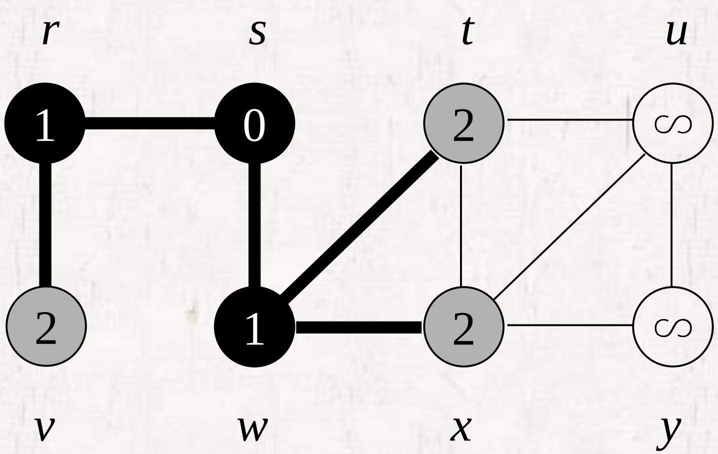
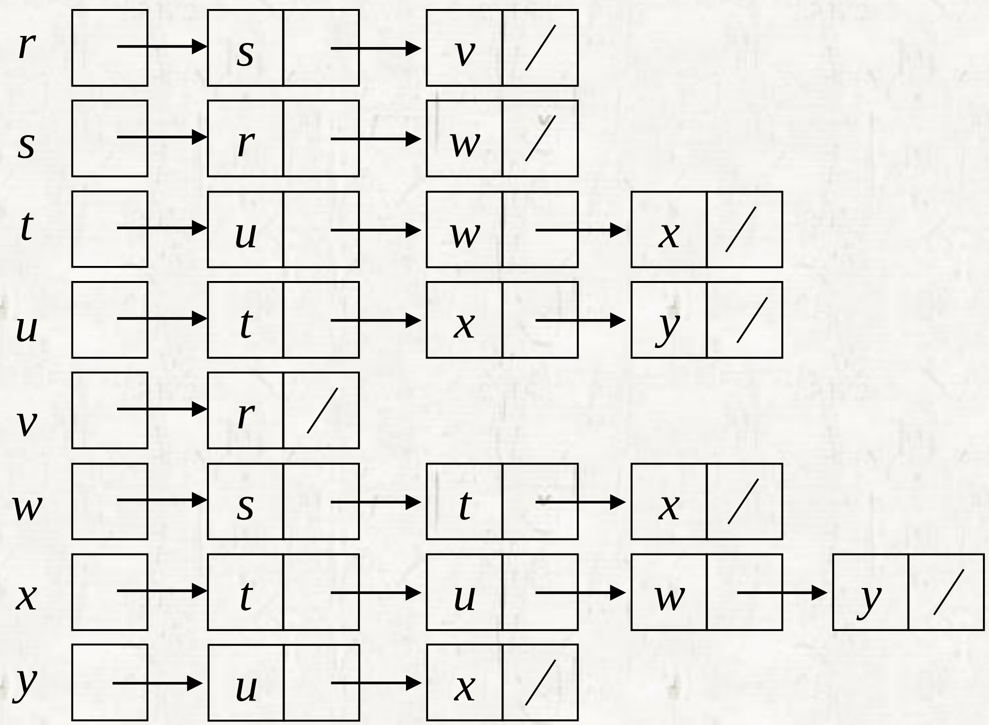


Q			
	v	*t*	*x*
	2	2	2



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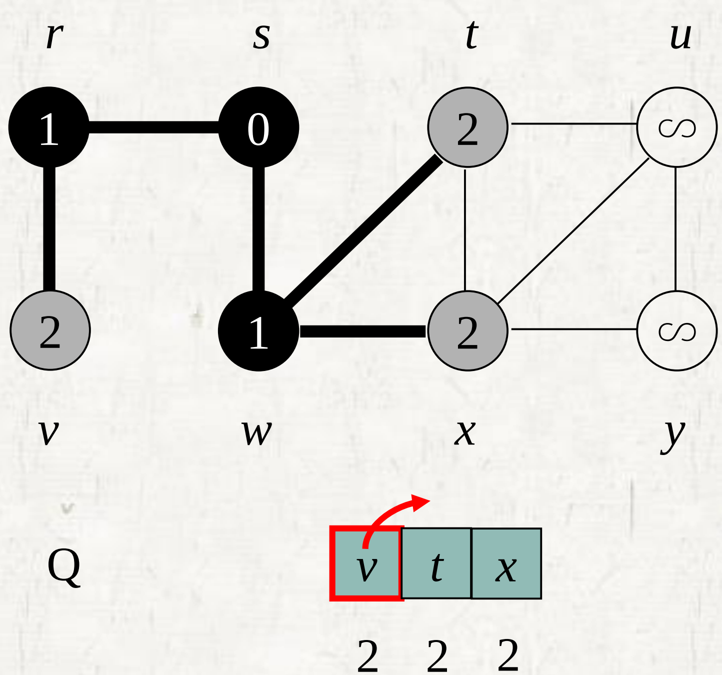

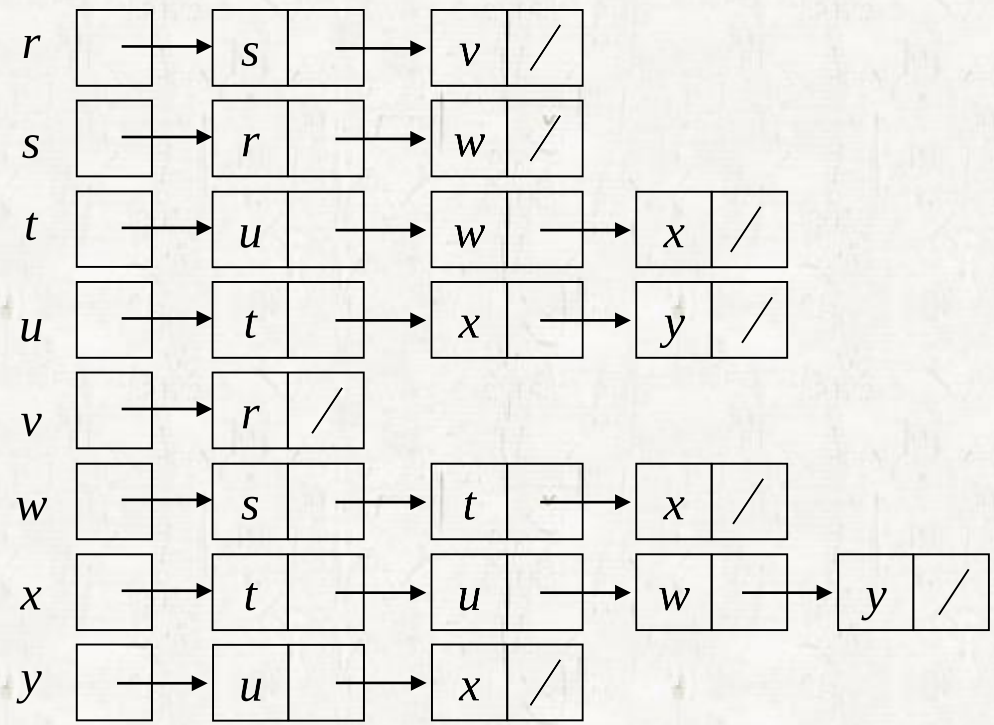

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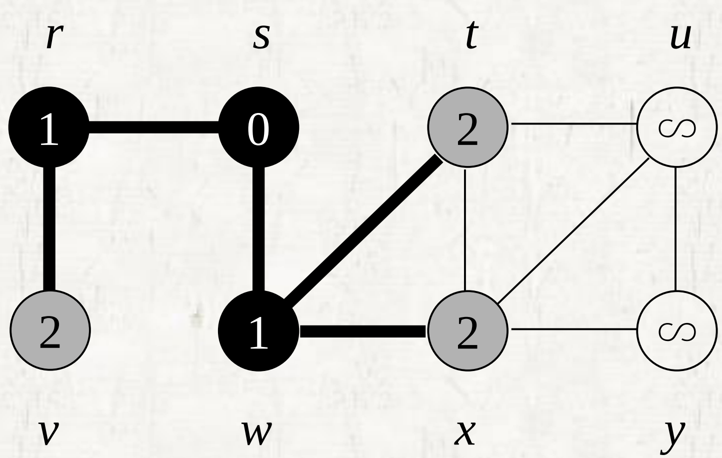
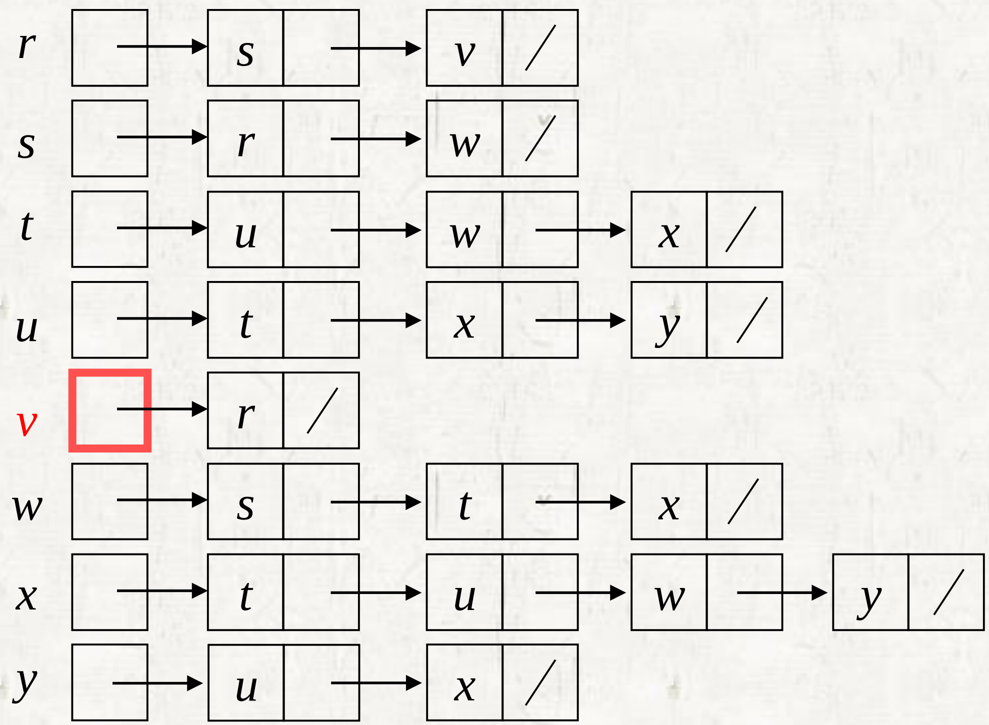


Q

t	x
2	2



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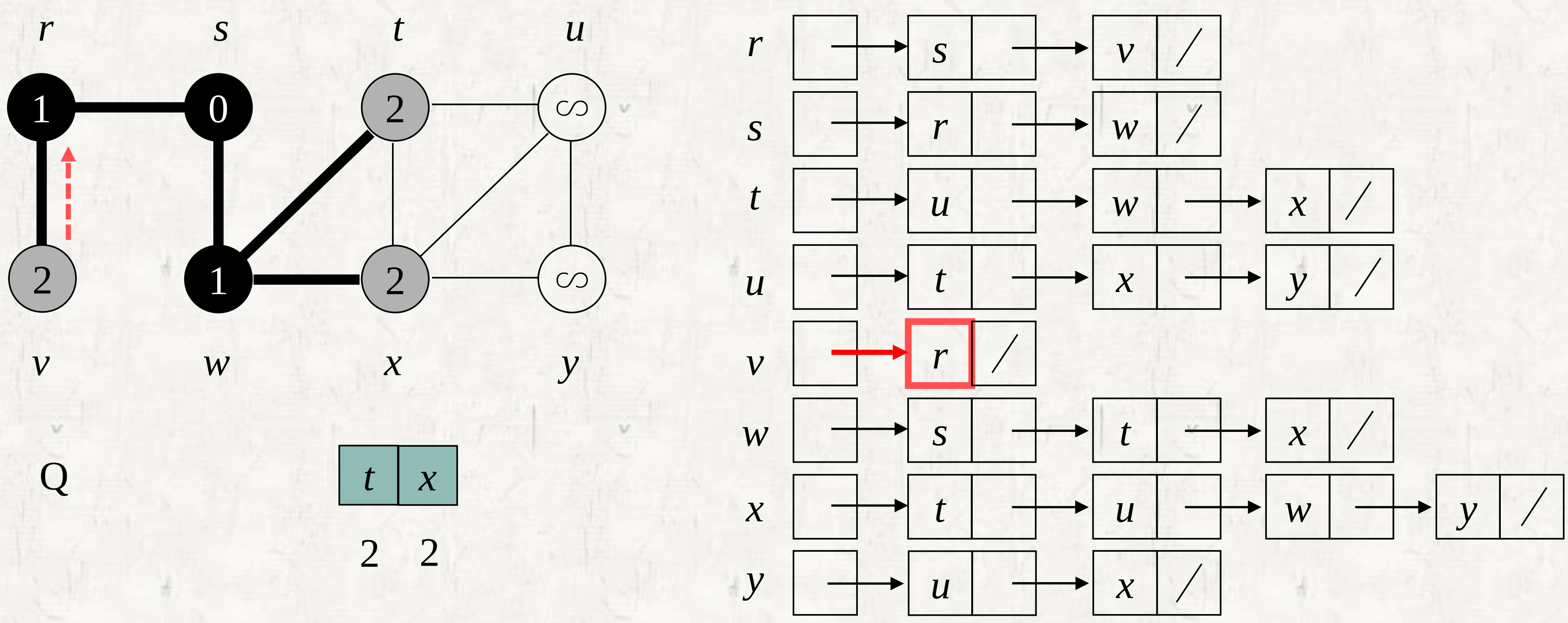

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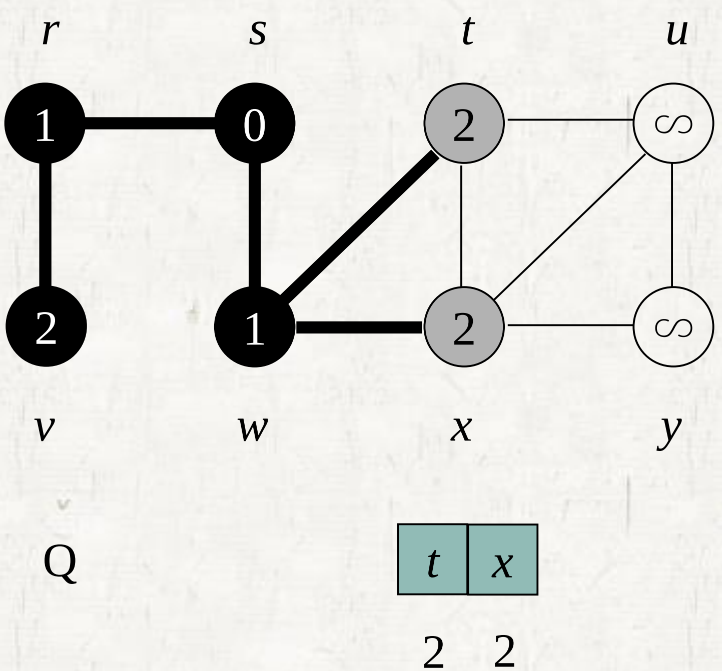

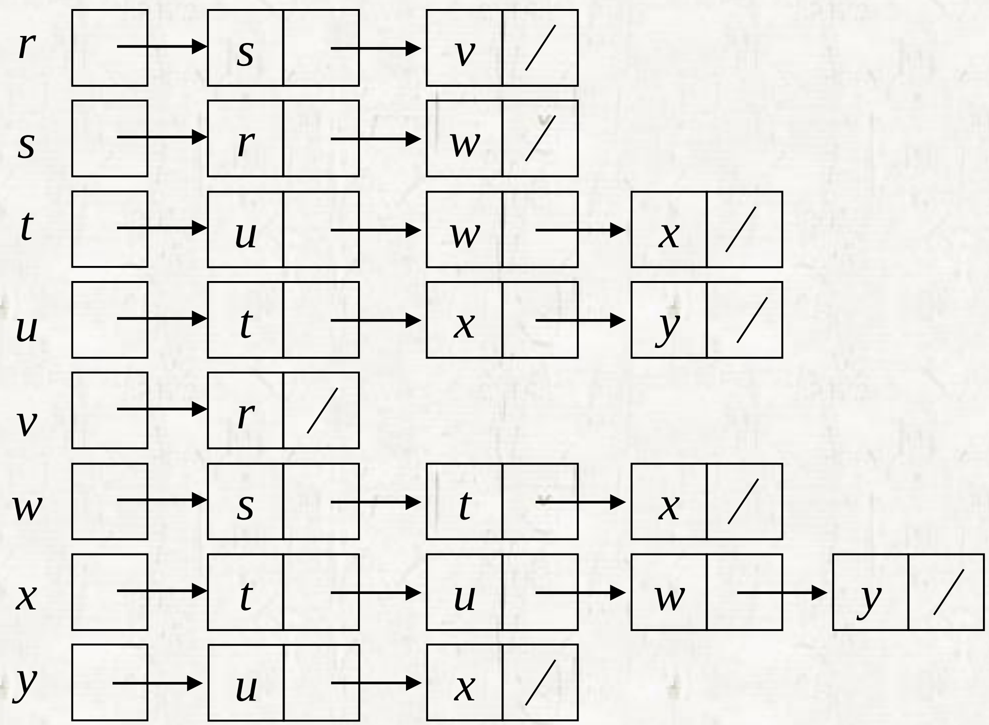

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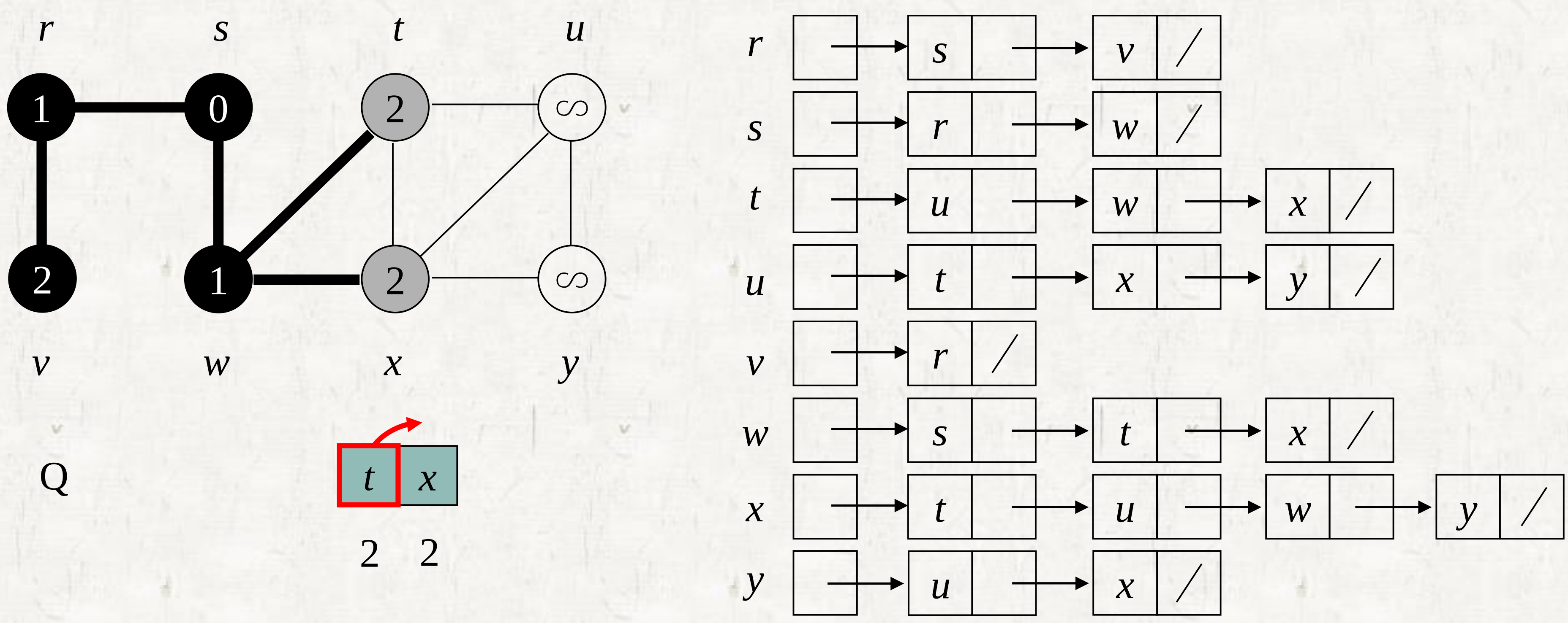

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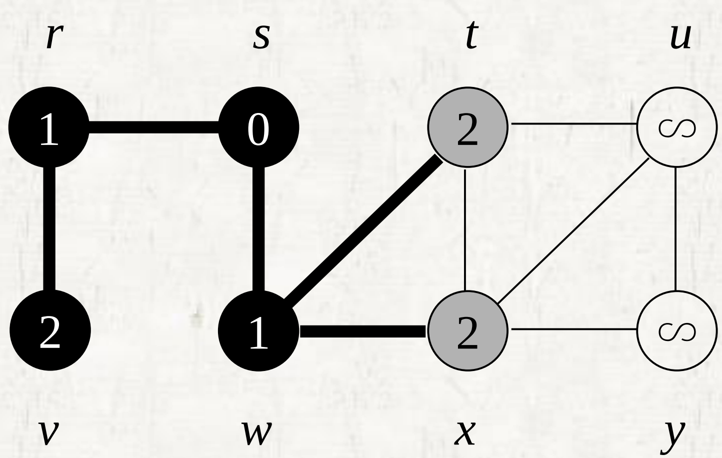
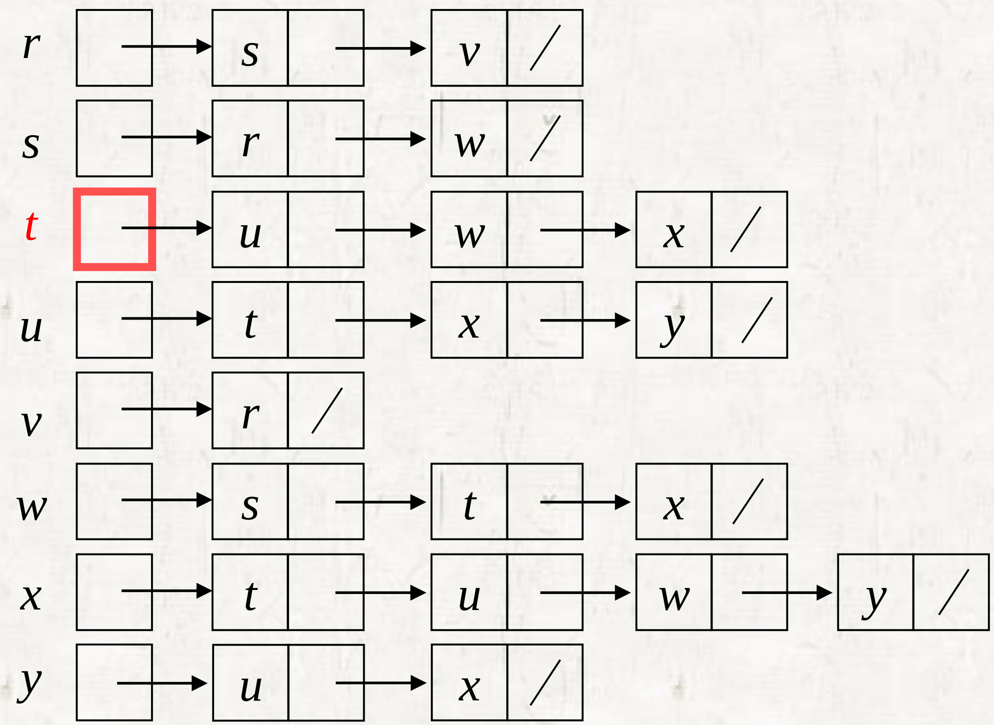


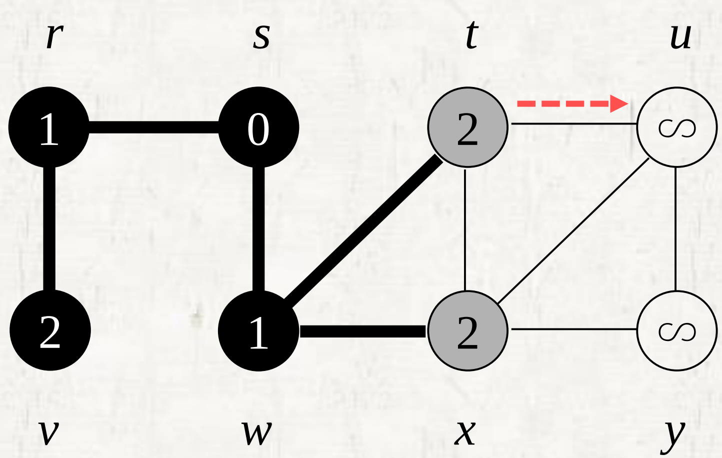


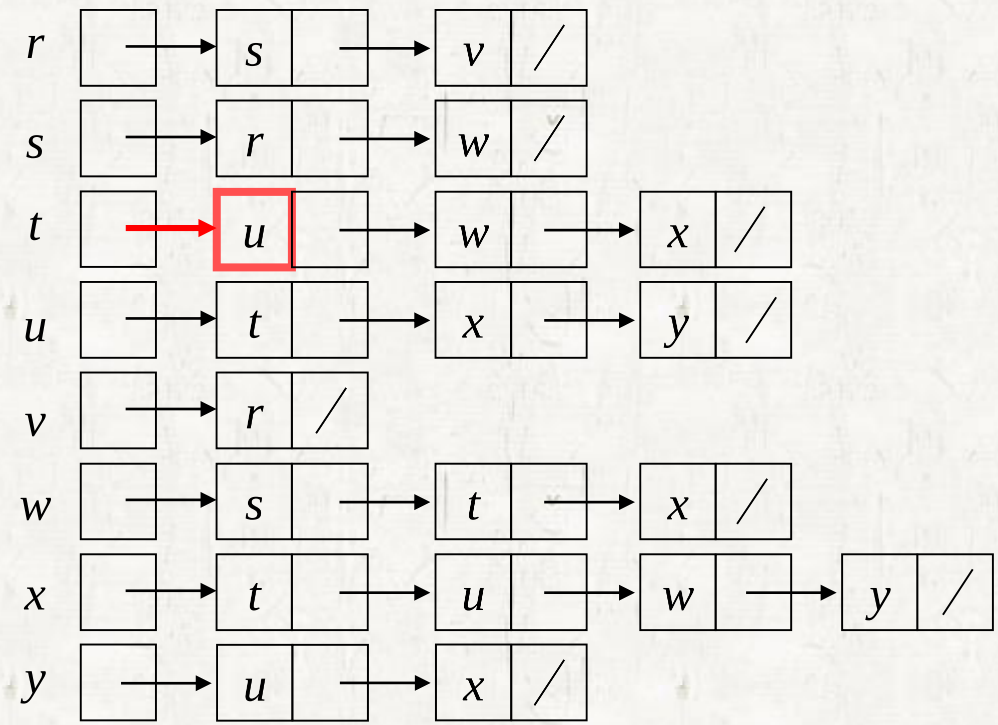
Q

x

2

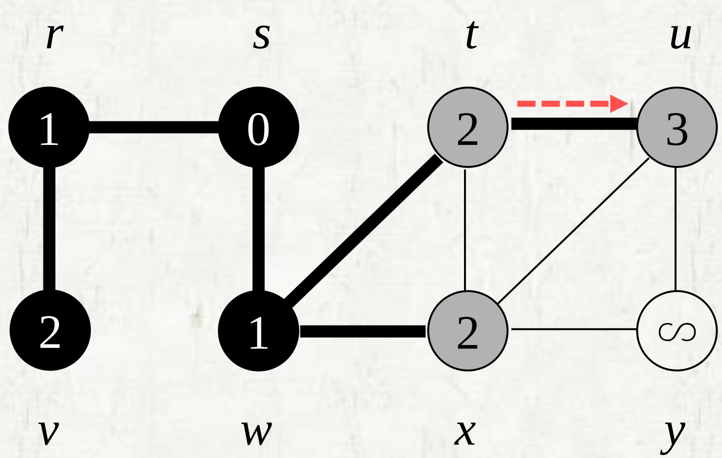
Breadth-first search

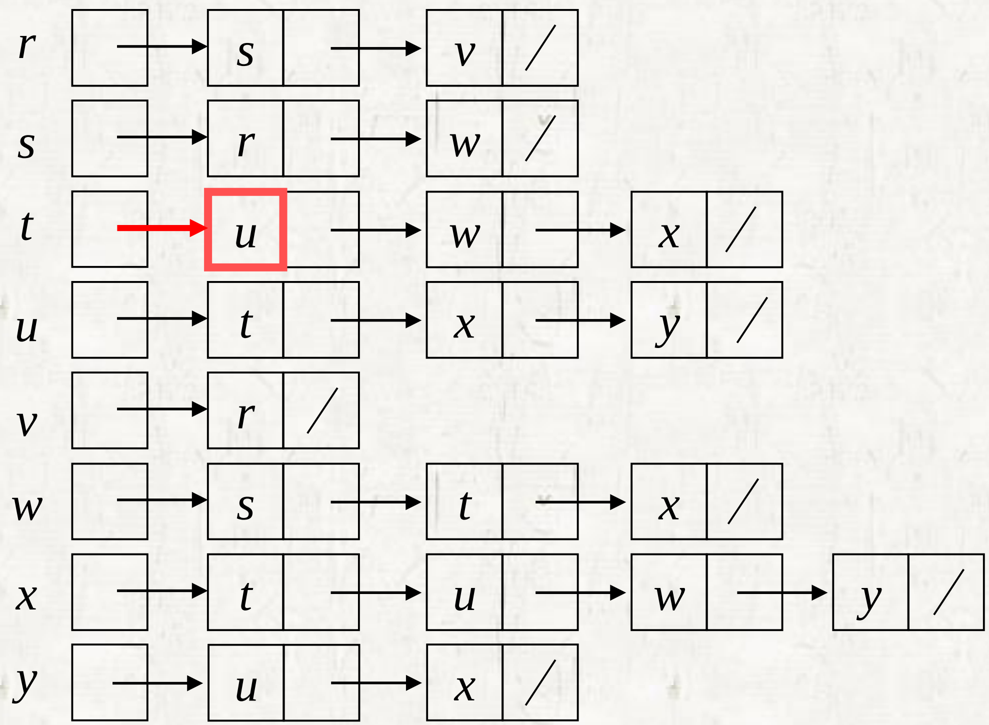




Q	x
	2

Breadth-first search





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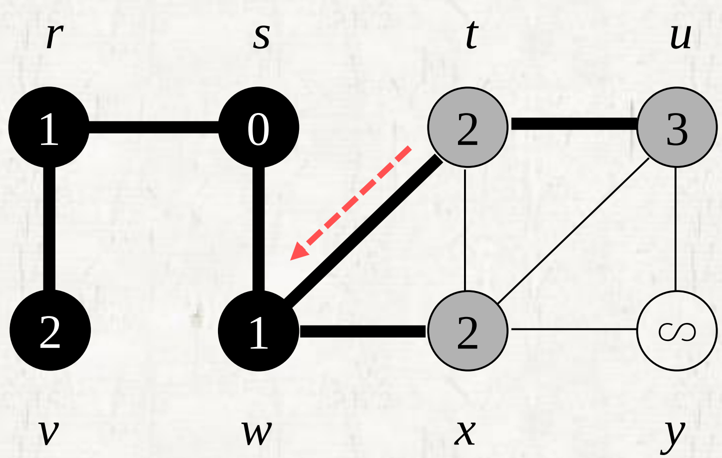
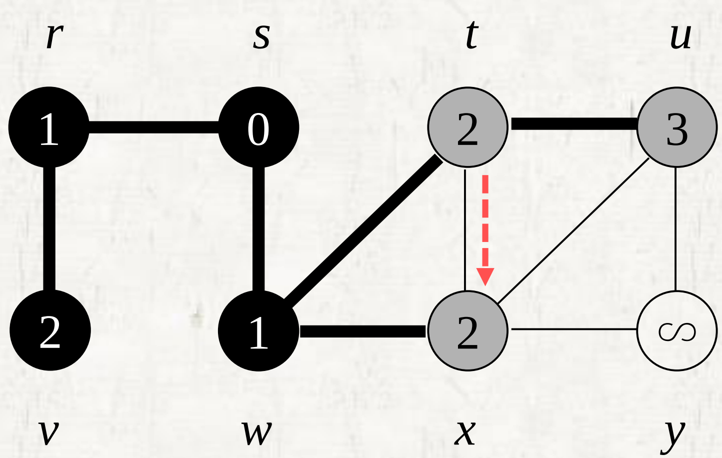


Q

x	*u*
2	3

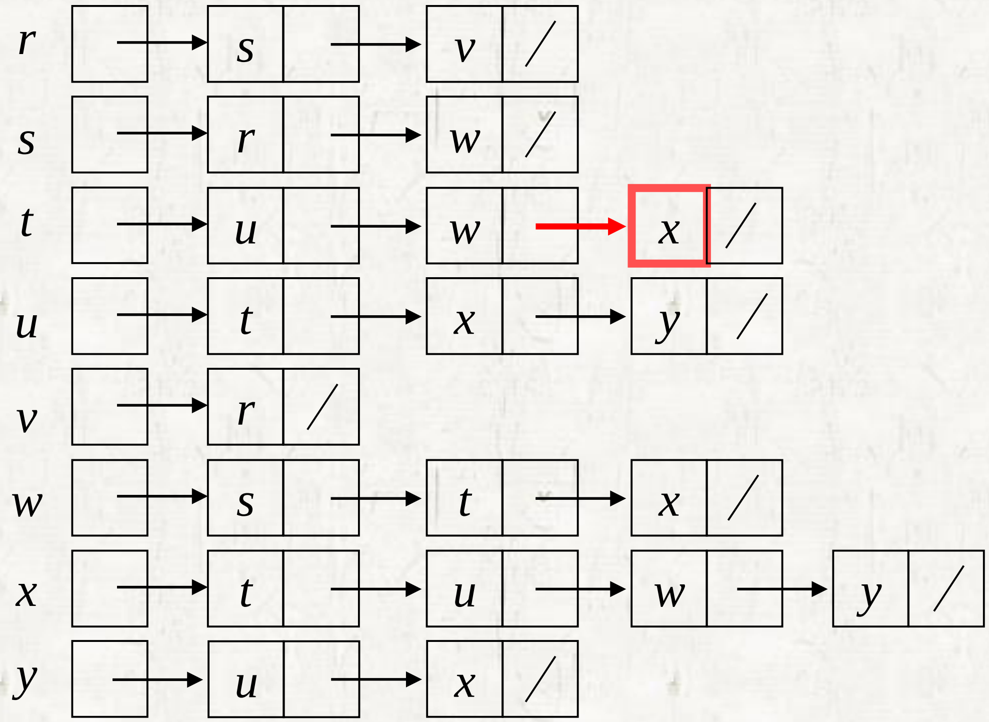
r	*s*	*v*		
s	*r*	*w*		
t	*u*	*w*	*x*	
u	*t*	*x*	*y*	
v	*r*			
w	*s*	*t*	*x*	
x	*t*	*u*	*w*	*y*
y	*u*	*x*		

Breadth-first search



Q

x	u
2	3



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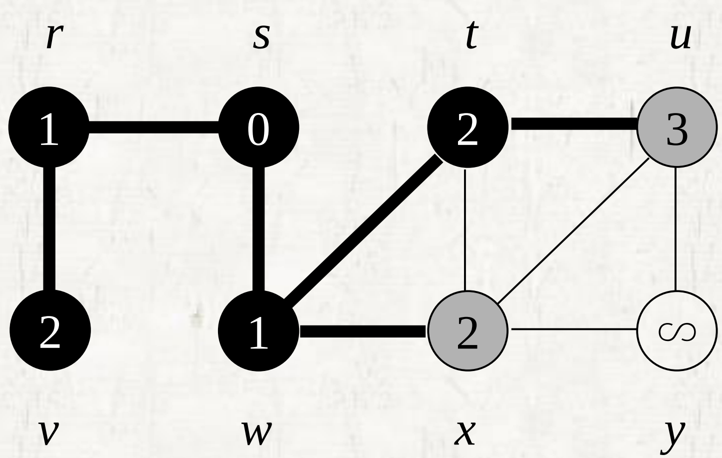
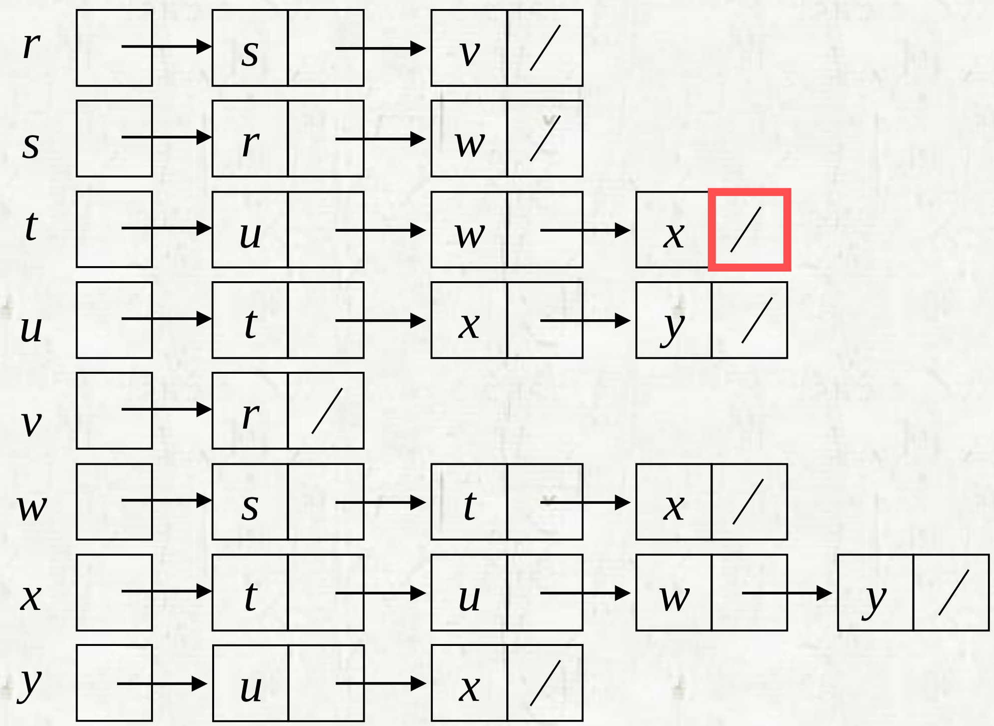
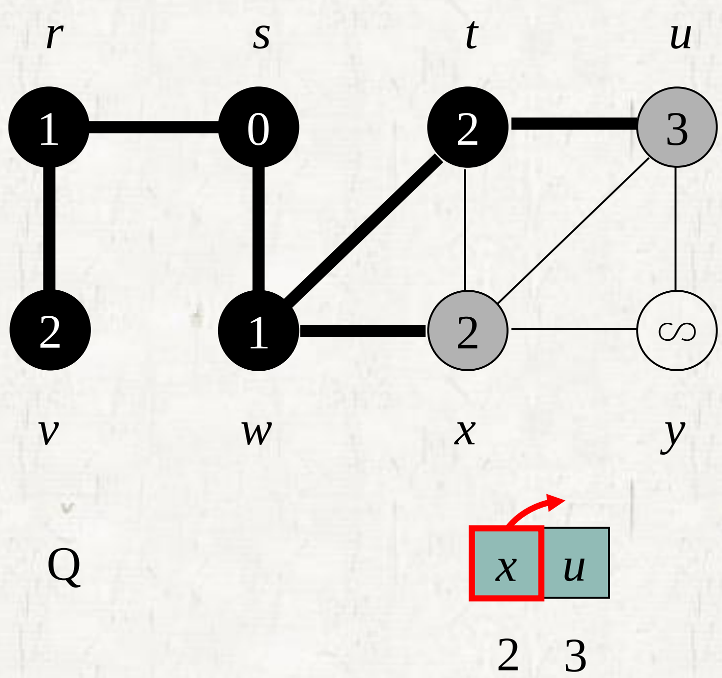


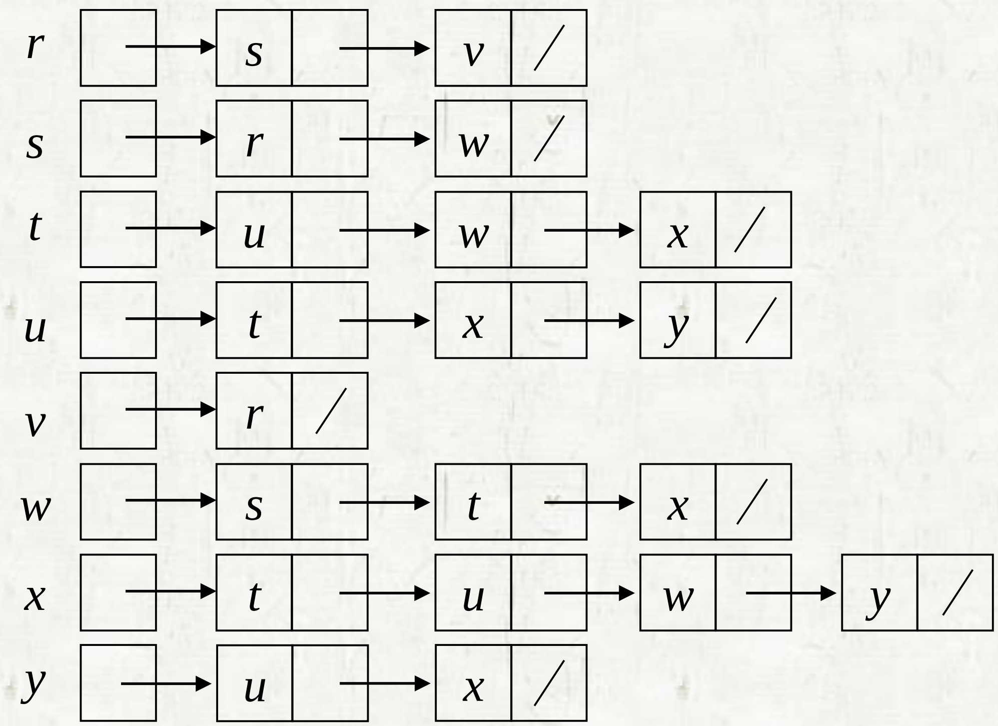


Q

x	u
2	3

Breadth-first search





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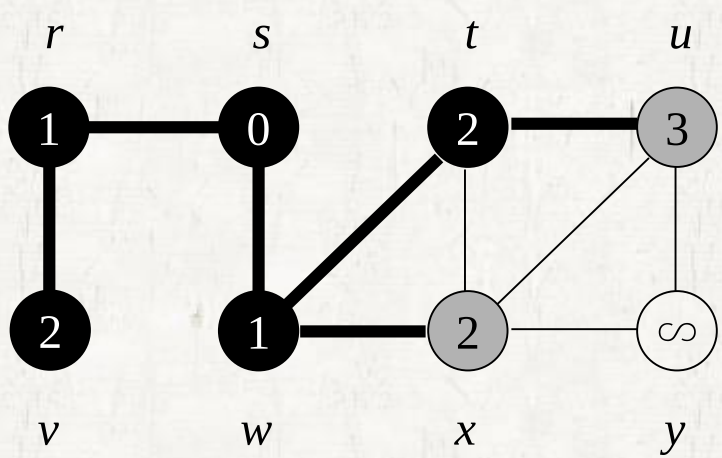
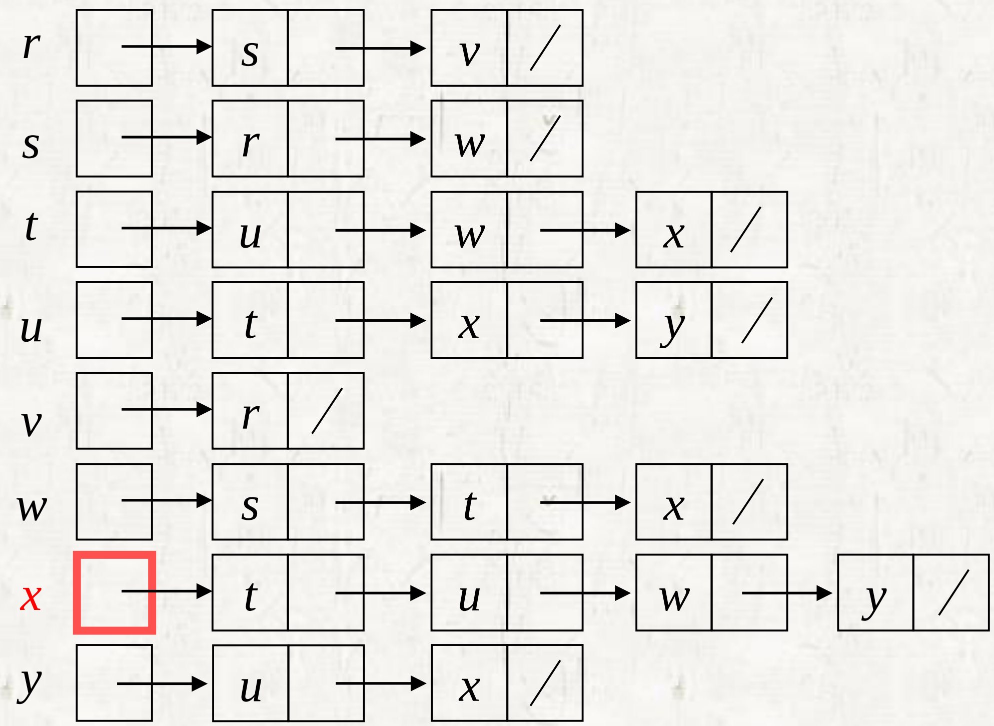


Q

u

3



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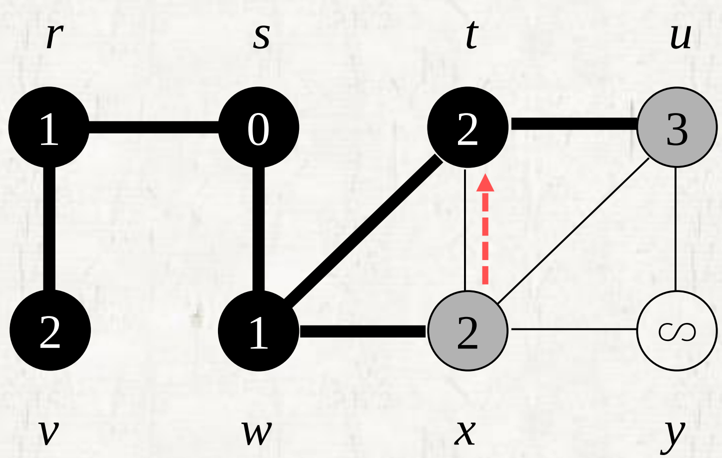
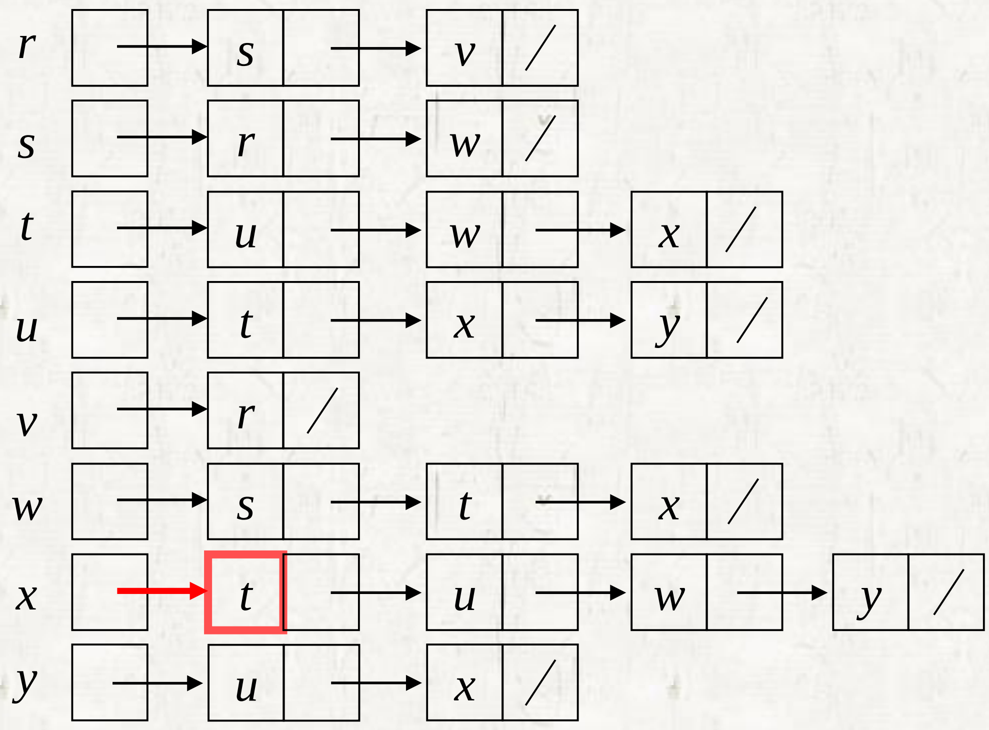


Q u

3



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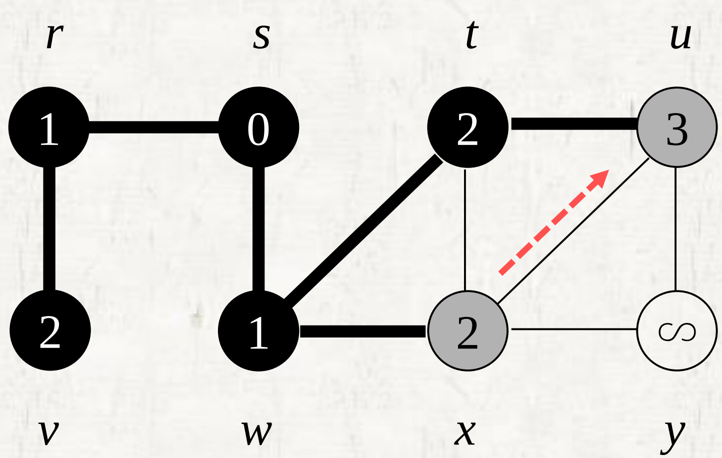
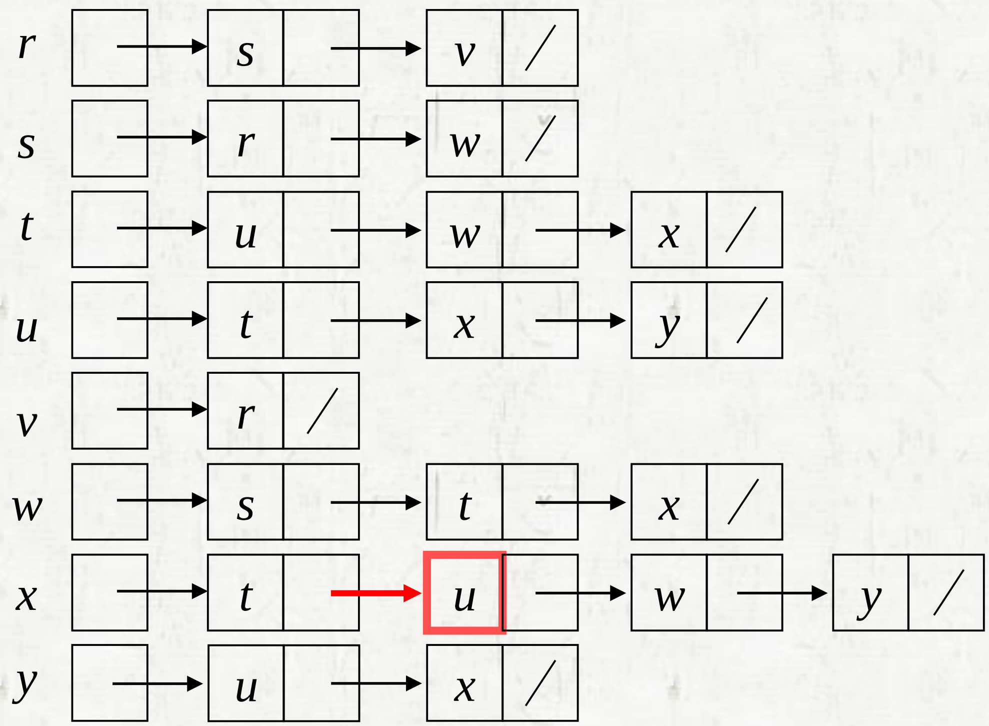


Q

u
3



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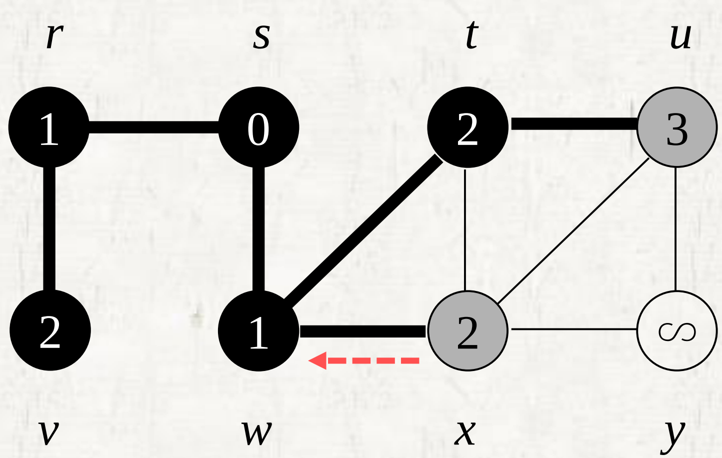
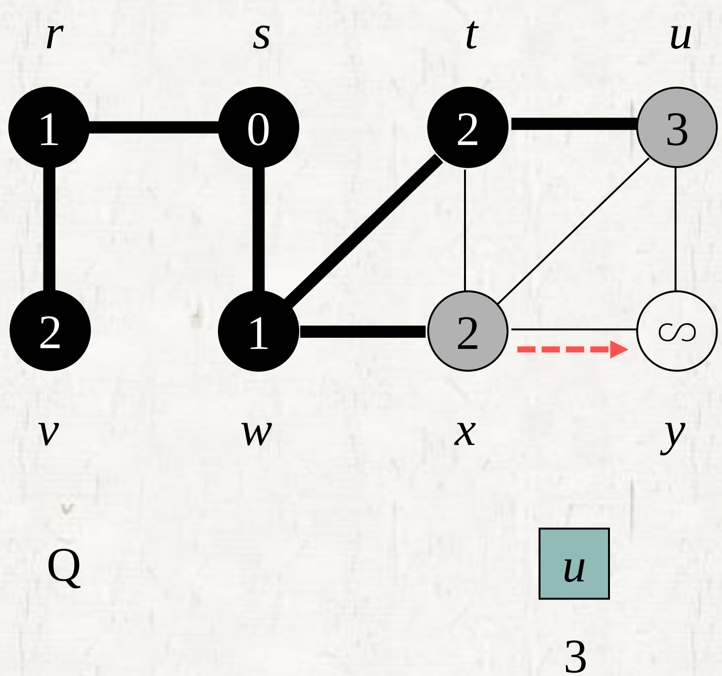


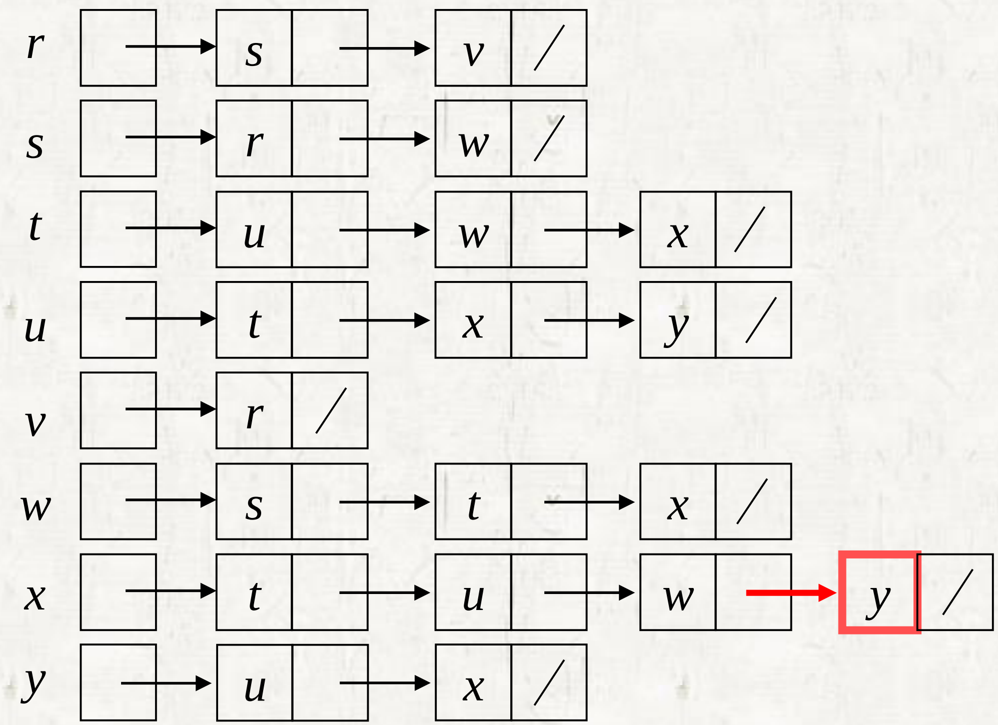
Q

u
3

r	s	v /		
s	r	w /		
t	u	w	x /	
u	t	x	y /	
v	r /			
w	s	t	x /	
x	t	u	w	y /
y	u	x /		

Breadth-first search





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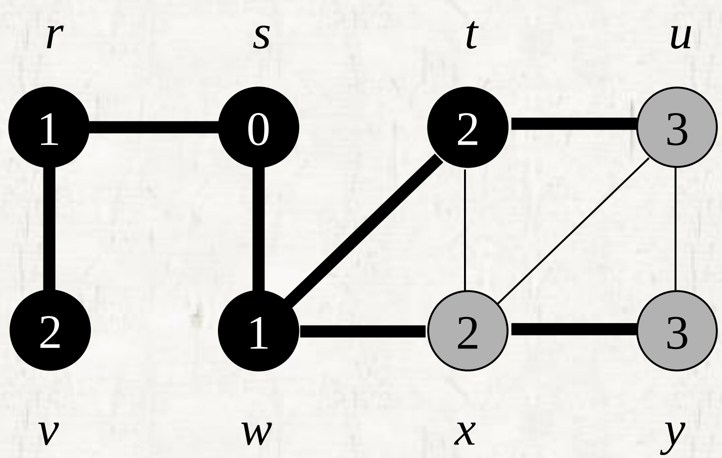
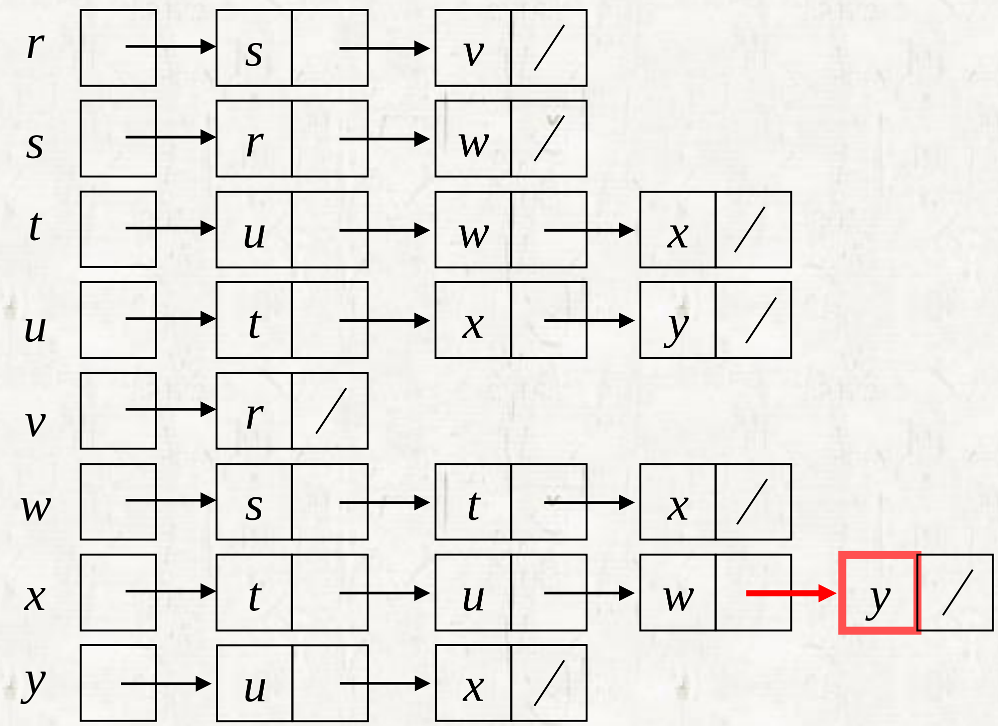


Q

u	*y*
3	3



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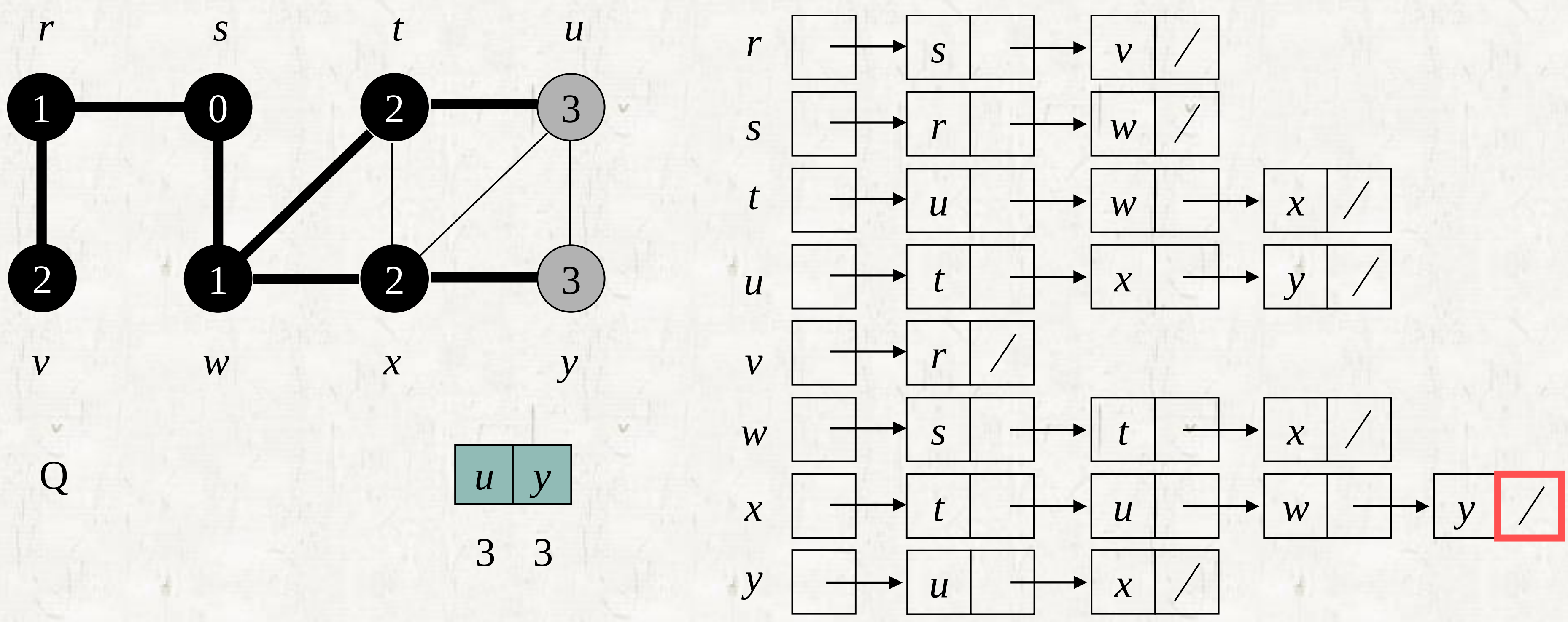

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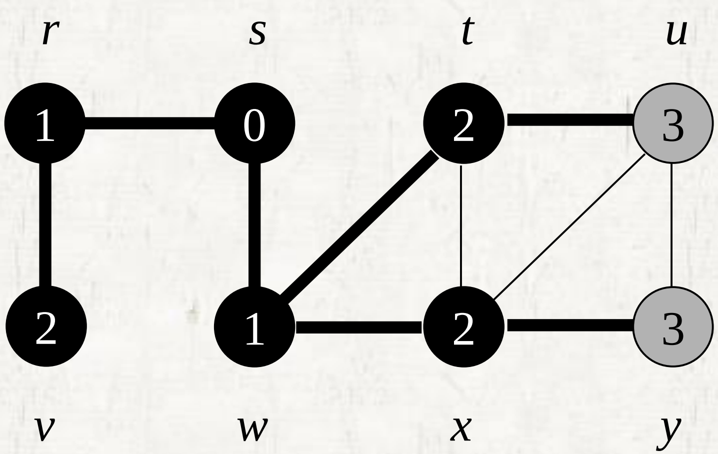
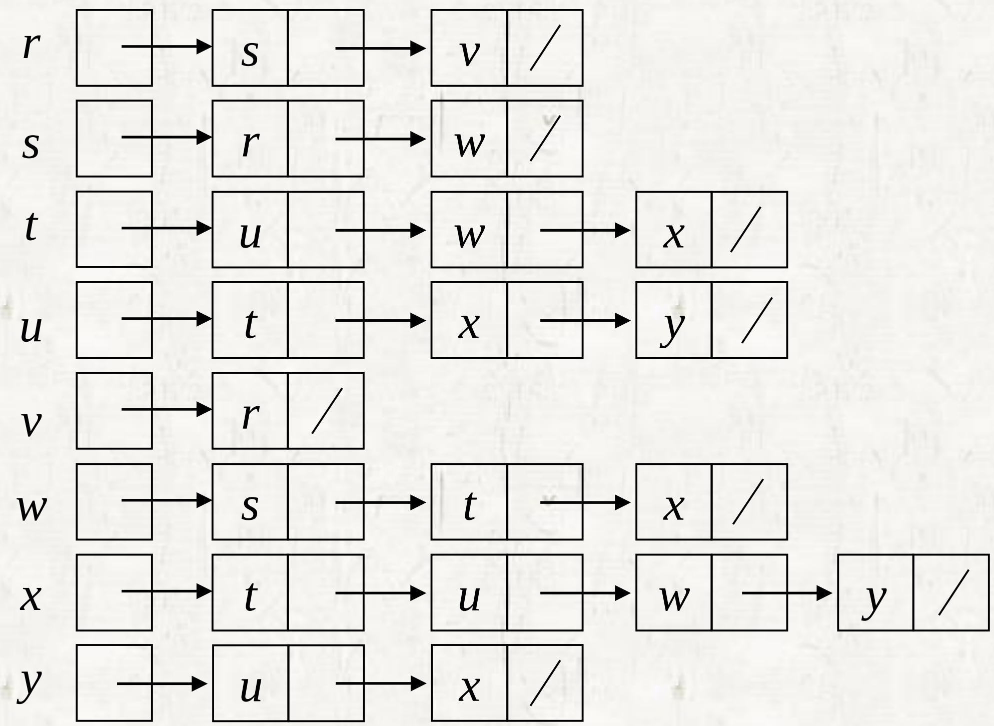
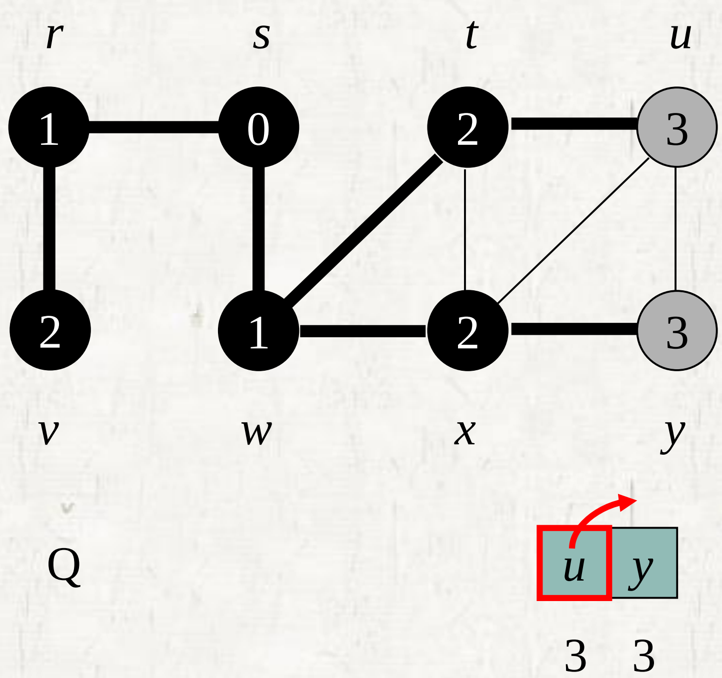


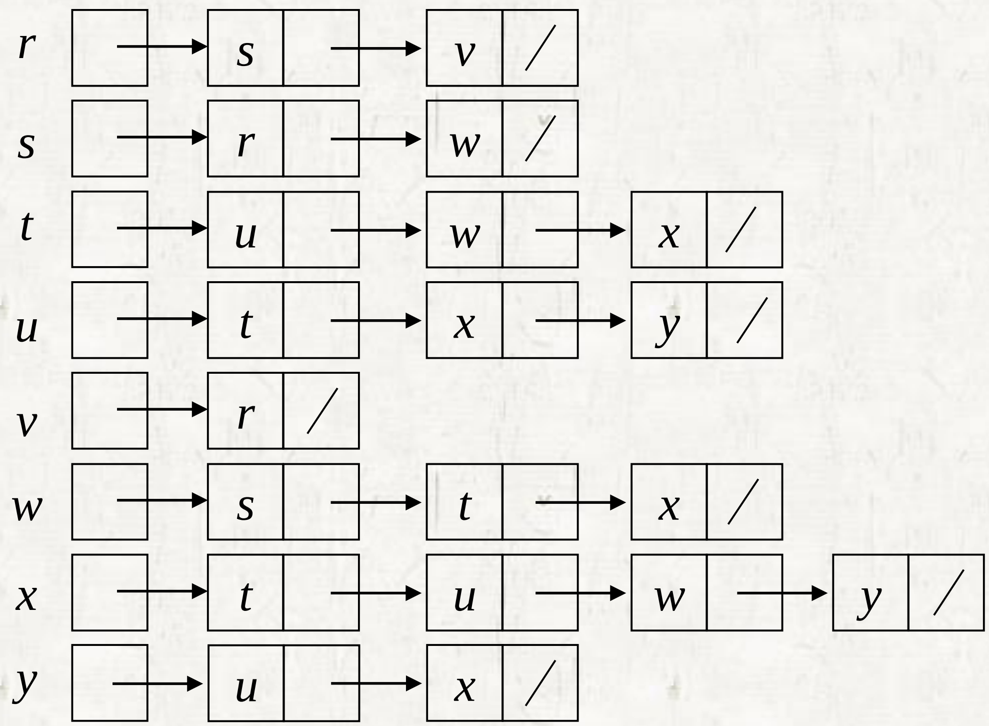


Q

u	*y*
3	3

Breadth-first search





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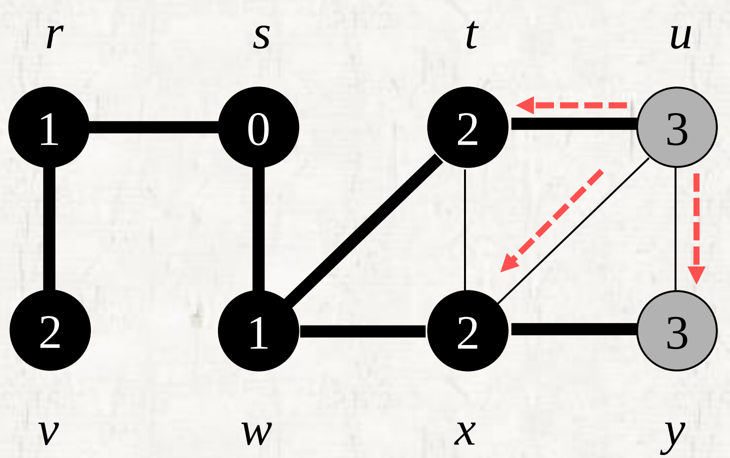
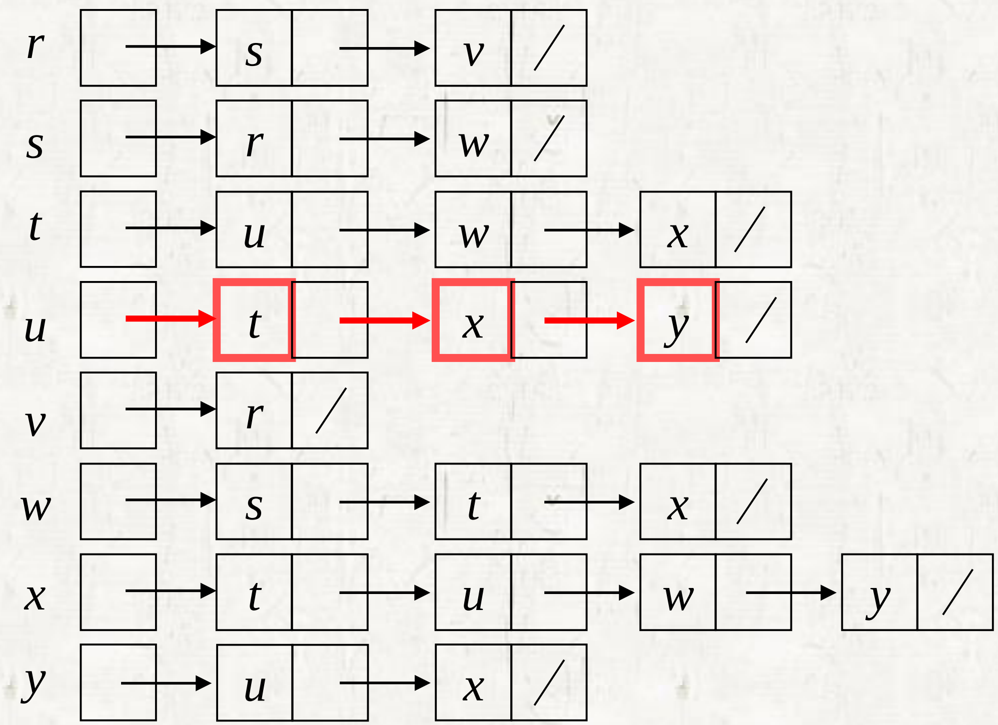


Q y

3



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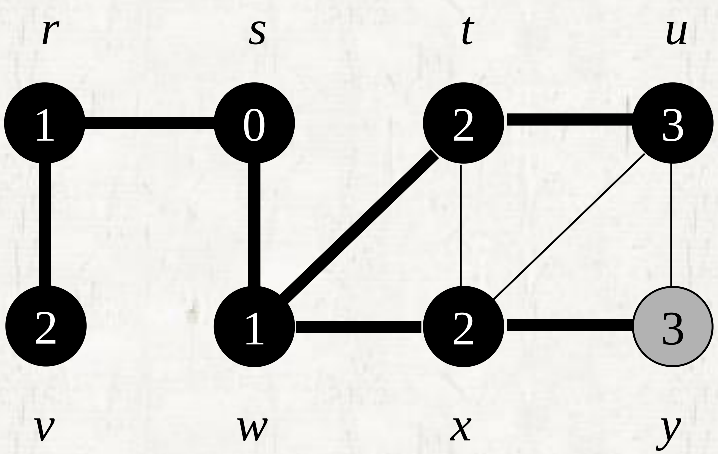
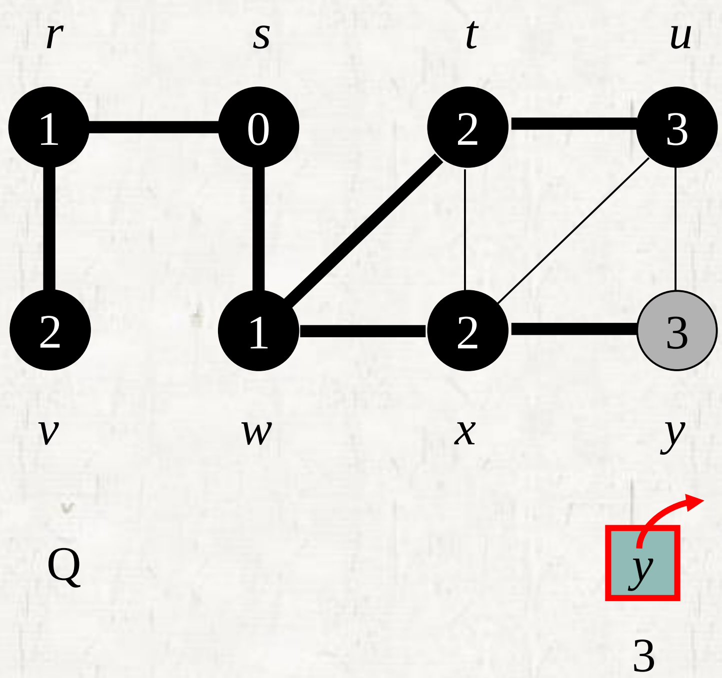


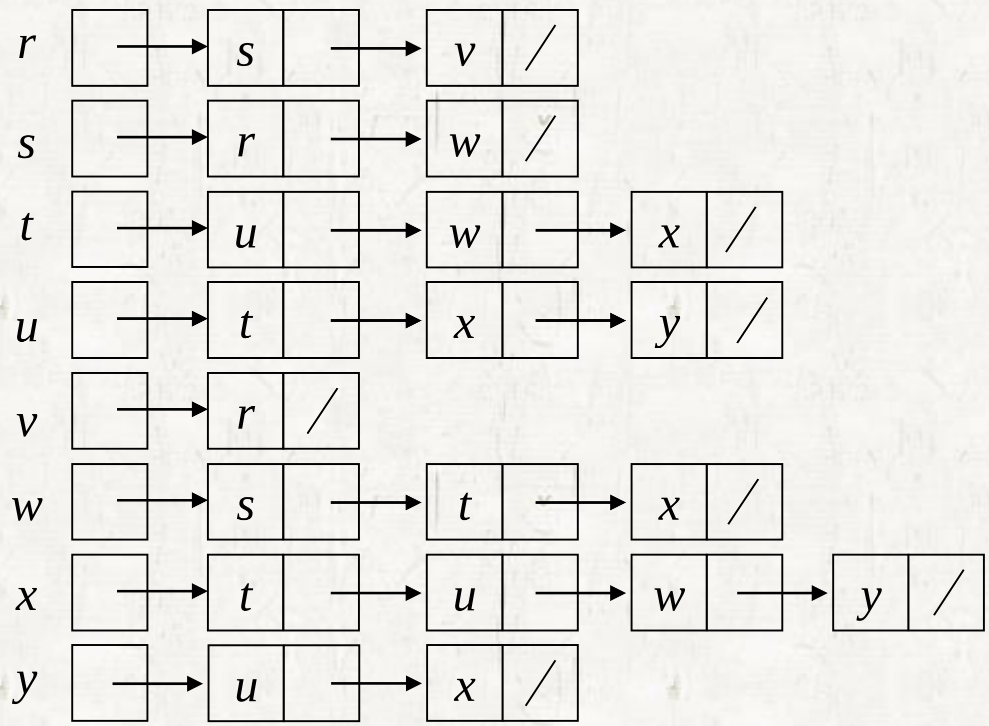
Q $\boxed{y}$

3

r		s	v /		
s		r	w /		
t		u	w	x /	
u		t	x	y /	
v		r /			
w		s	t	x /	
x		t	u	w	y /
y		u	x /		

Breadth-first search





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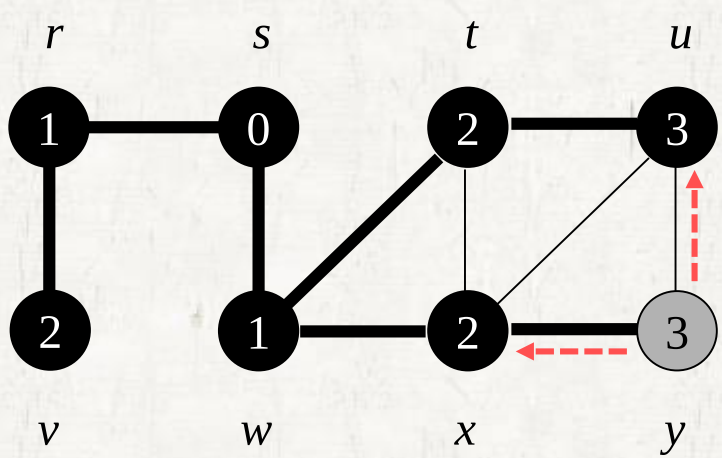

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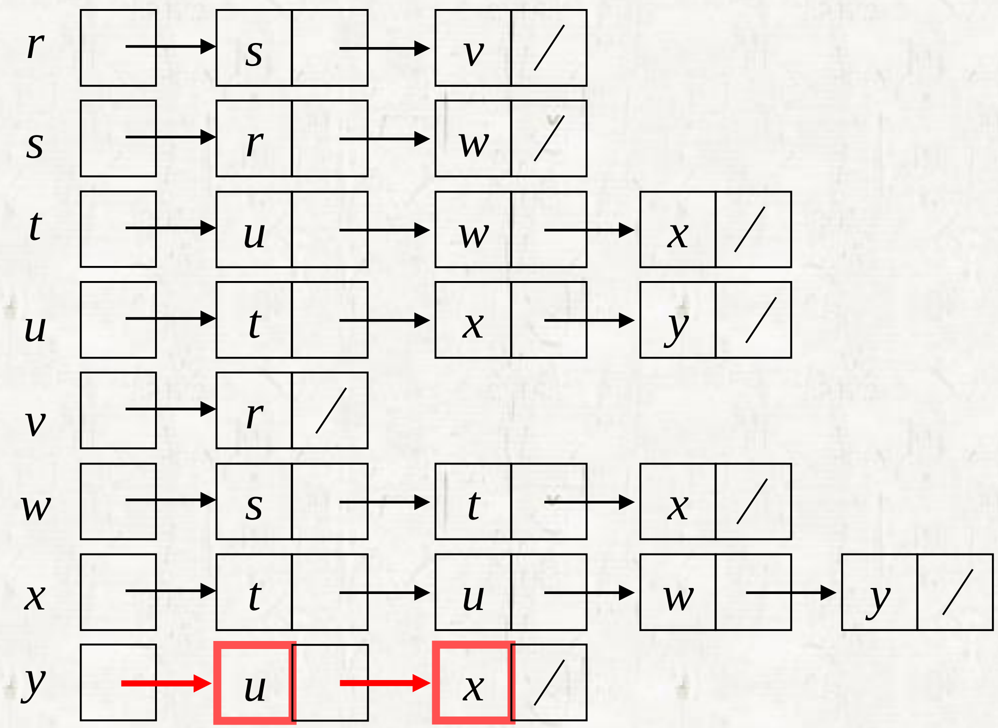

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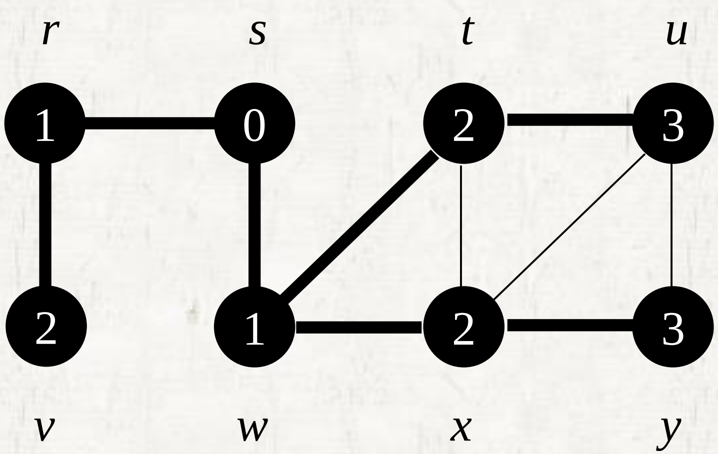
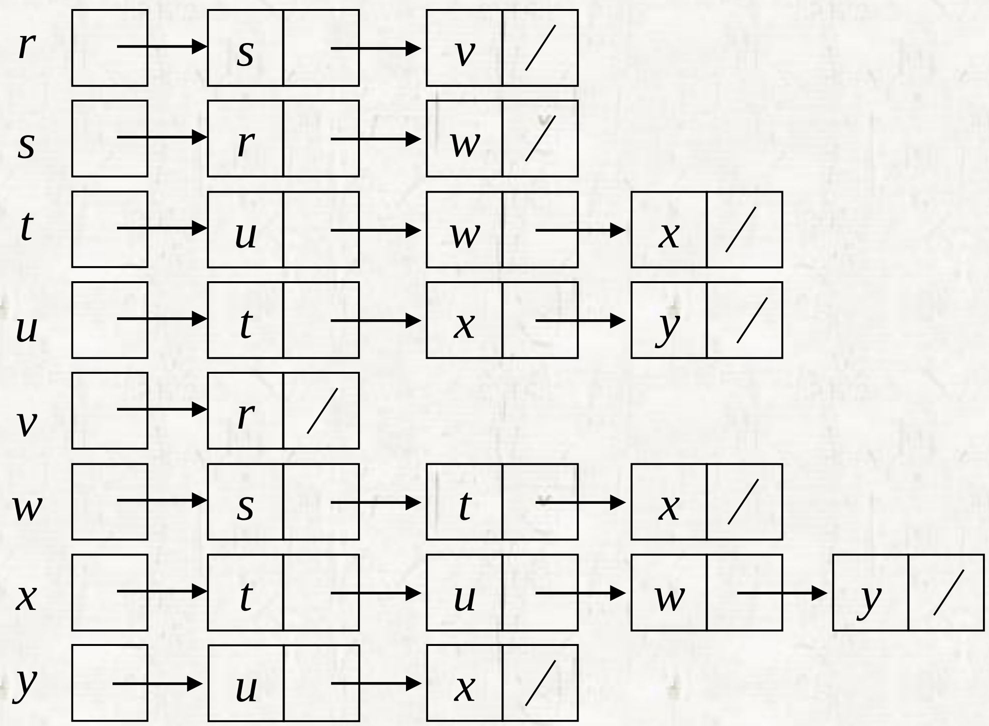


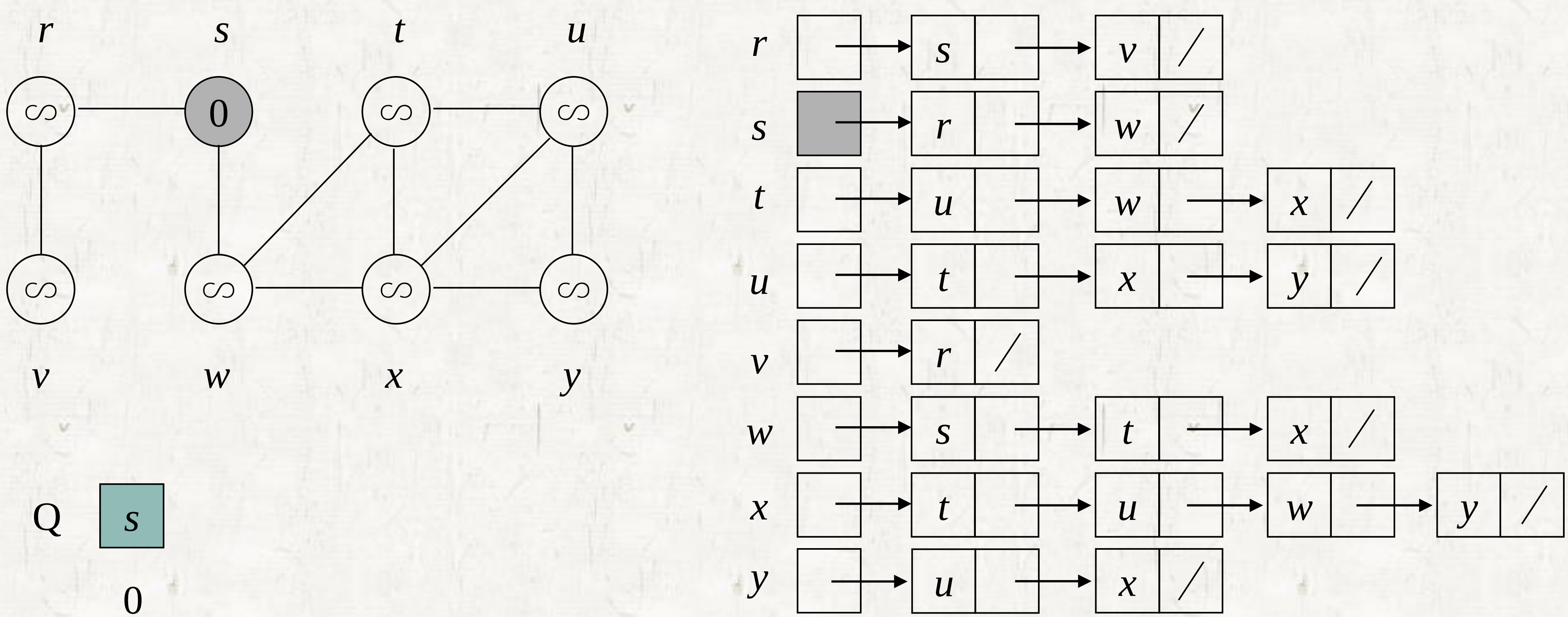


Q

Breadth-first search

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d  = ∞
        u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
      u = DEQUEUE(Q)
      for each v ∈ G.Adj[u]
            if  v.color == WHITE
                  v.color = GRAY
                  v.d = u.d + 1
                  v.π = u
                  ENQUEUE(Q, v)
      u.color = BLACK
```

Breadth-first search



Breadth-first search

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
      u = DEQUEUE(Q)
      for each v ∈ G.Adj[u]
            if v.color == WHITE
                  v.color = GRAY
                  v.d = u.d + 1
                  v.π = u
                  ENQUEUE(Q, v)
      u.color = BLACK
```

Initialization
$\Theta(V)$

Graph Exploration
$O(V + E)$

Breadth-first search

- **Running time**
  - **Initialization**: $\Theta(V)$
  - **Exploring the graph**: $O(V + E)$
    - A vertex is examined at most once.
    - An edge is explored at most twice.
  - **Overall:** $O(V + E)$

Self-study

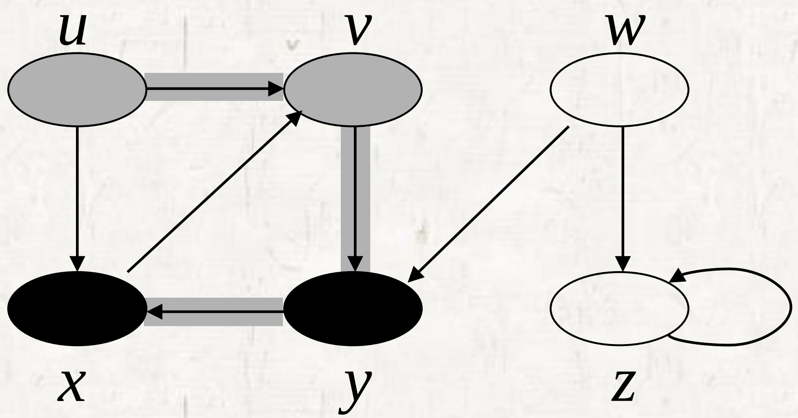
- **Exercise 22.2-4 (22.2-3 in the 2^{nd} ed.)**
  - The running time of BFS
    with adjacency matrix representation.
- **Exercise 22.2-6 (22.2-5 in the 2^{nd} ed.)**
  - Impossible breadth-first trees.
- **Exercise 22.2-7 (22.2-6 in the 2^{nd} ed.)**
  - Rivalry

Contents

- ***Graphs***
  - *Graphs basics*
  - *Graph representation*
- ***Searching a graph***
  - *Breadth-first search*
  - Depth-first search
- **Applications of depth-first search**
  - Topological sort

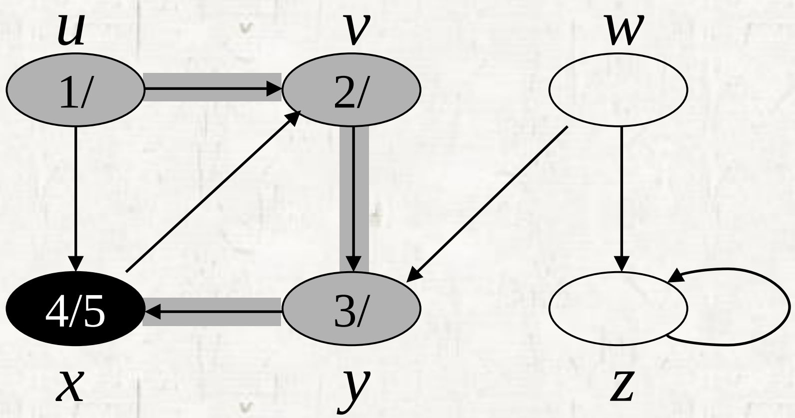
Depth-first search

- ***Colors of vertices***
  - Each vertex is initially ***white***. (not discovered)
  - The vertex is ***grayed*** when it is ***discovered.***
  - The vertex is ***blackened*** when it is ***finished***, that is, when its adjacency list has been examined completely.

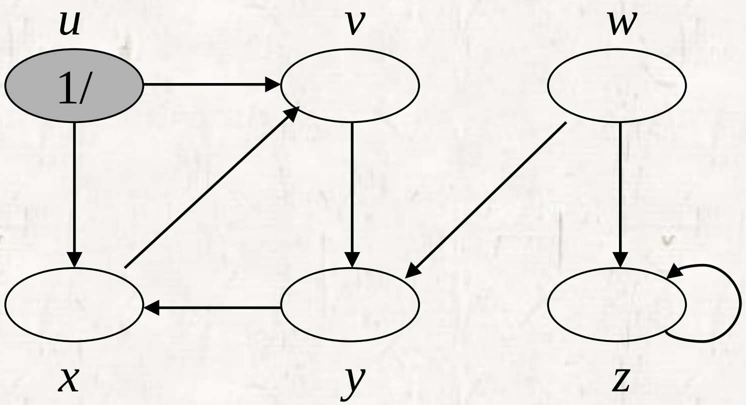


Depth-first search

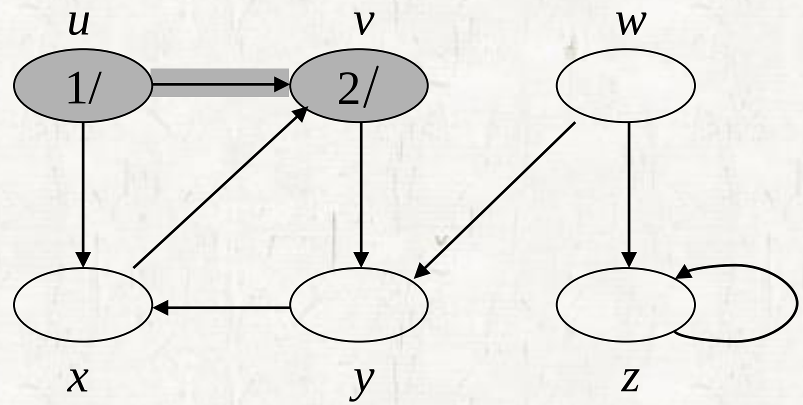
- ***Timestamps***
  - Each vertex *v* has two timestamps.
    - *v*.*d*: ***discovery time*** (when *v* is grayed)
    - *v*.*f*: ***finishing time*** (when *v* is blacken)



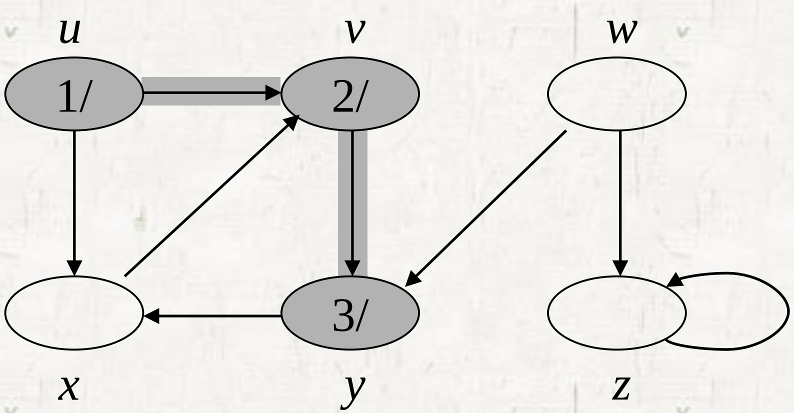
Depth-first search



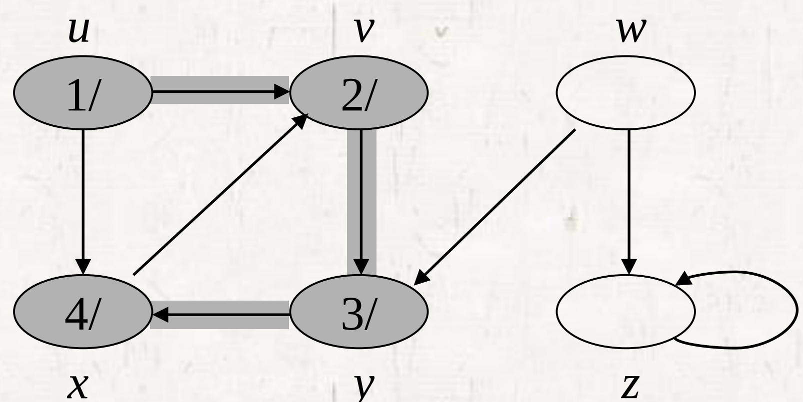
(a)



(b)

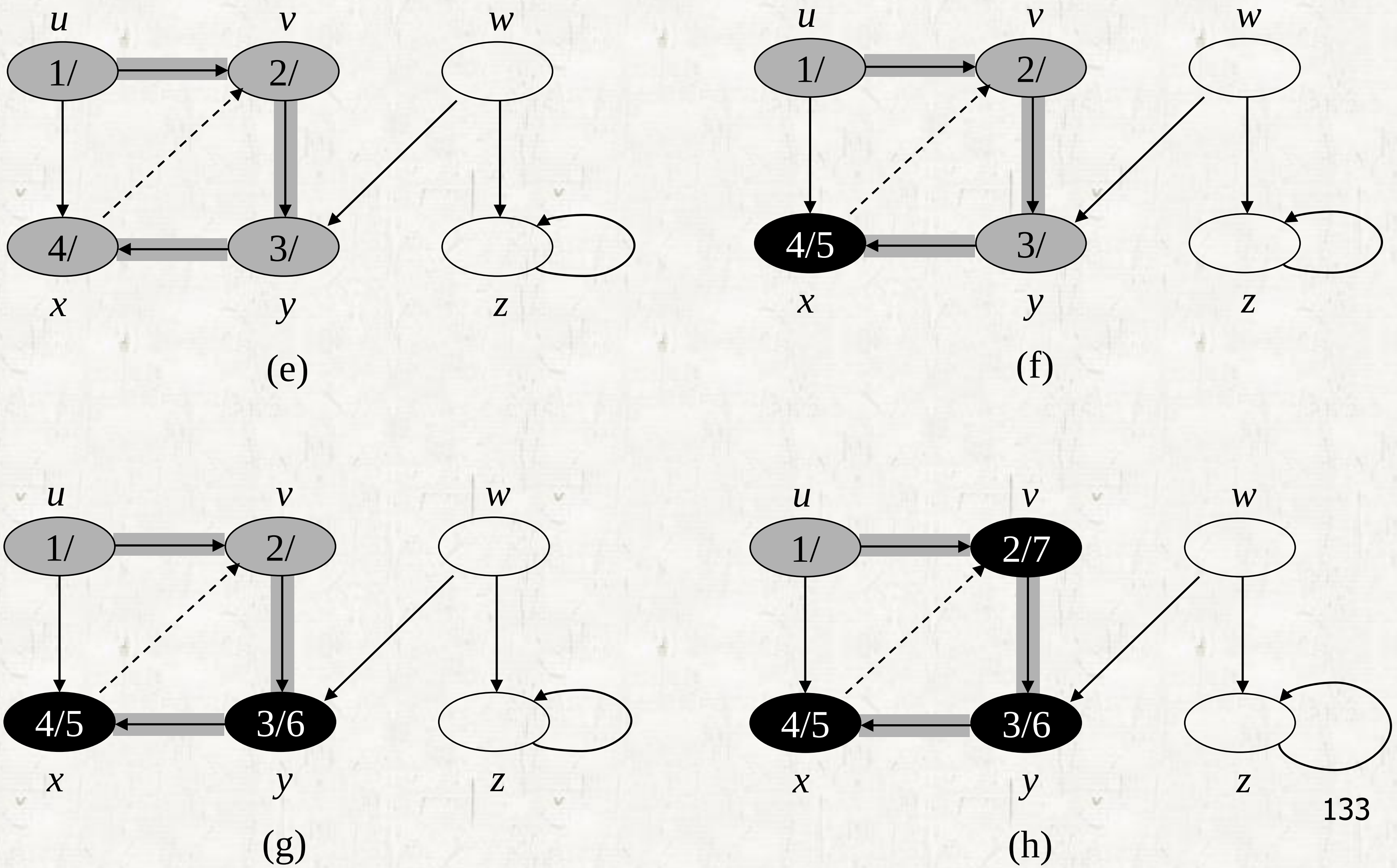


(c)

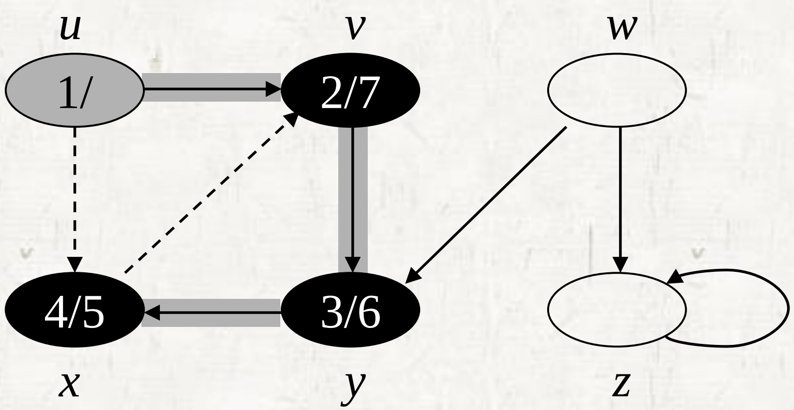


(d)

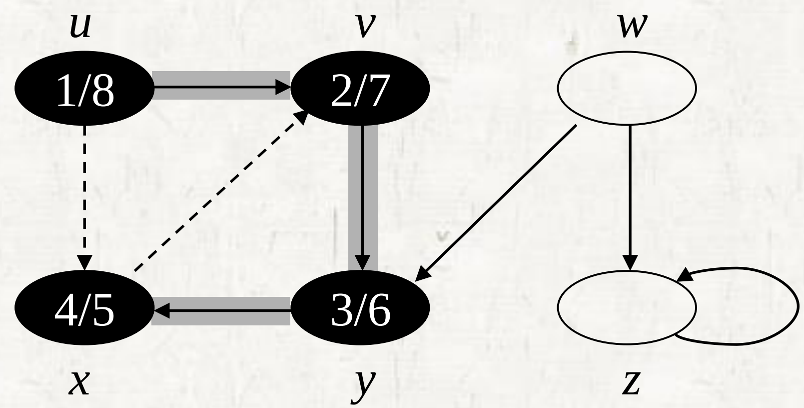
Depth-first search



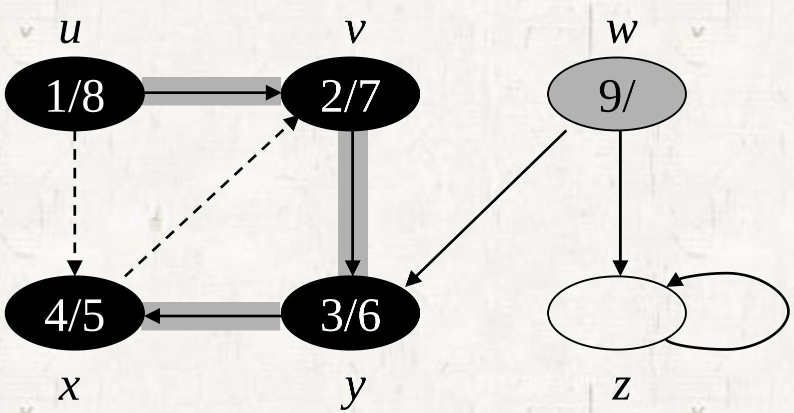
Depth-first search



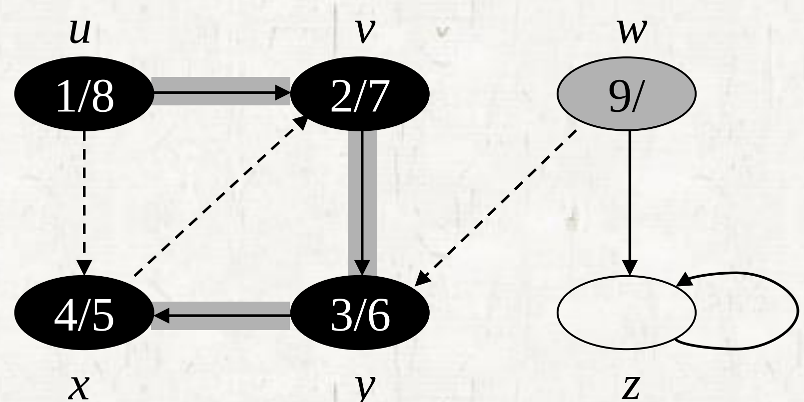
(i)



(j)

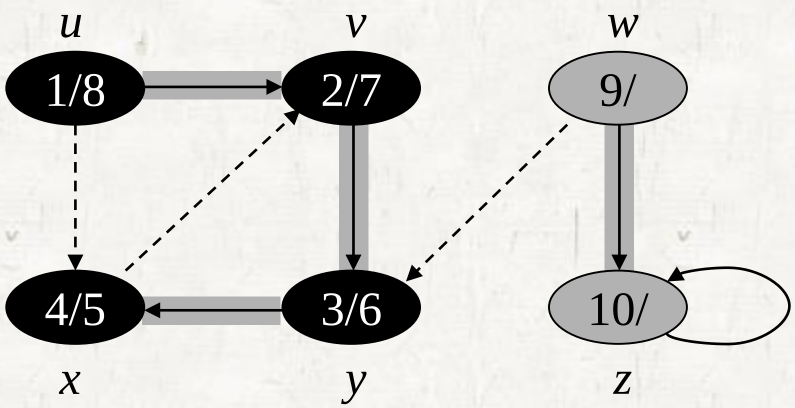


(k)

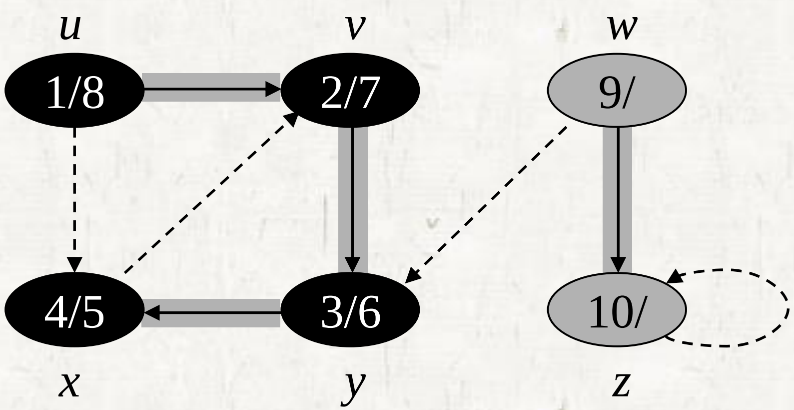


(l)

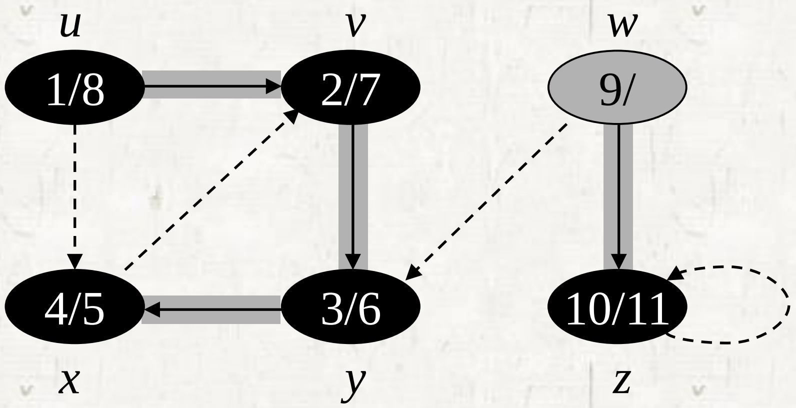
Depth-first search



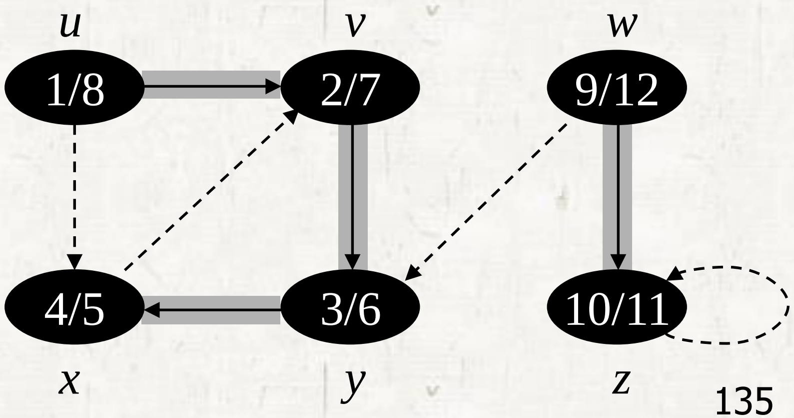
(m)



(n)



(o)



(p)

Depth-first search

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G,u)

DFS-VISIT(G,u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G,v)
u.color = BLACK
time = time + 1
u.f = time
```

Depth-first search

```
DFS(G)
    for each vertex u ∈ G.V
        u.color = WHITE
        u.π = NIL
    time = 0
    for each vertex u ∈ G.V
        if u.color == WHITE
            DFS-VISIT(G,u)

DFS-VISIT(G,u)
    time = time + 1
    u.d = time
    u.color = GRAY
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.π = u
            DFS-VISIT(G,v)
    u.color = BLACK
    time = time + 1
    u.f = time
```

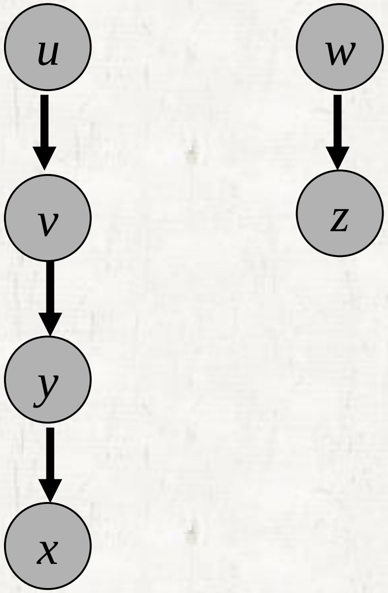
Initialization
$\Theta(V)$

Graph Exploration
$\Theta(V + E)$

DFS-VISIT
$\Theta(E)$

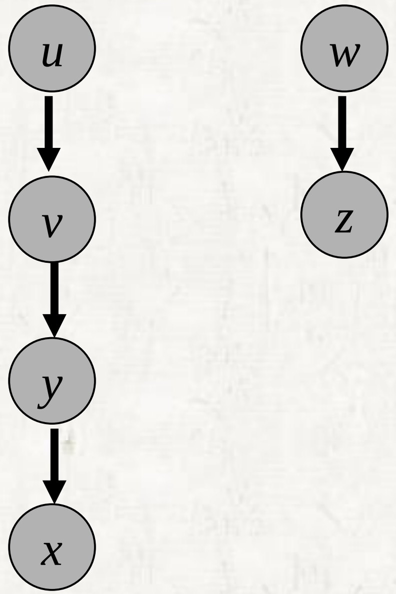
Depth-first search

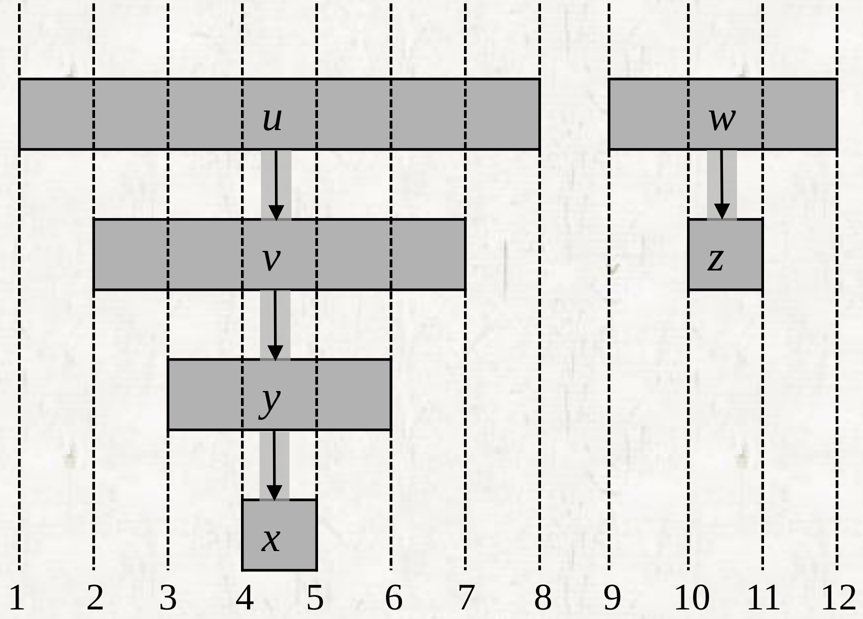
- The ***predecessor subgraph*** is a ***depth-first forest***.



Depth-first search

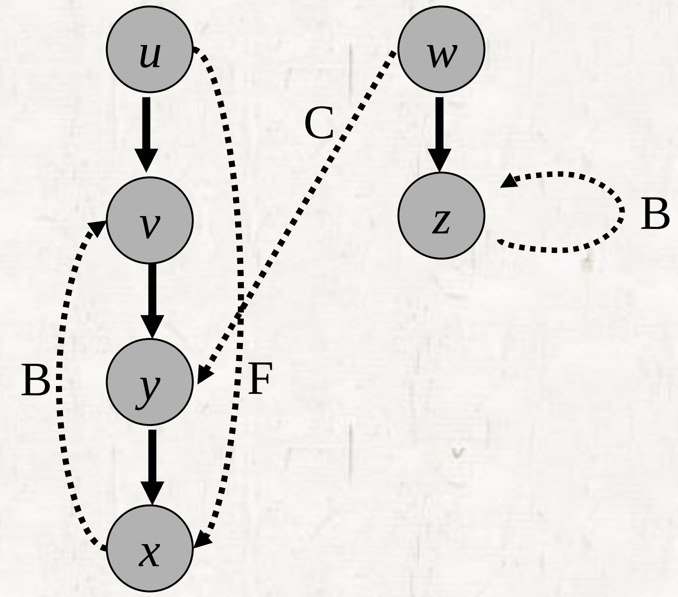
- ***Parenthesis theorem* (*for gray interval*)**
  - ***Inclusion***: The ancestor's includes the descendants'.
  - ***Disjoint***: Otherwise.





Depth-first search

- **Classification of edges**
  - ***Tree edges***
  - ***Back edges***
  - ***Forward edges***
  - ***Cross edges***



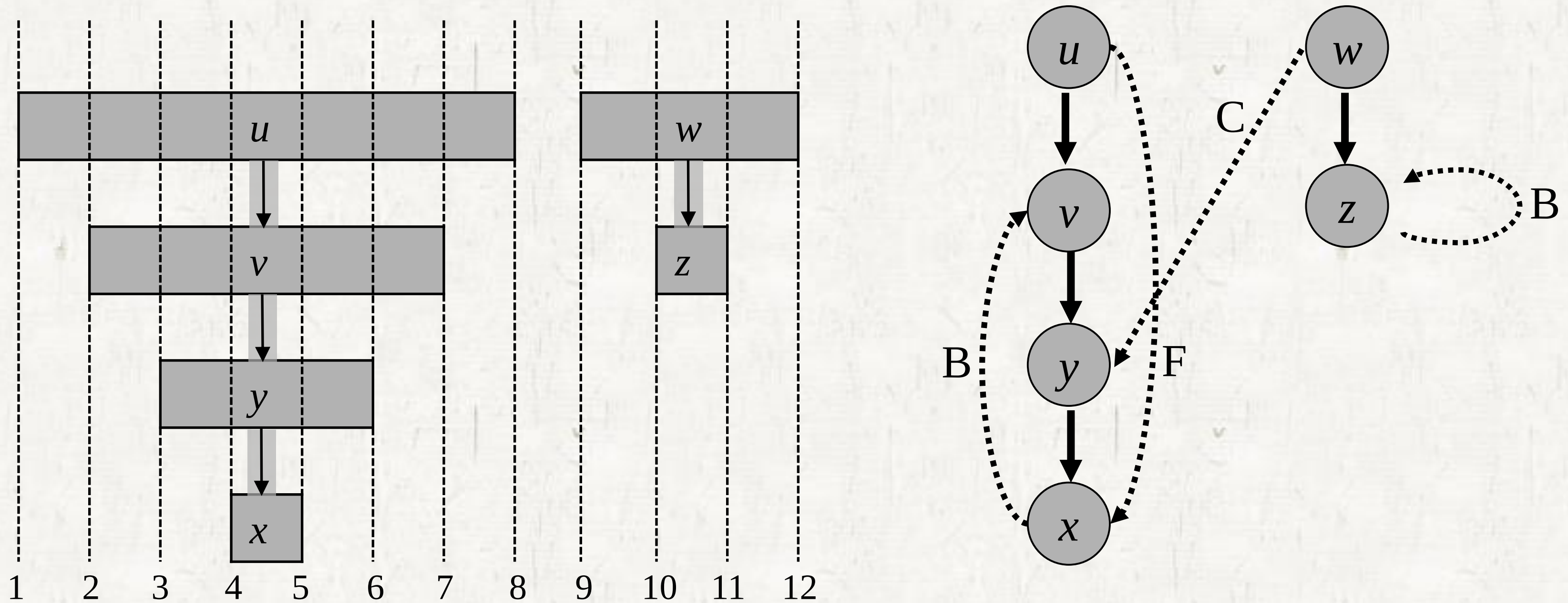
Depth-first search

- ***Tree edges***
  - Edges in a depth-first tree.
- ***Back edges*** ***(cycle)***
  - Edges from descendants to ancestors or self-loops
- ***Forward edges***:
  - Non-tree edges from ancestors to descendants
- ***Cross edges***:
  - All other edges, which are between vertices such that one vertex is not an ancestor of the other.

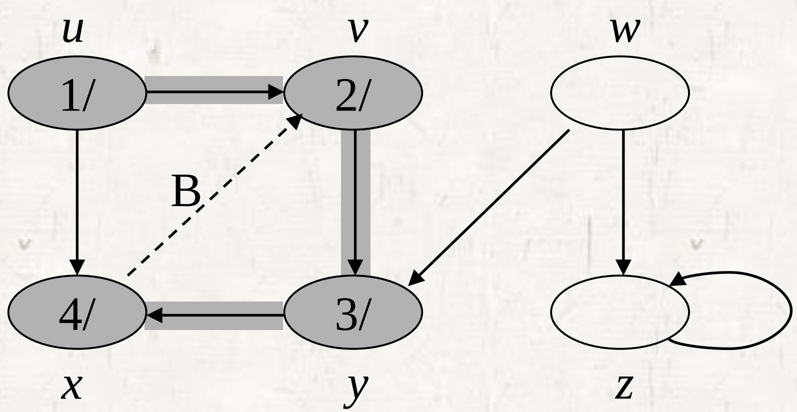
Depth-first search

- **Classification by the DFS algorithm**
  - Each edge (u, v) can be classified by the color of the vertex v that is reached when the edge is first explored:
    - ***white*** indicates a tree edge,
    - ***gray*** indicates a back edge, and
    - ***black*** indicates a forward or cross edge.
  - Forward and cross edges are classified by ***the inclusion of gray intervals of u and v***.

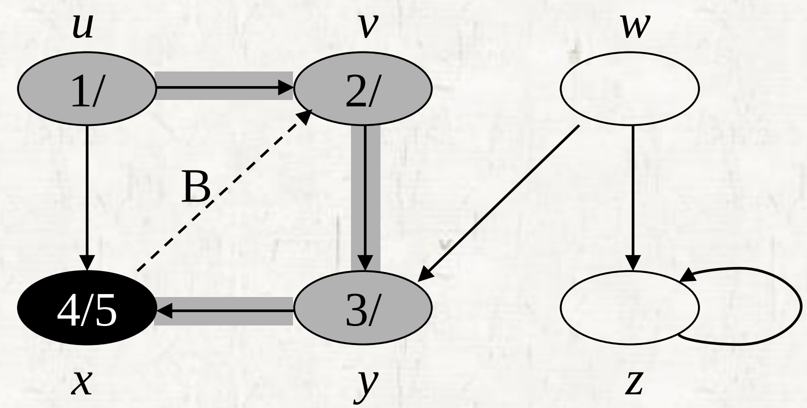
Depth-first search



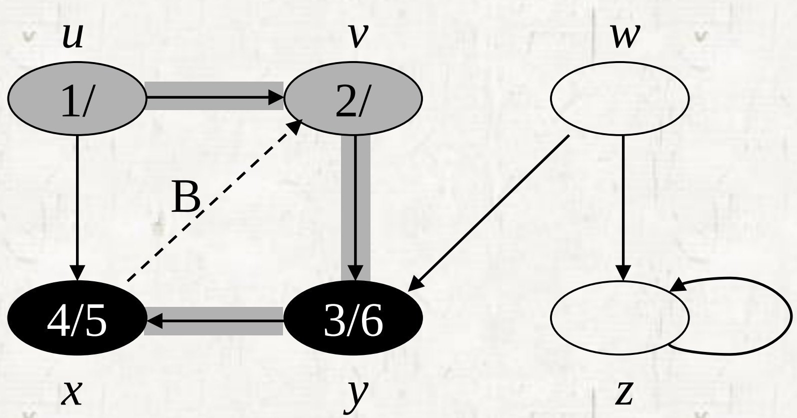
Depth-first search



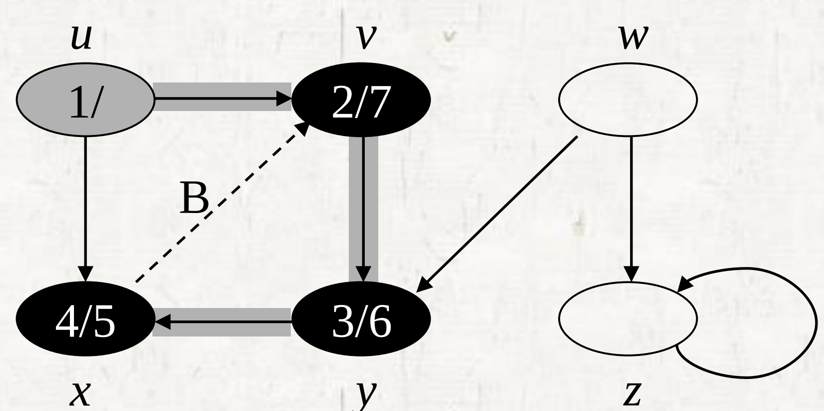
(e)



(f)

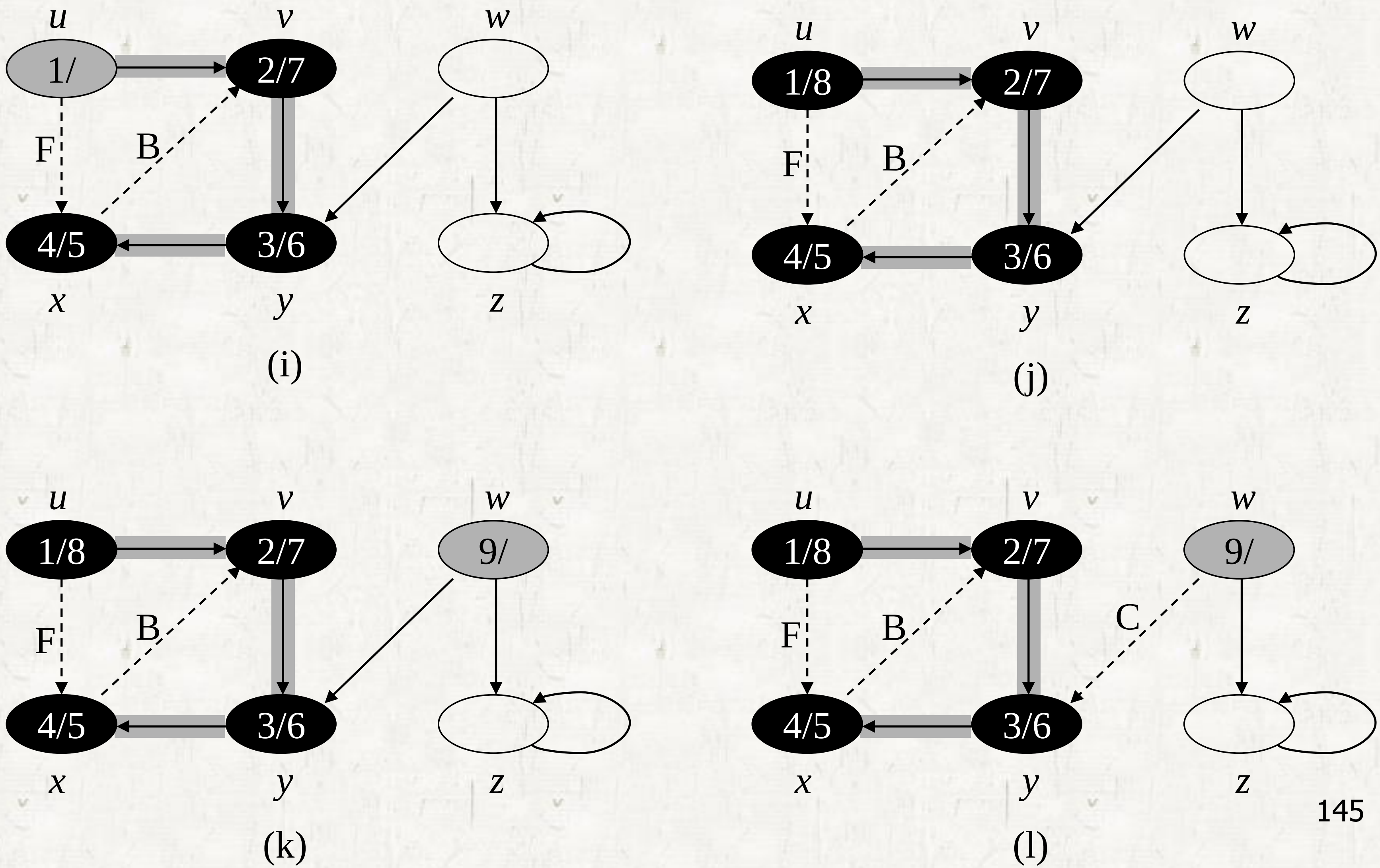


(g)

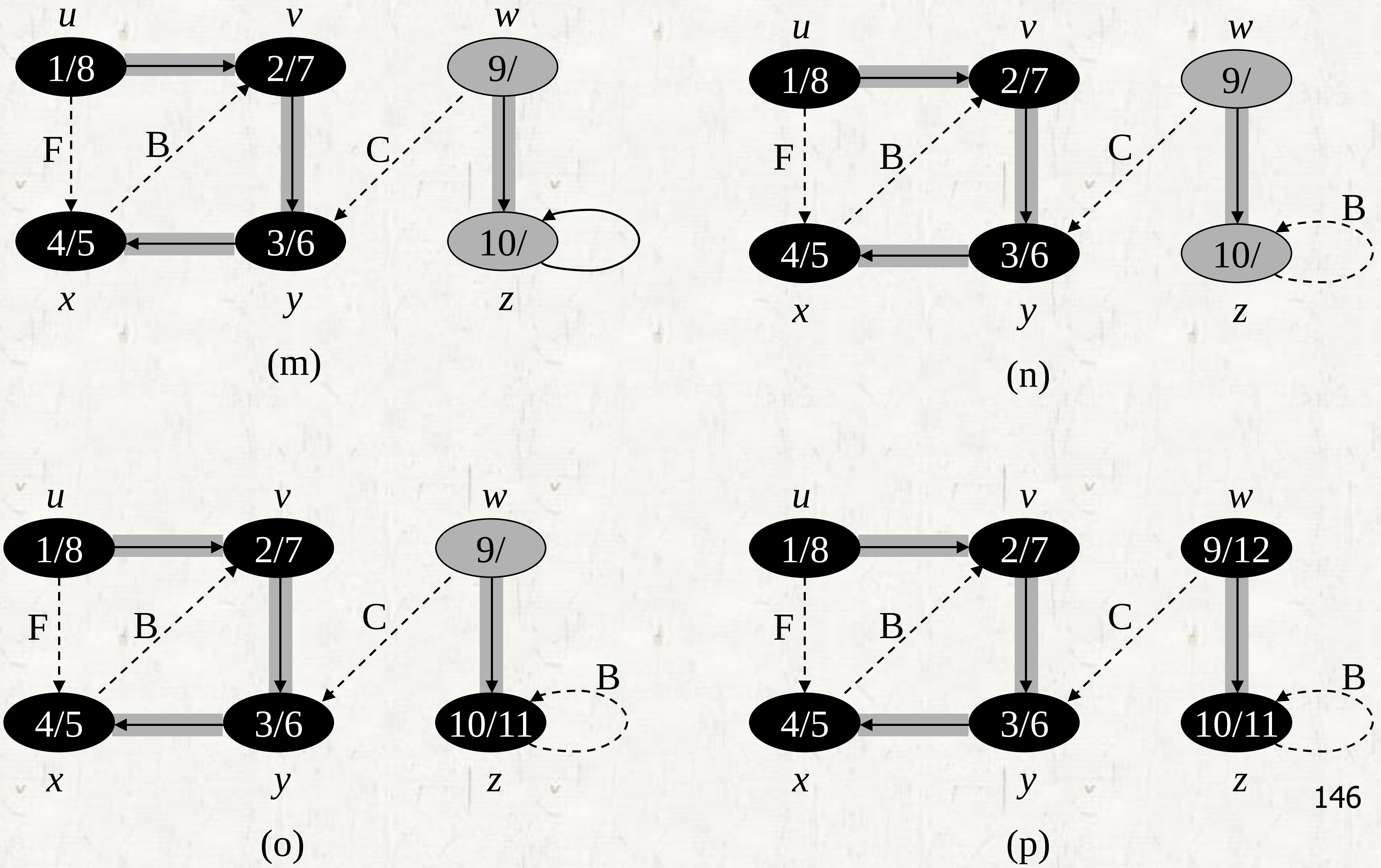


(h)

Depth-first search



Depth-first search



Depth-first search

- In a depth-first search of an ***undirected graph***, every edge of G is either a ***tree edge*** or a ***back edge***.
    - Forward edge?
    - Cross edge?
- Running Time
    - $\Theta(V+E)$

Self-study

- **Exercise 22.3-5 (22.3-4 in the 2nd ed.)**
  - Edge classification
- **Problem 22-2 *a*-*d***
  - Articulation points

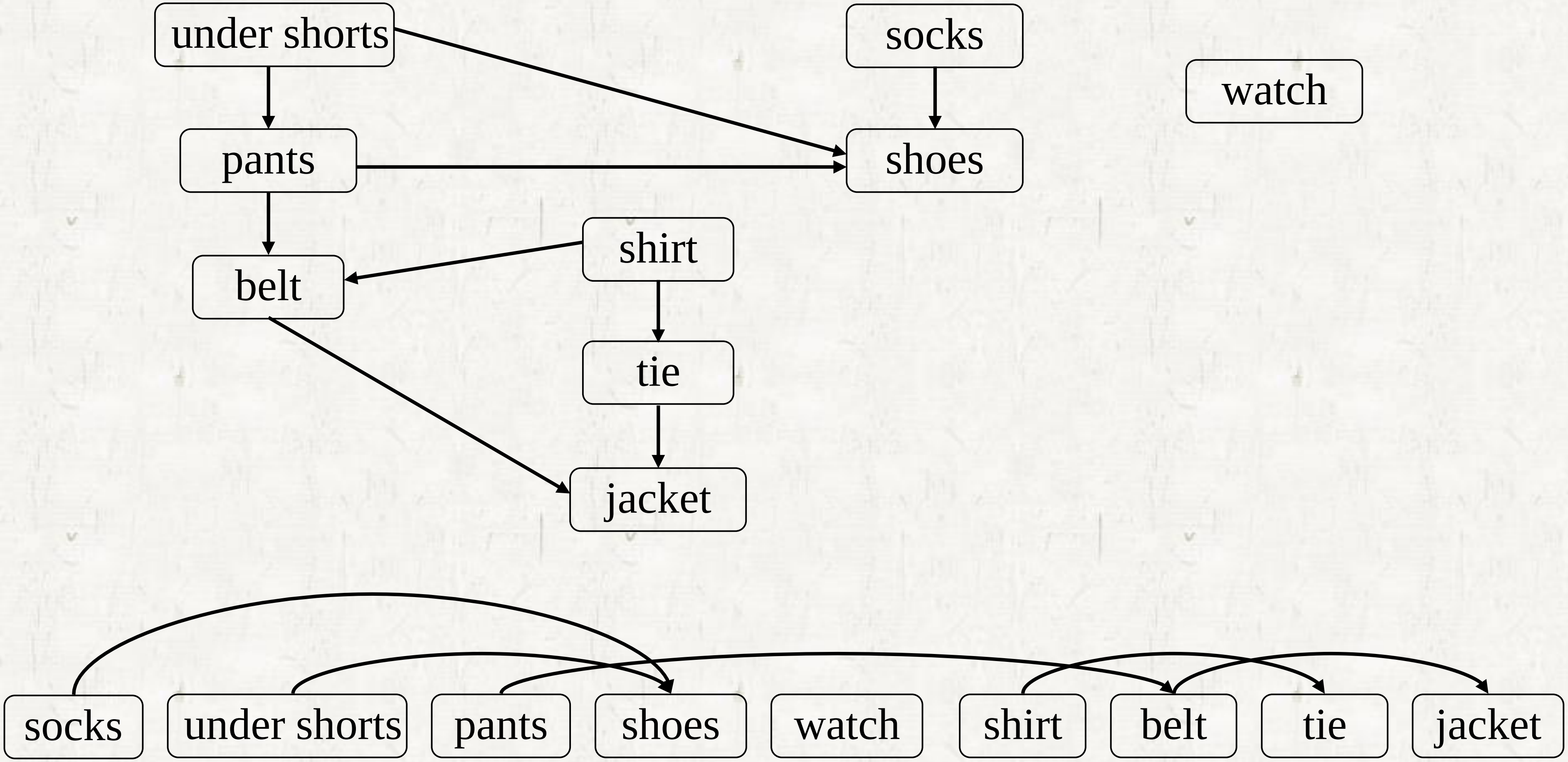
Contents

- ***Graphs***
  - *Graphs basics*
  - *Graph representation*
- ***Searching a graph***
  - *Breadth-first search*
  - *Depth-first search*
- **Applications of depth-first search**
  - Topological sort

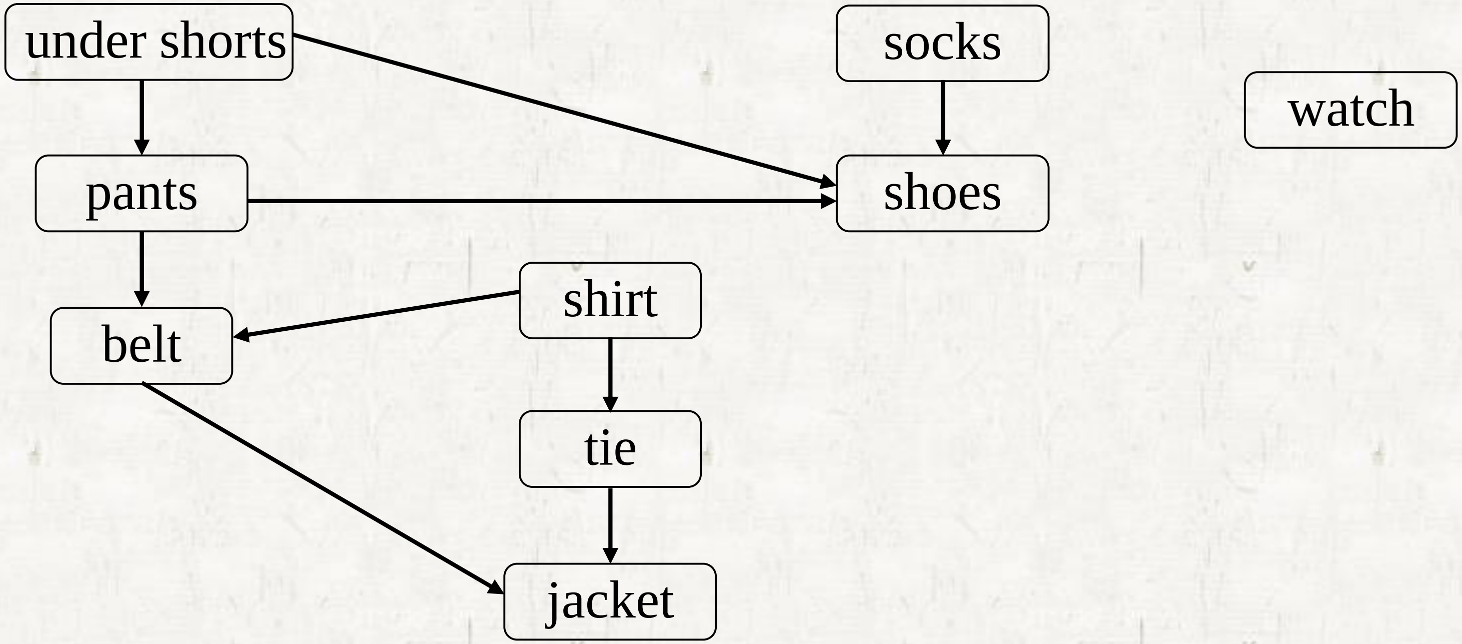
Topological sort

- ***Definition***
  - Given a DAG (directed acyclic graph), generate a linear ordering of all its vertices such that all edges go from left to right.

Topological sort



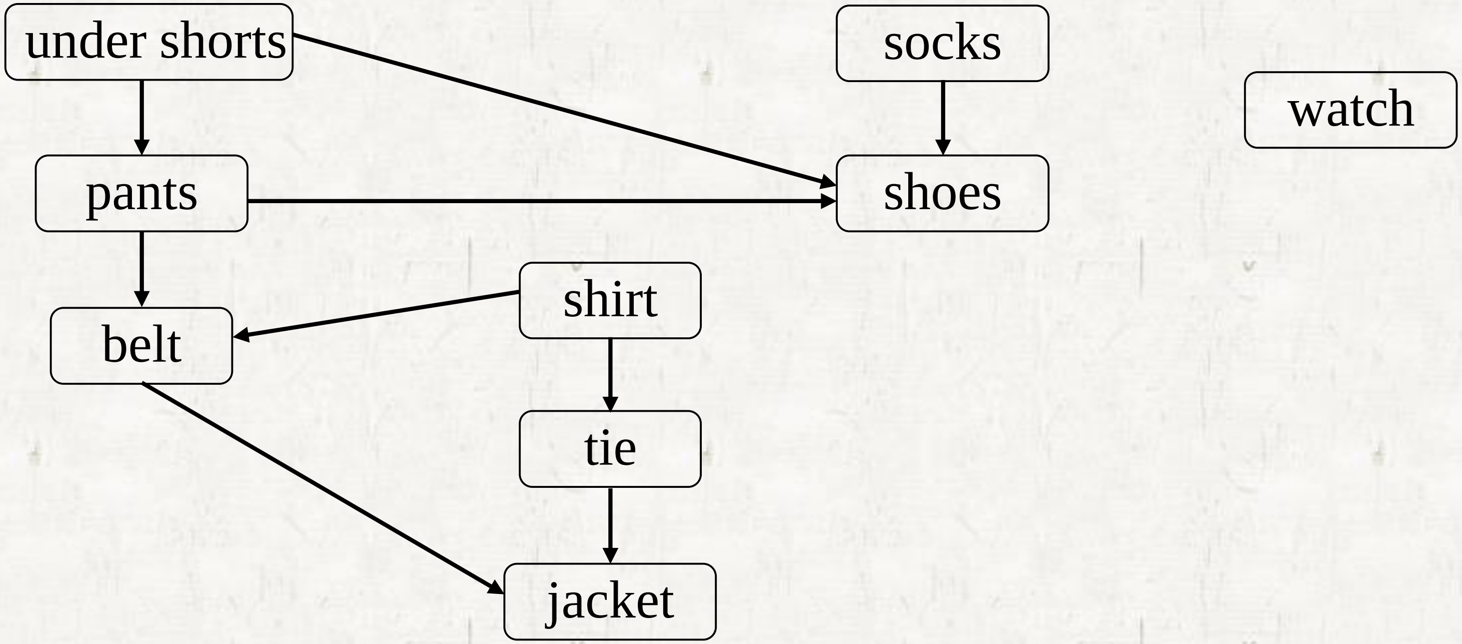
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
0	0	1	3	0	0	2	1	2

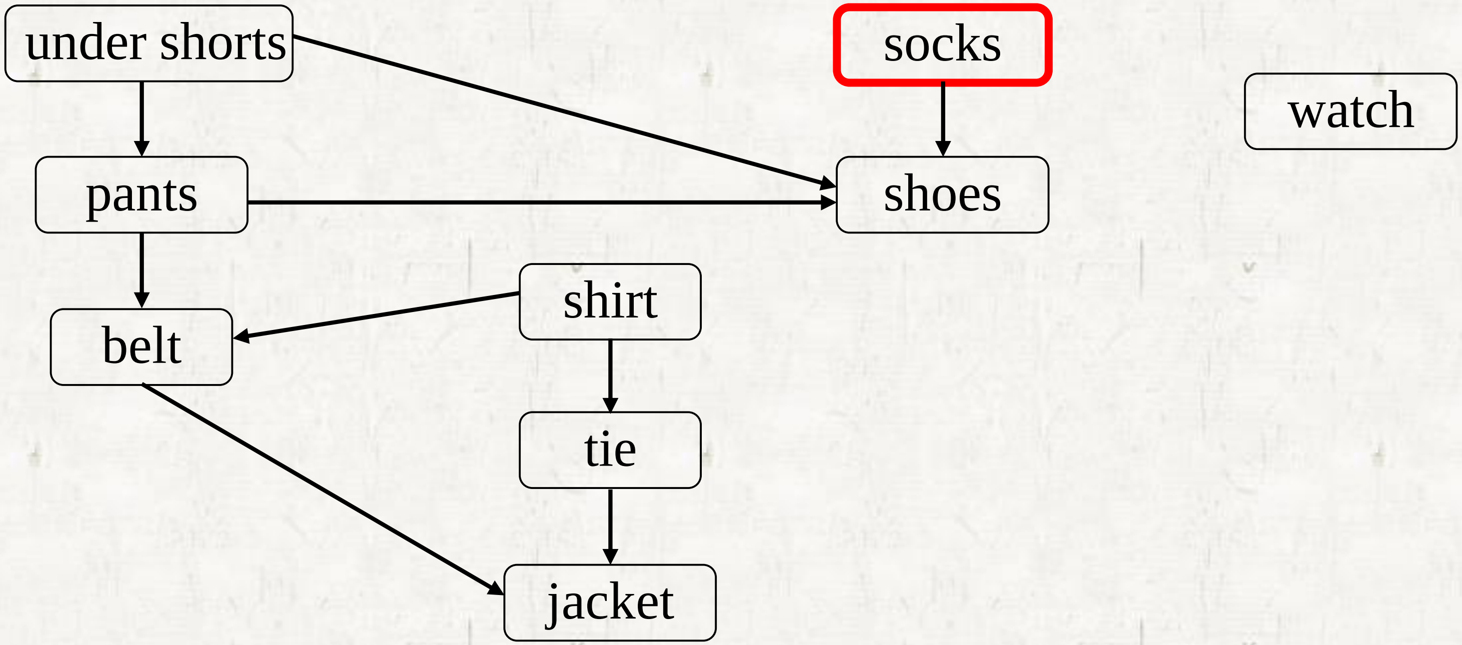
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
0	0	1	3	0	0	2	1	2

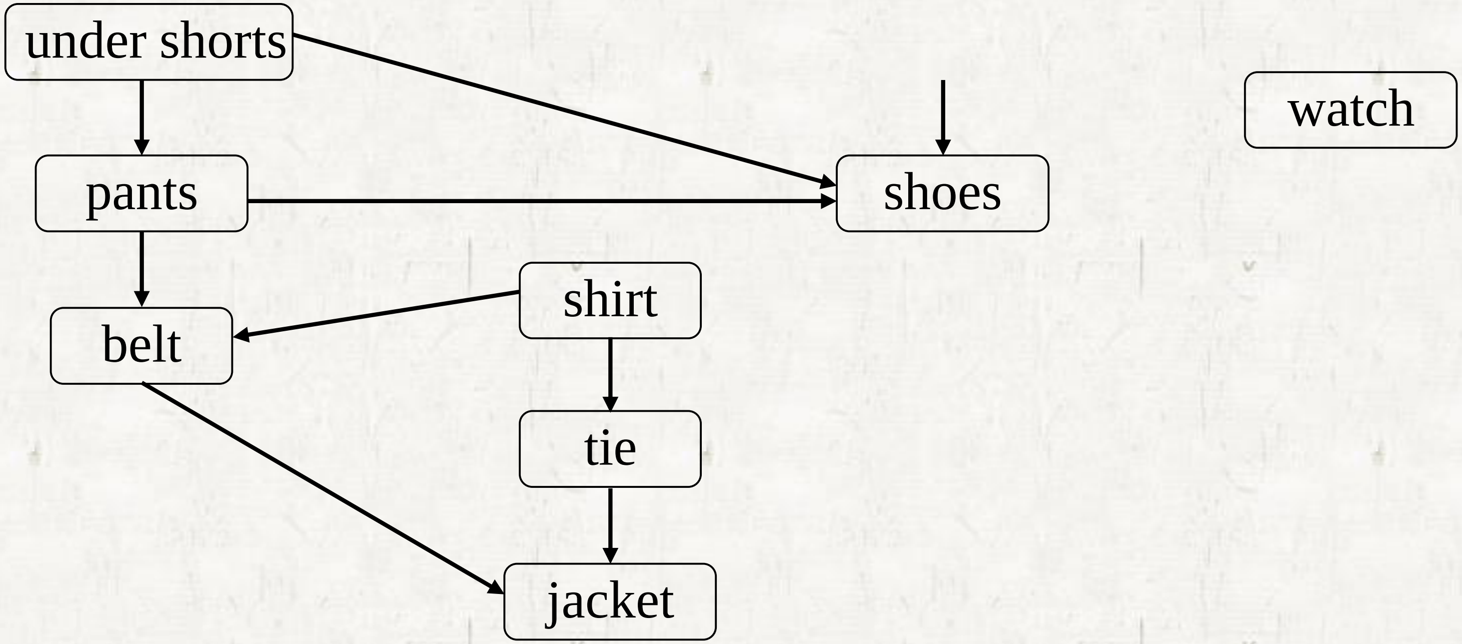
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
0	0	1	3	0	0	2	1	2

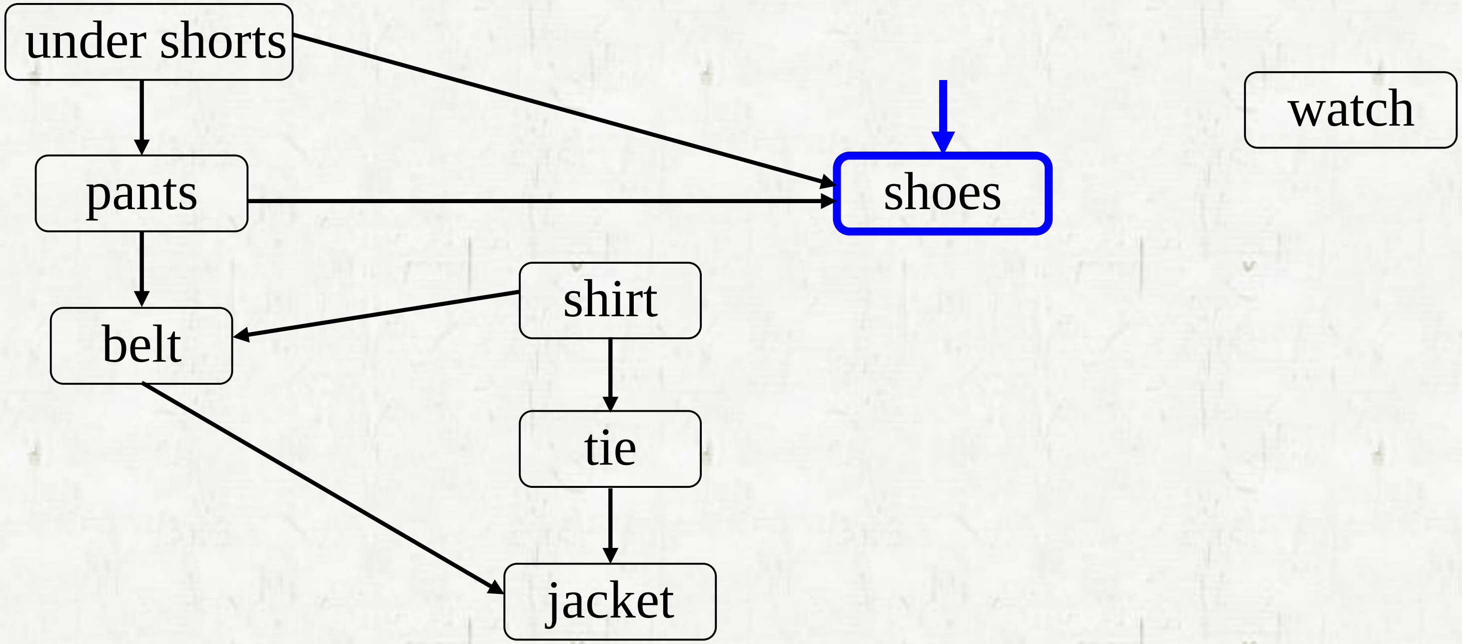
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	3	0	0	2	1	2

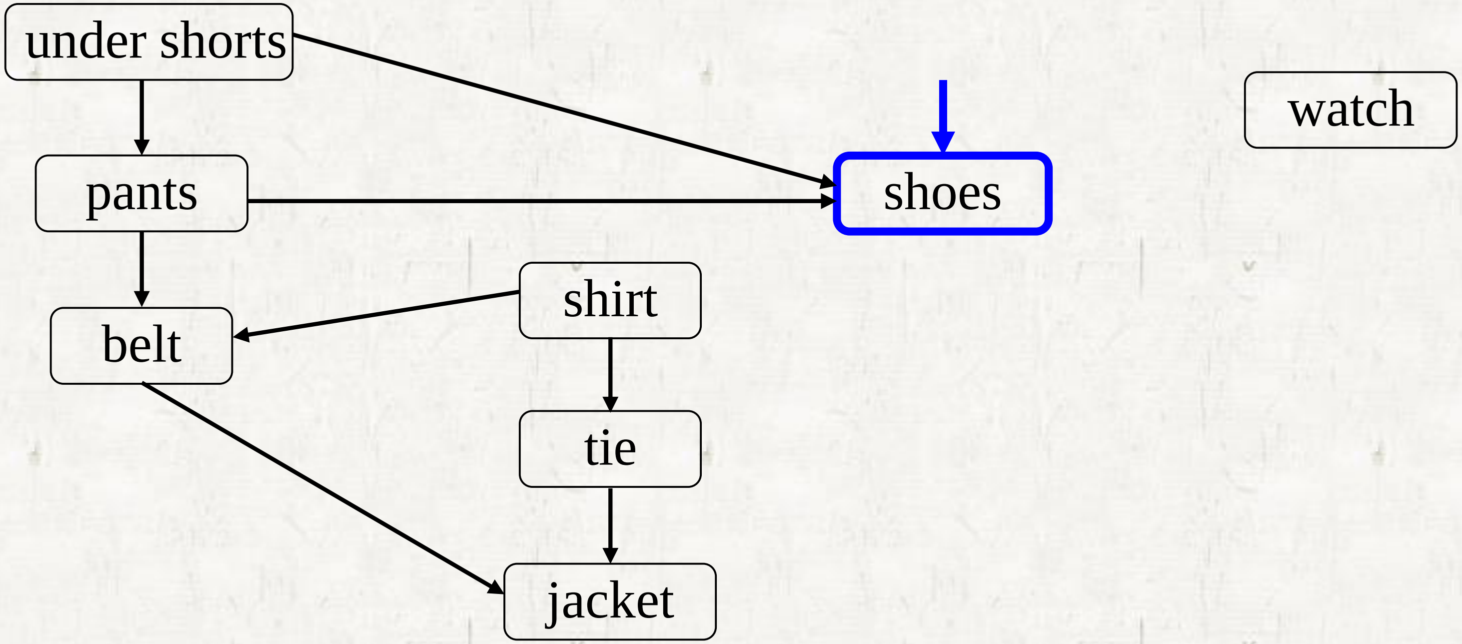
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	3	0	0	2	1	2

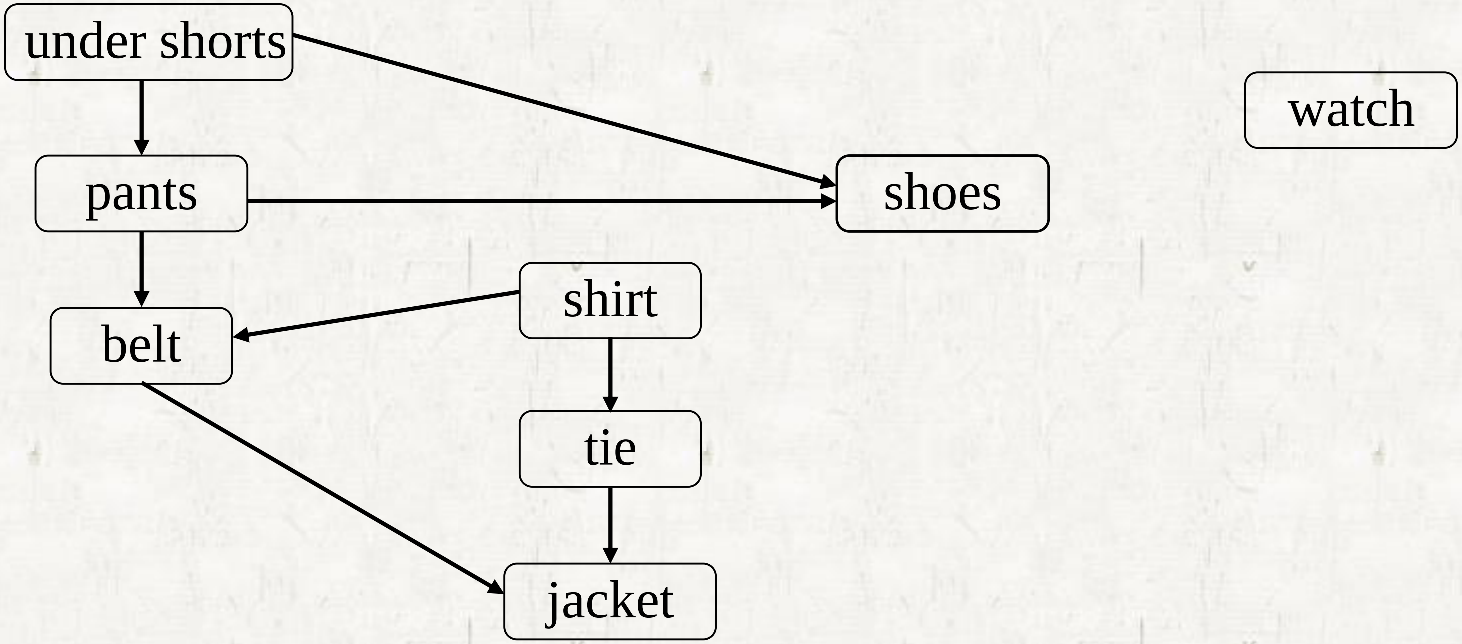
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	3 ➔ 2	0	0	2	1	2

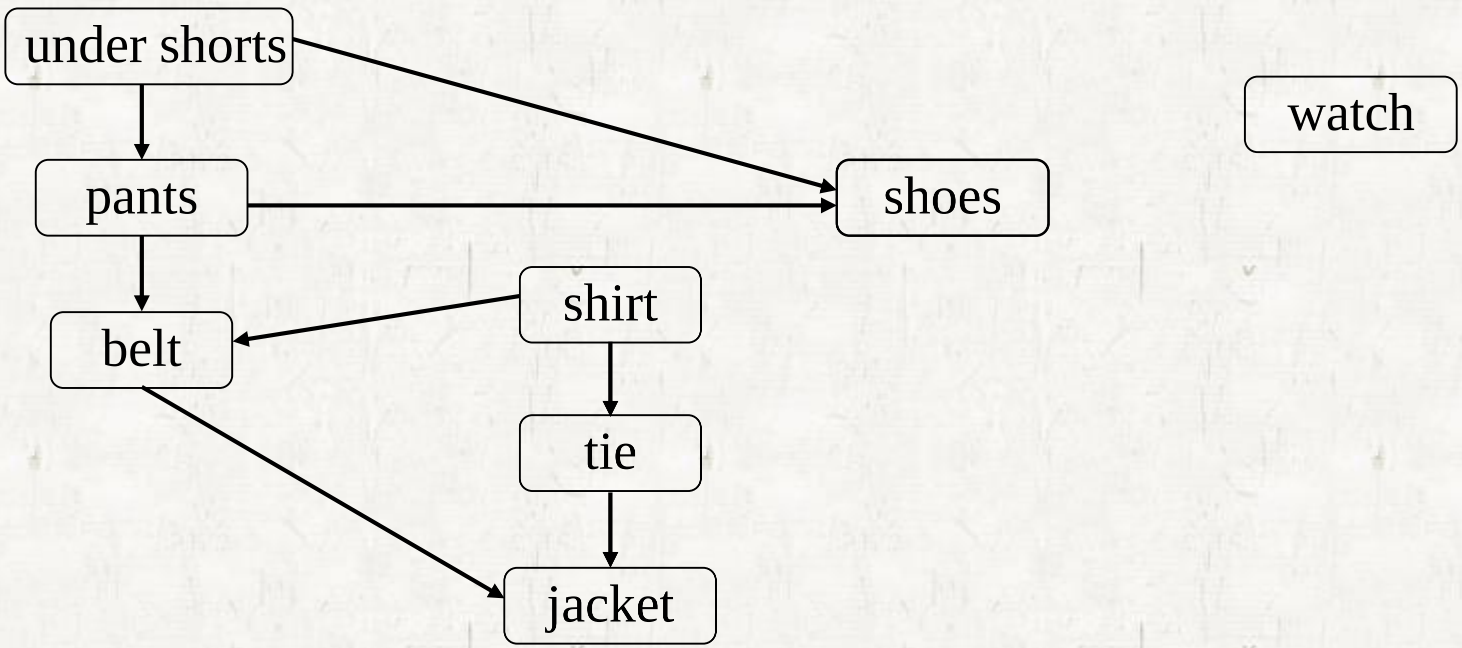
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	2	0	0	2	1	2

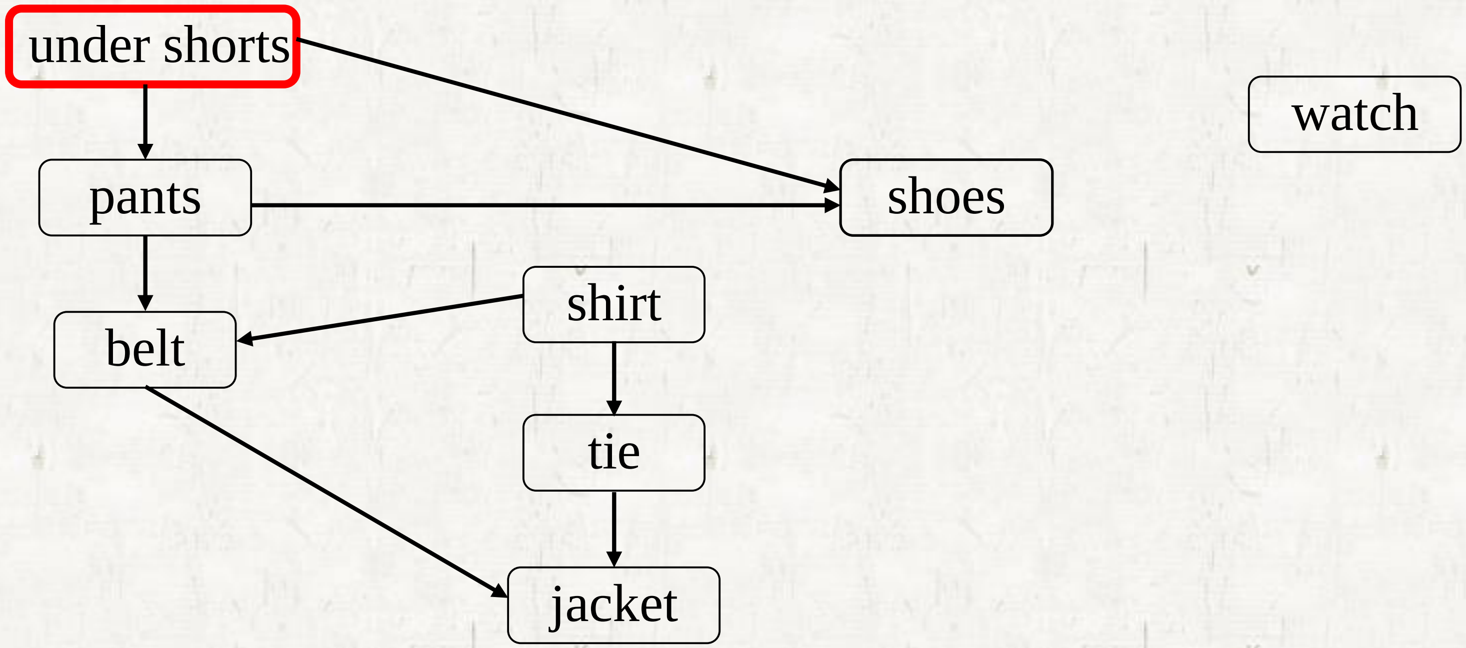
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	2	0	0	2	1	2

Topological sort

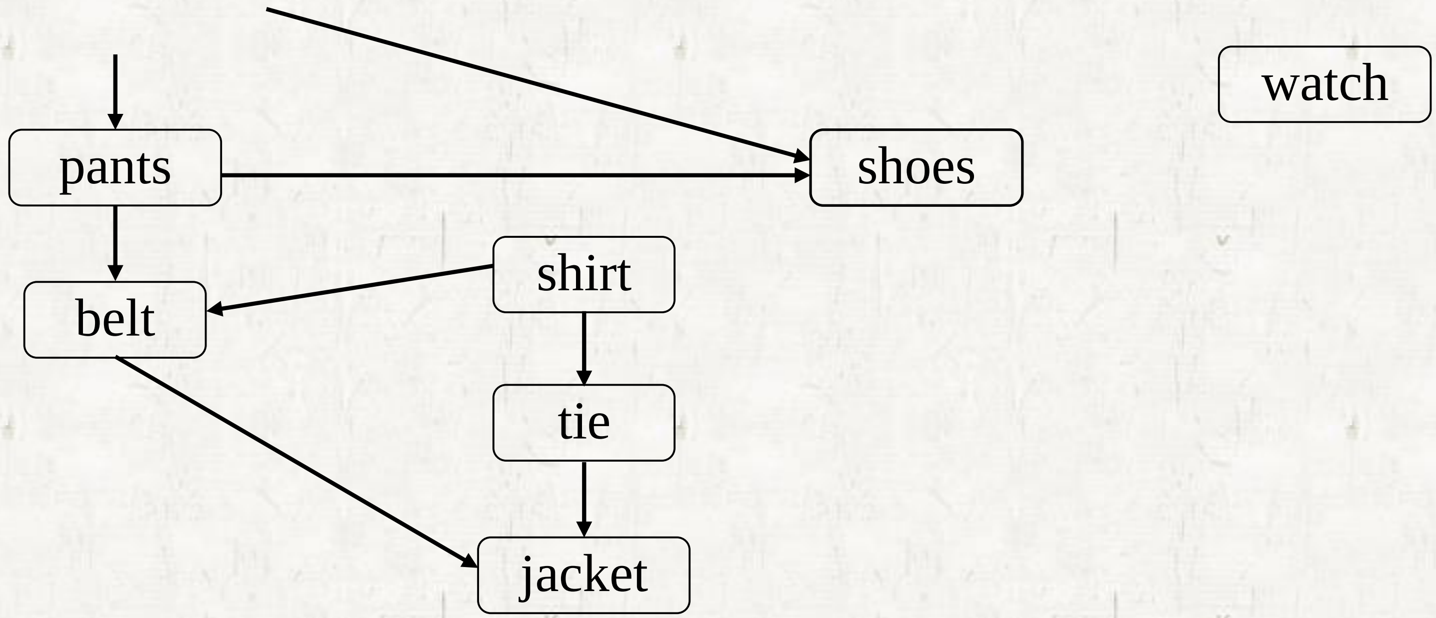


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	0	1	2	0	0	2	1	2

socks

Topological sort

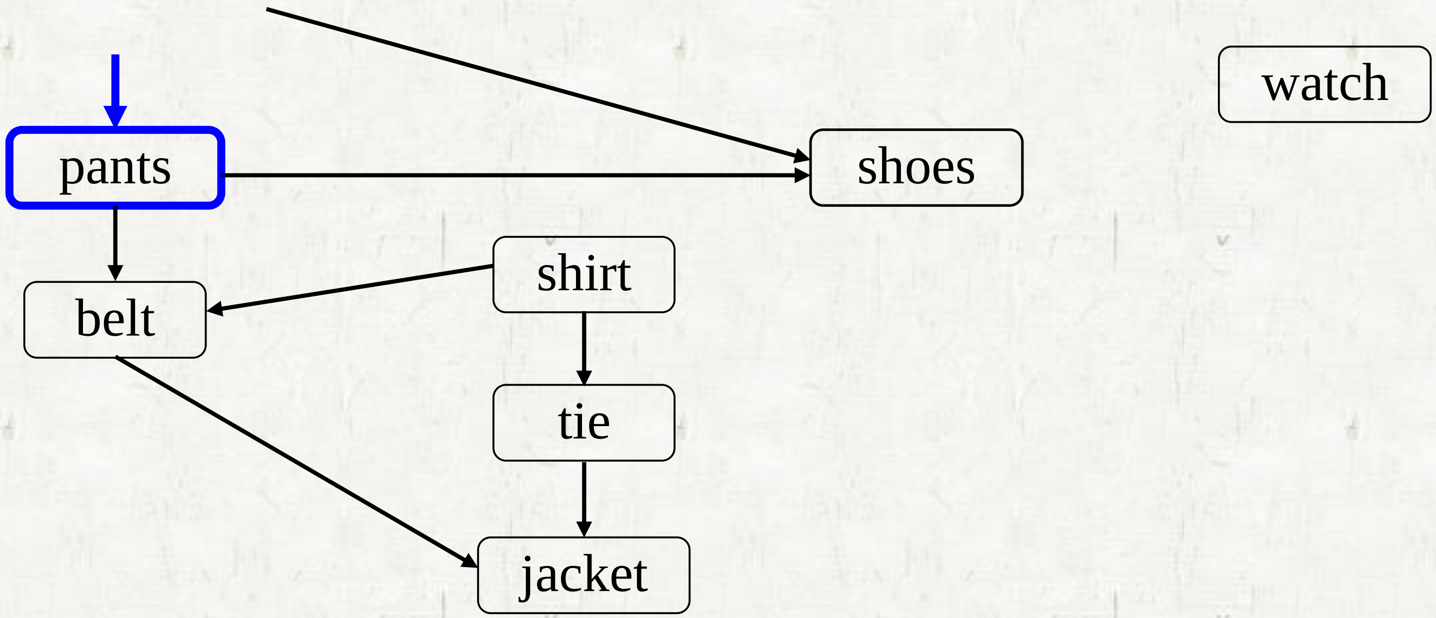


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	1	2	0	0	2	1	2

socks under shorts

Topological sort

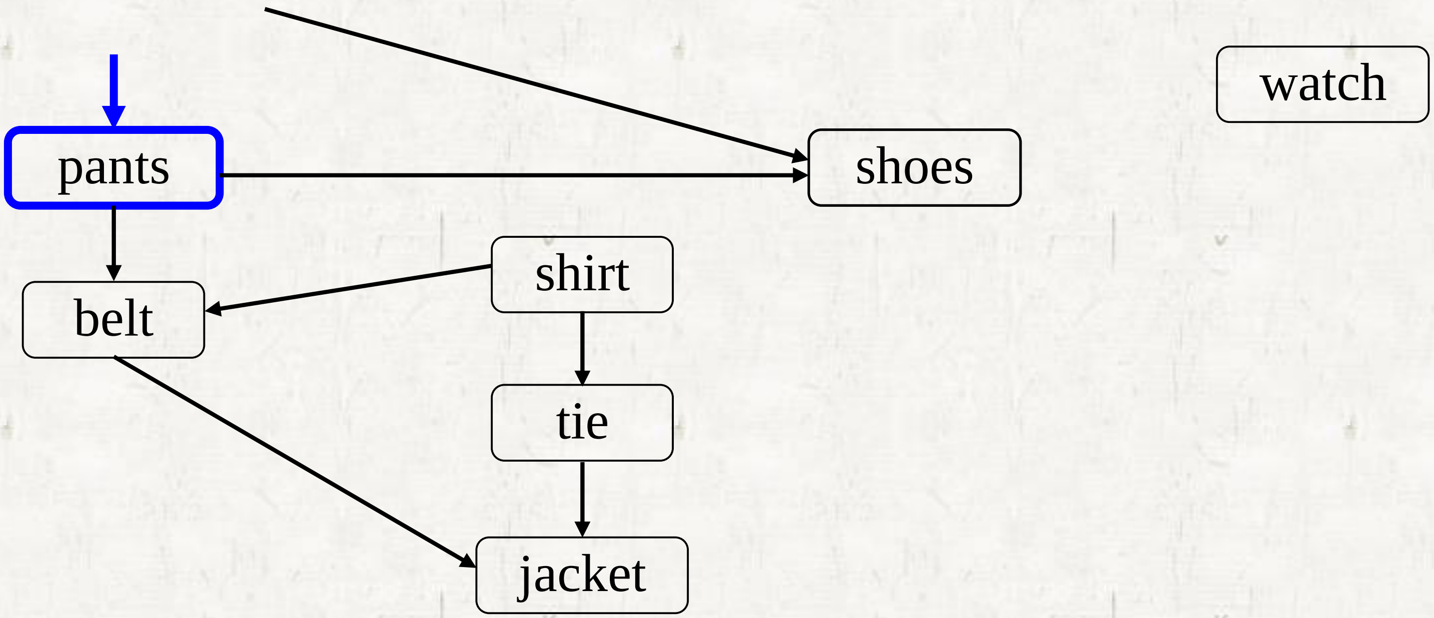


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	1	2	0	0	2	1	2

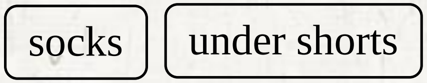
socks under shorts

Topological sort

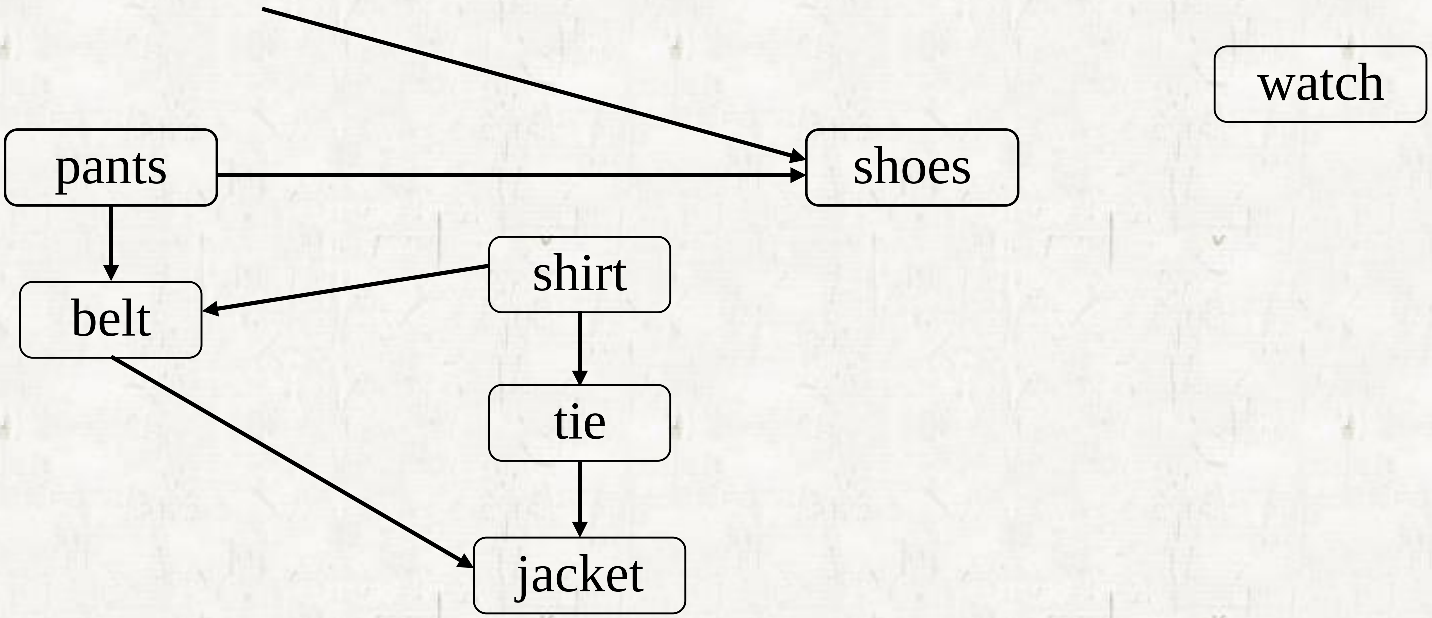


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	1 → 0	2	0	0	2	1	2

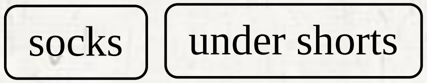


Topological sort

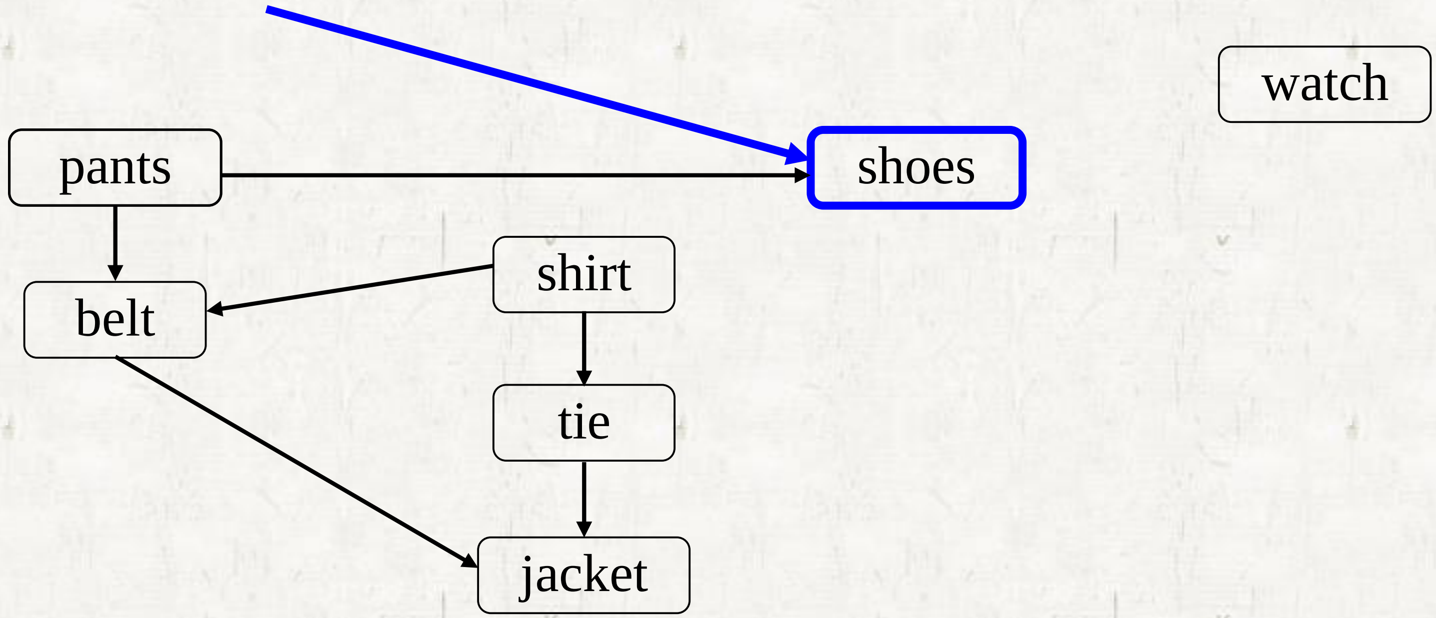


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	2	0	0	2	1	2



Topological sort

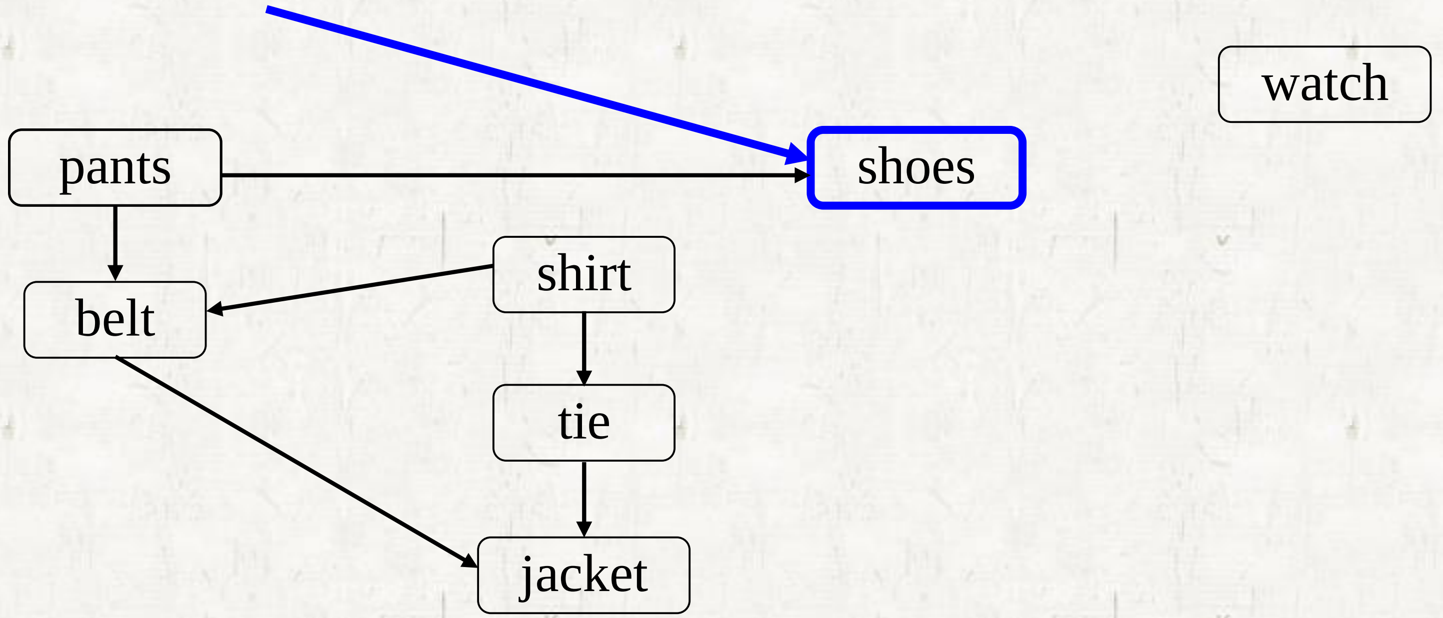


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	2	0	0	2	1	2

socks under shorts

Topological sort

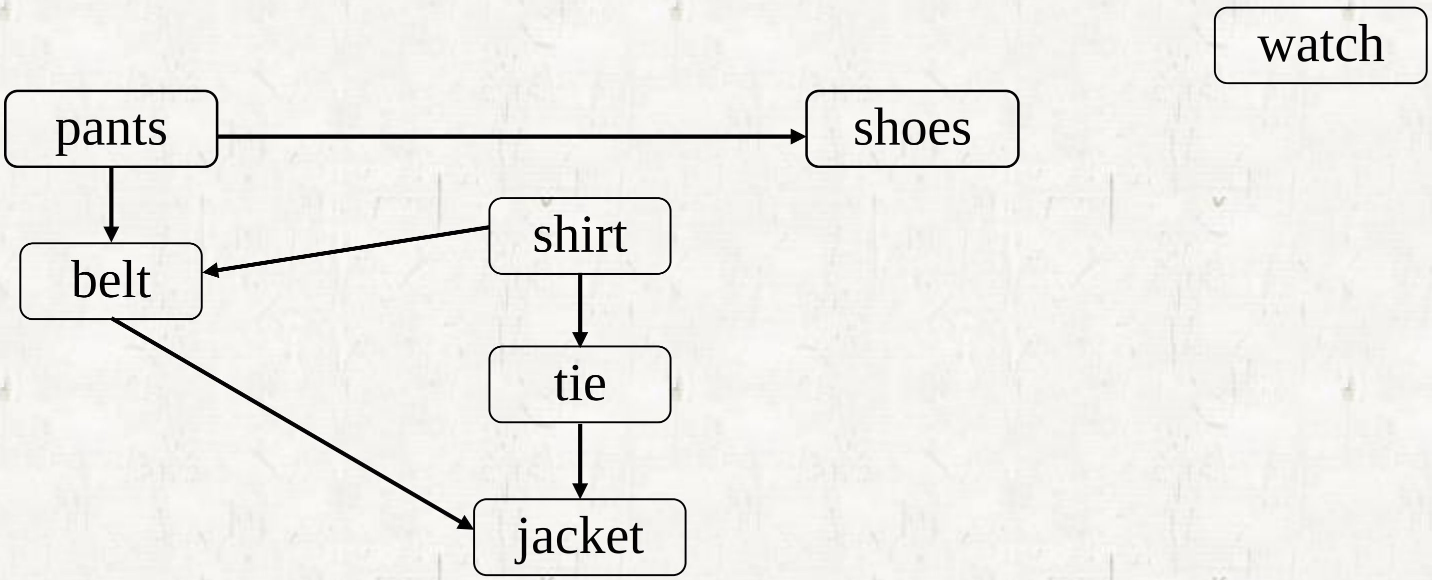


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	2 ➔ 1	0	0	2	1	2

socks under shorts

Topological sort

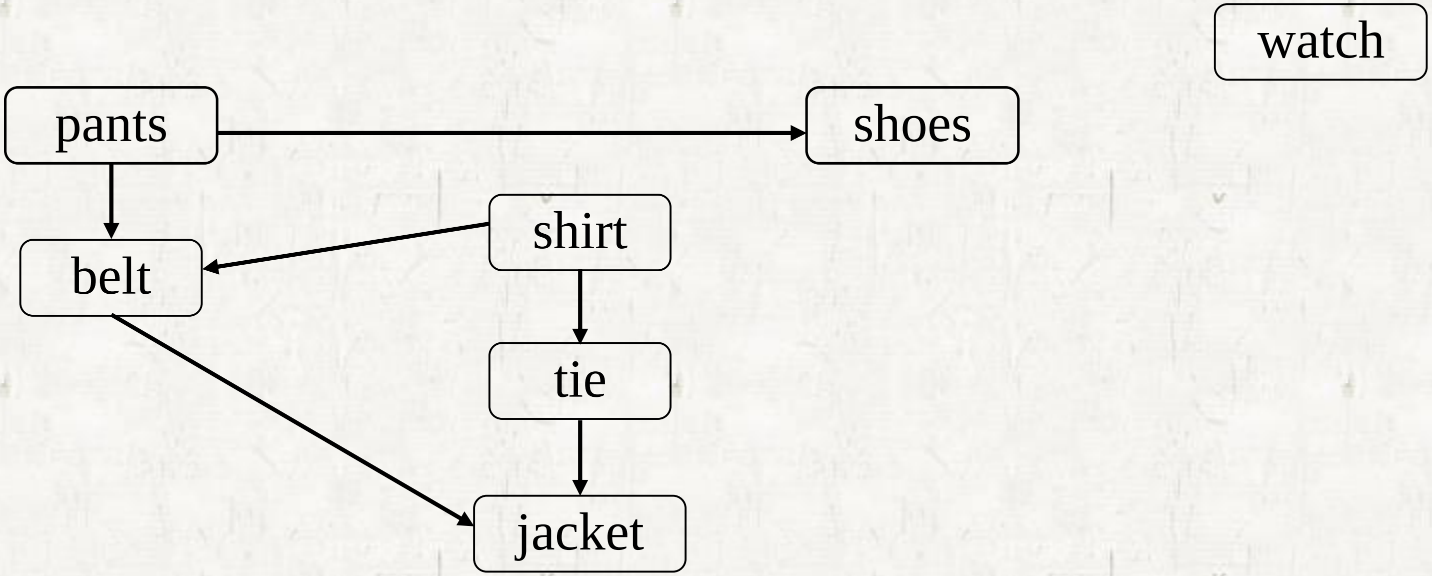


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	1	0	0	2	1	2

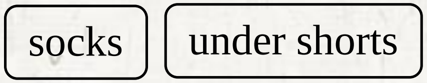
socks under shorts

Topological sort

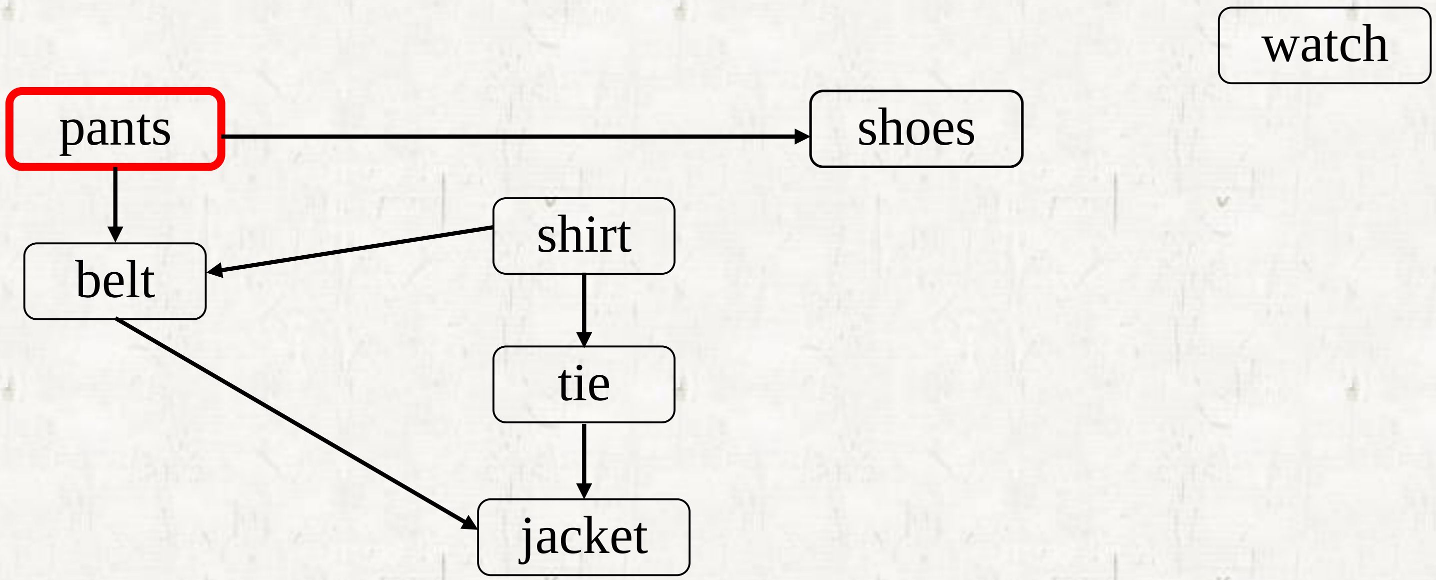


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	1	0	0	2	1	2



Topological sort

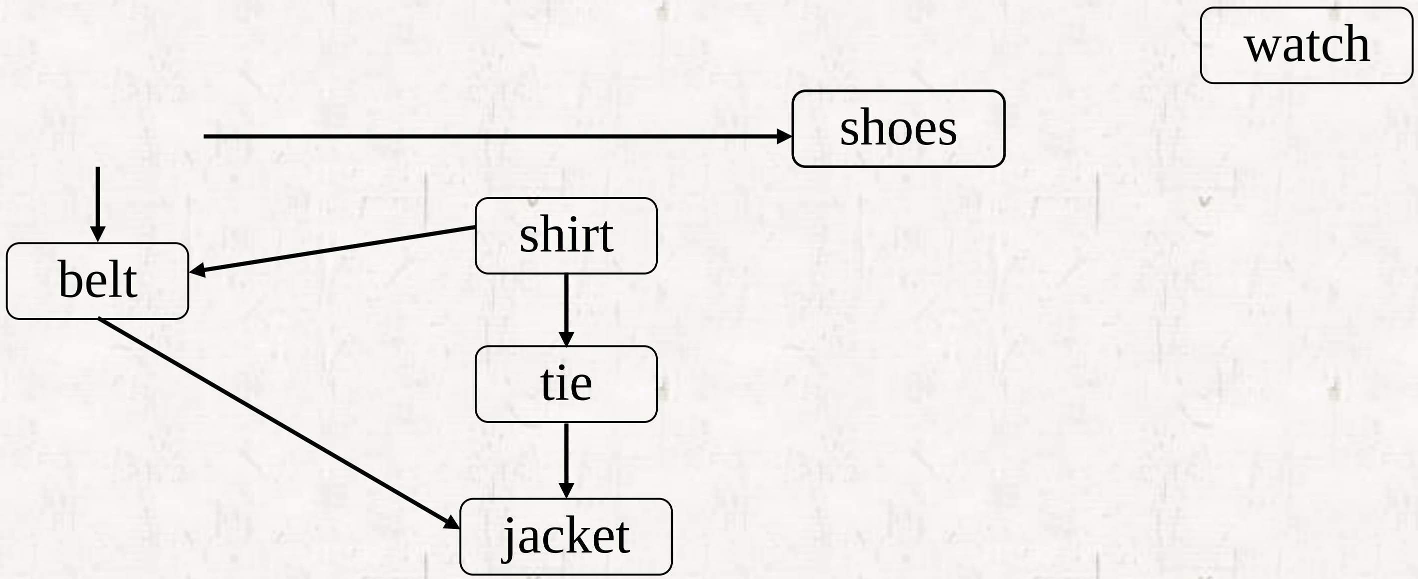


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	0	1	0	0	2	1	2

socks under shorts

Topological sort

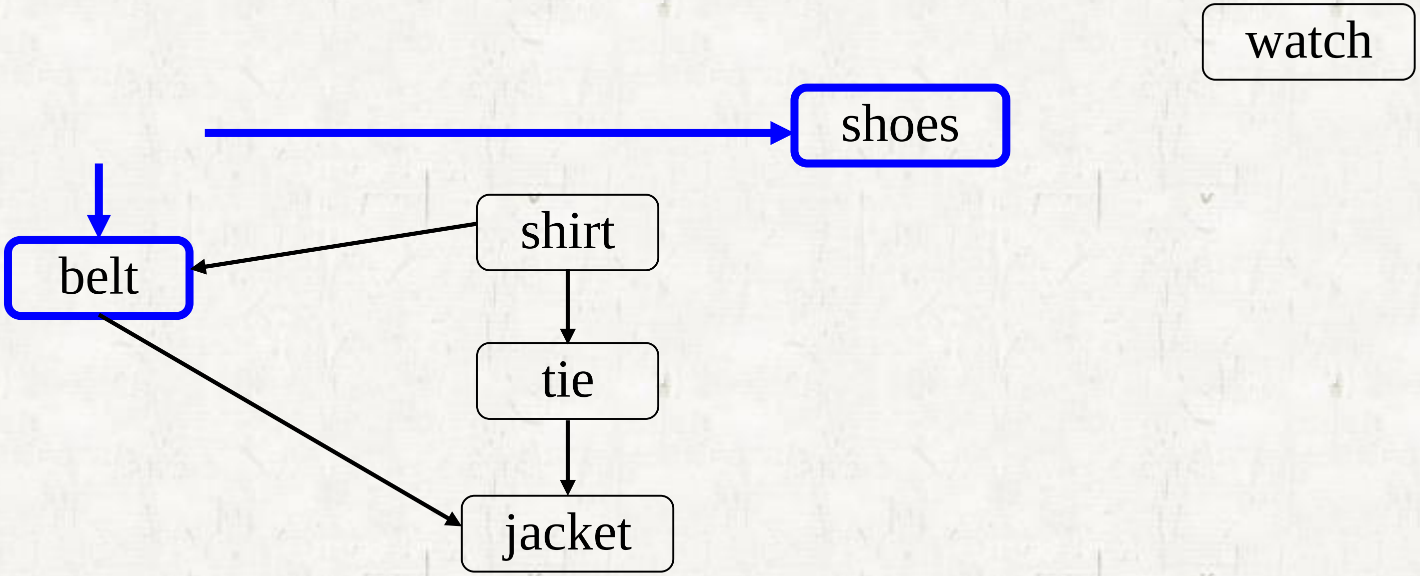


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	1	0	0	2	1	2

socks | under shorts | pants

Topological sort

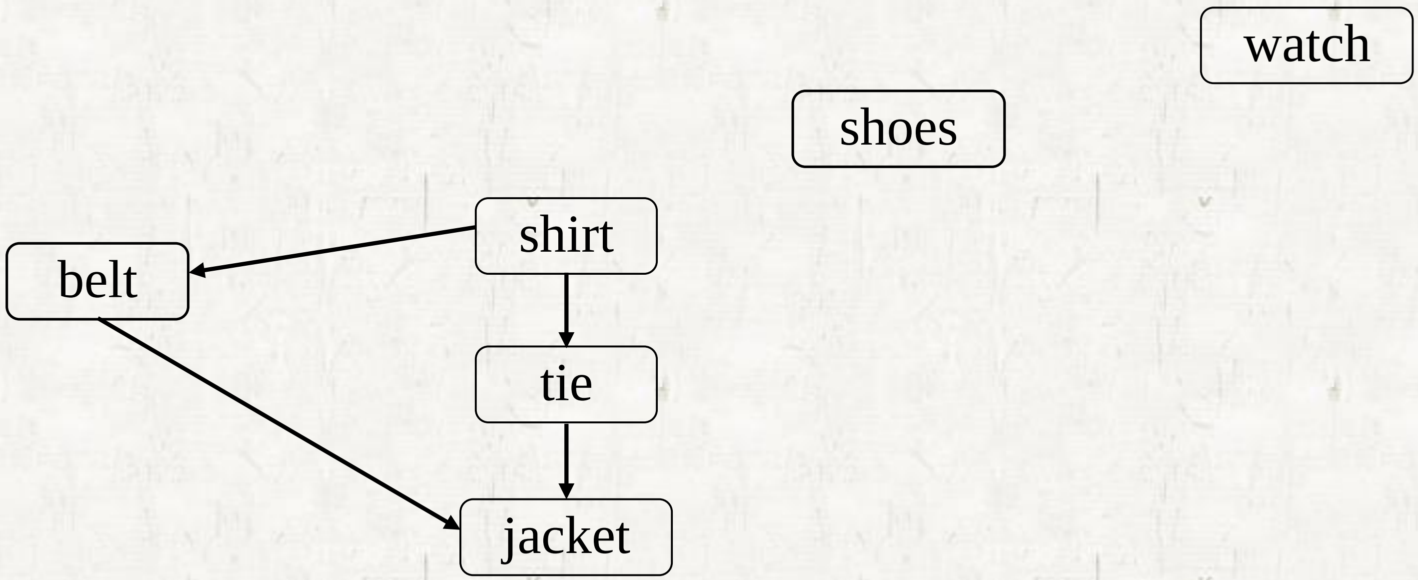


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	1 → 0	0	0	2 → 1	1	2

socks | under shorts | pants

Topological sort

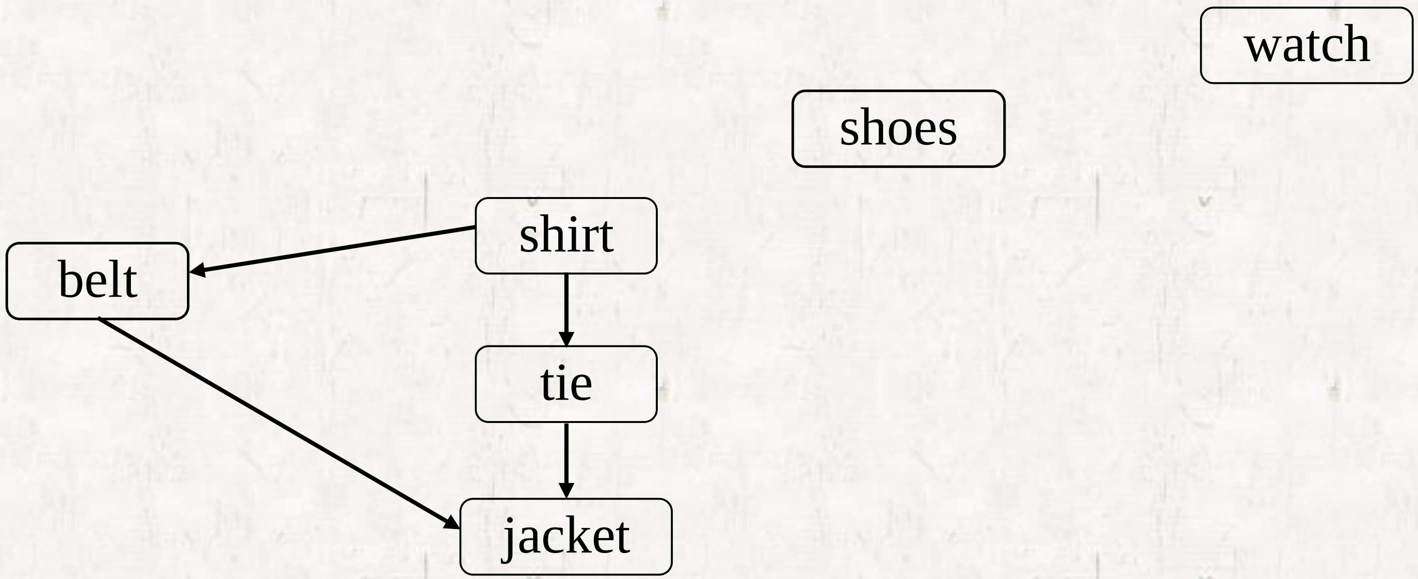


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	0	0	0	1	1	2

socks | under shorts | pants

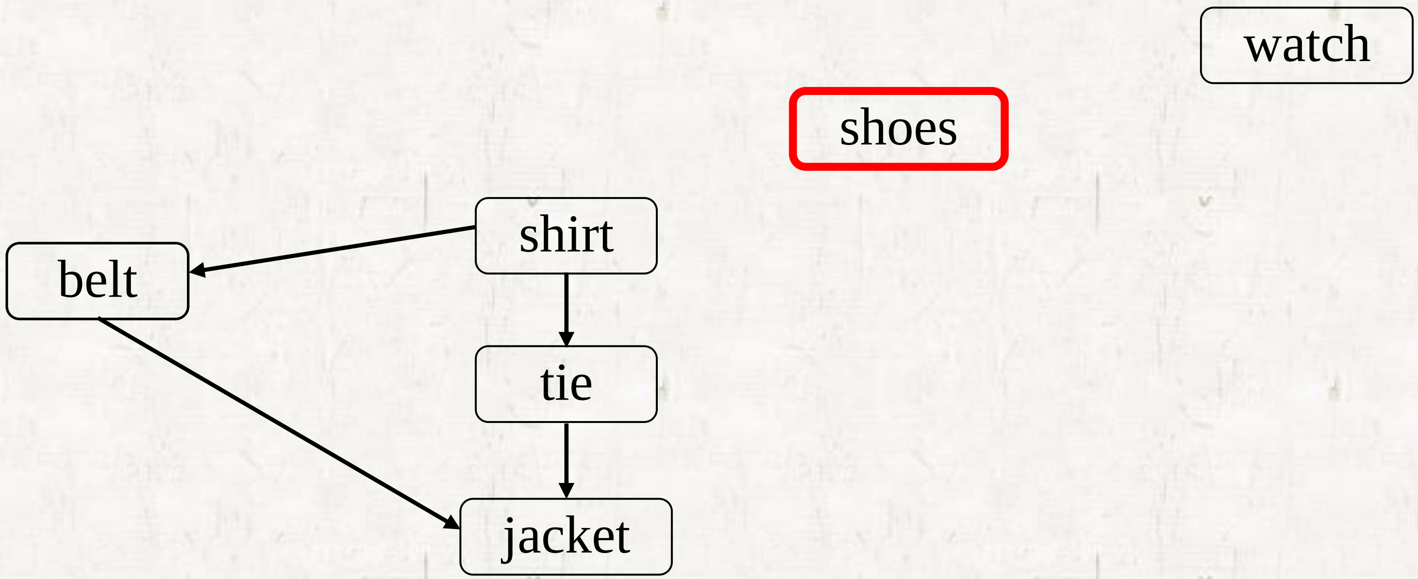
Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	0	0	0	1	1	2

Topological sort

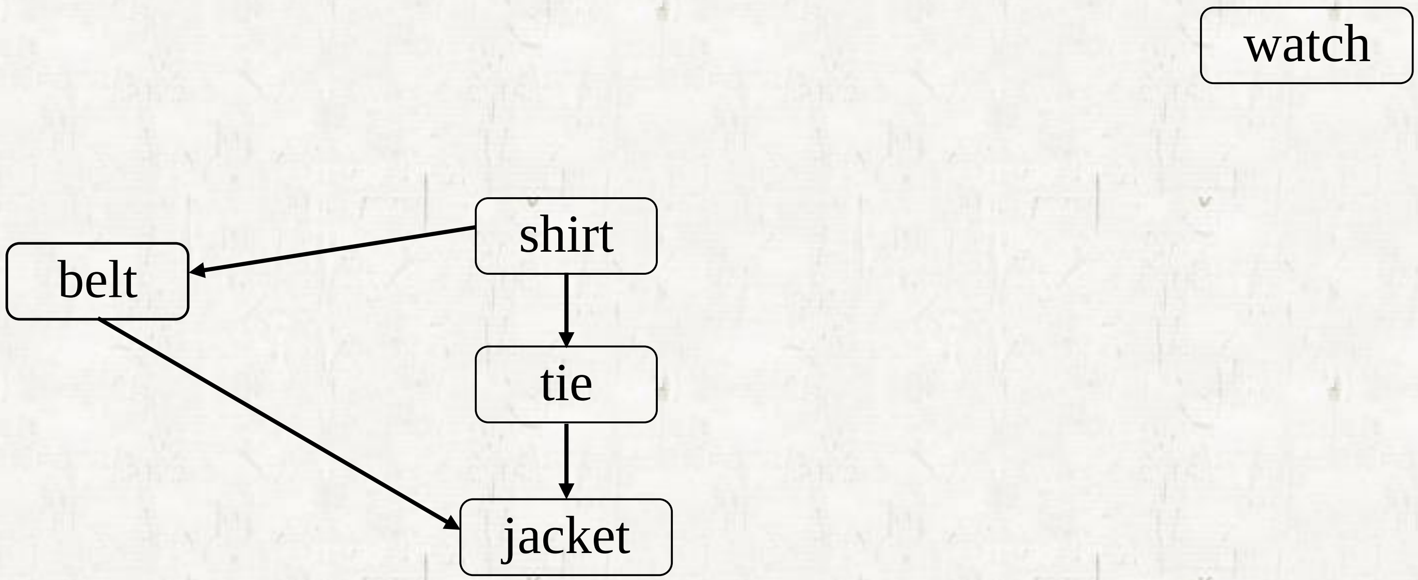


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	0	0	0	1	1	2

socks | under shorts | pants

Topological sort

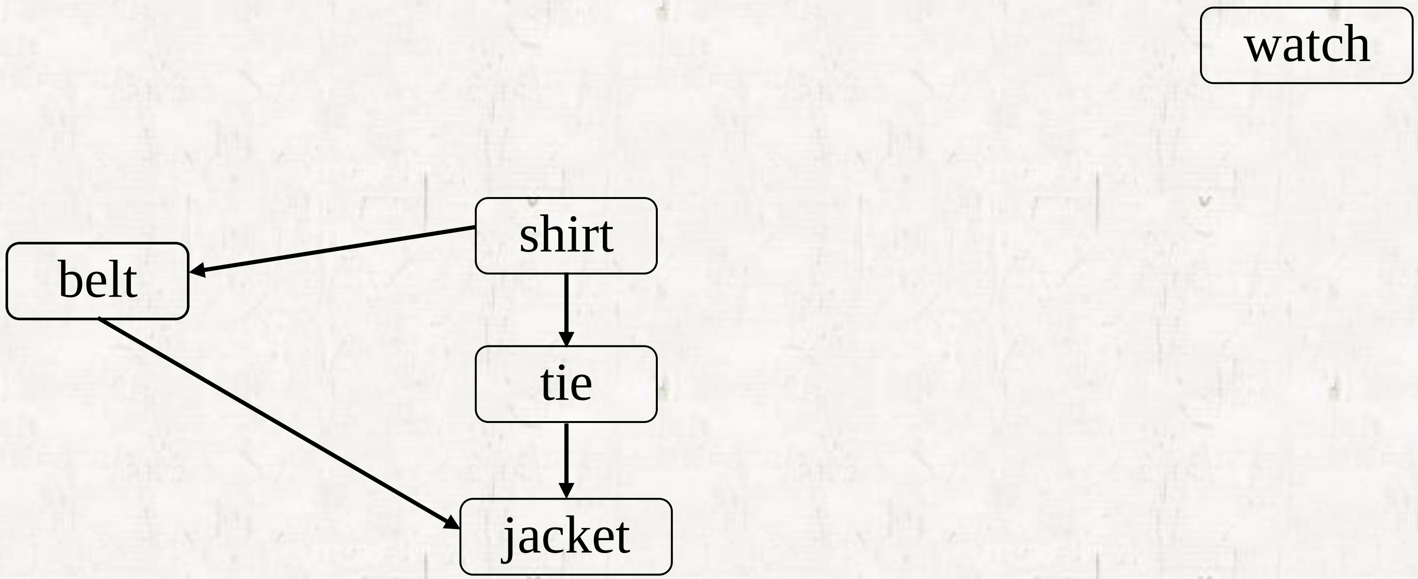


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	0	0	1	1	2

socks under shorts pants shoes

Topological sort

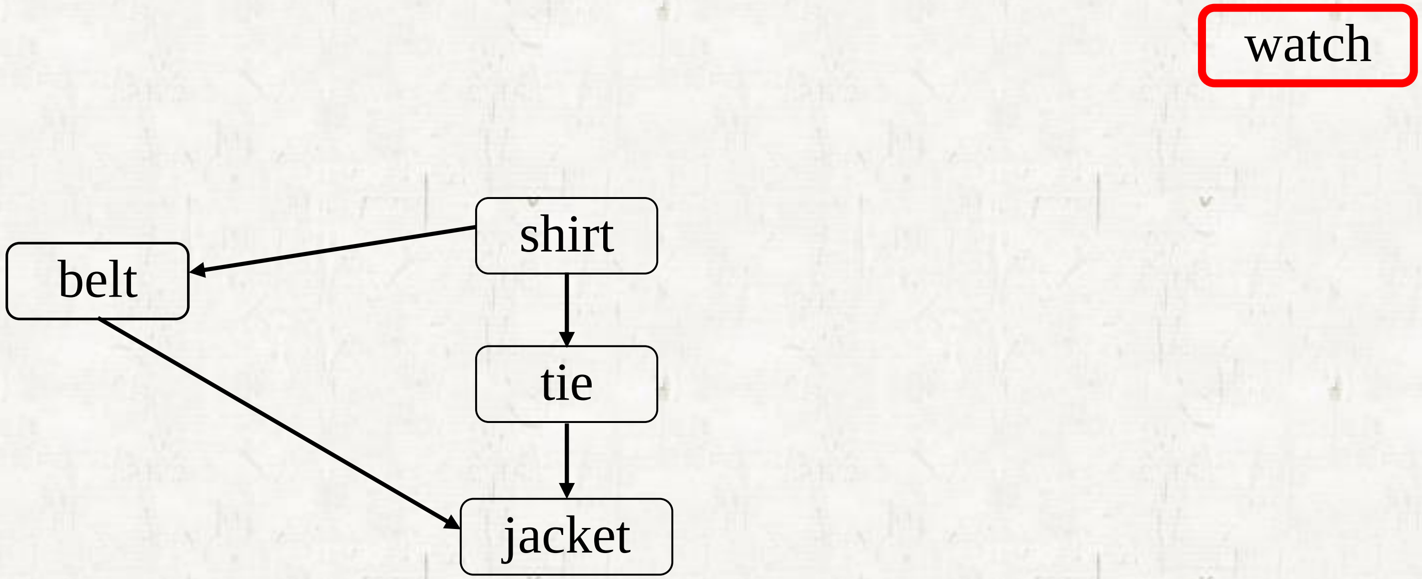


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	0	0	1	1	2

socks | under shorts | pants | shoes

Topological sort

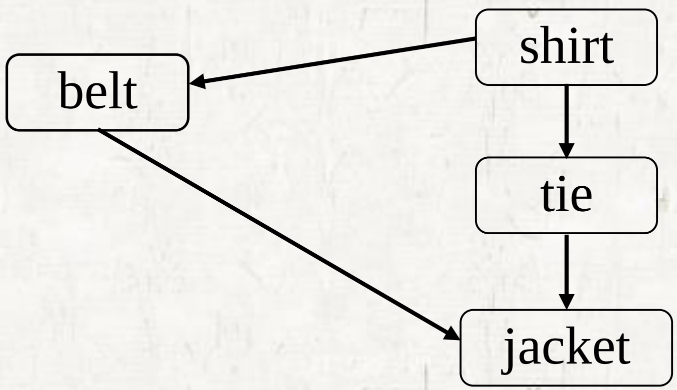


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	0	0	1	1	2

socks under shorts pants shoes

Topological sort

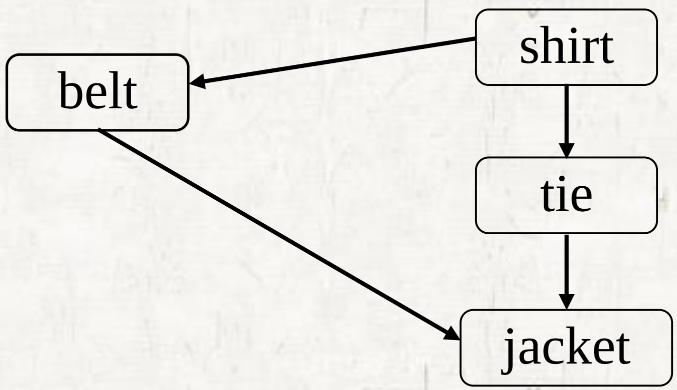


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	0	1	1	2

socks | under shorts | pants | shoes | watch

Topological sort

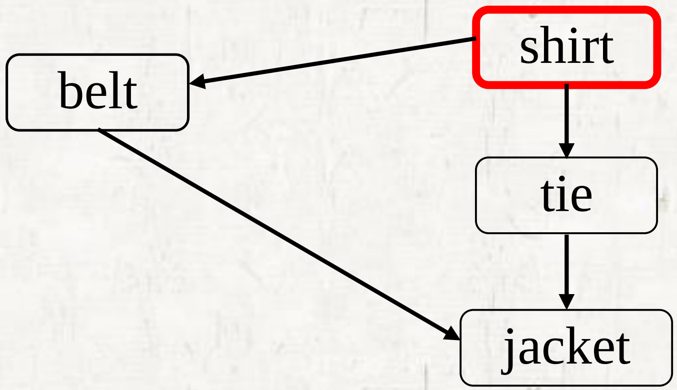


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	0	1	1	2

socks | under shorts | pants | shoes | watch

Topological sort

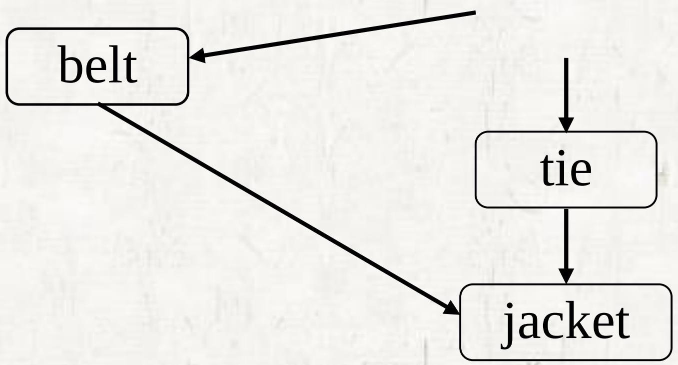


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	0	1	1	2

socks | under shorts | pants | shoes | watch

Topological sort

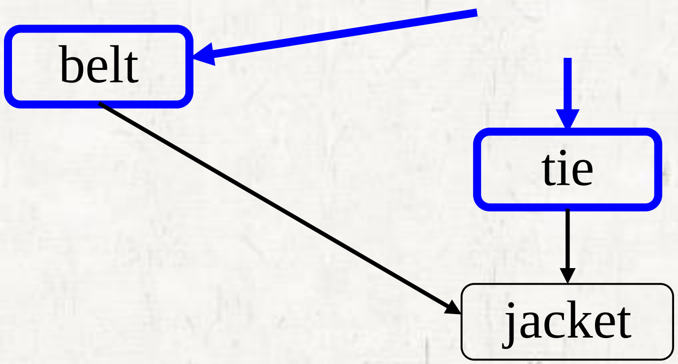


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	1	1	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort

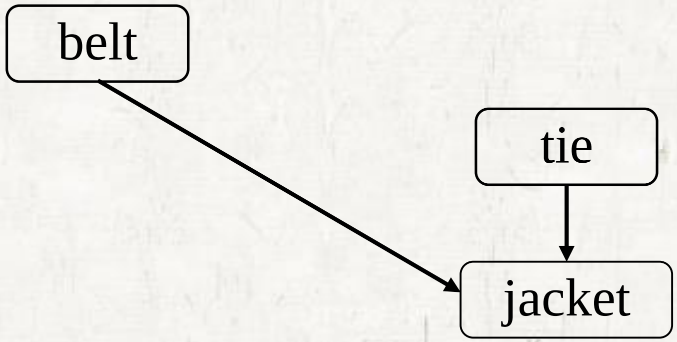


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	1 ➔ 0	1 ➔ 0	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort

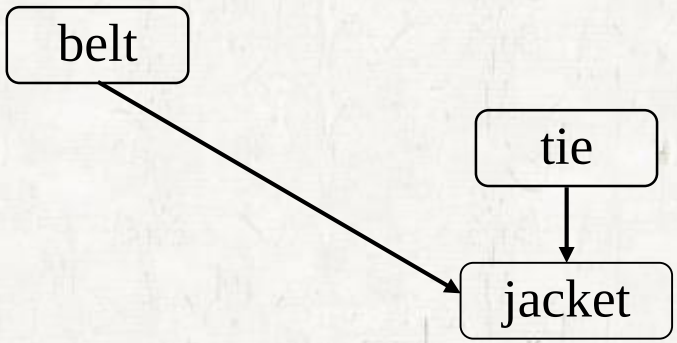


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	0	0	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort

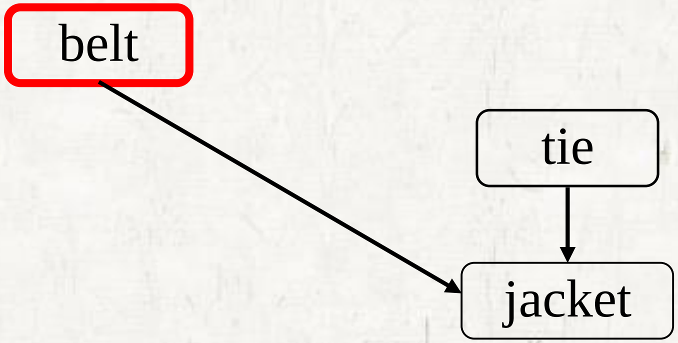


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	0	0	2

socks under shorts pants shoes watch shirt

Topological sort

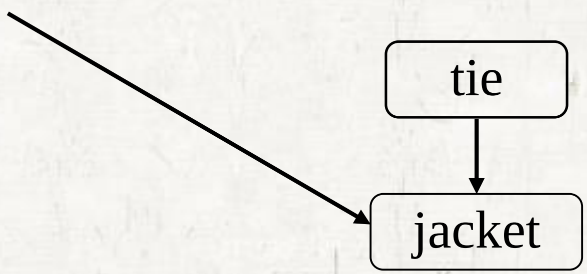


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	0	0	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort

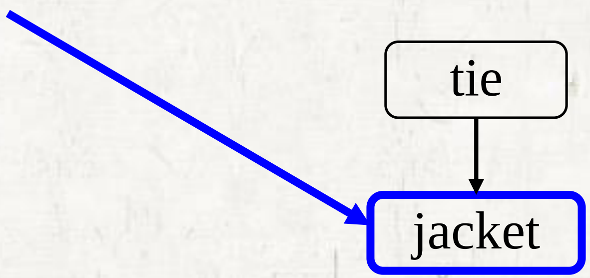


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	0	2

socks | under shorts | pants | shoes | watch | shirt | belt

Topological sort

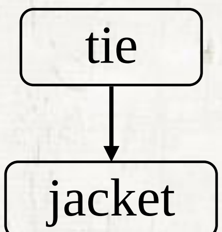


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	0	2 → 1

socks | under shorts | pants | shoes | watch | shirt | belt

Topological sort

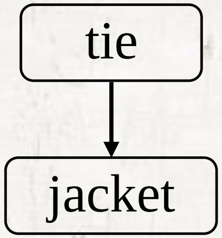


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	0	1

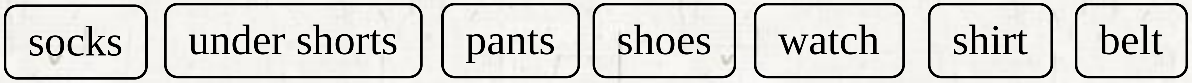
socks | under shorts | pants | shoes | watch | shirt | belt

Topological sort

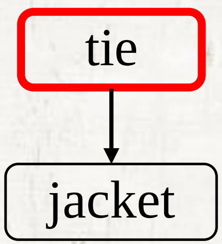


Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	0	1



Topological sort



Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	0	1

socks under shorts pants shoes watch shirt belt

Topological sort

jacket

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	1

socks | under shorts | pants | shoes | watch | shirt | belt | tie

Topological sort

jacket

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	1 → 0

socks | under shorts | pants | shoes | watch | shirt | belt | tie

Topological sort

jacket

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	0

socks under shorts pants shoes watch shirt belt tie

Topological sort

jacket

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	0

socks under shorts pants shoes watch shirt belt tie

Topological sort

jacket

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	0

socks | under shorts | pants | shoes | watch | shirt | belt | tie

Topological sort

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	-1

socks | under shorts | pants | shoes | watch | shirt | belt | tie | jacket

Topological sort

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	-1

socks under shorts pants shoes watch shirt belt tie jacket

Topological sort

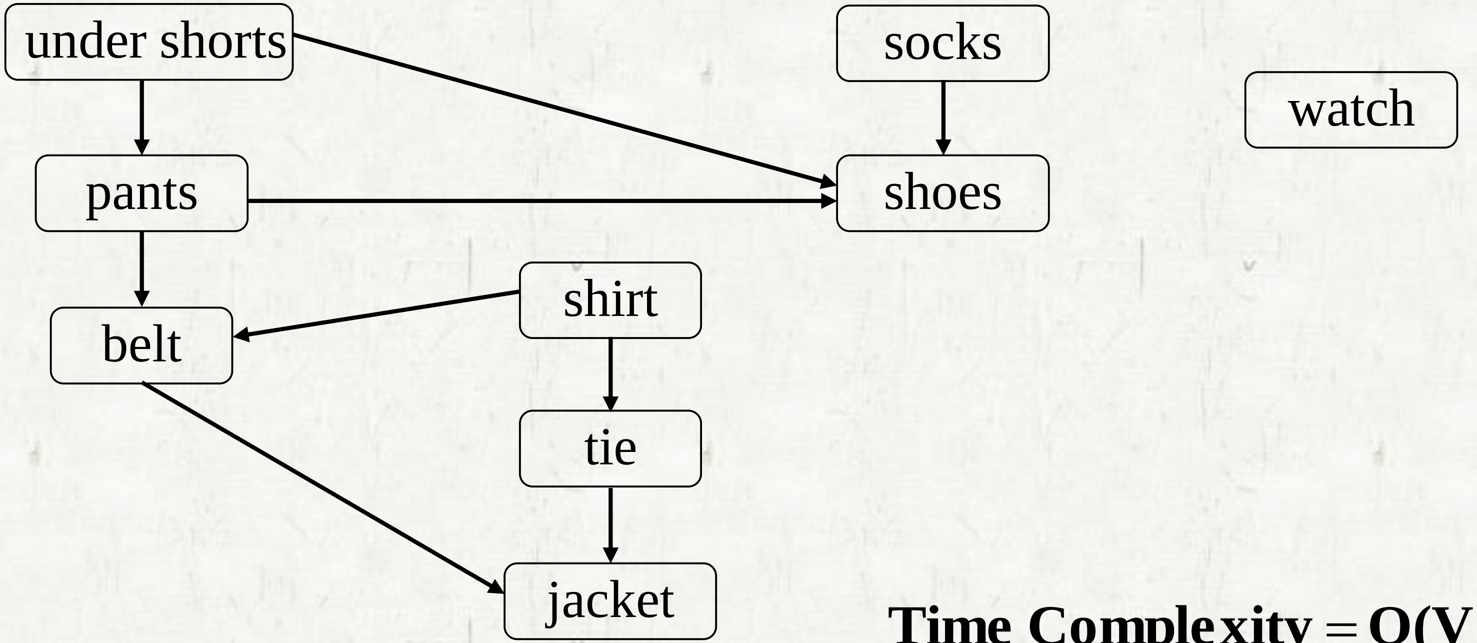
Time Complexity $= O(V^2)$

Indegree Array

socks	under shorts	pants	shoes	watch	shirt	belt	tie	jacket
-1	-1	-1	-1	-1	-1	-1	-1	-1

socks | under shorts | pants | shoes | watch | shirt | belt | tie | jacket

Topological sort

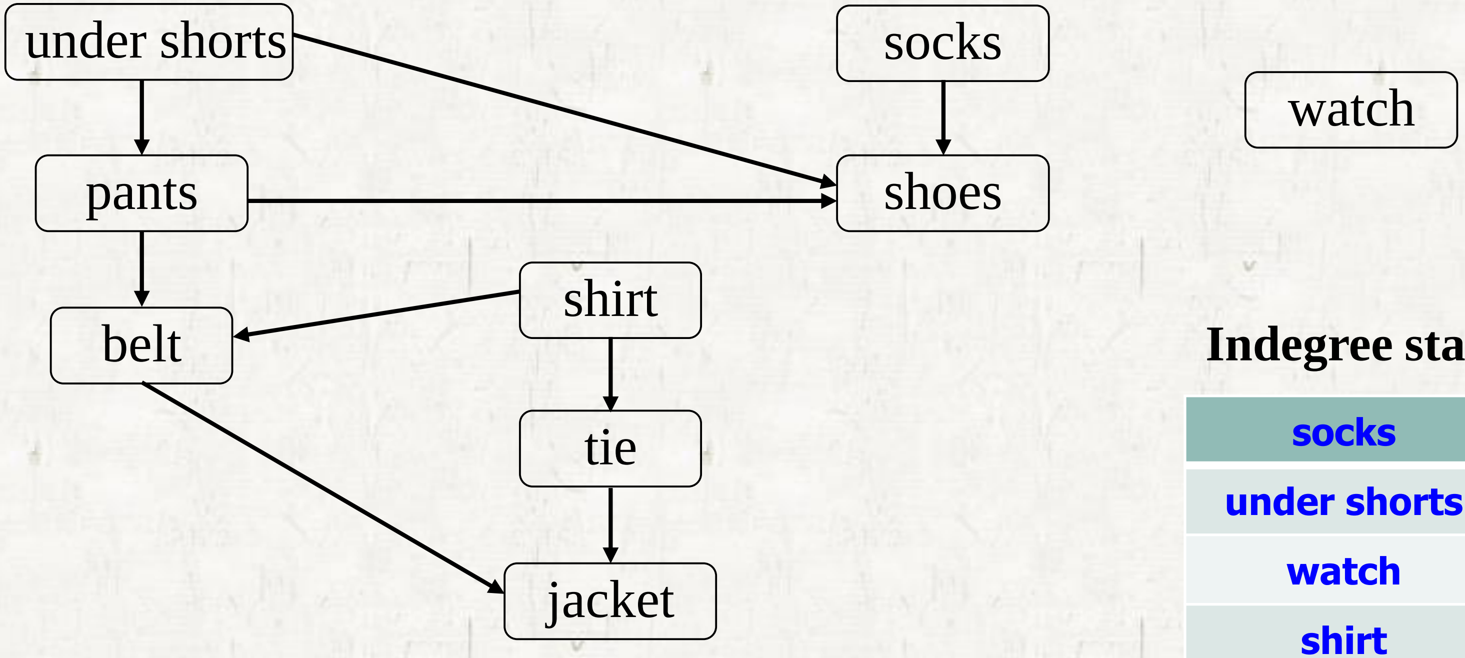


Time Complexity = O(V + E)

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	3	0	0	2	1	2

Topological sort



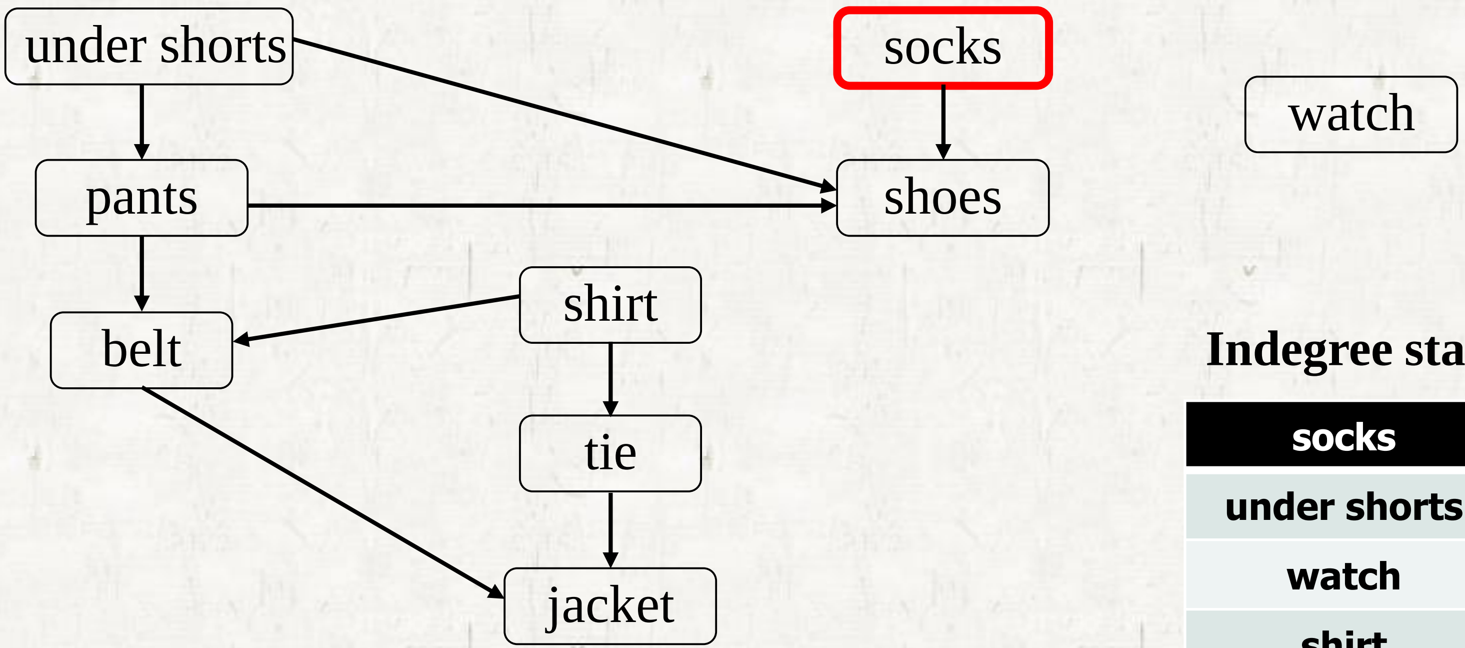
Indegree stack

socks	0
under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	3	0	0	2	1	2

Topological sort



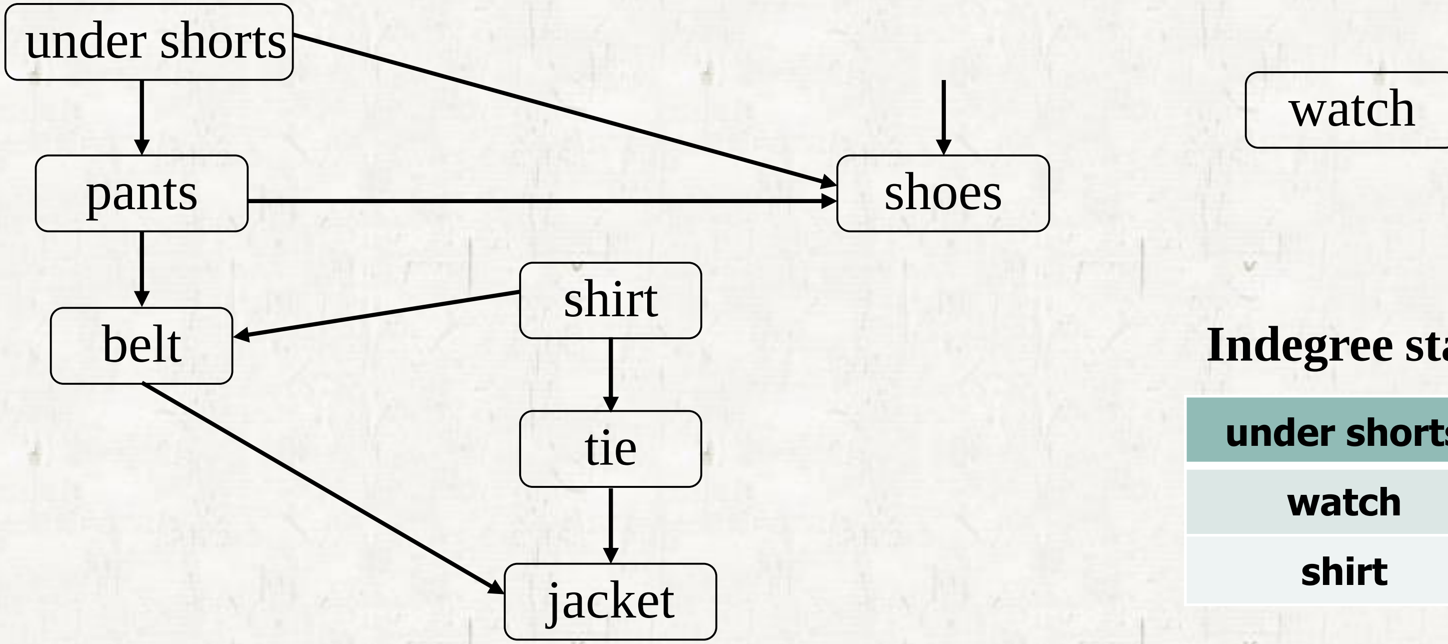
Indegree stack

socks	0
under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	3	0	0	2	1	2

Topological sort



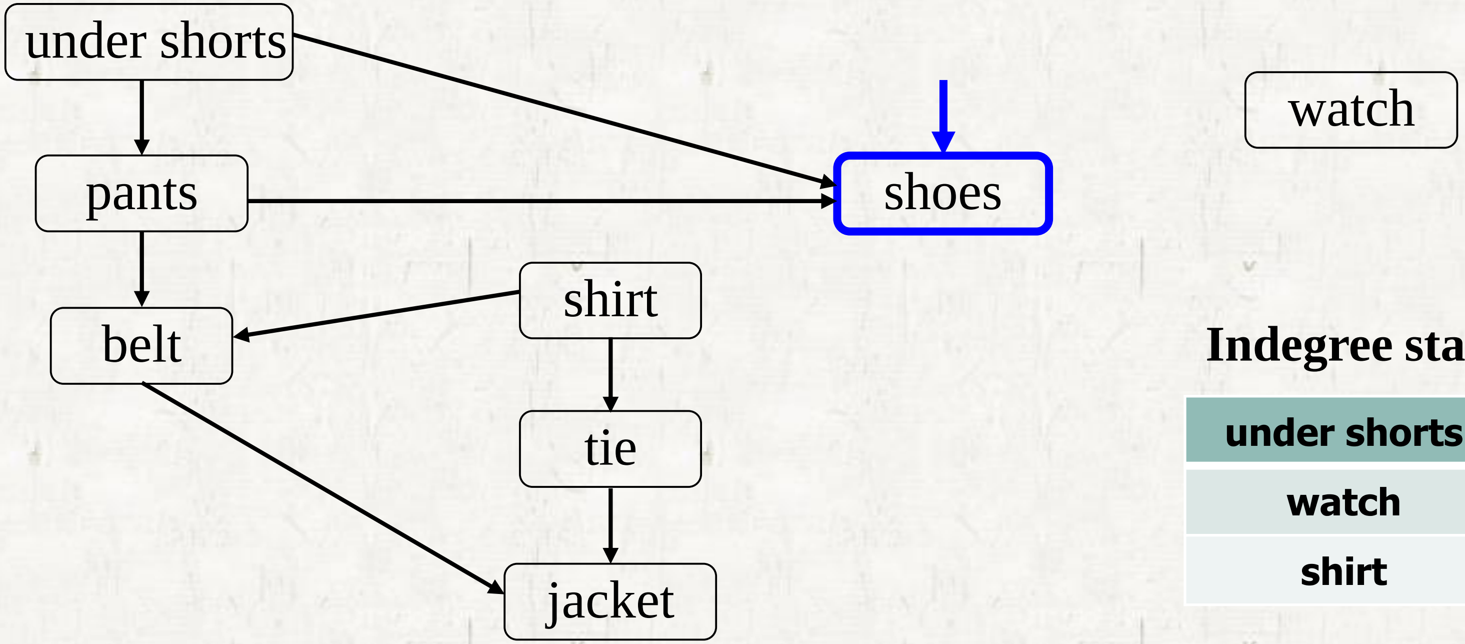
Indegree stack

under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	3	0	0	2	1	2

Topological sort



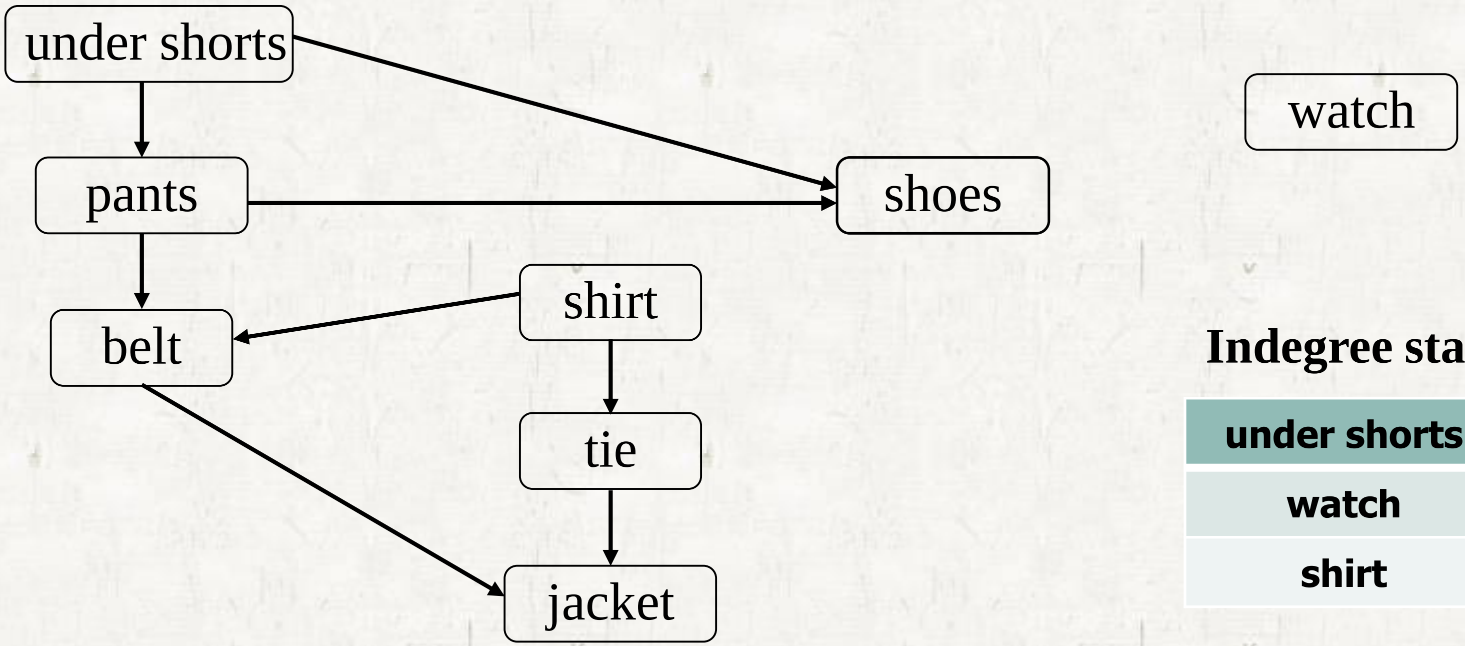
Indegree stack

under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	3 → 2	0	0	2	1	2

Topological sort



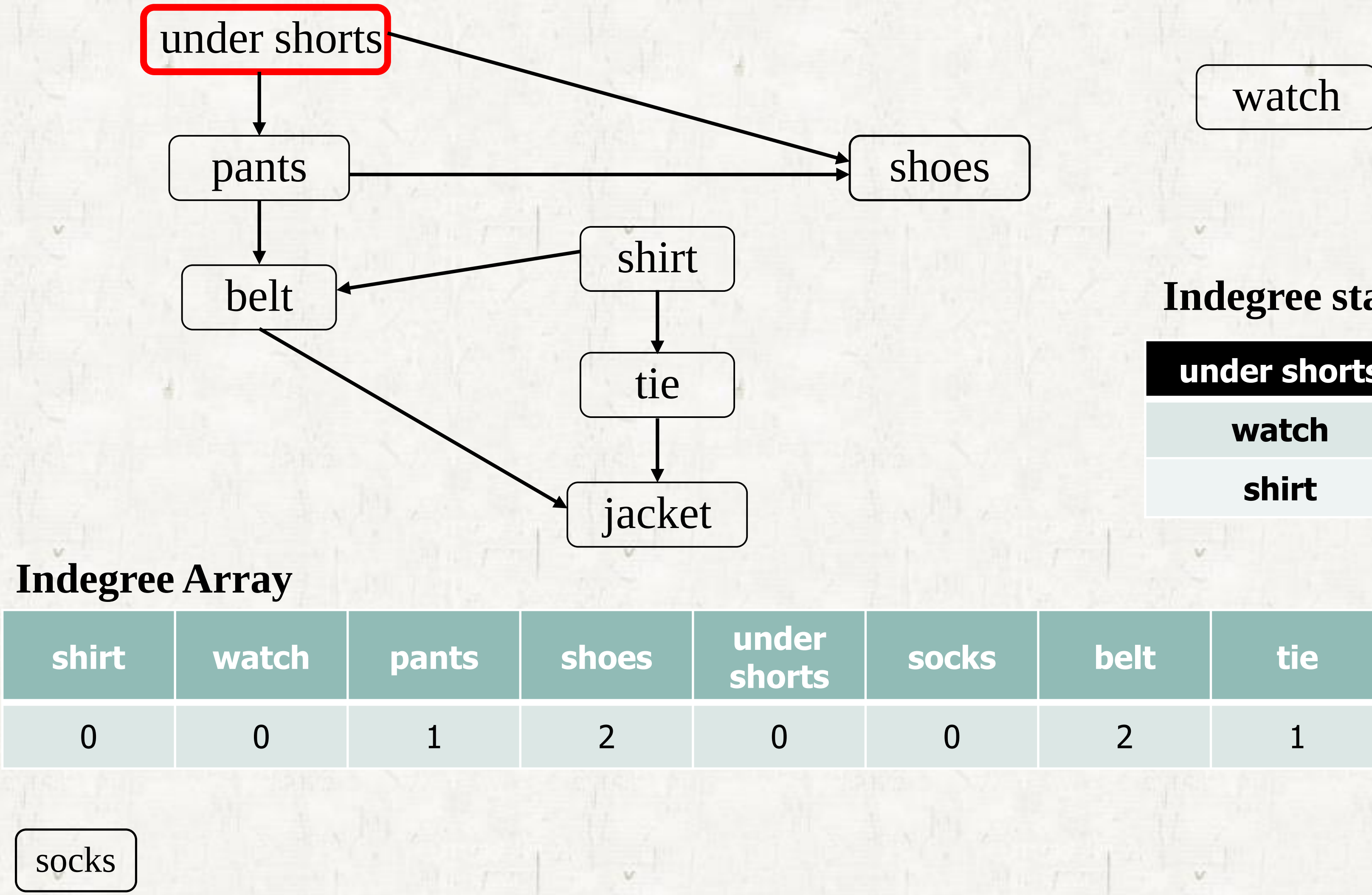
Indegree stack

under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	2	0	0	2	1	2

Topological sort



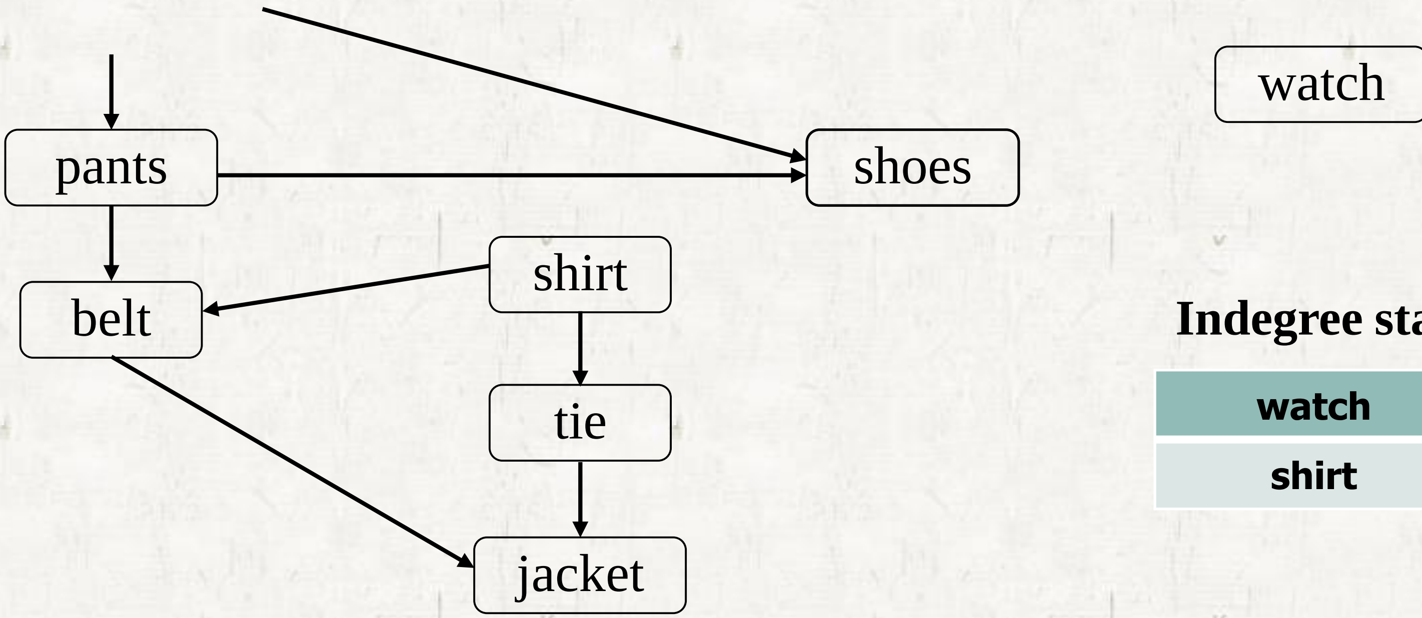
Indegree stack

under shorts	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	2	0	0	2	1	2

Topological sort



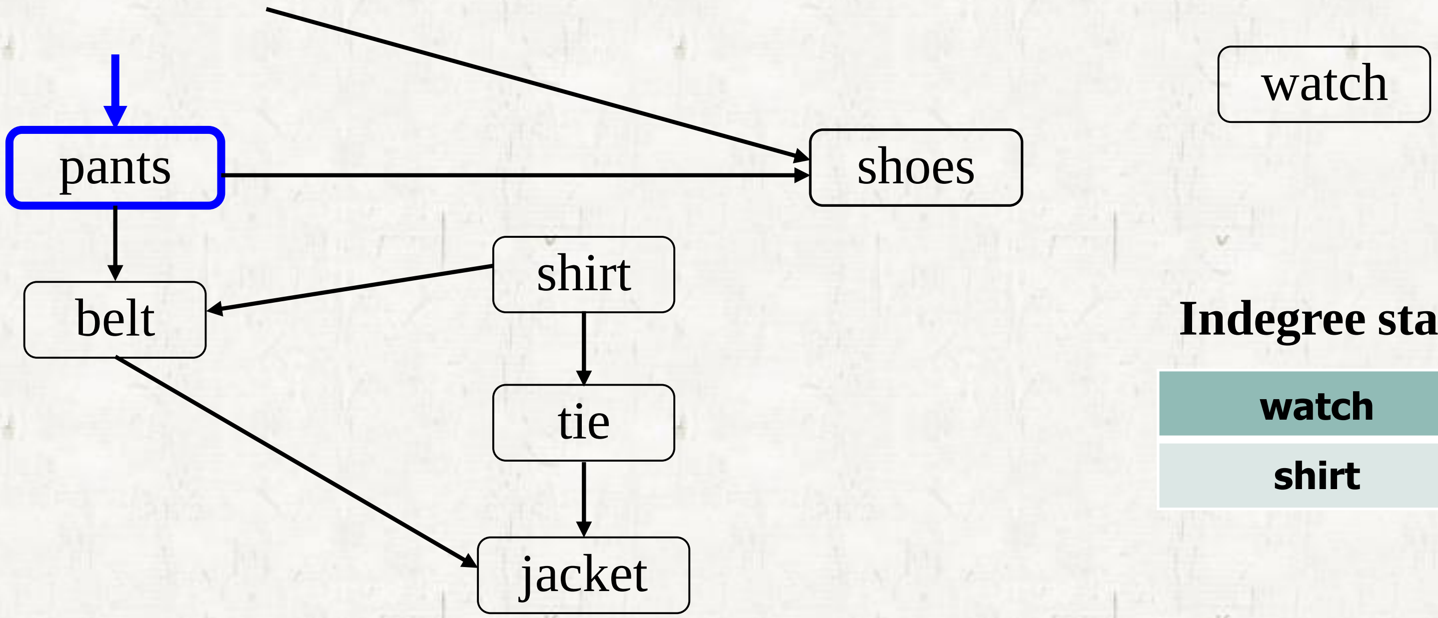
Indegree stack

watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1	2	0	0	2	1	2

Topological sort



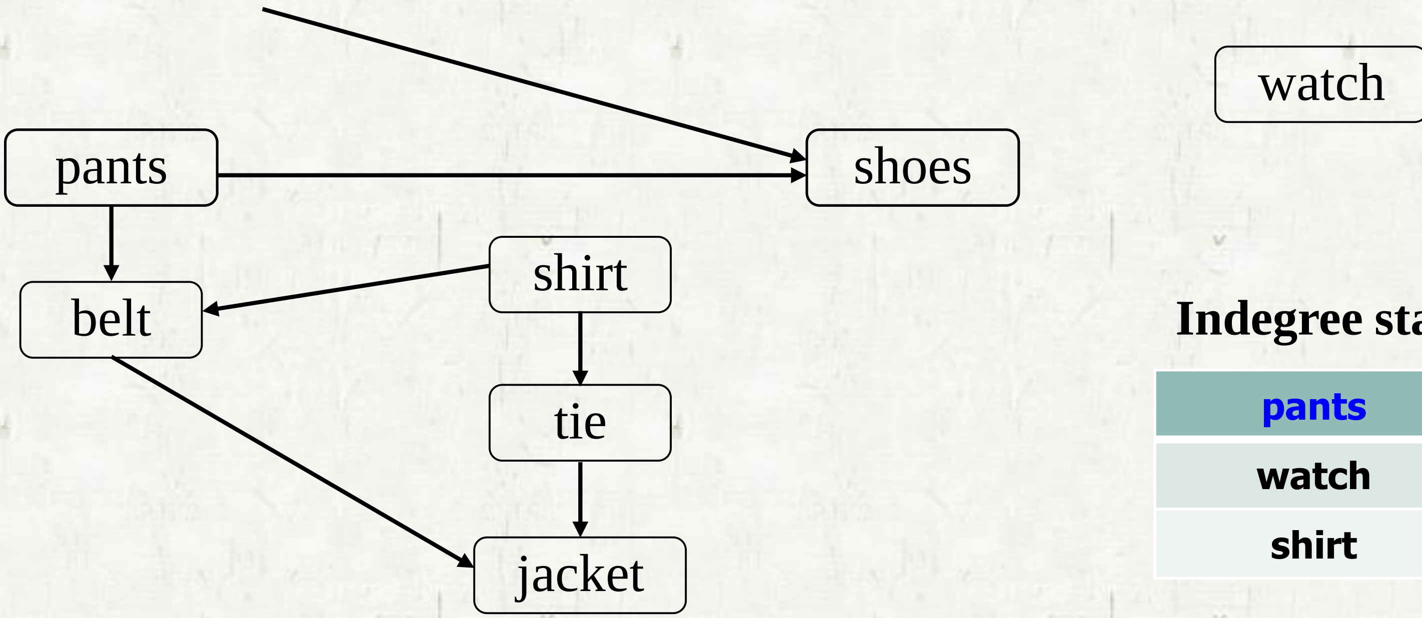
Indegree stack

watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	1 → 0	2	0	0	2	1	2

Topological sort



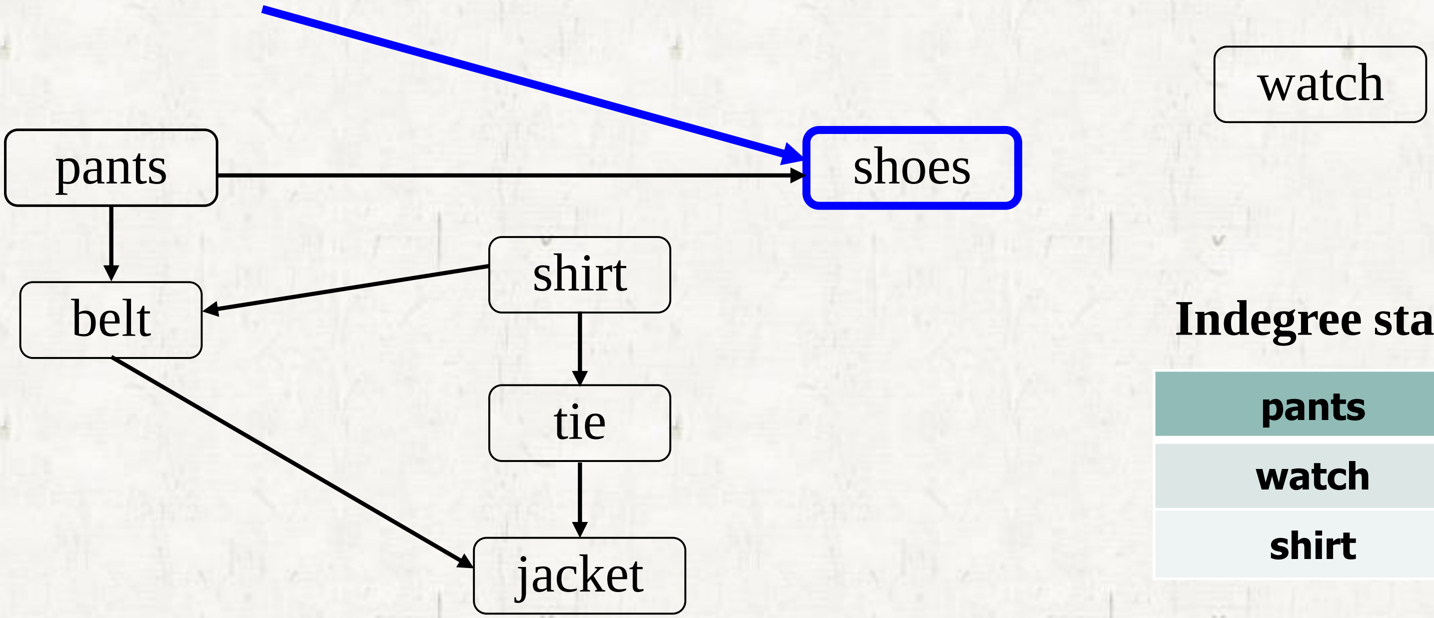
Indegree stack

pants	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	2	0	0	2	1	2

Topological sort



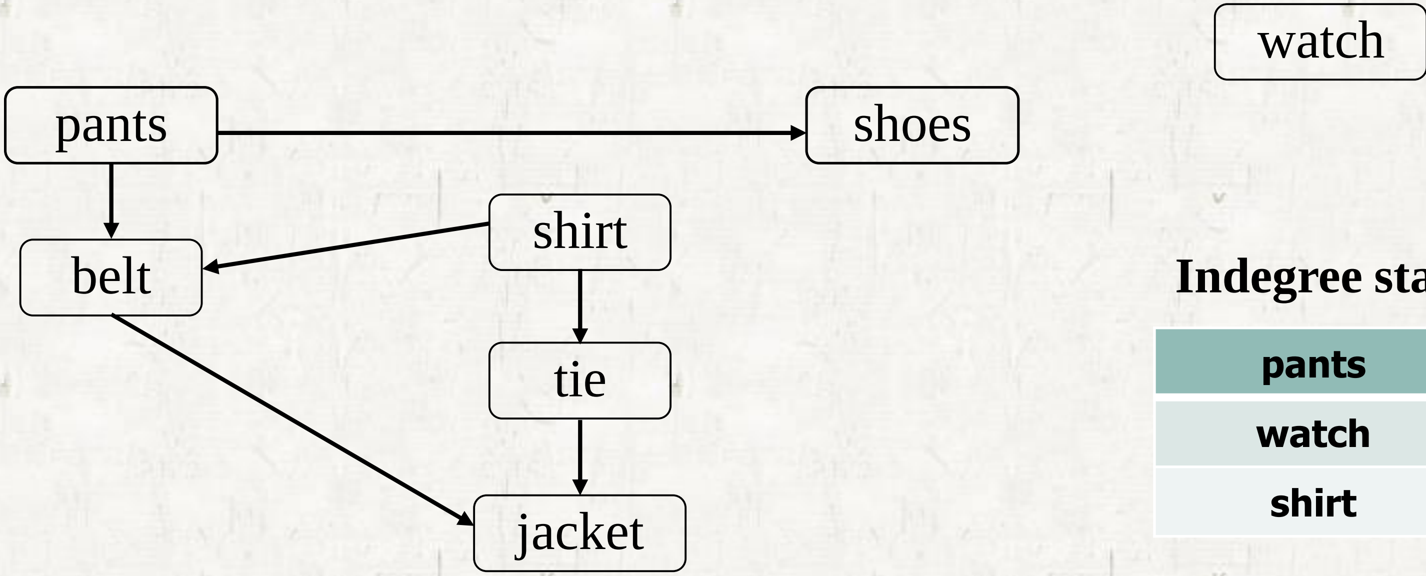
Indegree stack

pants	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	2 → 1	0	0	2	1	2

Topological sort



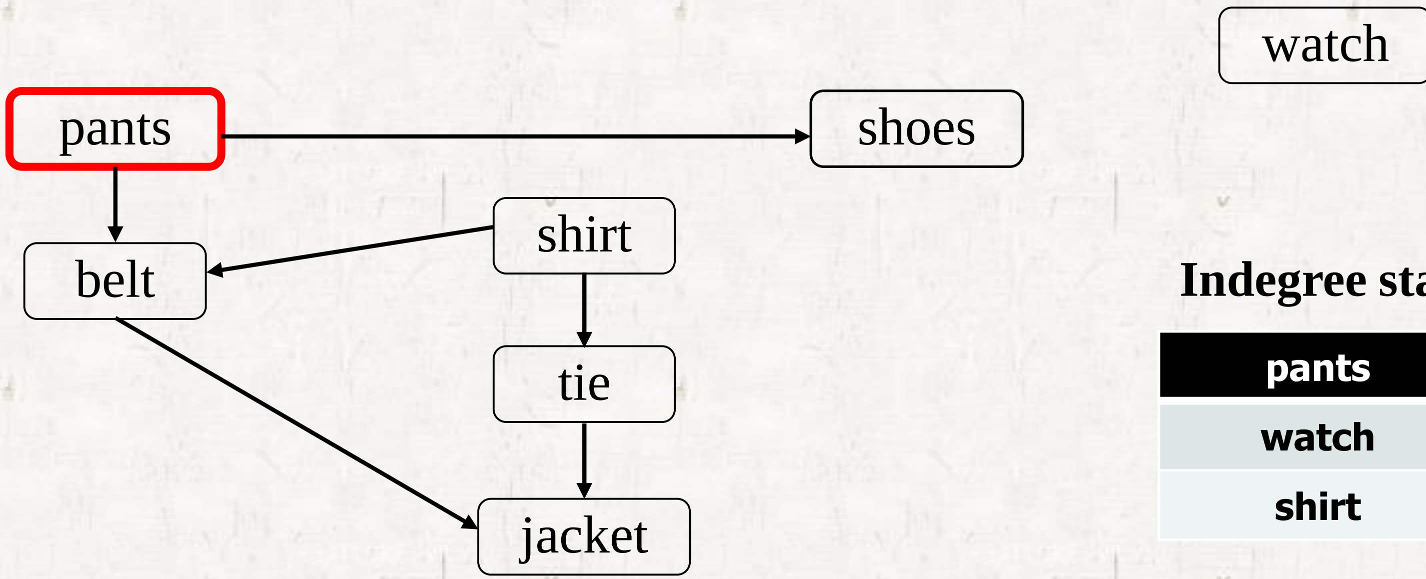
Indegree stack

pants	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	1	0	0	2	1	2

Topological sort



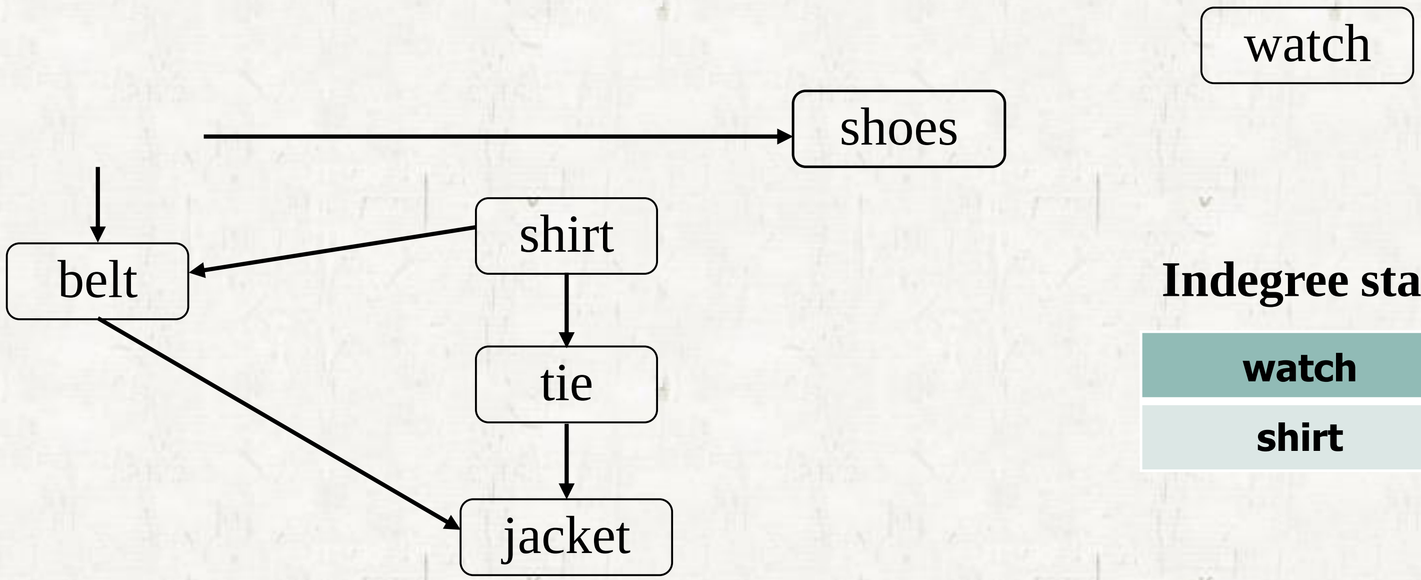
Indegree stack

pants	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	1	0	0	2	1	2

Topological sort



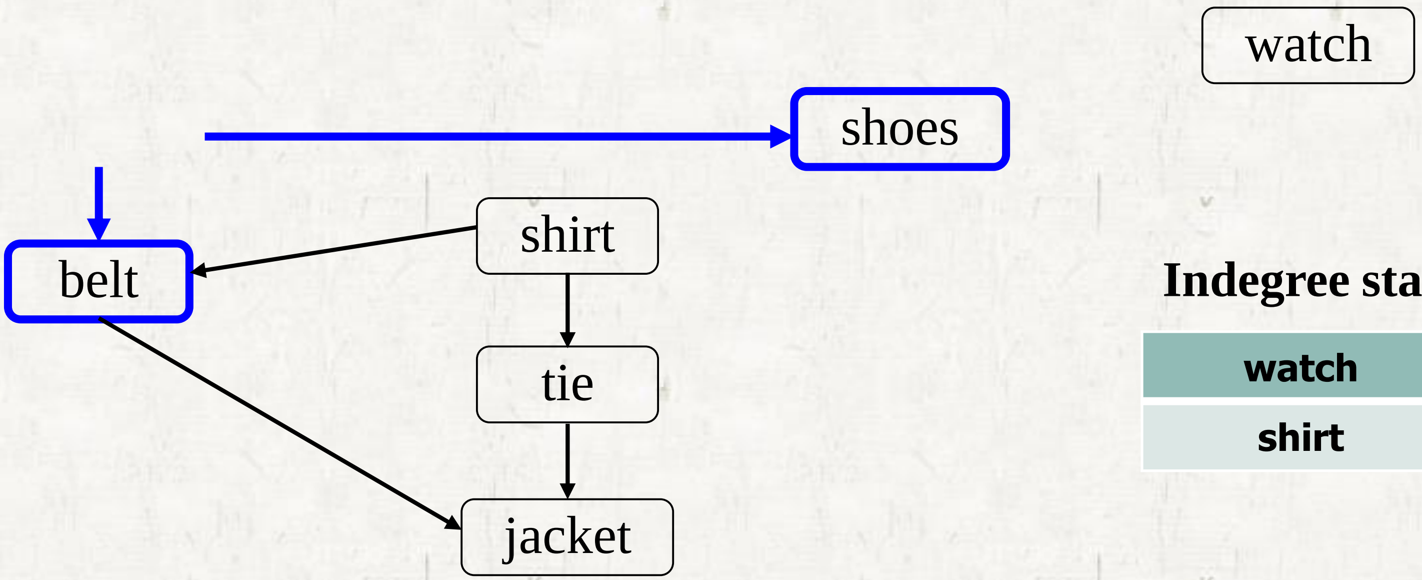
Indegree stack

watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	1	0	0	2	1	2

Topological sort



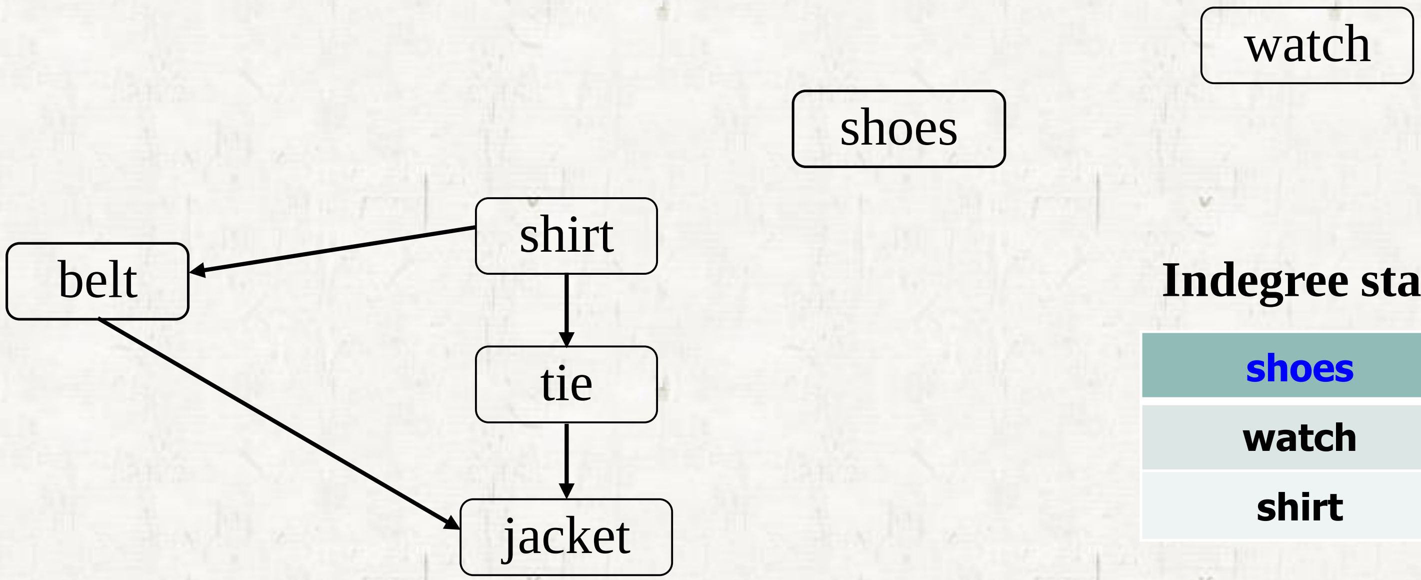
Indegree stack

watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	1 ➔ 0	0	0	2 ➔ 1	1	2

Topological sort



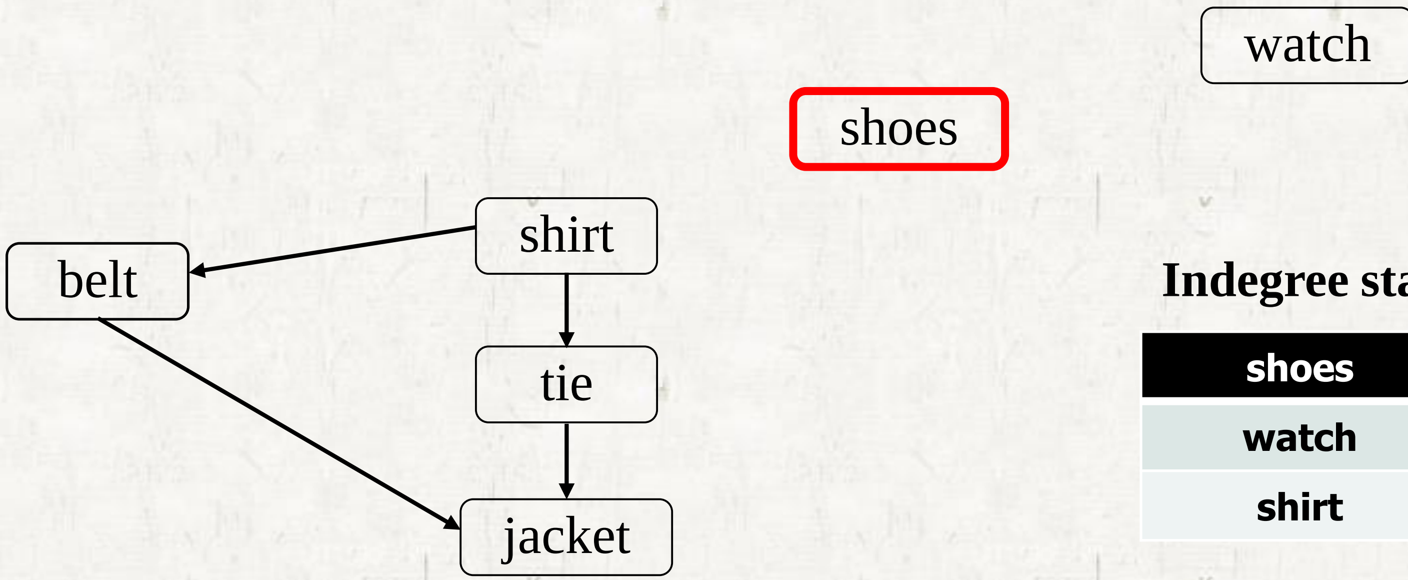
Indegree stack

shoes	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

Topological sort



Indegree stack

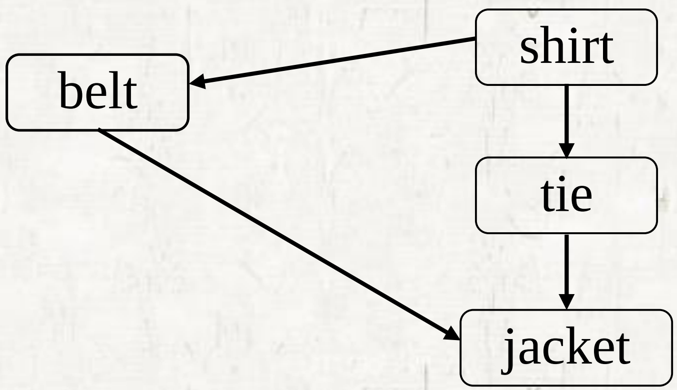
shoes	0
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

Topological sort

watch



Indegree stack

watch	0
shirt	0

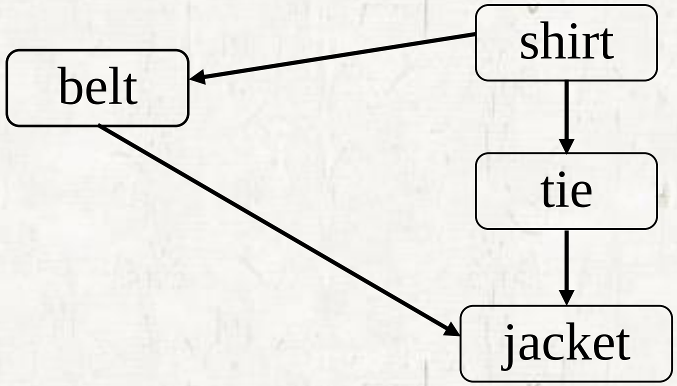
Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

socks under shorts pants shoes

Topological sort

watch



Indegree stack

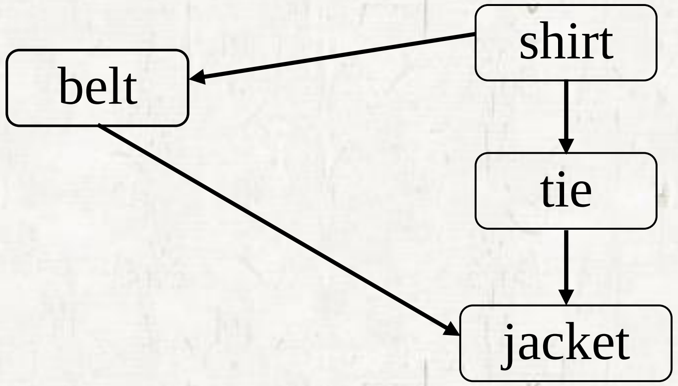
watch	0
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

socks under shorts pants shoes

Topological sort



Indegree stack

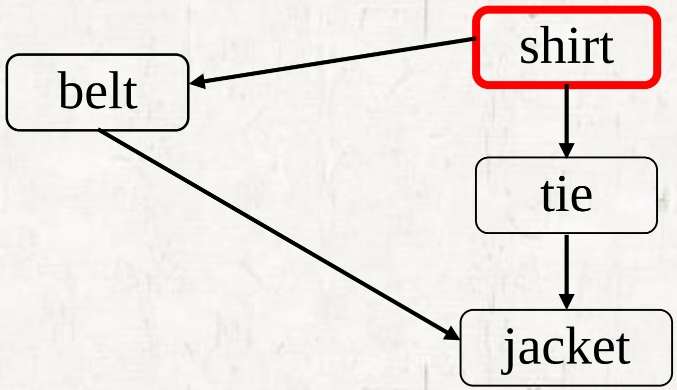
shirt	**0**

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

socks | under shorts | pants | shoes | watch

Topological sort



Indegree stack

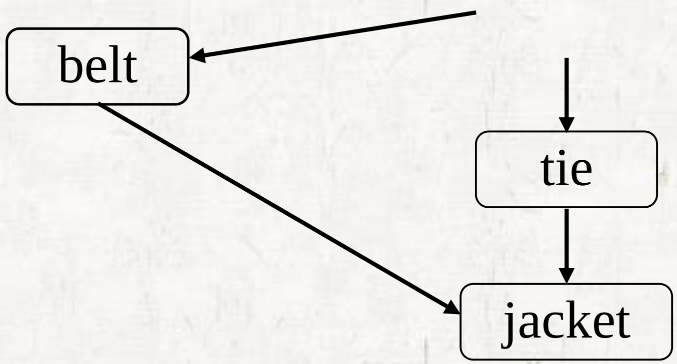
shirt	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

socks | under shorts | pants | shoes | watch

Topological sort



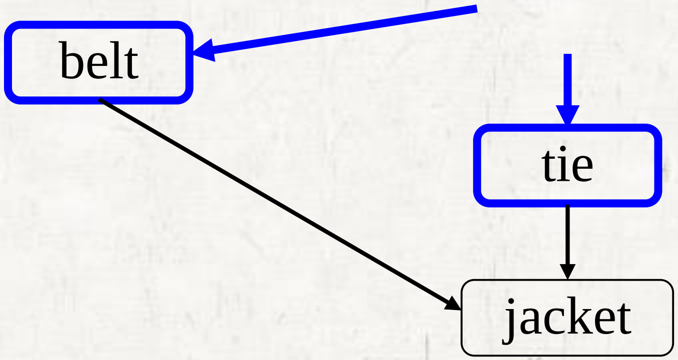
Indegree stack

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1	1	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort



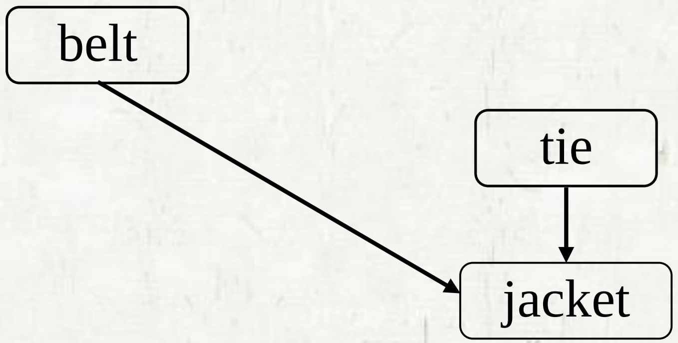
Indegree stack

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	1 → 0	1 → 0	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort



Indegree stack

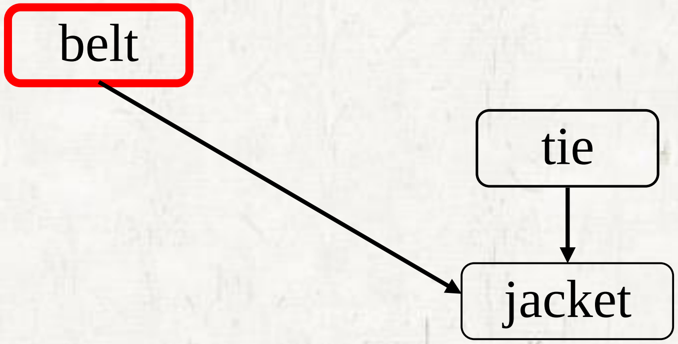
belt	0
tie	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	2

socks | under shorts | pants | shoes | watch | shirt

Topological sort



Indegree stack

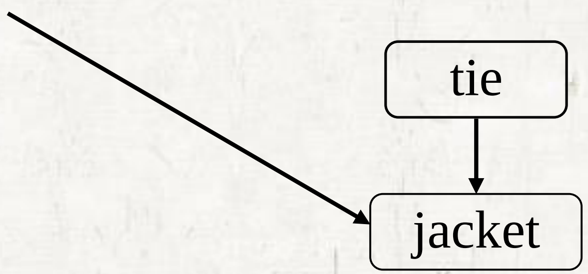
belt	0
tie	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	2

socks under shorts pants shoes watch shirt

Topological sort



Indegree stack

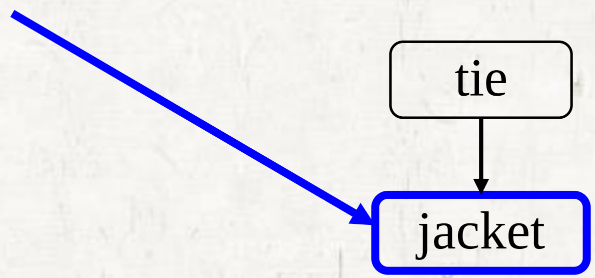
tie	**0**

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	2

socks under shorts pants shoes watch shirt belt

Topological sort



Indegree stack

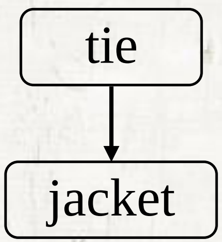
tie	**0**

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	2 → 1

socks under shorts pants shoes watch shirt belt

Topological sort



Indegree stack

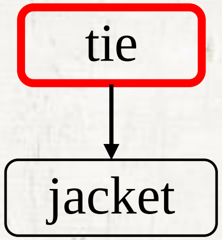
tie	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	1

socks | under shorts | pants | shoes | watch | shirt | belt

Topological sort

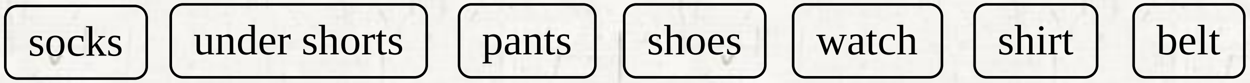


Indegree stack

tie	0

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	1



Topological sort

Indegree stack

jacket

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	1

socks | under shorts | pants | shoes | watch | shirt | belt | tie

Topological sort

Indegree stack

jacket

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	1 → 0

socks under shorts pants shoes watch shirt belt tie

Topological sort

Indegree stack

jacket	0

jacket

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	0

socks | under shorts | pants | shoes | watch | shirt | belt | tie

Topological sort

Indegree stack

jacket	**0**

jacket

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	0

socks under shorts pants shoes watch shirt belt tie

Topological sort

Indegree stack

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	0

socks under shorts pants shoes watch shirt belt tie jacket

Topological sort

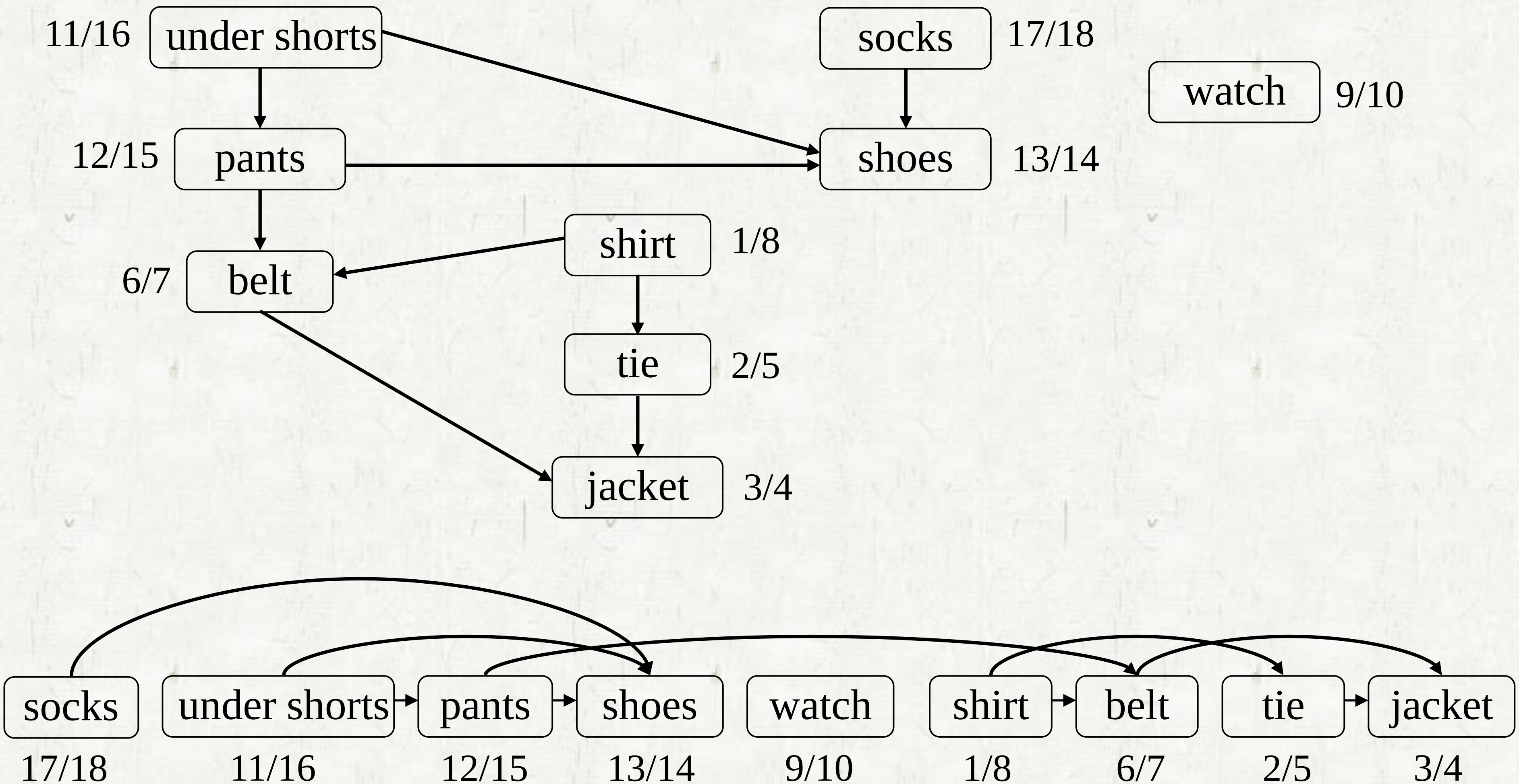
$$\textbf{Time Complexity} = \mathbf{O(V + E)}$$

Indegree Array

shirt	watch	pants	shoes	under shorts	socks	belt	tie	jacket
0	0	0	0	0	0	0	0	0

socks | under shorts | pants | shoes | watch | shirt | belt | tie | jacket

Topological sort



Topological sort

- ***Correctness***
  - If there is an edge from u to v, then $v.f < u.f$.
  - A directed graph G is ***acyclic*** if and only if a depth-first search of G yields ***no back edges***.

Topological sort

- ***Main ideas***
  - Successively place a node from the ***left*** with **0** ***in-degree***.
  - Successively place a node from the ***right*** with **0** ***out-degree***.
  - Run DFS on G and place the nodes from the ***right*** in the ***increasing order of the finishing time***.
  - $\Theta(V+E)$ time

Self-study

- **Exercise 22.4-2**
  - Computing the number of simple paths from s to t in linear time.
- **Exercise 22.4-3**
  - Cycle detection in an undirected graph.
- **Exercise 22.4-5**
  - Another topological sort algorithm.

Programming Assignment

- ***Depth-first search and its applications***
  - **Exercise 22.3-10 (22.3-9, 2nd ed.) (#1)**
    - Depth-first search with edge classification
  - **Exercise 22.3-12 (22.3-11, 2nd ed.) (#2)**
    - Connected component identification
  - **Topological sort (#3)**
    - The program should detect whether the input is a DAG or not.