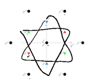
QCD-02 (symmetry & gauge fixing & vacua)

Sunday, February 19, 2023 12:43 PM

Symmetries in QCD

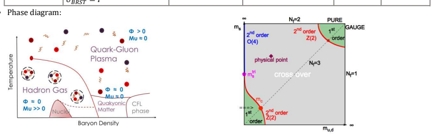
- Covariant derivative
  $D_\mu \equiv \partial_\mu + igA_\mu$ on field under fundamental representation
  $D_\mu \equiv \partial_\mu + ig[A_\mu,\ ]$ on field under adjoint representation
- Lagrangian:
  $\mathcal{L} = \bar{q}(i\partial\cdot\gamma - gA\cdot\gamma)q + \bar{q}Mq - \frac{1}{2}\mathrm{tr}\,F^2$
  $F^{\mu\nu} = \partial^{[\mu}A^{\nu]} + ig[A^\mu, A^\nu]$, $A^\mu \equiv A^\mu_a\tau^a$, $tr\,\tau^a\tau^b = \frac{1}{2}\delta^{ab}$
  $F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a - gf_{bca}A^\mu_b A^\nu_c$
- Members in the inner space:
  Spinor space: $q,\ \gamma$
  $\bar{q} \equiv q^{\%}\gamma^0$, $\bar{T} \equiv \gamma^0 T^{\%}\gamma^0$
  $P_{L/R} \equiv \frac{1}{2}(I \pm \gamma_5)$, $P^2_{L/R} = P_{L/R}$, $P_LP_R = 0$, $\bar{P}_{L/R} = P_{R/L}$, $P_{L/R}\gamma = \gamma P_{R/L}$, $\gamma_5 P_{L/R} = \pm P_{L/R}$
  $q_{L/R} \equiv P_{L/R}q$, $\bar{q}_{L/R}\left(\underbrace{\gamma\cdots\gamma}_{even\ \#\ of\ \gamma}\right)q_{L/R} = 0$, $\bar{q}_{L/R}\left(\underbrace{\gamma\cdots\gamma}_{odd\ \#\ of\ \gamma}\right)q_{R/L} = 0$
  Flavor space: q, M
  Color space: $q,\ A,\ F$



- Euler-Lagrangian equation:
  $(i\gamma^\mu\partial_\mu - gA\cdot\gamma + M)q = 0$
  $\partial_\nu F^{\mu\nu}_a = -g\bar{q}\gamma^\mu\tau_a q + gf_{abc}A_{\nu b}F^{\mu\nu}_c$
  If Lie-algebra is COMPACT → f_{abc} is totally anti-symmetric (see Section "Compactness & Totally …")
  $\to D_\nu F^{\mu\nu} = -g\bar{q}\gamma^\mu q$
  Gauss's Law (no dynamics): $D_i F^{0i}_a = -g\bar{q}\gamma^0\tau_a q$
  $(\partial^\nu\partial^\mu - g^{\mu\nu}\partial^2)A_{a\nu} = -g\bar{q}\gamma^\mu\tau_a q + gf_{abc}A_{\nu b}F^{\mu\nu}_c + gf_{bca}\partial_\nu(A^\mu_b A^\nu_c)$
  Projector with respect to momentum
- Canonical momentum density
  $\pi^i_A \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 A_{ai})} = -2F^{i0}_a$
  $\pi_q \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 q)} = iq^\dagger$
- *Key Observables (Conserved currents & order parameters):*
  - Wilson line (Lorentz scalar): $W(x,y) \equiv \mathcal{P}\exp[i\int_x^y dl\cdot A]$
  - Wilson loop (Lorentz scalar): $W_L \equiv \mathrm{tr}\exp[i\oint dl\cdot A]$
  - Polyakov loop (Well-defined in medium rest frame): $\Phi \equiv \mathcal{P}\exp[-\int_0^\beta d\tau A^0(-i\tau,\vec{x})]$
    $A^0(-i\beta,\vec{x}) = A^0(0,\vec{x})$
    $F_* = -T\ln\langle\mathrm{Tr}\Phi\rangle$
  - Quark condensate (Lorentz scalar): $\langle\bar{q}q\rangle = \langle\bar{q}_L q_R\rangle + \langle\bar{q}_R q_L\rangle$
  - Baryon current density (Lorentz vector): $\langle\bar{q}\gamma^\mu q\rangle = \langle\bar{q}_L\gamma^\mu q_L\rangle + \langle\bar{q}_R\gamma^\mu q_R\rangle$
  - Baryon axial-current density (Lorentz pseudo-vector): $\langle\bar{q}\gamma^\mu\gamma_5 q\rangle = \langle\bar{q}_L\gamma^\mu q_L\rangle - \langle\bar{q}_R\gamma^\mu q_R\rangle$
  - (Hilbert) Energy momentum tensor (Lorentz tensor):
    $T_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}} = -2\mathrm{tr}F_{\mu\sigma}F_\nu^{\ \sigma} + 2i\bar{q}D_{(\mu}\gamma_{\nu)}q - g_{\mu\nu}\left(-\frac{1}{2}\mathrm{tr}F^2 + i\bar{q}D\cdot\gamma q + \bar{q}Mq\right)$
  - Trace anomaly in Minkovsky 4-D (Lorentz scalar): $T^\mu_\mu = -2i\bar{q}D\cdot\gamma q - 4\bar{q}Mq \to -2\bar{q}Mq$
    In quantum level, gluon contributes as well due to anomalous dimension
- Symmetry:

Symmetry	Operation	Exact/Approximate	Breaking	Order parameter	Conserved Charge
$SU_c(3)$ Local	$A \to U_cAU_c^\dagger + ig^{-1}\partial U_cU_c^\dagger$ $F \to U_cFU_c^\dagger$ $q \to U_cq$ $W(x,y)$ $\to U_c(x)W(x,y)U_c^\dagger(y)$ $W_L \to W_L$	Exact			Color charge
Z_3	Same as the above, but $U(-i\beta,\vec{x}) = z_kU(0,\vec{x})$, $z_k^3 = 1$ So that $A'(-i\beta,\vec{x}) = A'(0,\vec{x})$ $Tr\Phi \to z_kTr\Phi$	Exact	@ high temperature	$Tr\ \Phi$; Monopole Condensate	
$SU_V(3)$ Global	$q \to U_Vq$	Approximate			flavor
$SU_A(3)$ Global	$q_L \to U_Aq_L$ $q_R \to U_A^\dagger q_R$ $2\Re U \equiv U_A + U_A^\dagger$ $2i\Im U \equiv U_A - U_A^\dagger$ $\Re U^2 + \Im U^2 = 1$ $q \to (\Re U_A + i\Im U_A\gamma_5)q$ $\bar{q}q \to \bar{q}(\Re U_A + i\Im U_A\gamma_5)^2q$	Approximate	By quark $\bar{q}Mq$ @ low temperature & chemical potential	$\bar{q}q$	Axial-flavor
$U_V(1)$ Global	$q \to e^{i\alpha}q$	Exact			Baryon charge
$U_A(1)$ Global	$q \to e^{i\alpha\gamma_5}q$ $\bar{q}q \to \bar{q}e^{2i\alpha\gamma_5}q$	Exactly Not due to chiral anomaly			
Conformal	Scaling + rotation: e.g., $g^{\mu\nu} \to \mathrm{diag}\{\lambda^0(x),\cdots\}$, $g^\mu_\mu \to \lambda(x)$	Only if M=0	By emergence of Λ_{QCD} @ low temperature & chemical potential	T^μ_μ	
BRST	$U_{BRST} = U_c(-c^a)$ with c being Grassmann number $U^2_{BRST} = I$	Auxiliary			

- Phase diagram:



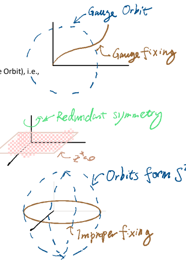
Compactness of Lie algebra & Totally anti-symmetric structure constants

- Lie algebra
  Is a linear algebra with basis = generator
- Inner product
  Given $X = X_a\tau_a$, $Y = Y_a\tau_a$, $\langle X\cdot Y\rangle \equiv \mathrm{tr}[XY] = \langle Y\cdot X\rangle$
- Metric
  $g_{ab} \equiv \mathrm{tr}[\tau_a\tau_b]$
  Under transformation: $\tau'_a = S_{ab}\tau_b$, $g \to g' = SgS^T$
  After transformation, it is possible to make the metric diagonal: $g_{ab} = \lambda_a\delta_{ab}$
- Compactness
  Lie algebra is compact only if $\lambda_a > 0,\ \forall a$
  Rescale the basis so that $g_{ab} = \lambda\delta_{ab}$ ($\lambda = 1/2$ under our choice for SU(3))
- Totally anti-symmetric structure constant
  Under the above choice of basis,
  $\mathrm{tr}[[\tau_a,\tau_b]\tau_c] = if_{abd}\mathrm{tr}[\tau_d\tau_c] = i\lambda f_{abc} \to$
  $f_{abc} = \frac{1}{i\lambda}\mathrm{tr}[\tau_a\tau_b\tau_c - \tau_b\tau_a\tau_c]$
  Invariant under $a \to b \to c \to a$

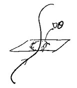
Differential form

- Differential wedge product
  $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$
  $dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \begin{cases} dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_n} & (if\ \{\nu_i\}\to\{\mu_i\}\text{ after even \#of permutation})\\ -dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_n} & (if\ \{\nu_i\}\to\{\mu_i\}\text{ after odd \#of permutation})\end{cases}$
  $dx^\mu \wedge dx_\mu = 0$
- Differential form of a
  - 0-form: a
  - 1-form: $a = a_\mu dx^\mu$
  - 2-form: $a = a_{\mu\nu}dx^\mu\wedge dx^\nu = a_{\nu\mu}dx^\nu\wedge dx^\mu = -a_{\nu\mu}dx^\mu\wedge dx^\nu = \frac{1}{2}a_{[\mu\nu]}dx^\mu\wedge dx^\nu$
  - n-form: $a = \frac{1}{n!}a_{[\mu_1\cdots\mu_n]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$
- Wedge product:
  $a\wedge b \equiv \frac{1}{n!}a_{[\mu_1\cdots\mu_n]}\frac{1}{m!}b_{[\nu_1\cdots\nu_m]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_m} = (-)^{mn}b\wedge a$ (n + m − form)
  $a\wedge(b\wedge c) = (a\wedge b)\wedge c$
- Differentiation:
  $d\wedge a \equiv \frac{1}{n!}\partial_{\mu_0}a_{[\mu_1\cdots\mu_n]}dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \frac{1}{(n+1)!}(\partial a)_{[\mu_0\cdots\mu_n]}dx^{\mu_0}\wedge\cdots\wedge dx^{\mu_n}$
  $d\wedge d\wedge a = 0$
  $d\wedge(f\wedge g) = df\wedge g + (-)^q f\wedge dg$
- Application on gauge field theory
  - Gauge field (1-form): $\mathcal{A} = A_\mu dx^\mu$, $\bar{\mathcal{A}}_{(n)} = ig^{-1}dU_nU_n^\dagger$
  - Covariant differentiation (1-form): $\mathcal{D} = d + ig\mathcal{A}$
  - EM tensor (2-form): $\mathcal{F} = \mathcal{D}\wedge\mathcal{A} = d\wedge\mathcal{A} + ig\mathcal{A}\wedge\mathcal{A}$ $\to \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu = \frac{1}{2}\partial_{[\mu}A_{\nu]}dx^\mu\wedge dx^\nu + \frac{ig}{2}[A^\mu,A^\nu]dx_\mu\wedge dx_\nu \to F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + ig[A_\mu,A_\nu]$
    $\bar{\mathcal{F}}_{(n)} = ig^{-1}d\wedge(dU_nU_n^\dagger) - ig^{-1}dU_nU_n^\dagger\wedge dU_nU_n^\dagger$
    $= -ig^{-1}(dU_n\wedge dU_n^\dagger) - ig^{-1}dU_nU_n^\dagger\wedge dU_nU_n^\dagger = -ig^{-1}dU_n\wedge(dU_n^\dagger + U_n^\dagger dU_nU_n^\dagger)$
    $d(U_n^\dagger U_n) = 0 \to (dU_n^\dagger + U_n^\dagger dU_nU_n^\dagger) = 0 \to \bar{\mathcal{F}}_{(n)} = 0$
  - Chern-Simons form:
    - 1-form: $\omega_1 \equiv ig\mathrm{Tr}\mathcal{A}$
    - 3-form: $\omega_3 \equiv -g^2\mathrm{Tr}\left[\mathcal{F}\wedge\mathcal{A} - \frac{ig}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right]$
    $d\wedge\omega_3 = -g^2\mathrm{Tr}\left[d\mathcal{F}\wedge\mathcal{A} + \mathcal{F}\wedge d\mathcal{A} - \frac{ig}{3}d\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} + \frac{ig}{3}\mathcal{A}\wedge d\mathcal{A}\wedge\mathcal{A} - \frac{ig}{3}\mathcal{A}\wedge\mathcal{A}\wedge d\mathcal{A}\right]$
    $= -g^2\mathrm{Tr}[d\mathcal{F}\wedge\mathcal{A} + \mathcal{F}\wedge d\mathcal{A} - ig\mathcal{A}\wedge\mathcal{A}\wedge d\mathcal{A}]$
    $= -g^2\mathrm{Tr}[igd\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} + \mathcal{F}\wedge d\mathcal{A}]$
    $= -g^2\mathrm{Tr}[igd\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} + \mathcal{F}\wedge\mathcal{F} - ig\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}]$
    $= -g^2\mathrm{Tr}[\mathcal{F}\wedge\mathcal{F} + g^2\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}]$
    Notice
    $\mathrm{Tr}[\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}] = \mathrm{Tr}[A^\mu A^\nu A^\rho A^\sigma]dx_\mu\wedge dx_\nu\wedge dx_\rho\wedge dx_\sigma$
    (due to cyclic of trace) $= \mathrm{Tr}[A^\sigma A^\mu A^\nu A^\rho]dx_\mu\wedge dx_\nu\wedge dx_\rho\wedge dx_\sigma$
    (due to odd # of permutation) $= -\mathrm{Tr}[A^\sigma A^\mu A^\nu A^\rho]dx_\sigma\wedge dx_\mu\wedge dx_\nu\wedge dx_\rho = 0$
    $d\wedge\omega_3 = -g^2\mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}]$

Gauge-fixing conditions



- A simple example:
  $Z = \int dxdydz\ L(x^2+y^2+z^2)$
  Invariant under $(x,y,z) \to (x',y',z') = R(\theta,\phi)(x,y,z)$
  - complete "gauge fixing": $y' = x'^{r_y}$, $z' = x'^{r_z}$ ($x'>0$)
    Gauge fixing condition is orthogonal to ALL (except for unity) SO(3) rotation (Gauge Orbit), i.e.,
    - $R(\theta,\phi)(x', x'^{r_y}, x'^{r_z}) \neq (x', x'^{r_y}, x'^{r_z}),\ \forall\theta,\phi \neq 0,0$
    - $\{R(\theta,\phi)(x', x'^{r_y}, x'^{r_z}) | \theta\in[0,\pi], \phi\in[0,2\pi]\} = R^3$
  - Incomplete "gauge fixing": $z'=0$
    redundant symmetry: SO(2) about z-axis (Gribov ambiguity)
  - Improper "gauge fixing": $x' = \cos\alpha$, $y' = \sin\alpha$, $z' = 0$
    - invariant under SO(2) about z-axis
    - $\{R(\theta,\phi)(x',y',z') | \theta\in[0,\pi], \phi\in[0,2\pi]\} = S^2 \neq R^3$
- Gauge fixing for QCD:
  Gauge transformation on $(q_c, \bar{q}_c, A_\mu, \cdots)$ covers all the field configurations
- Gallery of gauge-fixing conditions:
  - Coulomb gauge
    $\nabla\cdot A_c = 0$
  - Lorentz gauge
    $\partial_\mu A^\mu_c = 0$
    After gauge transformation: $A^\mu \to A'^\mu = U_cA^\mu_c U_c^\dagger + ig^{-1}\partial^\mu U_cU_c^\dagger$
    $\partial_\mu A'^\mu = \left[\partial_\mu U_cU_c^\dagger, A'^\mu\right] + ig^{-1}\partial^\mu\left(\partial_\mu U_cU_c^\dagger\right) \to ig^{-1}D^\mu\left(\partial_\mu U_cU_c^\dagger\right)$
    Redundant symmetry for $D^\mu\left(\partial_\mu U_cU_c^\dagger\right) = 0$
  - Background field gauge
    $\bar{D}_\mu A^\mu_c \equiv \partial_\mu A^\mu_c + ig\left[\bar{A}_\mu, A^\mu_c\right] = 0$
    After gauge transformation:
    $\bar{D}_\mu A'^\mu = \left[\bar{D}_\mu U_cU_c^\dagger, A'^\mu\right] + ig^{-1}\bar{D}^\mu\left(\partial_\mu U_cU_c^\dagger\right)$
    Redundant symmetry for $\left[\bar{D}_\mu U_cU_c^\dagger, A'^\mu\right] + ig^{-1}\bar{D}^\mu\left(\partial_\mu U_cU_c^\dagger\right) = 0$
  - Temporal gauge
    $A^0_c = 0$
    After gauge transformation:
    $A'^0 = ig^{-1}\partial^0 U_cU_c^\dagger$
    Redundant symmetry for $\partial^0 U_cU_c^\dagger = 0$
  - Gervais-Neveu gauge
    $\partial_\mu A^\mu_c + (gA^\mu_c A_{+\mu}) = 0$
  - Unitary gauge (for U(1) gauge theory with Higgs)
    - Lagrangian: $\mathcal{L} = D^\mu\phi^\dagger D_\mu\phi - \frac{1}{4}F^2 + V(\phi^\dagger\phi)$
      Invariant under $\phi \to \phi' = e^{i\alpha}\phi$, $A \to A' = A - g^{-1}\partial\alpha$
    - Fixing condition: $\phi_- = |\phi|$ (no imaginary + positive real, a stationary gauge fixing, good @ IR)
      For given $|\phi|$, photon looks massive with mass $= |\phi|$
      After gauge transformation: $\phi' = e^{i\alpha}|\phi| \neq \phi$, ($\alpha \neq 2n\pi$)
    - Complete, except @ the location where $|\phi| = 0$ ← Improper (invariant under U(1) trans.)
      Singularity @ $(t,\vec{r})$ given by $\Re\phi(t,\vec{r}) = 0$, $\Im\phi(t,\vec{r}) = 0$, to form a sheet in 4D or a string in 3D.
      Strings are everywhere in the phase where $\langle|\phi|\rangle = 0$
      Gauge dependent: location of string. Gauge independent: existence of string
    - Gauge fixed A field surrounding (very small $\vec{r}_\perp$) the string
      $\begin{pmatrix}\Re\phi(0,0,\vec{r}_\perp)\\ \Im\phi(0,0,\vec{r}_\perp)\end{pmatrix} = R\begin{pmatrix}r_x\\ r_y\end{pmatrix} + \mathcal{O}(\vec{r}_\perp^2)$
      By rotating & rescaling the coordinates, we might have $R = \varrho I$
      $\begin{pmatrix}\Re\phi\\ \Im\phi\end{pmatrix} = \varrho|\vec{r}_\perp|\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \mathcal{O}(\vec{r}_\perp^2)$
      Or $\phi = \varrho|\vec{r}_\perp|e^{i\theta} + \mathcal{O}(\vec{r}_\perp^2)$
      Gauge fixed $\phi_- = \varrho|\vec{r}_\perp| = e^{-i\theta}\phi$, $\vec{A}_- = \vec{A} + g^{-1}\nabla\theta$
      $\nabla\theta = \frac{(-r_y, r_x)}{r_\perp^2}$, $\oint dl\nabla\theta = 2\pi$
      $\oint d\vec{l}\cdot\vec{A}_- = \oint d\vec{l}\cdot\vec{A} + g^{-1}2\pi \leftarrow$ *magnetic flux tube*



  - Smoothed unitary gauge
    $\arg\phi + \kappa\partial\cdot A = 0$
    - Return to unitary gauge if $\kappa = 0$
    - Reduce to Lorentz gauge @ $\phi = 0$
  - Abelian gauge (1981, t'Hooft)
    - Introduce a scalar in the color adjoint representation: $\Phi = \sum_{\alpha=1}^{N_c^2-1}\Phi^\alpha\tau_\alpha$ (Hermitian + traceless)
      $N_c - 1$ among $N_c^2 - 1$ generators are diagonal.
      Under transformation: $\Phi \to \Phi' = U\Phi U^\dagger$
      Φ is taken as F_{12} in t'Hooft's original paper
    - Fixing condition: $\Phi_- = \mathrm{diag}[\lambda_1, \lambda_2\cdots\lambda_{N_c}]$, $\lambda_1 + \lambda_2 + \cdots + \lambda_{N_c} = 0$, λs are gauge invariant.
      $N_c^2 - N_c$ equations (restrict the non-diagonal elements to zero)
      Redundant symmetry: $U_R = \mathrm{diag}[e^{i\alpha_1}, e^{i\alpha_2}, \cdots, e^{i\alpha_{N_c}}]$, $U_R\Phi_-U_R^\dagger = \Phi_-$
      $1 = \det U_R \to \alpha_1 + \alpha_2 + \cdots + \alpha_{N_c} = 0$, $U_R \in U(1)^{N_c-1} \subset SU(N_c)$
    - Singularity determined by degeneracy, e.g., $\lambda_1(t,\vec{r}) = \lambda_2(t,\vec{r}) \Rightarrow \Phi^* = \Phi_{*0}\tau_0 + \cdots$ ($\Phi_{*3} = 0$)
      $\Phi_* = U_2\Phi_-U_2^\dagger$, $U_2 = \exp\sum_{a=1}^3 i\varphi_a\tau_a \in SU(2) \subset SU(N_c)$
      Located @ $\Phi^{1,2,3}(t,\vec{r}) = 0$ (Note: if any of them is non-zero, it will turn to finite Φ_{*3} after diagonalization), form a string in 4D or a dot in 3D.
      Gauge dependent: location of singularity Gauge independent: existence of singularity
    - Field configurations surrounding singularity (very small r):
      $\begin{pmatrix}\Phi_1(\vec{r})\\ \Phi_2(\vec{r})\\ \Phi_3(\vec{r})\end{pmatrix} = R\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \mathcal{O}(\vec{r}^2)$
      By rotating & rescaling the coordinates, we might have $R = \varrho I$
      $\begin{pmatrix}\Phi_1(\vec{r})\\ \Phi_2(\vec{r})\\ \Phi_3(\vec{r})\end{pmatrix} = \varrho r\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} + \mathcal{O}(\vec{r}^2)$ or
      $\Phi(r) = \begin{pmatrix}\lambda + \varrho r\cos\theta & \varrho r\sin\theta e^{-i\varphi} & \cdots\\ \varrho r\sin\theta e^{i\varphi} & \lambda - \varrho r\cos\theta & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix} = U_2\begin{pmatrix}\lambda + \varrho r & 0 & \cdots\\ 0 & \lambda - \varrho r & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix}U_2^\dagger = U_2\Phi_*(r)U_2^\dagger \to$
      $U_2 = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\varphi} & -\sin\frac{\theta}{2} & \cdots & 0\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{i\varphi} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}$
      Gauge fixed field:
      $A^\mu_* = U_2^\dagger AU_2 + ig^{-1}\partial^\mu U_2^\dagger U_2$, $\partial^\mu \to -\frac{\hat{\theta}}{r}\partial_\theta - \frac{\hat{\varphi}}{r\sin\theta}\partial_\varphi$
      $\vec{A} \equiv (g^{-1}\partial^\mu U_2^\dagger U_2 \to g^{-1}\left[(\sin\varphi\tau_1 + \cos\varphi\tau_2)\frac{\hat{\theta}}{r} + \left(-\cos\varphi\tau_1 + \sin\varphi\tau_2 + \frac{2\cos^2\frac{\theta}{2}}{\sin\theta}\tau_3\right)\frac{\hat{\varphi}}{r}\right] \leftarrow$ *Polyakov − t'Hooft Monopole*
      Extra-flux:
      $\int_0^{2\pi}d\varphi\ r\sin\theta\ ig^{-1}\partial^\mu U_2^\dagger U_2 = \frac{4\pi}{g}\cos^2\frac{\theta}{2}\tau_3 \leftarrow$ *diagonal & vanish @ south pole*
      $\nabla\times\vec{A} = \frac{\hat{r}}{gr^2}\left(-\cot\frac{\theta}{2}\cos\phi\tau_1 + \cot\frac{\theta}{2}\cos\phi\tau_1 - \tau_3\right) \leftarrow$ *hedgehog structure*

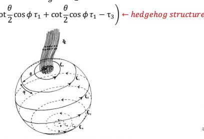
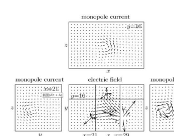


Fig. 1. The contour C on the sphere around the monopole. We displace it from C_0 to C_1, etc., until it shrinks at the bottom of the sphere. We require that there be no singularity at that point.



Nucl. Phys. B 79, 276 (1974) t'Hooft

Monopole condensate in the confined phase

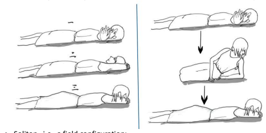
  - Maximum abelian gauge for SU(2)/ SU(3)
    $A^\mu = A^\mu_a\tau^a = (A^\mu_3, A^\mu_8)\cdot H + \sum_{a=1}^{N_c(N_c-1)/2}C^{\mu*}_aV_{+a} + C^\mu_aV_{-a}$
    - Measuring how large the "Off-diagonal" is:
      $[D^\mu, H] = ig[A^\mu, H] = ig\sum_{a=1}^{N_c(N_c-1)/2}C^{\mu*}_a[V_{+a}, H] + C^\mu_a[V_{-a}, H] = -ig\sum_{a=1}^{N_c(N_c-1)/2}w_a(C^{\mu*}_aV_{+a} - C^\mu_aV_{-a})$
      $R \equiv \mathrm{tr}\left([D^\mu, H]\cdot[D_\mu, H]\right) = -g^2\sum_{a,b}w_a\cdot w_b\mathrm{tr}\left[(C^{\mu*}_aV_{+a} - C^\mu_aV_{-a})(C^*_{b\mu}V_{+b} - C_{b\mu}V_{-b})\right]$
      $tr\ V_{+a}V_{+b} = trV_{-a}V_{-b} = 0$; $tr\ V_{+a}V_{-b} = \delta_{ab}$
      $R = 2g^2\sum_{a=1}^{N_c(N_c-1)/2}C^{\mu*}_aC^\mu_a$
    - Minimize "off-diagonal" or Maximum abelian:
      Minimize R ↔ Minimize the off-diagonal term ↔ Variation along the gauge obital = 0
      $R_U \equiv \mathrm{tr}\left([UD^\mu U^\dagger, H]\cdot[UD_\mu U^\dagger, H]\right) = \mathrm{tr}\left(U[D^\mu, U^\dagger HU]U^\dagger\cdot U[D_\mu, U^\dagger HU]U^\dagger\right) = \mathrm{tr}\left([D^\mu, U^\dagger HU]\cdot[D_\mu, U^\dagger HU]\right)$
      $U = I + i\chi = I + i\chi_a\tau_a$, $U^\dagger HU \approx H + i[H, \chi]$
      $R_U \approx R + 2i\mathrm{tr}\left([D^\mu, [H,\chi]]\cdot[D_\mu, H]\right)$
      $= R - 2i\mathrm{tr}\left([D^\mu, [D_\mu, H]]\cdot[H, \chi]\right)$
      $= R + 2i\mathrm{tr}\left(\chi\left[H_i, [D^\mu, [D_\mu, H_i]]\right]\right)$
      $\forall\chi_a$: $0 = R_U - R = 2i\mathrm{tr}\left(\chi\left[H_i, [D^\mu, [D_\mu, H_i]]\right]\right) \to$
    - Gauge fixing condition:
      $0 = \boxed{\left[H_i, [D^\mu, [D_\mu, H_i]]\right]}$
    - ? *How many eqns?*
    - ? *Relation between Abelian gauge:*
      For SU(2), $H = \{\tau^3\} \Rightarrow [D^\mu, [D_\mu, \tau_3]]$ is diagonal $\Rightarrow$ $\Phi = [D_\mu, [D_\mu, \tau_3]]$

Draft (3-form of pure gauge)

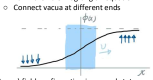
$dU_n = -n\sin n\theta\ Id\theta + in\cos n\theta\ \hat{x}\cdot\vec{\sigma}\ d\theta + i\sin n\theta\ \vec{\sigma}\cdot d\hat{x}$
$dU_nU_n^\dagger = (-n\sin n\theta\ Id\theta + in\cos n\theta\ \hat{x}\cdot\vec{\sigma}\ d\theta + i\sin n\theta\ \vec{\sigma}\cdot d\hat{x})(\cos n\theta\ I - i\sin n\theta\ \hat{x}\cdot\vec{\sigma})$
$= -\frac{n}{2}\sin2n\theta\ I\ d\theta + in\cos^2n\theta\ \hat{x}\cdot\vec{\sigma}\ d\theta + \frac{i}{2}\sin2n\theta\ \vec{\sigma}\cdot d\hat{x} + in\sin^2n\theta\ \hat{x}\cdot\vec{\sigma}\ d\theta + \frac{n}{2}\sin2n\theta\ I\ d\theta + \sin^2n\theta\left(\hat{x}\cdot d\hat{x}\ I + i(d\hat{x}\times\hat{x})\cdot\vec{\sigma}\right)$
$= n(i\ \hat{x}\cdot\vec{\sigma})\ d\theta + i\sin n\theta\left(\cos n\theta\ \vec{\sigma} + \sin n\theta\ (\hat{x}\times\vec{\sigma})\right)\cdot d\hat{x}$
$tr\ \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} = tr\ (ad\theta + b_id\hat{x}_i)\wedge(ad\theta + b_jd\hat{x}_j)\wedge(ad\theta + b_kd\hat{x}_k)$
$= tr\left([a, b_i]d\theta\wedge d\hat{x}_i + \frac{1}{2}[b_i, b_j]d\hat{x}_i\wedge d\hat{x}_j\right)\wedge(ad\theta + b_kd\hat{x}_k)$
$= tr\left(\left([a, b_i]b_j + \frac{1}{2}[b_i, b_j]a\right)d\theta\wedge d\hat{x}_i\wedge d\hat{x}_j\right)$
$= tr\left(\frac{3}{2}[b_i, b_j]ad\theta\wedge d\hat{x}_i\wedge d\hat{x}_j\right)$
$a = \frac{-n}{g}(\hat{x}\cdot\vec{\sigma})$
$b^i = -\frac{\sin n\theta}{g}\left(\cos n\theta\ \vec{\sigma} + \sin n\theta\ (\hat{x}\times\vec{\sigma})\right)^i$
$[b_i, b_j] = \frac{2\sin n\theta^2}{g^2}\left(\cos^2n\theta\ i\epsilon_{ijk}\sigma_k + i\cos n\theta\sin n\theta\left(\hat{x}^i\sigma^j - \hat{x}^j\sigma^i\right) + i\sin^2n\theta\ \epsilon_{ijk}\hat{x}_k(\hat{x}\cdot\vec{\sigma})\right)$
$\mathrm{tr}[b_i, b_j]a = \frac{-4ni\sin n\theta^2}{g^3}\epsilon_{ijk}\hat{x}_k$

QCD vacua & solitons

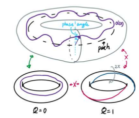
- Vacuum: Ground state of the theory
- Vacua: degenerated ground states which cannot transform continuously between each other without overcoming an energy barrier.



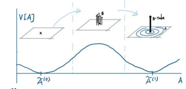
- Soliton, i.e., a field configuration:
  - Fulfill the Euler-Lagrangian equation
  - Connect vacua at different ends



- (Gauge) field configuration in ground state:
  $\bar{A}^\mu = ig^{-1}\partial^\mu UU^\dagger$, $\bar{F}^{\mu\nu} = 0$
  Group element U(x) is a map from space-time to group space
- $\pi_1(U(1))$ or $\pi_1(S^1)$
  - Transforming group element: $U_n(x) = e^{in\theta}$, $n\in Z$, defined on $R^2 - \{0\}$
    - $U_n(\theta = 2\pi) = U_n(\theta = 0)$
    - U_n is not homotopic to U_m, $\forall\ m \neq n$



  - A-Field configuration:
    $\bar{A}_{(n)}(\vec{x}) = \frac{n}{g}\frac{(-r_y, r_x)}{r_\perp^2}$
  - Flux
    $\oint d\vec{l}\cdot\bar{A}_{(n)} = \frac{n}{g}2\pi$
    Σ(っ ° Д ° ;)っ Finite flux with zero $\bar{F}^{\mu\nu}$???
    Flux must come from the field at origin !!!



  - Homotopy group
    transforming group elements $\{\cdots, U_{-1}, U_0, U_1, U_2, \cdots\}$ form a group
    - $U_i\ U_j = U_{i+j} \in \{\cdots, U_{-1}, U_0, U_1, \cdots\}$
    - $U_iU_{-i} = U_0 = I$
    - Labelled as $\pi_d(G)$, composed of the transforming group elements mapping from $S^d \to G$ (in this case, d=1, G=U(1) or S^1, so labelled as $\pi_1(S^1)$)
    - Vacua exist only if $\pi_d(G) \neq \{I\}$ (in this case, $\pi_1(S^1) = Z$)
  - Winding number
    If $\pi_d(G) = Z$, the integers are given by winding number, which is loop integrations of Chern-Simons d-form: $\oint_{S^d}\omega_d$
    In case of d=1,
    $\frac{1}{2\pi i}\oint\bar{\omega}_1^{(n)} = \frac{ig}{2\pi i}\oint\mathrm{tr}\bar{\mathcal{A}}_{(n)} = n$
- Table of homotopy group

Table 9.1 *Homotopy groups of a number of frequently encountered mappings. A blank entry means that no general statements can be made.*

	S^1	S^2	$S^{d>2}$	T^d	SU(2)	SU(N)
$k=1$	$\mathbb{Z}$			$\underbrace{\mathbb{Z}\ \cdots\ \mathbb{Z}}_{d}$		
$k=2$		$\mathbb{Z}$				
$k>2$		[a]	$k<d$: $k=d$: $\mathbb{Z}$ $k>d$:		$\pi_3(SU(2)) = \mathbb{Z}$	

[a] But $\pi_3(S^2) = \mathbb{Z}$.

- $\pi_3(SU(2))$ or $\pi_3(S^3)$
  - Geometry of SU(2)
    $U = e^{i\sum_{a=1}^3\varphi_a\tau_a} = e^{\frac{i}{2}\sum_{a=1}^3\varphi_a\sigma_a}$
    Since the product of the Pauli-matrices is the superposition of I & $\vec{\sigma}$ ⇒
    $U = \alpha I + i\vec{\beta}\cdot\vec{\sigma}$
    $1 = \det U = \alpha^2 + \vec{\beta}^2$
    $I = U^\dagger U = \left(|\alpha|^2 + \left|\vec{\beta}\right|^2\right)I + (-i\alpha\beta_k^* + i\alpha^*\beta_k + i\epsilon_{ijk}\beta_i\beta_k^*)\sigma_k$ $x^0 = |x|\cos\theta$
    ⇒ $\alpha, \vec{\beta} \in \Re$; $\alpha^2 + \vec{\beta}^2 = 1 \Rightarrow$ form a S^3 $\vec{x} = |x|\sin\theta\ \hat{x}$ ← 3-D
  - Transforming group element (in $R^4 - \{0\}$):
    $U_1(x) = \frac{x^0I + i\vec{x}\cdot\vec{\sigma}}{|x|} = \cos\theta\ I + i\sin\theta\ \hat{x}\cdot\vec{\sigma}$; $U_n(x) = U_1(x)^n = \cos n\theta\ I + i\sin n\theta\ \hat{x}\cdot\vec{\sigma}$
    Ill-defined for $|x| = 0$
  - A-field configuration
    $\bar{\mathcal{A}}_{(n)} = ig^{-1}dU_nU_n^\dagger = \frac{-n}{g}(\hat{x}\cdot\vec{\sigma})\ d\theta - \frac{\sin n\theta}{g}\left(\cos n\theta\ \vec{\sigma} + \sin n\theta\ (\hat{x}\times\vec{\sigma})\right)\cdot d\hat{x}$
    $\bar{\mathcal{A}}_{(1)} = \frac{1}{g|x|^2}\left[(\vec{x}\cdot\vec{\sigma})\ dx^0 - \left(x^0\vec{\sigma} + (\vec{x}\times\vec{\sigma})\right)\cdot d\vec{x}\right] \equiv -\frac{1}{g|x|^2}\eta^a_{\mu\nu}\sigma_ax^\nu dx^\mu$
    $\eta^a_{ij} = \epsilon^{aij}$, $\eta^a_{0i} = -\eta^a_{i0} = -\delta^a_i$
  - Winding number
    Chern-Simons 3-form:
    $\bar{\omega}_3^{(n)} \equiv \frac{ig^3}{3}\mathrm{Tr}\left[\bar{\mathcal{A}}_{(n)}\wedge\bar{\mathcal{A}}_{(n)}\wedge\bar{\mathcal{A}}_{(n)}\right]$
    Notice:
    $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi)$
    $\Rightarrow$ $(d\hat{x}_1, d\hat{x}_2, d\hat{x}_3) = (\cos\psi\cos\varphi, \cos\psi\sin\varphi, -\sin\psi)d\psi + (-\sin\psi\sin\varphi, \sin\psi\cos\varphi, 0)d\varphi$
    $\Rightarrow d\hat{x}_1\wedge d\hat{x}_2\wedge d\hat{x}_3 = 0$ due to appearance of $d\psi\wedge d\psi$ or $d\varphi\wedge d\varphi$
    $\bar{\omega}_3^{(n)} = 2n\sin^2n\theta\ \epsilon_{ijk}\hat{x}_kd\theta\wedge d\hat{x}_i\wedge d\hat{x}_j$
    $\oint_{S^3}\bar{\omega}_3^{(n)} = 12n\oint\sin^2n\theta\ \hat{x}_3d\theta\wedge d\hat{x}_1\wedge d\hat{x}_2$
    $= 12n\int_0^\pi d\theta\sin^2n\theta\wedge\int_0^\pi d\psi\cos^2\psi\sin\psi\wedge\int_0^{2\pi}d\varphi = 8n\pi^2$
    On the other hand,
    $\oint_{S^3}\bar{\omega}_3^{(n)} = \oint_{R^4}d\wedge\bar{\omega}_3^{(n)} = -g^2\oint_{R^4}\mathrm{Tr}\left[\bar{\mathcal{F}}_{(n)}\wedge\bar{\mathcal{F}}_{(n)}\right]$
    $\Rightarrow$ $Q \equiv -\frac{g^2}{8\pi^2}\oint_{R^4}\mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}] = n \leftarrow$ *topological charge*
    Topological charge must be contributed by $\mathcal{F}$ @ origin !!!
  - Instanton
    $\mathcal{A} = -\frac{1}{g(|x|^2 + \lambda^2)}\eta^a_{\mu\nu}\sigma_ax^\nu dx^\mu = \frac{\bar{\mathcal{A}}_{(1)}}{\left(\frac{\lambda}{|x|}\right)^2 + 1}$
    - When $|x| \to 0$, $\mathcal{A} \to \bar{\mathcal{A}}_{(0)} = const.$ $|x| \to \infty$, $\mathcal{A} \to \bar{\mathcal{A}}_{(1)}$
    - ? *Fulfill* Eular-Lagrangian equation: $D_\mu F^{\mu\nu} = 0$
    - ? Self-dual: $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$

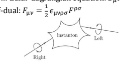
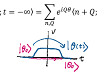


Figure 4.6: The (interaction with the) instanton flips the helicity of the fermion from right to left as shown in the text.

- $\pi_2(S^2)$
  See section "abelian gauge"
  Monopole condensate could be the reason of confinement
- θ −vacuum & Large gauge transformation
  - Large gauge transformation -- transform one vacuum to another: $|n+m\rangle = U_m|n\rangle$
  - Gauge symmetry requires: $\left[\hat{H}, U_i\right] = 0$ ⇒ TRUE vacuum should be the EIGENSTATE of U_i
  - θ-vacuum: $|\theta\rangle \equiv \sum_n e^{-in\theta}|n\rangle \Rightarrow$ $U_m|\theta\rangle = e^{im\theta}|\theta\rangle$
  - Transition amplitude:
    $\langle\theta; t=\infty|\theta; t=-\infty\rangle = \sum_{n,m}e^{i(m-n)\theta}\langle m; t=\infty|n; t=-\infty\rangle = \sum_{n,Q}e^{iQ\theta}\langle n+Q; t=\infty|n; t=-\infty\rangle$
    $= \sum_{n,\mathcal{F}}e^{-\frac{g^2i\theta}{8\pi^2}\oint_{R^4}\mathrm{Tr}[\mathcal{F}\wedge\mathcal{F}]}\langle n+Q; t=\infty|n; t=-\infty\rangle$
    θ-term relevant to strong CP violation
    θ → θ(x) ⇒ Axion



Similarly, ${}_\infty\langle P_1P_2|P_3P_4\rangle_{-\infty} \neq \delta(P_1-P_3)\delta(P_2-P_4) \pm (1\leftrightarrow2)$