Phase Transition-03 (Perturbative RG)

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Gaussian & Wilson-Fisher Fixed Points

Remind (Near a Fixed Point)

After RG: $u_i \to u_i' = b^{y_i} u_i$
For Infinitesimal Scaling: $b \sim 1 + d\lambda$; $d\lambda \ll 1$
$\Rightarrow u_i' = (1+d\lambda)^{y_i} u_i \sim u_i + y_i d\lambda u_i$
$\Rightarrow \frac{du_i}{d\lambda} = y_i u_i$

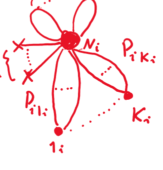
Schematic 2-parameters Example

- Two neighboring Fixed Points: $A = (0,0)$; $B = (\epsilon, 0)$.
- Ansatz Near A (Gaussian):
  $\frac{du_1}{d\lambda} = k u_1(u_1 - \epsilon) \Rightarrow y_1 = \frac{d}{du_1}\frac{du_1}{d\lambda}\Big|_{u_1\to 0} = -k\epsilon$
  (u_1 is relevant around A if $k\epsilon < 0$)
  $\frac{du_2}{d\lambda} = y u_2 + z u_1 u_2 \Rightarrow y_2 = \frac{d}{du_2}\frac{du_2}{d\lambda}\Big|_{u_1\to 0} = y$
- Near B (Wilson Fisher):
  $\frac{du_1}{d\lambda} = k u_1(u_1 - \epsilon) \Rightarrow y_1 = \frac{d}{du_1}\frac{du_1}{d\lambda}\Big|_{u_1\to \epsilon} = k\epsilon$
  (u_1 is irrelevant around B if $k\epsilon < 0$)
  $\frac{du_2}{d\lambda} = y u_2 + z u_1 u_2 \Rightarrow y_2 = \frac{d}{du_2}\frac{du_2}{d\lambda}\Big|_{u_1\to \epsilon} = y + z\epsilon$

A General Theory

- Hamiltonian: $\mathcal{H} = \mathcal{H}^* + \sum_i g_i a^{x_i} \sum_r O_i(r)$
  - $\mathcal{H}^*$: Hamiltonian of FREE theory
  - O_i: Interacting Terms
  - Gaussian Fixed Point: $\{g_i\} = \{0\}$
  - g_i are dimension-less
  - $\mathcal{O}(g_i) \sim \mathcal{O}(\epsilon)$
- Partition Function (Perturbative on g_i)
  $\mathcal{Z} = \mathrm{Tr}\, e^{-\mathcal{H}} = \mathrm{Tr}\, e^{-\mathcal{H}^* - \sum_i g_i a^{x_i} \sum_r O_i(r)}$
  $\sim \mathrm{Tr}\, e^{-\mathcal{H}^*}\left(1 - \sum_i g_i a^{x_i} \sum_r O_i(r) + \frac{1}{2}\sum_{i,j} g_i g_j a^{x_i + x_j} \sum_{r,r'} O_i(r) O_j(r') + \cdots\right)$
  $\sim \mathcal{Z}^*\left(1 - \sum_i g_i a^{x_i} \sum_r \langle O_i(r)\rangle + \frac{1}{2}\sum_{i,j} g_i g_j a^{x_i + x_j} \sum_{r,r'} \langle O_i(r) O_j(r')\rangle + \cdots\right)$
  After RG (Remind):
  - $a \to ba$
  - $\langle \mathcal{O}_1(r_1)\mathcal{O}_2(r_2)\cdots\mathcal{O}_n(r_n)\rangle_H \to b^{-x_1 - x_2 - \cdots - x_n}\left\langle \mathcal{O}_1\left(\frac{r_1}{b}\right)\mathcal{O}_2\left(\frac{r_2}{b}\right)\cdots\mathcal{O}_n\left(\frac{r_n}{b}\right)\right\rangle_{H'}$
  - $a^{x_i+x_j}\langle O_i(r) O_j(r')\rangle_H \to a^{x_i+x_j}\left\langle O_i\left(\frac{r}{b}\right) O_j\left(\frac{r'}{b}\right)\right\rangle_{H'}$
- Operator Product Expansion:
  For r, r' close enough, $O_i(r)O_j(r') = C_{ij}^k(r - r') O_k\left(\frac{r+r'}{2}\right)$
  $\langle O_i(r) O_j(r')\rangle = C_{ij}^k(r-r')\left\langle O_k\left(\frac{r+r'}{2}\right)\right\rangle$
  After RG:
  $\left\langle O_i\left(\frac{r}{b}\right) O_j\left(\frac{r'}{b}\right)\right\rangle b^{-x_i - x_j} = C_{ij}^k(r-r')\left\langle O_k\left(\frac{r+r'}{2b}\right)\right\rangle b^{-x_k}$
  $\left\langle O_i\left(\frac{r}{b}\right) O_j\left(\frac{r'}{b}\right)\right\rangle = C_{ij}^k\left(\frac{r-r'}{b}\right)\left\langle O_k\left(\frac{r+r'}{2b}\right)\right\rangle$
  $C_{ij}^k(r-r') = C_{ij}^k\left(\frac{r-r'}{b}\right) b^{x_k - x_i - x_j} \Rightarrow C_{ij}^k(r) = c_{ij}^k |r|^{x_k - x_i - x_j}$
- Continue Limit + OPE
  $\sum_r \to \int \frac{d^D r}{a^D}$
  $$\mathcal{Z} \sim \mathcal{Z}^*\left(1 - \sum_i g_i a^{x_i - D}\int d^D r\, \langle O_i(r)\rangle + \frac{1}{2}\sum_{i,j,k} g_i g_j c_{ijk} a^{x_i + x_j - 2D}\int_{|r-r'|>a} d^D(r-r') d^D\left(\frac{r+r'}{2}\right)\frac{\left\langle O_k\left(\frac{r+r'}{2}\right)\right\rangle}{|r-r'|^{x_i+x_j-x_k}} + \cdots\right)$$
  $$= \mathcal{Z}^*\left[1 - \sum_i a^{x_i - D}\left(g_i - \frac{1}{2}\sum_{j,k} g_j g_k c_{jk}^i a^{x_j + x_k - x_i - D}\int_{|r-r'|>a} d^D(r-r')\frac{1}{|r-r'|^{x_j+x_k-x_i}}\right)\int d^D r\, \langle O_i(r)\rangle + \cdots\right]$$
  After RG:
  - $a^\nu \to (ba)^\nu = (1+d\lambda)^\nu a^\nu \sim (1 + \nu d\lambda) a^\nu$
  - $\int_{|r|>a} d^D r \frac{1}{|r|^{x_i+x_j-x_k}} \to \int_{|r|>a(1+d\lambda)} d^D r \frac{1}{|r|^{x_i+x_j-x_k}} = \int_{|r|>a} d^D r \frac{1}{|r|^{x_i+x_j-x_k}} - \int_{a<|r|<a(1+d\lambda)} d^D r \frac{1}{|r|^{x_i+x_j-x_k}}$
    $= \int_{|r|>a} d^D r \frac{1}{|r|^{x_i+x_j-x_k}} - \frac{a^D S_D d\lambda}{a^{x_i+x_j-x_k}}$
    (S_D is solid angle of D-dimensional sphere)
  - $\mathcal{Z} \to \mathcal{Z}$
    $$\Rightarrow g_i - \frac{1}{2}\sum_{j,k} g_j g_k c_{jk}^i a^{x_j+x_k-x_i-D}\int_{|R|>a} d^D R\frac{1}{|R|^{x_j+x_k-x_i}} = b^{x_i - D}\left[g_i' - \frac{1}{2}\sum_{j,k} g_j' g_k' c_{jk}^i (ba)^{x_j+x_k-x_i-D}\left(\int_{|R|>a} d^D R\frac{1}{|R|^{x_j+x_k-x_i}} - \frac{a^D S_D d\lambda}{a^{x_i+x_j-x_k}}\right)\right]$$
    Ansatz: $g' = b^{D-x} g + \delta d\lambda \Rightarrow dg/d\lambda = (D-x)g + \delta$; $\delta \ll g$; $\delta(g=0) = 0 \Rightarrow \delta \sim \mathcal{O}(g^2)$
    $\Rightarrow 0 = b^{x_i - D}\delta_i d\lambda - \sum_{j,k} b^{x_k - D}\left(g_j \delta_k d\lambda c_{(jk)}^i\right)\int_{|R|>1} d^D R \frac{1}{|R|^{x_j+x_k-x_i}} + \frac{1}{2}\sum_{j,k} g_j g_k c_{jk}^i S_D d\lambda$
    Second term can be neglected since $g_j \delta_k d\lambda c_{(jk)}^i \sim \mathcal{O}(g^3) \Rightarrow$
    $\delta_i = -\frac{1}{2}\sum_{j,k} g_j g_k c_{jk}^i S_D \quad\Rightarrow\quad \frac{dg_i}{d\lambda} = (D - x_i) g_i - \frac{1}{2}\sum_{j,k} g_j g_k c_{jk}^i S_D$
    Similar to the form in the schematic example.
- Fixed points:
  $\frac{dg_i}{d\lambda} = \sum_k\left[(D - x_i)\delta_{ik} - \frac{1}{2}\sum_j g_j c_{jk}^i S_D\right] g_k = 0$
  - $\{g\} = \{0\}$ Gaussian Fixed Points
    Near Gaussian Fixed Points, $y_i = D - x_i$
  - $(D - x_i)\delta_{ik} + \frac{1}{2}\sum_j g_j c_{jk}^i S_D = 0$
    If $\sum_k c_{jk}^i p_{kl}^m = \delta_{jl}\mathbb{I}_{im}$
    $g_i = \frac{2}{S_D}\frac{\sum_k (D - x_k) p_{ki}^j}{\sum_k \mathbb{I}_{kj}} \quad\Leftarrow\quad$ Wilson − Fisher Fixed Point
    Near Wilson-Fisher Fixed Point:
    $y_i = (D - x_i)\left[1 - 2\frac{\mathbb{I}_{ij}}{\sum_l \mathbb{I}_{lj}}\right]$

Symmetry Factor for N-field theory



$$\prod_i^n \frac{1}{(P_{ii})!} C_{N_i}^2 C_{N_i - 2}^2 \cdots C_{N_i - 2P_{ii} + 2}^2 C_{N_i - 2P_{ii}}^{J_i}\prod_{k_i - cnct.}^{n_i} C_{N_i - 2P_{ii} - J_i - \sum_{j'=1_i}^{k_i} P_{ij'}}^{P_{ij}}\prod_{i\neq j}^n P_{ij}!$$
$$= \prod_i^n \frac{1}{(P_{ii})!}\frac{N_i(N_i-1)}{2}\frac{(N_i-2)(N_i-3)}{2}\cdots\frac{(N_i - 2P_{ii}+2)(N_i - 2P_{ii}+1)}{2}\frac{(N_i - 2P_{ii})!}{(N_i - 2P_{ii} - J_i)!J_i!}$$
$$\times\frac{(N_i - 2P_{ii} - J_i)!}{\left(N_i - 2P_{ii} - J_i - P_{i1_i}\right)!P_{i1_i}!}\cdots\frac{\left(N_i - 2P_{ii} - J_i - \sum_{k=1}^{n_i - 1} P_{ik_i}\right)!}{\left(N_i - 2P_{ii} - J_i - \sum_{k=1}^{n_i} P_{ik_i}\right)!P_{in_i}!}\prod_{i\neq j}^n P_{ij}!$$
$$= \prod_i^n\left[\frac{N_i!}{(P_{ii})!\,2^{P_{ii}} J_i!}\prod_{k_i=1}^{n_i}\frac{1}{P_{ik_i}!}\right]\prod_{i\neq j}^n P_{ij}!$$

$\prod^n \quad 1 \quad N_i! \quad \prod \quad 1$

Landau-Ginzburg-Wilson Model

Continuation of Ising Model

- Make discrete spin continuous $S = \pm 1 \to S \in \mathbb{R}$
  $\mathcal{H}[\{s_r\}] = \sum_r -J s_r s_{r+1} + h_r s_r \Rightarrow$
  $\mathcal{H} = -\frac{1}{2}\sum_{\vec r, \vec r'} J(|\vec r - \vec r'|) S(\vec r) S(\vec r') + \sum_{\vec r} h(\vec r) S(\vec r) + \lambda\left[S(\vec r)^2 - 1\right]^2$
  $\lambda\left[S(\vec r)^2 - 1\right]^2$ with postive λ for punishing the case where $|S|$ deviates from 1 a lot.
- Make Lattice continue
  - $\vec\delta \equiv \vec r - \vec r'$; $\vec R \equiv \frac{\vec r + \vec r'}{2}$; $S(\vec r \text{ or } \vec r') \sim S(\vec R) \pm \frac{\vec\delta}{2}\cdot\nabla S(\vec R) + \frac{1}{8}\delta^i\delta^j\partial_{i,j} S(\vec R)$
    $\sum_{\vec r, \vec r'} J(|\vec r - \vec r'|) S(\vec r) S(\vec r')$
    $\approx \sum_{\vec\delta, \vec R} J(\delta)\left[S_{\vec R}^2 - \left(\frac{\vec\delta}{2}\cdot\nabla S_{\vec R}\right)^2 + \frac{1}{4}S_{\vec R}\delta^i\delta^j\partial_{i,j}S_{\vec R}\right]$
    $= \sum_{\vec\delta, \vec R} J(\delta)\left[S_{\vec R}^2 - \frac{1}{2}\left(\vec\delta\cdot\nabla S_{\vec R}\right)^2\right]$
    Define: $J \equiv \sum_{\vec\delta} J(\delta)$; $J\Delta^2 a^2\delta_{ij} \equiv \frac{1}{2}\sum_{\vec\delta} J(\delta)\,\delta_i\delta_j$ ← isotropic;
    $[\Delta] = 0$ characterizes the average range (of sites) of the interaction.
    $\sum_{\vec r, \vec r'} J(|\vec r - \vec r'|) S(\vec r) S(\vec r') \sim \sum_{\vec r} J\left[S(\vec r)^2 - \Delta^2 a^2\left(\nabla S(\vec r)\right)^2\right]$
  - $\sum_{\vec r}\cdots \to \int \frac{d^D\vec r}{a^D}\cdots$
  - $\mathcal{H} \sim \int\frac{d^D\vec r}{a^D}\left\{-\frac{1}{2}J\left[S^2 - \Delta^2 a^2(\nabla S)^2\right] + h(\vec r)S(\vec r) + \lambda\left[S(\vec r)^2 - 1\right]^2\right\}$
    Redefine $S \to S' = \sqrt{J}\Delta a^{1-\frac{D}{2}} S$
    $\mathcal{H} \sim \int d^D\vec r\left\{\frac{1}{2}(\nabla S)^2 - \frac{1}{2}\left(1 + \frac{4\lambda}{J}\right)\frac{S^2}{\Delta^2 a^2} + \frac{h}{\sqrt{J}\Delta a^{1+\frac{D}{2}}}S + \frac{\lambda}{J^2\Delta^4 a^{4-D}}S^4 + const.\right\}$
    $\equiv \int d^D\vec r\left\{\frac{1}{2}(\nabla S)^2 + t a^{-2} S^2 + u a^{D-4} S^4 + h a^{-1-\frac{D}{2}} S\right\}$
- Dimension-analysis
  $[\mathcal{H}] = 0 \Rightarrow 0 = \left[d^D\vec r(\nabla S)^2\right] = 2 - d + 2[S] \Rightarrow [S] = \frac{D}{2} - 1$
  If $\left[d^D\vec r\lambda_n a^? S^n\right] = 0$ and $[\lambda_n] = 0$, then $0 = -D - ? + n(D/2 - 1) \Rightarrow ? = \left(\frac{n}{2} - 1\right)D - n$
  $[t] = [u] = [h] = 0$
- Gaussian Fixed Point
  - Location: $\{\lambda_n | n \in \mathbb{N}\} = \{t, u, h\} = 0$
  - After coarse-graining: $a \to ba$
    $\lambda_n \to \lambda_n b^{n - \left(\frac{n}{2} - 1\right)D} \quad\Leftrightarrow\quad \mathcal{H} \to \mathcal{H}$
    $t \to tb^2$; $\quad u \to ub^{4-d}$; $\quad h \to hb^{1+\frac{D}{2}}$
  - λ_n is Relevant if: $n - \left(\frac{n}{2} - 1\right)D > 0 \to n < \frac{2D}{D-2}$
  - Critical Exponents: $y_t = 2$, $y_h = 1 + \frac{D}{2}$
    $\alpha = 2 - D/y_t = 2 - \frac{D}{2}$
    $\beta = \frac{D - y_h}{y_t} = \frac{D-2}{4}$
    $\gamma = -\frac{D - 2y_h}{y_t} = 1$
    $\delta = \frac{1}{\frac{D}{y_h} - 1} = \frac{2+D}{D-2}$
    $\nu = \frac{1}{y_t} = \frac{1}{2}$
    $\eta = D - 2y_h + 2 = 0$

Wilson-Fisher Fixed Point

- Wick Contraction: $:S^2: \equiv S^2 - \langle S^2\rangle$; $:S^4: \equiv S^4 - 3\langle S^2\rangle S^2 = S^4 - 3\langle S^2\rangle :S^2: + 3\langle S^2\rangle^2$
  3: For a single S in S^4, there're 3 self-parterning options.
  Diagramatic meaning: Eliminate the self-cross lines.
  $\mathcal{H} = \int d^D\vec r\left\{\frac{1}{2}(\nabla S)^2 + h a^{-1-\frac{D}{2}} S + t a^{-2} S^2 + u a^{D-4} S^4\right\}$
  $= \int d^D\vec r\left\{\frac{1}{2}(\nabla S)^2 + h a^{-1-\frac{D}{2}} S + \left(t - 3u\langle S^2\rangle a^{D-2}\right)a^{-2} :S^2: + u a^{D-4} :S^4:\right\}$
  $\equiv \int d^D\vec r\left\{\frac{1}{2}(\nabla S)^2 + h a^{-1-\frac{D}{2}} S + t' a^{-2} :S^2: + u a^{D-4} :S^4:\right\}$
- Decomposition
  $\mathcal{H}^* = \int d^d\vec r\frac{1}{2}(\nabla S)^2$; $O_1 = S$; $O_2 =:S^2:$; $\quad O_4 =:S^4:$
- OPE
  - $G(r) \equiv \langle S(r_1) S(r_2)\rangle = r^{2-D}$; $r \equiv |\vec r_1 - \vec r_2|$; $\vec R \equiv \frac{\vec r_1 + \vec r_2}{2}$ (represented by ×−−− −diagrammatically)
  - $O_1(\vec r_1) O_1(\vec r_2) = S(\vec r_1) S(\vec r_2) = G(r)\,\mathbb{I} + :S^2(\vec R):$
  - $O_1(\vec r_1) O_2(\vec r_2) = S(\vec r_1) :S^2(\vec r_2): = 2\,G(r)\,S(\vec R) + :S^3(\vec R):$
    $2 = \frac{1!}{0!}\frac{2!}{1!}\frac{1}{1}$ $\qquad 1 = \frac{1!}{1!}\frac{2!}{2!}$
  - $O_2(\vec r_1) O_2(\vec r_2) =:S^2(\vec r_1)::S^2(\vec r_2): = 2G(r)^2\mathbb{I} + 4\,G(r) :S^2(\vec R): + :S^4(\vec R):$
    $2 = \frac{2!}{0!}\frac{2!}{0!}\frac{1}{2!}$; $\qquad 4 = \frac{2!}{1!}\frac{2!}{1!}\frac{1}{1!}$; $\qquad 1 = \frac{2!}{2!}\frac{2!}{2!}$
  - $O_2(\vec r_1) O_4(\vec r_2) =:S^2(\vec r_1)::S^4(\vec r_2): = 12 G(r)^2 :S^2(\vec R): + 8\,G(r) :S^4(\vec R): + :S^6(\vec R):$
    $12 = \frac{2!}{0!}\frac{4!}{2!}\frac{1}{2!}$ $\qquad 8 = \frac{2!}{1!}\frac{4!}{3!}\frac{1}{1}$ $\qquad 1 = \frac{2!}{2!}\frac{4!}{4!}$
  - $O_1(\vec r_1) O_4(\vec r_2) = S(\vec r_1) :S^4(\vec r_2): = 4\,G(r) :S^3(\vec R): + :S^5(\vec R):$
    $4 = \frac{1}{0!}\frac{4!}{3!}\frac{1}{1}$ $\qquad 1 = \frac{1!}{1!}\frac{4!}{4!}$
  - $O_4(\vec r_1) O_4(\vec r_2) = 24\,G(r)^4\,\mathbb{I} + 96\,G(r)^3 :S^2(\vec R): + 72\,G(r)^2 :S^4(\vec R): + 16\,G(r) :S^6(\vec R): + :S^8(\vec R):$
    $24 = \frac{4!}{0!}\frac{4!}{0!}\frac{1}{4!}$ $\qquad 96 = \frac{4!}{1!}\frac{4!}{1!}\frac{1}{3!}$ $\qquad 72 = \frac{4!}{2!}\frac{4!}{2!}\frac{1}{2!}$
- Beta function:
  - $\frac{dg_i}{d\lambda} = \sum_k\left[(D - x_i)\delta_{ik} - \frac{1}{2}\sum_j g_j c_{jk}^i S_D\right] g_k$
  - Seek $S(\vec R)$ in the R.H.S. Of $O_i(\vec r_1) O_j(\vec r_2)$
    $\frac{dh}{d\lambda} = \left(1 + \frac{D}{2}\right)h - \frac{S_D}{2}(2\times 2ht)$ ← 2: from cross terms $O_1(\vec r_1)O_2(\vec r_2) + O_2(\vec r_1)O_1(\vec r_2)$
  - Seek $:S^2(\vec R):$ in the R.H.S. Of $O_i(\vec r_1) O_j(\vec r_2)$
    $\frac{dt}{d\lambda} = 2t - \frac{S_D}{2}\left(h^2 + 4t^2 + 2\times 12tu + 96\,u^2\right)$
  - Seek $:S^4(\vec R):$ in the R.H.S. Of $O_i(\vec r_1) O_j(\vec r_2)$
    $\frac{du}{d\lambda} = (4 - D)u - \frac{S_D}{2}\left(t^2 + 2\times 8tu + 72\,u^2\right)$
  - Factor $S_D/2$ can be eliminated after redefining $\{h', t', u'\} \equiv S_D/2\{h, t, u\}$
- Wilson-Fisher Fixed Point ($D = 4 - \epsilon$)
  - An apparent fixed point: $\{t, h, u\} = \{0,0,0\}$
  - $du/d\lambda = 0 \Rightarrow u_* = \frac{2\epsilon}{72\,S_D} + \mathcal{O}(\epsilon^2)$ is another fixed point nearby ($\{t, h\} = 0$).
  - At Wilson-Fisher Fixed Point, $dt/d\lambda \sim \mathcal{O}(\epsilon^2) \to 0$
  - Around Wilson-Fisher Fixed Point:
    $\frac{dt}{d\lambda} = \left(2 - \frac{\epsilon}{3}\right)t \to y_t = 2 - \frac{\epsilon}{3}$
    $\frac{du}{d\lambda} = -\epsilon u \Leftarrow$ irrelevant for $\epsilon > 0$
  - Critical Exponents
    $4 - \epsilon \quad \epsilon \quad \epsilon$

$$\left(N_i - 2P_{ii} - J_i - P_{i1_i}\right)!\,P_{i1_i}! \quad \left(N_i - 2P_{ii} - J_i - \textstyle\sum_{k=1}^{n_i} P_{ik_i}\right)!\,P_{in_i}! \prod_{i\neq j} P_{ij}!$$

$$= \prod_i^n \left[\frac{N_i!}{(P_{ii})!\,2^{P_{ii}} J_i!} \prod_{k_i=1}^{n_i} \frac{1}{P_{ik_i}!}\right] \prod_{i\neq j}^{n} P_{ij}!$$

$$= \prod_{i=1}^n \frac{1}{(P_{ii})!} \frac{N_i!}{2^{P_{ii}} J_i!} \prod_{i\neq j} \frac{1}{P_{ij}!}$$

Symmetry Factor for $(\bar{\psi}\psi)^N$ theory

$$\prod_i^n \frac{1}{(P_{ii})!}\left(N(N-1)\cdots(N-P_{ii}+1)\right)^2 C_{N-P_{ii}}^{J_i} C_{N-P_{ii}}^{\bar{J}_i} \prod_{k_i-cnct.}^{n_i} C^{P_{ij}}_{N-P_{ii}-J_i-\sum_{j'=1_i}^{k_i} P_{ij'}} \prod_{\bar{k}_i-cnct.}^{\bar{n}_i} C^{\bar{P}_{ij}}_{N-P_{ii}-\bar{J}_i-\sum_{\bar{j}'=\bar{1}_i}^{\bar{k}_i} \bar{P}_{i\bar{j}'}} \prod_{i\neq j}^{n} P_{ij}!\,\bar{P}_{ij}!$$

$$= \prod_{i=1}^n \frac{1}{(P_{ii})!} \frac{[N!]^2}{J_i!\,\bar{J}_i!} \prod_{i\neq j} \frac{1}{P_{ij}!\,\bar{P}_{ij}!}$$

- Around Wilson-Fisher Fixed Point:

  $\frac{dt}{d\lambda} = \left(2 - \frac{\epsilon}{3}\right) t \; \rightarrow y_t = 2 - \frac{\epsilon}{3}$

  $\frac{du}{d\lambda} = -\epsilon u \; \Leftarrow$ irrelevant for $\epsilon > 0$

- Critical Exponents

  $\alpha = 2 - D/y_t = 2 - \frac{4-\epsilon}{2}\left(1 + \frac{\epsilon}{6}\right) = \frac{\epsilon}{6}$ $\leftarrow$ *small exponent*

  $\beta = \frac{D - y_h}{y_t} = \frac{2-\epsilon}{4}\left(1+\frac{\epsilon}{6}\right) = \frac{1}{2} - \frac{\epsilon}{6}$

  $\gamma = -\frac{D - 2y_h}{y_t} = 1 + \frac{\epsilon}{6}$

  $\delta = \frac{1}{\frac{D}{y_h} - 1} = \frac{6-\epsilon}{2-\epsilon} = 3 + \epsilon$

  $\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{\epsilon}{12}$

  $\eta = D - 2y_h + 2 = 0$ $\leftarrow$ *smaller exponent*

- Note on ϵ-expansion:
  For $\epsilon \neq 0$, the series will give an INACCURATE results if truncated at too high an order.