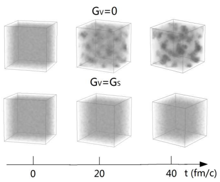
Phase Transition - 01 (1st-order)

Wednesday, July 19, 2023 10:15 AM

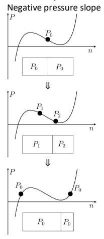
General Properties

Example:

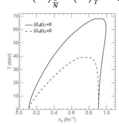


Inhomogeneity generated during the phase separation

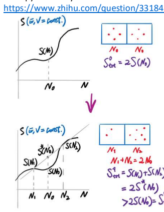
- Spinodal instability
  - Negative pressure slope



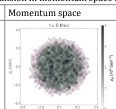
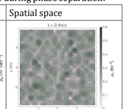
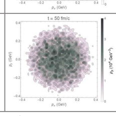
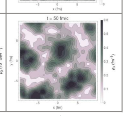
    - Isothermal spinodal: $\left(\frac{\partial P}{\partial n}\right)_T < 0$
    - Isentropic spinodal: $\left(\frac{\partial P}{\partial n}\right)_{\frac{S}{N}} < 0$
    - $\left(\frac{\partial P}{\partial n}\right)_{\frac{S}{N}} = \left(\frac{\partial P}{\partial n}\right)_T + \frac{(\partial_T P)_n^2}{n(\partial_T s)_n} > \left(\frac{\partial P}{\partial n}\right)_T$



  - Entropy convexity
    https://www.zhihu.com/question/331842176/answer/822361947

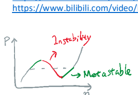


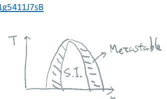
    - Entropy increment DOES NOT in contradiction with phase separation.
    - Phase seperation occurs when
      $\left(\frac{\partial^2 S}{\partial N^2}\right)_{E,V} > 0$ or $\left(\frac{\partial^2 F}{\partial N^2}\right)_{T,V} < 0$
      Inhomogenious system has a larger entropy (or smaller free energy) in such a case.
    - $\left(\frac{\partial^2 F}{\partial N^2}\right)_{T,V} < 0 \Leftrightarrow \frac{1}{N}\left(\frac{\partial P}{\partial n}\right)_T < 0$
    - Phase seperation ⇔ Particles occupy a smaller volumn
      Entropy increment ⇔ Particles occupy a larger phase space
      ⇒ Expansion in momentum space during phase separation:

	Momentum space	Spatial space
t=0		
t=50 fm/c		

  - A dynamical view (simple example with Skyrme potential)
    Basic Idea (Transport approach):
    - Solve the linearized Boltzmann equation about δf when the phase space distribution is SLIGHTLY deviated from the equilibrium distribution f_0
    - Obtain a linear equation of density fluctuation δn
    - Solve the linear equation ⇒ dispersion relation $\omega = \omega(k)$
    - Find the region where the frequency $\omega = i\gamma$ is imaginary
      ⇒ $e^{-i\omega t} \to e^{\gamma t}$ ⇒ Fluctuations grow exponentially with time.
    - $\omega = i\gamma$ is EQUIVALENT to $(\partial P/\partial n)_T < 0$

    See https://www.overleaf.com/7978565786cmqrphtzbjcw for details
- Meta-stable
  https://www.bilibili.com/video/BV1fE411G7Su
  https://www.bilibili.com/video/BV1g5411J7sB





Phase diagram (Based on NJL model)

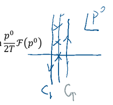
- (Eucilidean) Action:
$$\mathcal{S}_E[q,\bar{q}] = -\int_0^\beta d^4x \left\{\bar{q}[-\gamma^0\partial_\tau + i\gamma\cdot\nabla - m_0]q + G\sum_{a=0}^{3}[(\bar{q}q)^2 + (i\bar{q}\gamma_5\tau q)^2]\right\}$$
- Features:
  - Local interaction among u,d quarks
  - [G]=-2 ⇒ non-renormalizable. Explicit momentum cutoff is introduced
- Symmetry:

Symmetry	Operation	Exact/Approximate	Breaking	Order parameter	Conserved Charge
$U_V(1)$ Global	$q \to e^{i\theta}q$	Exact			Baryon
$SU_V(2)$ Global	$q \to U_V q$	Exact			flavor
$SU_A(2)$ Global	$q_L \to U_A q_L$ $q_R \to U_A^\dagger q_R$ $2\Re U \equiv U_A + U_A^\dagger$ $2i\Im U \equiv U_A - U_A^\dagger$ $\Re U^2 + \Im U^2 = 1$ $q \to (\Re U_A + i\Im U_A\gamma_5)q$ $\bar{q}q \to \bar{q}(\Re U_A + i\Im U_A\gamma_5)^2 q$	Approximate, Explicitly broken by $\bar{q}m_0 q$	By quark $\bar{q}Mq$ @ low temperature & chemical potential	$\bar{q}q$	Axial-flavor

- Partition function:
$$Z = \int \mathcal{D}[q,\bar{q}]\exp\left\{-\mathcal{S}_E[q,\bar{q}] + \mu\int_0^\beta d^4x\ \bar{q}\gamma^0 q\right\}\delta[q(0) + q(-i\beta)]$$
- Bosonization:
$$const. = \int D[\sigma]\, e^{-G\int d^4x\,\sigma^2} = \int D[\sigma]\, e^{-G\int d^4x\,(\sigma - \bar{q}q)^2}$$
$$const. = \int D[\vec{\pi}]\, e^{-G\int d^4x\,\vec{\pi}^2} = \int D[\pi]\, e^{-G\int d^4x\,(\vec{\pi} - i\bar{q}\gamma_5\tau q)^2}$$
$const. = \sqrt{\det G^{-1}}$ has NO temperature and chemical potential dependence
$$Z = \int \mathcal{D}[q,\bar{q}]D[\sigma]D[\vec{\pi}]\exp\left\{\int_0^\beta d^4x\,\{\bar{q}[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - m_0 + 2G\sigma + 2iG\gamma_5\vec{\pi}\cdot\vec{\tau}]q - G(\sigma^2 + \vec{\pi}^2)\}\right\}$$
Bilinear in $(q,\bar{q})$
$$Z = \int D[\sigma]D[\vec{\pi}]\exp\{-\mathcal{S}'_E[\sigma,\vec{\pi}]\}$$
$$\mathcal{S}'_E[\sigma,\vec{\pi}] = -\ln\int \mathcal{D}[q,\bar{q}]\exp\left\{\int_0^\beta d^4x\,\{\bar{q}[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - m_0 + 2G\sigma + 2iG\gamma_5\vec{\pi}\cdot\vec{\tau}]q - G(\sigma^2 + \vec{\pi}^2)\}\right\}$$
$$= \int_0^\beta d^4x\, G(\sigma^2 + \vec{\pi}^2) - \ln\int \mathcal{D}[q,\bar{q}]\exp\left\{\int_0^\beta d^4x\,\{\bar{q}[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - m_0 + 2G\sigma + 2iG\gamma_5\vec{\pi}\cdot\vec{\tau}]q\}\right\}$$
$$= \int_0^\beta d^4x\, G(\sigma^2 + \vec{\pi}^2) - \ln\det\{[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - m_0 + 2G\sigma + 2iG\gamma_5\vec{\pi}\cdot\vec{\tau}]\otimes\mathbb{I}_c\}$$
- Mean Field approx.
$\sigma \to \sigma^*$, $\vec{\pi} \to \vec{\pi}^*$, Both are C-numbers
$\mathcal{S}'_E[\sigma,\vec{\pi}] \to \mathcal{S}'_E(\sigma^*,\vec{\pi}^*) = \beta VG(\sigma^{*2} + \vec{\pi}^{*2}) - \ln\det\{[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}]\otimes\mathbb{I}_c\}$
Notice:
  - $Z = \int D[\sigma]D[\vec{\pi}]\exp\{-\mathcal{S}'_E[\sigma,\vec{\pi}]\} \to Z = \exp\{-\mathcal{S}'_E[\sigma^*,\vec{\pi}^*]\}$
  - $M = m_0 - 2G\sigma^*$ is effective quark mass
  - $-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}$ is a convolution operator ⇒
$$\ln\det\{[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}]\otimes\mathbb{I}_c\} = N_c\sum_n V\int\frac{d^3\vec{k}}{(2\pi)^3}\operatorname{tr}\ln\left[\gamma^0(i\omega_n + \mu) - \vec{\gamma}\cdot\vec{k} - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}\right]$$
Trick1: $\operatorname{tr}\ln A = \operatorname{tr}\ln[U diag\{a_1,\cdots,a_n\}U^{-1}] = \operatorname{tr}\{U\ln[diag\{a_1,\cdots,a_n\}]U^{-1}\} = \operatorname{tr}\ln[diag\{a_1,\cdots,a_n\}] = \ln\prod_i a_i$
$$\gamma^0(i\omega_n + \mu) - \vec{\gamma}\cdot\vec{k} - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau} = \begin{pmatrix} (i\omega_n + \mu - M)\mathbb{I}_2 & -\vec{\sigma}\cdot\vec{k} + 2iG\pi_z^*\mathbb{I}_2 & 0 & 2iG(\pi_x^* + i\pi_y^*)\mathbb{I}_2 \\ \vec{\sigma}\cdot\vec{k} + 2iG\pi_z^*\mathbb{I}_2 & (-i\omega_n - \mu - M)\mathbb{I}_2 & 2iG(\pi_x^* + i\pi_y^*)\mathbb{I}_2 & 0 \\ 0 & 2iG(\pi_x^* - i\pi_y^*)\mathbb{I}_2 & (i\omega_n + \mu - M)\mathbb{I}_2 & -\vec{\sigma}\cdot\vec{k} - 2iG\pi_z^*\mathbb{I}_2 \\ 2iG(\pi_x^* - i\pi_y^*)\mathbb{I}_2 & 0 & \vec{\sigma}\cdot\vec{k} - 2iG\pi_z^*\mathbb{I}_2 & (-i\omega_n - \mu - M)\mathbb{I}_2 \end{pmatrix}$$
Eigenvalues: $-M \pm \sqrt{(i\omega_n + \mu)^2 - \vec{k}^2 - 4G^2\vec{\pi}^{*2}}$ with degeneracy 4
$$\operatorname{tr}\ln\left[\gamma^0(i\omega_n + \mu) - \vec{\gamma}\cdot\vec{k} - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}\right] = 4\ln\left(M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2} - (i\omega_n + \mu)^2\right)$$
$$\ln\det\{[-\gamma^0(\partial_\tau + \mu) + i\gamma\cdot\nabla - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}]\otimes\mathbb{I}_c\} = 4N_c\sum_n V\int\frac{d^3\vec{k}}{(2\pi)^3}\ln\left(M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2} - (i\omega_n + \mu)^2\right)$$
$$= \frac{2N_c V}{T}\int_{C\downarrow + C\uparrow}\frac{dp^0}{2\pi i}\frac{d^3\vec{k}}{(2\pi)^3}\ln\left(M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2} - (p_0 + \mu)^2\right)\tanh\frac{p^0}{2T}$$
Trick2:
$$\int_{C\downarrow + C\uparrow}\frac{dp^0}{2\pi i}\ln\left(A - (p_0 + \mu)^2\right)\tanh\frac{p^0}{2T} = \int_{C\downarrow + C\uparrow}\frac{dp^0}{2\pi i}\int^A d\alpha\frac{1}{\alpha - (p_0 + \mu)^2}\tanh\frac{p^0}{2T}$$
$$= -\int^A d\alpha\ Res\left[\frac{1}{\alpha - (p_0 + \mu)^2}\tanh\frac{p^0}{2T}, p_0 = \pm\sqrt{\alpha} - \mu\right] = \int^A d\alpha\frac{1}{2\sqrt{\alpha}}\left(\tanh\frac{\sqrt{\alpha} - \mu}{2T} + \tanh\frac{\sqrt{\alpha} + \mu}{2T}\right)$$
$$= 2T\left[\ln\cosh\left(\frac{\sqrt{A} - \mu}{2T}\right) + \ln\cosh\left(\frac{\sqrt{A} + \mu}{2T}\right)\right]$$
$$\ln\det\{[-\gamma^0\partial_\tau + i\gamma\cdot\nabla - M + 2iG\gamma_5\vec{\pi}^*\cdot\vec{\tau}]\otimes\mathbb{I}_c\} = 4N_c V\int\frac{d^3\vec{k}}{(2\pi)^3}\left[\ln\cosh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} - \mu}{2T}\right) + \ln\cosh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} + \mu}{2T}\right)\right]$$
Thermodynamic potential: $\Omega \equiv -\frac{T}{V}\ln Z = \frac{T}{V}\mathcal{S}'_E[\sigma^*,\vec{\pi}^*]$
$$\Omega = G(\sigma^{*2} + \vec{\pi}^{*2}) - 4N_c T\int\frac{d^3\vec{k}}{(2\pi)^3}\left[\ln\cosh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} - \mu}{2T}\right) + \ln\cosh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} + \mu}{2T}\right)\right]$$
σ^* and $\vec{\pi}^*$ is obtained by minimize Ω ⇒ **GAP Equation**:
$$\frac{\partial\Omega}{\partial\vec{\pi}^*} = 0 \Rightarrow 2G\vec{\pi}^* + 2N_c\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2G\vec{\pi}^*}{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}}}\left[\tanh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} - \mu}{2T}\right) + \tanh\left(\frac{\sqrt{M^2 + \vec{k}^2 + 4G^2\vec{\pi}^{*2}} + \mu}{2T}\right)\right] \Rightarrow \vec{\pi}^* = 0$$
$$\frac{\partial\Omega}{\partial\sigma^*} = 0 \Rightarrow \sigma^* + 2N_c\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{M}{\sqrt{M^2 + \vec{k}^2}}\left[\tanh\left(\frac{\sqrt{M^2 + \vec{k}^2} - \mu}{2T}\right) + \tanh\left(\frac{\sqrt{M^2 + \vec{k}^2} + \mu}{2T}\right)\right] = 0$$
Once σ^* and $\vec{\pi}^*$ are determined, partition function and $(\partial P/\partial n)$ are thus obtained, which leads to the following **phase diagram**:
- Beyond MF approx. (several comments)
Pick up the fluctuations: $\sigma = \sigma^* + \sigma'$, $\pi = \pi^* + \pi' = \pi'$
$$\mathcal{S}'_E[\sigma',\vec{\pi}'] = \int_0^\beta d^4x\, G(\sigma_*^2 + 2\sigma_*\sigma' + \sigma'^2 + \vec{\pi}'^2) - \ln\det\{[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2G\sigma' + 2iG\gamma_5\vec{\pi}'\cdot\vec{\tau}]\otimes\mathbb{I}_c\}$$
Notice:
  - $-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2G\sigma' + 2iG\gamma_5\vec{\pi}'\cdot\vec{\tau}$ is NOT a convolution operator (x-coordinates enter via $\sigma'(x)$ and $\pi'(x)$) , but
  - Define: $\hat{S}^{-1} \equiv -\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M$, and $\hat{S}^{-1}S(x-y) = \delta(x-y)$ ⇒ $\tilde{S} = \left(\gamma^0(i\omega_n + \mu) - \vec{\gamma}\cdot\vec{k} - M\right)^{-1}\otimes\mathbb{I}_2$
    Homework: check
    - $\langle\bar{q}q\rangle = N_c trS(0) = TN_c\sum_n\int\frac{d^3\vec{k}}{(2\pi)^3}tr\tilde{S}$
    - **Gap equation ⇔ $\sigma^* = \langle\bar{q}q\rangle$**
  - $\ln\det\{[-\gamma^0(\partial_\tau - \mu) + i\gamma\cdot\nabla - M + 2G\sigma' + 2iG\gamma_5\vec{\pi}'\cdot\vec{\tau}]\otimes\mathbb{I}_c\} = N_c\operatorname{tr}\ln\left\{\hat{S}^{-1}\left[\mathbb{I} + \int dy\, S(x-y)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau})\right]\right\}$
$$= N_c\operatorname{tr}\ln\{\hat{S}^{-1}\} + N_c\operatorname{tr}\ln\left[\mathbb{I} + \int dy\, S(x-y)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau})\right]$$
$$+N_c\operatorname{tr}\left\{\int dy\, S(0)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau}) - \frac{1}{2}\int dxdy\, S(x-y)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau})S(y-x)(2G\sigma'(x) + 2iG\gamma_5\vec{\pi}'(x)\cdot\vec{\tau})\right\}$$
  - $\ln(\mathbb{I} + x) = x - \frac{1}{2}x\circ x + \cdots$ ⇒
$$N_c\operatorname{tr}\ln\left[\mathbb{I} + \int dy\, S(x-y)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau})\right]$$
$$= N_c\operatorname{tr}\left\{\int dy\, S(0)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau}) - \frac{1}{2}\int dxdy\, S(x-y)(2G\sigma'(y) + 2iG\gamma_5\vec{\pi}'(y)\cdot\vec{\tau})S(y-x)(2G\sigma'(x) + 2iG\gamma_5\vec{\pi}'(x)\cdot\vec{\tau}) + \cdots\right\}$$
$$= N_c\operatorname{tr}\left\{2GS(0)\int dy\,\sigma'(y) - 2G^2\int dxdy\, S(x-y)\sigma'(y)S(y-x)\sigma'(x) + 2G^2\int dxdy\, S(x-y)\gamma_5\vec{\pi}'(y)S(y-x)\gamma_5\vec{\pi}'(x) + \cdots\right\}$$
  - $\int_0^\beta d^4x\, G\sigma_*^2 - N_c\operatorname{tr}\ln\{\hat{S}^{-1}\} = \beta V\Omega_{MF}$
  - Gap equation ⇒ $\int_0^\beta d^4x\, G(2\sigma_*\sigma') - N_c\operatorname{tr}\left\{2GS(0)\int dy\,\sigma'(y)\right\} = 0$
$$\mathcal{S}'_E[\sigma',\vec{\pi}'] = \beta V\Omega_{MF} + \int_0^\beta d^4x\, G(\sigma'^2 + \vec{\pi}'^2) + 2G^2N_c\operatorname{tr}\left\{\int dxdy\, S(x-y)\sigma'(y)S(y-x)\sigma'(x) - \int dxdy\, S(x-y)\gamma_5\vec{\pi}'(y)S(y-x)\gamma_5\vec{\pi}'(x) + \cdots\right\}$$

Feynman Rules Review:

- Transform to Eucilidean:
  - In action: $i\int dt\, L \to \int_0^\beta d\tau\, L$; $\partial_0 \to i\partial_\tau \Rightarrow p^0 \sim i\partial_0 \to -\partial_\tau \sim i\omega_n$
  - $i\omega_n = 2n\pi iT$ (Bosonic: $i\omega_n\beta = 1 \Rightarrow G(\beta) = G(0)$) or $(2n+1)\pi iT$ (Fermionic: $i\omega_n\beta = -1 \Rightarrow S(\beta) = -S(0)$)
  - $\int\frac{dp^0}{2\pi} \to T\sum_n \square$
- Residue Magic
  - $\frac{1}{2}\tanh\frac{z}{2T}$ has poles at $(2n+1)\pi iT$, with residue equal to T
  - $\frac{1}{2}\coth\frac{z}{2T}$ has poles at $2n\pi iT$, with residue equal to T
  - $\int\frac{dp^0}{2\pi}\mathcal{F}(p^0) \to T\sum_n\mathcal{F}(i\omega_n) \to \frac{1}{2}\int_{C\downarrow + C\uparrow}\frac{dp^0}{2\pi i}\tanh\frac{p^0}{2T}\mathcal{F}(p^0)$ or $\frac{1}{2}\int_{C\downarrow + C\uparrow}\frac{dp^0}{2\pi i}\coth\frac{p^0}{2T}\mathcal{F}(p^0)$
  - As $T \to 0$, $\tanh x \to \coth x \to \operatorname{sgn} x$, reduce to Wick's rotation.
  - $\tanh\frac{z + 2\pi niT}{2T} = \coth\frac{z + (2n+1)\pi iT}{2T} = \tanh\frac{z}{2T}$
  - $\coth\frac{z + 2\pi niT}{2T} = \tanh\frac{z + (2n+1)\pi iT}{2T} = \coth\frac{z}{2T}$



- Propagator:
$$\tilde{S}(p) = -(p^\mu\gamma_\mu - M)^{-1} = -\frac{p^\mu\gamma_\mu + M}{p^2 - M^2}$$
- Gaussian path integration (for fermion):
$$Z \equiv \int D[q,\bar{q}]\, e^{\bar{q}\cdot A\cdot q} = \det A = e^{tr\ln A} \to \exp\left(\sum_n\int\frac{d^3\vec{k}}{(2\pi)^3}\ln\tilde{A}_n(\vec{k})\right)$$

Spinodal Unstable Modes

Linear Response Thoery (see lecture notes on plasma)

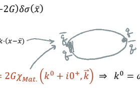
- Decomposition of Hamiltonian: $\hat{H} = \widehat{H_0} + \hat{H}' \leftarrow H' \ll H$ is perturbation
- Decompostion of time-evolution operator:
$\mathcal{U}(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\, H(t)} = \mathcal{U}_0(t_2,t_1)\mathcal{U}_1(t_2,t_1)$
$\mathcal{U}_0(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\, H_0(t)}$
$\mathcal{U}_1(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\, H_I'(t)}$; $\mathcal{O}_I(t) \equiv \mathcal{U}_0^\dagger(t,t_0)\mathcal{O}_S\mathcal{U}_0(t,t_0)$
- Linear response:
$$\mathcal{U}_1(t_2,t_1) \approx I - i\int_{t_1}^{t_2}dt\ H_I'(t)$$
$$\overline{\mathcal{O}(t)} = \operatorname{tr}[\hat{\rho}_S(t)\mathcal{O}_S] = \operatorname{tr}[\mathcal{U}(t,-\infty)\rho_0\mathcal{U}^\dagger(t,-\infty)\mathcal{O}_S] = \operatorname{tr}\left[\rho_0\mathcal{U}_1^\dagger(t,-\infty)\mathcal{U}_0^\dagger(t,-\infty)\mathcal{O}_S\mathcal{U}_0(t,-\infty)\mathcal{U}_1(t,-\infty)\right]$$
$$= \operatorname{tr}\left[\rho_0\mathcal{U}_1^\dagger(t,-\infty)\mathcal{O}_I(t)\mathcal{U}_1(t,-\infty)\right] = \text{expct. w/o perturbation} \to \operatorname{tr}[\rho_0\mathcal{O}_I(t)] - i\int_{-\infty}^t d\bar{t}\,[\mathcal{O}_I(t), H_I'(\bar{t})]$$
$$\overline{\delta\mathcal{O}(t)} = -i\int_{-\infty}^{\infty}d\bar{t}\,\theta(t - \bar{t})[\mathcal{O}_I(t), H_I'(\bar{t})]$$

Spectral Representation

- Green functions **(In thermal equilibrium)** :
  - $G_+(x-y) \equiv \langle\phi_a(x)\phi_b(y)\rangle = \operatorname{Tr}[\rho_0\phi_a(x)\phi_b(y)]$
  - $G_-(x-y) \equiv \pm\langle\phi_b(y)\phi_a(x)\rangle = \pm\operatorname{Tr}[\rho_0\phi_b(y)\phi_a(x)]$
  - $\rho(x-y) \equiv G_+(x,y) - G_-(x,y)$
  - $\Delta(\tau,\vec{x}) = G_+(i\tau,\vec{x})$ $(\tau \in [0,\beta])$ ← Matsubara Green's Function
  - $G_R(x-y) = \theta(x^0 - y^0)\rho(x-y)$ ← Retarded Green's Function
- *Kubo-Martin-Schwinger Relation:*
$G_+(x^0,\vec{x}) = Z^{-1}\operatorname{Tr}\left[e^{-\beta\bar{H}}\phi_a(x)\phi_b(0)\right] = Z^{-1}\operatorname{Tr}\left[e^{ii\beta\bar{H}}\phi_a(x)e^{-ii\beta\bar{H}}e^{-\beta\bar{H}}\phi_b(0)\right]$
$= Z^{-1}\operatorname{Tr}\left[e^{-\beta\bar{H}}\phi_b(0)\phi_a(x^0 + i\beta,\vec{x})\right] = \pm G_-(x^0 + i\beta,\vec{x})$
⇒ $\tilde{G}_+(k) = \int d^4x\, e^{ik\cdot x}G_+(x) = \pm\int d^4x\, e^{ik\cdot x}G_-(x^0 + i\beta,\vec{x}) = \pm e^{\beta k^0}\int d^4x\, e^{ik\cdot x}G_-(x) = \pm e^{\beta k^0}\tilde{G}_-(k)$
$\tilde{\rho}(k) = \tilde{G}_+(k) - \tilde{G}_-(k) = \left(\pm e^{\beta k^0} - 1\right)\tilde{G}_-(k)$
- Spectral representation:
  - $\tilde{G}_-(k) = \frac{1}{\pm e^{\beta k^0} - 1}\tilde{\rho}(k) = \pm f(k^0)\tilde{\rho}(k)$
  - $\tilde{G}_+(k) = \frac{\pm e^{\beta k^0}}{\pm e^{\beta k^0} - 1}\tilde{\rho}(k) = \left(1 \pm f(k^0)\right)\tilde{\rho}(k)$
  - $\tilde{G}_R(k) = \int d^4x\, e^{ik\cdot x}\theta(x^0)\rho(x) = \int d^4x\, e^{ik\cdot x}\left\{i\int\frac{d\omega}{2\pi}\frac{e^{-i\omega x^0}}{\omega + i0^+}\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\tilde{\rho}(q)\right\}$
    $= i\int\frac{dq^0}{2\pi}\frac{\tilde{\rho}(q^0,\vec{k})}{k^0 - q^0 + i0^+}$
  - Exercise: prove that
    $\tilde{\Delta}(i\omega_n,\vec{k}) = -\int\frac{dk^0}{2\pi}\frac{\rho(k)}{i\omega_n - k_0}$ ⇒
    $\tilde{G}_R(k) = -i\tilde{\Delta}(k^0 + i0^+,\vec{k})$

Instability

- Observable: $\bar{q}q \sim \sigma$
- Perturbation: $H_I' = \int d^3\vec{x}\ \bar{q}q\delta M$
  - δM the fluctuating force acting on a quark under M.F. approx.
  - δM is treated CLASSICALLY
  - $\delta M = -2G\delta\sigma$
- Using linear response theory:
$$\overline{\delta\sigma(x)} = -i\int_{-\infty}^{\infty}d^4\bar{x}\,\theta(x^0 - \bar{x}^0)[\bar{q}_I q_I(x), \bar{q}_I q_I(\bar{x})](-2G)\delta\sigma(\bar{x})$$
Define:
$$\chi_{ret.} \equiv \int d^4(x - \bar{x})\ \theta(x^0 - \bar{x}^0)[\bar{q}_I q_I(x), \bar{q}_I q_I(\bar{x})]e^{ik\cdot(x - \bar{x})}$$



After Fourier transform:
$(1 - 2iG\chi_{ret.}(k))\widetilde{\delta\sigma}(k) = 0$ ⇒ $1 = 2iG\chi_{ret.}(k) = 2G\chi_{Mat.}(k^0 + i0^+,\vec{k})$ ⇒ $k^0 = \omega(|\vec{k}|)$
Instability ⇔ imaginary k^0