Phase Transition - 02 (Criticality)

Wednesday, July 19, 2023 1:18 PM

Scaling & Scaling Invariant

- Zooming out = coarse graining (3x3 pixels -> 1 big pixel) + rescaling (1 big pixel -> 1 normal pixel): *Information lost*

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magick dragon_sm.gif    -resize 64x64\>  shrink_dragon.gif
magick terminal.gif  -resize 64x64\>  shrink_terminal.gif
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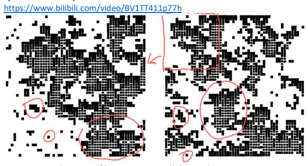
- Inverse process
  https://waifu2x.udp.jp/index.html
  https://www.zhihu.com/question/522277185/answer/2431196144
- Side Story（Application of scaling）
  Q: How do you justify atomism in acient greece?
  A: Differences between atomism & the apeiron scenario:

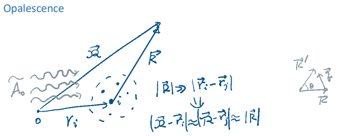
Magnification	Atomism	Apeiron
X1	Transparent liquid with bubbles / waves	Transparent liquid with bubbles / waves
X100	Transparent liquid with bubbles / waves	Transparent liquid with bubbles / waves
$\times 10\cdots 0$	Flying and colliding atoms	Transparent liquid with bubbles / waves

  1) This world, in general, is NOT scaling invariant.
  2) Appearance of anomalous dimension.
  https://www.zhihu.com/question/306019540/answer/3141217013
- systems at critical point
  https://www.bilibili.com/video/BV1TT411p77h



  - Given a density profile, one can not tell the resolution.
  - Distrubution of the cluster size $f(x)$:
    Before zooming out: f(1)=100, f(10)=5, f(10)/f(1)=1/20
    After zooming out: f(1)=100, f(10)=5, f(10)/f(1)=1/20
    $\frac{f(kx)}{f(x)} = g(k) \Rightarrow f(x) = Ax^{-\lambda}; g(k) = k^{-\lambda}$
  - Insight from the video:
    Zooming out not only changes the spin density profile, but also drive the 'temperature' away from T_c effectively.

Critical Opalescence



- Amplitude:
  - From source to particle, $\vec{A}_0 \to \vec{A}_0 \exp\left(i\,\vec{k}\cdot\vec{r}_i\right)$
  - From particle to observer, after scattering with particle $\vec{k} \to \vec{k}' \equiv \vec{k} + \vec{q}$

$$\vec{A}_0 \exp\left(i\,\vec{k}\cdot\vec{r}_i\right) \to \frac{\vec{A}_0 \exp\left(i\,\vec{k}\cdot\vec{r}_i\right)\exp\left(i\,\vec{k}'\cdot\left(\vec{\mathcal{R}}-\vec{r}_i\right)\right)}{\left|\vec{\mathcal{R}}-\vec{r}_i\right|} \sim \frac{\vec{A}_0 \exp\left(-i\,\vec{q}\cdot\vec{r}_i\right)\exp\left(i\,\vec{k}'\cdot\vec{\mathcal{R}}\right)}{R}$$

- Intensity:

$$\frac{\mathrm{d}I(\vec{q})}{dS} \approx \left|\sum_i \frac{\vec{A}_0 \exp\left(-i\,\vec{q}\cdot\vec{r}_i\right)\exp\left(i\,\vec{k}'\cdot\vec{\mathcal{R}}\right)}{R}\right|^2 = \frac{\left|\vec{A}_0\right|^2}{R^2}\sum_{i,j}\exp\left(-i\,\vec{q}\cdot\left(\vec{r}_i-\vec{r}_j\right)\right)$$

$$= \frac{\left|\vec{A}_0\right|^2}{R^2}\int d\vec{r}\,d\vec{r}'\exp\left(-i\,\vec{q}\cdot\left(\vec{r}-\vec{r}'\right)\right)\sum_{i,j}\delta(\vec{r}-\vec{r}_i)\,\delta\left(\vec{r}'-\vec{r}_j\right)$$

  Notice that:
  Density of a single particle located at $\vec{r}^*$ is $n(\vec{r}) = \delta\left(\vec{r}-\vec{r}^*\right)$
  Density of multiple single particles located at $\{\vec{r}^*\}$ is $n(\vec{r}) = \sum_i \delta\left(\vec{r}-\vec{r}_i^*\right)$

$$\sum_{i,j}\delta(\vec{r}-\vec{r}_i)\,\delta\left(\vec{r}'-\vec{r}_j\right) = \langle n(\vec{r})n(\vec{r}')\rangle \equiv G(\vec{r},\vec{r}') \overset{translational\ symmetry}{=} G(\vec{r}-\vec{r}')$$

$$\int d\vec{r}\,d\vec{r}'\,G(\vec{r},\vec{r}') = \int d\vec{r}\,d\vec{r}'\sum_{i,j}\delta(\vec{r}-\vec{r}_i)\,\delta\left(\vec{r}'-\vec{r}_j\right) = N^2$$

$$\frac{dI(\vec{q})}{dS} \approx \frac{\left|\vec{A}_0\right|^2}{R^2}\int d\vec{r}\,d\vec{r}'\exp\left(-i\,\vec{q}\cdot\left(\vec{r}-\vec{r}'\right)\right)G(\vec{r}-\vec{r}') = \frac{\left|\vec{A}_0\right|^2}{R^2}V\int d\vec{r}\exp(-i\,\vec{q}\cdot\vec{r})\,G(\vec{r}) \equiv \frac{\left|\vec{A}_0\right|^2}{R^2}V\tilde{G}(\vec{q})$$

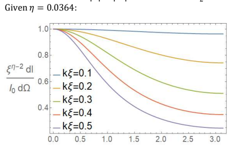
- Critical opalescence
  Near CEP:

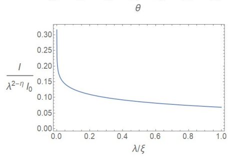
$$G(r) \propto \frac{1}{4\pi}\frac{e^{-\frac{r}{\xi}}}{r^{1+\eta}} \Rightarrow$$

$$V\tilde{G}(\vec{q}) \propto \int\frac{d\vec{r}}{4\pi}e^{-i\vec{q}\cdot\vec{r}}\frac{e^{-\frac{r}{\xi}}}{r^{1+\eta}} = \frac{1}{2}\xi^{2-\eta}\int_0^\infty dr\,r^2\int_{-1}^{1}dx\,e^{-i\xi qrx}\frac{e^{-r}}{r^{1+\eta}}$$

$$= \xi^{2-\eta}\int_0^\infty dr\frac{\sin\xi qr}{\xi q}\frac{e^{-r}}{r^\eta} = \frac{\Gamma(1-\eta)}{(1+(\xi q)^2)^{\frac{1-\eta}{2}}}\frac{\xi^{2-\eta}}{\xi q}\sin[(1-\eta)\arctan\xi q]$$

  As $\eta \to 0$, $V\tilde{G}(\vec{q}) \propto \frac{\xi^2}{1+(\xi q)^2}$
  Elastic Scattering $\Rightarrow \left|\vec{k}\right| = \left|\vec{k}'\right| \Rightarrow q = 2|k|\sin\frac{\theta}{2}$
  Given $\eta = 0.0364$:





  As approaching CEP
  - Scattering becomes more forward
  - Intensity diverges at CEP.
- Partonic Critical Opalescence
  See arXiv:2208.14297

Renormalization Group Formalism

Basic Idea:

- Partition function preserves after scaling $\Rightarrow$ Effective Hamiltonian after scaling
- Parameters in Hamiltonian varies under scaling
- Existence of fixed point
- Nearby the unstable fixed point, varying some parameters (called relevant parameters) are equivalent to varying the thermal parameters (e.g., temperature, magnetic field, cheimical potential...) $\Rightarrow$
- Obtain the dependence of partition function on the thermal parameters nearby the unstable fixed point (critical point). $\Rightarrow$ critical exponents

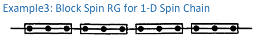
Example1: central limit theorem

https://www.zhihu.com/question/571460160/answer/3135590699

Example2: 1-d Percolation model

Chpt. 7 in《相变与临界现象》——于渌、郝柏林

Example3: Block Spin RG for 1-D Spin Chain



- Hamiltonian:
  $H[\{s_r\}] = -J\sum_r s_r s_{r+1}$
  $s = \pm 1$
- Partition function (j=J/T; low T-> large j) :
  $Z = \sum_{\{s_r\}} e^{-\beta H} = \sum_{\{s_r\}} e^{j\sum_r s_r s_{r+1}}$
- Projection operator $\mathcal{P}(s';\{s_i\}_{block})$
  - $\sum_{s'}\mathcal{P}(s';\{s_i\}_{block}) = 1$
  - $Z = \sum_{\{s_r\}}\sum_{\{s'\}}\mathcal{P}(s';\{s_i\}_{block})\,e^{j\sum_r s_r s_{r+1}} = \sum_{\{s'\}} e^{-\beta H'[\{s'\}]}$
    $e^{-\beta H'[\{s'\}]} = \sum_{\{s_r\}}\mathcal{P}(s';\{s_i\}_{block})\,e^{j\sum_r s_r s_{r+1}}$
  - In this example (b=3), $\mathcal{P}(s';\{s_{i-1},s_i,s_{i+1}\}) = \delta_{s' s_i}$
    Justification:
    - chance of $s_{i-1} = s_{i+1} = -s_i$ is $\frac{e^{-2j}}{e^{-2j}+2+e^{2j}} \ll \frac{1}{4}$ (supressed)
    - Suitable for 1-d only
- Effective Hamiltonian:

$$e^{-\beta H'[\{s'\}]} = \sum_{\{s_r\}}\delta_{s_i' s_{3i}}e^{j\sum_r s_r s_{r+1}}$$

$$= \sum_{\{s_r\}}\delta_{s_i' s_{3i}}e^{j\sum_i(s_{3i}s_{3i+1}+s_{3i+1}s_{3i+2}+s_{3i+2}s_{3i+3})}$$

$$= \sum_{\{s_r\}}e^{j\sum_i(s_i' s_{3i+1}+s_{3i+1}s_{3i+2}+s_{3i+2}s_{i+1}')}$$

  trick: $e^{js_i s_{i+1}} = \cosh j + s_i s_{i+1}\sinh j$

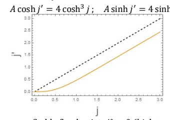
$$e^{-\beta H'[\{s'\}]} = \sum_{\{s_r\}}\prod_i(\cosh j + s_i' s_{3i+1}\sinh j)(\cosh j + s_{3i+1}s_{3i+2}\sinh j)(\cosh j + s_{i+1}' s_{3i+2}\sinh j)$$

$$= \sum_{\{s_r\}}\prod_i\left[(\cosh^3 j + s_i' s_{i+1}'\sinh^3 j) + linear\ terms\ of\ s_{3i+?}\right] \leftarrow vanish\ after\ summation$$

$$= \prod_i 4(\cosh^3 j + s_i' s_{i+1}'\sinh^3 j) = \prod_i A(\cosh j' + s_i' s_{i+1}'\sinh j') = A^{N'}e^{\sum_i j' s_i' s_{i+1}'}$$

$$\beta H'[\{s'\}] = -j'\sum_i s_i' s_{i+1}' - N'\ln A$$

  Second term: contribution of UV physics
- RG flow of j
  $A\cosh j' = 4\cosh^3 j$; $A\sinh j' = 4\sinh^3 j \Rightarrow A = 4\sqrt{\cosh^6 j - \sinh^6 j}$; $\boldsymbol{j' = \mathrm{arctanh}\tanh^3 j}$



  - Stable fixed point: $j^* = 0$ (high temperature)
  - Unstable fixed point $j^* = \infty$ ($T = 0$)
  - No phase transition for 1-D spin chain $\Rightarrow$ Disappearing of order under a certain dimension d_l
    - $d_l = 1$ for discrete symmetry
    - $d_l = 2$ for continuous symmetry
- Correlation length
  - $\xi = \xi(H) = \xi(z); z \equiv \tanh j; z' = z^3$
  - $\xi(z') = \frac{\xi(z)}{3}$
  - $\xi(z^b) = \frac{\xi(z)}{b} \Rightarrow \xi \propto \frac{1}{\ln z} = \frac{1}{\ln\tanh j}$

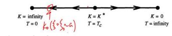
Review on Critical Exponents:

- Heat capacity: $C \sim A_\pm|t|^{-\alpha}$ ($\alpha_+ = \alpha_-, h = 0$)
- Order parameter:
  - $\langle O\rangle \propto (-t)^\beta$ ($h = 0$)
  - $\langle O\rangle \propto |h|^{1/\delta}$ ($t = 0$)
- Susceptibility: $\chi \propto |t|^{-\gamma}$ ($h = 0$)
- Correlation length: $\xi \propto |t|^{-\nu}$ (h=0)
- Correlation EXACTLY @ CEP: $G(r) \sim r^{-(d-2+\eta)}$
- Relaxation: $\tau \propto \xi^z$ (critical slowing down)

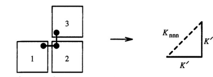
Renormalization Group Formalism

Extension: Schematic Block Spin RG for higher dimension

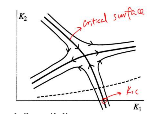
- RG flow (for single parameter)
  - Stable fixed point: $j^* = 0$ (high temperature)
  - At low temperatures, $j' \sim j$ for 1-d
    - Reason: spins align almost in the same direction,
      $j' = -\beta H_i(s_i' = 1, s_{i+1}' = 1) \sim j\langle s_{3i+1}\rangle_{s_i'=1}\langle s_{3i+2}\rangle_{s_{i+1}'=1} \sim j$
    - Extension to higher dimension:
      $j' = -\beta H_{i,l}(s_{i,l}' = 1, s_{i+1,l}' = 1) \sim \sum_{\bar{l}} j\langle s_{3i+1,\bar{l}}\rangle_{s_{i,l}'=1}\langle s_{3i+2,\bar{l}}\rangle_{s_{i+1,l}'=1} \sim 3^{d-1}j$
    - $j' > j$ for $d \geq 2 \Rightarrow$ stable fixed point at $j \to \infty$
  - Unstable fixed point j_c in middle
    - Flow from unstable fixed point to stable ones
    - Flow slowly around j_c



- RG flow (for multiple parameters)
  - Emerge of next-leading terms



  - Theoreical (or parameter) space
  - Evolution arround fixed point



    $\{K'\} = R(\{K\})$
    $\{K^*\} = R(\{K^*\})$
    For $K = K^* + \delta K$
    $\{K'\} = R(\{K^*\} + \{\delta K\}) = R(\{K^*\}) + R'(\{K^*\})\{\delta K\} = \{K^*\} + R'(\{K^*\})\{\delta K\}$
    $\{\delta K'\} = R'(\{K^*\})\{\delta K\}$
    After diagnolization,
    $u_i' = b^{y_i}u_i$
  - Relevant parameters
    - $y > 0$
    - # of relevant parameters = # of thermal parameters (=2)
  - Irrelavant parameters
    - $y < 0$
    - Critical surface that seperating the parameter space
  - For ising model:
    - $\delta j = j - j_c = J\left(\frac{1}{T} - \frac{1}{T_c}\right) \approx -\frac{Jt}{T_c} \propto t \Rightarrow u_t \propto t$
    - $u_h \propto h$
  - Relation with beta function
    Consider an infinitesimal scaling: $b = 1 + d\lambda$; $d\lambda \ll 1$
    Variation of length scale: $\Delta L = L\,b - L = d\lambda L$
    Beta function: $\beta(K) = \mu\frac{dK}{d\mu} = -L\frac{dK}{dL} = -\frac{dK}{d\lambda}$
    $\overrightarrow{K'} = \vec{K} - \vec{\beta}\left(\vec{K}\right)d\lambda$
    Near CEP: $\vec{\beta}\left(\overrightarrow{K^*}\right) = 0$
    $\overrightarrow{\delta K'} = \overrightarrow{\delta K} - \vec{\beta}\left(\overrightarrow{K^*} + \delta\vec{K}\right)d\lambda = \overrightarrow{\delta K} - \left[\vec{\beta}\left(\overrightarrow{K^*}\right) + \nabla_K\vec{\beta}\left(\overrightarrow{K^*}\right)\cdot\delta K + \cdots\right]d\lambda = \left[I - d\lambda\nabla_K\vec{\beta}\left(\overrightarrow{K^*}\right)\right]\cdot\{\delta K\}$
    $R'(\{K^*\}) = I - d\lambda\nabla_K\vec{\beta}(\{K^*\})$
    If $R'(\{K^*\})$ is diagonalized
    $b^{y_i} = (1 + d\lambda)^{y_i} \approx 1 + y_i d\lambda = 1 - d\lambda\partial_{K_i}\beta_i(\{K^*\}) \Rightarrow y_i = -\partial_{K_i}\beta_i(\{K^*\}) \to eigenvalue\ of\ -\partial_{K_i}\beta_j(\{K^*\})$
    y_i can be calculated perturbatively with field theory !!
- Free energy per site
  $f(H) \equiv \frac{F}{NT} = \frac{1}{N}\ln Z(H)$
  After zooming out, $N \to N' = b^{-d}N$;
  $Z(u_t, u_h, N) = e^{Nf(u_t,u_h)} = Z(u_t', u_h', N') = e^{N'f(u_t',u_h')} \Rightarrow$
  $f(u_t, u_h) = b^{-d}f(b^{y_t}u_t, b^{y_h}u_h) = \cdots = b^{-nd}f(b^{ny_t}u_t, b^{ny_h}u_h)$
  After n steps of coarse graining, u_t evolves to a CERTAIN point u_{t0} besides the fixed point, with $u_{t0} = b^{ny_t}u_t \to b^n = (u_{t0}/u_t)^{1/y_t}$
  $f(u_t, u_h) = (u_{t0}/u_t)^{-d/y_t}f\left(u_{t0}, (u_{t0}/u_t)^{y_h/y_t}u_h\right)$
  Since u_{t0} is a CONSTANT, $f\left(u_{t0}, (u_{t0}/u_t)^{y_h/y_t}u_h\right) = \phi\left(u_h u_t^{-y_h/y_t}\right) \Rightarrow$
  $f(u_t, u_h) \propto u_t^{d/y_t}\phi\left(u_h u_t^{-y_h/y_t}\right) \sim t^{d/y_t}\phi\left(ht^{-y_h/y_t}\right)$
- Correlation function
  - Generating function $Z \to Z[\hbar] = Ze^{\sum_r \hbar(r)s(r)}$
    $\hbar$ is an AUXILARY & LOCAL magnetic field $\Rightarrow \hbar' = b^{y_h}\hbar$
    $G(r_1, r_2; t, h) \equiv \langle s(r_1)s(r_2)\rangle = \left.\frac{\partial^2\ln Z[\hbar]}{\partial\hbar(r_1)\partial\hbar(r_2)}\right|_{\hbar=0}$
  - Scaling rule of spin-spin correlation function:
    $\ln Z[\hbar] = \cdots + \frac{1}{2}\sum_{r_1,r_2}\hbar(r_1)\,G(r_1, r_2; t, h)\hbar(r_2) + \cdots \approx \cdots + \frac{N^2}{2}\hbar(r_1)G(r_1, r_2; t, h)\hbar(r_2) + \cdots$
    $= \ln Z'[\hbar'] = \cdots + \frac{1}{2}\sum_{r_1',r_2'}\hbar'(r_1')\,G(r_1', r_2'; t', h')\hbar'(r_2') + \cdots \approx \cdots + \frac{N'^2}{2}\hbar'(r_1')G(r_1', r_2'; t', h')\hbar'(r_2') + \cdots$
    With $r' = r/b$ & $N' = N/b^d \Rightarrow$
    $G(r_1, r_2; t, h) = b^{-2(d-y_h)}G\left(\frac{r_1}{b}, \frac{r_2}{b}; b^{y_t}t, b^{y_h}h\right) = \cdots = b^{-2n(d-y_h)}G\left(\frac{r_1}{b^n}, \frac{r_2}{b^n}; b^{ny_t}t, b^{ny_h}h\right)$
    Consider a CERTAIN point beside the fixed point, with $t_0 = b^{ny_t}t \to b^n = (t_0/t)^{1/y_t}$
    $G(r; t, h) = \left(\frac{t_0}{t}\right)^{\frac{-2(d-y_h)}{y_t}}G\left(r\left(\frac{t}{t_0}\right)^{1/y_t}; t_0, \left(\frac{t_0}{t}\right)^{y_h/y_t}h\right) \propto t^{\frac{2(d-y_h)}{y_t}}g\left(rt^{1/y_t}; ht^{-y_h/y_t}\right)$
  - Extension to other observables:
    Suppose $H = \cdots + \sum_{r,i}\mathcal{O}_i(r)u_i + \cdots$
    $\langle\mathcal{O}_1(r_1)\mathcal{O}_2(r_2)\cdots\mathcal{O}_n(r_n)\rangle_H = b^{-nd+y_1+\cdots+y_n}\left\langle\mathcal{O}_1\left(\frac{r_1}{b}\right)\mathcal{O}_2\left(\frac{r_2}{b}\right)\cdots\mathcal{O}_n\left(\frac{r_n}{b}\right)\right\rangle_{H'}$
- Critical Exponents
  - Heat capacity: $C \propto \partial^2 f/\partial t^2\big|_{h=0} \propto t^{d/y_t-2}\phi(0) \Rightarrow \alpha = 2 - d/y_t$
  - Order parameter:
    - At $h = 0$, $\langle M\rangle \propto \partial f/\partial h \propto \phi'(0)t^{\frac{d-y_h}{y_t}} \Rightarrow \beta = \frac{d-y_h}{y_t}$
    - Assume $\phi(x) \propto x^\kappa + \cdots \to f \propto h^\kappa t^{\frac{d-\kappa y_h}{y_t}} + \cdots \Rightarrow \langle M\rangle \propto \partial f/\partial h \propto h^{\kappa-1}t^{\frac{d-\kappa y_h}{y_t}} + \cdots$
      If at $t = 0$ and $h \neq 0$, a FINITE magnatization is expected, one demand $\frac{d-\kappa y_h}{y_t} = 0 \to \kappa = \frac{d}{y_h} \Rightarrow \frac{1}{\delta} = \frac{d}{y_h} - 1$
  - Susceptibility at h=0: $\chi \propto \partial^2 f/\partial h^2 \propto \phi''(0)\,t^{\frac{d-2y_h}{y_t}} \Rightarrow \gamma = -\frac{d-2y_h}{y_t}$
  - Correlation length:
    $\frac{r}{\xi} \sim rt^{\frac{1}{y_t}} \Rightarrow \xi \propto t^{\frac{-1}{y_t}} \Rightarrow \nu = \frac{1}{y_t}$
  - Correlation at critical point ($t = 0, h = 0$):
    Assume $g\left(rt^{1/y_t}; ht^{-y_h/y_t}\right) \propto \left(rt^{1/y_t}\right)^\zeta + \cdots \to G(r) \propto r^\zeta t^{\frac{\zeta+2(d-y_h)}{y_t}} + \cdots$
    If at CEP, a FINITE correlation function is expected, one demand $\frac{\zeta+2(d-y_h)}{y_t} = 0 \to \zeta = -2d + 2y_h \Rightarrow G(r) \propto r^{-2d+2y_h}$; $\eta = d - 2y_h + 2$
  - Relations among the critical exponents
    - $\alpha + 2\beta + \gamma = 2 - d/y_t + \frac{2d-2y_h}{y_t} - \frac{d-2y_h}{y_t} = 2$
    - $\alpha + \beta(1+\delta) = 2 - d/y_t + \frac{d-y_h}{y_t}\left(\frac{y_h}{d-y_h} + 1\right) = 2$
    - $\alpha = 2 - d\nu$
    - $\nu(2-\eta) = \frac{1}{y_t}(2y_h - d) = \gamma$