QCD-01 (Reference + Overview + category)

Sunday, February 12, 2023 10:01 AM

References:
- <Quantum Chromodynamics> by Walter Greiner
- <Introduction to High-energy Heavy-ion Collisions> by Cheuk-Yin Wong
- [2212.11107] 50 Years of Quantum Chromodynamics (arxiv.org)
- <Weak Interactions> by Howard Georgi
- [hep-th/0010225v1] Monopoles, Instantons and Confinement (arxiv.org)
- <QCD sum rules, a modern perspective>
- 王一研究宇宙的个人空间_哔哩哔哩_bilibili

Pre-request:
QFT-I
Group theory and Lie algebra

Convention:
- $\hbar = c = 1 = 197 MeV \cdot fm$
- $g^{\mu\nu} = diag\{1,-1,-1,-1\}$
- % means dagger
- $\langle \vec{p}|\vec{p}'\rangle = \left[a_p, a_{p'}^{\%}\right]_{\pm} = (2\pi)^3\delta^3(\vec{p}-\vec{p}')$
- $\int \frac{d^3\vec{p}}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}| = I$ is Lorentz invariant
  So is $\int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2-m^2)\theta(p^0) = \int \frac{d^3\vec{p}}{2E_p(2\pi)^3}$
  $\rightarrow \sqrt{2E_p}\sum_s |p,s\rangle$ is Lorentz invarinat
  If a^μ is a Lorentz vector, so is $2\sqrt{E_pE_p'}\sum_{s,s'}\langle p,s|a^\mu|p',s'\rangle$
- $T^{(\mu\nu)} \equiv \frac{1}{2}(T^{\mu\nu}+T^{\nu\mu})$
- $T^{[\mu\nu]} \equiv T^{\mu\nu} - T^{\nu\mu}$

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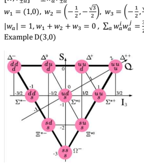
Representation of (FLAVOR) SU(3) group

- SU(3) Group: $U = e^{i\phi_a\tau_a}$, summation on a from 1 to 8
  $U^{\%}U = I$, $\det U = 1$
- Center (subgroup composed of elements commuting with all):
  $Z_3 = \{e^{2n\pi i/3}I_3\}$, $\det[e^{2n\pi i/3}I_3] = 1$
- Generator τ_a satisfies:
  $[\tau_a,\tau_b] = i f_{abc}\tau_c$, $\{\tau_a,\tau_b\} = \frac{1}{3}\delta_{ab}I + d_{abc}\tau_c$, $\tau^{\%} = \tau$
  f_{abc} & d_{abc} are independent of representation
- Casimir Operator:
  - Product of the generators
  - Commute with all the generators, e.g., J^2 in SU(2) or SO(3)
  - Characterize the irreducible representation of the group.
  - Two Casimir operators for SU(3):
    $C_1 = \sum_a \tau_a\cdot\tau_a$, $C_2 = \sum_{abc} d_{abc}\tau_a\tau_b\tau_c = \sum_{a,b}\{\tau_a,\tau_b\}\tau_a\tau_b - \frac{1}{3}C_1$
    Irreducible representation for SU(3) D(p,q) is labelled by TWO numbers:
    ? p: # of quarks; q: # of anti-quarks.
    $C_1 = \frac{1}{3}(p^2+q^2+3p+3q+pq)$
    $C_2 = \frac{1}{18}(p-q)(3+p+2q)(3+q+2p)$
    $p \leftrightarrow q$, $C_1 \rightarrow C_1$, $C_2 \rightarrow -C_2$, (Rev. Mod. Phys. 38, 215 (1966) - Dynamical Symmetry in Particle Physics (aps.org))
- Dimension of the irreducible representation:
  $d(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$
- Fundamental representation D(1,0): $\tau_a = \lambda_a/2$, where λ_a is the Gell-Mann matrices.
  $C_1 = \frac{4}{3}I_3$, $C_2 = 10/9\, I_3$
- Catan-Weyl basis
  - Ladder operators (switch the states in the flavor space):
    $V_{\pm1} = \tau_1 \pm i\tau_2$, $V_{\pm2} = \tau_4 \mp i\tau_5$, $V_{\pm3} = \tau_6 \pm i\tau_7$
    Under D(1,0)
    $V_{\pm1}$: $u \leftrightarrow d$ $V_{\pm2}$: $u \leftrightarrow s$ $V_{\pm3}$: $s \leftrightarrow d$
  - z-components:
    $I_3 = \tau_3$, (isospin) $Y = \frac{2}{\sqrt{3}}\tau_8$ (hypercharge)
    $[I_3, Y] = 0 \rightarrow$ They share eigenstates
    Under D(1,0), $I_3 = diag\{\frac{1}{2},-\frac{1}{2},0\}$, $Y = diag\{\frac{1}{3},\frac{1}{3},-\frac{2}{3}\}$
  - Irreducible representation: $|t,y\rangle$
    $I_3|t,y\rangle = t|t,y\rangle$; $Y|t,y\rangle = y|t,y\rangle$
- Root vectors
  Diagonal generators H = (τ_3,τ_8)
  $[H, V_{\pm a}] = \pm w_a V_{\pm a}$
  $w_1 = (1,0)$, $w_2 = \left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, $w_3 = \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
  $|w_a| = 1$, $w_1+w_2+w_3 = 0$, $\sum_a w_a^i w_a^j = \frac{3}{2}\delta_{ij}$
- Example D(3,0)

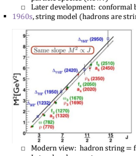


Quark Bare Masses (from Chiral EFT):

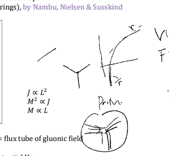
- Lagrangian for Goldstone (pseudo-scalar) boson under EXACT chiral symmetry
  $\mathcal{L}_{ps} = \frac{f^2}{4}\mathrm{tr}[\partial_\mu U^{\%}\partial^\mu U]$, $U = \exp\left(\frac{2i\varpi^a\tau_a}{f}\right)$
  $[f] = [\varpi] = 1 \rightarrow f = f_0\Lambda_{QCD}$
  Invariant under $U \rightarrow LUR^{\%}$
  If $L = R \rightarrow$ normal GLOBAL $SU_f^V(3)$ transformation
  If $L = R^{\%} \rightarrow$ chiral GLOBAL $SU_f^A(3)$ transformation
  Leading order:
  $\mathrm{tr}[\partial_\mu\varpi^a\tau_a\partial^\mu\varpi^b\tau_b] = \frac{1}{2}\mathrm{tr}[\partial_\mu\varpi^a\partial^\mu\varpi^b\{\tau_a,\tau_b\}] = \frac{1}{2}\partial_\mu\varpi^a\partial^\mu\varpi_a + O\left(\frac{\partial}{f}\right)$
- Extension of chiral QCD:
  $\mathcal{L}_{QCD} \rightarrow \mathcal{L}_{QCD}(m_Q = 0) + \bar{q}_L s q_R + \bar{q}_R s^{\%} q_L$
  $[q_{L/R}] = 3/2$; $[s]=1$
  Invariant under $q_L \rightarrow Lq_L$, $q_R \rightarrow Rq_R$, $s \rightarrow LsR^{\%}$
- Lowest order extension of Goldstone Lagrangian
  $\mathcal{L}_{ps} \rightarrow \mathcal{L}_{ps} + v\frac{f^2}{4}\mathrm{tr}[s^{\%}U + sU^{\%}]$
  $[v] = 1$
  Invariant under both the normal and chiral flavor SU(3) transformation
- Breaks the chiral symmetry EXPLICITLY.
  $s \rightarrow M_Q + s'$, $M_Q = \mathrm{diag}\{m_u, m_d, m_s\}$
  $v\frac{f^2}{4}\mathrm{tr}[s^{\%}U + sU^{\%}] \approx v\frac{f^2}{4}\mathrm{tr}[M_Q(U+U^{\%})] + \cdots$
  Leading order:
  $\frac{vf^2}{2}(m_u+m_d+m_s) - v\mathrm{tr}[M_Q\tau_a\tau_b]\varpi^a\varpi^b + O(f^{-2})$
  $\rightarrow$ Mass of the a-flavor pseudo-scalar meson: $M_a^2 = 2v\mathrm{tr}[M_Q\tau_a\tau_a]$
- List of the squared meson mass:
  $\frac{v}{2}\Big\{m_u+m_d, m_u+m_d, m_u+m_d, m_u+m_s, m_u+m_s, m_s+m_d, m_s+m_d, \frac{1}{3}(m_u+m_d+4m_s)\Big\}$
  pion, $K^\pm$, $K^0(\bar{K}^0)$, η
- Including the EM correction to $m_{\pi^\pm}$ and $m_{K^\pm}$
  If neglect the mass difference between d/s quarks at a reasonable energy scale, the EM corrections on both the $\pi^\pm$ and $K^\pm$ are considered same.
  $m_{\pi^\pm}^2 = \frac{v}{2}(m_u+m_d) + \Delta_{EM}$
  $m_{\pi^0}^2 = \frac{v}{2}(m_u+m_d)$
  $m_{K^\pm}^2 = \frac{v}{2}(m_u+m_s) + \Delta_{EM}$
  $m_{K^0/\bar{K}^0}^2 = \frac{v}{2}(m_s+m_d)$
  $m_\eta^2 = \frac{v}{6}(m_u+m_d+4m_s)$
- Break of isospin symmetry:
  $m_u = v^{-1}(2m_{\pi^0}^2 - m_{\pi^\pm}^2 + m_{K^\pm}^2 - m_{K^0}^2)$
  $m_d = v^{-1}(m_{\pi^\pm}^2 - m_{K^\pm}^2 + m_{K^0}^2)$
  $m_s = v^{-1}(-m_{\pi^\pm}^2 + m_{K^\pm}^2 + m_{K^0}^2)$
  $m_u : m_d : m_s \approx 1 : 1.78 : 36.03$
  $m_u < m_d$
  - Not due to EM interaction since u carries more charge → Isospin symmetry breaks down
  - Proton is more stable than neutron in vacuum.
  - Isospin/flavor symmetry approximately preserved since both the u/d/s masses are much smaller than Λ_{QCD}
- η' mass & break down of $U_A(1)$ symmetry.
  - Generator: $\tau^0 \equiv \frac{1}{\sqrt{6}}I_3$
  - Mass from chiral EFT:
    $M_{\eta'}^2 = 2v\mathrm{tr}[M_Q\tau_0\tau_0] = \frac{v}{3}(m_u+m_d+m_s) \approx 0.53M_\eta^2$
  - In reality:
    $M_\eta \approx 548$ MeV $< M_{\eta'} \approx 958$ MeV $\rightarrow$
    $U_A(1)$ is not even an approximate symmetry of QCD.
  - How:
    $\partial\cdot j_5 \neq 0$ due to chiral anomaly

Overview of Pre-QCD Era:

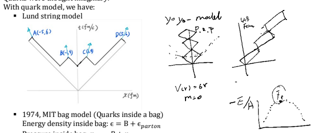
- Discovery of the strong force:
  - 1932, discovery of neutron, by Chadwick
    Strong force required to bound the positively charged nucleus
  - 1932, Isospin symmetry (not exact) proposed, by Heisenberg
  - Model the strong force:
    - Potential models preserving isospin symmetry -> shell model of nucleus
    - (Local) field theory, 1935, nuclear force mediated by pion, Yukawa (汤川秀树)
      Interaction: $i g\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\pi}\psi \rightarrow \frac{f}{m}\bar{\psi}\gamma_5\vec{\tau}\cdot\bar{\psi}\gamma^\mu\partial_\mu(\vec{\pi}\psi) + h.c.$
      Obstacles:
      - Too large coupling constant & non-realistic interaction
      - 1950-60s, (local) field theory became less popular than S-matrix theory + Regge pole (1959, Regge):
        $\langle \alpha_f|S|\beta_f\rangle \equiv (\alpha, +\infty|\beta, -\infty)$, $S^{\%}S = 1$
        $S = \frac{\Gamma(l - g(E))}{\Gamma(l + g(E))}e^{i\pi l}$
        Poles @ $l - g(E) = -n \in N$ corresponds to the bound states
        Later development: boundary conformal theory in AdS space
  - 1954, Non-Abelian Gauge theory of nuclear force, by Yang and Mills
    Proposed earlier by W. Pauli. However, since gauge boson was considered massless, he thought the gauge theory describes long-range interactions only.
  - 1960, small pion mass (137 MeV) and large nucleon mass (939MeV) due to spontaneous chiral symmetry breaking, by Nambu (2008 Nobel Prize)
    - Chiral symmetry suggests vanishing nucleon bare mass (parameter)
    - Pion plays the role of massless Goldstone boson
    - $m_\pi \sim 137$ MeV → Chiral symmetry is NOT Exact
      from chiral effective theory + Goldstone theory:
      Goldstone field = order parameter $(\bar{\psi}\psi) \times$ Generator $(\gamma_5\tau^a$ in flavor$)$
      Mass of Goldstone fields:
      $M_a^2 = \frac{8v^3}{f^2}tr[M_q\tau^a\tau^a]$
      Gellmann-Okubo relation: $3m_\eta^2 + m_\pi^2 = 4m_K^2$ (1962, Gellmann, Okuba)
- Hadrons are not fundamental:
  - 1940, neutron has non-vanishing magnetic moment, by Luis W. Alvarez from UCBerkely & F. Bloch from Stanford
  - 1950s-1960s, a large amount of hadron species, including strange particles, were discovered. (Particle zoo)
  - 1953, discovery of strangeness, by M. Gellmann & K. Nishijima
    Isospin symmetry => (approximate) flavor (once named as chocolate, vanilla, strawberry) symmetry
  - Models, under S-matrix formalism, other than quark model:
    - 1962, Bootstrap model, by Geoffrey Chew, et. al:
      - Hadrons are composed of hadrons
      - Nuclear democracy (no hadron is more fundamental than others)
      - With the particle spectrum obtained via Regge trajectory, formulate the S-matrix for infinitely many particle species (How?)
      - Later development: conformal bootstrap
    - 1960s, string model (hadrons are strings), by Nambu, Nielsen & Susskind



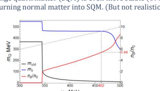
$J \propto L^2$
$M^2 \propto J$
$M \propto L$



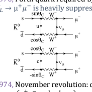
      - Modern view: hadron string = flux tube of gluonic field
      - Later development:
        - suggest the confining potential $V \propto r$
        - String fragment mechanism in PYTHIA
        - String theory
  - 1961, Eightfold way, by Gellmann
    Organize the hadrons by isospin I_3 & hypercharge Y = B + S
    Triumph: predict Ω^- (sss) (discovered in 1963)
  - 1964, Current algebra, by Gellmann
    - Semi-leptons decays are related with currents, e.g., $\langle 0|j_5^\mu|\pi^-\rangle = f_\pi p^\mu$
    - They follow commutation relation:
      $[j_0^a(\vec{x}), j_0^b(\vec{y})] = i f_{abc}\delta(\vec{x}-\vec{y})j_0^c(\vec{x})$
      $[j_0^a(\vec{x}), j_{50}^b(\vec{y})] = i f_{abc}\delta(\vec{x}-\vec{y})j_{50}^c(\vec{x})$
      $[j_{50}^a(\vec{x}), j_{50}^b(\vec{y})] = i f_{abc}\delta(\vec{x}-\vec{y})j_0^c(\vec{x})$
  - 1964, quark model, by M. Gellmann and Zwig
    Problem: no one have ever seen a single quark (with fractional electron charge)
    Quarks were thought imaginary.
    With quark model, we have:
    - Lund string model



    - 1974, MIT bag model (Quarks inside a bag)
      Energy density inside bag: $\epsilon = B + \epsilon_{parton}$
      Pressure inside bag: $p = -B + p_{parton}$
      Predictions:
      - Cold stable quark matter
        @ zero temperature, $\epsilon \ll k_F^4$, $n \ll k_F^3 \rightarrow \epsilon \propto n^{4/3}$
        $E/A = B/n + \frac{3}{4}n^{1/3}$ have minumum @ certain density.
        Strange quark matter (SQM) is even more stable (1984, Witten) -> strangelet that destroy the world by turning normal matter into SQM. (But not realistic since....)

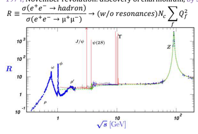


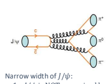
      - Quark deconfinement and form QGP if $p_{parton} > B$.
  - 1968, Deep inelastic scatterings & Bjorken scaling => Point-like partons inside hadron, by Bjorken & Feynmann
  - 1970, Forth quark required by GIM mechanism, by S. Glashow, J. Iliopoulos, L. Maiani
    $K_L \rightarrow \mu^+\mu^-$ is heavily suppressed due to the cancellation of the two processes:



$$\begin{pmatrix} d_W \\ s_W \end{pmatrix} = \begin{pmatrix} \cos\theta_C & -\sin\theta_C \\ \sin\theta_C & \cos\theta_C \end{pmatrix}\begin{pmatrix} d_M \\ s_M \end{pmatrix}$$

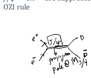
  - 1974, November revolution: discovery of charmonium, by S. Ting, et. al., @ SPEAR & SLAC
    $R \equiv \frac{\sigma(e^+e^- \rightarrow hadron)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \rightarrow (w/o\ resonances)\, N_c\sum_q Q_q^2$



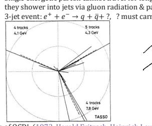


Narrow width of J/ψ:
1. J/ψ is NOT comprised by hadrons
2. True reason:
   a. $J/\psi \rightarrow D(q\bar{c})\bar{D}$ is forbidden since $M_{J/\psi} > 2M_D \sim 3.6$ GeV
   b. $J/\psi \rightarrow 3\pi$ -- are suppressed due to OZI rule

- Towards quantum chromodynamics
  - 1964, propose of color, by Han & Nambu
    - Wavefunction of $\Delta^{++}(uuu)$ or $\Omega^-(sss)$ symmetric in flavor & spin & space anti-symmetric over all.
    - Neutral pion EM decay
      $\Gamma_{theo} \sim \frac{1}{9}\Gamma_{exp} \sim \frac{1}{N_c^2}\Gamma_{exp}$
    - Hadron in color singlet state
      Baryon: $|B\rangle = 1/\sqrt{6}\,\epsilon_{abc}|abc\rangle$
      Meson: $|M\rangle = 1/\sqrt{3}(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle)$



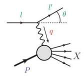
  - Discovery of gluon
    - Indirect evidence from DIS (1970)
      Quark contributes only 45% of the momentum fraction of proton,
      Calculated via light-cone current algebra, by H. Fritzsch & M. Gellmann
    - 1979, Direct evidence from 3-jet events, by Sau Lan Wu @ TASSO, DESY
      - 1977, Jet predicted, by Sterman & Weinberg
        Single energetic parton cannot travel for long distance due to confinement, they shower into jets via gluon radiation & particle excitation.
      - 3-jet event: $e^+ + e^- \rightarrow q + \bar{q} + ?$, ? must carry color charge and thus confined.



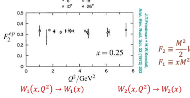
  - Birth of QCD! (1973, Harald Fritzsch, Heinrich Leutwyler, M. Gell-Mann)

Q: Is the progress in fundamental research slowing down compared to 50 years ago?

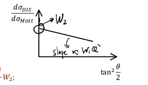
Deep-Inelastic scattering ($e^- + P \rightarrow e^- + X$)



- Kinematics
  - Momentum of the lepton: $l \rightarrow l'$; $l^2 = l'^2 = m^2 \sim 0$
  - Momentum of the initial nucleon: p, $p^2 = M^2$
  - Total momentum of the final hadron X: $\sum_{i=1}^{N_X} p_i' = p'$
  - Momentum transfer: $q = l - l' = p' - p$ (space-like);
    $Q^2 \equiv -q^2 = -(l-l')^2 = 2l\cdot l' - 2m^2 = -2l\cdot q \approx 2E_lE_{l'}(1-\cos\theta) = 4E_lE_{l'}\sin^2\frac{\theta}{2} > 0$
  - $x \equiv \frac{Q^2}{2p\cdot q}$
  - $\Delta M^2 \equiv p'^2 - p^2 = q\cdot(p+p') = q\cdot(2p+q) \rightarrow Q^2 = 2p\cdot q - \Delta M^2$
    $x = 1 - \frac{\Delta M^2}{2p\cdot q}$
    $\Delta M^2 = (q+p)^2 - p^2 = q^2 + 2p\cdot q \leq 2p\cdot q$
    $\rightarrow x > 0$, $x = 1$ *for Elastic scattering*
  - Fixed target (FT) frame: $p = (M, \vec{0})$
    - $p\cdot q = Mq^0 = M(E_l - E_l') > 0$
    - $x \approx \frac{E_lE_{l'}(1-\cos\theta)}{M(E_l - E_l')} \rightarrow E_l' \approx \frac{Mx}{1-\cos\theta + Mx/E_l}$ *for FIXED x*
    - $E_l'^2 dE_l' d\Omega \approx \frac{2\pi Q^2E_l'}{4ME_lx^2}dQ^2dx$
- Scattering amplitude:
  $\mathcal{M}_{fi} \equiv \left\langle l', X\left|\mathcal{T}e^{-i\int d^4x\mathcal{H}_{int}(x)}\right| l, N(p)\right\rangle$
  For EM interaction, $\mathcal{H}_{int} = A\cdot J \leftarrow$ *Electric current*
  The intial & final states are approximately the direct product of the free particle states:
  e.g., $|l,X\rangle \approx |l\rangle \otimes |X\rangle$
  $\mathcal{M}_{fi} \approx \left\langle l', X\left| I - i\int d^4x A\cdot J - \int d^4x d^4y\,\mathcal{T}D^{\mu\nu}(y-x)J_\mu(x)J_\nu(y) + \cdots\right| l, N(p)\right\rangle$
  With $D^{\mu\nu}(y-x) \equiv \langle 0|\mathcal{T}A^\mu(x)A^\nu(y)|0\rangle \leftarrow |0\rangle$ is the vacuum of photon
  The initial and final states are considered ORTHOGONAL, i.e., $\langle l|l'\rangle = \langle X|N(p)\rangle = 0$,
  Hence the first two terms vanish, and the third term is of the LEADING order:
  $\mathcal{M}_{fi} \approx -\int d^4xd^4y\,D^{\mu\nu}(y-x)\langle l'|J_\mu(x)|l\rangle\langle X|J_\nu(y)|N(p)\rangle$
  $= -\int\frac{d^4q}{(2\pi)^4}d^4xd^4y\,\tilde{D}^{\mu\nu}(q)e^{iq\cdot x}\left\langle l'\left|e^{i\hat{P}\cdot x}J_\mu(0)e^{-i\hat{P}\cdot x}\right|l\right\rangle e^{-iq\cdot y}\langle X|e^{i\hat{P}\cdot y}J_\nu(0)e^{-i\hat{P}\cdot y}|N(p)\rangle$
  $= -(2\pi)^4\int d^4q\,\tilde{D}^{\mu\nu}(q)\delta^4(l'-l+q)\langle l'|J_\mu(0)|l\rangle\delta^4(p'-p-q)\langle X|J_\nu(0)|N(p)\rangle$
  $= (2\pi)^4\delta^4(l+p-l'-p')\mathcal{T}_{fi}$
  $\mathcal{T}_{fi} = -\tilde{D}^{\mu\nu}(l-l')\langle l'|J_\mu(0)|l\rangle\langle X|J_\nu(0)|N(p)\rangle$
- The current matrix element
  - Orthogonal to momentum transfer due to charge conservation
    $\partial_\mu J^\mu(x) = 0$
    $0 = \partial_\mu[\langle l'|J^\mu(x)|l\rangle] = \partial_\mu\left[\left\langle l'\left|e^{i\hat{P}\cdot x}J^\mu(0)e^{-i\hat{P}\cdot x}\right|l\right\rangle\right] = \partial_\mu e^{i(l'-l)\cdot x}\langle l'|J^\mu(0)|l\rangle = ie^{i(l'-l)\cdot x}(l'-l)_\mu\langle l'|J^\mu(0)|l\rangle$
    $q_\mu\langle l'|J^\mu(0)|l\rangle = 0$
  - Normalization to the total Charge
    $\langle p,s|\hat{Q}|X,S'\rangle = q\langle p,s|X,S'\rangle = q(2\pi)^3\delta^3(p-p_X')\delta_{ss'}\delta_{N_X,1}$
    $= \left\langle p,s\left|\int d^3\vec{x}\,J^0(x)\right|X,S'\right\rangle = \int d^3\vec{x}\left\langle p,s\left|e^{-i\hat{P}\cdot\vec{x}}J^0(t,\vec{0})e^{i\hat{P}\cdot\vec{x}}\right|X,S'\right\rangle = \int d^3\vec{x}\,e^{ix\cdot(p-\Sigma_X p_X')}\left\langle p,s\left|J^0(t,\vec{0})\right|X,S'\right\rangle = (2\pi)^3\delta^3(p-p_X')\left\langle p,s\left|J^0(t,\vec{0})\right|X,S'\right\rangle$
    $\rightarrow \langle p,s|J^0(0)|p,S'\rangle = q\delta_{ss'}\delta_{N_X,1}$
  - The current matrix element for lepton (w/o quantum correction on lepton-photon vertex):
    Electric current for lepton: $J^\mu = -e\bar{\psi}_+\gamma^\mu\psi_+$
    $\psi_+(x) = (2\pi)^{-3}\sum_\sigma\int\frac{d^3\vec{p}}{\sqrt{2E_p}}[u(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x}]$
    $\langle l',s'|J_\mu(0)|l,s\rangle$
    $= -\frac{e}{(2\pi)^6}\sum_{\sigma,\sigma'}\int\frac{d^3\vec{p}}{\sqrt{2E_p}}\frac{d^3\vec{p}'}{\sqrt{2E_p'}}\bar{u}(\vec{p},\sigma)\gamma^\mu u(\vec{p}',\sigma')\langle l',s'|a^{\%}(\vec{p},\sigma)a(\vec{p}',\sigma')|l,s\rangle$
    $= -\frac{e}{2\sqrt{E_lE_l'}}\bar{u}(\vec{l}',s')\gamma^\mu u(\vec{l},s)$
- Under Lorentz gauge
  $\tilde{D}^{\mu\nu}(q) = -\frac{i}{q^2+i0^+}\left(g^{\mu\nu} + (\alpha-1)\frac{q^\mu q^\nu}{q^2}\right)$
  The second term vanishes in the product with the current matrix element.
  $\mathcal{T}_{fi} = \frac{i}{q^2+i0^+}\langle l'|J_\mu(0)|l\rangle\langle X|J^\mu(0)|N(p)\rangle$
- Spin-averaged inclusive cross section in the FT-frame:
  $d\sigma \equiv \frac{(2\pi)^4}{|\vec{v}_l|}\frac{d^3\vec{l}'}{(2\pi)^3}\frac{1}{4}\sum_{X,spin}\int\prod_{i=1}^{N_X}\frac{d^3\vec{p}_i'}{(2\pi)^3}\delta^4\left(l+p-l'-\sum_{i=1}^{N_X}p_i'\right)|\mathcal{T}_{fi}|^2$
  $= \frac{(2\pi)^4}{q^4|\vec{v}_l|}\frac{d^3\vec{l}'}{(2\pi)^3}\frac{1}{4}\sum_{X,spin}\langle l'|J_\mu(0)|l\rangle\langle l|J_\nu(0)|l'\rangle\int\prod_{i=1}^{N_X}\frac{d^3\vec{p}_i'}{(2\pi)^3}\delta^4\left(l+p-l'-\sum_{i=1}^{N_X}p_i'\right)\langle N(p)|J^\mu(0)|X\rangle\langle X|J^\nu(0)|N(p)\rangle$
  - 1/4 comes from the average of the initial spin
  - $d\sigma$ is Lorentz invariant (see e.g., Weinberg Vol.1, 3.4.3)
  - $L_{\mu\nu}(l,l') \equiv \frac{4E_lE_l'}{2}\sum_{spin}\langle l'|J_\mu(0)|l\rangle\langle l|J_\nu(0)|l'\rangle = \frac{e^2}{2}\sum_{spin}\bar{u}(l')\gamma_\mu u(l)\bar{u}(l)\gamma_\nu u(l') = \frac{e^2}{2}\mathrm{Tr}[u(l)\bar{u}(l)\gamma_\nu u(l')\bar{u}(l')\gamma_\mu]$
    $= \frac{e^2}{2}\mathrm{Tr}[(l\cdot\gamma - m)\gamma_\nu(l'\cdot\gamma - m)\gamma_\mu] = \frac{e^2}{2}\left[\mathrm{Tr}[(l\cdot\gamma)\gamma_\nu(l'\cdot\gamma)\gamma_\mu] + m^2\,\mathrm{Tr}[\gamma_\nu\gamma_\mu]\right]$
    $= 2e^2[l_\mu l_\nu' + l_\nu l_\mu' - g_{\mu\nu}(l\cdot l' - m^2)] = 2e^2[2l_\mu l_\nu - l_\mu q_\nu - l_\nu q_\mu + g_{\mu\nu}l\cdot q]$
  - $W^{\mu\nu}(p,p') \equiv \frac{2E_p}{2}\sum_{X,spin}\int\prod_{i=1}^{N_X}\frac{d^3\vec{p}_i'}{(2\pi)^3}(2\pi)^4\delta^4\left(q+p-\sum_{i=1}^{N_X}p_i'\right)\langle N(p)|J^\mu(0)|X\rangle\langle X|J^\nu(0)|N(p)\rangle$
    - Orthogonality: $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$
    - Normalize to total charge (e): $W^{00}(p,p) \equiv 2e^2E_p(2\pi)\delta(E_p - E_p')$
  $d\sigma = \frac{1}{q^4|\vec{v}_l|}\frac{d^3\vec{l}'}{(2\pi)^3}\frac{1}{8E_lE_l'M}L_{\mu\nu}(l,q)W^{\mu\nu}(p,q)$
  - Projector Operator:
    - $\Delta^{\mu\nu}(q) \equiv -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}$
    - $q_\mu\Delta^{\mu\nu}(q) = q_\nu\Delta^{\mu\nu}(q) = 0$
    - $\Delta^{\mu\nu}(q)\Delta_{\nu\tau}(q) = -\Delta_\tau^\mu(q)$, $\Delta_\mu^\mu = -3$
    - *In the 'Imaginary' rest frame of q,* $\Delta^{\mu\nu}(q) \rightarrow diag\{0,1,1,1\}$
    - $a_\perp^\mu \equiv \Delta^{\mu\nu}(q)a_\nu = -a^\mu + q^\mu\frac{a\cdot q}{q^2}$, $q\cdot a_\perp = 0$
    - $l_\perp^\mu = -l^\mu + \frac{q^\mu}{2}$; $l_\perp^2 = m^2 + \frac{Q^2}{4} \approx \frac{Q^2}{4}$
    - $p_\perp^\mu = -p^\mu - \frac{q^\mu}{2x}$; $p_\perp^2 = M^2 + \frac{Q^2}{4x^2}$
    - $l_\perp\cdot p_\perp = l\cdot p - \frac{Q^2}{4x} \xrightarrow{in\ \mathrm{FT-frame}} l_\perp\cdot p_\perp = Ml^0 - \frac{Q^2}{4x}$
  - $L_{\mu\nu}(l,q) = 2e^2\left[2\left(l_{\perp\mu} - \frac{q_\mu}{2}\right)\left(l_{\perp\nu} - \frac{q_\nu}{2}\right) + l_{\perp\mu}q_\nu + l_{\perp\nu}q_\mu - q^\mu q^\nu - \Delta_{\mu\nu}l\cdot q + \frac{q^\mu q^\nu}{q^2}l\cdot q\right]$
    $= 2e^2\left[2l_{\perp\mu}l_{\perp\nu} - (q_\mu l_{\perp\nu} + q_\nu l_{\perp\mu}) + \frac{1}{2}q_\mu q_\nu + l_{\perp\mu}q_\nu + l_{\perp\nu}q_\mu - q^\mu q^\nu - \Delta_{\mu\nu}l\cdot q + \frac{1}{2}q_\mu q_\nu\right]$
    $= 2e^2\left[2l_{\perp\mu}l_{\perp\nu} + \frac{Q^2}{2}\Delta_{\mu\nu}\right]$
  - Due to orthogonality, the $\mu\nu$ indices of $W^{\mu\nu}(p,q)$ MUST come from $p_\perp^\mu$ and $\Delta^{\mu\nu}(q)$, So,
    $W^{\mu\nu}(p,q) = e^2[p_\perp^\mu p_\perp^\nu W_2(x,Q^2) + \Delta^{\mu\nu}(q)Q^2W_1(x,Q^2)]$
    $W^{00}(p,0) = e^2E_p^2W_2(x,0) = 2e^2E_p(2\pi)\delta(E_p - E_p') \rightarrow W_2(x,0) = 8\pi\delta\left(E_p^2 - E_p'^2\right) = 8\pi\delta(p'^2 - M^2)$; $W_1(x,0)$ *is FINITE*
  $d\sigma = \frac{1}{q^4|\vec{v}_l|}\frac{d^3\vec{l}'}{(2\pi)^3}\frac{1}{8E_lE_l'M}2e^2[2l_{\perp\mu}l_{\perp\nu} - \Delta_{\mu\nu}l\cdot q]e^2[p_\perp^\mu p_\perp^\nu W_2(p\cdot q, q^2) + \Delta^{\mu\nu}(q)Q^2W_1(p\cdot q, q^2)]$
  $= \frac{1}{q^4|\vec{v}_l|}\frac{d^3\vec{l}'}{(2\pi)^3}\frac{e^4}{4E_lE_l'M}\{[2(l_\perp\cdot p_\perp)^2 + p_\perp^2(l\cdot q)]W_2(x,Q^2) - [2l_\perp^2 + 3(l\cdot q)]Q^2W_1(x,Q^2)\}$
  $\approx \frac{1}{Q^4}\frac{dE_l'E_l'^2d\Omega}{(2\pi)^3}\frac{e^4}{4E_lE_l'M}\left\{\left[2\left(Ml^0 - \frac{Q^2}{4x}\right)^2 - \left(M^2 + \frac{Q^2}{4x^2}\right)\frac{Q^2}{2}\right]W_2(x,Q^2) + Q^4W_1(x,Q^2)\right\}$
  $\approx \frac{1}{Q^4}\frac{dE_l'E_l'^2d\Omega}{(2\pi)^3}\frac{e^4}{4E_lE_l'M}\left\{\left[2M^2E_l^2 - \frac{ME_lQ^2}{x} - \frac{M^2Q^2}{2}\right]W_2(x,Q^2) + Q^4W_1(x,Q^2)\right\}$
  $\approx \frac{1}{Q^4}\frac{dE_l'E_l'^2d\Omega}{(2\pi)^3}\frac{e^4}{4E_lE_l'M}\left\{\left[2M^2E_l^2 - 2M^2E_l(E_l - E_l') - 2M^2E_lE_l'\sin^2\frac{\theta}{2}\right]W_2(x,Q^2) + 4E_lE_l'\sin^2\frac{\theta}{2}Q^2W_1(x,Q^2)\right\}$
  $\approx \frac{\alpha^2}{8\pi\sin^4\frac{\theta}{2}}\frac{dE_l'd\Omega}{E_l^2M}\left\{\frac{M^2}{2}\cos^2\frac{\theta}{2}W_2(x,Q^2) + \sin^2\frac{\theta}{2}Q^2W_1(x,Q^2)\right\}$
- Comparison: Rosenbluth cross section (see e.g., Weinberg Vol.1, 10.6.2) for ELASTIC scatterings:
  $\frac{d\sigma_{Ros}}{d\Omega} = \frac{\alpha^2}{4\sin^4\frac{\theta}{2}}\frac{E_l'}{E_l^3}\left\{\left(\frac{4M^2G_E^2 + Q^2G_M^2}{4M^2+Q^2}\right)\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}G_M^2\sin^2\frac{\theta}{2}\right\}$
  $G_E \rightarrow$ Electric form factor (Fourier tranformation of charge denstiy) $\rightarrow W_2$ is electric.
  $G_M \rightarrow$ Magnetic form factor (Fourier tranformation of magnetic moment denstiy) $\rightarrow W_1$ is magnetic (related to spin)
  $Q^2 \rightarrow 0$, $\frac{d\sigma_{Ros}}{d\Omega} \rightarrow \frac{d\sigma_{Mott}}{d\Omega} = \frac{\alpha^2}{4E_l^2\sin^4\frac{\theta}{2}}G_E^2\cos^2\frac{\theta}{2}$
  $\frac{d\sigma_{DIS}}{d\sigma_{Mott}} = \frac{1}{2\pi}\frac{dE_l'}{MG_E^2}\left\{\frac{M^2}{2}W_2(x,Q^2) + \tan^2\frac{\theta}{2}Q^2W_1(x,Q^2)\right\}$
- Bjorken scaling

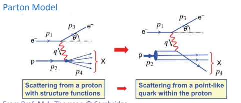


$F_2 \equiv \frac{M^2}{2}W_2$;
$F_1 \equiv xM^2W_1$



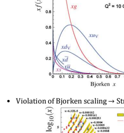
$W_1(x,Q^2) \rightarrow W_1(x)$ $W_2(x,Q^2) \rightarrow W_2(x)$

Parton Model



From Prof. M.A. Thomson @ Cambridge

- Model assumption:
  $|N(p)\rangle = |\mathcal{P}(k)\rangle \otimes |rest\rangle$
  $|X\rangle = |\mathcal{P}(k+q)\rangle \otimes |rest\rangle$
  Interactions among the partons are neglected ← Asymptotic freedom
- Kinematics:
  In the proton Infinite momentum (IM) frame,
  $p \approx (E_p, 0, 0, E_p)$, $E_p \gg M$, Notice: $p^2 = M^2 \neq 0$!
  Parton initially moves in the direction of proton: $k \approx \xi p \rightarrow \xi p + q$
  $m_Q^2 = (\xi p + q)^2 = \xi^2p^2 + 2\xi p\cdot q + q^2 = m_Q^2 + 2\xi p\cdot q + q^2$
  $\rightarrow \xi = \frac{Q^2}{2p\cdot q} = x$!!!
  Deep inelastic scatterings probe the partons with small momentum share.
- Hadronic Form Factor:
  $W^{\mu\nu}(p,p') \equiv \frac{2E_p}{2}\sum_{spin}\int\frac{d^3\vec{k}'}{(2\pi)^3}(2\pi)^4\delta^4(q+\xi p-k')\langle\mathcal{P}(\xi p)|J^\mu(0)|\mathcal{P}(k')\rangle\langle\mathcal{P}(k')|J^\nu(0)|\mathcal{P}(\xi p)\rangle$
  $= \frac{4\xi E_pE_k'}{2\times 2\xi E_k'}(2\pi)\delta(q^0 + \xi E_p - k_0')\sum_{spin}\langle\mathcal{P}(\xi p)|J^\mu(0)|\mathcal{P}(k')\rangle\langle\mathcal{P}(k')|J^\nu(0)|\mathcal{P}(\xi p)\rangle$
  $= \frac{1}{2\xi E_k'}(2\pi)\delta(q^0 + \xi E_p - k_0')L^{\mu\nu}(\xi p, \xi p + q)$
  $= \frac{1}{\xi}(2\pi)\delta\left((q^0 + \xi E_p)^2 - k_0'^2\right)2q^2\left[2\xi^2p_{\perp\mu}p_{\perp\nu} + \frac{Q^2}{2}\Delta_{\mu\nu}\right]$
  $= (2\pi)\delta(k'^2 - m_Q^2)q^2\left[4\xi p_{\perp\mu}p_{\perp\nu} + \frac{Q^2}{\xi}\Delta_{\mu\nu}\right]$
  $\rightarrow W_2(x,Q^2) = 8\pi(q/e)^2\delta(k'^2 - m_Q^2)x$, $W_1(x,Q^2) = 2\pi(q/e)^2\delta(k'^2 - m_Q^2)\frac{1}{x}$
  Bjorken scaling!!
  $W_2(x,Q^2) = 4x^2W_1(x,Q^2)$ Callan-Gross relation.
- Parton distribution function:
  Introduce $f_{\mathcal{P}}(x)$ denoting the # of $\mathcal{P}$-flavor parton sharing the momentum xp
  $\rightarrow W_2(x,Q^2) = 8\pi\left(\frac{q}{e}\right)^2\delta(k'^2 - m_Q^2)xf_{\mathcal{P}}(x)$,
  $W_1(x,Q^2) = 2\pi(q/e)^2\delta(k'^2 - m_Q^2)\frac{1}{x}f_{\mathcal{P}}(x)$
  $\int_0^1 dx\, f_{\mathcal{P}}(x) \leftarrow$ # of $\mathcal{P}$-flavor parton inside proton
  $\int_0^1 dx\, xf_{\mathcal{P}}(x) \leftarrow$ momentum fraction taken by $\mathcal{P}$-flavor parton on average
  A *hint* for the existence of the gluon
  $\sum_{\mathcal{P}=Quarks}\int_0^1 dx\, xf_{\mathcal{P}}(x) < 1$



$\int dx f(x) = 1$
$\int dx\, x f(x) = \frac{1}{3}$
$\sum_p \int dx\, x f_p(x) = 1$

- Violation of Bjorken scaling → Strong interactions @ small x