QCD-07 (OPE & sum rule)

Tuesday, May 23, 2023 7:10 AM

Ref:
QCD SUM RULES, A MODERN PERSPECTIVE | At The Frontier of Particle Physics (worldscientific.com)

Target

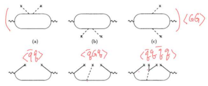
Bridge the **mesonic properties** and the **non-perturbative QCD quantities** ($\langle\bar qq\rangle$, $\langle F^a_{\mu\nu}F_a^{\mu\nu}\rangle,\cdots$) via:

$$\Pi_{\mu\nu}(q)=i\int d^4x\,\langle\mathcal T\,j_\mu(x)j_\nu(0)\rangle e^{iq\cdot x}=(q_\mu q_\nu-g_{\mu\nu}q^2)\Pi(q^2)=q^2\Delta_{\mu\nu}(q)\Pi(q^2)$$

$$\Delta_{\mu\nu}=-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2};\ \Delta^\mu_\mu=-3;\ \ q\cdot\Pi=0$$

From QCD perspective

- Operator Product Expansion (OPE)
  - In coordinate space:
    Once x and y are **close enough**,
    Given a complete set of local operators $\{\mathcal O_i\}$ composed of field operator & gradients
    $$\mathcal O_i(x)\mathcal O_j(y)=\sum_k C_{ij}^k(x-y)\mathcal O_k(y)$$
  - In frequency space:
    Once $|q^2|$ is **large enough**,
    $$\int d^4x\,\langle\mathcal O_i(x)\mathcal O_j(0)\rangle e^{iq\cdot x}=\sum_k\tilde C_{ij}^k(q^2)\langle\mathcal O_k(0)\rangle$$
- OPE in QCD
  - Factorization:
    $$\Pi(q^2)=\sum_d\tilde C_d(q^2)\langle\mathcal O_d(0)\rangle$$
    $\mathcal O\in\{\mathcal I,\bar qq,F^a_{\mu\nu}F_a^{\mu\nu},\cdots\}$
    $d\equiv\dim(\mathcal O)\in\{0,3,4,\cdots\}$
    If $|q^2|$ is **large enough,**
    $\tilde C_d(q^2)$ is perturbative; $\langle\mathcal O_d(0)\rangle$ is IR & non-perturbative.
  - Diagrammatic view (examples):
    - $d=0,\ \mathcal O_0=\mathcal I,\Rightarrow\ \tilde C_0(q^2)=\Pi_{pert.}(q^2)$
    - $d=3,\ \mathcal O_3=\bar qq,\ j^\mu=\sum_i\bar q_i\gamma^\mu q_i$ (i is color & flavor indices)



$$\Pi^{d=3}_{\mu\nu}(q^2)=i\int d^4x\,\langle\mathcal T\,j_\mu(x)j_\nu(0)\rangle e^{iq\cdot x}$$
$$=i\int d^4x\sum_{i,j}\langle\mathcal T\,\bar q_i(x)\gamma_\mu q_i(x)\bar q_j(0)\gamma_\nu q_j(0)\rangle\,e^{iq\cdot x}$$
$$=-i\int d^4x\sum_{i,j}\mathrm{tr}\langle\mathcal T\,\gamma_\mu q_i(x)\bar q_j(0)\gamma_\nu q_j(0)\bar q_i(x)\rangle\,e^{iq\cdot x}$$
$$\approx-i\int d^4x\sum_{i,j}\mathrm{tr}[\langle\mathcal T\,\gamma_\mu S_{ij}(x)\gamma_\nu q_j(0)\bar q_i(x)\rangle+\langle\mathcal T\,\gamma_\mu q_i(x)\bar q_j(0)\gamma_\nu S_{ji}(-x)\rangle]\,e^{iq\cdot x}$$

Non perturbative physics ⇔ IR physics ⇒
q_i treated as **external and slowly varying** source: $q(x)\approx q(0)+x\cdot Dq(0)+\mathcal O(x^2)$
Notice: $\partial\cdot j=0$, and hence $q\cdot\Pi=0$ are **violated** under the above approximation, since $\partial_\mu q=D_\mu q(0)\neq0$
Violation vanishes @ chiral limit since $\gamma\cdot Dq\sim mq\to0$

$$\Pi^{d=3}_{\mu\nu}(q^2)\approx-i\int d^4x\sum_{i,j}\mathrm{tr}[\gamma_\mu S_{ij}(x)\gamma_\nu\langle q_j(0)\bar q_i(0)\rangle+\langle q_i(0)\bar q_j(0)\rangle\gamma_\nu S_{ji}(-x)\gamma_\mu]\,e^{iq\cdot x}$$
$$-i\int d^4x\sum_{i,j}\mathrm{tr}\left[\gamma_\mu S_{ij}(x)\gamma_\nu\left\langle q_j(0)\bar q_i(0)\overleftarrow{D_\alpha}\right\rangle+\langle D_\alpha q_i(0)\bar q_j(0)\rangle\gamma_\nu S_{ji}(-x)\gamma_\mu\right]x^\alpha e^{iq\cdot x}$$

$$\Pi^{d=3}_{\mu\nu}(q^2)\approx-i\sum_{i,j}\mathrm{tr}[\gamma_\mu\tilde S_{ij}(q)\gamma_\nu\langle q_j(0)\bar q_i(0)\rangle+\langle q_i(0)\bar q_j(0)\rangle\gamma_\nu\tilde S_{ji}(-q)\gamma_\mu]\qquad\Leftarrow\ \pm i\partial_{q_\alpha}e^{\mp iq\cdot x}=e^{\mp iq\cdot x}x^\alpha$$
$$-\sum_{i,j}\mathrm{tr}\left[\gamma_\mu\partial_{q_\alpha}\tilde S_{ij}(q)\gamma_\nu\left\langle q_j(0)\bar q_i(0)\overleftarrow{D_\alpha}\right\rangle+\langle D_\alpha q_i(0)\bar q_j(0)\rangle\gamma_\nu\partial_{q_\alpha}\tilde S_{ji}(-q)\gamma_\mu\right]$$

Assumptions: $\langle q_j(0)\bar q_i(0)\rangle=\mathcal A\mathcal I\delta_{ij}$, $\langle iD_\alpha q_i(0)\bar q_j(0)\rangle=\mathcal B\gamma_\alpha\delta_{ij}$, $\left\langle q_j(0)\bar q_i(0)\overleftarrow{D_\alpha}i\right\rangle=\mathcal B'\gamma_\alpha\delta_{ij}$
Physical meaning of $\mathcal A$: quark condensate $\langle\bar qq\rangle\equiv\sum_i\langle\bar q_iq_i\rangle=-\mathrm{tr}\sum_i\langle q_i\bar q_i\rangle=-4N_cN_f\mathcal A$
Dirac equation: $i\gamma^\alpha D_\alpha q=mq\Rightarrow\ 4\mathcal B=-4\mathcal B'=m\mathcal A$

$$\Pi^{d=3}_{\mu\nu}(q^2)\approx\frac{4N_cN_f\mathcal A}{q^2-m^2}2g_{\mu\nu}+\frac{iN_cN_fm\mathcal A}{4}\mathrm{tr}[-\gamma_\mu\partial_{q_\alpha}\tilde S(q)\gamma_\nu\gamma_\alpha+\gamma_\alpha\gamma_\nu\partial_{q_\alpha}\tilde S(-q)\gamma_\mu]$$
$$=\frac{4N_cN_fm\mathcal A}{q^2-m^2}2g_{\mu\nu}-\frac{4N_cN_fm\mathcal A}{(q^2-m^2)}g_{\mu\nu}-\frac{4N_cN_fm\mathcal A}{(q^2-m^2)^2}(2q_\mu q_\nu-g_{\mu\nu}q^2)\qquad i\partial_{q_\alpha}\left(\frac{\pm q_\beta\gamma^\beta+m}{q^2-m^2}\right)=\frac{\pm i\gamma^\alpha}{q^2-m^2}-\frac{2iq^\alpha(\pm q\cdot\gamma+m)}{(q^2-m^2)^2}$$
$$=\frac{2m\langle\bar qq\rangle}{(q^2-m^2)^2}\left(q_\mu q_\nu-g_{\mu\nu}q^2+\frac{m^2}{2}g_{\mu\nu}\ \leftarrow\text{from violation of }\partial\cdot j=0\right)$$

At **chiral limit** $|q^2|\gg m^2$, $\Pi^{d=3}(q^2)=\frac{2m\langle\bar qq\rangle}{q^4}$

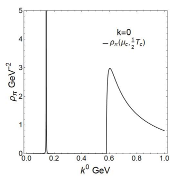
- More terms:
$$\Pi(q^2)=\Pi_{pert.}(q^2)+\frac{2m\langle\bar qq\rangle}{q^4}+\frac{\alpha_s\langle F_a^{\mu\nu}F^a_{\mu\nu}\rangle}{12\pi\,q^4}+\cdots$$
From dimensional analysis, at **chiral limit**: $C_d(q^2)\propto q^{-d}$ for **even d**.

Extension of spectral representation to finite-T

- Green functions **(In thermal equilibrium)** :
  - $G_+(x-y)\equiv\langle\phi_a(x)\phi_b(y)\rangle=\mathrm{Tr}[\rho_0\phi_a(x)\phi_b(y)]$
  - $G_-(x-y)\equiv\pm\langle\phi_b(y)\phi_a(x)\rangle=\pm\mathrm{Tr}[\rho_0\phi_b(y)\phi_a(x)]$
  - $\rho(x-y)\equiv G_+(x,y)-G_-(x,y)$
  - $\Delta(\tau,\vec x)=G_+(i\tau,\vec x)$ $(\tau\in[0,\beta])$
  - $G_R(x-y)=i\theta(x^0-y^0)\rho(x-y)$
  - $G_A(x-y)=i\theta(y^0-x^0)\rho(x-y)$
  - $G_F(x-y)=i\theta(x^0-y^0)G_+(x-y)+i\theta(y^0-x^0)G_-(x-y)$
- *Kubo-Martin-Schwinger Relation:*
$$G_+(x^0,\vec x)=Z^{-1}\mathrm{Tr}\left[e^{-\beta H}\phi_a(x)\phi_b(0)\right]=Z^{-1}\mathrm{Tr}\left[e^{ii\beta H}\phi_a(x)e^{-ii\beta H}e^{-\beta H}\phi_b(0)\right]$$
$$=Z^{-1}\mathrm{Tr}\left[e^{-\beta H}\phi_b(0)\phi_a(x^0+i\beta,\vec x)\right]=\pm G_-(x^0+i\beta,\vec x)$$
$$\Rightarrow\ \tilde G_+(k)=\int d^4x\,e^{ik\cdot x}G_+(x)=\pm\int d^4x\,e^{ik\cdot x}G_-(x^0+i\beta,\vec x)=\pm e^{\beta k^0}\int d^4x\,e^{ik\cdot x}G_-(x)=\pm e^{\beta k^0}\tilde G_-(k)$$
$$\tilde\rho(k)=\tilde G_+(k)-\tilde G_-(k)=\left(\pm e^{\beta k^0}-1\right)\tilde G_-(k)$$
- Spectral properties:
  - $\tilde G_-^{ab}(k)=\pm\sum_{m,n}\int d^4x\,e^{ik\cdot x}\mathcal P_n\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(x)|n\rangle=\pm\sum_{m,n}\mathcal P_n\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(0)|n\rangle(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)\theta(p_n^2)\theta(p_m^2)$
  - $\tilde G_+^{ab}(k)=\sum_{m,n}\mathcal P_m\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(0)|n\rangle(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)\theta(p_n^2)\theta(p_m^2)$
  - $\tilde\rho_{ab}(k)=\sum_{m,n}(\mathcal P_m\mp\mathcal P_n)\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(0)|n\rangle(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)\theta(p_n^2)\theta(p_m^2)$
    $=Z^{-1}\sum_{m,n}\left(e^{-\beta p_m^0}\mp e^{-\beta p_n^0}\right)\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(0)|n\rangle(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)\theta(p_n^2)\theta(p_m^2)$
    $=Z^{-1}\sum_{m,n}e^{-\beta p_n^0}\left(e^{\beta k^0}\mp1\right)\langle n|\phi_b(0)|m\rangle\langle m|\phi_a(0)|n\rangle(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)\theta(p_n^2)\theta(p_m^2)=\left(\pm e^{\beta k^0}-1\right)\tilde G_-^{ab}(k)$
  - Symmetry:
    $k\to-k\ \Leftrightarrow\ n\leftrightarrow m$: $\tilde G^{ab}_-(-k)=\pm\tilde G^{ba}_+(k)$; $\tilde\rho_{ab}(-k)=\mp\tilde\rho_{ba}(k)$;
  - Hermitian: $\tilde\rho_{ab}(k)=\tilde\rho^*_{ba}(k)$
  - Positivity:
    Once the operators are **IDENTICAL**:
    $$\tilde\rho(k)=\sum_{m,n}(\mathcal P_m\mp\mathcal P_n)\,|\langle n|\phi(0)|m\rangle|^2(2\pi)^4\delta(k+p_m-p_n)\theta(p_n^0)\theta(p_m^0)$$
    For $k^0>0$, $p_m^0<p_n^0$, $\mathcal P_m>\mathcal P_n$: $\tilde\rho(k)>0$
    For $k^0<0$, $\tilde\rho(k)=\mp\tilde\rho(-k)$
    (opposite if $\mathcal P$ increase with energy)
- Spectral representation:
  - $\tilde G_-(k)=\frac{1}{\pm e^{\beta k^0}-1}\tilde\rho(k)=\pm f(k^0)\tilde\rho(k)$
  - $\tilde G_+(k)=\frac{\pm e^{\beta k^0}}{\pm e^{\beta k^0}-1}\tilde\rho(k)=\left(1\pm f(k^0)\right)\tilde\rho(k)$
  - $\tilde G_F(k)=i\int d^4x\,e^{ik\cdot x}\{\theta(x^0)G_+(x)+\theta(-x^0)G_-(x)\}$
    $$=i\int d^4x\,e^{ik\cdot x}\left\{i\int\frac{d\omega}{2\pi}\frac{e^{-i\omega x^0}}{\omega+i0^+}\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\left(1\pm f(q^0)\right)\tilde\rho(q)\pm i\int\frac{d\omega}{2\pi}\frac{e^{i\omega x^0}}{\omega+i0^+}\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}f(q^0)\tilde\rho(q)\right\}$$
    $$=-\left\{\int\frac{dq^0}{2\pi}\frac{\left(1\pm f(q^0)\right)\tilde\rho(q^0,\vec k)}{k^0-q^0+i0^+}\pm\int\frac{dq^0}{2\pi}\frac{f(q^0)\tilde\rho(q^0,\vec k)}{q^0-k^0+i0^+}\right\}$$
    $$=-\int\frac{dq^0}{2\pi}\frac{\tilde\rho(q^0,\vec k)}{k^0-q^0+i0^+}\mp if(k^0)\tilde\rho(k)$$
    $\mathrm{Im}\,\tilde G_F(k)=\left(\frac12\mp f(k^0)\right)\tilde\rho(k)$
    Back to vacuum, $f\to0$
    $\tilde\rho(k)=2\mathrm{Im}\tilde G_F(k)$
    Consitent with vacuum conclusion!
  - $\tilde G_R(k)=-\int\frac{dq^0}{2\pi}\frac{\tilde\rho(q^0,\vec k)}{k^0-q^0+i0^+}$
  - $\tilde G_A(k)=-\int\frac{dq^0}{2\pi}\frac{\tilde\rho(q^0,\vec k)}{q^0-k^0+i0^+}$

From hadronic perspective:

- Kaellen-Lehmann spectral representation (Ref.: Chpt10.7 in Weinberg I)
  - $$\langle j_\mu(x)j_\nu(0)\rangle=\sum_n\langle0|j_\mu(x)|n\rangle\langle n|j_\nu(0)|0\rangle=\sum_n\langle0|e^{i\hat P\cdot x}j_\mu(0)e^{-i\hat P\cdot x}|n\rangle\langle n|j_\nu(0)|0\rangle$$
    $$=\sum_n\langle0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle e^{-ip_n\cdot x}\theta(p_n^0)\theta(p_n^2)\ \ \Leftarrow p_n^0\geq0\ (g.s.\,energy)\ \&\ p_n^2>0\ (\text{time like})$$
    $$=\int\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p-p_n)\sum_n\langle0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle e^{-ip_n\cdot x}\theta(p_n^0)\,\theta(p_n^2)$$
    $$=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(p^0)\theta(p^2)\sum_n\langle0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle(2\pi)^4\delta^4(p-p_n)\equiv\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(p^0)\theta(p^2)\Delta_{\mu\nu}(p)\rho(p^2)\quad\Leftarrow\ \partial\cdot\langle0|j|n\rangle=0$$
    $$=\int_0^\infty d\mu^2\,\delta(p^2-\mu^2)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(p^0)\theta(p^2)\Delta_{\mu\nu}(p)\rho(p^2)$$
    $$=\int_0^\infty d\mu^2\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(p^0)\delta(p^2-\mu^2)\Delta_{\mu\nu}(p)\rho(\mu^2)\equiv\int_0^\infty d\mu^2\,\mathrm D^+_{\mu\nu}(x;\mu^2)\rho(\mu^2)$$
  - $$\langle j_\nu(0)j_\mu(x)\rangle=\int_0^\infty d\mu^2\,\mathrm D^-_{\nu\mu}(x;\mu^2)\rho(\mu^2)$$
    $$\mathrm D^-_{\nu\mu}(x;\mu^2)\equiv\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\theta(p^0)\delta(p^2-\mu^2)\Delta_{\nu\mu}(p)\xrightarrow{\mathrm{p\to-p}}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(-p^0)\delta(p^2-\mu^2)\Delta_{\nu\mu}(-p)$$
  - $$\langle[j_\mu(x),j_\nu(0)]\rangle=\int_0^\infty d\mu^2\,[\mathrm D^+_{\mu\nu}(x;\mu^2)-\mathrm D^-_{\mu\nu}(x;\mu^2)]\rho(\mu^2)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\Delta_{\mu\nu}(p)\,\mathrm{sgn}(p^0)\theta(p^2)\,\rho(p^2)$$
    $\Rightarrow$ Causality ($[j_\mu(x),j_\nu(0)]=0$ for $\mathrm x^2<0$)
  - $$\langle\mathcal Tj_\mu(x)j_\nu(0)\rangle\equiv\theta(x^0)\langle j_\mu(x)j_\nu(0)\rangle+\theta(-x^0)\langle j_\nu(0)j_\mu(x)\rangle$$
    $$\theta(t)=i\int\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega+i0^+}$$
    $$\Pi_{\mu\nu}(q)=i\int d^4x\,\langle\mathcal T\,j_\mu(x)j_\nu(0)\rangle e^{iq\cdot x}=i\int d^4x\,\{\theta(x^0)\langle j_\mu(x)j_\nu(0)\rangle+\theta(-x^0)\langle j_\nu(0)j_\mu(x)\rangle\}e^{iq\cdot x}$$
    $$=-\int_0^\infty d\mu^2\,\rho(\mu^2)\int d^4x\left\{\int\frac{d\omega}{2\pi}\frac{e^{-i\omega x^0}}{\omega+i0^+}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(p^0)\delta(p^2-\mu^2)\Delta_{\mu\nu}(p)+\int\frac{d\omega}{2\pi}\frac{e^{i\omega x^0}}{\omega+i0^+}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\theta(-p^0)\delta(p^2-\mu^2)\Delta_{\nu\mu}(-p)\right\}e^{iq\cdot x}$$
    $$=-\int_0^\infty d\mu^2\,\rho(\mu^2)\left\{\int_0^\infty\frac{dp^0}{2\pi}\frac{\delta(p_0^2-\vec q^2-\mu^2)\Delta_{\mu\nu}(p^0,\vec q)}{q^0-p^0+i0^+}+\int_{-\infty}^0\frac{dp^0}{2\pi}\frac{\delta(p_0^2-\vec q^2-\mu^2)\Delta_{\nu\mu}(-p^0,-\vec q)}{p^0-q^0+i0^+}\right\}$$
    $$=\frac{-1}{2\pi}\int_0^\infty d\mu^2\frac{\rho(\mu^2)}{2\sqrt{\vec q^2+\mu^2}}\left\{\frac{\Delta_{\mu\nu}\left(\sqrt{\vec q^2+\mu^2},\vec q\right)}{q^0-\sqrt{\vec q^2+\mu^2}+i0^+}+\frac{\Delta_{\nu\mu}\left(-\sqrt{\vec q^2+\mu^2},-\vec q\right)}{-\sqrt{\vec q^2+\mu^2}-q^0+i0^+}\right\}$$
    $$=\frac{-1}{2\pi}\int_0^\infty d\mu^2\frac{\Delta_{\mu\nu}\left(\sqrt{\vec q^2+\mu^2},\vec q\right)\rho(\mu^2)}{q^2-\mu^2+i0^+}$$
    $$\Pi(q^2)=-\frac{1}{3q^2}\Pi^\mu_\mu=\frac{-1}{2\pi}\int_0^\infty d\mu^2\frac{\rho(\mu^2)}{\mu^2}\left(\frac{1}{q^2-\mu^2+i0^+}-\frac{1}{q^2}\right)$$
  - Spectral density
    - $\rho(p^2)=-\frac13\sum_n\langle0|j_\mu(0)|n\rangle\langle n|j^\mu(0)|0\rangle(2\pi)^4\delta^4(p-p_n)=-\frac13\sum_n|\langle0|j_\mu|n\rangle|^2(2\pi)^4\delta^4(p-p_n)\in\mathrm R$ $\left(-\frac13\text{ comes from }\Delta^\mu_\mu=-3\right)$
    - $\frac{1}{q^2-\mu^2+i0^+}=P\frac{1}{q^2-\mu^2}-i\pi\delta(q^2-\mu^2)\Rightarrow$
      $\rho(q^2)\theta(q^2)=2\,q^2\mathrm{Im}\Pi(q^2)\quad\Rightarrow\quad\Pi(q^2)\in R$ for $\mathrm q^2<0$
    - If $|\mathrm n\rangle$ is a stable bound (hadron) state labelled as $|\mathrm p_V,\mathrm s\rangle$ with a momentum p_V and a spin s
      Summing over these bound states: $\sum_n\to\frac{\int d^3\vec p_V}{(2\pi)^3}\sum_s$
      $$\rho_h(p^2)\equiv-\frac13\int\frac{d^3\vec p_V}{(2\pi)^3}\sum_s\langle0|j_\mu(0)|p_V,s\rangle\langle p_V,s|j^\mu(0)|0\rangle\,(2\pi)^4\delta^4(p-p_V)$$
      Notice, from the tensor analysis: $\sqrt{2E_{p_V}}\langle0|j^\mu(0)|p_V,s\rangle=\epsilon^\mu_s(\vec p_V)f_Vm_V$
      - $\sqrt{2E_{p_V}}\langle0|j^\mu(0)|p_V\rangle$ is a Lorentz 4-vector (see lecture note QCD-01)
      - $Dim\left[\sqrt{2E_{p_V}}\langle0|j^\mu(0)|p_V,s\rangle\right]=2$
      - $\partial\cdot j=0=\partial_\mu\langle0|j^\mu(x)|p_V,s\rangle=\partial_\mu\langle0|e^{i\hat P\cdot x}j^\mu(0)e^{-i\hat P\cdot x}|p_V,s\rangle=\partial_\mu e^{-ip_V\cdot x}\langle0|j^\mu(0)|p_V,s\rangle\ \Rightarrow\ p_V\cdot\langle0|j(0)|p_V,s\rangle=0$
      - ϵ_V is the **Polariztion vector** of $|V\rangle$, *satisfying* 1) $\epsilon\cdot p=0$; 2) $\sum_s\epsilon^\mu_s\epsilon^{*\nu}_s=\Delta^{\mu\nu}$
      - f_V is a Lorentz **scalar** called **Meson Decay Constant**.
      - $Dim[f_V]=1$
      - $|\langle0|j_\mu|p_V,s\rangle|^2$ describes the chance of the decay process $V\to$ final state with current j
      $$\rho_h(p^2)=\int\frac{d^3\vec p_V}{(2\pi)^32E_{p_V}}m_V^2f_V^2(2\pi)^4\delta^3(\vec p-\vec p_V)\delta(p^0-E_{p_V})$$
      $$=2\pi m_V^2f_V^2\delta(p^2-m_V^2)\theta(p^0)$$
    - $\rho(\mu^2)=\rho_h(\mu^2)+\rho_{cont.}(\mu^2)\theta(\mu^2-s_0)$
      $\rho_h(\mu^2)$: stable hadron, non-perturbative
      $\rho_{cont.}(\mu^2)$: free quarks, perturbative at large μ^2, threshold energy square $s_0>m_V^2$



- Conclusion:
$$\Pi(q^2)=-f_V^2\left(\frac{1}{q^2-m_V^2+i0^+}-\frac{1}{q^2}\right)-\frac{1}{2\pi}\int_{s_0}^\infty ds\frac{\rho(s)}{s}\left(\frac{1}{q^2-s+i0^+}-\frac{1}{q^2}\right)\ \leftarrow\text{UV divergence}$$

Quark-Hadron Duality

- Borel transformation:
  Definition:
  $$\mathcal B[\mathrm F(-q^2)]=\lim_{n\to\infty}\frac{(-q^2)^{n+1}}{n!}\frac{d^n}{d^nq^2}\mathrm F(-q^2)\Big|_{-q^2\to nM^2}$$
  $$\mathcal B\left[\frac{1}{(q^2-s)^k}\right]=\lim_{n\to\infty}\frac{(-q^2)^{n+1}}{n!}\left[(-)^n\frac{(k+n-1)!}{(k-1)!}(q^2-s)^{-k-n}\right]\Bigg|_{-q^2\to nM^2}$$
  $$=\lim_{n\to\infty}(-)^{-k}\frac{(k+n-1)!}{n!\,(k-1)!}\frac{(nM^2)^{n+1}}{(nM^2+s)^{k+n}}=-\frac{(-M^2)^{1-k}}{(k-1)!}e^{-\frac{s}{M^2}}$$
- Borel transformation on Kaellen-Lehmann form:
  $$\mathcal B[\Pi_{KL}(-q^2)]=\mathcal B\left[-f_V^2\left(\frac{1}{q^2-m_V^2+i0^+}-\frac{1}{q^2}\right)-\frac{1}{2\pi}\int_{s_0}^\infty ds\frac{\rho(s)}{s}\left(\frac{1}{q^2-s+i0^+}-\frac{1}{q^2}\right)\right]$$
  $$=f_V^2\left[\exp\left(-\frac{m_V^2}{M^2}\right)-1\right]+\frac{1}{2\pi}\int_{s_0}^\infty ds\frac{\rho(s)}{s}\left[\exp\left(-\frac{s}{M^2}\right)-1\right]$$
- Borel transformation on OPE expansion:
  $$\mathcal B[\Pi_{OPE}(-q^2)]=\mathcal B\left[\Pi_{pert.}+\frac{2m\langle\bar qq\rangle}{q^2}+\frac{\alpha_s\langle F^a_{\mu\nu}F_a^{\mu\nu}\rangle}{12\pi q^2}+\cdots\right]=\mathcal B[\Pi_{pert.}]+\left(2m\langle\bar qq\rangle+\frac{\alpha_s\langle F^a_{\mu\nu}F_a^{\mu\nu}\rangle}{12\pi}\right)\frac{1}{M^2}+\cdots$$
- Quark-hadron duality
  $\rho(s)=2\,s\mathrm{Im}\Pi(s)$; **as $\mathbf{s_0}$ is sufficently large,** $\rho(s)\approx2\,s\mathrm{Im}\Pi_{\mathrm{pert.}}(s)$ for $\mathrm s>\mathrm s_0$
  $$f_V^2\left[\exp\left(-\frac{m_V^2}{M^2}\right)-1\right]=\mathcal B[\Pi_{pert.}]-\frac{1}{\pi}\int_{s_0}^\infty ds\,\mathrm{Im}\Pi_{\mathrm{pert.}}(s)\left[\exp\left(-\frac{s}{M^2}\right)-1\right]+\left(2m\langle\bar qq\rangle+\frac{\alpha_s\langle F^a_{\mu\nu}F_a^{\mu\nu}\rangle}{12\pi}\right)\frac{1}{M^2}+\cdots$$