QCD-03 (Quantization & Chiral anomaly)

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Review of QM

- Wavefunction:
  ⟨basis|state⟩
- Density matrix:
  - $\hat{\rho} = \sum_{ij} \rho_{ij}|i\rangle\langle j|$ ($\leftarrow$ *basis*), $\quad \rho_{ij} = \langle i|\hat{\rho}|j\rangle$
  - $Tr\hat{\rho} \equiv \sum_{i} \langle i|\hat{\rho}|i\rangle = 1$
  - Pure state: $\hat{\varrho} = |\psi\rangle\langle\psi|$, $\varrho_{ij} = \psi_i\psi_j^*$, $\hat{\varrho}^2 = \hat{\varrho}$
  - Under diagonalized representation: $\hat{\rho} = \sum_i p_i|i\rangle\langle i|$
    $p_i \to$ propbability for occupying the basis state $|i\rangle$
  - $\overline{O} = \mathrm{tr}[\hat{\rho}\hat{O}]$
  - In thermal equilibrium (defined in M.R.F), $\hat{\rho} = Z^{-1}e^{-\beta(\hat{H}-\mu\hat{N})}$
    $Z = \mathrm{tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$
- Pictures

	Heisenberg	Schrödinger
State	$\lvert\psi(t)\rangle_H = \lvert\psi(0)\rangle_H$	$\lvert\psi(t)\rangle_S = e^{-i\hat{H}t}\lvert\psi(0)\rangle_S$
Basis	$\lvert a_n, t\rangle_H = e^{i\hat{H}t}\lvert a_n, 0\rangle_H$	$\lvert a_n, t\rangle_S = \lvert a_n, 0\rangle_S$
Operator	$O_H(t) = e^{i\hat{H}t}O_H(0)e^{-i\hat{H}t}$	$O_S(t) = O_S(0)$
Density matrix	$\rho_H(t) = \rho_H(0)$	$\rho_S(t) = e^{-i\hat{H}t}\rho_S(0)e^{i\hat{H}t}$

  Wave function & density matrix components are independent of picture.
- q-representation:
  $\hat{q}_H(t)|q,t\rangle_H = q|q,t\rangle_H$
  $\hat{q}$: (position or field) operator
  q: Eigenvalue & parameter in state, independent of time
  $\langle q'|q\rangle = \delta(q'-q), \qquad \int dq\, |q\rangle\langle q| = I$
- p-representation:
  $\hat{p}_H(t)|p,t\rangle_H = p(t)|p,t\rangle_H$
  $\hat{p} \equiv \frac{\partial\hat{L}}{\partial(\partial_t\hat{q})}; \quad [\hat{p},\hat{q}] = i; \quad \langle q,t|p,t\rangle = e^{ip\cdot q}$
  $\int \frac{dp}{2\pi}|p\rangle\langle p| = I$

Quantization of Fermion

- Field decomposition
  $\hat{\psi} = \hat{\psi}_+ + \hat{\psi}_-$
  $\hat{\psi}_+ = \sum_{p,s} u_{p,s}(x)a_{p,s}\,; \quad \hat{\psi}_- = \sum_{p,s} \bar{v}_{p,s}(x)b^\dagger_{p,s}$
  $\hat{\psi}_+^2 = \hat{\psi}_-^2 = 0$
  Eigenstate of $\hat{\psi}$ is the DIRECT PRODUCT of those of $\hat{\psi}_\pm$
  Eigenstate of $\widehat{\psi_\pm}$ is the DIRECT PRODUCT of those of $a_{p,s}$ & $b^\dagger_{p,s}$
- Coherent states I:
  Ansatz: Eigenstate of $\hat{a}_{p,s}$ is the superposition of $|0\rangle$ & $|p,s\rangle$, i.e.,
  $|\eta_{p,s}\rangle = \alpha_{p,s}|0\rangle + \eta_{p,s}|p,s\rangle$
  $a_{p,s}|\eta_{p,s}\rangle = \eta_{p,s}|0\rangle = \eta_{p,s}\langle\eta_{p,s}|0\rangle\,|\eta_{p,s}\rangle + |\perp\rangle$
  $0 = |\perp\rangle = \eta_{p,s}|0\rangle - \eta_{p,s}\langle\eta_{p,s}|0\rangle\,|\eta_{p,s}\rangle = \eta_{p,s}\left(1-|\alpha_{p,s}|^2\right)|0\rangle - \alpha^*_{p,s}\eta^2_{p,s}|p,s\rangle \Rightarrow$
  - $|\alpha_{p,s}| = 1$
  - $\alpha_{p,s}$ is chosen as 1 (states are in the projectile Hilbert space)
  - $\alpha^*_{p,s}\eta^2_{p,s} = 0 \;\to\; \eta^2_{p,s} = 0 \leftarrow$ Grassmann Number

  In summary,
  $|\eta_{p,s}\rangle = |0\rangle + \eta_{p,s}|p,s\rangle$
  $a_{p,s}|\eta_{p,s}\rangle = \eta_{p,s}|\eta_{p,s}\rangle$
- Coherent states II:
  Ansatz: Eigenstate of $\hat{b}^\dagger_{p,s}$ is the superposition of $|0\rangle$ & $|\bar{p},\bar{s}\rangle$, i.e.,
  $|\bar{\eta}_{p,s}\rangle = \bar{\eta}_{p,s}|0\rangle + \alpha_{p,s}|\bar{p},\bar{s}\rangle$
  $b^\dagger_{p,s}|\bar{\eta}_{p,s}\rangle = \bar{\eta}_{p,s}|\bar{p},\bar{s}\rangle = \bar{\eta}_{p,s}\langle\bar{\eta}_{p,s}|\bar{p},\bar{s}\rangle\,|\bar{\eta}_{p,s}\rangle + |\perp\rangle$
  $0 = |\perp\rangle = \bar{\eta}_{p,s}|\bar{p},\bar{s}\rangle - \bar{\eta}_{p,s}\langle\bar{\eta}_{p,s}|\bar{p},\bar{s}\rangle\,|\bar{\eta}_{p,s}\rangle = \bar{\eta}_{p,s}\left(1-|\alpha_{p,s}|^2\right)|\bar{p},\bar{s}\rangle - \alpha^*_{p,s}\bar{\eta}^2_{p,s}|0\rangle \Rightarrow$
  - $|\alpha_{p,s}| = 1$
  - $\alpha_{p,s}$ is chosen as 1 (states are in the projectile Hilbert space)
  - $\alpha^*_{p,s}\bar{\eta}^2_{p,s} = 0 \;\to\; \bar{\eta}^2_{p,s} = 0 \leftarrow$ Grassmann Number

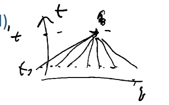
  In summary,
  $|\bar{\eta}_{p,s}\rangle = \bar{\eta}_{p,s}|0\rangle + |\bar{p},\bar{s}\rangle$
  $b^\dagger_{p,s}|\bar{\eta}_{p,s}\rangle = \bar{\eta}_{p,s}|\bar{\eta}_{p,s}\rangle$
- Eigenstate of the field operator:
  $$|\psi\rangle \equiv \prod_i \otimes |\eta_{p_i,s_i}\rangle \prod_j \otimes |\bar{\eta}_{p_j,s_j}\rangle$$
  $$\hat{\psi}|\psi\rangle = \psi|\psi\rangle; \quad \psi = \sum_{p,s} u_{p,s}(x)\eta_{p,s} + \bar{v}_{p,s}(x)\bar{\eta}_{p,s} \leftarrow \textit{Grassmann \#}$$

Grassmann Algebra:

- Convention:
  Normal # represented by Latin: $a, b, c \cdots$
  Grassmann # represented by Greek: $\alpha, \beta, \gamma \cdots$
- Anti-commuting & Grassmann function
  - $a\eta = \eta a$
  - $\eta\zeta = -\zeta\eta \to \eta^{n\geq 2} = 0$
  - $\gamma(\alpha\beta) = (\alpha\beta)\gamma \to$
    product of even # of Grassmann # is Normal
    Product of odd # of Grassmann is Grassmann
  - $f(\eta) = f(0) + \eta f'(0) \leftarrow$ Exact!
- Left & Right derivative
  - $\frac{d}{d\eta}$ is Grassmann
  - $\frac{\overrightarrow{d}}{d\eta}\eta = -\eta\frac{\overleftarrow{d}}{d\eta} = 1$
  - $\frac{\overrightarrow{\partial}}{\partial\eta_i}(\eta_j\eta_k) = \frac{\overrightarrow{\partial}}{\partial\eta_i}\eta_j\eta_k - \eta_j\frac{\overrightarrow{\partial}}{\partial\eta_i}\eta_k = \delta_{ij}\eta_k - \eta_j\delta_{ik}$
- Integration:
  -

Quantization of QCD

Path integral formalism (for a 1-D oscillator → a field),



- Our purpose
  - Evolution of the wave function:
    $$\psi(q,t) = \langle q,t|\psi\rangle_H = \int dq'\,\langle q,t|q',t_0\rangle_H\psi(q',t_0)$$
  - Transition amplitude
    $$\langle\psi,t|\psi,t_0\rangle_H = \int dq dq'\,\psi^*(q',t)\langle q',t|q,t_0\rangle_H\psi(q,t_0)$$
    Notice: $|\psi,t\rangle$ and $|\psi,t_0\rangle$ are DIFFERENT states. Both are the eigenstates of some operaters defined at t and t_0
  - Partition function
    $$Z = \mathrm{tr}\,e^{-\beta(\hat{H}-\mu\hat{N})} = \int dq\left\langle q\left|e^{-\beta(\hat{H}-\mu\hat{N})}\right|q\right\rangle = \int dq\,\langle q,-i\beta|q,0\rangle_{H'}$$
    $$\langle q,-i\beta|_{H'} = \langle q,0|e^{-i\hat{H}'(-i\beta)}, \qquad \hat{H}' = \hat{H} - \mu\hat{N}$$
- Hypothesis: $\hat{H} = \sum \hat{q}\cdots\hat{q}\hat{p}\cdots\hat{p}$
- Kernel calculation
  $$\langle q,t|q',t_0\rangle_H = \int_{t_0<t_1\cdots<t_{N-1}<t} dq_1\cdots dq_{N-1}\,\langle q,t|q_{N-1},t_{N-1}\rangle\langle q_{N-1},t_{N-1}|q_{N-2},t_{N-2}\rangle\cdots\langle q_1,t_1|q',t_0\rangle$$
  $$= \int_{t_0<t_1\cdots<t_{N-1}<t_N=t} dq_0\cdots dq_N\,\langle q,t|q_N,t\rangle\langle q_N,t_N|q_{N-1},t_{N-1}\rangle\cdots\langle q_1,t_1|q_0,t_0\rangle\langle q_0,t_0|q',t_0\rangle$$
  $\to \delta(q'-q_0)$
  $$\langle q_{i+1},t_{i+1}|q_i,t_i\rangle = \int\frac{dp_i}{2\pi}\langle q_{i+1},t_{i+1}|p_i,t_i\rangle\langle p_i,t_i|q_i,t_i\rangle$$
  $$= \int\frac{dp_i}{2\pi}\left\langle q_{i+1},t_i\left|e^{-i\hat{H}(t_{i+1}-t_i)}\right|p_i,t_i\right\rangle e^{-ip_i\cdot q_i}$$
  $$\approx \int\frac{dp_i}{2\pi}\langle q_{i+1},t_i|\hat{I} - i\hat{H}\delta t + \cdots|p_i,t_i\rangle e^{-ip_i\cdot q_i}$$
  $$= \int\frac{dp_i}{2\pi}\langle q_{i+1},t_i|p_i,t_i\rangle(1 - iH(q_{i+1},p_i)\delta t + \cdots)e^{-ip_i\cdot q_i}$$
  $$\approx \int\frac{dp_i}{2\pi}e^{ip_i\cdot q_{i+1}}\,e^{-iH(q_{i+1},p_i)\delta t}e^{-ip_i\cdot q_i} = \int\frac{dp_i}{2\pi}\exp\{i[p_i\cdot\dot{q}_i - H(q_{i+1},p_i)]\delta t\}$$
  $\dot{q}$ is NOT governed by the classical evolution!!
  $H(p,q)$ is an ORDINARY function of the eigenvalues p & q.
  $$\langle q,t|q',t_0\rangle_H = \int dq_0\cdots dq_N\frac{dp_0}{2\pi}\cdots\frac{dp_{N-1}}{2\pi}\,\langle q,t|q_N,t\rangle\exp\left\{\sum_{i=0}^{N-1} i[p_i\cdot\dot{q}_i - H(q_{i+1},p_i)]\delta t\right\}\langle q_0,t_0|q',t_0\rangle$$
  $$= \int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]\langle q,t|q_N,t\rangle\exp\left\{i\int_{t_0}^{t}d\bar{t}\,[p\cdot\dot{q} - H(p,q)]\right\}\langle q_0,t_0|q',t_0\rangle$$
- Quantities of interest:
  - Transition amplitude:
    $$\langle\psi,t|\psi,t_0\rangle_H = \int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]\psi^*(q_N,t)\exp\left\{i\int_{t_0}^{t}d\bar{t}\,[p\cdot\dot{q} - H(p,q)]\right\}\psi(q_0,t_0)$$

    If $|\psi\rangle \to |VAC\rangle$ is the ground state of a harmonic oscillator (at t and t_0),
    $\langle q,t|VAC\rangle = \mathcal{N}\exp\left(-\frac{\kappa}{2}q^2\right)$ *(see Weinberg, Vol I, 9.2 for details)*
    $$\langle VAC,t|VAC,t_0\rangle_H = |\mathcal{N}|^2\int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]\exp\left\{-\frac{\kappa}{2}q(t)^2 - \frac{\kappa}{2}q(t_0)^2 + i\int_{t_0}^{t}d\bar{t}\,[p\cdot\dot{q} - H(p,q)]\right\}$$
    ? $\lim_{\epsilon\to 0^+}\epsilon\int_{-\infty}^{\infty}d\bar{t}\,f(\bar{t})e^{-\epsilon|\bar{t}|} = f(\infty) + f(-\infty)$
    $$\langle VAC,\infty|VAC,-\infty\rangle_H = \lim_{\epsilon\to 0^+}|\mathcal{N}|^2\int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]\exp\left\{i\int_{-\infty}^{\infty}d\bar{t}\,[p\cdot\dot{q} - H(p,q) + i\epsilon q^2]\right\}$$
  - Partition function:
    $$Z = \int dq_0\cdots dq_N\frac{dp_0}{2\pi}\cdots\frac{dp_{N-1}}{2\pi}\exp\left\{i\int_0^{-i\beta}d\bar{t}\,[p\cdot\dot{q} - H(p,q)]\right\}\delta(q_N - q_0)$$
    $$= \int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]\exp\left\{\int_0^\beta d\tau\left[ip\cdot\frac{dq}{d\tau} - H(p,q)\right]\right\}\delta\big(q(-i\beta) - q(0)\big)$$
  - Expectation of observables:
    - $\langle VAC,t|\mathcal{T}\hat{O}(t'_1)\cdots\hat{O}(t'_N)|VAC,t_0\rangle_H$
      - *Hypothesis:* $\hat{O} = \hat{q}\cdots\hat{q}\hat{p}\cdots\hat{p}$
      - *if* $t_i < t'_I < t_j$:
        $$\langle q_{i+1},t_{i+1}|\hat{O}(t'_I)|q_i,t_i\rangle \approx \int\frac{dp_i}{2\pi}O(q_{i+1},p_i)\langle q_{i+1},t_{i+1}|p_i,t_i\rangle\langle p_i,t_i|q_i,t_i\rangle$$
      - $\mathcal{T}$: time-order: $t'_1 > \cdots > t'_N$
        If NOT in time-order, say $t'_1 < t'_2$, then:
        $\hat{O}(t'_1)\langle q_{i+1},t_{i+1}|\hat{O}(t_2)|q_i,t_i\rangle$, $\hat{O}(t'_1)$ cannot be sandwidged between the basises.
      - $\langle VAC,t|\mathcal{T}\hat{O}(t'_1)\cdots\hat{O}(t'_N)|VAC,t_0\rangle_H$
        $$= \lim_{\epsilon\to 0^+}|\mathcal{N}|^2\int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]O(q(t'_1),p(t'_1))\cdots O(q(t'_N),p(t'_N))\exp\left\{i\int_{-\infty}^{\infty}d\bar{t}\,[p\cdot\dot{q} - H(p,q) + i\epsilon q^2]\right\}$$
    - $tr[\hat{\rho}_0\mathcal{T}_\tau\hat{O}(\tau'_1)\cdots\hat{O}(\tau'_N)]$
      $$= Z^{-1}\int\mathcal{D}[q]\mathcal{D}\left[\frac{p}{2\pi}\right]O(q(\tau'_1),p(\tau'_1))\cdots O(q(\tau'_N),p(\tau'_N))\exp\left\{\int_0^\beta d\tau\left[ip\cdot\frac{dq}{d\tau} - H(p,q)\right]\right\}\delta\big(q(-i\beta) - q(0)\big)$$
- Extension to field theory:
  degrees of freedom: $\hat{q}(t) \to \hat{\phi}(x)$; $\hat{p}(t) \to \varpi(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}$
  representation: $\hat{q}(t)|q,t\rangle = q|q,t\rangle \;\to\; \hat{\phi}(x)|\phi,t\rangle = \phi(\vec{x})|\phi,t\rangle$ (matress senario or ...COHERENT STATE....)
  $\hat{H}(\hat{p},\hat{q}) \to \hat{H}[\hat{\phi},\hat{\pi}] = \int d^3\vec{x}\,\hat{\mathcal{H}}(\hat{\phi},\hat{\pi})$
  - Transition amplitude:
    $$\langle VAC,\infty|VAC,-\infty\rangle_H = \lim_{\epsilon\to 0^+}|\mathcal{N}|^2\int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]\exp\left\{i\int d^4x\,[\varpi_a\partial_t\phi_a - \mathcal{H}(\varpi_a,\phi_a) + i\epsilon\phi_a^2]\right\}$$
  - Partition function:
    $$Z = \int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]\exp\left\{\int_0^\beta d\tau d^3\vec{x}\,[i\varpi_a\,\partial_\tau\phi_a - \mathcal{H}(\varpi_a,\phi_a)]\right\}\delta[\phi_a(-i\beta) - \phi_a(0)]$$
  - Expectation of observables:
    - $\langle VAC,\infty|\mathcal{T}\hat{O}(x'_1)\cdots\hat{O}(x'_N)|VAC,-\infty\rangle_H$
      $$= \lim_{\epsilon\to 0^+}|\mathcal{N}|^2\int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]O(x'_1)\cdots O(x'_N)\exp\left\{i\int d^4x\,[\varpi_a\partial_t\phi_a - \mathcal{H}(\varpi_a,\phi_a) + i\epsilon\phi_a^2]\right\}$$
      $$= \lim_{\epsilon\to 0^+}|\mathcal{N}|^2\frac{\delta}{i\delta\mathcal{J}(x'_1)}\cdots\frac{\delta}{i\delta\mathcal{J}(x'_N)}\int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]\exp\left\{i\int d^4x\,[\varpi_a\partial_t\phi_a - \mathcal{H}(\varpi_a,\phi_a) + i\epsilon\phi_a^2 + \mathcal{J}O]\right\}\Bigg|_{\mathcal{J}\to 0}$$
    - $\overline{\mathcal{T}_\tau\hat{O}(x'_1)\cdots\hat{O}(x'_N)}\Big|_{eq}$
      $$= Z^{-1}\int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]O(x'_1)\cdots O(x'_N)\exp\left\{\int_0^\beta d\tau d^3\vec{x}\,[i\varpi_a\,\partial_\tau\phi_a - \mathcal{H}(\varpi_a,\phi_a)]\right\}\delta[\phi_a(-i\beta) - \phi_a(0)]$$
      $$= Z^{-1}\frac{\delta}{\delta\mathcal{J}(x'_1)}\cdots\frac{\delta}{\delta\mathcal{J}(x'_N)}\int\mathcal{D}[\phi]\mathcal{D}\left[\frac{\varpi}{2\pi}\right]\exp\left\{\int_0^\beta d\tau d^3\vec{x}\,[i\varpi_a\,\partial_\tau\phi_a - \mathcal{H}(\varpi_a,\phi_a) + \mathcal{J}O]\right\}\delta[\phi_a(-i\beta) - \phi_a(0)]\Bigg|_{\mathcal{J}\to 0}$$
  - Polyakov loop:
    $$\overline{\mathrm{tr}\,\Phi} \equiv \overline{\mathrm{tr}\,\mathcal{T}_\tau\exp\left[-\int_0^\beta d\tau A^0(-i\tau,\vec{x})\right]} = Z^{-1}\int\mathcal{D}[A]\mathcal{D}\left[\frac{\pi_A}{2\pi}\right]\exp\left\{\int_0^\beta d\tau d^3\vec{x}'\,[i\pi^i_{A_a}\,\partial_\tau A^i_a - \mathcal{H}(\pi_A,A) - \mathrm{tr}\,A^0\delta^3(\vec{x}-\vec{x}')]\right\}\delta[A(-i\beta) - A(0)] = \frac{Z'}{Z}$$
    $$Z' \equiv \int\mathcal{D}[A]\mathcal{D}\left[\frac{\pi_A}{2\pi}\right]\exp\left\{\int_0^\beta d\tau d^3\vec{x}'\,[i\pi^i_{A_a}\,\partial_\tau A^i_a - \mathcal{H}'(\pi_A,A)]\right\}\delta[A(-i\beta) - A(0)]$$
    $\mathcal{H}'(\pi_A,A) = \mathcal{H}(\pi_A,A) + trA^0\delta^3(x) \leftarrow$ energy increment due to a color source
    $\overline{\mathrm{tr}\,\Phi} \equiv \exp[\beta(T\ln Z' - T\ln Z)] = \exp[\beta\Delta F]$
  - Fermion field