QCD-09 (pQCD @ finite T)

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Feynman Rules Review:

- Transform to Eucilidean:
  - In action: $i\int dt\, L \to \int_0^\beta d\tau\, L;\quad \partial_0 \to i\partial_\tau - i\mu \Rightarrow p^0 \sim i\partial_0 \to -\partial_\tau + \mu \sim i\omega_n + \mu$
  - $i\omega_n = 2n\pi i T$ (Bosonic: $i\omega_n\beta = 1 \Rightarrow G(\beta) = G(0)$) or $(2n+1)\pi i T$ (Fermionic: $i\omega_n\beta = -1 \Rightarrow S(\beta) = -S(0)$)
  - $\int \frac{dp^0}{2\pi} \to T\sum_n \square$
- Residue Magic
  - $\frac{1}{2}\tanh\frac{z}{2T}$ has poles at $(2n+1)\pi i T$, with residue equal to T
  - $\frac{1}{2}\coth\frac{z}{2T}$ has poles at $2n\pi i T$, with residue equal to T
  - $\int \frac{dp^0}{2\pi}\mathcal{F}(p^0) \to T\sum_n \mathcal{F}(i\omega_n + \mu) \to \frac{1}{2}\int_{C\downarrow+C\uparrow}\frac{dp^0}{2\pi i}\tanh\frac{p^0}{2T}\mathcal{F}(p^0+\mu)$ *or* $\frac{1}{2}\int_{C\downarrow+C\uparrow}\frac{dp^0}{2\pi i}\coth\frac{p^0}{2T}\mathcal{F}(p^0+\mu)$
  - As $T\to 0$, $\tanh x \to \coth x \to \operatorname{sgn} x$, reduce to Wick's rotation.
  - $\tanh\frac{z+2\pi n i T}{2T} = \coth\frac{z+(2n+1)\pi i T}{2T} = \tanh\frac{z}{2T}$
  - $\coth\frac{z+2\pi n i T}{2T} = \tanh\frac{z+(2n+1)\pi i T}{2T} = \coth\frac{z}{2T}$
- Propagator:
  - Quark:

    $\tilde{S}(p) = -\left(p^\mu\gamma_\mu + \mu\gamma^0 - M\right)^{-1} = -\frac{p^\mu\gamma_\mu + M}{p^2 - M^2}$
  - *Gluon:*

    $\tilde{D}^{\mu\nu}(p) = p^{-2}\left[g^{\nu\mu} + (\xi - 1)\frac{p^\nu p^\mu}{p^2}\right]$
  - Ghost:

    $\tilde{G}(p) = p^{-2}$
- *Vertex*
  - Gluon-Quark Vertex

    $V_{\mu;a}^{ij;\alpha\beta} = -g\tau_{ij}^a\gamma_\mu^{\alpha\beta}$
  - 3-Gluon Vertex

    $V_{\alpha\beta\gamma}^{abc}(p,q,r) = -igf_{abc}\left[g_{\alpha\beta}(q-p)_\gamma + g_{\beta\gamma}(r-q)_\alpha + g_{\alpha\gamma}(p-r)_\beta\right]$ ← momentum of outgoing gluon
  - 4-Gluon Vertex

    $V_{\alpha\beta\gamma\delta}^{abcd} = -g^2\left[f_{abh}f_{cdh}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) + f_{ach}f_{dbh}\left(g_{\alpha\delta}g_{\beta\gamma} - g_{\gamma\delta}g_{\alpha\beta}\right) + f_{adh}f_{bch}\left(g_{\alpha\beta}g_{\gamma\delta} - g_{\delta\beta}g_{\alpha\gamma}\right)\right]$
  - Ghost-Gluon Vertex

    $V_{abc;\mu} = -igf_{abc}p_\mu$ ← momentum of outgoing ghost

QED Plasma

Photon Self-energy (General Properties)

$$\Pi_{\mu\nu}(q) = \int d^4x \left\langle \mathcal{T}_\tau\, j_\mu(x) j_\nu(0)\right\rangle e^{iq\cdot x}$$

- Projectors:

  A new 4-vector involved $\Rightarrow u^\mu \Leftarrow$ flow velocity of the medium

  (a new frame is specified $\Rightarrow u = (1,\vec{0}) \Leftarrow$ medium rest frame)

  Further decompose the projector:

  $\boldsymbol{\Delta^{\mu\nu}(p) \equiv -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} = \Delta_L^{\mu\nu}(p,u) + \Delta_T^{\mu\nu}(p,u)}$

  $\boldsymbol{\Delta_L^{\mu\nu} \equiv \frac{\Delta^\mu{}_\alpha\Delta^\nu{}_\beta u^\alpha u^\beta}{u\cdot\Delta\cdot u}};\qquad \boldsymbol{\Delta_T = \Delta - \Delta_L}$

  $\Delta\cdot\Delta = -\Delta;\quad \Delta\cdot\Delta_L = \Delta_L\cdot\Delta = -\Delta_L;\quad \Delta_L\cdot\Delta_L = -\Delta_L;\quad \Delta_L\cdot\Delta_T = 0;\quad \Delta_T\cdot\Delta_T = -\Delta_T$

  In medium rest frame:

  $$\Delta_L^{\mu\nu}(p) = \frac{\Delta^\mu{}_0\Delta^\nu{}_0}{\Delta^{00}} = \frac{\left(-g_0^\mu + \frac{p^\mu p_0}{p^2}\right)\left(-g_0^\nu + \frac{p^\nu p_0}{p^2}\right)}{-1 + p_0^2/p^2} = \vec{p}^{-2}\left(-g_0^\mu p^2 + p^\mu p_0\right)\left(-g_0^\nu + \frac{p^\nu p_0}{p^2}\right)$$

  $$= \vec{p}^{-2}\left(g_0^\mu g_0^\nu p^2 - p_0 p^\mu g_0^\nu - p_0 p^\nu g_0^\mu + \frac{p_0^2}{p^2}p^\mu p^\nu\right) = \begin{pmatrix} \frac{\vec{p}^2}{p^2} & \frac{p^0 p^\nu}{p^2} \\ \frac{p^0 p^\mu}{p^2} & \frac{p_0^2}{p^2\vec{p}^2}p^\mu p^\nu \end{pmatrix}$$

  $$\Delta_T^{\mu\nu}(p) = \Delta^{\mu\nu} - \Delta_L^{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} - \vec{p}^{-2}\left(g_0^\mu g_0^\nu p^2 - p_0 p^\mu g_0^\nu - p_0 p^\nu g_0^\mu + \frac{p_0^2}{p^2}p^\mu p^\nu\right)$$

  $$= -g^{\mu\nu} - \vec{p}^{-2}\left(g_0^\mu g_0^\nu p^2 - p_0 p^\mu g_0^\nu - p_0 p^\nu g_0^\mu + p^\mu p^\nu\right) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\mu\nu} - \frac{p^\mu p^\nu}{\vec{p}^2}\end{pmatrix}$$

  In M.R.F., Δ_T erases time component & takes the SPATIAL component perpendicular to $\vec{p}$
- Decomposition of $\Pi^{\mu\nu}$

  $\Pi^{\mu\nu}(p) = \Delta_L^{\mu\nu}(p)\Pi_L(p) + \Delta_T^{\mu\nu}(p)\Pi_T(p)$

  $\Pi_L(p) = \Delta_{\mu\nu}^L(p)\Pi^{\mu\nu}(p);\qquad \Pi_T(p) = \frac{1}{2}\Delta_T^{\mu\nu}(p)\Pi_{\mu\nu}(p);\quad \Delta_{\mu\nu}(p)\Pi^{\mu\nu}(p) = \Pi_L(p) + 2\Pi_T(p)$

QED plasma

Photon Self-energy to 1-Loop (Transverse)

$$\delta\Pi_T = \frac{1}{2}\Delta_T^{\mu\nu}(k)\Pi_{\mu\nu}(k)$$

$$= -4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(|\vec{p}_-|^2 - (\vec{p}_-\cdot\hat{k})^2\right) - \vec{k}\cdot\vec{p}_- + k^0E_-}{E_-(k^0+E_--E_+)(k^0+E_-+E_+)}f\left(\frac{E_-}{T}\right)$$

$$-4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(|\vec{p}_+|^2 - (\vec{p}_+\cdot\hat{k})^2\right) + \vec{k}\cdot\vec{p}_+ - k^0E_+}{(E_+-k^0-E_-)(E_+-k^0+E_-)E_+}f\left(\frac{E_+}{T}\right)$$

$$= -4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\vec{p}^2 - (\vec{p}\cdot\hat{k})^2 + \frac{\vec{k}^2}{2} - \vec{p}\cdot\vec{k} + k^0E_-}{E_-(k^0+E_--E_+)(k^0+E_-+E_+)}f\left(\frac{E_-}{T}\right)$$

$$-4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\vec{p}^2 - (\vec{p}\cdot\hat{k})^2 + \frac{\vec{k}^2}{2} + \vec{k}\cdot\vec{p} - k^0E_+}{(E_+-k^0-E_-)(E_+-k^0+E_-)E_+}f\left(\frac{E_+}{T}\right)$$

Substitution:

$$\varepsilon \equiv \frac{1}{2}(E_++E_-);\quad \omega \equiv E_+-E_-;\quad E_\pm = \varepsilon \pm \frac{1}{2}\omega;\quad \varepsilon\omega = \frac{1}{2}(E_+^2-E_-^2) = \vec{p}\cdot\vec{k};\quad \varepsilon^2 + \frac{1}{4}\omega^2 = \frac{1}{2}(E_+^2+E_-^2) = \vec{p}^2 + \frac{\vec{k}^2}{4} + M^2;$$

$$\int d^3\vec{p} = 2\pi\int dp\vec{p}^2\int_{-1}^1 dz = 2\pi\int d\varepsilon d\omega\ \vec{p}^2\left|\frac{\partial(\varepsilon,\omega)}{\partial(p,z)}\right|^{-1} = \frac{2\pi}{|\vec{k}|}\int d\varepsilon d\omega\, E_+E_- = \frac{2\pi}{|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)$$

$$\varepsilon_0 \equiv \sqrt{|\vec{k}|^2 + \frac{M^2}{4}};\quad \omega_0 \equiv |\vec{k}|\sqrt{1 - \frac{4M^2}{4\varepsilon^2 - |\vec{k}|^2}}$$

$$\delta\Pi_T = -\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)\frac{\varepsilon^2 + \frac{1}{4}(\omega^2+\vec{k}^2) - M^2 - \left(\frac{\varepsilon\omega}{|\vec{k}|}\right)^2 - \frac{1}{2}k^0\omega - \varepsilon\omega + k^0\varepsilon}{\left(\varepsilon - \frac{1}{2}\omega\right)(k^0-\omega)(k^0+2\varepsilon)}f\left(\frac{\varepsilon - \frac{1}{2}\omega}{T}\right)$$

$$-\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)\frac{\varepsilon^2 + \frac{1}{4}(\omega^2+\vec{k}^2) - M^2 - \left(\frac{\varepsilon\omega}{|\vec{k}|}\right)^2 - \frac{1}{2}k^0\omega + \varepsilon\omega - k^0\varepsilon}{\left(\varepsilon + \frac{1}{2}\omega\right)(k^0-\omega)(k^0-2\varepsilon)}f\left(\frac{\varepsilon + \frac{1}{2}\omega}{T}\right)$$

$$= -\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)\left[\frac{\left(\varepsilon^2 + \frac{\vec{k}^2}{4}\right)\left(1 - \frac{\omega^2}{\vec{k}^2}\right) - M^2}{k^0-\omega} - \frac{\omega}{2}\right]\left[\frac{f\left(\frac{\varepsilon - \frac{1}{2}\omega}{T}\right)}{\left(\varepsilon - \frac{1}{2}\omega\right)(k^0+2\varepsilon)} + \frac{f\left(\frac{\varepsilon + \frac{1}{2}\omega}{T}\right)}{\left(\varepsilon + \frac{1}{2}\omega\right)(k^0-2\varepsilon)}\right]$$

$$+\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\,\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)\left[\frac{f\left(\frac{\varepsilon - \frac{1}{2}\omega}{T}\right)}{\left(\varepsilon - \frac{1}{2}\omega\right)(k^0+2\varepsilon)} - \frac{f\left(\frac{\varepsilon + \frac{1}{2}\omega}{T}\right)}{\left(\varepsilon + \frac{1}{2}\omega\right)(k^0-2\varepsilon)}\right]$$

$$\delta\Pi_T(k_0 = 0) = \frac{2g^2T^2}{\pi^2}\int_{\frac{M}{2T}}^\infty dx\sqrt{1 - \frac{M^2}{x^2T^2}}\,xf(x) + \mathcal{O}\left(\frac{\vec{k}^2}{T^2}\right) \xrightarrow{M\to 0}\frac{1}{6}g^2T^2$$

QED Plasma

Photon Self-energy to 1-Loop (Longitudinal)

$$\delta\Pi_L(k) = \Delta_{\mu\nu}^L(k)\delta\Pi^{\mu\nu}(k)$$

$$= -4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\left(E_p^2 + \frac{k_0^2}{2}\right)\frac{\vec{k}^2}{k^2} + \frac{2k_0^2}{k^2}(\vec{p}\cdot\hat{k})^2 - (\vec{p}\cdot\vec{k})\left(\frac{3k_0^2+\vec{k}^2}{k^2}\right) + k^0E_-\left(1 - \frac{2\left(2\vec{p}\cdot\vec{k} - \vec{k}^2\right)}{k^2}\right)}{E_-(k^0+E_--E_+)(k^0+E_-+E_+)}f\left(\frac{E_-}{T}\right)$$

$$-4g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\left(E_p^2 + \frac{k_0^2}{2}\right)\frac{\vec{k}^2}{k^2} + \frac{2k_0^2}{k^2}(\vec{p}\cdot\hat{k})^2 + (\vec{p}\cdot\vec{k})\left(\frac{3k_0^2+\vec{k}^2}{k^2}\right) - \left(1 + \frac{2\left(2\vec{p}\cdot\vec{k} + \vec{k}^2\right)}{k^2}\right)k^0E_+}{(E_+-k^0-E_-)(E_+-k^0+E_-)E_+}f\left(\frac{E_+}{T}\right)$$

$$= -\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\frac{\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)}{k^0-\omega}\left\{\left[2\left(\varepsilon^2 + \frac{1}{4}\omega^2 - \frac{1}{4}\vec{k}^2 + \frac{k_0^2}{2}\right)\frac{\vec{k}^2}{k^2} + \frac{2k_0^2}{k^2\vec{k}^2}\varepsilon^2\omega^2\right]\left[\frac{f\left(\frac{E_-}{T}\right)}{E_-(k^0+2\varepsilon)} + \frac{f\left(\frac{E_+}{T}\right)}{E_+(k^0-2\varepsilon)}\right] - \frac{4\varepsilon\omega k^0}{k^2}\left[\frac{f\left(\frac{E_-}{T}\right)}{(k^0+2\varepsilon)} + \frac{f\left(\frac{E_+}{T}\right)}{(k^0-2\varepsilon)}\right]\right\}$$

$$-\frac{g^2}{\pi^2|\vec{k}|}\int_{\varepsilon_0}^\infty d\varepsilon\int_{-\omega_0}^{\omega_0}d\omega\frac{\left(\varepsilon^2 - \frac{1}{4}\omega^2\right)}{k^0-\omega}\left\{\left[\varepsilon\omega\left(\frac{3k_0^2+\vec{k}^2}{k^2}\right)\right]\left[\frac{f\left(\frac{E_+}{T}\right)}{E_+(k^0-2\varepsilon)} - \frac{f\left(\frac{E_-}{T}\right)}{E_-(k^0+2\varepsilon)}\right] - k^0\left[1 + \frac{2\vec{k}^2}{k^2}\right]\left[\frac{f\left(\frac{E_+}{T}\right)}{(k^0-2\varepsilon)} - \frac{f\left(\frac{E_-}{T}\right)}{(k^0+2\varepsilon)}\right]\right\}$$

$$\delta\Pi_L(k_0 = 0) = -\frac{2g^2T^2}{\pi^2}\int_{\frac{M}{2T}}^\infty dx\sqrt{1 - \frac{M^2}{x^2T^2}}\,x^2f'(x) + \mathcal{O}\left(\frac{\vec{k}^2}{T^2}\right)\xrightarrow{M\to 0}\frac{1}{3}g^2T^2$$

QED Plasma

Linear Response Thoery (see lecture notes on plasma)

- Decomposition of Hamiltonian: $\hat{H} = \widehat{H_0} + \hat{H}' \leftarrow H' \ll H$ is perturbation
- Decompostion of time-evolution operator:

  $\mathcal{U}(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\,H(t)} = \mathcal{U}_0(t_2,t_1)\mathcal{U}_1(t_2,t_1)$

  $\mathcal{U}_0(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\,H_0(t)}$

  $\mathcal{U}_1(t_2,t_1) \equiv \mathcal{T}e^{-i\int_{t_1}^{t_2}dt\,H_I'(t)};\quad \mathcal{O}_I(t) \equiv \mathcal{U}_0^\dagger(t,t_0)\mathcal{O}_S\mathcal{U}_0(t,t_0)$
- Linear response:

  $\mathcal{U}_1(t_2,t_1) \approx I - i\int_{t_1}^{t_2}dt\ H_I'(t)$

  $\overline{\mathcal{O}(t)} = \mathrm{tr}[\hat{\rho}_S(t)\mathcal{O}_S] = \mathrm{tr}[\mathcal{U}(t,-\infty)\rho_0\mathcal{U}^\dagger(t,-\infty)\mathcal{O}_S] = \mathrm{tr}\left[\rho_0\mathcal{U}_1^\dagger(t,-\infty)\mathcal{U}_0^\dagger(t,-\infty)\mathcal{O}_S\mathcal{U}_0(t,-\infty)\mathcal{U}_1(t,-\infty)\right]$

  $= \mathrm{tr}\left[\rho_0\mathcal{U}_1^\dagger(t,-\infty)\mathcal{O}_I(t)\mathcal{U}_1(t,-\infty)\right] =$ expct. w/o perturbation $\to \mathrm{tr}[\rho_0\mathcal{O}_I(t)] - i\int_{-\infty}^t d\bar{t}\,[\mathcal{O}_I(t), H_I'(\bar{t})]$

  $\overline{\delta\mathcal{O}(t)} = -i\int_{-\infty}^\infty d\bar{t}\,\theta(t-\bar{t})[\mathcal{O}_I(t), H_I'(\bar{t})]$
- Debye Screening:

  Perturbed by an external charge density: $H' = \int d^3\vec{x}\, g\rho_{cl}A^0$

  $\overline{\delta A^0(x)} = -i\int_{-\infty}^\infty d^4\bar{x}\,\theta(t-\bar{t})[A_I^0(x), A_I^0(\bar{x})]\rho_{cl} \Rightarrow$

  $\delta\widetilde{A^0}(k) = -\frac{1}{k^2 + \delta\Pi^{00}(k)}\widetilde{\rho_{cl}}(k)$

  For $\rho_{cl} = Q\delta^3(\vec{x})$; $\tilde{\rho}_{cl} = Q\delta(k^0)$

  $\delta\widetilde{A^0}(k) = \frac{Q\delta(k^0)}{\vec{k}^2 - \delta\Pi^{00}(k)} \approx \frac{Q\delta(k^0)}{\vec{k}^2 - \frac{4g^2}{T}\int\frac{d^3\vec{p}}{(2\pi)^3}f'\left(\frac{E}{T}\right)} = \frac{Q\delta(k^0)}{\vec{k}^2 + m_D^2}$

  $m_D^2 = -\frac{4g^2}{T}\int\frac{d^3\vec{p}}{(2\pi)^3}f'\left(\frac{E}{T}\right)$ ← *Debye's Screening Mass*
- Plasma Oscillation

  Perturbation: $H' = \int d^3\vec{x}\, gj\cdot\delta A^{cl}$

  $\overline{\delta j^\mu(x)} = -ig\int_{-\infty}^\infty d^4\bar{x}\,\theta(t-\bar{t})[j_I^\mu(x), j_I^\nu(\bar{x})]\overline{\delta A_\nu^{cl}(\bar{x})}$

  $\delta\widetilde{j^\mu}(k) = -\frac{1}{g}\Pi^{\mu\nu}(k)\delta\widetilde{A_\nu^{cl}}(k) = -\frac{1}{g}\Delta_L^{\mu\nu}\Pi_L(k)\delta\widetilde{A_\nu^{cl}}(k) - \frac{1}{g}\Delta_T^{\mu\nu}\Pi_L(k)\delta\widetilde{A_\nu^{cl}}(k) \Rightarrow$

  $k\cdot\delta\tilde{j} = 0 \Rightarrow \delta\widetilde{j^\mu}(k) = \delta j_L^\mu + \delta j_T^\mu$

  On the other hand,

  $\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\mu = -gj^\nu \Rightarrow$

  $k^2\Delta^{\mu\nu}(k)\delta\widetilde{A_\mu}(k) = -g\delta\widetilde{j^\nu}(k) \Rightarrow k^2\Delta_{L/T}^{\mu\nu}(k)\delta\widetilde{A_\mu}(k) = -g\delta\widetilde{j_{L/T}^\nu}(k)$

  So,

  $\Delta_{L/T}^{\mu\nu}\Pi_{L/T}(k)\delta\widetilde{A_\nu^{cl}}(k) = k^2\Delta_{L/T}^{\mu\nu}(k)\delta\widetilde{A_\mu^{cl}}(k)$
  - Transverse mode:

    $0 = k^2 - \Pi_T(k) \approx k^2 + \frac{g^2}{6}T^2 + \frac{g^2}{4\pi^2}\ln\frac{T}{|\vec{k}|}\left\{k_0^2 + \vec{k}^2\right\} + \frac{5}{8}\frac{g^2}{\pi^2}\vec{k}^2$
  - Logitudinal mode:

    $0 = k^2 - \Pi_L(k) \approx k^2 + \frac{g^2}{6k^4}\left(k_0^4 - 5k_0^2\vec{k}^2 - 2\vec{k}^4\right)T^2 + \frac{g^2}{8\pi^2k^4}\vec{k}^2\left[k_0^4 + 12\vec{k}^2k_0^2 + 3\vec{k}^4\right] + \frac{g^2}{4\pi^2k^4}\left[\vec{k}^6 - k_0^6 + 5\vec{k}^2k_0^2\left(\vec{k}^2 + k_0^2\right)\right]\ln\frac{T}{|\vec{k}|}$

QED Plasma

Photon Self-energy (1-Loop Calc.)

Notation: $p_\pm \equiv p \pm \frac{k}{2}$; $\quad E_\pm \equiv E_{p\pm\frac{k}{2}}$; $\quad \tilde{p}_\pm \equiv \left(E_\pm, \vec{p} \pm \frac{\vec{k}}{2}\right) \equiv E_\pm u + \vec{p}_\pm$; $\quad \bar{p}_\pm \equiv \left(-E_\pm, \vec{p}\pm\frac{\vec{k}}{2}\right) \equiv -E_\pm u + \vec{p}_\pm$; $\quad \widetilde{p_\pm^2} = M^2 = \bar{p}_\pm^2$

$$\Pi^{\mu\nu}\left(k^0 = 2\pi niT, \vec{k}\right) = -(-g)^2\int_{C\downarrow+C\uparrow}\frac{d^4p}{(2\pi)^4i}\mathrm{tr}[\gamma^\mu S(p_-)\gamma^\nu S(p_+)]\frac{1}{2}tanh\frac{p^0 + \frac{k^0}{2}}{2T}$$

$$= -g^2\int_{C\downarrow+C\uparrow}\frac{d^4p}{(2\pi)^4i}\frac{\mathrm{tr}\left[\gamma^\mu\left((p_-)_\sigma\gamma^\sigma + M\right)\gamma^\nu\left((p_+)_\rho\gamma^\rho + M\right)\right]}{\left((p_-)^2 - M^2\right)\left((p_+)^2 - M^2\right)}\frac{1}{2}\tanh\frac{p^0 + \frac{k^0}{2}}{2T}$$

$$= -g^2\int_{C\downarrow+C\uparrow}\frac{d^4p}{(2\pi)^4i}\frac{\mathrm{tr}\left[\gamma^\mu(p_-)_\sigma\gamma^\sigma\gamma^\nu(p_+)_\rho\gamma^\rho\right] + \mathrm{tr}[\gamma^\mu\gamma^\nu]M^2}{\left((p_-)^2 - M^2\right)\left((p_+)^2 - M^2\right)}\frac{1}{2}\tanh\frac{p^0 + \frac{k^0}{2}}{2T}$$

$$= -4g^2\int_{C\downarrow+C\uparrow}\frac{d^4p}{(2\pi)^4i}\frac{p_-^\mu p_+^\nu + p_+^\mu p_-^\nu - g^{\mu\nu}p_+\cdot p_- + g^{\mu\nu}M^2}{\left(\left(p^0 - \frac{k^0}{2} + \mu\right)^2 - E_-^2\right)\left(\left(p^0 + \frac{k^0}{2} + \mu\right)^2 - E_+^2\right)}\frac{1}{2}\tanh\frac{p^0 + \frac{k^0}{2}}{2T}$$

Medium Modification:

$\Pi^{\mu\nu} = \Pi^{\mu\nu}(T = 0) + \delta\,\Pi^{\mu\nu}$

$$\delta\,\Pi^{\mu\nu} = -4g^2\int_{C\downarrow+C\uparrow}\frac{d^4p}{(2\pi)^4i}\frac{p_-^\mu p_+^\nu + p_+^\mu p_-^\nu - g^{\mu\nu}p_+\cdot p_- + g^{\mu\nu}M^2}{\left(p^0 - \frac{k^0}{2} - E_-\right)\left(p^0 - \frac{k^0}{2} + E_-\right)\left(p^0 + \frac{k^0}{2} - E_+\right)\left(p^0 + \frac{k^0}{2} + E_+\right)}\frac{1}{2}\left(\tanh\frac{p^0 + \frac{k^0}{2}}{2T} - \operatorname{sgn}\mathrm{Re}p^0\right)$$

$$= 2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_-^\mu\tilde{p}_-^\nu + \tilde{p}_-^\mu k^\nu + k^\mu\tilde{p}_-^\nu - g^{\mu\nu}k\cdot\tilde{p}_-}{2E_-(k^0+E_--E_+)(k^0+E_-+E_+)}\left(\tanh\frac{E_- - \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_-^\mu\bar{p}_-^\nu + \bar{p}_-^\mu k^\nu + k^\mu\bar{p}_-^\nu - g^{\mu\nu}k\cdot\bar{p}_-}{2E_-(k^0-E_--E_+)(k^0-E_-+E_+)}\left(\tanh\frac{E_- + \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_+^\mu\tilde{p}_+^\nu - \tilde{p}_+^\mu k^\nu - k^\mu\tilde{p}_+^\nu + g^{\mu\nu}k\cdot\tilde{p}_+}{2E_+(k^0+E_--E_+)(k^0-E_--E_+)}\left(\tanh\frac{E_+ - \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_+^\mu\bar{p}_+^\nu - \bar{p}_+^\mu k^\nu - k^\mu\bar{p}_+^\nu + g^{\mu\nu}k\cdot\bar{p}_+}{(2E_+)(k^0+E_-+E_+)(k^0-E_-+E_+)}\left(\tanh\frac{E_+ + \mu}{2T} - 1\right)$$

Substitution: $\vec{p} \to -\vec{p}$; $\quad E_+ \leftrightarrow E_-$; $\quad \bar{p}_\pm \to -\bar{p}_\mp$; $\quad \tanh\frac{x}{2} - 1 = -2f(x) \Rightarrow$

$$\delta\Pi^{\mu\nu} = 2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_-^\mu\tilde{p}_-^\nu + \tilde{p}_-^\mu k^\nu + k^\mu\tilde{p}_-^\nu - g^{\mu\nu}k\cdot\tilde{p}_-}{2E_-(k^0+E_--E_+)(k^0+E_-+E_+)}\left(\tanh\frac{E_- + \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_-^\mu\bar{p}_-^\nu + \bar{p}_-^\mu k^\nu + k^\mu\bar{p}_-^\nu - g^{\mu\nu}k\cdot\bar{p}_-}{2E_-(k^0-E_--E_+)(k^0-E_-+E_+)}\left(\tanh\frac{E_- - \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_+^\mu\tilde{p}_+^\nu - \tilde{p}_+^\mu k^\nu - k^\mu\tilde{p}_+^\nu + g^{\mu\nu}k\cdot\tilde{p}_+}{2E_+(k^0+E_--E_+)(k^0-E_--E_+)}\left(\tanh\frac{E_+ + \mu}{2T} - 1\right)$$

$$+2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_+^\mu\bar{p}_+^\nu - \bar{p}_+^\mu k^\nu - k^\mu\bar{p}_+^\nu + g^{\mu\nu}k\cdot\bar{p}_+}{(2E_+)(k^0+E_-+E_+)(k^0-E_-+E_+)}\left(\tanh\frac{E_+ - \mu}{2T} - 1\right)$$

Taking Average:

$$\delta\Pi^{\mu\nu} = -2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_-^\mu\tilde{p}_-^\nu + \tilde{p}_-^\mu k^\nu + k^\mu\tilde{p}_-^\nu - g^{\mu\nu}k\cdot\tilde{p}_-}{2E_-(k^0+E_--E_+)(k^0+E_-+E_+)}\left(f\left(\frac{E_- - \mu}{T}\right) + f\left(\frac{E_- + \mu}{T}\right)\right)$$

$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_-^\mu\bar{p}_-^\nu + \bar{p}_-^\mu k^\nu + k^\mu\bar{p}_-^\nu - g^{\mu\nu}k\cdot\bar{p}_-}{2E_-(k^0-E_--E_+)(k^0-E_-+E_+)}\left(f\left(\frac{E_- - \mu}{T}\right) + f\left(\frac{E_- + \mu}{T}\right)\right)$$

$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_+^\mu\tilde{p}_+^\nu - \tilde{p}_+^\mu k^\nu - k^\mu\tilde{p}_+^\nu + g^{\mu\nu}k\cdot\tilde{p}_+}{2E_+(k^0+E_--E_+)(k^0-E_--E_+)}\left(f\left(\frac{E_+ - \mu}{T}\right) + f\left(\frac{E_+ + \mu}{T}\right)\right)$$

$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_+^\mu\bar{p}_+^\nu - \bar{p}_+^\mu k^\nu - k^\mu\bar{p}_+^\nu + g^{\mu\nu}k\cdot\bar{p}_+}{(2E_+)(k^0+E_-+E_+)(k^0-E_-+E_+)}\left(f\left(\frac{E_+ - \mu}{T}\right) + f\left(\frac{E_+ + \mu}{T}\right)\right)$$

$$= -2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(k_0^2 + E_-^2 - E_+^2\right)\left(4\vec{p}^\mu\vec{p}^\nu - \vec{k}^\mu\vec{k}^\nu + 2g^{\mu\nu}\vec{k}\cdot\vec{p}_- + 2\vec{p}_-^\mu u^\nu k^0 + 2\vec{p}_-^\nu u^\mu k^0\right) - 4E_-^2\left(2k^0u^\mu\vec{p}^\nu + 2k^0u^\nu\vec{p}^\mu - u^\mu u^\nu\left(E_-^2 - E_+^2\right)\right) + 4k_0^2E_-^2\Delta_u^{\mu\nu}}{2E_-\left(k_0^2 - (E_- - E_+)^2\right)\left(k_0^2 - (E_- + E_+)^2\right)}\left(f\left(\frac{E_- - \mu}{T}\right) + f\left(\frac{E_- + \mu}{T}\right)\right)$$

$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(k_0^2 - E_-^2 + E_+^2\right)\left(4\vec{p}^\mu\vec{p}^\nu - \vec{k}^\mu\vec{k}^\nu - 2k^0\vec{p}_+^\mu u^\nu - 2k^0\vec{p}_+^\nu u^\mu - 2g^{\mu\nu}\vec{k}\cdot\vec{p}_+\right) + 4E_+^2\left(2k^0u^\mu\vec{p}^\nu + 2k^0u^\nu\vec{p}^\mu + u^\mu u^\nu\left(-E_-^2 + E_+^2\right)\right) + 4k_0^2E_+^2\Delta_u^{\mu\nu}}{2E_+\left(k_0^2 - (E_- - E_+)^2\right)\left(k_0^2 - (E_- + E_+)^2\right)}\left(f\left(\frac{E_+ - \mu}{T}\right) + f\left(\frac{E_+ + \mu}{T}\right)\right)$$

QED Plasma

Chiral and High-T limit

- $M \to 0;\ \Rightarrow\ \varepsilon_0 \sim \omega_0 \to |\vec{k}|$
- $T \gg |\vec{k}| \sim \omega_0 \Rightarrow \int_{-\omega_0}^{\omega_0}d\omega\ \mathcal{F}(\omega)\, f\left(\frac{??-\omega}{T}\right) \approx \omega_0\left[\mathcal{F}(\omega_0)\, f\left(\frac{??-\omega_0}{T}\right) + \mathcal{F}(-\omega_0)\, f\left(\frac{??+\omega_0}{T}\right)\right]$
- Transverse Self-energy:

  $$\delta\Pi_T \approx -\frac{g^2}{2\pi^2}\int_{\varepsilon_0}^\infty d\varepsilon\frac{\left(4\varepsilon^2 - |\vec{k}|^2\right)}{\left(4\varepsilon^2 - k_0^2\right)}\varepsilon|\vec{k}|\left[\frac{f\left(\frac{\varepsilon + \frac{1}{2}|\vec{k}|}{T}\right)}{\left(\varepsilon + \frac{1}{2}|\vec{k}|\right)} - \frac{f\left(\frac{\varepsilon - \frac{1}{2}|\vec{k}|}{T}\right)}{\left(\varepsilon - \frac{1}{2}|\vec{k}|\right)}\right] + \frac{g^2}{\pi^2}\int_{\varepsilon_0}^\infty d\varepsilon\,\varepsilon^2\frac{\left(4\varepsilon^2 - |\vec{k}|^2\right)}{\left(4\varepsilon^2 - k_0^2\right)}\left[\frac{f\left(\frac{\varepsilon + \frac{1}{2}|\vec{k}|}{T}\right)}{\left(\varepsilon + \frac{1}{2}|\vec{k}|\right)} + \frac{f\left(\frac{\varepsilon - \frac{1}{2}|\vec{k}|}{T}\right)}{\left(\varepsilon - \frac{1}{2}|\vec{k}|\right)}\right]$$

  $$\approx \frac{g^2T^2}{\pi^2}\int_0^\infty dx\frac{8x^3f[x] + 2x\left(f[x] - 2xf'[x] + \frac{x^2}{2}f''[x]\right)\frac{\vec{k}^2}{T^2}}{\left(4x^2 - \frac{k_0^2}{T^2}\right)}$$

  $$\approx \frac{g^2}{6}T^2 + \frac{g^2}{4\pi^2}\ln\frac{T}{|\vec{k}|}\left\{k_0^2 + \vec{k}^2\right\} + \frac{5}{8}\frac{g^2}{\pi^2}\vec{k}^2$$
- Longitudinal Self-energy:

  $$\delta\Pi_L \approx -\frac{g^2T^2}{\pi^2k^2}\int_{\frac{\varepsilon_0}{T}}^\infty dx\frac{\left(x^2 - \frac{\vec{k}^2}{4T^2}\right)}{\left(\frac{k_0^2}{T^2} - 4x^2\right)}\left\{\left[4\left(2x^2 + \frac{k_0^2}{2T^2}\right)\frac{\vec{k}^2}{k^2} + 2x^2\right]\left[\left(k_0^2 - 2xT|\vec{k}|\right)\frac{f\left(x - \frac{|\vec{k}|}{2T}\right)}{\left(x - \frac{|\vec{k}|}{2T}\right)} + \left(k_0^2 + 2xT|\vec{k}|\right)\frac{f\left(x + \frac{|\vec{k}|}{2T}\right)}{\left(x + \frac{|\vec{k}|}{2T}\right)}\right]\right\}$$

  $$+\frac{g^2T^2}{\pi^2k^2}\int_{\frac{\varepsilon_0}{T}}^\infty dx\frac{\left(x^2 - \frac{\vec{k}^2}{4T^2}\right)}{\left(\frac{k_0^2}{T^2} - 4x^2\right)}\left\{\frac{8xk_0^2|\vec{k}|}{k^2}\left[-\left(2xT - |\vec{k}|\right)f\left(x - \frac{|\vec{k}|}{2T}\right) + \left(2xT + |\vec{k}|\right)f\left(x + \frac{|\vec{k}|}{2T}\right)\right]\right\}$$

  $$-\frac{g^2T^2}{\pi^2k^2}\int_{\frac{\varepsilon_0}{T}}^\infty dx\frac{\left(x^2 - \frac{\vec{k}^2}{4T^2}\right)}{\left(\frac{k_0^2}{T^2} - 4x^2\right)}\left\{\left[\frac{2x|\vec{k}|}{T}\left(\frac{3k_0^2 + \vec{k}^2}{k^2}\right)\right]\left[\left(k_0^2 + 2xT|\vec{k}|\right)\frac{f\left(x + \frac{|\vec{k}|}{2T}\right)}{\left(x + \frac{|\vec{k}|}{2T}\right)} - \left(k_0^2 - 2xT|\vec{k}|\right)\frac{f\left(x - \frac{|\vec{k}|}{2T}\right)}{\left(x - \frac{|\vec{k}|}{2T}\right)}\right]\right\}$$

  $$+\frac{g^2T^2}{\pi^2k^2}\int_{\frac{\varepsilon_0}{T}}^\infty dx\frac{\left(x^2 - \frac{\vec{k}^2}{4T^2}\right)}{\left(\frac{k_0^2}{T^2} - 4x^2\right)}\left\{2k_0^2\left[1 + \frac{2\vec{k}^2}{k^2}\right]\left[\left(2x + \frac{|\vec{k}|}{T}\right)f\left(x + \frac{|\vec{k}|}{2T}\right) + \left(2x - \frac{|\vec{k}|}{T}\right)f\left(x - \frac{|\vec{k}|}{2T}\right)\right]\right\}$$

  $$\approx \frac{g^2T^2\vec{k}^2}{2k_0^2} + \frac{4g^2T^2}{\pi^2}\int_{\frac{\varepsilon_0}{T}}^\infty dx\left\{\frac{x^3\left[f[x] + f''[x]\frac{\vec{k}^2}{8T^2}\right]}{\left(\frac{k_0^2}{T^2} - 4x^2\right)}\right\}$$

$$\delta\Pi^{\mu\nu}=-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_-^{\mu}\tilde{p}_-^{\nu}+\tilde{p}_-^{\mu}k^{\nu}+k^{\mu}\tilde{p}_-^{\nu}-g^{\mu\nu}k\cdot\tilde{p}_-}{2E_-\left(k^0+E_--E_+\right)\left(k^0+E_-+E_+\right)}\left(f\left(\frac{E_--\mu}{T}\right)+f\left(\frac{E_-+\mu}{T}\right)\right)$$
$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_-^{\mu}\bar{p}_-^{\nu}+\bar{p}_-^{\mu}k^{\nu}+k^{\mu}\bar{p}_-^{\nu}-g^{\mu\nu}k\cdot\bar{p}_-}{2E_-\left(k^0-E_--E_+\right)\left(k^0-E_-+E_+\right)}\left(f\left(\frac{E_--\mu}{T}\right)+f\left(\frac{E_-+\mu}{T}\right)\right)$$
$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\tilde{p}_+^{\mu}\tilde{p}_+^{\nu}-\tilde{p}_+^{\mu}k^{\nu}-k^{\mu}\tilde{p}_+^{\nu}+g^{\mu\nu}k\cdot\tilde{p}_+}{2E_+\left(k^0+E_--E_+\right)\left(k^0-E_--E_+\right)}\left(f\left(\frac{E_+-\mu}{T}\right)+f\left(\frac{E_++\mu}{T}\right)\right)$$
$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{2\bar{p}_+^{\mu}\bar{p}_+^{\nu}-\bar{p}_+^{\mu}k^{\nu}-k^{\mu}\bar{p}_+^{\nu}+g^{\mu\nu}k\cdot\bar{p}_+}{\left(2E_+\right)\left(k^0+E_-+E_+\right)\left(k^0-E_-+E_+\right)}\left(f\left(\frac{E_+-\mu}{T}\right)+f\left(\frac{E_++\mu}{T}\right)\right)$$
$$\delta\Pi^{\mu\nu}=-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(k_0^2+E_-^2-E_+^2\right)\left(4\vec{p}^{\mu}\vec{p}^{\nu}-\vec{k}^{\mu}\vec{k}^{\nu}+2g^{\mu\nu}\vec{k}\cdot\vec{p}_-+2\vec{p}_-^{\mu}u^{\nu}k^0+2\vec{p}_-^{\nu}u^{\mu}k^0\right)-4E_-^2\left(2k^0u^{\mu}\vec{p}^{\nu}+2k^0u^{\nu}\vec{p}^{\mu}-u^{\mu}u^{\nu}\left(E_-^2-E_+^2\right)\right)+4k_0^2E_-^2\Delta_u^{\mu\nu}}{2E_-\left(k_0^2-\left(E_--E_+\right)^2\right)\left(k_0^2-\left(E_-+E_+\right)^2\right)}\left(f\left(\frac{E_--\mu}{T}\right)+f\left(\frac{E_-+\mu}{T}\right)\right)$$
$$-2g^2\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\left(k_0^2-E_-^2+E_+^2\right)\left(4\vec{p}^{\mu}\vec{p}^{\nu}-\vec{k}^{\mu}\vec{k}^{\nu}-2k^0\vec{p}_+^{\mu}u^{\nu}-2k^0\vec{p}_+^{\nu}u^{\mu}-2g^{\mu\nu}\vec{k}\cdot\vec{p}_+\right)+4E_+^2\left(2k^0u^{\mu}\vec{p}^{\nu}+2k^0u^{\nu}\vec{p}^{\mu}+u^{\mu}u^{\nu}\left(-E_-^2+E_+^2\right)\right)+4k_0^2E_+^2\Delta_u^{\mu\nu}}{2E_+\left(k_0^2-\left(E_--E_+\right)^2\right)\left(k_0^2-\left(E_-+E_+\right)^2\right)}\left(f\left(\frac{E_+-\mu}{T}\right)+f\left(\frac{E_++\mu}{T}\right)\right)$$

$2\tilde{p}_-^{\mu}\tilde{p}_-^{\nu}+\tilde{p}_-^{\mu}k^{\nu}+k^{\mu}\tilde{p}_-^{\nu}-g^{\mu\nu}k\cdot\tilde{p}_-+2\bar{p}_-^{\mu}\bar{p}_-^{\nu}+\bar{p}_-^{\mu}k^{\nu}+k^{\mu}\bar{p}_-^{\nu}-g^{\mu\nu}k\cdot\bar{p}_-$
$=2\left(E_-u+\vec{p}_-\right)^{\mu}\left(E_-u+\vec{p}_-\right)^{\nu}+\left(E_-u+\vec{p}_-\right)^{\mu}k^{\nu}+k^{\mu}\left(E_-u+\vec{p}_-\right)^{\nu}-g^{\mu\nu}k\cdot\left(E_-u+\vec{p}_-\right)$
$+2\left(-E_-u+\vec{p}_-\right)^{\mu}\left(-E_-u+\vec{p}_-\right)^{\nu}+\left(-E_-u+\vec{p}_-\right)^{\mu}k^{\nu}+k^{\mu}\left(-E_-u+\vec{p}_-\right)^{\nu}-g^{\mu\nu}k\cdot\left(-E_-u+\vec{p}_-\right)$
$=4E_-^2u^{\mu}u^{\nu}+4\vec{p}_-^{\mu}\vec{p}_-^{\nu}+2\vec{p}_-^{\mu}k^{\nu}+2\vec{p}_-^{\nu}k^{\mu}+2g^{\mu\nu}\vec{k}\cdot\vec{p}_-$
$=4E_-^2u^{\mu}u^{\nu}+4\vec{p}^{\mu}\vec{p}^{\nu}-\vec{k}^{\mu}\vec{k}^{\nu}+2\vec{p}_-^{\mu}u^{\nu}k^0+2\vec{p}_-^{\nu}u^{\mu}k^0+2g^{\mu\nu}\vec{k}\cdot\vec{p}_-$

$2\tilde{p}_-^{\mu}\tilde{p}_-^{\nu}+\tilde{p}_-^{\mu}k^{\nu}+k^{\mu}\tilde{p}_-^{\nu}-g^{\mu\nu}k\cdot\tilde{p}_--\left(2\bar{p}_-^{\mu}\bar{p}_-^{\nu}+\bar{p}_-^{\mu}k^{\nu}+k^{\mu}\bar{p}_-^{\nu}-g^{\mu\nu}k\cdot\bar{p}_-\right)$
$=2\left(E_-u+\vec{p}_-\right)^{\mu}\left(E_-u+\vec{p}_-\right)^{\nu}+\left(E_-u+\vec{p}_-\right)^{\mu}k^{\nu}+k^{\mu}\left(E_-u+\vec{p}_-\right)^{\nu}-g^{\mu\nu}k\cdot\left(E_-u+\vec{p}_-\right)$
$-2\left(-E_-u+\vec{p}_-\right)^{\mu}\left(-E_-u+\vec{p}_-\right)^{\nu}-\left(-E_-u+\vec{p}_-\right)^{\mu}k^{\nu}-k^{\mu}\left(-E_-u+\vec{p}_-\right)^{\nu}+g^{\mu\nu}k\cdot\left(-E_-u+\vec{p}_-\right)$
$=4E_-u^{\mu}\left(\vec{p}^{\nu}+\frac{k^0}{2}u^{\nu}\right)+4E_-u^{\nu}\left(\vec{p}^{\mu}+\frac{k^0}{2}u^{\mu}\right)-2g^{\mu\nu}E_-k^0$
$=4E_-u^{\mu}\vec{p}^{\nu}+4E_-u^{\nu}\vec{p}^{\mu}-2\left(g^{\mu\nu}-2u^{\mu}u^{\nu}\right)E_-k^0$

$\left(k_0+\Delta\right)\left(k_0+E\right)=k_0^2+E_-^2-E_+^2+2k_0E_-$
$\left(k_0-\Delta\right)\left(k_0-E\right)=k_0^2+E_-^2-E_+^2-2k_0E_-$

$2\tilde{p}_+^{\mu}\tilde{p}_+^{\nu}-\tilde{p}_+^{\mu}k^{\nu}-k^{\mu}\tilde{p}_+^{\nu}+g^{\mu\nu}k\cdot\tilde{p}_++2\bar{p}_+^{\mu}\bar{p}_+^{\nu}-\bar{p}_+^{\mu}k^{\nu}-k^{\mu}\bar{p}_+^{\nu}+g^{\mu\nu}k\cdot\bar{p}_+$
$=2\left(E_+u+\vec{p}_+\right)^{\mu}\left(E_+u+\vec{p}_+\right)^{\nu}-\left(E_+u+\vec{p}_+\right)^{\mu}k^{\nu}-k^{\mu}\left(E_+u+\vec{p}_+\right)^{\nu}+g^{\mu\nu}k\cdot\left(E_+u+\vec{p}_+\right)$
$+2\left(-E_+u+\vec{p}_+\right)^{\mu}\left(-E_+u+\vec{p}_+\right)^{\nu}-\left(-E_+u+\vec{p}_+\right)^{\mu}k^{\nu}-k^{\mu}\left(-E_+u+\vec{p}_+\right)^{\nu}+g^{\mu\nu}k\cdot\left(-E_+u+\vec{p}_+\right)$
$=4E_+^2u^{\mu}u^{\nu}+4\vec{p}_+^{\mu}\vec{p}_+^{\nu}-2\vec{p}_+^{\mu}k^{\nu}-2\vec{p}_+^{\nu}k^{\mu}-2g^{\mu\nu}\vec{k}\cdot\vec{p}_+$
$=4E_+^2u^{\mu}u^{\nu}+4\vec{p}^{\mu}\vec{p}^{\nu}-\vec{k}^{\mu}\vec{k}^{\nu}-2k^0\vec{p}_+^{\mu}u^{\nu}-2k^0\vec{p}_+^{\nu}u^{\mu}-2g^{\mu\nu}\vec{k}\cdot\vec{p}_+$

$2\tilde{p}_+^{\mu}\tilde{p}_+^{\nu}-\tilde{p}_+^{\mu}k^{\nu}-k^{\mu}\tilde{p}_+^{\nu}+g^{\mu\nu}k\cdot\tilde{p}_+-\left(2\bar{p}_+^{\mu}\bar{p}_+^{\nu}-\bar{p}_+^{\mu}k^{\nu}-k^{\mu}\bar{p}_+^{\nu}+g^{\mu\nu}k\cdot\bar{p}_+\right)$
$=2\left(E_+u+\vec{p}_+\right)^{\mu}\left(E_+u+\vec{p}_+\right)^{\nu}-\left(E_+u+\vec{p}_+\right)^{\mu}k^{\nu}-k^{\mu}\left(E_+u+\vec{p}_+\right)^{\nu}+g^{\mu\nu}k\cdot\left(E_+u+\vec{p}_+\right)$
$-2\left(-E_+u+\vec{p}_+\right)^{\mu}\left(-E_+u+\vec{p}_+\right)^{\nu}+\left(-E_+u+\vec{p}_+\right)^{\mu}k^{\nu}+k^{\mu}\left(-E_+u+\vec{p}_+\right)^{\nu}-g^{\mu\nu}k\cdot\left(-E_+u+\vec{p}_+\right)$
$=4E_+u^{\mu}\vec{p}_+^{\nu}+4E_+u^{\nu}\vec{p}_+^{\mu}-2E_+u^{\mu}k^{\nu}-2E_+u^{\nu}k^{\mu}+2g^{\mu\nu}E_+k\cdot u$
$=4E_+u^{\mu}\vec{p}^{\nu}+4E_+u^{\nu}\vec{p}^{\mu}+2g^{\mu\nu}E_+k^0-4E_+k^0u^{\mu}u^{\nu}$

$\left(k_0+\Delta\right)\left(k_0-E\right)=k_0^2-E_-^2+E_+^2-2k_0E_+$
$\left(k_0-\Delta\right)\left(k_0+E\right)=k_0^2-E_-^2+E_+^2+2k_0E_+$

$\left(k_0^2-E_-^2+E_+^2\right)\left(4\vec{p}^{\mu}\vec{p}^{\nu}-\vec{k}^{\mu}\vec{k}^{\nu}-2k^0\vec{p}_+^{\mu}u^{\nu}-2k^0\vec{p}_+^{\nu}u^{\mu}-2g^{\mu\nu}\vec{k}\cdot\vec{p}_+\right)+4E_+^2\left(2k^0u^{\mu}\vec{p}^{\nu}+2k^0u^{\nu}\vec{p}^{\mu}+u^{\mu}u^{\nu}\left(-E_-^2+E_+^2\right)\right)+4k_0^2E_+^2\Delta_u^{\mu\nu}$