ZAD.1. Rozwiąż równania różniczkowe

(a) (4p) $(1+x^2)y' - x^2 y = e^x$

(b) (4p) $y'' - 4y' + 4y = (2x+4)e^{2x}$

a) $(1+x^2)y' - x^2 y = e^x$

$v(x) = e^{\int \frac{-x^2}{1+x^2}dx} = e^{\operatorname{arctg} x - x}$

$y' - \frac{x^2 y}{1+x^2} = \frac{e^x}{1+x^2}$

$\int \frac{-x^2}{1+x^2}dx = \operatorname{arctg} x - x$

$(y \cdot e^{\operatorname{arctg} x - x})' = \frac{e^{\operatorname{arctg} x}}{1+x^2}$

$\int \frac{e^{\operatorname{arctg} x}}{1+x^2}dx = \left\{ \begin{array}{l} u = \operatorname{arctg} x \\ du = \frac{1}{1+x^2}dx \end{array} \right\} = \int e^u du = e^u + C = e^{\operatorname{arctg} x} + C$

$y \cdot e^{\operatorname{arctg} x - x} = e^{\operatorname{arctg} x} + C_1$

$y = e^x + \frac{C_1}{e^{\operatorname{arctg} x - x}} = e^x + C_2$

b) $y'' - 4y' + 4y = (2x+4)e^{2x}$

$\delta = 2$

$1°$ $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r_1 = r_2 = 2$

$2°$ $\varphi = (Ax+B)x^2 e^{2x} = (Ax^3 + Bx^2)e^{2x}$

$\varphi' = (3Ax^2 + 2Bx)e^{2x} + 2e^{2x}(Ax^3 + Bx^2)$

$\varphi'' = (6Ax + 2B)e^{2x} + 2e^{2x}(3Ax^2 + 2Bx) + 4e^{2x}(Ax^3 + Bx^2) + 2e^{2x}(3Ax^2 + 2Bx)$

$y_0 = C_1 e^{2x} + C_2 x e^{2x}$

$2°$ cd. można obustronnie skrócić $e^{2x}$

$L = (6Ax + 2B) + \cancel{2(3Ax^2 + 2Bx)} + \cancel{4(Ax^3 + Bx^2)} + \cancel{2(3Ax^2 + 2Bx)} +$
$- \cancel{4(3Ax^2 + 2Bx)} - \cancel{8(Ax^3 + Bx^2)} + \cancel{4(Ax^3 + Bx^2)}$

$P = 2x + 4$

$6A = 2$, $A = \frac{1}{3}$

$2B = 4$, $B = 2$

$\Rightarrow y_s = \left(\frac{1}{3}x^3 + 2x^2\right)e^{2x}$

$y = C_1 e^{2x} + C_2 x e^{2x} + \left(\frac{1}{3}x^3 + 2x^2\right)e^{2x}$

ZAD.2. (a)*(1p)* Podaj warunek konieczny istnienia pochodnej funkcji $f(x)$ w $x = x_0$.

(b)*(2p)* Korzystając z definicji, oblicz pochodną funkcji $f(x) = \cos x$.

(c)*(6p)* Znajdź asymptoty oraz punkty przegięcia i zbadaj wklęsłość/wypukłość wykresu funkcji

$$f(x) = x \cdot \operatorname{arc\,ctg}(x^3)$$

a) jeżeli funkcja jest różniczkowalna w $x_0$ to jest ciągła w otoczeniu tego punktu

b) $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} =$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot \sin\left(\frac{h}{2}\right)}{h} = \lim_{h\to 0} \frac{\sin\left(x+\frac{h}{2}\right)\cdot \sin\frac{h}{2}}{-\frac{h}{2}} = -\sin x$$

c) $f(x) = x \cdot \operatorname{arccot}(x^3)$

$$f'(x) = \operatorname{arccot} x^3 + 3x^2 \cdot \frac{-1}{1+(x^3)^2} \cdot x = \operatorname{arccot} x^3 - \frac{3x^3}{1+x^6}$$

$$f''(x) = 3x^2 \cdot \left(-\frac{1}{1+(x^3)^2}\right) - \frac{9x^2\cdot(1+x^6) - 6x^5 \cdot 3x^3}{(1+x^6)^2} =$$

$$= \frac{-3x^2\cdot(1+x^6) - 9x^2\cdot(1+x^6) + 18x^8}{(1+x^6)^2} = \frac{-12x^2(1+x^6)+18x^8}{(1+x^6)^2} =$$

$$= \frac{-12x^2 + 6x^8}{(1+x^6)^2} = \frac{-6x^2(2-x^6)}{(1+x^6)^2}$$

$D = \mathbb{R} \Rightarrow$ brak asymptot pionowych

$$a_1 = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \operatorname{arccot} x^3 = 0$$

$$b_1 = \lim_{x\to\infty} (f(x) - ax) = \lim_{x\to\infty} x \operatorname{arccot} x^3 = \lim_{x\to\infty} \frac{\operatorname{arccot} x^3}{\frac{1}{x}} \overset{H}{=}$$

$$= \lim_{x\to\infty} \frac{3x^2 \cdot \frac{-1}{1+x^6}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3x^4}{x^6+1} = 0$$

asymptota pozioma prawostronna $y = 0$

$$a_2 = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \operatorname{arccot} x^3 = \pi$$

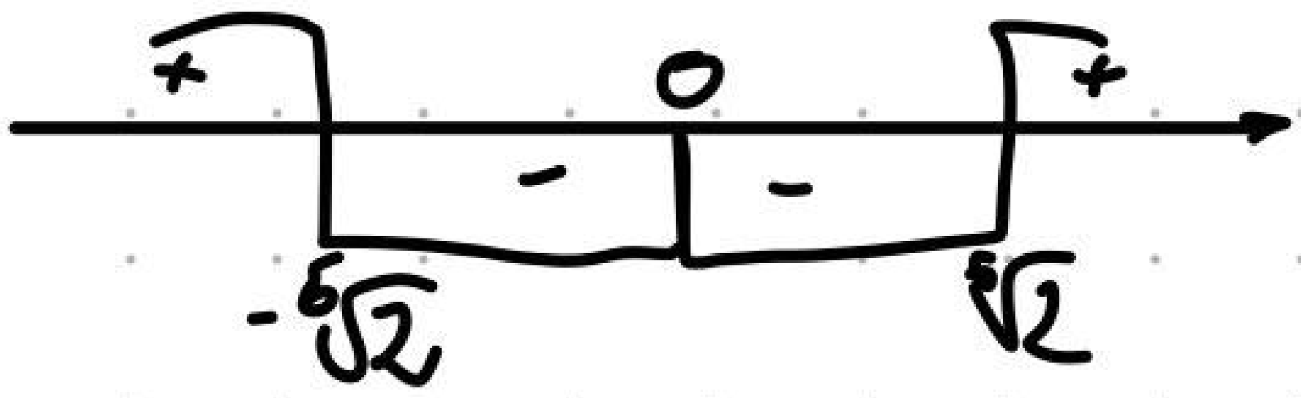
$$b = \lim_{x \to -\infty} (f(x) - ax) = \lim_{x \to -\infty} x(\operatorname{arccot} x^3 - \pi) =$$

$$= \lim_{x \to -\infty} \frac{\operatorname{arccot} x^3 - \pi}{\frac{1}{x}} \overset{H}{=} \lim_{x \to -\infty} \frac{3x^2 \cdot \frac{-1}{1+x^6}}{-\frac{1}{x^2}} = \lim_{x \to -\infty} \frac{3x^4}{1+x^6} = 0$$

asymptota lewostr. ukośna: $y = \pi x$

$$f''(x) = \frac{-6x^2(2 - x^6)}{(1+x^6)^2}$$

$f''(x) = 0 \iff x \in \{0, \sqrt[6]{2}, -\sqrt[6]{2}\}$




$2 = x^6$

$x = \sqrt[6]{2} \lor x = -\sqrt[6]{2}$

f. jest wyp dla $x \in (-\infty, -\sqrt[6]{2})$ oraz $x \in (\sqrt[6]{2}, \infty)$

f. jest wklęsta dla $x \in (-\sqrt[6]{2}, 0)$ i $x \in (0, \sqrt[6]{2})$

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$$\begin{array}{l} \quad\ x+2 \\ \hline (x^4 - x^3 - 2x^2 + 8x + 16) : (x^3 - 3x^2 + 4x) = x+2 \text{ r. } 16 \\ -(x^4 - 3x^3 + 4x^2) \\ \hline 0 + 2x^3 - 6x^2 + 8x + 16 \\ -\ (2x^3 - 6x^2 + 8x) \\ \hline \qquad\qquad 16 \end{array}$$

(a) (3p) $\int_0^{\pi/2} \frac{1}{3+2\cos x}\,dx$

(b) (5p) $\int \frac{x^4-x^3-2x^2+8x+16}{x^3-3x^2+4x}\,dx$

a) $\int_0^{\frac{\pi}{2}} \frac{1}{3+2\cos x}\,dx = \left\{ \begin{array}{l} tg\frac{x}{2} = u \qquad \cos x = \frac{1-u^2}{1+u^2} \\ dx = \frac{2}{1+u^2}\,du \end{array} \right\} =$

$= \int_0^{tg\frac{\pi}{4}} \frac{2}{1+u^2} \cdot \frac{1}{3+2\frac{1-u^2}{1+u^2}}\,du = \int_0^{tg\frac{\pi}{4}} \frac{2}{1+u^2} \cdot \frac{1}{\frac{2-2u^2+3+3u^2}{1+u^2}}\,du = \int_0^{tg\frac{\pi}{4}} \frac{2\cdot\cancel{(1+u^2)}}{\cancel{(1+u^2)}(5+u^2)}\,du =$

$= 2\int_0^{tg\frac{\pi}{4}} \frac{1}{u^2+5} = \left[2\cdot\frac{1}{\sqrt5}\,arctg\frac{u}{\sqrt5}\right]_0^{tg\frac{\pi}{4}} = \left[\frac{2}{\sqrt5}\,arctg\frac{tg\frac{x}{2}}{\sqrt5}\right]_0^{\frac{\pi}{2}} = \frac{2}{\sqrt5}\cdot arctg\frac{1}{\sqrt5}$

b) $\int \frac{x^4-x^3-2x^2+8x+16}{x^3-3x^2+4x}\,dx = \int x+2+\frac{16}{x(x^2-3x+4)}\,dx =$

$= \frac{x^2}{2} + 2x + 4\ln|x| + \frac{12}{\sqrt7}\cdot arctg\frac{2x-3}{\sqrt7} - 2\ln|x^2-3x+4| + C$

$\int x+2\,dx = \frac{x^2}{2} + 2x + C$

$\int \frac{16}{x(x^2-3x+4)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{(x^2-3x+4)}\,dx = \int \frac{4}{x} + \frac{-4x+12}{x^2-3x+4}\,dx$

$16 = A(x^2-3x+4) + (Bx+C)\cdot x$

$x=0: \quad 16 = 4A \Rightarrow A = 4$

$x^2: \quad 0 = A+B = B = -4$

$x=1: \quad 16 = 8 + C - 4 \Rightarrow C = 12$

$x^2 - 3x + \frac{9}{4}$

$-3 = 2b \qquad 4 - \frac{9}{4} = \frac{7}{4}$

$b = -\frac{3}{2}$

$(x-\frac{3}{2})$

$\int \frac{4}{x}\,dx = 4\ln|x| + C$

$-2\int \frac{2x-6}{x^2-3x+4}\,dx = -2\int \frac{2x-3}{x^2-3x+4}\,dx - 2\int \frac{-3}{x^2-3x+4}\,dx = -2\ln|x^2-3x+4| +$

$+ 6\cdot\int \frac{1}{x^2-3x+4}\,dx = -2\ln|x^2-3x+4| + 6\cdot\frac{2}{\sqrt7}\,arctg\frac{2x-3}{\sqrt7} + C$

$\hookrightarrow \int \frac{1}{(x-\frac{3}{2})^2+\frac{7}{4}}$

ZAD.4. *(5p)* Oblicz objętość bryły powstałej z obrotu wokół osi $OX$ obszaru ograniczonego przez $y = 0$ i $y = \frac{\sqrt{\ln x}}{x+1}$ dla $x \geq 1$

$V = \int \pi \cdot f^2(x)$

$$\int_1^{\infty} \pi \frac{\ln x}{(x+1)^2} dx = \pi \int_1^{\infty} \frac{\ln x}{(x+1)^2} dx = \pi \lim_{t \to \infty} \int_1^t \frac{\ln x}{x^2+2x+1} dx \ominus$$

$$\int \frac{\ln x}{(x+1)^2} dx = \left\{ \begin{array}{ll} u = \ln x & v' = \frac{1}{(x+1)^2} \\ u' = \frac{1}{x} & v = -\frac{1}{x+1} \end{array} \right\} = -\frac{\ln x}{x+1} + \int \frac{1}{x(1+x)} dx = -\frac{\ln x}{x+1} + \ln|x| - \ln|x+1| + C$$

$$\int \frac{1}{x(1+x)} dx = \int \frac{A}{x} + \frac{B}{1+x} dx = \ln|x| - \ln|x+1| + C$$

$1 = A(1+x) + Bx$

$x=0: \ 1 = A$

$x=1: \ 1 = 2 + B \Rightarrow B = -1$

$$\ominus \pi \lim_{t \to \infty} \left[ -\frac{\ln x}{x+1} + \ln|x| - \ln|x+1| \right]_1^t = \pi \lim_{t \to \infty} \left[ -\frac{\ln x}{x+1} + \ln \frac{x}{x+1} \right]_1^t =$$

$$\pi \left( 0 + 0 + 0 - \ln \frac{1}{2} \right) = -\pi \ln\left(\frac{1}{2}\right) = \pi \ln 2$$

ZAD.5. (a) *(1p)* Podaj warunek konieczny zbieżności nieskończonego szeregu liczbowego

(b) *(3p)* Znajdź przedział zbieżności szeregu

$$\sum_{n=4}^{\infty} \frac{\sqrt[3]{n^2-3}\,(x-2)^n}{n^2+3}$$

(c) *(4p)* Oblicz sumę szeregu

$$\sum_{n=2}^{\infty} \frac{(n+2)(-1)^n}{3^n}$$

a) jeżeli szereg $\sum_{n=0}^{\infty} a_n$ jest zbieżny, to $\lim_{n\to\infty} a_n = 0$ !

b) $\sum_{n=4}^{\infty} \frac{\sqrt[3]{n^2-3}\,(x-2)^n}{n^2+3}$

$$\underbrace{(n^2-3)^{\frac{1}{3n}\to 0}}_{1}$$

$$-1 < x-2 < 1$$
$$1 < x < 3$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\sqrt[3]{n^2-3}\,(x-2)^n}{n^2+3}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{\sqrt[3]{n^2-3}}\;|(x-2)|}{\underbrace{\sqrt[n]{n}}_{1}\cdot\underbrace{\sqrt[n]{n+\frac{3}{n}}}_{1}} = |x-2|$$

$$x=3: \sum_{n=4}^{\infty} \frac{\sqrt[3]{n^2-3}\cdot 1^n}{n^2+3} = \frac{\sqrt[3]{n^2-3}}{n^2+3} = a_n$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(n^2-3)^{\frac{1}{3}}}{n^2+3}}{\frac{1}{n^{\frac{4}{3}}}} = \lim_{n\to\infty} n^{\frac{4}{3}} \cdot \frac{\sqrt[3]{n^2-3}}{n^2+3} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{\frac{4}{3}}}\cdot n^{\frac{2}{3}}\sqrt[3]{1-\frac{3}{n^2}}}{\cancel{n^2}+3} = \lim_{n\to\infty} \frac{\sqrt[3]{1-\frac{3}{n^2}}}{1+\frac{3}{n^2}} = 1 \Rightarrow \text{szeregi są jednocześnie zbieżne } (0<L<\infty)$$

($b_n$ to sz. Dirichleta $p=\frac{4}{3}$)

$$x=1: \sum_{n=4}^{\infty} \frac{\sqrt[3]{n^2-3}\,(-1)^n}{n^2+3}$$

$$a_n = \left|\frac{\sqrt[3]{n^2-3}\,(-1)^n}{n^2+3}\right| = \frac{\sqrt[3]{n^2-3}}{n^2+3} \Rightarrow \text{bezwzględnie zbieżny jw}$$

$$\underline{\underline{x \in \langle 1, 3 \rangle}}$$

(c) *(4p)* Oblicz sumę szeregu

$$\sum_{n=2}^{\infty} \frac{(n+2)(-1)^n}{3^n}$$

c) $\sum_{n=2}^{\infty} \frac{(n+2)(-1)^n}{3^n} = \sum_{n=2}^{\infty} (n+2)\left(-\frac{1}{3}\right)^n$ $\qquad t = -\frac{1}{3}$

$\sum_{n=0}^{\infty} (n+2)\cdot t^n = \sum_{n=0}^{\infty} (n+2)\, t^{n+1} \cdot \frac{1}{t} = \frac{1}{t}\sum_{n=0}^{\infty} (n+2)\, t^{n+1} = \frac{1}{t}\sum_{n=0}^{\infty} \left(t^{n+2}\right)' \overset{|t|<1}{=}$

$\frac{1}{t}\left(\sum_{n=0}^{\infty} t^{n+2}\right)' = \frac{1}{t}\left(\frac{t^2}{1-t}\right)' = \frac{1}{\cancel{t}}\left(\frac{2\cancel{t}(1-t)+t^{\cancel{2}}}{(1-t)^2}\right) = \frac{2-t}{(1-t)^2}$

$\sum_{n=2}^{\infty} (n+2)\left(-\frac{1}{3}\right)^n = \frac{2+\frac{1}{3}}{\left(\frac{4}{3}\right)^2} - 2 + \frac{1}{3}\cdot 3 = \frac{\frac{21}{9}}{\frac{16}{9}} - 1 = \frac{21}{16} - 1 = \underline{\underline{\frac{5}{16}}}$