Kartka 1

Zad.1. Rozwiąż równania różniczkowe

(a) *(4p)* $(1+x^2)y' - x^2y = e^x$

(b) *(4p)* $y'' - 4y' + 4y = (2x+4)e^{2x}$

Kartka 2

Zad.2. (a)*(1p)* Podaj warunek konieczny istnienia pochodnej funkcji $f(x)$ w $x = x_0$.

(b)*(2p)* Korzystając z definicji, oblicz pochodną funkcji $f(x) = \cos x$.

(c)*(6p)* Znajdź asymptoty oraz punkty przegięcia i zbadaj wklęsłość/wypukłość wykresu funkcji

$$f(x) = x \cdot \operatorname{arc\,ctg}(x^3)$$

Kartka 3

Zad.3. Oblicz całki

(a) *(3p)* $\displaystyle\int_0^{\pi/2} \frac{1}{3+2\cos x}\,dx$

(b) *(5p)* $\displaystyle\int \frac{x^4 - x^3 - 2x^2 + 8x + 16}{x^3 - 3x^2 + 4x}\,dx$

Kartka 4

Zad.4. *(5p)* Oblicz objętość bryły powstałej z obrotu wokół osi OX obszaru ograniczonego przez $y = 0$ i $y = \dfrac{\sqrt{\ln x}}{x+1}$ dla $x \geq 1$

Zad.5. (a) *(1p)* Podaj warunek konieczny zbieżności nieskończonego szeregu liczbowego

(b) *(3p)* Znajdź przedział zbieżności szeregu

$$\sum_{n=4}^{\infty} \frac{\sqrt[3]{n^2-3}\,(x-2)^n}{n^2+3}$$

(c) *(4p)* Oblicz sumę szeregu

$$\sum_{n=2}^{\infty} \frac{(n+2)(-1)^n}{3^n}$$

Zad.6. *(+2p; sprawdzane po zaliczeniu egzaminu)* Dana jest funkcja $w = f(x(u,s), y(u,t), z(s,t))$. Podaj wzór na pochodną $\dfrac{\partial f}{\partial s}$

1.

a) $(1+x^2)y' - x^2 y = e^x$

$$y' - \frac{x^2}{x^2+1}\, y = \frac{e^x}{x^2+1}$$

$$u = e^{-\int \frac{x^2}{x^2+1}} = e^{-\int 1 - \frac{1}{x^2+1}\, dx} = e^{-x+\operatorname{arctg} x}$$

$$\left(y \cdot e^{-x+\operatorname{arctg} x}\right)' = \frac{e^{\operatorname{arctg} x}}{x^2+1}$$

$$y \cdot e^{-x+\operatorname{arctg} x} = \int \frac{e^{\operatorname{arctg} x}}{x^2+1}\, dx = e^{\operatorname{arctg} x} + C \qquad \Big|\cdot e^{x-\operatorname{arctg} x}$$

$$y = e^x + C \cdot e^{x-\operatorname{arctg} x}$$

b) $y'' - 4y' + 4y = (2x+4)e^{2x}$

1 $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2 \quad \Rightarrow \quad y_1 = e^{2x}$

$y_2 = x e^{2x}$

2 $\varphi = (Ax^3 + Bx^2)e^{2x}$

$$\varphi' = (Ax^3 + Bx^2) \cdot 2 \cdot e^{2x} + (3Ax^2 + 2Bx) \cdot e^{2x}$$

$(3Ax^2+2Bx) \cdot 4 \cdot e^{2x}$

$$\varphi'' = (Ax^3 + Bx^2) \cdot 4e^{2x} + (3Ax^2 + 2Bx) \cdot 2e^{2x} + (3Ax^2 + 2Bx)2e^{2x} + (6Ax + 2B) \cdot e^{2x}$$

$$e^{2x}\left(\cancel{4(Ax^3+Bx^2)} \cancel{+4(3Ax^2+2Bx)} + 6Ax + 2B \cancel{-8(Ax^3+Bx^2)} \cancel{-4(3Ax^2+2Bx)} \cancel{+4(Ax^3+Bx^2)}\right) = (2x+4) \cdot e^{2x}$$

$6A = 2$

$2B = 4$

$$\begin{cases} A = \frac{1}{3} \\ B = 2 \end{cases}$$

$$\varphi = \left(\frac{1}{3}x^3 + 2x^2\right) \cdot e^{2x}$$

2.

b) $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{-2\cdot\sin\left(\frac{x+h-x}{2}\right)\cdot\sin\left(\frac{x+h+x}{2}\right)}{h} = \lim_{h\to 0} \frac{-2\cdot\sin\left(x+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h} =$$

$$= \lim_{h\to 0} - \frac{\overset{\to 1}{\sin\left(\frac{h}{2}\right)}\cdot\sin\left(x+\overset{\to 0}{\frac{h}{2}}\right)}{\frac{h}{2}} = -\sin x$$

c) $f(x) = x\cdot \operatorname{arctg}(x^3)$

$$f'(x) = \operatorname{arctg}(x^3) + x\cdot 3x^2\cdot\frac{1}{x^6+1} = \operatorname{arctg} x^3 + \frac{x^3}{x^6+1}$$

$$f''(x) = 3x^2\cdot\frac{1}{x^6+1} + \frac{3x^2(x^6+1) - 6x^5\cdot x^3}{(x^6+1)^2}$$

3.

a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3+2\cos x}\,dx$

$$\int \frac{1}{3+2\cos x}\,dx = \left| \begin{array}{l} u = \operatorname{tg}\frac{x}{2} \Rightarrow \frac{x}{2} = \operatorname{arctg} u \Rightarrow dx = \frac{2}{u^2+1}\,du \\ \Rightarrow \cos x = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{u^2+1} \end{array} \right| = \int \frac{1}{3+\frac{2-2u^2}{u^2+1}}\cdot\frac{2}{u^2+1}\,du =$$

$$= \int \frac{\cancel{u^2+1}}{3u^2+3+2-2u^2}\cdot\frac{2}{\cancel{u^2+1}}\,du = 2\int\frac{1}{u^2+5}\,du = \frac{2}{\sqrt{5}}\operatorname{arctg}\left(\frac{\operatorname{tg}\left(\frac{x}{2}\right)}{\sqrt{5}}\right) + C$$

b) jest na dole

4.

$$V = \int_1^{\infty} \pi\cdot\left(\frac{\sqrt{\ln x}}{x+1}\right)^2 dx$$

$$\int \frac{\ln x}{(x+1)^2}\,dx = \left| \begin{array}{ll} u = \ln x & v' = (x+1)^{-2} \\ u' = \frac{1}{x} & v = -1(x+1)^{-1} \end{array} \right| = \frac{-\ln x}{x+1} + \int \frac{1}{x(x+1)}\,dx =$$

$$= \frac{-\ln x}{x+1} + \int \frac{1}{x} + \frac{-1}{x+1}\,dx = -\frac{\ln x}{x+1} + \ln|x| - \ln|x+1| + C$$

$$V = \left[\pi\cdot\left(-\frac{\ln x}{x+1} + \ln x - \ln(x+1)\right)\right]_1^{\infty} = \lim_{T\to\infty}\left[-\pi\frac{\ln x}{x+1} + \pi\ln\left(\frac{x}{x+1}\right)\right]_1^T =$$

$$\lim_{T\to\infty}\left[-\pi\frac{\ln T}{T+1} + \pi\ln\frac{T}{T+1} + \pi\frac{\ln 1}{2} - \pi\ln\left(\frac{1}{2}\right)\right] = \pi\left[\lim_{T\to\infty}\left(-\frac{\ln T}{T+1}\right) + \lim_{T\to\infty}\left(\ln\left(\frac{T}{T+1}\right)\right) - \lim_{T\to\infty}\left(\ln\frac{1}{2}\right)\right]$$

$=0$

gówno, $\frac{\infty}{\infty}$ $\quad$ $\ln(1)$ $\quad$ $\ln\frac{1}{2}$

$$\lim_{T\to\infty}\left(-\frac{\ln T}{T+1}\right) \overset{L'H}{=} \lim_{T\to\infty}\left(-\frac{\frac{1}{T}}{1+0}\right) = \lim_{T\to\infty}\left(-\frac{1}{T}\right) = 0$$

$$\pi\left[\lim_{T\to\infty}\left(-\frac{\ln T}{T+1}\right)+\lim_{T\to\infty}\left(\ln\left(\frac{T}{T+1}\right)\right)-\lim_{T\to\infty}\left(\ln\frac{1}{2}\right)\right]=\pi\cdot\left[0+\ln 1-\ln\frac{1}{2}\right]=\pi\cdot\ln 2$$

5.

b) $\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}\,(x-2)^n}{n^2+3}$

$$\lim_{n\to\infty}\left|\frac{\sqrt[3]{n^2+2n-2}\,(x-2)^n(x-2)}{n^2+2n+4}\cdot\frac{n^2+1}{(x-2)^n\sqrt[3]{n^2+3}}\right|=\lim_{n\to\infty}\left|\sqrt[3]{\frac{n^2+2n-2}{n^2+2n+4}}\cdot\frac{n^2+1}{n^2+2n+2}\cdot(x-2)\right|=$$

$$=\left|\sqrt[3]{1}\cdot 1\cdot(x-2)\right|=|x-2|$$

$|x-2|<1$

$-1<x-2<1$

$1<x<3$

dla $x=3$: $\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}\cdot 1^n}{n^2+3}=\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}}{n^2+3}$

$a_n=\frac{n^{\frac{2}{3}}}{n^2}=\frac{1}{n^{\frac{4}{3}}}$ ← sz. Dirichleta, $p=\frac{4}{3}$, zb.

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2-3}}{n^2+3}\cdot\frac{n^2}{n^{\frac{2}{3}}}=\lim_{n\to\infty}\frac{n^2}{n^2+3}\cdot\frac{\sqrt[3]{n^2-3}}{n^{\frac{2}{3}}}=\lim_{n\to\infty}\left(\frac{1}{1+\frac{3}{n^2}}\right)\cdot\frac{\cancel{n^{\frac{2}{3}}}\cdot\sqrt[3]{1-\frac{3}{n^2}}}{\cancel{n^{\frac{2}{3}}}\cdot 1}=\left[\frac{1}{1}\cdot\sqrt[3]{1}\right]=1\in(0,\infty)$$

↓

badany szereg też jest zbieżny

dla $x=1$: $\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}\cdot(-1)^n}{n^2+3}$

$B=\sum_{n=4}^{\infty}\left|\frac{\sqrt[3]{n^2-3}\cdot(-1)^n}{n^2+3}\right|=\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}}{n^2+3}$ ← szereg zbieżny, udowodnione tu

czyli $\sum_{n=4}^{\infty}\frac{\sqrt[3]{n^2-3}\cdot(-1)^n}{n^2+3}$ jest zbieżny bezwzględnie $\Rightarrow$ $x\in\langle 1,3\rangle$

6. XD

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b) $\int \frac{x^4-x^3-2x^2+8x+16}{x^3-3x^2+4x}dx = \int \frac{x^4-3x^3+4x^2+2x^3-6x^2+8x+16}{x^3-3x^2+4x}dx =$

$= \int x+2+\frac{16}{x(x^2-3x+4)} =$

$\frac{16}{x(x^2-3x+4)} = \frac{A}{x} + \frac{Bx+C}{x^2-3x+4} = \frac{4(x^2-3x+4)+x(-4x+12)}{x(x^2-3x+4)} = \frac{\cancel{4x^2}-12x+16-\cancel{4x^2}+12}{\cancel{\quad}}$

$A \leftarrow 4 \qquad -4\cdot(1x-3)$

$= \frac{1}{2}x^2+2x+\int \frac{4}{x} - \frac{4x-12}{x^2-3x+4}dx = \frac{1}{2}x^2+2x+4\ln|x| - \int 2\cdot\frac{2x-3}{x^2-3x+4} - 6\cdot\frac{1}{x^2-3x+4}d$

$= \frac{1}{2}x^2+2x+4\ln|x|-2\ln|x^2-3x+4| + 6\int\frac{1}{x^2-3x+4}dx =$

$= \frac{1}{2}x^2+2x+4\ln|x|-2\ln|x^2-3x+4| + \frac{12}{\sqrt{7}}\operatorname{arctg}\left(\frac{2}{\sqrt{7}}x-\frac{3}{\sqrt{7}}\right)+C$

$\int \frac{1}{x^2-3x+4}dx = \int \frac{1}{(x-\frac{3}{2})^2+\frac{7}{4}}dx = \frac{2}{\sqrt{7}}\operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(x-\frac{3}{2}\right)\right)+C$