20MT30017
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Third Subtask

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For Gaussian Discriminant Analysis (GDA), we have to show that

$$p(y=1 \mid x; \phi, \mu_0, \mu_1, \Sigma) = \frac{1}{1+e^{-(\theta^T x + \theta_0)}}$$

where $\theta \in \mathbb{R}^n$ and $\theta_0 \in \mathbb{R}$ are some appropriate functions of $\phi, \mu_0, \mu_1 and \Sigma$.
Given,

$$p(y) = \begin{cases} \phi & \text{if } y = 1 \\ 1-\phi & \text{if } y = 0 \end{cases}$$
$$p(x \mid y=0) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1} (x-\mu_0)\right)$$
$$p(x \mid y=1) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1} (x-\mu_1)\right)$$

Using the following abbreviations for the expressions

$$\alpha \equiv -\frac{1}{2}(x-\mu_1)^T \Sigma^{-1} (x-\mu_1)$$
$$\beta \equiv -\frac{1}{2}(x-\mu_0)^T \Sigma^{-1} (x-\mu_0)$$
$$\gamma \equiv \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$

The joint probability distribution equation boils down to,

$$p(x \mid y=1) = \gamma e^{\alpha}$$
$$p(x \mid y=0) = \gamma e^{\beta}$$

Bayes formula gives

$$\begin{aligned} p(y=1 \mid x) &= \frac{p(x \mid y=1)p(y=1)}{p(x \mid y=1)p(y=1) + p(x \mid y=0)p(y=0)} \\ &= \frac{\gamma e^{\alpha}\phi}{\gamma e^{\alpha}\phi + \gamma e^{\beta}(1-\phi)} \\ &= \frac{e^{\alpha}\phi}{e^{\alpha}\phi + e^{\beta}(1-\phi)} \\ &= \frac{1}{1+\frac{1-\phi}{\phi}e^{\beta-\alpha}} \\ &= \frac{1}{1+e^{-1(\alpha-\beta+\ln\phi-\ln(1-\phi))}} \end{aligned}$$

Therefore,

$$p(y=1 \mid x) = \frac{1}{1+e^{-(\alpha-\beta+\ln\phi-\ln(1-\phi))}} \tag{1}$$

Note that,

$$
\begin{aligned}
\alpha-\beta &= -\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)+\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\\
&= -\frac{1}{2}\big[x^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu_1 - \mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \big(x^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0\big)\big]\\
&= -\frac{1}{2}\big[- x^T\Sigma^{-1}\mu_1 - \mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \big(- x^T\Sigma^{-1}\mu_0 - \mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0\big)\big]\\
&= -\frac{1}{2}\big[- (x^T\Sigma^{-1}\mu_1)^T - \mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \big(- (x^T\Sigma^{-1}\mu_0)^T - \mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0\big)\big]\\
&= -\frac{1}{2}\big[- \mu_1^T\Sigma^{-1}x - \mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \big(- \mu_0^T\Sigma^{-1}x - \mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0\big)\big]\\
&= -\frac{1}{2}\big[- 2\mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \big(- 2\mu_0^T\Sigma^{-1}x + \mu_0^T\Sigma^{-1}\mu_0\big)\big]\\
&= -\frac{1}{2}\big[- 2\mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 + 2\mu_0^T\Sigma^{-1}x - \mu_0^T\Sigma^{-1}\mu_0\big]\\
&= -\frac{1}{2}\big[2\mu_0^T\Sigma^{-1}x - 2\mu_1^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \mu_0^T\Sigma^{-1}\mu_0\big]\\
&= -\frac{1}{2}\big[2(\mu_0-\mu_1)^T\Sigma^{-1}x + \mu_1^T\Sigma^{-1}\mu_1 - \mu_0^T\Sigma^{-1}\mu_0\big]\\
&= (\mu_1-\mu_0)^T\Sigma^{-1}x + \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1
\end{aligned}
$$

So,

$$
\begin{aligned}
\alpha-\beta+\ln\phi-\ln(1-\phi) &= (\mu_1-\mu_0)^T\Sigma^{-1}x + \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 + \ln\Big(\frac{\phi}{1-\phi}\Big)\\
&= \big(\Sigma^{-1}(\mu_1-\mu_0)\big)^T x + \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 - \ln\Big(\frac{1-\phi}{\phi}\Big)
\end{aligned}
$$

Note that, $(\Sigma^{-1})^T \equiv \Sigma^{-1}$ because the covariance matrix is constant times Identity Matrix

Defining,

$$
\begin{aligned}
\theta &\equiv \Sigma^{-1}(\mu_1-\mu_0)\\
\theta_0 &\equiv \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 - \ln\Big(\frac{1-\phi}{\phi}\Big)
\end{aligned}
$$

Equation (1) becomes

$$
p(y=1 \mid x) = \frac{1}{1+e^{-y(\alpha-\beta+\ln\phi-\ln(1-\phi))}} = \frac{1}{1+e^{-y(\theta^T x+\theta_0)}}
$$

which is in the form of the logistic function.