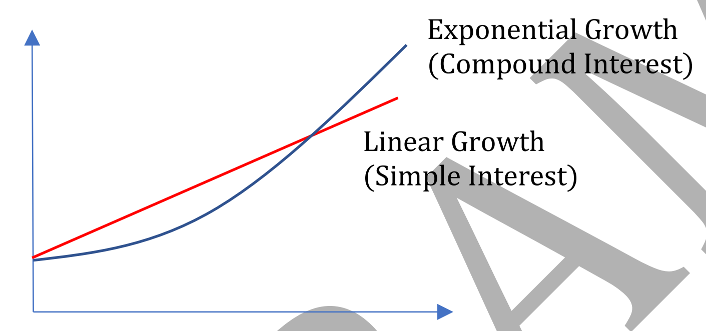
STAT203: Financial Mathematics

Contents

Chapter 1: Interest Measurement

Overview of Interest Theory

- Assume we want to grow our money with just a **SINGLE deposit**
- Our initial deposit is known as the **principal amount** and how much earn on that is known as the **interest payments** (As compensation for the loss of use of money)
- There are two types of growth we can experience:
    - **Linear Growth** → Simple Interest
    - **Exponential Growth** → Compound Interest
    - Simple Interest initially earns more than Compound interest. But in the long run, **because Interest will earn interest**, Compound Interest will earn more
- Compound Interest powerful because the interest is essentially reinvested at the same rate. **If we were to withdraw the interest each period, we would have simple interest**



Mathematical Notation

Amount function, $A(t)$

- $A(t) = A(0) * a(t)$
- Represents the total amount of money in that fund **at that point of time**

Accumulation Function, $a(t)$

- Represents the **change in money from time 0 to time t**
    - **Simple Interest (Linear) →** $a(t) = 1 + it$
    - **Compound Interest (Polynomial) →** $a(t) = (1 + i)^t$
    - **Double/Triple growth (Integer)** → $a(t) = 2 \text{ or } 3$
- It is a **special case of the amount function** where the initial amount is 1. Thus, by multiplying any initial amount to it, we obtain the total amount.
    - Additionally, it also means that **any formula that uses $A(t)$** in both numerator and denominators can be **substituted with $a(t)$**
- If our **starting time is not 0**, we use a more general expression of the amount function $a_k(t-k)$ where it represents the **change in value from time 0 to time k**
    - $a_k(t-k) = \frac{a(t)}{a(k)} \rightarrow \boldsymbol{Intuitive\ through\ recursion\ \&\ forward\ rates}$

Interest Function, $I(t)$

- $I(t) = A(t) - A(t-1) = A(0)[a(t) - a(t-1)]$
- Since the only change in the fund is interest, we can find it by taking the **difference in amounts** across two periods

Visualizing via Timelines

- To save on memory work, it is important that we understand **how formulas are derived** so that we can apply it to any context
- The first thing we should do for any question is to **draw a timeline** of the cashflows
- There are some important points we should remember:
  - Cash Inflows or Outflows should be **strictly on one side** of the timeline respectively
  - The units of time can be in **any units** (Years, Months) & with **any interpretation** (End or start of the period). The important thing is to **label it and be consistent**
  - It is good practice to **mark out the comparison date** we want to use so that we do not get confused along the way
  - Subtracting time depends on the cashflows we want to consider:
    - By default, it **assumes that only one of the end points** has a cashflow
    - We need to **add 1 period if both end points** has a cashflow

Effective Interest

- Unlike the simplified example earlier, most amount functions don't have smooth curves
- The fund grows at different rates at different times during the period. We can **summarize the entire growth into an effective interest rate for that period**
- This creates a **standardized measure of growth** during the period. Two funds with the same effective interest will lead to the same amounts, regardless of their volatility
- These interest rates can be Yearly, monthly etc. The **key is to be consistent** with the time periods you use throughout the same calculations:
  - Convert time periods to be expressed in the **same way as effective interest**
  - Convert effective interest to be expressed in the **same way as time periods**

$$i_t = \frac{\textbf{Fund growth } \textit{during the period } (Interest)}{\boldsymbol{Initial\ fund\ value}\ at\ the\ start\ of\ period} = \frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}$$

Deriving Amount Function:

$i_t = \dfrac{A(t) - A(t-1)}{A(t-1)}$
$i_t * A(t-1) = A(t) - A(t-1)$
$A(t) = A(t-1) + i_t * A(t-1)$
$A(t) = A(t-1)[1 + i_t]$

We can recursively add to this formula…
$A(t) = A(t-2) * [1 + i_{t-1}] * [1 + i_t]$
$A(t) = A(t-3) * [1 + i_{t-2}] * [1 + i_{t-1}] * [1 + i_t]$
$\vdots$
$A(t) = A(0) * [1 + i_1] * [1 + i_2] * \dots [1 + i_t]$

If we have a **constant effective interest** across all periods,
$A(t) = A(0) * [1 + i]^t$

More generally, if we start at time k and end at time t,
$A(t) = A(k) * [1 + i]^{t-k}$

Expressing Interest Rates

- Typically expressed as a **Percentage**, where $1\% = 0.01$
- For extremely small interest rates, it could be expressed as **Basis Points** instead, where $1\ BP = 0.01\% = 0.00001$

Time Value of Money

- "A dollar today is worth more than a dollar in the future"
- This is based off the principal of interest. If you invested the dollar today, you should be getting $(1+i_t)^t$ in the future. Conversely, a dollar in the future is worth $\frac{1}{(1+i_t)^t}$ today
- Bringing values into the future is known as **Accumulating** which results in the **Future Value.** Conversely, bringing future ones into the present is known as **Discounting** which results in the **Present Value**
- The key is that when working with cashflows across different time periods, we should **bring them to the same point in time, be it present or future to compare them**

$$\boldsymbol{Present\ Value, A(k)} = \frac{A(t)}{[1+i_t]^{t-k}} = A(t) * [1+i_t]^{-(t-k)}$$

$$\boldsymbol{Future\ Value, A(t)} = A(k) * [1+i_t]^{t-k}$$

- Notice that the formulas are almost identical with the main difference being the starting value and the exponent
- The FV has a **Positive Exponent**, signifying that we are **going forward in time** while the PV has a **Negative Exponent**, signifying that we are **going backwards in time**
- Since we use discounting and compounding so often, we use the **variable** $\boldsymbol{v = \frac{1}{a(t)}}$ to represent the **Discount Factor** so that we simplify our workings

Effective Rate of Discount

- Instead of Effective Interest Rates which measure the Interest relative to the starting fund value, another standardized way of measuring growth is **Effective rate of Discount**, which measures **Interest relative to the ending fund value instead**
- They are simply **an alternative measure to Interest Rates**, and can be used in a similar fashion to accumulate or discount values
- **DO NOT confuse it with the discount factor.** The discount factor is simply a notation we use to simplify the equation while discount rate is a completely new concept

<u>Effective Interest Rate</u>	<u>Effective Discount Rate</u>
$i_t = \frac{A(t)-A(t-1)}{\boldsymbol{A(t-1)}} = \frac{a(t)-a(t-1)}{a(t-1)}$	$d_t = \frac{A(t)-A(t-1)}{\boldsymbol{A(t)}} = \frac{a(t)-a(t-1)}{a(t)}$
If we fairly assume that the fund is growing, $A(t) > A(t-1)$. Thus, $i_t > d_t$.	
$\boldsymbol{FV} = A(k) * [1+i_t]^{t-k}$ $\boldsymbol{PV} = A(t) * [1+i_t]^{-(t-k)}$	$\boldsymbol{FV} = A(k) * [1-d_t]^{-(t-k)}$ $\boldsymbol{PV} = A(t) * [1-d_t]^{t-k}$
$A(k) * [1+i_t]^{t-k}$ → 0 —— t ← $A(t) * [1+i_t]^{-(t-k)}$	$A(k) * [1-d_t]^{-(t-k)}$ → 0 —— t ← $A(t) * [1-d_t]^{t-k}$
Interest rate naturally goes forward in time (Positive exponent when accumulating) while **Discount rate naturally goes backward** in time (Positive Exponent when Discounting)	

Another way to think about it is that Interest rates are usually paid at the end of the period, while discount rates is interest that is paid at the start of the period instead. But both are equivalent:

$$d = \frac{i}{1+i} = i * \frac{1}{1+i} = iv$$

Compounding Frequency & Nominal Rates

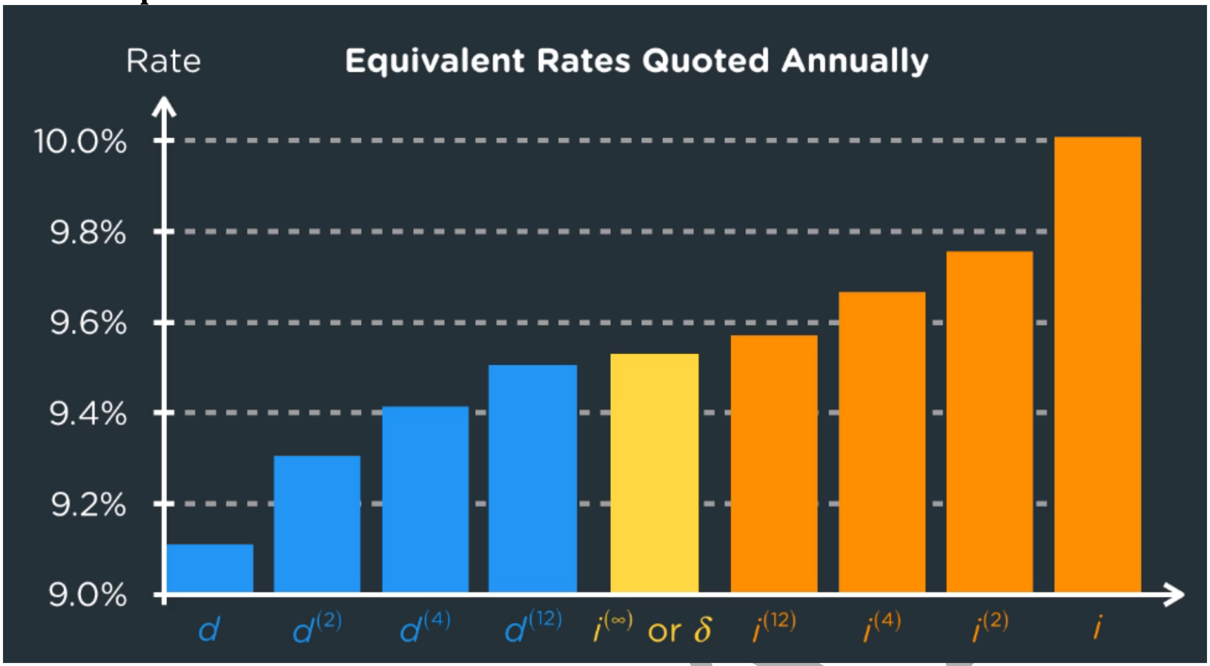
- Compounding annually is just one of the ways we can compound. We could **compound interest Monthly, Quarterly, semi-annually** etc
- However, quoting effective rates with different timespans can become confusing. Thus, we multiply to make them reflect an annual **nominal rate** instead:
    - Monthly rates multiply by 12; Quarterly rate multiply by 4
- The word nominal means **"in name only"**, which implies that these rates are just placeholders. If we actually compound 12 months using a monthly rate VS compounding once using the nominal rate, we get two different values
- This is because the **nominal rate is in name only and not to be used**. Unfortunately, most interest rates are quoted in nominal terms. We always need to **remember that to convert nominal rates to effective rates.**

	Interest Rates	**Discount Rate**
Annual Effective Rate	i	d
Annual Nominal Rate (Compounded m times a year, every $\frac{12}{m}$ months)	$i^{(m)}$	$d^{(m)}$
Monthly Effective Rate (Any time other than Annual)	j	k
Convert Nominal to m-Effective	$j = \frac{i^{(m)}}{m}$	$k = \frac{d^{(m)}}{m}$
Convert m to Annual Effective	$(1+j)^m = (1+i)$ $i = (1+j)^m - 1$	$(1-k)^m = (1-d)$ $d = 1-(1-k)^m$
Convert Annual to Nominal (Sub in any m)	$\left(1+\frac{i^{(m)}}{m}\right)^m = (1+i)$ $i^{(m)} = m\left((1+i)^{\frac{1}{m}} - 1\right)$	$\left(1-\frac{d^{(m)}}{m}\right)^m = (1-d)$ $d^{(m)} = m\left(1-(1-d)^{\frac{1}{m}}\right)$
Interesting Result (Nominal = Nominal)	$\frac{1}{m} = \frac{1}{d^{(m)}} - \frac{1}{i^{(m)}}$	

Definition of m

- m refers to the **number of times interest is compounded within a year**
    - Compound once a year $\rightarrow m = 1$
        - EG. One year annual effective rate
    - Compound multiple times a year $\rightarrow m > 1$
        - EG. 1-year nominal rate, compounded monthly
    - Compound once in a few years $\rightarrow m < 1$
        - EG. 4-year nominal rate, compounded yearly

Visual Representation



- Assuming the **SAME effective rate of interest**, the above diagram provides a visual representation of the values of the various rates
- Effective rates are always at the **tail ends** – They move toward the centre as the compounding frequency for each type of rate, with infinite compounding (Force of interest) as the midpoint between both
    - **Nominal Interest Rate is always lower than the equivalent effective rate**. This allows companies to give the false perception of lower interest rates. This is why nominal rates are often used in the industry (Vice versa for Discount)
- Use this as a sense check on the values calculated – whether or not it should be higher or lower

Force of Interest

- As the compounding frequency increases, we can see that both the interest and discount rates move towards the centre of the graph
- If the **compounding frequency m, tends to infinity**, both the discount and interest rates tend to a central rate as well, known as the **Force of Interest, δ**. It represents how much the fund accumulates at that **instantaneous time t**

Deriving Force of Interest

Constant Force of Interest (Constants)	Variable Force of Interest (Polynomials)
By definition, if m tends to infinity: $i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$ $\lim_{m\to\infty} i^{(m)} = \ln(1+i)$ $\delta = i^{(\infty)} = \ln(1+i)$ $e^{\delta} = (1+i)$ $\therefore a(t) = (1+i)^t = e^{\delta t}$ This **only holds for $(\boldsymbol{0}, \boldsymbol{t})$**. If we are calculating the forward rate, this formula is not true.	By definition, it is instantaneous: $\delta = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}$ $\int_0^t \delta_r \, dr = \ln\big(a(t)\big)$ $\therefore a(t) = (1+i)^t = e^{\int_0^t \delta_r \, dr}$ If simple interest instead, $\delta = \frac{a'(t)}{a(t)} = \frac{i}{1+it}$

Dealing with Variable Force of Interest

- Always find the **accumulation function FIRST**
- Take note that the limits of integration $(0, t)$ is for a standard accumulation function $a_0(t)$. For a more general accumulation function $a_k(t-k)$ the limits would be (k, t)
- Always consider logic when dealing with Variable Force – the interest function should always be different for different years and periods
  - When your start time is fixed, integrating from $(0, t)$ makes sense
  - But when even the start time is not fixed (Yearly interest payments) then $(t, t+1)$ would be better
  - If you ever see a constant output while having a variable force, something is wrong

Equivalent Rates

- As alluded to earlier, if two funds have the same starting and ending values, they will have the **same Accumulation Function regardless which types of rates were used**
- We get an all-in-one formula that we convert from one to another:

$$a(t) = (1+i)^t = (1-d)^{-t} = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d(m)}{m}\right)^m = e^{\delta t} = e^{\int_0^t \delta_r \, dr} = v^{-t}$$

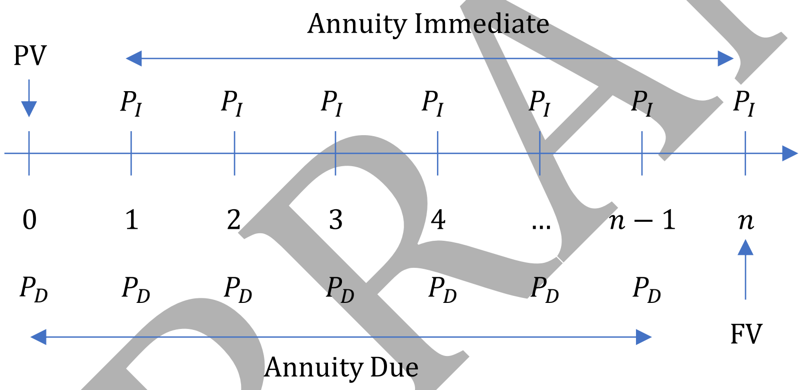
There is no equivalent Compound – Simple Interest rate.

- We can convert between simple interest rates across time periods using the same principle – equating accumulation functions
- The conversion will simplify to become just multiplying by a factor to convert the rate

Chapter 2: Level Annuities

Types of Annuities

- An annuity is a series of **level payments at equal intervals**
- Our goal is determining the **Present or Future Value** of the Annuity. It is the **SUM** of the Present or Future value of **EVERY** payment
- There are two main types of annuities:
  - **Immediate** → Payments made at the **end of the period (Arrears)**
    - We denote the PV as $a_{\overline{n|}\,i}$ and FV as $s_{\overline{n|}\,i}$
  - **Due** → Payments are made at the **start of the period (Advance)**
    - We denote the PV as $\ddot{a}_{\overline{n|}\,i}$ and FV as $\ddot{s}_{\overline{n|}\,i}$
- We calculate the **PV at the start of the first period** and the **FV at the end of the last period**, regardless of the type of Annuity
  - $a_{\overline{n|}\,i}$ is calculated **one period before the first the payment** while $s_{\overline{n|}\,i}$ is calculated **on the period of the last payment**
  - Conversely, $\ddot{a}_{\overline{n|}\,i}$ is calculated on the **period of the first payment** while $\ddot{s}_{\overline{n|}\,i}$ is calculated **one period after the last payment**
  - Notice how both are symmetrical on a timeline:



Important:

- Remember that the **start of the period is also the end of the previous period**. This means that given any set of Cashflows, we can **treat it as either an Immediate or Due**, depending on the comparison date we want to use
- The cashflows above are one cycle above for illustration purposes to drive home the idea that they have different comparison dates

Deriving Annuity Formulas:

- Since payments are level, we can **factorise out the common payments P** to make the expressions simpler. However, we have to multiply it back for our final answers
- Effectively, this allows us to derive the formulas **assuming payments of 1**

	<u>Annuity Immediate</u>	<u>Annuity Due</u>
PV	Since we discount to one period before the first payment, we **discount the first payment (Start from 1)** $a_{\overline{n}\rceil\, i} = v + v^1 + v^2 + \cdots v^n$ $a_{\overline{n}\rceil\, i} = \sum_1^n v^n$ $a_{\overline{n}\rceil\, i} = \frac{v - v^{n+1}}{1 - v}$ $a_{\overline{n}\rceil\, i} = \frac{v(1 - v^n)}{1 - v}$ $a_{\overline{n}\rceil\, i} = \frac{1 - v^n}{i}$	Since we discount to the period of the first payment, we **don't discount the first payment (Start from 0)** $\ddot{a}_{\overline{n}\rceil\, i} = v^0 + v^1 + v^2 + \cdots v^{n-1}$ $\ddot{a}_{\overline{n}\rceil\, i} = \sum_0^{n-1} v^n$ $\ddot{a}_{\overline{n}\rceil\, i} = \frac{1 - v^n}{1 - v}$ $\ddot{a}_{\overline{n}\rceil\, i} = \frac{1 - v^n}{d}$
FV	Since we accumulate to the period of the last payment, we **don't accumulate the last payment (End with 0)** $s_{\overline{n}\rceil\, i} = (1+i)^{n-1} + \cdots (1+i)^0$ $s_{\overline{n}\rceil\, i} = \sum_0^{n-1} (1+i)^n$ $s_{\overline{n}\rceil\, i} = \frac{1 - (1+i)^n}{1 - (1+i)}$ $s_{\overline{n}\rceil\, i} = \frac{1 - (1+i)^n}{-i}$ $s_{\overline{n}\rceil\, i} = \frac{(1+i)^n - 1}{i}$	Since we accumulate to the one period after the last payment, we **accumulate the last payment (End with 1)** $\ddot{s}_{\overline{n}\rceil\, i} = (1+i)^n + \cdots (1+i)^1$ $\ddot{s}_{\overline{n}\rceil\, i} = \sum_1^n (1+i)^n$ $\ddot{s}_{\overline{n}\rceil\, i} = \frac{(1+i) - (1+i)^{n+1}}{1 - (1+i)}$ $\ddot{s}_{\overline{n}\rceil\, i} = \frac{(1+i)[1 - (1+i)^n]}{-i}$ $\ddot{s}_{\overline{n}\rceil\, i} = \frac{(1+i)^n - 1}{d}$
Both	Alternatively, if we know either the Present or Future value, we can simply Accumulate or Discount that by the number of periods to amount to obtain the other: $s_{\overline{n}\rceil\, i} = a_{\overline{n}\rceil\, i}(1+i)^n$ $\ddot{s}_{\overline{n}\rceil\, i} = \ddot{a}_{\overline{n}\rceil\, i}(1+i)^n$	
Other	$Sum\ of\ Geometric\ Series = \frac{First\ term - First\ Omitted\ Term\ (\text{"Next" term})}{1 - Factor}$ • First omitted term is **zero for an infinite series** (No omission) $\frac{v}{1-v} = \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}} = \frac{\frac{1}{1+i}}{\frac{i}{1+i}} = \frac{1}{i}$ $1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i} = d$ We can easily remember the denominator for each by ***i* for Immediate and *d* for Due**	

Linking the two kinds of Annuities

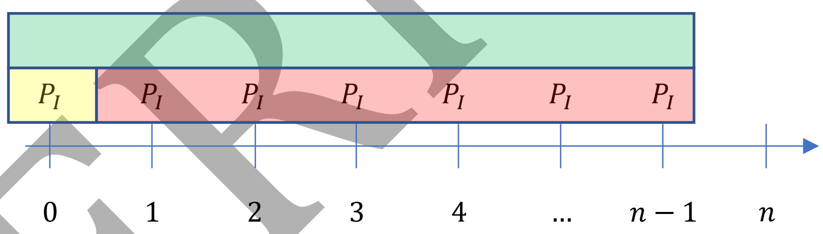
- The financial calculator naturally assumes that the Annuities we calculate are Annuity Immediate (Need to convert for Due)
- If we know how to convert Immediate to Due, we can simply use the calculator and apply the conversion which will save a lot of time
- Both methods are viable, but usually the first method is easier and more direct. But the magic of the second approach is that it **does not require you to know the interest rate**

Via the payment amount

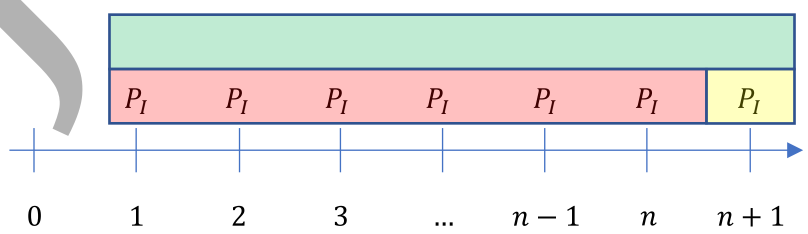
- If we compare the payments for Immediate and Due, we see the regardless FV or PV, the payments for Annuity Due are **always greater by a factor of** $(\mathbf{1+i})$
- Thus, we can go from Immediate to Due by multiplying $(1+i)$
  - $\ddot{a}_{n\rceil\, i} = (1+i)a_{n\rceil\, i}$
  - $\ddot{s}_{n\rceil\, i} = (1+i)s_{n\rceil\, i}$
- Alternatively, we can also use this by **changing the payment amount by** $(\mathbf{1+i})$ in the calculator to directly calculate Annuity Due

Via the number of payments

- The difference between the two is that an **Annuity Due has additional payment at time 0** and **one less payment at time** $\boldsymbol{n}$
- Opposite effects on both because they are comparing in different time periods
- Given any series of cashflows, we can convert between the two by:
- **Present Value**
  - **Additional payment at time 0** → Add a one additional payment $(+1)$
  - **One less payment at time** $\boldsymbol{n}$ → Consider one less payment $(a_{n-1\rceil\, i})$
  - $\therefore \ddot{a}_{n\rceil\, i} = 1 + a_{n-1\rceil\, i}$



- **Future Value**
  - **Additional payment at time 0** → Consider additional payment $(s_{n+1\rceil\, i})$
  - **One less payment at time** $\boldsymbol{n}$ → Subtract one less payment (-1)
  - $\therefore \ddot{s}_{n\rceil\, i} = s_{n+1\rceil\, i} - 1$



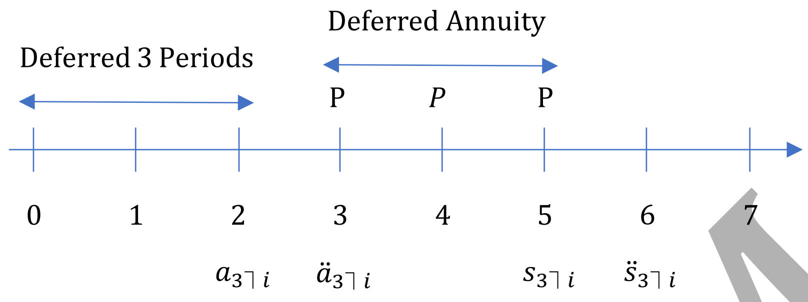
Easy way to remember

- Annuity Due is always the subject the equation; Annuity Immediate is not
- If the RHS time is PLUS/MINUS then the it must be MINUS/PLUS 1 (Opposite)

<u>Special Annuity Cases</u>

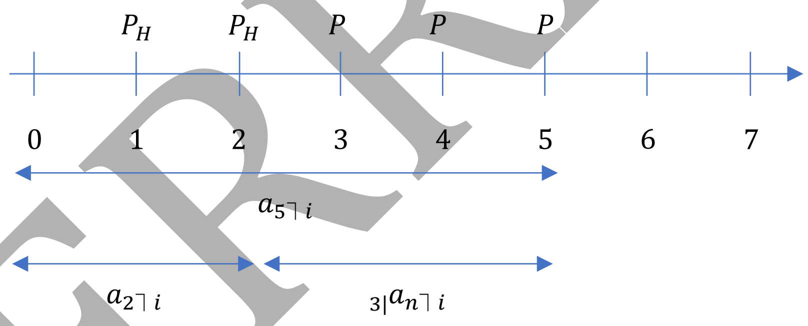
Deferred Annuities

- Variant where the annuity is deferred and **starts at some later date** (not time 0)
- We denote the present value of an annuity deferred by m periods as ${}_{m|}a_{\overline{n}|\,i}$



Calculating Values Deferred Annuities

- **Standard Method**
  - If we already know the PV/FV of the normal annuity (Any of the 4 times) we can simply discount that value to the time we want
  - ${}_{m|}a_{\overline{n}|\,i} = v^m * \ddot{a}_{\overline{n}|\,i}$ (Or any of the other values)
- **Differences in Annuities**
  - Consider the cashflows involved in terms of their raw equation
  - We can take a larger annuity minus a smaller annuity to obtain the deferred annuity that we want



Calculating values in the middle of the annuity payments

- All our previous formulas helped us to find the value of the annuity at the start or end
- We have no formula to help us to find the value of the annuity in the **middle of it**
- Similarly, we can use either:
  - **Simple method** → Accumulate discount a value from the start/end respectively
  - **Sum of annuities**
    - Split the annuity into two at the period we want to compare
    - This artificially creates two annuities with end points at the time we want to compare, allowing us to sum together their values to obtain the value of the overall annuity

Balloon & Drop Payments

- Sometimes instead of finding PV/FV, we are tasked to find the number of payments
- Since this is a calculated value, we often **got non-integer values** (EG. 15.42). This means that there are **15 standard payments and a fractional payment**
    - **Balloon Payment** → Combine the last standard and fractional payment together
    - **Drop Payment** → Treat the fractional payment as its own payment

Let X be the fractional payment:

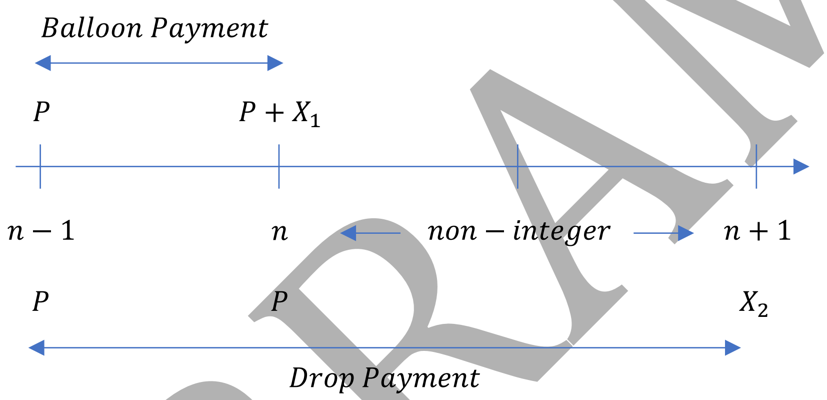
$\boldsymbol{Balloon\ Payment} = P + X$

$PV_{Balloon} = a_{\overline{n|}\,i} + X * v^n$

$\boldsymbol{Drop\ Payment} = X(1 + i)$

$PV_{Drop} = a_{\overline{n|}\,i} + X(1 + i) * v^{n+1}$

- Since the fractional payment was treated as its own payment in the **next period,** we have **to account for the time value of money** and accumulate it accordingly

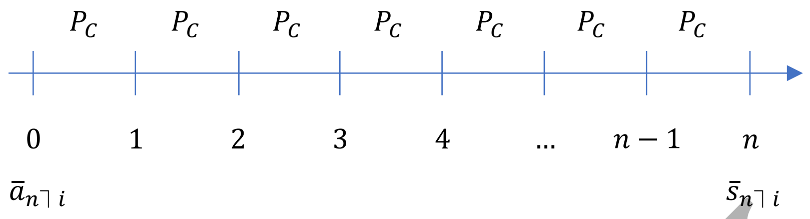


Varying Interest Rates

- So far, we have used just one fixed effective rate to discount or accumulate annuities
- More realistically, interest rates are bound to change across periods
- Assume that we have <u>two periods</u> with rates i_1 & i_2, $i_1 \neq i_2$. There are two methods to use:
- **Market Portfolio Method:**
    - **End of Period 1** → Accumulate first payment by $(1 + i_1)$
    - **End of Period 2** → Accumulate both payments by $(1 + i_2)$
    - **Result** → $FV = P(1 + i_1)(1 + i_2) + P(1 + i_2)$
- **Yield-Curve Method:**
    - **End of Period 1** → Accumulate first payment by $(1 + i_1)$
    - **End of Period 2** → Accumulate payments by $(1 + i_1)$ and $(1 + i_2)$ respectively
    - **Result** → $FV = P(1 + i_1)^2 + P(1 + i_2)$
- Market Portfolio is more intuitive. The key difference is how the interest of the first payment (More generally, all previous periods) is treated
    - **Market Portfolio** accumulates according to **rate of the current period**
    - **Yield Curve** accumulates according to the **rate when that payment was made**
- We can know which method to use based on how the question defines interest rates, be it according to the time of payment or prevailing period
- Either way, we will need to manually map out each payment and discount/accumulate them according to their corresponding interest rates

Continuous Annuities

- Instead of making payments at fixed time intervals like the start or end of the period, payments are made **continuously throughout the period**, totalling up to the same amount
- Similarly, we denote the PV/FV of such payments with $\bar{a}_{\overline{n|}\,i}$ and $\bar{s}_{\overline{n|}\,i}$



Deriving Continuous Annuity Formulas

<table>
<tr><th><u>Present Value</u></th><th><u>Future Value</u></th></tr>
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$\bar{a}_{\overline{n|}\,i} = \int_0^n e^{-\delta t}\ dt$

$\bar{a}_{\overline{n|}\,i} = \left[\frac{e^{-\delta t}}{-\delta}\right]_0^n$

$\bar{a}_{\overline{n|}\,i} = \frac{e^{-\delta n}}{-\delta} + \frac{1}{\delta}$

$\bar{a}_{\overline{n|}\,i} = \frac{1 - e^{-\delta n}}{\delta}$

$\bar{a}_{\overline{n|}\,i} = \frac{1 - v^n}{\delta}$

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$\bar{s}_{\overline{n|}\,i} = \int_0^n e^{\delta t}\ dt$

$\bar{s}_{\overline{n|}\,i} = \left[\frac{e^{\delta t}}{\delta}\right]_0^n$

$\bar{s}_{\overline{n|}\,i} = \frac{e^{\delta n}}{\delta} - \frac{1}{\delta}$

$\bar{s}_{\overline{n|}\,i} = \frac{e^{n\delta} - 1}{\delta}$

$\bar{s}_{\overline{n|}\,i} = \frac{(1+i)^n - 1}{\delta}$

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Similarly, we factorise out constant **CONTINOUS payments of P** which assumes a continuous rate of cashflows of 1 for the proof

The **accumulation function for the force of interest** is $a(t) = e^{\delta t}$, which represents the accumulated or discounted cashflows at each time t. If δ is variable, we change the accumulation factor to $e^{\int_k^t \delta_r\, dr}$ and perform the same steps.

Since we want the sum of the discounted or accumulated cashflows from $(0, n)$, we integrate our general accumulation function from that same limit. It is the same idea as our discrete annuities, but we use **Integrals instead of Sigma** to perform the sum.

The limit of integration should follow the accumulation function used:

- If we use the **standard accumulation function $\boldsymbol{a(t)}$**, then use $(0, n)$
- If we use the **general accumulation function $\boldsymbol{a_k(t-k)}$**, then use $(0, t-k)$

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</table>

Comparing all three annuity types

<table>
<tr><th><u>Annuity Due</u></th><th><u>Annuity Immediate</u></th><th><u>Continuous Annuity</u></th></tr>
<tr><td>$\ddot{a}_{\overline{n|}\,i} = \frac{1 - v^n}{d}$</td><td>$a_{\overline{n|}\,i} = \frac{1 - v^n}{i}$</td><td>$\bar{a}_{\overline{n|}\,i} = \frac{1 - v^n}{\delta}$</td></tr>
<tr><td>Since d is the smallest rate, this produces the largest PV</td><td>Since i is the largest rate, this produces the smallest PV</td><td>Since δ is the midpoint rate, it produces the midpoint PV</td></tr>
<tr><td></td><td colspan="2">$\bar{a}_{\overline{n|}\,i} = \frac{1}{\delta} a_{\overline{n|}\,i}$</td></tr>
<tr><td colspan="2">$\ddot{a}_{\overline{n|}\,i} = \frac{i}{d} a_{\overline{n|}\,i} = (1+i) a_{\overline{n|}\,i}$</td><td></td></tr>
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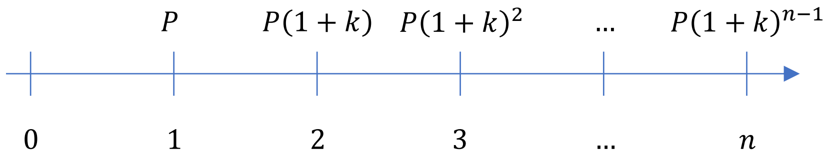
Chapter 2.5: Non-level Annuities

Geometric Annuities

- Payments change according to a Geometric Series (By a common factor k)



Deriving Geometric Formula (Immediate)

- Since the original annuity proof **already made use of the Geometric series**, increasing payments by a geometrically simply just changes the factor we use in the proof

$$PV_{Geometric} = v + (1+k)v^2 + \cdots (1+k)^{n-1}v^n$$
$$PV_{Geometric} = v\left[1 + (1+k)v + \cdots + \left((1+k)v\right)^{n-1}\right]$$
$$PV_{Geometric} = v\left[\frac{1-(1+k)^n v^n}{1-(1+k)v}\right]$$
$$PV_{Geometric} = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\frac{1+k}{1+i}}\right]$$
$$PV_{Geometric} = v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{\frac{i-k}{1+i}}\right]$$
$$PV_{Geometric} = \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}$$

- This only holds for $i \neq k$, **derive the formula manually again** to see the interaction. The terms will cancel each other, **becoming a standard summation problem**
- If k is negative, remember that the denominator nets out to become a plus instead

Converting non-level to Level

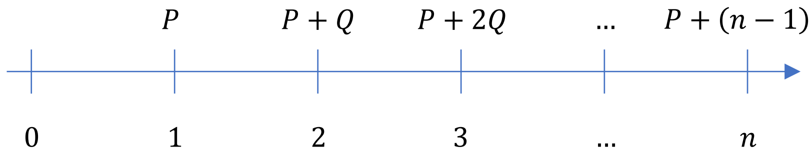
- Looking at the proof, we notice that it is **VERY similar to the level version**
- Since the Financial Calculator only allows level payments, **converting a non-level annuity to a level one** would save us a lot of time
- The key is noticing that we can substitute the common factors for each other:

$i > k$	$i < k$
$(1+k)v = \frac{1+k}{1+i} < 1$ $v' = \frac{1+k}{1+i}$ $1+i' = \frac{1+i}{1+k}$ $i' = \frac{1+i}{1+k} - 1$	$(1+k)v = \frac{1+k}{1+i} > 1$ $1+i' = \frac{1+k}{1+i}$ $i' = \frac{1+k}{1+i} - 1$

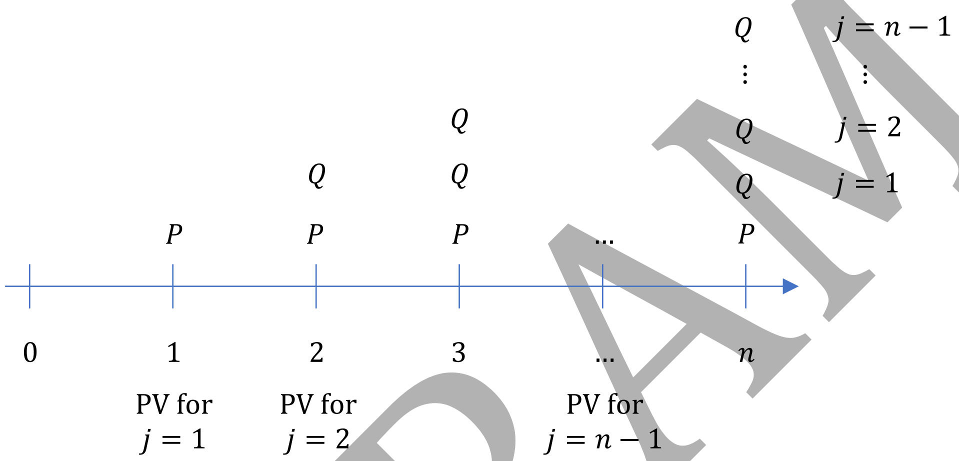
- Depending on the relative magnitude of i and k, we can determine if the factor is closer to the Present Value, or the Future Value formula & choose the appropriate conversion
- When converting from FV to PV (vice versa), **remember to use $(\mathbf{1}+\boldsymbol{i}')$** not $(1+i)$
- This method is preferred when we have to **solve for $\boldsymbol{i}$ or $\boldsymbol{k}$** as it simplifies the algebra. When solving for i' or v', **do not be alarmed by weird results**

Arithmetic Annuities

- Payments change according to an Arithmetic Progression (By fixed constant Q)
- **Be careful when using AP Formula** – The formula **assumes the first payment occurs at time is 1**, but sometimes your payments start at time 0 so an **adjustment is needed**



Deriving Arithmetic Formula (Immediate)



- **ALWAYS split up the payments** into their individual components to better visualize it
- We notice that we can form several different level annuities:
    - One level annuity with n payments of P
    - $j-1$ **deferred** level annuities with $n-j$ payments of Q
- We consider the PV at time 0:

$$PV_{Arithmetic} = P * a_{\overline{n|}\,i} + Q * \sum_{j=1}^{n-1} v^j * \frac{1-v^{n-j}}{i}$$

$$PV_{Arithmetic} = P * a_{\overline{n|}\,i} + Q * \sum_{j=1}^{n-1} \frac{v^j - v^n}{i}$$

$$PV_{Arithmetic} = P * a_{\overline{n|}\,i} + Q\left(\sum_{j=1}^{n-1} \frac{v^j}{i} - \frac{nv^n}{i}\right)$$

$$PV_{Arithmetic} = P * a_{\overline{n|}\,i} + Q\left(\frac{a_{\overline{n|}\,i}}{i} - \frac{nv^n}{i}\right)$$

$$PV_{Arithmetic} = a_{\overline{n|}\,i}\left(P + \frac{Q}{i}\right) - \frac{Qnv^n}{i}$$

- Using the Financial Calculator, we can quickly calculate this value by setting $\left(P+\frac{Q}{i}\right)$ as the Payment and $\left(\frac{Qnv^n}{i}\right)$ as the Future Value
- Understanding this proof is important as this **concept of splitting annuities is extremely important** when dealing with Arithmetic Annuities

Special case: Unit Increasing

- When $P = Q = 1$. The condition can be met **for any $P = Q = x$**, as we can simply factor out x to obtain $P = Q = 1$, but **remember to multiply back x**
- First payment starts out at 1 and increases by 1 each period **until n** in the last period

When $P = Q = 1$,

$$(Ia)_{n\rceil\, i} = a_{n\rceil\, i} + \left(\frac{a_{n\rceil\, i}}{i} - \frac{nv^n}{i}\right)$$

$$(Ia)_{n\rceil\, i} = \frac{i * a_{n\rceil\, i} + a_{n\rceil\, i} - nv^n}{i}$$

$$(Ia)_{n\rceil\, i} = \frac{a_{n\rceil\, i}(1+i) - nv^n}{i}$$

$$(Ia)_{n\rceil\, i} = \frac{\ddot{a}_{n\rceil\, i} - nv^n}{i}$$

Special case: Unit Decreasing

- When $P = n$ and $Q = -1$
- First payment starts out at n and decreases by 1 each period **until 1** in the last period

When $P = \text{n}, Q = -1$,

$$(Da)_{n\rceil\, i} = n * a_{n\rceil\, i} - \left(\frac{a_{n\rceil\, i}}{i} - \frac{nv^n}{i}\right)$$

$$(Da)_{n\rceil\, i} = n * \left(\frac{1-v^n}{i}\right)_n - \left(\frac{a_{n\rceil\, i}}{i} - \frac{nv^n}{i}\right)$$

$$(Da)_{n\rceil\, i} = \frac{(1-v^n) - (a_{n\rceil\, i} - nv^n)}{i}$$

$$(Da)_{n\rceil\, i} = \frac{n - nv^n - a_{n\rceil\, i} + nv^n}{i}$$

$$(Da)_{n\rceil\, i} = \frac{n - a_{n\rceil\, i}}{i}$$

Continuously Increasing Annuities

- Annuities that **increase continuously** rather than at discrete times
- Rate of payment can be described as a **function of t** (Since t is always increasing as well)

Deriving Continuously increasing Annuities

- Similar to the level version, we integrate the payments and the accumulation function:

$$(\bar{I}a)_{n\rceil\, i} = \int_0^n f(t) * e^{\delta t}\, dt$$

$$PV = \int_0^n f(t) * e^{\int_0^t \delta_r\, dr}\, dt$$

Tabular integration:

Differentiate	Integrate
$f(t)$	$(+)$ $\int e^{\delta t}$
$f'(t)$	$(-)$ $\int\int e^{\delta t}$
$\vdots$	$(\pm)$ $\vdots$
1	$\int \dots \int e^{\delta t}$

Perpetuities

- Annuities whose payments **go on forever** (No end date)
- All formulas can be derived by letting $\boldsymbol{n} \to \infty$

Level Annuities

Annuity Due	Annuity Immediate	Continuous Annuity
$\ddot{a}_{\overline{n\vert}\, i} = \dfrac{1 - v^n}{d}$	$a_{\overline{n\vert}\, i} = \dfrac{1 - v^n}{i}$	$\bar{a}_{\overline{n\vert}\, i} = \dfrac{1 - v^n}{\delta}$
$\ddot{a}_{\overline{\infty\vert}\, i} = \dfrac{1}{d}$	$a_{\overline{\infty\vert}\, i} = \dfrac{1}{i}$	$\bar{a}_{\overline{\infty\vert}\, i} = \dfrac{1}{\delta}$

Non-level Annuities

Geometric Annuities	Arithmetic Annuities
$PV_{Geometric} = \dfrac{1 - \left(\frac{1+k}{1+i}\right)^n}{i - k}$ $PV_{Geometric,\infty} = \dfrac{1}{i - k}$ **If $\boldsymbol{k}$ is negative**, the denominator nets out to **become a plus instead**	$PV_{Arithmetic} = P * a_{\overline{n\vert}\, i} + Q\left(\dfrac{a_{\overline{n\vert}\, i}}{i} - \dfrac{nv^n}{i}\right)$ $PV_{Arithmetic,\infty} = P * a_{\overline{\infty\vert}\, i} + Q\left(\dfrac{a_{\overline{\infty\vert}\, i}}{i}\right)$ $PV_{Arithmetic,\infty} = P * \dfrac{1}{i} + Q\left(\dfrac{\frac{1}{i}}{i}\right)$ $PV_{Arithmetic,\infty} = \dfrac{P}{i} + \dfrac{Q}{i^2}$

Important concepts for Perpetuities

- **The difference between two perpetuities is always an Annuity**. Since both payments go on forever, the **difference is simply when they start**
- The present value of a Perpetuity **at any time** is always the same. This is because the payments go forever, so it **doesn't matter when it starts**
- We can approximate Perpetuities in the TVM by setting the **number of years to be 9999,** which is sufficiently large to approximate such cashflows

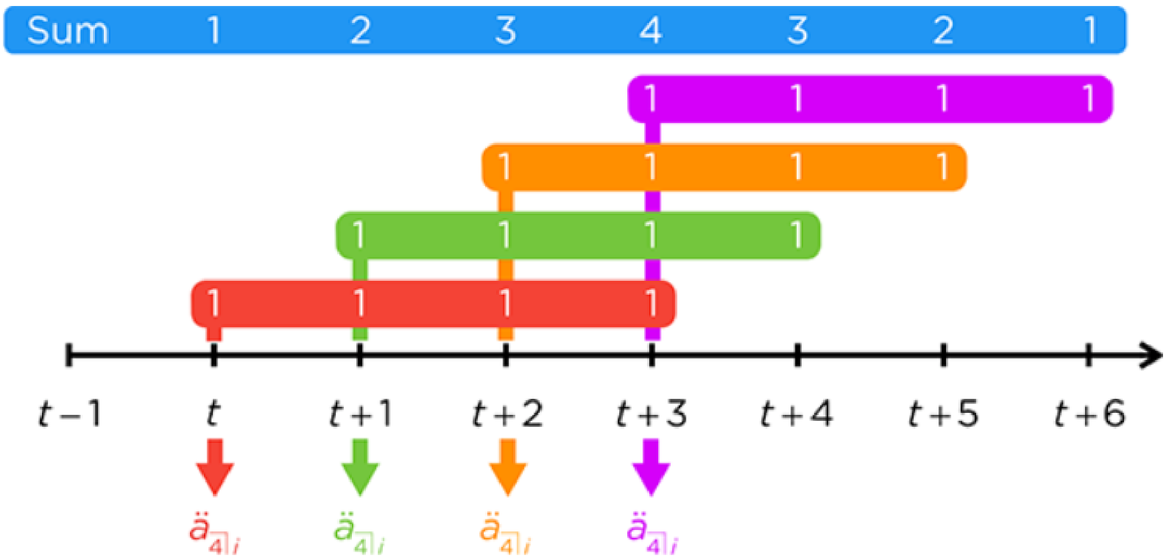
Other Important Annuity Patterns

Block Payments (For any increase/decrease – very powerful concept)

- Annuities that pay out in blocks. Payments within blocks are the same but each block has a different set of payments
- We can treat each block as a deferred annuity, but a faster way is to add/subtract them
- If we **start from the date furthest** from the comparison date and add the **change in payment levels,** the final result is the PV/FV of the entire annuity

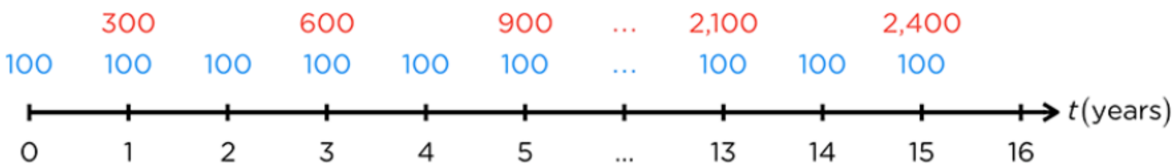
Repeat Reverse Annuities (For increases/decreases of 1)

- Annuities that start of increasing, reach a peak, and decrease back to the start
- We could split into an increasing and decreasing annuity, but another faster method would be to **recognise that they are all the same deferred annuity**:



Odd-Even Split (For alternating increases)

- Annuities that follow a cyclical pattern (Increase in odd and decrease in even years)
- We can **split them into a level annuity and an increasing annuity with different intervals:**

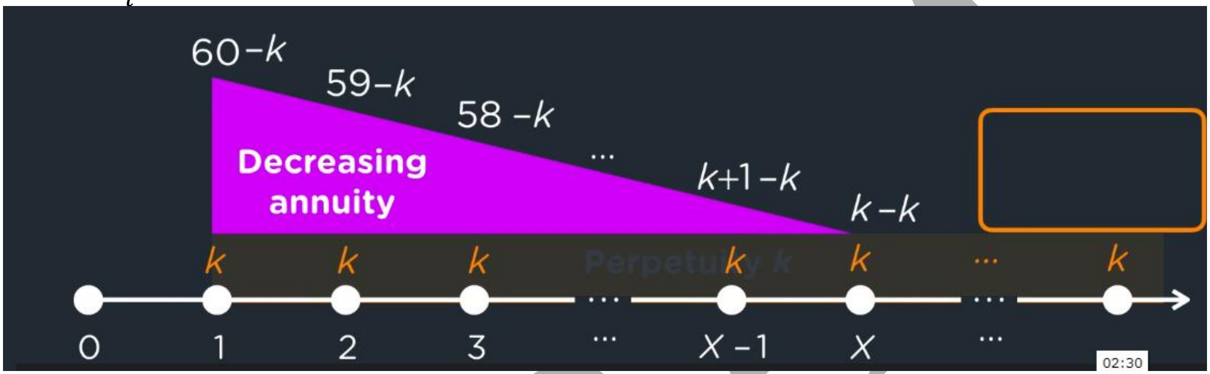


Annuities into Perpetuities:

Decreasing Annuity into a Perpetuity:

- Annuities that decrease till a certain level then becomes a level perpetuity
- We could treat each block as a decreasing annuity and deferred perpetuity
- But a faster way would be to treat them as a **perpetuity and unit decreasing annuity**:

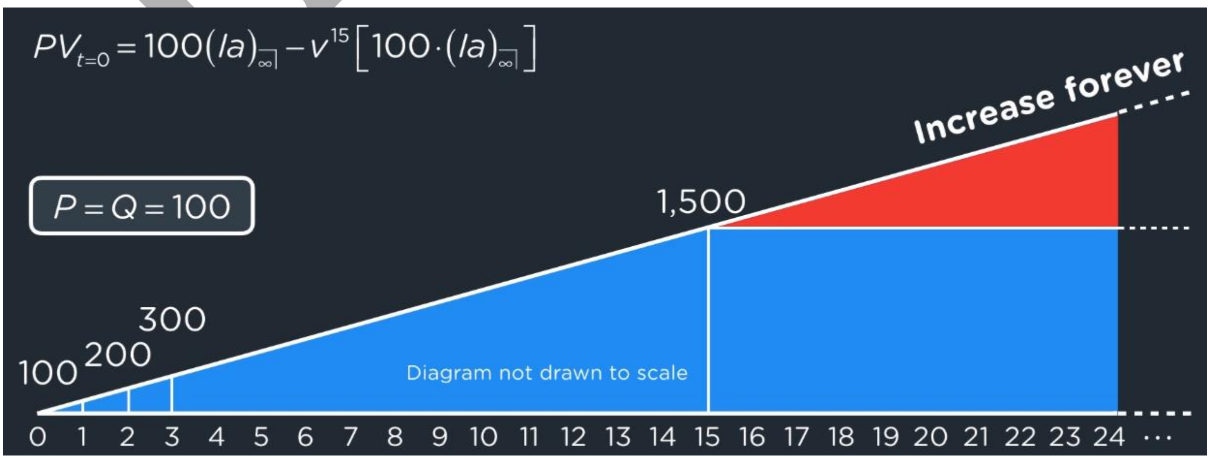
$$PV = \frac{k}{i} + (Da)_{\overline{n-k}|i}$$
$$PV = \frac{k}{i} + \frac{n - k + a_{\overline{n}|i}}{i}$$
$$PV = \frac{n + a_{\overline{n}|i}}{i}$$



Increasing Annuity into perpetuity

- Annuities that increase till a certain level then become a level perpetuity
- We could treat each block as an increasing annuity then deferred perpetuity
- But a faster way would be to treat them as the **difference of Unit Increasing Perpetuities**:

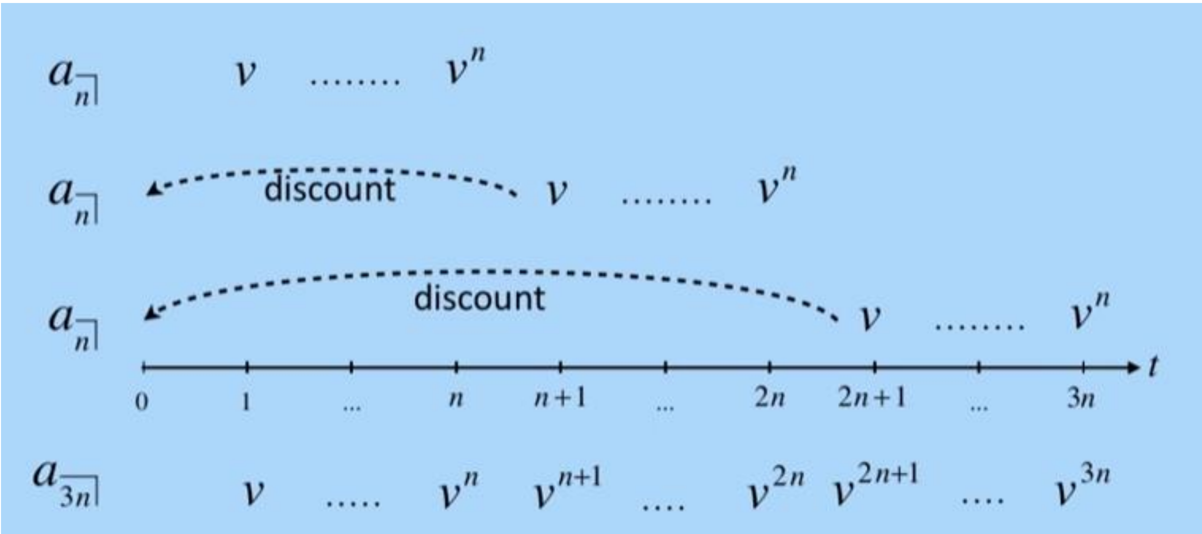
$$PV = (Ia)_{\overline{\infty}|i} - v^n (Ia)_{\overline{\infty}|i}$$
$$PV = (Ia)_{\overline{\infty}|i}(1 - v^n)$$
$$PV = \left(\frac{1}{i} + \frac{1}{i^2}\right)(1 - v^n)$$
$$PV = \left(\frac{1+i}{i^2}\right)(1 - v^n)$$
$$PV = \frac{1}{i} * \frac{1 - v^n}{d}$$
$$PV = \frac{\ddot{a}_{\overline{n}|i}}{i}$$



Annuities with unknown number of payments

- Annuities that can be split into different smaller annuities
- We could treat it as one large annuity, but we may not have all the information needed (Typically, this is used for questions where n is not stated)
- But a faster method would be to treat each section as its own Annuity (Based on what information we know) and then discount them accordingly:

$a_{3n\rceil\, i} = a_{n\rceil\, i} + v^{n} * a_{n\rceil\, i} + v^{2n} * a_{n\rceil\, i}$
$a_{Xn\rceil\, i} = a_{n\rceil\, i} + v^{n} * a_{n\rceil\, i} + v^{2n} * a_{n\rceil\, i} + \cdots v^{Xn} * a_{n\rceil\, i}$

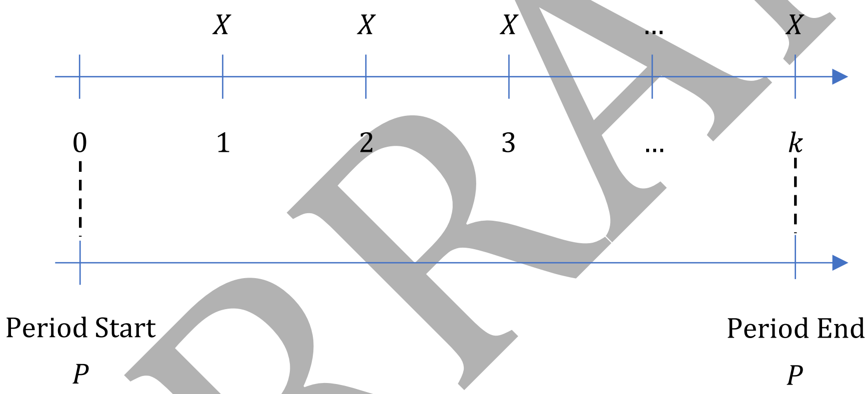


Matching Payment and Interest frequencies

- For us to use ALL of the above annuity formulas, the rates we use MUST MATCH the frequency of the payments (EG. Monthly interest with Monthly Payments)
- If they are not in sync, we have two methods to make them sync
    - **Converting Interest Rates**
        - Using the principle of Equivalent Rates (Chapter 1)
        - Generally preferred, but not applicable for symbolic questions
    - **Converting Payments**
        - Lower to higher payment frequency → Expand payments
        - Higher to lower payment frequency → Collapse payments

Lower to Higher payment frequency

- Payments occur **less frequently** than the effective rate given
    - EG. Quarterly Payments but Monthly Interest → **Convert to monthly payment**
- We can split the payments into k smaller payments of X to be aligned with the given rate
- We must keep the PV/FV of these payments to be equal to the original payment. We can use the level annuity formulas to calculate the smaller payment X
- Since X matches the given rates, we can use both in our annuity formulas

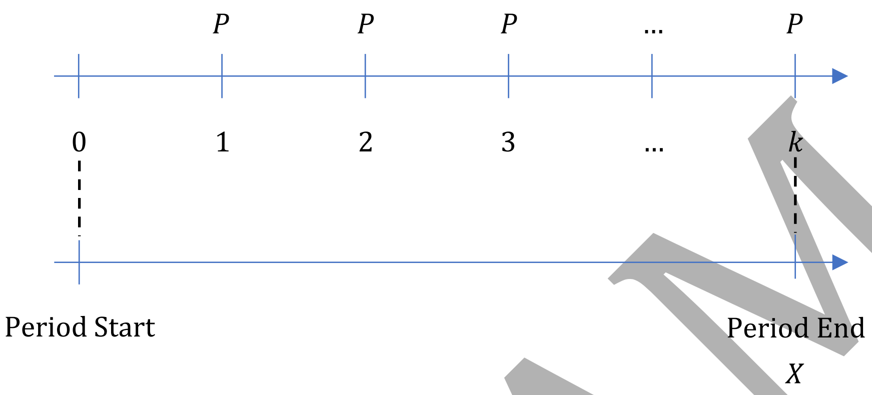


Deriving Formulas

Payment at the start of period	Payment at the end of Period
$P = X * a_{\overline{k\vert}\,i}$ $X = \frac{P}{a_{\overline{k\vert}\,i}}$	$P = X * s_{\overline{k\vert}\,i}$ $X = \frac{P}{s_{\overline{k\vert}\,i}}$
$PV = X * a_{\overline{n\vert}\,i}$ $PV = \frac{P}{a_{\overline{k\vert}\,i}} * a_{\overline{n\vert}\,i}$ $PV = P * \frac{a_{\overline{n\vert}\,i}}{a_{\overline{k\vert}\,i}}$	$PV = X * a_{\overline{n\vert}\,i}$ $PV = \frac{P}{s_{\overline{k\vert}\,i}} * a_{\overline{n\vert}\,i}$ $PV = P * \frac{a_{\overline{n\vert}\,i}}{s_{\overline{k\vert}\,i}}$
$FV = X * s_{\overline{n\vert}\,i}$ $FV = \frac{P}{a_{\overline{k\vert}\,i}} * s_{\overline{n\vert}\,i}$ $FV = P * \frac{a_{\overline{n\vert}\,i}}{a_{\overline{k\vert}\,i}}$	$FV = X * s_{\overline{n\vert}\,i}$ $FV = \frac{P}{s_{\overline{k\vert}\,i}} * s_{\overline{n\vert}\,i}$ $FV = P * \frac{s_{\overline{n\vert}\,i}}{s_{\overline{k\vert}\,i}}$

Higher to Lower payment frequency

- Payments occur more frequently than the effective rate given
    - EG. Monthly payments but yearly interest given → Convert to Yearly payment
- We can combine k smaller payments into one payment of X be aligned with the given rate
- We must keep the PV/FV of these payments to be equal to the original payment. We can use the level annuity formulas to calculate the larger payment X
- Since X matches the given rates, we can use both in our annuity formulas



Deriving Formulas

We first show the equivalent rates:

$$\left(1 + i^{(m)}\right)^k = (1 + i)$$

$$i = \left(1 + i^{(m)}\right)^k - 1$$

We then accumulate to the end of the period:

$$X = P * s_{\overline{n|}\, i}$$

$$X = P * \frac{\left(1 + i^{(m)}\right)^k - 1}{i^{(m)}}$$

$$X = P * \frac{i}{i^{(m)}}$$

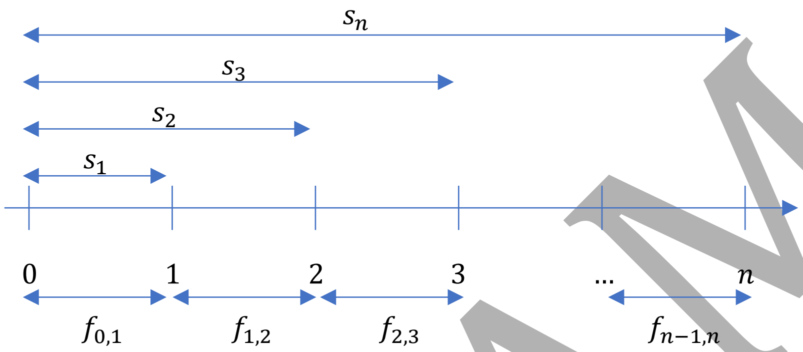
$$PV = X * a_{\overline{n|}\, i}$$

$$PV = P * \frac{i}{i^{(m)}} * a_{\overline{n|}\, i}$$

Chapter 3: Spot, Forward and Interest Swaps

Spot & Forward Rates

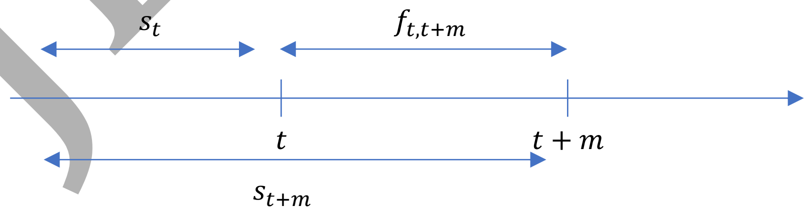
- Rates offered are likely to change in accordance with the **investment horizon.** Different periods of investment will result in different rates
- There are two main kinds of rates:
  - **Spot** (s_t)→ From **today to some future date** $(\mathbf{0}\ \boldsymbol{to}\ \boldsymbol{t})$
  - **Forward** $(f_{t-1,t})$→ From a **future date to 1 year ahead** $(\boldsymbol{t-1}\ \boldsymbol{to}\ \boldsymbol{t})$



Linking the two rates

- Both rates are quoted **Annual Effective.** This is important for Spot rates, as it spans across more than one period:
  - $a(t) \neq (1+s_t)$
  - $a(t) = (1+s_t)^t$
- Using the concept of Equivalent Rates, we can convert between the two by:
  - $(1+s_t)^t = (1+s_{t-1})^{t-1}(1+f_t)$
  - **OR** $(1+s_t)^t = (1+f_1)(1+f_2)\dots(1+f_t)$
  - By definition, $s_1 \equiv f_1$
- Using these formulas, we can form the accumulation function and solve cashflow problems using first principles by manually accumulating/discounting every cashflow
- Forward Rates calculated this way are known as the **Implied Rates.** They are an **estimate** of what will be the **actual 1-year spot rate during that time**, known as the Quoted Rates. Where possible, we should use the Quoted Rate over the implied forward rates.

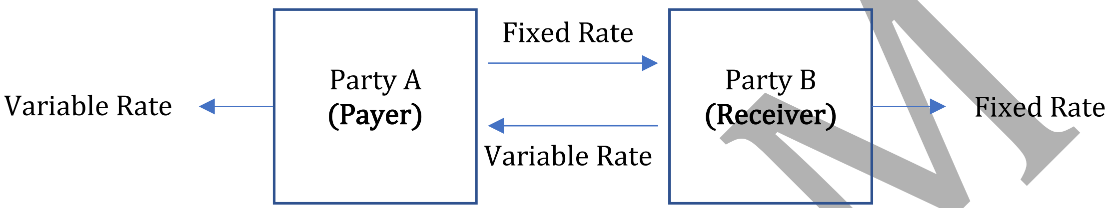
Forward rates lasting more than one period



- They are like spot rates in the sense that they last more than one period. Thus, we can use the same principle of equivalent rates to calculate them:
  - **Using spot rates:** $(1+s_{t+m})^{t+m} = (1+s_t)^t * \left(1+f_{t,t+m}\right)^m$
  - **Using forward rates:** $\left(1+f_{t,t+m}\right)^m = \left(1+f_{t,t+1}\right) * \dots \left(1+f_{t+m-1,t+m}\right)$
- Sometimes, we are given forward rates for cashflows that occur before the forward rate period. This means that we should assume the cashflow is **reinvested** at that forward rate and we should find the value after reinvestment

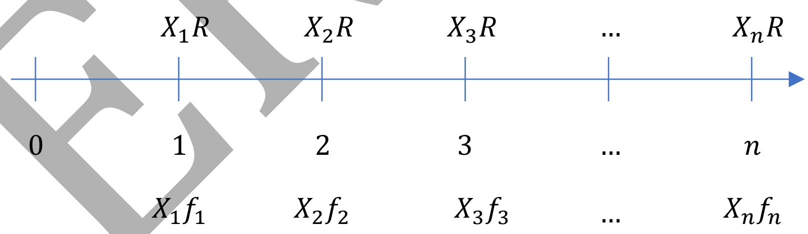
Interest Rate Swaps

- Two **counterparties** currently have two kinds of liabilities with different style of interests:
    - Party A → Variable Interest
    - Party B → Fixed Interest
- Each party would now like the other style of payment for various reasons:
    - Hedge against interest rates
    - Predictability of payments
- One solution would be to **Swap their interest payments.** However, since they can't directly swap their contract, they can achieve a similar outcome by paying for the other party:
    - Party A will **pay the fixed rate** to party B (**Payer**)
    - Party B will pay the variable rate to party A (**Receiver**)



Swap Mechanics

- Fixed payments will be made at a **constant Swap Rate** ($\boldsymbol{R}$) at **periodic intervals** known as the **Settlement Period**
- Variable payments will be made at the prevailing rates at the time, modelled by using the current **Forward Rates** ($\boldsymbol{f_n}$)
- Since both parties may not have identical principals, the payments are made according to some common **Notional Amount** ($\boldsymbol{X_n}$)
    - Constant Notional → **Level Swap**
    - Increasing Notional → **Accreting Swap**
    - Decreasing Notional → **Amortizing Swap**



To ensure that the deal is fair, the **PV of both Cashflows** must be the same:

$$PV(Fixed\ Payments) = PV(Variable\ Payments)$$

$$\frac{X_1R}{(1+s_1)} + \cdots \frac{X_nR}{(1+s_n)^n} = \frac{X_1f_1}{(1+s_1)} + \cdots \frac{X_nf_n}{(1+s_n)^n}$$

Deferred Swaps

- Similar concept to a **Deferred Annuity.** However, when an SOA says $\boldsymbol{n}$**-year deferred**, $\boldsymbol{m}$**-year** interest rate swaps, the actual swap occurs for $\boldsymbol{n-m}$ **years** after $\boldsymbol{n}$ years
- In other words, the **swap period defined by SOA includes the deferral as well**

Level Swap Shortcuts

- There are two shortcuts available for **Level Swaps.** If the swap is Accreting or Amortizing, we will have to perform the calculations manually
- **The first shortcut is factoring out X**:
  - If the swap is level, $X_1 = X_2 = \cdots = X_n$, which allows it to be factored out from both sides of the equation and cancelled
  - This simplifies the calculation to just using the interest rates
- **The second is to use the Bond pricing formula:**
  - This method uses an entirely different approach to calculating the swap rate
  - We **consider the net cashflows for Party A**, including their current obligation. We notice that the resulting cashflows looks very similar to that of a bond, thus we can make use of that formula to calculate R as well!

Diagram 1

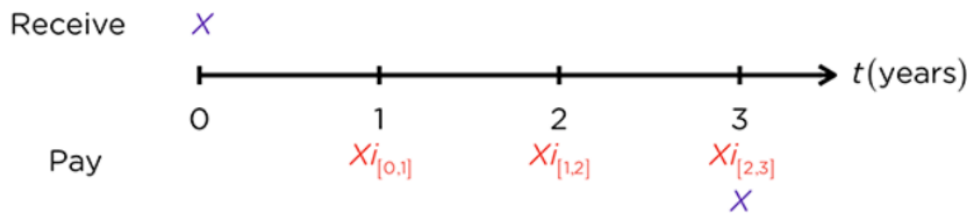


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Diagram 2

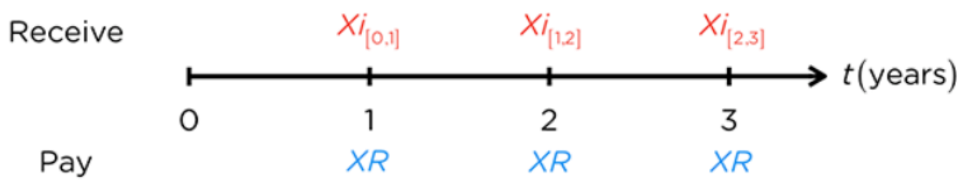

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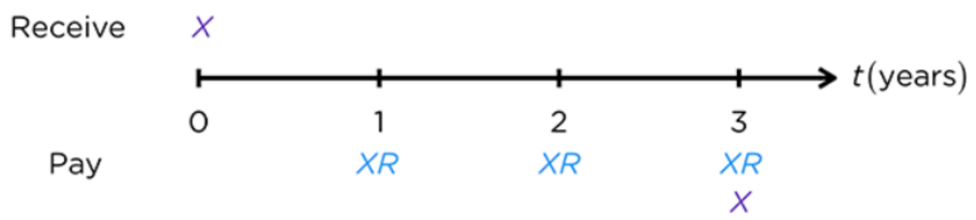
Diagram 3



Bond Pricing Formula:

$$Price\ of\ Bond = PV(Coupon\ Payments) + PV(Principal)$$

$$X = \frac{XR}{(1+s_1)} + \cdots + \frac{XR}{(1+s_n)^n} + \frac{X}{(1+s_n)^n}$$

Since level swap, we can factorise & cancel out X:

$$1 = R\left(\frac{1}{(1+s_1)} + \cdots + \frac{1}{(1+s_n)^n}\right) + \frac{1}{(1+s_n)^n}$$

If it is a **deferred swap**, we all we need to do is to **discount the LHS to the first period** as well.

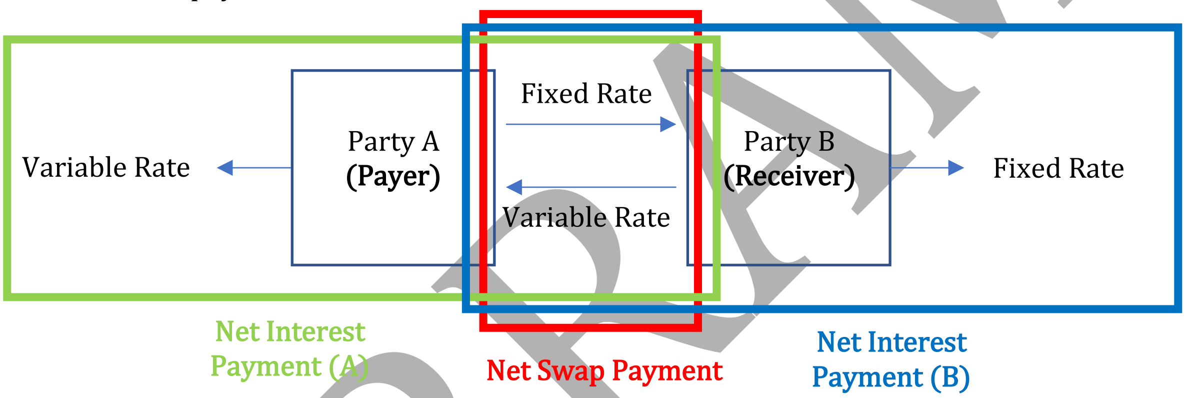
Other Swap Calculations

Net Swap Payments

- Since both parties pay each other on the same day, it is redundant to make payments
- Instead, it is more efficient for one party to just make a **Net Swap Payment** instead
- It is important to **consider whose perspective we are taking**: An outflow for one party is an Inflow for another

Net Interest Payment

- We want to consider how much one party pays in total on the settlement date
- There are two cashflows from their perspective:
  - Net Swap Payment
  - Original Interest Payment
- The Net Interest Payment is simply the sum of these two cashflows, which is how much they end up paying in total
- Similarly, it is also important to consider whose perspective when computing the net interest payment



Market Value

- One may choose to leave the position before the end of the Swap Term
- In order to sell it to another person, we must know the market value of the contract
- We can calculate it by computing the **Net Present Value of all future Net Swap Payments**
- By definition, the Market Value of a contract at time 0 will be 0 because $\boldsymbol{NPV = 0}$
  - However, due to interest rates changing (Our projected forward rate not being the actual rate come the appropriate time), the **market value of the contract at some later time is not likely to be 0**
  - We should always use the prevailing **Spot Rates** at the time we are calculating the market value instead of the forward rates we predicted in the past
- It is important to consider whose perspective we are looking at, as one person's gain is another person's loss

Why interest rate swaps?

- Typically, the payer of the swap rate currently has an existing loan with interest payments based on some floating rate
- The **payer** believes that interest rates will rise, hence wants to pay a fixed swap rate to **hedge against these rising interest rates**
- The receiver believes that interest rates will fall, hence is willing to pay the lower variable rate

Chapter 4: Rates of Return

One Period Return Rates

Time Weighted Rate of Return (TWRR)

Let R be sub-period return rate:

$$R_n = \frac{Balance\ before\ new\ transaction}{Balance\ after\ previous\ transaction} = \frac{B_n^{Before}}{B_{n-1}^{After}}$$

Using the concept of equivalent rates, we can solve for the TWRR
$(R_1)(R_2)\dots(R_n) = (1 + i_{TWRR})$

Balance **before** transaction	0	B_1^{Before}	...	B_{n-1}^{Before}	B_n^{After}
Transaction (Injection/Withdrawal)	Initial Deposit	First Transaction	...	Final Transaction	
Balance **after** transaction	B_0^{After}	B_1^{After}	...	B_{n-1}^{After}	
Sub-period return rate		R_1	R_2	R_{n-1}	R_n

Dollar Weighted Rate of Return (DWRR)

B_{Start}	CF_1	CF_2	CF_3	...	B_{end}
0	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	...	1

DWRR is essentially the **IRR of the fund**:

$$B_{Start}(1 + i_{DWRR})^1 + CF_1(1 + i_{DWRR})^{\frac{11}{12}} + \cdots + CF_n(1 + i_{DWRR})^{\frac{(1-n)}{n}} = B_{End}$$

- However, as seen earlier, IRR is **numerically complicated** to compute
- Thus, we consider a **Simple Interest Approximation** instead:

$$B_{Start}(1 + i_{DWRR}) + CF_1\left(1 + \frac{11}{12} i_{DWRR}\right) + \cdots + CF_n\left(1 + \frac{(1-n)}{n} i_{DWRR}\right) = B_{End}$$

$$i_{DWRR} = \frac{B_{End} - B_{Start} - \sum CF_t}{B_{Start} + \sum CF_t(1-t)}$$

- **For time spans of exactly one-year**, this simple interest approximation will be equivalent to the one-year IRR of the fund
  - Can be verified using Desmos by plotting $y = (1 + i_{IRR})^x$ and $y = (1 + i_{DWRR}x)$
  - For any $i_{IRR} = i_{DWRR}$, the intersection point will always occur at $x = 1$
  - Phrased another way, as long as $x = t = 1$, $i_{DWRR} = i_{IRR}$
- Notice that this formula is extremely **similar to the regular interest formula**. This means that **if the cashflows are sufficiently small**, the simple interest approximation method of DWRR can be used to estimate the true annual effective rate

Note: Investment Income refers to the interest earned on all the cashflows during the period.

Differences between each measure:

- **TWRR** makes use of **Compound Interest** while **DWRR** uses **simple interest approximation**
- **TWRR** is a measure of the **Fund Managers Performance**, while **DWRR** is a measure of the **Overall Performance of the Fund**
  - TWRR removes the effect of the timing and additional transactions. This is similar to how a **fund manager does not control over when they receive funds and how much**, thus is used to approximate the fund manager performance
- If DWRR $>$ TWRR, it means that the **timing** of deposits/withdrawals was very good – **deposited right before a high growth period** after or **withdrawn before a crash**

Different Timespans

- The above formulas use timespans of **one year** to compute the rates
- Thus, the calculated TWRR and DWRR are naturally assumed to be annual 1-year rate
- However, some questions may specify a **different timespan** – 6 months or 2-year
  - The most important thing is to MAINTAIN the formula
  - For TWRR, maintain the LHS as just $(1 + i)$ and for DWRR, keep the coefficients of i to be under 1 (Proportion till the end of the given time)
- This way, the rates calculated will be the 6-month or 2-year rate
- Using the **concept of equivalent rates**, we will need to convert these 6-month or 2-year rates into annual effective 1-year rates (**True TWRR/DWRR)**

Multiple periods return rates

Arithmetic Mean

- Given multiple one period returns $(R_1 \dots R_n)$, the arithmetic mean rate of return is the average of all of them
- It is a measure of the **average fund performance if you were to hold it for one year.** Some say that this measure is **overly optimistic for long time horizons**

$$R_{Arithmetic} = \frac{R_1 + \cdots R_n}{n}$$

Geometric Mean (AKA Cumulative Annual Growth Rate – CAGR)

- Given multiple one period returns $(R_1 \dots R_n)$, the geometric mean rate of return is the annualized effective rate of all of them
- It is a measure of the **expected annualized rate if you were to hold it for n years.** Some say that this is **overly pessimistic for short time horizons**

$$(1 + R_1) \dots (1 + R_n) = (1 + R_{Geometric})^n$$

Important: To find the variance of these return rates, we should use the **Sample Variance formula**:

$$s^2 = \frac{1}{n-1}\sum(r_i - \bar{r})$$

Discounted Cashflow Analysis

Net Present Value (NPV)

- It is the **difference between the PV of Cash Inflows and Outflows**. It thus measures the raw value created for the firm
- $NPV = PV(Inflows) - PV(Outflows)$

Internal Rate of Return (IRR)

- The required **interest rate that makes the NPV of a project zero**. It represents the annual rate of return that will be earned
- $NPV = 0 \rightarrow Solve\ for\ i\ using\ Financial\ Calc$
    - Can also be used to **solve quadratic equations** involving v

Other Measures of Return (FNCE101)

Dollar Returns

- A measure of the **absolute return** on an investment over a period of time
- Poor measure of return since there is no indication if it is high or low

$$Dollar\ Return = Income\ from\ Investment + Capital\ Gain\ or\ Loss$$

Percentage Return

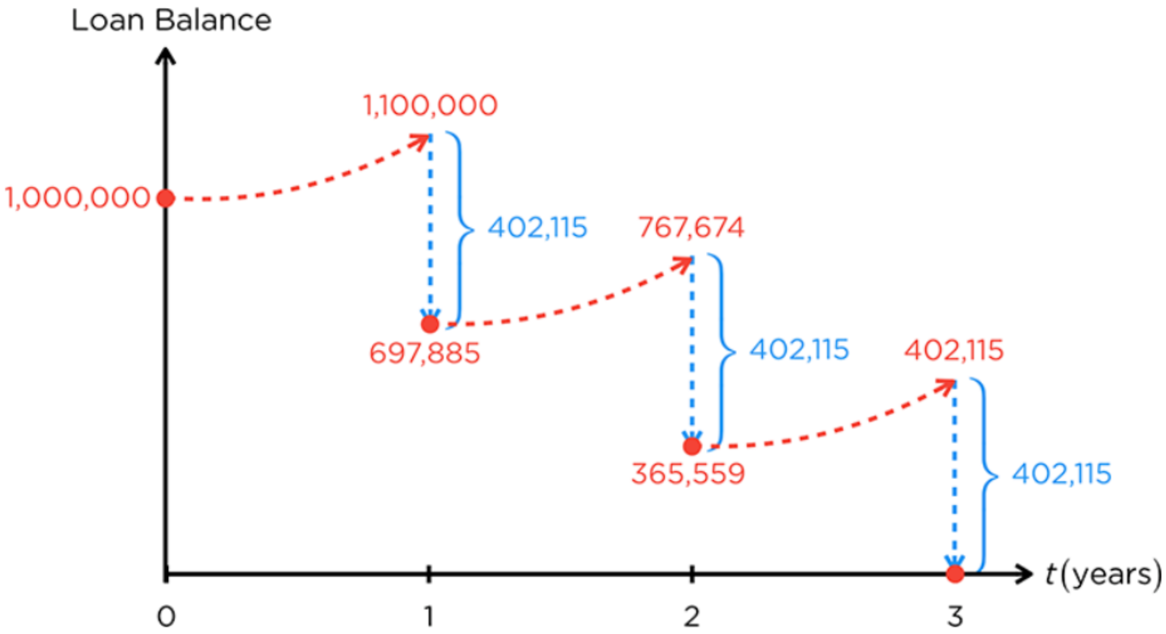
- Measure of the **percentage return** on an investment over a period of time
- Better measure as it considers the beginning values of the assets

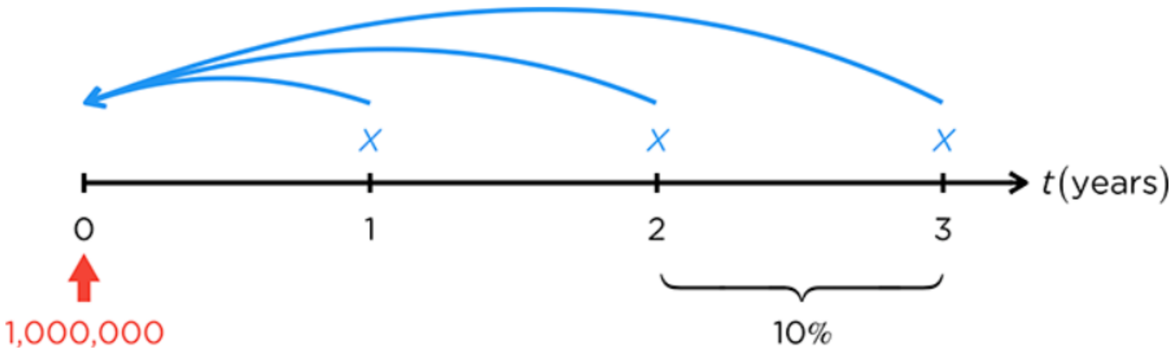
$$Percentage\ Return = Dividend\ Yield + Capital\ Gain\ Yield$$
$$Percentage\ Return = \frac{Dividend}{Beginning\ Price} + \frac{Capital\ Gain\ or\ Loss}{Beginning\ Price}$$

Chapter 5: Loans

What is a loan?

- When we borrow money from the bank or another company, we are taking a **Loan**
- The **outstanding loan balance earns interest**, increasing the amount we have to pay for each period we do not fully repay the loan
- We consider the standard scenario where loans are **repaid at level amounts at regular intervals** (Forming a Level Annuity)
    - Like all other problems we have dealt with till now, the key principle is that the $\boldsymbol{PV(Loan\ Amount) = PV(Loan\ Payments)}$
- There are two components to each loan repayment:
    - Interest repaid
    - Principal repaid
- Each repayment made will **first pay off any interest earned so far**, and the **remaining amount will pay off the principal**
- We are interested in calculating the not only the repayment amount but the **various components of the loan** at each time to better understand the status of Debt





Loan Calculations

Loan Variables

- B_t to denote the **Outstanding Balance,** where B_0 is the principal
- R_t to denote the **Repayment Amount** (Assumed to be Constant)
- I_t to denote the amount that was used to pay off **Interest**
- P_t to denote the amount that was used to pay off the **principal**

Three ways to Calculate Outstanding Loan Balance

Recursive Method

Using first principals and manually calculating the change in each period:

$B_t = B_{t-1} - P_t$
$B_t = B_{t-1} + I_t - R$
$B_t = B_{t-1} + B_{t-1} * i - R$
$B_t = B_{t-1}(1 + i) - R$

Applying recursion,

$B_t = (B_{t-2}(1 + i) - R)(1 + i) - R$
$\vdots$
$B_t = B_0(1 + i)^t - R - \cdots R(1 + i)^{t-1}$
$B_t = B_0(1 + i)^t - R * s_{\overline{n|}\,i}$

Retrospective Method

The outstanding balance is the Time adjusted **difference of the principal & what was already paid**:

$B_t = B_0(1 + i)^t - R * s_{\overline{n|}\,i}$

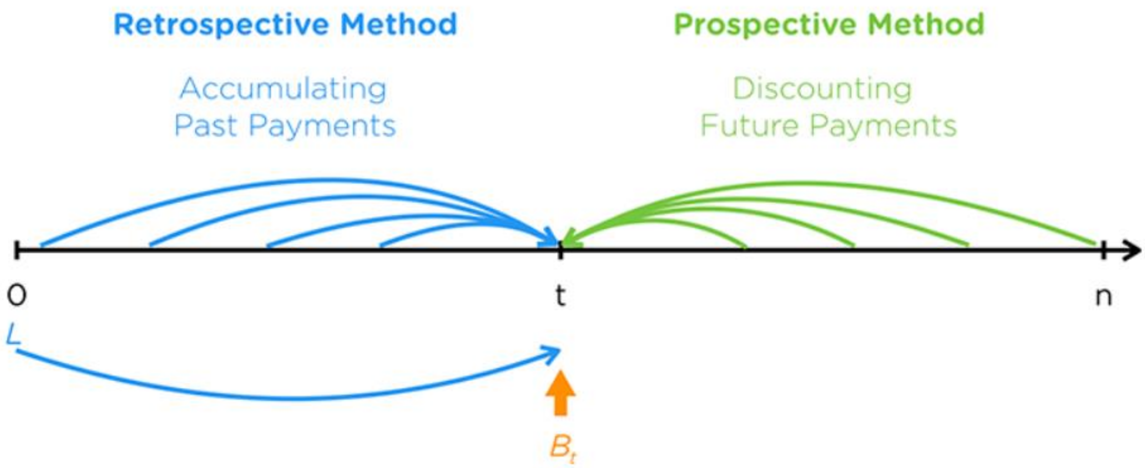
- Notice that this is the **same result as the Recursive Method**; they are the same

Prospective Method

The outstanding balance is the **PV of all future loan payments**:

$B_t = R * a_{\overline{n-t|}\,i}$

- Note that this method can also be used if we have a future loan balance as well
- By discounting the **Future Loan Balance** and **any payments between then and the current time**, we obtain the Current Loan Balance



All three methods are equivalent and will lead to the same results. We need to know all three because it **depends on what information the question gives us.** It is faster to use a particular method if the question has already given us the relevant information.

Loan Amortization

- In finance, Amortization means to **spread payments** (of a Loan, in this case)
- The goal is to create an **Amortization Schedule** that shows how each **Repayment** is used to pay off both the **Interest** and **Principal in each period**, slowly reducing our **Outstanding Balance** to 0 at the end of the Loan
- For simplicity, we factor out all payments to 1 (Similar to our Annuity Proofs). Remember to **multiply by the payment amount** for actual questions

Recursive/Retrospective Schedule

- Using the Recursive Method, we can determine each component in any given period
- However, this method is **best done in Excel** when the calculations can be performed quickly. For the exam, this **may not be the best method**

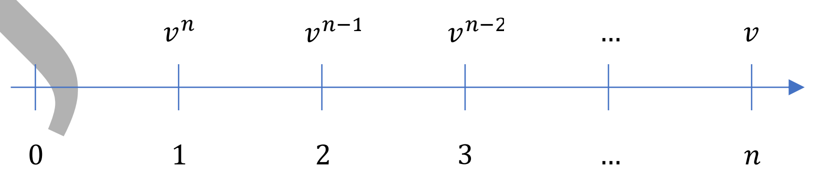
Period	Repayment	Interest Repaid $I_t = B_{t-1} * i$	Principal Repaid $P_t = R_t - I_t$	Loan Balance $B_t = B_{t-1} - P_t$
0				B_0
1	1	$B_0 * i$	$1 - B_0 * i$	$B_0(1+i) - 1$
⋮	⋮	⋮	⋮	⋮
n	1	$B_{n-1} * i$	$1 - B_{n-1} * i$	$B_0(1+i)^t - s_{\overline{n}|}$
Total	n	ΣI_t	$n - \Sigma I_t$	

Prospective Schedule

- Using the prospective method instead, we notice that the Loan Balance forms an Annuity
- Using this Annuity formula, we can see how the schedule becomes simplified:

Period	**Repayment** Factor to 1	**Interest Repaid** $I_t = B_{t-1} * i$ $I_t = 1 - v^{n-t+1}$	**Principal Repaid** $P_t = 1 - I_t$ $P_t = v^{n-t+1}$	**Loan Balance** $B_t = a_{\overline{n-t}|\,i}$
0				$a_{\overline{n}|\,i}$
1	1	$1 - v^n$	v^n	$a_{\overline{n-1}|\,i}$
⋮	⋮	⋮	⋮	⋮
n	1	$1 - v$	v	$a_{\overline{0}|\,i}$
Total	n	$n - a_{\overline{n}|\,i}$	$a_{\overline{n}|\,i}$	

- From this insight, we can see that **all we need is the interest rate** and **repayment** amount to identify any component
- If we are given the **principal repaid in one period**, we can use the **accumulate or discount** it accordingly to find the principal repaid in another
- Principal Payments are proportional – $\frac{Principal_t}{Principal_{t+2}} = \frac{Principal_{t+n}}{Principal_{t+n+2}} = v^2$
- The principal payments can also be viewed in a **timeline:**



Varying Interest Rates

- If interest rates change midway, the **remaining schedule has to be changed** as well
- In particular, we have to **recalculate the remaining number of payments needed.** Using this, we can calculate the remaining loan balance after the interest rate change
- However, the **shortcut method still works – just use the appropriate interest rates at the time when they change**

Other Payment Patterns

Level Principal Payments

- Instead of having level overall payments, this method of payment ensures that the **principle decreases by a flat amount** in each period
  - We can determine the amount at which it decreases by taking the **loan balance divided by the number of remaining payments**
- Given this information, we are able to **easily calculate the outstanding balance**, and thus the **Interest payment** and subsequently the **overall repayment amount each period**

$$Principal\ Repayment\ each\ period = \frac{B_0}{n}$$

$$Outstanding\ Balance\ each\ period = \frac{n-t+1}{n} B_0$$

$$Total\ Repayment\ each\ period = i\left(\frac{n-t}{n}\right) B_0 + \frac{B_0}{n}$$

- It is important to note that the **outstanding loan balance forms a decreasing arithmetic progression**. We can make use of this to easily calculate total interest and hence total amount paid:
  - $Total\ Interest = i * (B_0 + B_1 + B_2 + \cdots 0)$
  - $Sum\ of\ Arithmetic\ progression = \frac{N}{2}(First\ Term + Last\ Term)$
  - $Total\ Amount\ Paid = Total\ Interest + Loan\ Amount$
- **A few key insights:**
  - Since the **outstanding balance falls each period**, the interest falls each period as well. This means that the **repayments fall each period**
  - The repayment amount will be **initially higher** than the flat repayment amount, but will fall and become smaller than it
  - Since the total repayment amount falls over time, this means that you will **pay lesser compared to the typical method**

Non-Level Payments

- Payments that follow an **arithmetic or geometric annuity**
- Use their respective PV/FV formulas to **find the loan balance at every time**
- With the various loan balances, we should focus on using the **recursive method**:
  - $Balance_{t+1} = Balance_t - P_{t+1}$
  - $P_{t+1} = Balance_{t+1} - Balance_t$
- Questions may look the same, but the **approach is completely different**
- If the question has a **balloon/drop payment involved**, always calculate the remaining balance in the **period before the balloon/drop**. Then **ROLL FORWARD the balance** to obtain the Ballooned or Dropped Payment

Missing or Additional Payments

- The general schedule above assumes that payments are made every period. However, if **payments are missed or additional payments are made**, then the Loan will take **longer/shorter to amortize**
  - If payments are **missed**, then the Loan balance at the end of the usual loan period will still be positive → **How much?**
  - If **additional payments** were made, then the Loan Balance will hit 0 before the usual loan period → **How Much/When?**
- In either case, we can easily visualize the problem by **drawing out the timeline** to better visualize the problem

<u>Missing Payments</u>	<u>Additional Payments</u>
The key idea is that the **present/future value** will **NOT tally** with the original payment schedule. Using this idea, the **difference in the two values** is the remaining amount of the loan. A **key shortcut** understanding is that if payments are missed, the outstanding Loan Balance will be the **sum of the accumulated value of any missing payments** Using the same concept, if we are given the Outstanding Balance at various points instead, we can **roll forward the loan balance assuming no payments were made.** We then use **Retrospective Method** to calculate the remaining Loan Balance instead.	The key idea is that the **Present/future value MUST tally** with the original payment schedule. Using this idea, we can form two equations and **set them equal** to each other to **solve for the missing value** we are interested in. Given the additional payment amount, we can solve for the new ending time. Given the new ending time instead, we can solve for the additional payment amount.

Payments that scale with Interest

- Some loan payments will **scale based on the interest amount**
  - EG. Equal to 100% of interest payments. 150% of interest payments etc.
- There are some special insights to take note of:
  - **100% of interest** → No principal is repaid; loan balance stays the same
  - **150% of interest** → Balance falls by $(150 - 100)\% * Interest$ every period
  - **90% of interest** → Balance increases by $(100 - 90)\% * Interest$ every period

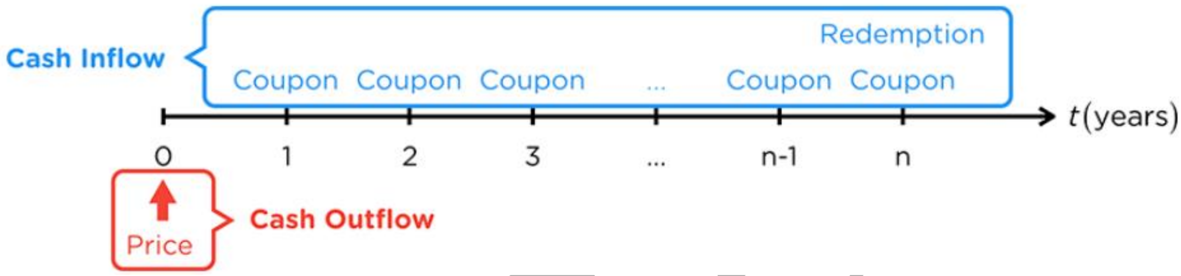
Other situations:

- **Different Interest Rates for Interest payments and discount** → Draw timeline
- **Cost of loan** → Reduces the principal amount to control the yield rate

Chapter 6: Bonds and Stocks

What is a Bond?

- A bond is a special type of **Loan:** The party who issues the bond (Bond Issuer) borrows money from the party who purchases the bought the bond (Bond Investor)
- However, Bonds have a **different payment structure**:
  - **Every period → Coupon payments**
    - These coupon payments are based off interest on the **Face Amount/Par Value** of the bond, which is simply a notional amount for calculations
  - **Last period → Redemption Value**
    - If not specified otherwise, the Redemption value is **equal to the Face Amount/Par Value.** It is **paid on-top** of the coupon payment
- Thus, unlike a Loan who has Interest and Principal payments every period, Bonds repay the interest every period and **only repays the principal at the end in a lump sum**



Bond Valuation

Let us use the following notations:

- $\boldsymbol{P}$ is the **Price of the Bond**
- $\boldsymbol{r}$ is the **Coupon Rate** (Quoted Annual Nominal Rate)
- $\boldsymbol{F}$ is the **Face Amount**
- $\boldsymbol{C}$ is the **Call Value/Redemption Value**
- $\boldsymbol{i}$ is the interest rate equivalent to **Yield to Maturity** (Quoted Annual Nominal Rate)

$$Bond\ Price = PV(Coupons) + PV(Redemption\ Value)$$
$$\therefore \boldsymbol{P = Fr * a_{\overline{n}|\,i} + C * v^n}$$

Dirty Prices

- When valuing a Bond, we typically assume that the bond is being valued **6-months or 1-year intervals** from the issue date. Prices valued this way are known as the **Quoted Price**
- However, if we purchase a Bond, it is likely that we will buying **it in between valuation dates.** We will thus have to adjust the price, known as the **Dirty Price**
- The idea is that we must **compensate the previous bondholder** for the additional time after the previous coupon payment by paying him a **corresponding portion of the Coupon**

$$\boldsymbol{Dirty\ Price} = Quoted\ Price + Accrued\ Interest$$
$$\boldsymbol{Accrued\ Interest} = \frac{Number\ of\ Days\ since\ Coupon}{Number\ of\ days\ between\ Coupons} * Coupon$$

$$\boldsymbol{Alternative\ Dirty\ Price} = Quoted\ Price * (1+i)^{\frac{Number\ of\ Days\ since\ Coupon}{Number\ of\ days\ between\ Coupons}}$$

Bond Premium & Discount

We recall the Annuity Immediate formula:

$$a_{n\rceil i} = \frac{1 - v^n}{i}$$
$$v^n = 1 - i * a_{n\rceil i}$$

Using this formula, we can **rewrite our Bond Price formula:**

$$P = Fr * a_{n\rceil i} + C * v^n$$
$$P = Fr * a_{n\rceil i} + C * (1 - i * a_{n\rceil i})$$
$$P = Fr * a_{n\rceil i} + C - Ci * a_{n\rceil i}$$
$$P = C + (Fr - Ci) * a_{n\rceil i}$$

- Where $(Fr - Ci) * a_{n\rceil i}$ is the Premium/Discount of the Bond

We consider the difference in the Price of the Bond & the Redemption Value $P - C$,

Premium Bond	Discount Bond	Par Bond
$P - C > 0$ $(Fr - Ci) * a_{n\rceil i} > 0$ $(Fr - Ci) > 0$ $Fr > Ci$ $\boldsymbol{r > i}$	$P - C < 0$ $(Fr - Ci) * a_{n\rceil i} < 0$ $(Fr - Ci) < 0$ $Fr < Ci$ $\boldsymbol{r < i}$	$P - C = 0$ $(Fr - Ci) * a_{n\rceil i} = 0$ $(Fr - Ci) = 0$ $Fr = Ci$ $r = i$
Coupon payments received are **larger than the expected amount they would receive from a similar par bond.** This is additional **periodic income**, which is why the bond costs more than par.	Coupon payments received are **smaller than the expected amount they would receive from a similar par bond.** This is **insufficient income**, which is why the bond costs less than par.	Coupon payments received are **equal to the amount expected,** thus the Bond has no premium or discount. $\boldsymbol{Price = Face = Call}$

Bond Amortization

- There are some **terminology differences** between Loan and Bonds:
  - **Coupon Payment** → Repayment
  - **Book Value** → Outstanding Balance
  - **Write-up/down or Discount/Premium Amortization** → Principal Repaid
    - Also known as the Accumulation of Discoumt/Premium
- The goal is to create an Amortization Schedule that shows how each **Coupon Payment** is used to **offset the Premium/Discount** in each period, slowly bringing the **Book Value to the Redemption Value** (Usually equal to Par Value) at the end of the Loan
- Similarly, each **coupon first pays off interest**, then the remaining amount is used to offset the premium/discount on the bond, which **decreases/increases the book value** to Par
- All types of questions that apply to loans **ALSO apply to Bond Amortization!**

Period	Coupon $F * r$	Interest Repaid $B_{t-1} * i$ $Fr + (Fr - Ci) * v^{n-t+1}$	Write-up/down $\lvert Fr - B_{t-1} * i \rvert$ $\lvert (Fr - Ci)v^{n-t+1} \rvert$	Book Value $C + (Fr - Ci)a_{n-t\rceil i}$
0				$C + (Fr - Ci)a_{n\rceil i}$
1	Fr	$Fr + (Fr - Ci) * v^n$	$\lvert (Fr - Ci)v^n \rvert$	$C + (Fr - Ci)a_{n-1\rceil i}$
⋮	⋮	⋮	⋮	⋮
n	Fr	$Fr + (Fr - Ci) * v$	$\lvert (Fr - Ci)v \rvert$	$C + (Fr - Ci)a_{0\rceil i}$
Total				

Special Type of Bonds

Zero Coupon Bonds (Pure discount Bonds)

- Bonds that do not pay coupons, only the redemption value at the back
- Due to only having one cashflow at the end of t periods, the yield of the zero-coupon bond is **essentially the Spot Rate** for that length of time. Thus, given the price of any zero-coupon bond, we can calculate the spot rate
- However, we **may not be given a zero-coupon bond directly**. Like the interest rate swap shortcut, **by buying and selling regular bonds in a portfolio**, the **net cashflows per period may be zero**, essentially creating a zero-coupon bond.

Floating Rate Bonds

- Bonds whose **coupon rates vary with some external index.** Their values rise when interest rises as well, thus they **suffer less interest rate risk**

Callable Bonds

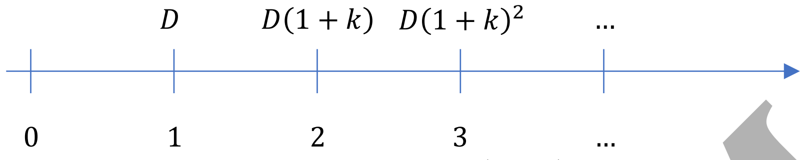
- Bonds where there is a **choice** to fully repay the bond on an earlier date instead
  - Choosing to repay the Bond earlier is known as **Calling the Bond.** The redemption value at the time is known as the **Call Price**
  - To ensure that the Investor is properly compensated for lost coupon payments if the Bond is called, the **Call Price is higher the earlier the Bond is called**
- **Callable bonds** are good for Issuers, as their flexible nature allows them to **take advantage of falling interest rates** (Issue new one at lower cost) and allows them **to eliminate obligations at will** with the main disadvantage being the **higher Coupon Rates**
- **Put Provisions** are good for Investors, but not issuer, as they may have to **buy back the bond at an unattractive price**, but they allow the issuer to use **lower coupon rates**

Scenario Testing

- Callable Bond **cashflows are uncertain** as they are at the discretion of the Bond issuer
- Thus, the main approach to work with Callable bonds is to **Scenario test – Consider every possible outcome** at form an analysis from that
- There are two main types:
  - **Given Price, determine Yield**
    - Use the Fin Calc to quickly reverse engineer the Yields for each scenario
    - We focus on the **Lowest Yield – Yield to Worst**
    - This represents the worst-case scenario for the investor
  - **Given Yield, determine Price**
    - Use the Fin Calc to quickly reverse engineer to price for each scenario
    - We focus on the **LOWEST Price – Maximum amount they should pay**
    - This represents the highest price an investor will have to pay to guarantee at least **Yield to Worst**, regardless which scenario ends up happening
      - Price and Yield have an **inverse relationship**
      - Thus, by choosing the lowest price, it means that even if another scenario were to occur, we would earn even higher than YTW
      - It is also known as the **maximum price**, because if we increase price beyond this point, other scenarios may be fine, but this scenario will lead to a situation worse than YTW, that means that overall YTW is no longer guaranteed
  - We can **eliminate scenarios to test by using the shortcuts:**
    - YTW always occurs at the **end points** – First & Last year that the **call price changes** (Including maturity)
    - YTW is the **First end date for Premium** & **Last for Discount Bonds** for **each group** of call dates with the same call price

Stock Valuation: Dividend Discount Model

- Like all other financial instruments, we have seen so far, we can value a stock by taking the **Present Value of its future cashflows (Dividends)**
    - If we **expect to sell the stock** in the future, we treat that as a cashflow as well
- Since corporate lifespans are infinite, these dividend payments can occur infinitely, forming a **perpetuity.** Depending on the type of stock, Dividends may remain constant or continue to grow every period, forming a **growing perpetuity**



$$PV = \frac{D}{i-k} = \frac{D_0(1+k)}{i-k}$$

- The math behind the calculation is a simple application of the growing perpetuity concept
- There are a few things to look out for:
    - Questions may give you the **current** dividends paid. Since the dividends grows, we can find the time 1 dividend by multiplying by the growth factor
        - Given the dividend or stock value at any one point of time, we can simply multiply it by the growth factor to determine the value at another time
    - If the growth rate changes midway, we consider a **two-stage model**, which simply means we now have an **Annuity and Deferred Perpetuity**

Chapter 7: Bond Yield & Term Structure of Interest Rates

Measures of Bond Yield

Current Yield & Nominal Yield

- Simple measure of the potential return of the bond – **How much** an investor would expect to make if they held the bond **for a year**
- However, it is not an accurate reflection of the potential gain of the bond:
  - It does not take into the time value of money
  - It does not consider capital gain or loss (Discount/Premiums)
  - It is based on Quoted Prices while the actual investment is Dirty Prices

$$Current\ Yield = \frac{\boldsymbol{Annual}\ Coupon\ Payments}{Quoted\ Price}$$
$$Nominal\ Yield = \frac{\boldsymbol{Annual}\ Coupon\ Payments}{Face\ value} = Coupon\ rate$$

Yield to Maturity

- When we price bonds, we are supposed to use **prevailing spot rates** at the time (Obtained from yield curve) for more accurate representation of the time value of money
- However, in practice the **term structure is not observable**, but the **transaction prices are**. Thus, if we solve for the discount rate that sets the discounted cashflows to the transaction price, we obtain the **Yield to Maturity**
- It is the **IRR for a bond** – which measures the **average annual rate** we would receive on the bond should we **hold it to maturity.** It is a sort of **"weighted average" of the spot rates**
- For this course, we assume the yield curve to be flat. Thus, the interest rate used to discount is constant, **thus is also the YTM** of the bond
- However, there are some issues with using YTM to evaluate Bonds:
  - There is **no closed form solution** for YTM. It can only be calculated using **numerical methods**
  - Although YTM is an "average" of the spot rates, it also changes with the coupon rate, which makes it an **"impure" measure**
  - It also **assumes** that the bond will be held to **maturity (Most are traded)** as well as that the **coupons are reinvested (Interest on Interest)** at the same rate (Hard)

Par Yield

- An alternative measure is par yield, which is the **coupon rate that makes the bond trade at par** under the current term structure **($\boldsymbol{Price = Face = Call}$)**
- However, it is an **inaccurate measure** – It is more of a **summary of the existing term structure** rather than measures of a potential return of a bond

Holding Period Yield

- Measure of the return we get by holding a bond for a period of time (Assuming it is redeemed at the end of the period)
- Based on the **idea of interest –** It takes the difference between the ending value and starting value, accounting for any additional income (Coupons) as well

$P_0(1+i)^t = P_t + FV(Coupons)$
Solve for i as the t-period holding period yield

Spot Rates and the Term Structure

- The plot of the various spot rates and against time is known as the **Yield Curve** & the mathematical relationship between the two is known as the **Term Structure of Interest Rates**
- Empirically, the yield curve has been shown to **take many shapes**:
    - Upward, Downward, Flat etc
    - Upward Yield Curves are the most common. However, for this course, we assume a **flat yield curve** unless stated otherwise (Spot rates are constant throughout)
- If the yield curve is not flat, we do have a few options to determine the yield curve:

Discretely Compounded Yield Curve

- It is called Discrete because it only calculates points of the yield curve at discrete points (Typically 6-month intervals)
- It uses the Prices of Bonds of various lengths to determine to reverse engineer the prevailing spot rates at the time
    - The first period bond **must be a zero-coupon bond**. All other bonds can be regular annual/semi-annual coupon bonds
- This method of calculating is also known as a **bootstrap method**:
    - Calculate the first spot rate using the zero-coupon bond
    - If the second bond has a coupon, **substitute in the spot rate calculated before** to discount it and then solve for the second spot rate
    - Repeatedly and recursively do this until all spot rates have been found
- If all Bonds were zero-coupon bonds, then this method would be a lot easier

Continuously Compounded Yield Curve

- We now consider **continuous interest** and **time points.** One key assumption we make is that the **force of interest is constant between two successive valuation dates**
- It is based on the concept of **spot rates and forward rates.** By combining an initial spot rate and multiple forward rates, we are able to obtain spot rates for any period
- We then form a **piecewise function** based on the time period to form the yield curve

Linking Spot Rates and Force of Interest:

$$e^{s_t * t} = e^{\int_0^t \delta_t}$$

$$s_t = \frac{1}{t}\int_0^t \delta_t$$

Recursive nature (Splitting integration)

$$e^{s_{t_2} * t_2} = e^{-\int_0^{t_2} \delta_{t_2}}$$

$$e^{s_{t_2} * t_2} = e^{-\int_0^{t_1} \delta_{t_1} + \int_{t_1}^{t_2} \delta_{t_2 - t_1}}$$

$$s_{t_2} = \frac{1}{t_2}\left(\int_0^{t_1} \delta_{t_1} + \int_{t_1}^{t_2} \delta_{t_2 - t_1}\right)$$

In practice, we solve for the various δ using bond prices:

- The same key concept of splitting the force of interest intervals applies

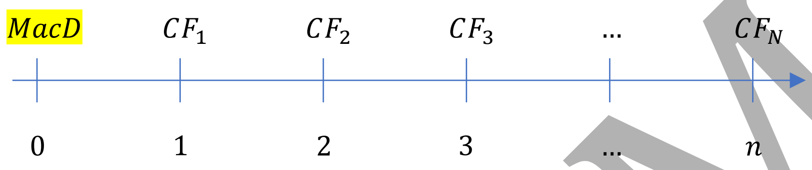
<u>Zero Coupon bonds</u>	<u>Regular Coupon Bonds</u>
$P_t = C * e^{-\int_0^t \delta_t}$	$P_t = (C + Fr) * e^{-\int_0^t \delta_t} + Fr * e^{-\int_0^{t-1} \delta_{t-1}}$

Chapter 8: Bond Management

Macaulay Duration (MacD)

- It is the **Weighted Average** of the **time taken in YEARS** for cashflows to occur, weighted based on the **relative size of the PV of future cashflows.** It calculates **average amount of time needed to get breakeven on your investment**
- Based on this, Cashflows that are **Larger** and **Occur Earlier** (Don't need to discount as much) will **influence the calculation more**. We can **use this intuition to check** our calculations to see if they make sense

Deriving Macaulay Duration



$$\therefore MacD = \frac{1 * PV + 2 * PV_2 + \cdots + n * PV_n}{PV_1 + PV_2 + \cdots PV_N} = \frac{\sum t * v^t * CF_t}{\sum v^t * CF_t} = -\frac{P'(\delta)}{P(\delta)}$$

- If all **cashflows are level** ($CF_1 = \cdots CF_n$), they can be **factorized out** from both the numerator and denominator, effectively cancelling out
- Building on this, consider two sets of Cashflows:
    - Set A: 1, 2, 3, 4…
    - Set B: 100, 200, 300, 400…
- We can factorise out 100 from Set B, which will cancel out and result in the same cashflows as Set A. Thus, MacD is **dependent not on the size but the pattern of cashflows**

Other Time Frequencies

- The typical duration formulas use full years (1,2,3…)
- However, some questions (especially semi-annual bonds) have cashflows semi-annually
- We can treat the semi-annual timings as a full year, then **convert it back to half years at the end!**
- Alternatively, we can **treat them as 0.5 right from the get-go**

Macaulay Duration Shortcuts

Zero-Coupon Bond (Single Cashflow)	n
Level Annuity (Loans)	$\frac{(Ia)_{\overline{n}\vert i}}{a_{\overline{n}\vert i}}$
Standard Bond*	$\frac{Fr * (Ia)_{\overline{n}\vert i} + nCv^n}{Fr * a_{\overline{n}\vert i} + Cv^n}$

*Note that a **deferred Bond** cannot use this formula directly; an adjustment will have to be made

Par Bond ($P = F = C, i = r$)	**Common Stock (Increasing Perpetuity)**
$MacD = \frac{Fr * (Ia)_{\overline{n}\vert i} + nCv^n}{Fr * a_{\overline{n}\vert i} + Cv^n}$ $MacD = \frac{i * (Ia)_{\overline{n}\vert i} + nv^n}{i * a_{\overline{n}\vert i} + v^n}$ $MacD = \frac{\ddot{a}_{\overline{n}\vert i} - nv^n + nv^n}{1 - v^n + v^n}$ $MacD = \ddot{a}_{\overline{n}\vert i}$	$P = (i-k)^{-1}$ $P' = -(1-k)^{-2}$ $ModD = -\frac{P'}{P} = -\frac{(i-k)^{-2}}{-(1-k)^{-1}} = (i-k)^{-1}$ $MacD = ModD * (1+i) = \frac{1+i}{i-k}$

Modified Duration

- As its name suggests, ModD is simply a **modification of MacD**
- The longer we must wait to get back our money, the larger the window for those cashflows to be affected by interest rates hence higher sensitivity to them
- Thus, we can modify MacD to directly approximate the **Sensitivity of the change in price of a Bond to changes in Interest Rates.** Thus, ModD measures the **percentage decrease in the value of the security per unit increase in interest rates (Price Risk)**

Deriving Modified Duration

$$Price\ of\ a\ Security, P(i) = \sum v^t * CF_t = \sum (1+i)^{-t} * CF_t$$

$$Changes\ in\ Price, P'(i) = \sum -t * (1+i)^{-t-1} * CF_t = \sum -t * v^{t+1} * CF_t$$

$$\therefore ModD = -\frac{P'(i)}{P(i)} = -\frac{\sum -t * v^{t+1} * CF_t}{\sum v^t * CF_t} = \frac{\sum t * v^{t+1} * CF_t}{\sum v^t * CF_t} = v * \frac{\sum t * v^t * CF_t}{\sum v^t * CF_t} = \frac{MacD}{1+i}$$

- If interest compounds continuously, we can show a similar result for MacD as well
- Note that if **no specification** was made, Duration refers to **Macaulay Duration**

Portfolio Duration

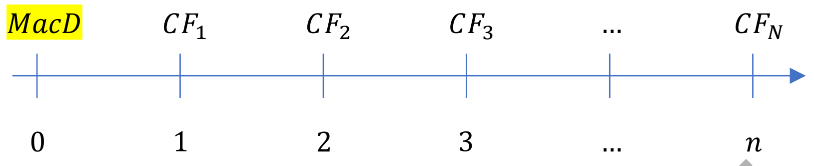
- Consider a portfolio of Bonds. We can **aggregate the cashflows** for the entire portfolio and then **calculate the Duration using the same method**
- However, this can be time consuming. If we already know the **individual Durations** for each component, we can quickly calculate the Duration of the entire portfolio by using their **relative weights** (Similar to Modern Portfolio Theory)
- Important: The weights MUST be **time-value adjusted**

$$MacD_{Portfolio} = w_X * MacD_X + w_Y * MacD_Y + w_Z * MacD_Z + \cdots$$

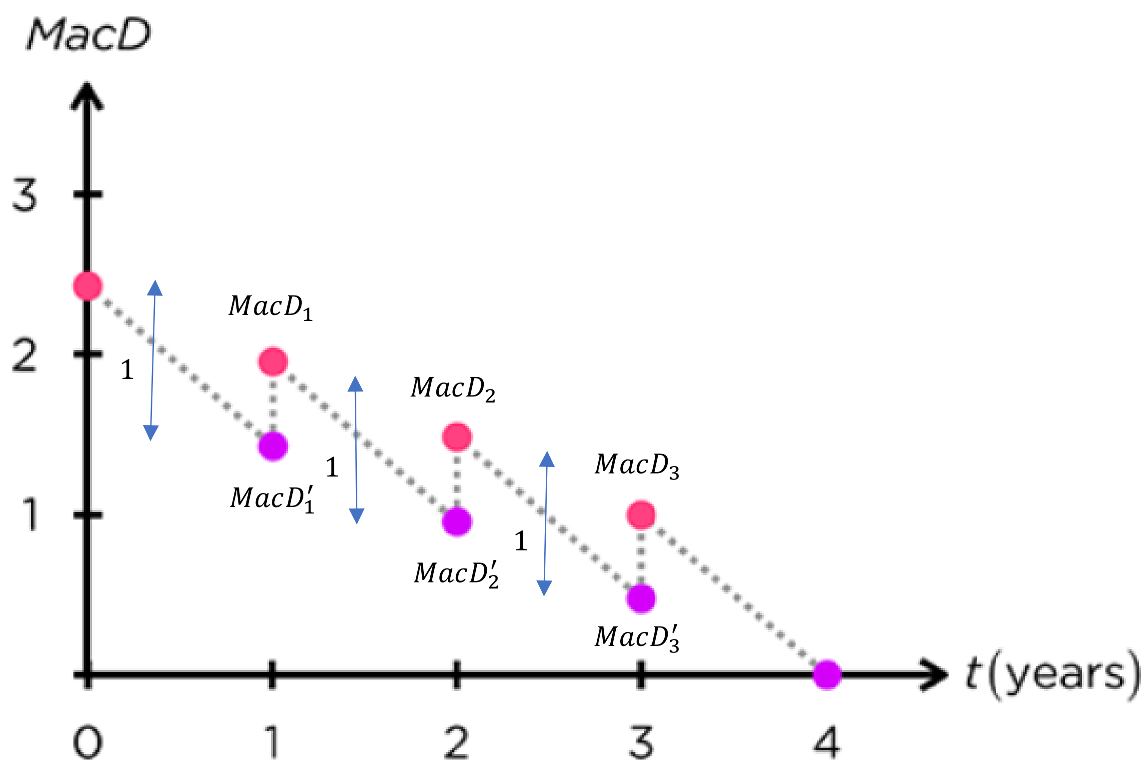
- This same portfolio method **applies for Convexity** as well

Different Comparison date

- The base definition of Duration **assumes that it is calculated at time 0**
- We can generalize the formula and calculate them on any date by understanding that **t represents the time till the cashflow occurs from the comparison date**
- However, we can **link the durations across time** which may help us save time

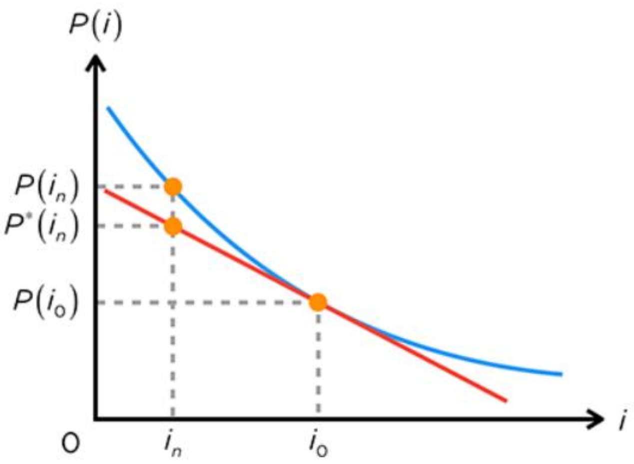


- If we were to move the comparison one period later, there are two different scenarios we must consider:
  - **Calculate Duration BEFORE the payment ($\boldsymbol{MacD'_t}$)**
    - Under this scenario, the **cashflows remain the same** as compared to the original. The only thing that changed was time
    - Thus, the average time taken for cashflows just moved forward a period:
    - $MacD'_t = MacD_{t-1} - 1$
  - **Calculate Duration AFTER the payment ($\boldsymbol{MacD_t}$)**
    - Since the payment has occurred, both the **cashflows and timing are different** as compared to the original
    - Thus, we cannot form a relationship & must be manually recalculated
  - For any given period, $\boldsymbol{MacD_t > MacD'_t}$
    - This is because $MacD_t$ removes the earliest cashflow, effectively removing the smallest value from the calculation
    - Since MacD is an average, **removing the smallest value will force the average to increase in value as compared to $\boldsymbol{MacD'_t}$**



Using Durations for Estimations

- Given the **only** current price of a Bond, we can **estimate** the new price of the bond for a specified change in interest rates using Durations
- We consider a graph of the price of the Bond. Without knowing the equation for Price, we can approximate the new price **using the tangent at the current price**:



Firstly, we consider the **first principles interpretation of Tangent/Slope**,

$$P'(i_0) = \frac{P^*(i_n) - P(i_0)}{i_n - i_0}$$

Next, we use the **ModD formula**,

$$ModD = -\frac{P'(i_0)}{P(i_0)}$$

$$P'(i_0) = -P(i_0) * ModD$$

Lastly, **we combine the two** and rearrange,

$$\frac{P^*(i_n) - P(i_0)}{i_n - i_0} = -P(i_0) * ModD$$

$$P^*(i_n) - P(i_0) = (-P(i_0) * ModD) * (i_n - i_0)$$

$$\boldsymbol{P^*(i_n) = P(i_0) - P(i_0)(i_n - i_0)(ModD)}$$

First Order Modified Approximation	First Order Macaulay Approximation
$P^*(i_n) = P(i_0) - P(i_0)(i_n - i_0)(ModD)$	$P(i_n) \approx P(i_0)\left(\frac{1+i_0}{1+i_n}\right)^{MacD}$

- Proof for Macaulay variant is algebraically challenging and is not shown
- Generally, Macaulay Approximation is **slightly more accurate** than the Modified one

An alternative (Intuitive) way to think about Approximation

- Premise → Macaulay and Modified Duration **represents time**
- Looking at the formulas for the approximation, we notice that we have seen them before:
    - **Macaulay** → $Interest^{Time}$ → **Compound Interest**
    - **Modified** → $Interest * Time$ → **Simple Interest**
- If interest rates were constant, we can simply use the original accumulation function. However, since interest rates changed, we **need to make a correction to the original**
- Macaulay approximation is a **Compound Interest Correction,** while Modified Approximation is a **Simple Interest Correction**
- Following the nature of each type of interest, Macaulay (Compound Interest) uses **Multiplication/Division** while **Modified** (Simple Interest) used **Addition/Subtraction**

Convexity

- Notice that we are **using a Line to approximate a Curve**. Thus, the first order approximations are good for **small changes** in price where the curve is relatively flat over that length
- Since the curve is Convex, the line will always **under-approximate the price** of the bond. This under-approximation becomes greater the larger the change in interest rates as the curve begins to **bend more**
- We can thus use the **Curvature to approximate this inherent error** (Convexity) and account for it to obtain a **more accurate approximation**

Derivation

<table>
<tr><th><u>Modified Convexity</u></th><th><u>Macaulay Convexity</u></th></tr>
<tr>
<td>
$ModC = \frac{P''(i)}{P(i)}$

$P(i) = \sum (1+i)^{-t} * CF_t$
$P'(i) = \sum -t * (1+i)^{-t-1} * CF_t$

$P''(i) = \sum -t * (-t-1) * (1+i)^{-t-2} * CF_t$
$P''(i) = \sum t * (t+1) * (1+i)^{-(t+2)} * CF_t$

$\therefore \boldsymbol{ModC} = \frac{\sum t * (t+1) * v^{t+2} * CF_t}{\sum v^t * CF_t}$
$\equiv \boldsymbol{ModC} = \frac{\sum t * (t+1) * v^t * CF_t}{\sum v^t * CF_t} * v^2$
</td>
<td>
$MacC = \frac{P''(\delta)}{P(\delta)}$

$P(i) = \sum e^{-\delta t} * CF_t$
$P'(i) = \sum -t * e^{-\delta t} * CF_t$

$P''(i) = \sum -t * -t * e^{-\delta t} * CF_t$
$P''(i) = \sum t^2 * e^{-\delta t} * CF_t$

$\therefore \boldsymbol{MacC} = \frac{\sum t^2 * v^t * CF_t}{\sum v^t * CF_t}$

MacC for a single cashflow is $\boldsymbol{n^2}$
</td>
</tr>
<tr>
<td colspan="2">
<u>Portfolio Convexity (Same as before)</u>
$MacC_{Portfolio} = w_X * MacC_X + w_Y * MacC_Y + w_Z * MacC_Z + \cdots$

Note: If **not specified**, Convexity always refers to **Modified Convexity**
</td>
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</table>

<u>Additional Formulas</u>

Linking Duration and Convexity:

$ModC = v^2 * (MacC + MacD)$

Second Order Modified Approximation:

$$P(i_n) \approx P(i_0) - P(i_0) * (i_n - i_0)(ModD) + P(i_0) * \frac{(i_n - i_0)^2}{2}(ModC)$$

Second Order Macaulay Approximation:

$$P(i_n) \approx P(i_0)\left(\frac{1+i_0}{1+i_n}\right)^{MacD} * \left[1 + \left(\frac{i - i_0}{1+i_0}\right)^2 \left(\frac{MacC - MacD^2}{2}\right)\right]$$

Immunization

- Companies often make payments (Liability Cashflows) and receive payments from their investments (Asset Cashflows)
- The goal of any company is to ensure that they have **enough Asset Cashflows to cover their Liability Cashflows**. In other words, the **Present Value** of Asset Cashflows should be more than Liability Cashflows. We call the excess a **Surplus.**
    - $Surplus = PV(CF_{Assets}) - PV(CF_{Liabilities})$
- However, this Surplus is prone to **interest rate risk**:
    - When interest rates fall or rise, the present value of assets might fall, present value of liabilities might rise or both
    - This causes the surplus to become negative which may put the company in a state of financial distress
- The goal of any company is to thus find the **combination of securities such that no matter how interest rates change, the surplus will always increase or stay the same**. Essentially, they are **immunizing their surplus** from the effects of interest rates

Redington Immunization (Duration Matching)

- **Three assumptions needed**:
    - Interest rates for all maturities are identical (Flat Yield Curve)
    - A change in interest rates affects all maturities (Parallel shift)
    - A change in interest rates do not affect cashflows (Fixed cashflows)
- Under these three assumptions, a portfolio is **immunized from small changes** in interest rates if the following **three conditions** are met:
    - $\boldsymbol{PV(Assets) = PV(Liabilities)} \Leftrightarrow \boldsymbol{P_A = P_L}$
        - They are the same because PV is equivalent to price
    - $\boldsymbol{Duration_{Asset} = Duration_{Liability}} \Leftrightarrow \boldsymbol{P'_A = P'_L}$
        - Both Macaulay and Modified duration can be used here
        - Since prices are the same, both their **denominators cancel out**, thus we only need to consider the **first derivative**
    - $\boldsymbol{Convexity_{Asset} > Convexity_{Liability}} \Leftrightarrow \boldsymbol{P''_A > P''_L}$
        - Similarly, both Macaulay and Modified convexity applies. Since prices are the same, we only consider their **second derivative**
- We can view this relationship **graphically:**
    - Since prices are the same, both Asset and Liability price curves intersect at the current interest rate
    - Since duration is the same, they have the same tangent at the intersection point
    - Since Asset Convexity is greater, it curves higher than Liability
- Given the same start point but asset curves more, the Asset will always **increase more and decrease less** than a similar change in Liabilities, ensuring surplus is protected
- Thus, we can **use the first two conditions to solve** for the optimal portfolio, and then use the **third condition to check** if the solution is valid, if necessary

Another way to express the conditions:

- We can also express the conditions in terms of Net Present Value (NPV)
- **First condition** states that the present values must match, $\boldsymbol{NPV = 0}$
- **Second condition** states that the first derivatives must match, $\boldsymbol{NPV' = 0}$
- **Third condition** states that the second derivative of inflows must be higher, $\boldsymbol{NPV'' > 0}$

Full Immunization

- Similarly, it has the **same assumptions as Redington**:
  - Flat Yield Curve, Parallel Shifts & Fixed Cashflows
- It also has **similar conditions** as Redington, with only the last being different:
  - $PV(Assets) = PV(Liabilities) \Leftrightarrow P_A = P_L$
  - $Duration_{Asset} = Duration_{Liability} \Leftrightarrow P'_A = P'_L$
  - **There must be one Asset Cashflow BEFORE and AFTER a Liability Cashflow**
    - There can be more than one Liability, but each must be sandwiched between two Asset cashflows
    - They do not have to be unique; there can be **infinitely many Liabilities between two Assets**
    - Cashflows also include small ones such as Bond Coupons
- The intuition behind it is **more qualitative:**
  - Another way to think about Immunization is for the firm to is for cashflows to be worth the amount they expected, regardless of market conditions
  - Imagine there was a rise in interest after the first asset cashflow. The value of liabilities would rise, but so would the Asset Cashflow that comes after that, mitigating the risk of interest rate changes. Vice-versa applies as well.
- Comparing both types of Immunization:
  - **Full immunization covers ALL** changes in interest rates, while **Redington covers only small interest rate changes**
  - Full immunization is a **stricter version** of Redington Immunization – Full is a **subset of Redington immunization. If Full Immunization conditions are met, it can also be used for a Redington problem**

Immunization Shortcut

- Find the duration of **EACH asset** and the **overall duration of the liabilities**. Using the **portfolio duration method**, equate both liabilities together:
  - $w_1 * MacD_{Asset\ 1} + w_2 * MacD_{Asset\ 2} + \cdots = MacD_{Liabilities}$
  - Solve for the weights of the portfolio, where $w_2 = 1 - w_1 + \cdots$
- Find the **overall PV of the liabilities**. Multiply the calculated weights by the PV of the liabilities, to obtain the **PV of each of the assets** used. **Multiply by interest** accordingly to find the value of the assets when their cashflows occur
- This process can be done as **all liabilities at once**, or for **individual liabilities**
  - If we are given the Asset Cashflows, we can find its PV and calculate the weight using by comparing it to the PV of the liabilities (Reverse order)

Alternative: Exact Cashflow Matching

- Redington and Full Immunization require very hard assumptions that are unrealistic in the real world
- This provides a simpler, more intuitive method by **matching the timing and amount** of Asset and Liability Cashflows
- This method **STILL immunizes against ALL changes in interest rates**
- Assume we have three bonds:
  - $Bond_1$ matures in 1 year, with Face F_1, Coupon Rate R_1 and YTM_1
  - $Bond_n$ matures in n years, with Face F_n, Coupon Rate R_n and YTM_n
  - $Bond_N$ matures in N years, with Face F_N, Coupon Rate R_N and YTM_N

	Period 1	**Period n**	**Period N**
$Bond_1$	F_1R_1	-	-
$Bond_n$	F_nR_n	$F_n + F_nR_n$	-
$Bond_N$	F_NR_N	F_NR_N	$F_N + F_NR_N$
	$Liability_1$	$Liability_n$	$Liability_N$

Methodology:
We use the **Longest Bond to match the Longest Liability**:
$F_N + F_NR_N = Liability_N$

If the **liability is larger than the Face and Redemption** of a single bond, then we need to **buy multiple bonds**. Let x be the number of **units to buy**, where x can be a non-integer:
$x_N(F_N + F_NR_N) = Liability_N$

Fill in the coupons for previous period, then **repeat** for the next longest liability:
$x_n(F_n + F_nR_n) + x_N(F_NR_N) = Liability_n$
$\vdots$

The total cost of exact matching is the number of **units multiplied by the price** of each bond:
$Total\ Cost = x_1 * P_1 + x_n * P_n + x_N * P_N$

- Note that each individual price is calculated using their respective YTM
- If the coupons are all **zero-coupon bonds,** then the total cost of matching is simply the present value of the liabilities (no need to do the above steps)

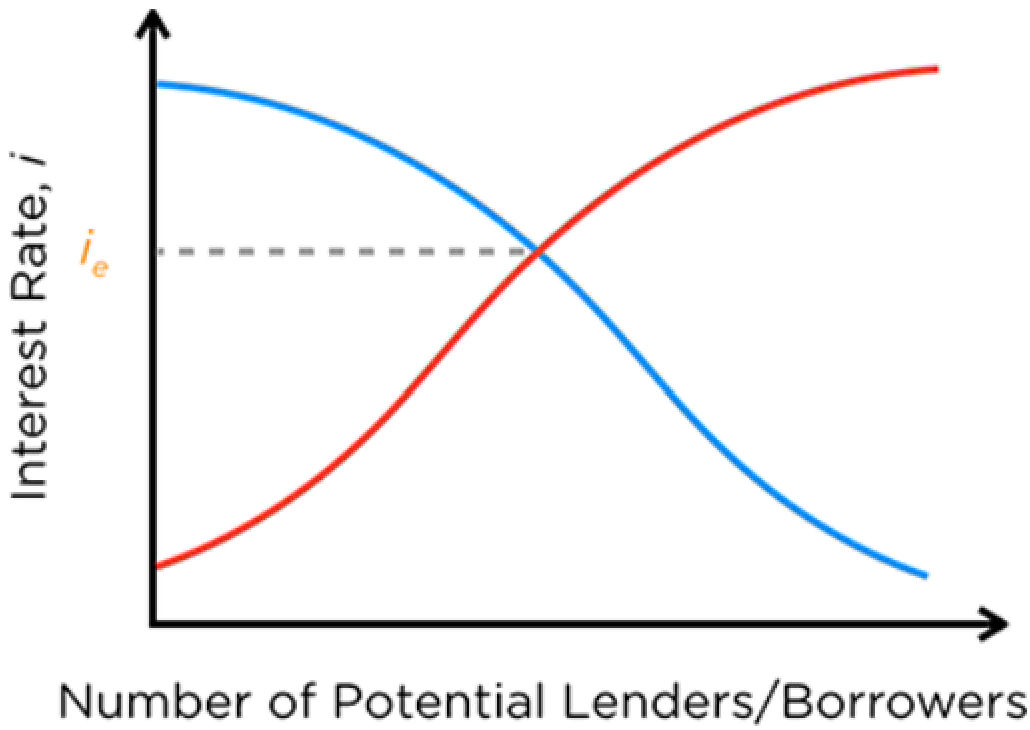
Generally, there **are two kinds of problems** given:

- **Face Value of the bonds are given**. In this case, you NEED to consider the UNITS of bonds of to buy since the Par Value is predetermined
- **Face Value of the Bonds are NOT given**. In this case, you can directly solve for the "par" value of the bond needed
- Both of these are the SAME. Both are calculating the true par value invested. Method 2 directly calculates this while method 1 has to consider the units of the bond because there is a fixed par value

Chapter 9: Determinants of Interest Rates

How are Interest rates determined?

- Interest rates can be viewed as the **cost of borrowing capital**. Like all things in economics, we can determine the equilibrium by considering the **supply and demand of capital**
- There are two perspectives we must consider:
    - **People with money (Lenders)**
        - They can either choose to **spend it now, or lend the money** out & receive interest payments
        - The **higher the interest, the more likely these people are to lend** the money. Thus, the money supply curve is **upward sloping**
    - **People without money (Borrowers)**
        - They can either choose to **refrain from spending, or borrow the money** & pay interest
        - The **lower the interest, the more likely these people are to borrow money**. Thus, the money demand curve is **downward sloping**
- By plotting out these two curves, the point at where they intersect is the equilibrium interest rate for the economy
- This is an **over-simplified** explanation, as there are other factors that affect it as well:
    - **Demand for capital** → Higher when economy is expanding or when new technology is discovered; people want more capital in these cases to take advantage of opportunities
    - **Supply of capital** → Affected by Time Preference, Risk Appetite, Inflation & individual characteristics of securities



Supply and Demand Curves of Money

Central Banks and Interest Rates

What is the central bank?

- It functions as a **bank to other commercial banks** in the country
- The central bank requires every commercial bank to deposit and maintain a specific amount with it, known as the **bank reserve requirement**
  - This reserve is to **ensure that these banks have enough capital** so that they do not fall into financial distress
  - The **amount required varies from bank to bank** as well as other situational factors

What if banks fail to meet this requirement?

- Banks may **fail to hit the requirements** for a variety of reasons, the most common being that they were too aggressive in lending money which results in more outflows than inflows during that period
- They have two options to meet the shortfall in capital:
  - Borrow from the **Central Bank** at the **discount rate**
  - Borrow from other **Commercial Banks** at the **federal funds rate**
- The central bank will always say yes to lending, which is why it is known as a **lender of last resort**. However, borrowing money directly from them **reflects poorly on the bank**, as it indicates difficulty in borrowing from other institutions due to **credibility issues**
- This may result in more **scrutiny over the bank**, which is why they **prefer to borrow from another commercial bank** instead

Impact on Interest rates

- The part of the central bank that oversees this process is the **Federal Open Market Committee (FOMC)**
- They set the **target interest rates** that they would like **for the federal funds rate**, and then they will buy and **sell T-bills** in the secondary market to influence the federal funds rate
  - This **affects the money demand and supply for T-bills**, which changes the discount rate, which makes borrowing from T-bills vs other banks **more or less desirable**
  - It is in the **banks best interest to lend money out**, thus they will **adjust the fed funds rate to ensure that they are more competitive (LOWER) than the discount rate** to ensure that other banks will borrow from them rather than the CB
- If the **discount rate increases**, **federal fund rate increases**, it becomes **more expensive for Banks to borrow money** hence **more costly for them to fall below the reserve requirement**
- They will become **less aggressive** and **not lend out as much money**, reducing money supply and hence **increasing interest rates**

Treasury Bills

<table>
<tr><th><u>US Treasury Bills</u></th><th><u>Canadian Treasury Bills</u></th></tr>
<tr><td>$Quoted\ rate = \frac{360}{N} * \frac{I}{C}$</td><td>$Quoted\ Rate = \frac{365}{N} * \frac{I}{P}$</td></tr>
<tr><td>Based on the state of the US economy</td><td>Based on the level of economic activity in Canada, including Supply/Demand of CND</td></tr>
<tr><td colspan="2">They use different approximations for the number of days in the year – US assumes each month has 30 days ($30 * 12 = 360$) while Canada uses the conventional 365 days. Also, the US uses the redemption value while Canada uses the price.

US bonds (Liquid) are traded more than Canadian Bonds (Ill-liquid). This lower liquidity risk means that US bonds yield a lower rate. All else equal, this also means US has a lower price.</td></tr>
</table>

Components of Interest Rate

- We can decompose interest rates into several smaller components:
  - **Real-Risk Free Rate** → Compensation for **deferred consumption**
  - **Maturity Risk Premium** → Compensation for the risk of longer-term investments
  - **Default Risk Premium** → Compensation for the risk of money not being paid
  - **Inflation Risk Premium** → Compensation for loss of purchasing power due
  - **Liquidity Risk Premium** → Compensation for added cost of converting to cash
- This is consistent with what we learnt in FNCE101 – The interest rates we observe is the combination of the Risk-Free Rate + Various Risk Premiums **(Reward for risk)**

Discrete Rates	**Continuous Rates**
$(1+R)^t = [(1+r_1)(1+r_2)\dots(1+r_n)]^t$ $R = (1+r_1)(1+r_2)\dots(1+r_n) - 1$	$e^{Rt} = e^{r_1 t} * e^{r_2 t} * \dots e^{r_n t}$ $R = r_1 + r_2 + \cdots r_n$

- Both ways of determining the various components are correct. However, for **simplicity**, we prefer to use the **Continuous Rates**
- Some parts of this section are purely theoretical while others involve some calculation to actually determine the value of the component r

Maturity Risk Premium

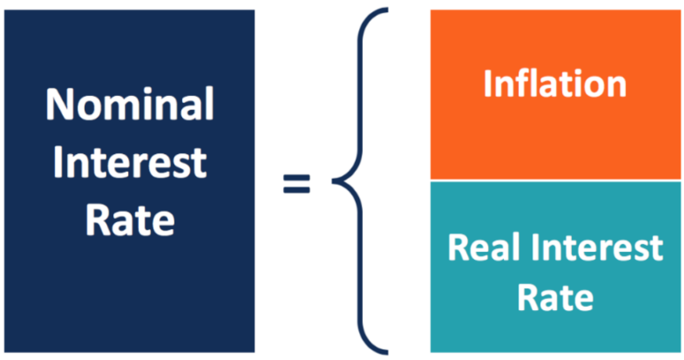
- All else equal, longer-term investments will have a higher interest rate
- There are several explanations:
  - **Market Segmentation Theory** → Different investors have **clearly defined and different investment horizons**. This results in different supply & demand factors for different time horizons, resulting in different rates
  - **Preferred Habitat Theory →** Builds off the above theory but says that the segments are **not written in stone**. These segments are just the preferred segment for investors. Given **better rates in another segment, these investors will move**
  - **Liquidity Preference Theory** → Lenders typically **prefer to lend for a shorter time**; thus, they require higher rates to lend for longer times
  - **Expectation Theory →** The longer the period of time, the more risk there is for interest rates to vary, which is why longer investments require a higher rate
    - It also means that the current forward rates are an **unbiased estimator of the future spot rates** ($EG.\ f_1\ now\ is\ s_1\ one\ year\ from\ now$)

Default Risk Premium

- All else equal, investments with a higher chance of not paying have higher interest rates
- There are two scenarios when it comes to defaulting:
  - Defaults **with no recovery** (Pay some smaller amount instead)
  - Defaults **with recovery** (Pay nothing)
- Consider the **lenders perspective**. In either case, they want to **get the same amount**, no matter what the scenario. Thus, they charge an appropriate default risk premium:
  - $Total\ \# * Loan\ Received = \#\ Non-Default * X + \#\ Default * Recovery$
  - X represents the **amount that needs to be RECEIVED** ($\boldsymbol{Loan * Interest}$)
  - $X = Loan\ amount * e^{rt}$
  - Solve for $\boldsymbol{r}$ **as our interest needed** → **Difference** between this and the original interest rate is the **default risk premium**

Inflation Risk Premium

- Inflation is the rise in the prices of goods and services over time
- Inflation is approximated via tracking one of the following indices:
    - **Consumer Price Index** (CPI) → Basket of typical consumer items
    - **Producer Price Index** (PPI) → Basket of typical producer items
- Inflation **decreasing the purchasing power** of money. Thus, Lenders of money want to be compensated via higher interest payments that scale with inflation. However, Borrowers are unwilling to pay this extra amount
- We see this pan out in two different scenarios:
    - **Lenders issue Loan with Inflation Protection for themselves**
        - The amount of interest that the borrowers pay scales with inflation. To incentivise borrowers, they lower they base interest by the cost of protection (c)
        - $R = r - c + i_{Actual}$
        - In other words, Lenders are **willing to receive a lower amount of base interest to pass on the risk of inflation to the borrower**. In finance terms, since they experience **less risk, they require a lower reward** $(r - c)$
    - **Lenders issue Loan without Inflation Protection for themselves**
        - Not all borrowers are willing to bear the inflation risk themselves. They would **rather pay an extra but certain amount to the lender** to compensate them for bearing the inflation risk instead
        - $R = r + i_{Expected} + i_{Unexpected}$
        - In other words, Lenders are **willing to pay a higher base interest to retain the risk of inflation with the lender**. In finance terms, since the lender experiences **more risk, they have a higher reward** $(r + i_e + i_{ue})$
    - The problem is that the there **is no guarantee that the amount loaded is equivalent to the actual inflation rate experienced**. It could be better, or it could be worse
- In finance/economics terms, we can view them in two different ways:
    - **With** inflation protection → **Real Interest Rate** $(\boldsymbol{r - c})$
    - **Without** inflation protection → **Nominal Interest Rate** $(\boldsymbol{r + i_e + i_{ue}})$
    - In a **perfect world**, where inflation can be predicted perfectly $(i_e + i_{ue} = i_a)$, then the following equation holds true (But the idea is there):



Other Points

- Bonds in the US are usually **issued at nominal rates** rather than real rates
- US tax is linked to the inflation rates

Liquidity Risk Premium

- All else equal, an investment that is **less-liquid** (Harder to convert to cash) have **higher interest rates**
- Liquidity is defined as the speed and ease at which an asset can be converted to cash, without significant loss in value
- The risk posed by an ill-liquid investment is that **if it has to be quickly converted to cash, it will lose a significant amount of its value**
- Thus, in order to **compensate for this** loss of value, ill-liquid investments demand higher interest rates

Financial Calculator Tips

Overview of using the BA II Plus

- It is a calculator with **pre-programmed financial formulas**. While this saves you the time of remembering and writing out formulas, you **must still understand how the formulas work to know which to use and what to input**
- Its main usage should be for these **financial calculations only.** While it can perform more general calculations, it is recommended to stick to a scientific calculator for those

Basic settings to toggle beforehand

- By default, the calculator only shows values up to **2 decimal places.** Due to rounding errors in intermediate steps, this is not ideal for a math paper. We can change the number of decimal places displayed:
    - 2nd . → DEC → 9
    - This sets the calculator to **FLOAT**, which means that it **displays up to 8** decimal places if available, otherwise up to the number of non-zero places
- The calculator only allows for **one step of calculations at a time and evaluates them immediately.** It can be annoying to calculate a value using an equation this way, as one would have to use brackets (which can't be easily seen) to input successive steps. We can change the way the calculator evaluates a series of instructions:
    - 2nd . → CHN → 2nd ENTER
    - This sets the calculator to **Algebraic Operating System** which follows the BODMAS format of evaluating equations

Basic calculator features

- **Financial Functions** – Since the calculator already has formulas built in, a large part of the process is **about assigning values** to these variables. There are two different ways:
    - **Value to Variable** → Key in the VALUES then press the VARIABLE
        - CPT → TARGET VARIABLE
    - **Variable to Value** → Press the VARIABLE then key in the VALUES & press ENTER
        - TARGET VARIABLE → CPT
    - Always ensure that the TARGET VARIABLE is set to 0
    - In all cases, we can confirm that the value has been properly assigned to the variable when we see a **full stop after the value**
    - Once we are done with the function, we should use 2ND CE|C or 2nd FV to reset all other variables
- **Storing Variables** – It is not practical or memorize intermediate values. We can store them in one of the 9 memory slots and recall them at any time:
    - VALUE → STO → 1-9 (Stores a value in the chosen memory slot)
    - RCL → 1-9 (Recalls a value from the chosen memory slot)
    - 2nd 0 → M1-9 → VALUE → ENTER (View current stored data and manually edit

Positive and Negative Signs

- We use **Positive** signs for **Cash Inflows** and **Negative** signs for **Cash Outflows.** We can toggle between positive and negative using +|−
- We can determine whether they are Inflows and Outflows by choosing whose perspective we want to take **- Investor or the Fund Manager**
- As an **Investor**, you **pay** an initial amount along with periodic payments. At the end, we **receive** our returns. PV and PMT are thus **negative**, while FV is **positive.**
- As the **Fund Manager**, you **receive** an initial amount along with periodic payments. At the end, you pay back the total returns. PV and PMT are thus **positive**, while FV is **negative**

Time Value of Money

- The function can be used to determine the present value of any annuities and Bonds

N → Number of periods

- As the name suggests, this can represent any time period (Months/Years etc)
- For perpetuities, we can input any number sufficiently large number (~999)

I/Y → Effective Interest per period **(Integer)**
PV → Present Value
PMT → Payment for Level **Annuity Immediate (Arrears)**

- By **default, the option is payments at the end of the period**. We can change this by going to 2nd PMT → 2nd ENTER which sets the mode to Beginning of the period instead

FV → Future Value

Amortization Schedule

- This mode can only be accessed after filling in the relevant TVM variables for a bond

2nd PV → Amortization Mode

- **P1** → Starting Period
- **P2** → Ending Period
    - Both should be the same value to find amortization in a specific period
- **BAL** → Remaining Balance of the Loan after P2
- **PRN** → Sum of the Principal paid from P1 to P2
- **INT** → Sum of the interest paid from P1 to P2

Cashflow Function

CF → Cashflow Mode

- **CF0** → Cashflow at time 0
    - This is typically the **initial investment** amount & hence should be a **negative value**
- **C01** → Cashflow at time 1
- **F01** → Frequency of C01 (How many periods it repeats – For perpetuities, put 999)
- ⋮

NPV → Net Present Value Mode

- **I** → Interest Rate **(Integer, in %)**

IRR → Internal Rate of Return

Interest Rate Conversion (Nominal to Effective)

2nd 2 → Enter Interest rate conversion mode

- **EFF** → Effective Interest (**EAR** - Effective Annual Rate)
- **NOM** → Nominal Interest Rate (**APR** – Annual Percentage Rate)
    - It is always assumed to be compounded 12 times a year unless stated otherwise
- **C/Y** → Compounding Frequency over a year (m)