coaching actuaries

Exam IFM

updated 12/20/21

INTRODUCTION TO DERIVATIVES

Reasons for Using Derivatives

- To manage risk
- To speculate
- To reduce transaction cost
- To minimize taxes / avoid regulatory issues

Bid-ask Spread

- *Bid price:* The price at which market-makers will buy and end-users will sell.
- *Ask/Offer price:* The price at which market-makers will sell and end-users will buy.
- Bid-ask spread = Ask price – Bid price
- *Round-trip transaction cost:* Difference between what you pay and what you receive from a sale using the same set of bid/ask prices.

Payoff and Profit

- *Payoff:* Amount that one party would have if completely cashed out.
- *Profit:* Accumulated value of cash flows at the risk-free rate.

Long vs. Short

- A *long* position in an asset benefits from an *increase* in the price of the asset.
- A *short* position in an asset benefits from a *decrease* in the price of the asset.

Short-Selling

Process of short-selling:

- Borrow an asset from a lender
- Immediately sell the borrowed asset and receive the proceeds (usually kept by lender or a designated 3rd party)
- Buy the asset at a later date in the open market to repay the lender (close/cover the short position)

Haircut: Additional collateral placed with lender by short-seller. It belongs to the short-seller.

Interest rate on haircut is called:

- *short rebate* in the stock market
- *repo rate* in the bond market

Reasons for short-selling assets:

- Speculation – To speculate that the price of a particular asset will decline.
- Financing – To borrow money for additional financing of a corporation.
- Hedging – To hedge the risk of a long position on the asset.

Option Moneyness

- *In-the-money:* Produce a *positive* payoff (not necessarily positive profit) if the option is exercised immediately.
- *At-the-money:* The spot price is approximately *equal* to the strike price.
- *Out-of-the-money:* Produce a *negative* payoff if the option is exercised immediately.

Option Exercise Styles

- *European-style options* can only be exercised at *expiration.*
- *American-style options* can be exercised at *any time* during the life of the option.
- *Bermudan-style options* can be exercised during *bounded periods* (i.e., specified periods during the life of the option).

Zero-coupon Bond (ZCB)

Buying a risk-free ZCB = Lending at risk-free rate

Selling a risk-free ZCB = Borrowing at risk-free rate

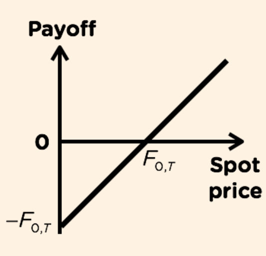
Payoff on a risk-free ZCB = ZCB's maturity value

Profit on a risk-free ZCB = 0

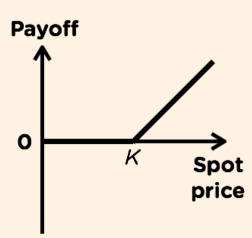
FORWARD CONTRACTS, CALL OPTIONS, AND PUT OPTIONS

Contract	Position in Contract	Description	Position in Underlying	Payoff	Profit	Maximum Loss	Maximum Gain	Strategy
Forward	Long Forward	Obligation to *buy* at the forward price	Long	$S_T - F_{0,T}$	Payoff	$F_{0,T}$	∞	Guarantee/lock in purchase price of underlying
	Short Forward	Obligation to *sell* at the forward price	Short	$F_{0,T} - S_T$	Payoff	∞	$F_{0,T}$	Guarantee/lock in sale price of underlying
Call	Long Call	Right (but not obligation) to *buy* at the strike price	Long	$\max[0, S_T - K]$	Payoff $- AV(\text{Prem.})$	$AV(\text{Prem.})$	∞	Insurance against high underlying price
	Short Call	Obligation to *sell* at the strike price if the call is exercised	Short	$-\max[0, S_T - K]$	Payoff $+ AV(\text{Prem.})$	∞	$AV(\text{Prem.})$	Sells insurance against high underlying price
Put	Long Put	Right (but not obligation) to *sell* at the strike price	Short	$\max[0, K - S_T]$	Payoff $- AV(\text{Prem.})$	$AV(\text{Prem.})$	$K - AV(\text{Prem.})$	Insurance against low underlying price
	Short Put	Obligation to *buy* at the strike price if the put is exercised	Long	$-\max[0, K - S_T]$	Payoff $+ AV(\text{Prem.})$	$K - AV(\text{Prem.})$	$AV(\text{Prem.})$	Sells insurance against low underlying price

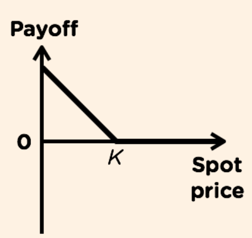
Long forward



Long call



Long put



OPTION STRATEGIES

Naked Writing vs. Covered Writing

- If an option writer *does not have* an offsetting position in the underlying asset, then the option position is said to be *naked*.
- If an option writer *has* an offsetting position in the underlying asset, then the option position is said to be *covered*.

Floor, Cap, Covered Call, Covered Put

Key Relationship: Call – Put = Stock – Bond

Rearranging, we have:

- Floor = + Stock + Put (guarantee a minimum selling price for stock)
- Write a covered put = – Floor = – Stock – Put
- Cap = – Stock + Call (guarantee a maximum purchase price for stock)
- Write a covered call = – Cap = + Stock – Call

Graphing Payoffs from First Principles

Create a payoff table separated by the strike regions, and then graph the total payoff in each region accordingly.

Shortcut Method for Graphing Payoff of All Calls or All Puts

- For *calls*, go *left-to-right* on payoff diagram, and evaluate slope of the payoff diagram at each strike price.

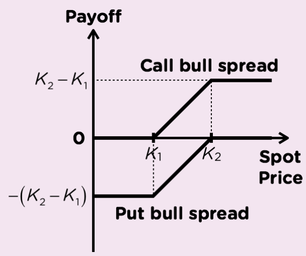
  Going *left-to-right* means that a *positive* slope is one that *increases left-to-right*, and a *negative* slope is one that *decreases left-to-right*.
- For *puts*, go *right-to-left* on payoff diagram, and evaluate slope of the payoff diagram at each strike price.

  Going *right-to-left* means that a *positive* slope is one that *increases right-to-left*, and a *negative* slope is one that *decreases right-to-left*.

Bull Spread

Call Bull: Long call (K_1) + Short call (K_2), $K_1 < K_2$

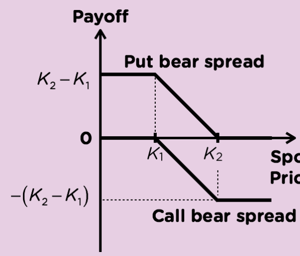
Put Bull: Long put (K_1) + Short put (K_2), $K_1 < K_2$



Bear Spread (opposite of bull spread)

Call Bear: Short call (K_1) + Long call (K_2), $K_1 < K_2$

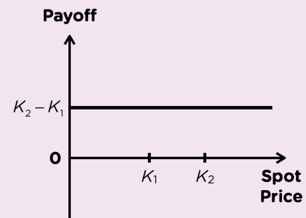
Put Bear: Short put (K_1) + Long put (K_2), $K_1 < K_2$



Box Spread

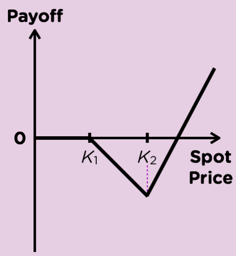
Long call (put) bull spread +

Long put (call) bear spread



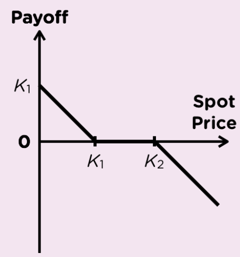
Ratio Spread

Long and short an unequal number of calls/puts with different strike prices



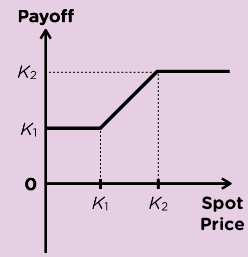
Collar

Long put (K_1) + Short call (K_2), $K_1 < K_2$



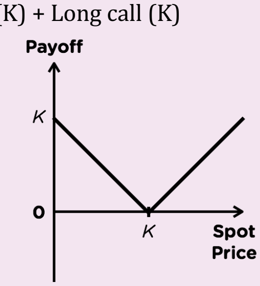
Collared Stock

Long collar + Long stock



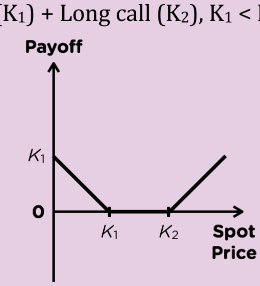
Straddle

Long put (K) + Long call (K)



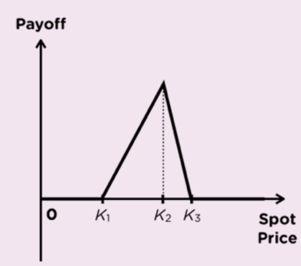
Strangle

Long put (K_1) + Long call (K_2), $K_1 < K_2$



Butterfly Spread (Symmetric/Asymmetric)

Buy high- and low-strike options. Sell middle-strike option. Quantity sold = Quantity bought.



Example - For 3 strike prices 30, 43, 46:

- Buy $46 - 43 = 3$ options with strike 30
- Buy $43 - 30 = 13$ options with strike 46
- Sell $46 - 30 = 16$ options with strike 43

Any multiple of (buy 3, sell 16, buy 13) would work.

FORWARDS

4 Ways to Buy a Share of Stock

Ways	Payment Time	Receive Stock at Time	Payment
Outright purchase	0	0	S_0
Fully leveraged purchase	T	0	S_0e^{rT}
Prepaid forward contract	0	T	$F_{0,T}^P(S)$
Forward contract	T	T	$F_{0,T}(S)$

Relationship between $F_{t,T}(S)$ and $F_{t,T}^P(S)$

$F_{t,T}(S)$ = Accumulated Value of $F_{t,T}^P(S)$

$= F_{t,T}^P(S) \cdot e^{r(T-t)}$

Dividend Structure	$F_{t,T}^P(S)$
No Divs	S_t
Discrete Divs	$S_t - PV_{t,T}(\text{Divs})$
Continuous Divs	$S_t e^{-\delta(T-t)}$

Dividend Structure	$F_{t,T}(S)$
No Divs	$S_t e^{r(T-t)}$
Discrete Divs	$S_t e^{r(T-t)} - AV_{t,T}(\text{Divs})$
Continuous Divs	$S_t e^{(r-\delta)(T-t)}$

Forward premium $= \dfrac{F_{0,T}}{S_0}$

Annualized forward premium rate

$= \dfrac{1}{T} \ln \dfrac{F_{0,T}}{S_0}$

Synthetic Forward

Synthetic *long* forward is created by:

- buying a stock and borrowing money (i.e., selling a bond), or
- buying a call and selling a put at the same strike.

Synthetic *short* forward is the opposite, created by:

- selling a stock and lending money (i.e., buying a bond), or
- selling a call and buying a put at the same strike.

Arbitrage

A transaction which generates a positive cash flow either today or in the future by simultaneous buying and selling of related assets, with no net investment or risk.

Arbitrage strategy: "Buy Low, Sell High."

Cash-and-Carry

The actual forward is overpriced.
Short actual forward + Long synthetic forward

Reverse Cash-and-Carry

The actual forward is underpriced.
Long actual forward + Short synthetic forward

FUTURES

Futures Compared to Forward

- Traded on an exchange
- Standardized (size, expiration, underlying)
- More liquid
- Marked-to-market and often settled daily
- Minimal credit risk
- Price limit is applicable

Features of Futures Contract

Notional Value = # Contracts × Multipler × Futures price

$Bal_t = Bal_{t-1} \cdot e^{rh} + Gain_t$

where

- $Gain_t$ = # Contracts × Multipler × Price Change_t (for *long* position)
- $Gain_t$ = − # Contracts × Multipler × Price Change_t (for *short* position)
- Price Change_t = Future Price_t − Future Price_{t-1}

Margin Call

- *Maintenance margin*: Minimum margin balance that the investor is required to maintain in margin account at all times
- *Margin call*: If the margin balance falls below the maintenance margin, then the investor will get a request for an additional margin deposit. The investor has to add more funds to bring the margin balance back to the <u>initial</u> margin.

PUT-CALL PARITY (PCP)

PCP for Stocks

$C(S,K) - P(S,K) = F_{t,T}^P(S) - Ke^{-r(T-t)}$

PCP for Futures

$C(F,K) - P(F,K) = Fe^{-r(T-t)} - Ke^{-r(T-t)}$

PCP for Bonds

$C(B,K) - P(B,K) = F_{t,T}^P(B) - Ke^{-r(T-t)}$

where

$F_{t,T}^P(B) = B_t - PV_{t,T}(\text{Coupons})$

B_t = Bond price at time t

PCP for Exchange Options

$C(A,B)$	$P(A,B)$
receive A, give up B	give up A, receive B

$C(A,B) - P(A,B) = F_{t,T}^P(A) - F_{t,T}^P(B)$

$C(A,B) = P(B,A)$

PCP for Currency Options

Use the generalized PCP for exchange option.

For example, the prepaid forward price for 1 yen denominated in dollars is:

$$\$F_{0,T}^P(¥1) = \left(x_0 \frac{\$}{¥}\right)(¥1e^{-r_{¥}T}) = \$x_0e^{-r_{¥}T}$$

Alternatively:

$S_0 \to x_0 \quad r \to r_d \quad \delta \to r_f$

$C_d(f,K) - P_d(f,K) = x_0e^{-r_fT} - Ke^{-r_dT}$

where x_0 is in d/f

COMPARING OPTIONS

Bounds of Option Prices

Call and Put

$S \geq C_{Amer} \geq C_{Eur} \geq \max(0, F^P(S) - Ke^{-rT})$

$K \geq P_{Amer} \geq P_{Eur} \geq \max\left(0, Ke^{-rT} - F^P(S)\right)$

European vs. American Call

$F^P(S) \geq C_{Eur} \geq \max(0, F^P(S) - Ke^{-rT})$

$S \geq C_{Amer} \geq \max(0, S - K)$

European vs. American Put

$Ke^{-rT} \geq P_{Eur} \geq \max\left(0, Ke^{-rT} - F^P(S)\right)$

$K \geq P_{Amer} \geq \max(0, K - S)$

Early Exercise of American Option

$PV(\text{Interest on strike}) = K(1 - e^{-rT})$

$PV(\text{Divs}) = S\left(1 - e^{-\delta T}\right)$

American Call

- Nondividend-paying stock
  - Early exercise is never optimal.
  - $C_{Amer} = C_{Eur}$
- Dividend-paying stock
  - It is rational to early exercise if:
    $PV(\text{Divs}) >$
    $PV(\text{Interest on strike}) + \text{Implicit Put}$
  - It may be rational to early exercise if:
    $PV(\text{Divs}) > PV(\text{Interest on strike})$

American Put

It is rational to early exercise if:

$PV(\text{Interest on strike}) >$
$PV(\text{Divs}) + \text{Implicit Call}$

It may be rational to early exercise if:

$PV(\text{Interest on strike}) > PV(\text{Divs})$

Strike Price Effects

For $K_1 < K_2 < K_3$:

Call

$C(K_1) \geq C(K_2) \geq C(K_3)$

$C(K_1) - C(K_2) \leq K_2 - K_1$

European: $C(K_1) - C(K_2) \leq PV(K_2 - K_1)$

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

Put

$P(K_1) \leq P(K_2) \leq P(K_3)$

$P(K_2) - P(K_1) \leq K_2 - K_1$

European: $P(K_2) - P(K_1) \leq PV(K_2 - K_1)$

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}$$

Time Until Expiration

For $T_1 < T_2$:

$C_{Amer}(S, K, T_1) \leq C_{Amer}(S, K, T_2)$

$P_{Amer}(S, K, T_1) \leq P_{Amer}(S, K, T_2)$

For a nondividend-paying stock:

$C_{Eur}(S, K, T_1) \leq C_{Eur}(S, K, T_2)$

This is also *generally* true for European call options on dividend-paying stocks and European puts, with some exceptions.

BINOMIAL MODEL

Option Pricing: Replicating Portfolio

An option can be replicated by *buying* Δ shares of the underlying stock and *lending B* at the risk-free rate.

$\Delta = e^{-\delta h}\left(\frac{V_u - V_d}{S(u - d)}\right) \quad B = e^{-rh}\left(\frac{uV_d - dV_u}{u - d}\right)$

$V = \Delta S + B$

	Call	Put
Δ	+	−
B	−	+

To replicate a *call*, buy shares and borrow money.

To replicate a *put*, sell shares and lend money.

Option Pricing: Risk-neutral Valuation

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

$V_0 = e^{-rh} \cdot E^*[\text{Payoff}]$
$= e^{-rh}[(p^*)V_u + (1 - p^*)V_d]$

$S_0 e^{(r-\delta)h} = (p^*)S_u + (1 - p^*)S_d$

Constructing a Binomial Tree

General Method

$u = \frac{S_u}{S_0} \qquad d = \frac{S_d}{S_0}$

Standard Binomial Tree

This is the *usual method* in McDonald based on *forward prices*.

$u = e^{(r-\delta)h + \sigma\sqrt{h}} \qquad d = e^{(r-\delta)h - \sigma\sqrt{h}}$

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}$$

Probability

For n periods, let k be the number of "up" jumps needed to reach an ending node. Then, the risk-neutral probability of reaching that node is given by:

$$\binom{n}{k}(p^*)^k(1 - p^*)^{n-k}, k = 0, 1, \ldots, n$$

No-Arbitrage Condition

Arbitrage is possible if the following inequality is not satisfied:

$0 < p^* < 1 \quad \Leftrightarrow \quad d < e^{(r-\delta)h} < u$

Option on Currencies

Substitutions: $S_0 \rightarrow x_0 \quad r \rightarrow r_d \quad \delta \rightarrow r_f$

$u = e^{(r_d - r_f)h + \sigma\sqrt{h}} \qquad d = e^{(r_d - r_f)h - \sigma\sqrt{h}}$

$$p^* = \frac{e^{(r_d - r_f)h} - d}{u - d}$$

Option on Futures Contracts

$F_{t,T_F} = S_t e^{(r-\delta)(T_F - t)}$

T = Expiration date of the option

T_F = Expiration date of the futures contract

$T \leq T_F$

Substitutions: $S_t \rightarrow F_{t,T_F} \quad \delta \rightarrow r$

$u_F = e^{\sigma\sqrt{h}} \qquad d_F = e^{-\sigma\sqrt{h}}$

$$p^* = \frac{1 - d_F}{u_F - d_F}$$

$$\Delta = \frac{V_u - V_d}{F(u_F - d_F)}$$

$B = e^{-rh}[p^* V_u + (1 - p^*)V_d]$

LOGNORMAL MODEL

Normal vs. Lognormal

$X \sim N(m, v^2) \Leftrightarrow Y = e^X \sim LogN(m, v^2)$

- $E[Y] = e^{m+0.5v^2}$
- $Var[Y] = (E[Y])^2[e^{v^2} - 1]$

$X = m + v \cdot Z, Z \sim N(0,1)$

$N(-a) = 1 - N(a)$

Two important properties of lognormal:

- It cannot be negative.
- The product of two lognormal is a lognormal.

Lognormal Model for Stock Prices

Assume the current time is time t:

For $T > t, \ln\left[\frac{S_T}{S_t}\right] \sim N[m, v^2]$

- $m = (\alpha - \delta - 0.5\sigma^2)(T - t)$
- $v^2 = \sigma^2(T - t)$
- $\frac{S_T}{S_t} \sim LogN(m, v^2)$

For $T > t, \ln[S_T] \sim N[m, v^2]$

- $m = \ln S_t + (\alpha - \delta - 0.5\sigma^2)(T - t)$
- $v^2 = \sigma^2(T - t)$
- $S_T \sim LogN(m, v^2)$

$E[S_T] = E[S_T|S_t] = S_t e^{(\alpha-\delta)(T-t)}$

$Var[S_T] = Var[S_T|S_t] = (E[S_T])^2(e^{v^2} - 1)$

$S_T = S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma\sqrt{T-t}\cdot Z}, Z \sim N(0,1)$

$Cov(S_t, S_T) = E\left[\frac{S_T}{S_t}\right] \cdot Var[S_t|S_0]$

To find the p^{th} percentile of S_T:

1. Determine the corresponding p^{th} percentile of the standard normal random variable Z.
2. Substitute the resulting value of Z into the expression for S_T.

$$\begin{aligned}\text{Median} &= \text{50th percentile} \\ &= S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)} \\ &= E[S_T] \cdot e^{-0.5\sigma^2(T-t)}\end{aligned}$$

Prediction Interval

The $100\%(1 - p)$ prediction interval is given by S_T^L and S_T^U such that $Pr[S_T^L < S_T < S_T^U] = 1 - p$.

$Pr[Z < z^L] = \frac{p}{2} \Rightarrow z^L = N^{-1}\left(\frac{p}{2}\right)$

$z^U = -z^L = -N^{-1}\left(\frac{p}{2}\right)$

$S_T^L = S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma\sqrt{T-t}\cdot z^L}$

$S_T^U = S_t e^{(\alpha-\delta-0.5\sigma^2)(T-t)+\sigma\sqrt{T-t}\cdot z^U}$

Probability

$Pr[S_T < K] = N(-\hat{d}_2)$

$Pr[S_T > K] = N(+\hat{d}_2)$

$$\hat{d}_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

Conditional and Partial Expectation

$$\begin{aligned}E[S_T|S_T < K] &= \frac{PE[S_T|S_T < K]}{Pr[S_T < K]} \\ &= \frac{S_t e^{(\alpha-\delta)(T-t)}N(-\hat{d}_1)}{N(-\hat{d}_2)}\end{aligned}$$

$$\begin{aligned}E[S_T|S_T > K] &= \frac{PE[S_T|S_T > K]}{Pr[S_T > K]} \\ &= \frac{S_t e^{(\alpha-\delta)(T-t)}N(+\hat{d}_1)}{N(+\hat{d}_2)}\end{aligned}$$

$$\hat{d}_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (\alpha - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

Expected Option Payoffs

$$\begin{aligned}E[\text{Call Payoff}] = S_t e^{(\alpha-\delta)(T-t)}N(\hat{d}_1) \\ - KN(\hat{d}_2)\end{aligned}$$

$$\begin{aligned}E[\text{Put Payoff}] = KN(-\hat{d}_2) \\ - S_t e^{(\alpha-\delta)(T-t)}N(-\hat{d}_1)\end{aligned}$$

True Pricing

To calculate option price, discount the true expected option payoff at the expected rate of return on the option:

$$V_0 = e^{-\gamma T}E[\text{Payoff}]$$

Risk-Neutral Pricing

Assume $\alpha = \gamma = r$.

To calculate option price, discount the risk-neutral expected option payoff at the risk-free rate:

$$V_0 = e^{-rT}E^*[\text{Payoff}]$$

Estimating Return and Volatility

Given $S_0, S_1, \ldots, S_n$, where the observations are at intervals of length h, we can estimate the lognormal parameters as follows:

1. Calculate the continuously compounded returns:
$$r_i = \ln\frac{S_i}{S_{i-1}} \quad i = 1,2, \ldots, n$$
2. Calculate the sample mean of the returns:
$$\bar{r} = \frac{\sum_{i=1}^n r_i}{n}$$
3. Estimate the standard deviation of returns by taking the square root of the sample variance of the returns:
$$\hat{\sigma}_h = \sqrt{\frac{\sum_{i=1}^n (r_i - \bar{r})^2}{n - 1}}$$
4. Annualize the estimate of the standard deviation:
$$Var\left[\ln\frac{S_{t+h}}{S_t}\right] = \hat{\sigma}^2 h = \hat{\sigma}_h^2$$
$$\Rightarrow \hat{\sigma} = \frac{\hat{\sigma}_h}{\sqrt{h}}$$
5. Annualize the estimate of the expected return:
$$E\left[\ln\frac{S_{t+h}}{S_t}\right] = \left(\hat{\alpha} - \delta - \frac{1}{2}\hat{\sigma}^2\right)h = \bar{r}$$
$$\Rightarrow \hat{\alpha} = \frac{\bar{r}}{h} + \delta + \frac{1}{2}\hat{\sigma}^2$$

THE BLACK-SCHOLES (BS) FORMULA

BS Formula's Assumptions:

- Continuously compounded returns on the stock are normally distributed and independent over time. There are no sudden jumps in the stock price.
- Volatility is known and constant.
- Future dividends are known.
- The risk-free rate is known and constant (i.e., the yield curve is flat).
- There are no taxes or transaction costs.
- Short-selling is allowed at no cost.
- Investors can borrow and lend at the risk-free rate.

Generalized BS Formula

Assume the current time is time 0 and the options expire at time T:

$C = F^P(S) \cdot N(d_1) - F^P(K) \cdot N(d_2)$

$P = F^P(K) \cdot N(-d_2) - F^P(S) \cdot N(-d_1)$

$$d_1 = \frac{\ln\left(\frac{F^P(S)}{F^P(K)}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{F^P(S)}{F^P(K)}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\frac{\mathrm{Var}\{\ln[S_t]\}}{t}}, \quad 0 < t \le T$$

$$= \sqrt{\frac{\mathrm{Var}\{\ln[F_{t,T}(S)]\}}{t}}, \quad 0 < t \le T$$

$$= \sqrt{\frac{\mathrm{Var}\{\ln[F^P_{t,T}(S)]\}}{t}}, \quad 0 < t \le T$$

The generalized BS formula can be applied to various assets, including stocks, futures contracts, and currencies.

If the stock pays **discrete** dividends, then the volatility of the **prepaid forward price** should be used as the volatility parameter.

For a stock that pays continuous dividends, the generalized BS formula can be written as:

$C = S_0 e^{-\delta T} \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$

$P = Ke^{-rT} \cdot N(-d_2) - S_0 e^{-\delta T} \cdot N(-d_1)$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

Options on Futures Contract

Use the generalized BS formula in conjunction with the appropriate prepaid forward prices.

The prepaid forward price for a futures contract is just the present value of the futures price:

$F^P_{0,T}(F) = Fe^{-rT}$

Alternatively:

$S_0 \to F_{0,T_F} \quad \delta \to r$

$C = F_{0,T_F}\, e^{-rT} \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$

$P = Ke^{-rT} \cdot N(-d_2) - F_{0,T_F}\, e^{-rT} \cdot N(-d_1)$

$$d_1 = \frac{\ln\left(\frac{F_{0,T_F}}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{F_{0,T_F}}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Options on Currencies

Use the generalized BS formula in conjunction with the appropriate prepaid forward prices.

For example, the prepaid forward price for 1 yen denominated in dollars is:

$$\$F^P_{0,T}(¥1) = \left(x_0 \frac{\$}{¥}\right)(¥1e^{-r_¥ T}) = \$x_0 e^{-r_¥ T}$$

Alternatively:

$S_0 \to x_0 \quad r \to r_d \quad \delta \to r_f$

$C = x_0 e^{-r_f T} \cdot N(d_1) - Ke^{-r_d T} \cdot N(d_2)$

$P = Ke^{-r_d T} \cdot N(-d_2) - x_0 e^{-r_f T} \cdot N(-d_1)$

$$d_1 = \frac{\ln\left(\frac{x_0}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{x_0}{K}\right) + \left(r_d - r_f - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

OPTION GREEKS

Greek	Definition	Long Call	Long Put
Δ	$\frac{\partial V}{\partial S}$	+	−
Γ	$\frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$	+	+
θ	$\frac{\partial V}{\partial t}$	− *	−*
Vega	$\frac{\partial V}{\partial \sigma}$	+	+
ρ	$\frac{\partial V}{\partial r}$	+	−
ψ	$\frac{\partial V}{\partial \delta}$	−	+

* θ is usually negative.

Note: For short positions, just reverse the signs.

Option Greeks Formulas

The formulas for the six option Greeks for both call and put options under the BS framework as well as $N'(x)$ will be provided on the exam.

$\Delta_C = e^{-\delta(T-t)}N(d_1),\ 0 \le \Delta_C \le 1$

$\Delta_P = -e^{-\delta(T-t)}N(-d_1),\ -1 \le \Delta_P \le 0$

$\Delta_C - \Delta_P = e^{-\delta(T-t)}$

$$\Gamma_C = \Gamma_P = \frac{\exp(-\delta(T-t))N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$\theta_C = \delta S e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$$

$\theta_P = \theta_C + rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$

$\mathrm{Vega}_C = \mathrm{Vega}_P = Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$

$\rho_C = (T-t)Ke^{-r(T-t)}N(d_2)$

$\rho_P = -(T-t)Ke^{-r(T-t)}N(-d_2)$

$\psi_C = -(T-t)Se^{-\delta(T-t)}N(d_1)$

$\psi_P = (T-t)Se^{-\delta(T-t)}N(-d_1)$

$$N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Risk Premium

The *risk premium* of an asset is defined as the excess of the expected return of the asset over the risk-free return:

- Risk $\text{Premium}_{\text{Option}} = \gamma - r$
- Risk $\text{Premium}_{\text{Stock}} = \alpha - r$

$\gamma - r = \Omega(\alpha - r)$

$\sigma_{\text{Option}} = |\Omega|\sigma_{\text{Stock}}$

Sharpe Ratio

$$\phi_{\text{Option}} = \frac{\text{Option's risk premium}}{\text{Option's volatility}} = \frac{\gamma - r}{\sigma_{\text{Option}}}$$

$$\phi_{\text{Stock}} = \frac{\text{Stock's risk premium}}{\text{Stock's volatility}} = \frac{\alpha - r}{\sigma_{\text{Stock}}}$$

$\phi_C = \phi_{\text{Stock}};\quad \phi_P = -\phi_{\text{Stock}}$

Elasticity

$$\Omega = \frac{\%\text{ change in option price}}{\%\text{ change in stock price}} = \frac{\Delta \cdot S}{V}$$

$\Omega_C \geq 1;\ \ \Omega_P \leq 0$

Portfolio Greek & Elasticity

Greek for a portfolio = sum of the Greeks

Elasticity for a portfolio = weighted average of the elasticities

$$\Omega_{Port} = \frac{\Delta_{Port} \cdot S}{V_{Port}} = \sum_{i=1}^{n} \omega_i \Omega_i$$

$\gamma_{Port} - r = \Omega_{Port}(\alpha - r)$

Delta-Gamma-Theta Approximation

$V_{t+h} \approx V_t + \Delta_t \epsilon + \frac{1}{2}\Gamma_t \epsilon^2 + \theta_t h;\ \epsilon = S_{t+h} - S_t$

DELTA & GAMMA HEDGING

Overnight Profit

A delta-hedged portfolio has 3 components:

- Buy/sell options
- Buy/sell stocks
- Borrow/lend money (sell/buy bond)

Overnight profit is the sum of:

- Profit on options bought/sold
- Profit on stocks bought/sold
- Profit on bond

Alternatively, overnight profit is the sum of:

- Gain on options, ignoring interest
- Gain on stocks, ignoring interest
- Interest on borrowed/lent money

For a market-maker who writes an option and delta-hedges the position, the market-maker's profit from time t to $t + h$ is:

$$= \Delta_t(S_{t+h} - S_t) - (V_{t+h} - V_t) - (e^{rh} - 1)(\Delta_t S_t - V_t)$$

$$\approx -\frac{1}{2}\epsilon^2 \Gamma_t - h\theta_t - (e^{rh} - 1)(\Delta_t S_t - V_t)$$

where $\epsilon = S_{t+h} - S_t$

If h is small, then $e^{rh} - 1 \approx rh$.

Breakeven

If the price of the underlying stock changes by one standard deviation over a short period of time, then a delta-hedged portfolio does not produce profits or losses.

Assuming the BS framework, given the current stock price, S, the two stock prices after a period of h for which the market-maker would break even are:

$$S \pm S\sigma\sqrt{h}$$

Multiple Greeks Hedging

$\Delta_{Stock} = 1$; all other Greeks of the stock $= 0$

To hedge multiple Greeks, set the sum of the Greeks you are hedging to zero.

ACTUARIAL-SPECIFIC RISK MANAGEMENT

Options Embedded in Insurance Products

- A *guaranteed minimum death benefit (GMDB)* guarantees a minimum amount will be paid to a beneficiary when the policyholder dies.
- A *guaranteed minimum accumulation benefit (GMAB)* guarantees a minimum value for the underlying account after some period of time, even if the account value is less.
- A *guaranteed minimum withdrawal benefit (GMWB)* guarantees that upon the policyholder reaching a certain age, a minimum withdrawal amount over a specified period will be provided.
- A *guaranteed minimum income benefit (GMIB)* guarantees the purchase price of a traditional annuity at a future time.

GMDB with a Return of Premium Guarantee

- A guarantee which returns the greater of the account value and the original amount invested:
  $$\max(S_T, K) = S_T + \max(K - S_T, 0)$$
- The embedded option is a *put* option. Its value is:
  $$E[P(T_x)] = \int_0^\infty P(t) f_{T_x}(t)\,dt$$

Earnings-Enhanced Death Benefit

- Pays the beneficiary an amount based on the increase in the account value over the original amount invested, e.g., $40\% \cdot \max(S_T - K, 0)$.
- The embedded option is a *call* option. Its value is:
  $$E[C(T_x)] = \int_0^\infty C(t) f_{T_x}(t)\,dt$$

GMAB with a Return of Premium Guarantee

- Similar to GMDB with ROP guarantee, but the benefit is *contingent* on the policyholder surviving to the end of the guarantee period.
- The embedded option is a *put* option. Its value is:
  $$P(m) \cdot \Pr[T_X^* \geq m]$$

Mortgage Loan as Put

For an uninsured position, the loss to the mortgage lender is $\max(B + C^* - R, 0)$, where:

- B is the outstanding loan balance at default
- C^* is the lender's total settlement cost
- R is the amount recovered on the sale of property

This is a put payoff with $K = B + C^*$ and $S = R$.

Static vs. Dynamic Hedging

Static/hedge-and-forget: Buy options and hold to expiration

Dynamic: Frequently buy/sell assets and/or derivatives with the goal of matching changes in the value of guarantee

Hedging of Catastrophic Risk

Catastrophe bond: A bond issued to investors where repayments and principal payments are contingent on there not being a catastrophe which causes large losses for the insurer. Thus, investors who buy these bonds face the risk of not receiving coupon payments or repayment of their principal. In general, cat bondholders typically receive higher interest rates for taking on this risk.

EXOTIC OPTIONS

Asian Option

$\bar{S} = \begin{cases} A(S) & \text{arithmetic average} \\ G(S) & \text{geometric average} \end{cases}$

$A(S) = \dfrac{\sum_{t=1}^{N} S_t}{N}$ $\qquad G(S) = \left(\prod_{t=1}^{N} S_t\right)^{\frac{1}{N}}$

$G(S) \le A(S)$

	Average Price	Average Strike
$\text{Payoff}_{\text{Call}}$ $\text{Payoff}_{\text{Put}}$	$\max[0, \bar{S} - K]$ $\max[0, K - \bar{S}]$	$\max[0, S - \bar{S}]$ $\max[0, \bar{S} - S]$

The value of an average price Asian option is less than or equal to the value of an otherwise equivalent ordinary option.

As N increases:

- Value of average price option decreases
- Value of average strike option increases

Barrier Option

Three types:

- *Knock-in*
  Goes into existence if barrier is reached.
- *Knock-out*
  Goes out of existence if barrier is reached.
- *Rebate*
  Pays fixed amount if barrier is reached.

Down vs. Up:

- If $S_0 <$ barrier:
  Up-and-in, up-and-out, up rebate
- If $S_0 >$ barrier:
  Down-and-in, down-and-out, down rebate

Knock-in + Knock-out = Ordinary Option
Barrier option ≤ Ordinary Option
Special relationships:

- If $S_0 \le$ barrier ≤ strike:
  Up-and-in Call = Ordinary Call
  Up-and-out Call = 0
- If $S_0 \ge$ barrier ≥ strike:
  Down-and-in Put = Ordinary Put
  Down-and-out Put = 0

Compound Option

A *compound call* allows the owner to *buy* another option at the strike price.
A *compound put* allows the owner to *sell* another option at the strike price.

The value of the underlying option at time t_1
$= V(S_{t_1}, K, T - t_1)$
The value of the compound call at time t_1
$= \max[0, V(S_{t_1}, K, T - t_1) - x]$
The value of the compound put at time t_1
$= \max[0, x - V(S_{t_1}, K, T - t_1)]$
where

- K is the strike of the underlying option
- x is the strike of the compound option
- T is the maturity of the underlying option
- t_1 is the maturity of the compound option

Put-call parity for compound options:

- $\text{CallonCall} - \text{PutonCall} = C_{\text{Eur}} - xe^{-rt_1}$
- $\text{CallonPut} - \text{PutonPut} = P_{\text{Eur}} - xe^{-rt_1}$

Gap Option

K_1 : Strike Price
K_2 : Trigger Price

K_1 determines the amount of the nonzero payoff.
K_2 determines whether the option will have a nonzero payoff.

$$\text{Payoff}_{\text{Gap Call}} = \begin{cases} 0, & S_T \le K_2 \\ S_T - K_1, & S_T > K_2 \end{cases}$$

$$\text{Payoff}_{\text{Gap Put}} = \begin{cases} K_1 - S_T, & S_T < K_2 \\ 0, & S_T \ge K_2 \end{cases}$$

Negative payoffs are possible.

$\text{GapCall} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$
$\text{GapPut} = K_1 e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$
where d_1 and d_2 are based on K_2
$\text{GapCall} - \text{GapPut} = S_0 e^{-\delta T} - K_1 e^{-rT}$

Exchange Option

$C(A, B) = F^P(A) \cdot N(d_1) - F^P(B) \cdot N(d_2)$
$P(A, B) = F^P(B) \cdot N(-d_2) - F^P(A) \cdot N(-d_1)$

$$d_1 = \frac{\ln\left(\frac{F^P(A)}{F^P(B)}\right) + \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}}$$

$d_2 = d_1 - \sigma\sqrt{T - t}$

$$\sigma = \sqrt{\frac{\operatorname{Var}\left[\ln\frac{A}{B}\right]}{t}} = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\text{Cov}_{A,B}} = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}$$

Maxima and Minima

- $\max(A, B) = \max(0, B - A) + A$
  $\max(A, B) = \max(A - B, 0) + B$
- $\max(cA, cB) = c \cdot \max(A, B) \quad c > 0$
  $\max(cA, cB) = c \cdot \min(A, B) \quad c < 0$
- $\max(A, B) + \min(A, B) = A + B$
  $\Rightarrow \min(A, B) = -\max(A, B) + A + B$

Forward Start Option

For a call option expiring at time T whose strike is set on future date t to be XS_t:

$C(S_t, XS_t, T - t)$
$= S_t e^{-\delta(T-t)} N(d_1) - XS_t e^{-r(T-t)} N(d_2)$
$= S_t[e^{-\delta(T-t)} N(d_1) - Xe^{-r(T-t)} N(d_2)]$

$$d_1 = \frac{\ln\left(\frac{S_t}{XS_t}\right) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = \frac{\ln\left(\frac{1}{X}\right) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$d_2 = d_1 - \sigma\sqrt{T - t}$

The time-0 value of the forward start option is:

$$V_0 = F_{0,t}^P(S) \times [e^{-\delta(T-t)} N(d_1) - Xe^{-r(T-t)} N(d_2)]$$

Same analysis applies to a put option.

Chooser Option

For an option that allows the owner to choose at time t whether the option will become a European call or put with strike K and expiring at time T:

$$V_t = \max[C(S_t, K, T-t), P(S_t, K, T-t)]$$
$$= e^{-\delta(T-t)} \max\left[0, Ke^{-(r-\delta)(T-t)} - S_t\right] + C(S_t, K, T-t)$$

The time-0 value is:

$$V_0 = e^{-\delta(T-t)} \cdot P\left(S_0, Ke^{-(r-\delta)(T-t)}, t\right) + C(S_0, K, T)$$

Lookback Option

An option whose payoff at expiration depends on the maximum or minimum of the stock price over the life of the option.

Type	Payoff
Standard lookback call	$S_T - \min(S)$
Standard lookback put	$\max(S) - S_T$
Extrema lookback call	$\max[0, \max(S) - K]$
Extrema lookback put	$\max[0, K - \min(S)]$

Standard lookback options are known as lookback options with a *floating* strike price.

Extrema lookback options are known as lookback options with a *fixed* strike price.

Shout Option

An option that gives the owner the right to lock in a minimum payoff exactly once during the life of the option, at a time that the owner chooses. When the owner exercises the right to lock in a minimum payoff, the owner is said to shout to the writer.

S^* is the value of the stock at the time when the option owner shouts to the option writer.

Payoff for a shout call

$$= \begin{cases} \max[S_T - K, S^* - K, 0] & \text{if exercised} \\ \max[S_T - K, 0] & \text{if not exercised} \end{cases}$$

Payoff for a shout put

$$= \begin{cases} \max[K - S_T, K - S^*, 0] & \text{if exercised} \\ \max[K - S_T, 0] & \text{if not exercised} \end{cases}$$

Rainbow Option

An option whose payoff depends on two or more risky assets.

Hedging Strategies Using Exotic Options

- *Forward start options* are useful for hedging guarantees that will come into effect during the payout period of a GMWB while the variable annuity is still in the accumulation period.
- *Chooser options* are useful hedging tools for variable annuities with two-sided guarantees, e.g., a GMDB with a return-of-premium guarantee and an earnings-enhanced death benefit equal to 35% of any account value gains.
- *Lookback options* are useful for hedging variable annuity guarantees where the guarantee value is periodically recalculated as the greater of the account value and the existing guarantee value.
- *Shout options* are useful for hedging variable annuity guarantees in situations where the guarantee value is recalculated at the discretion of the policyholder.
- *Rainbow options* are useful hedging tools when policyholders can hold multiple assets in their accounts and the guarantee applies to the account as a whole rather than individual assets in the account.

MEAN-VARIANCE PORTFOLIO THEORY

For Corporate Finance questions, unless you are told otherwise, assume that interest rates are annual effective, consistent with the Berk/DeMarzo text.

Risk and Return of a Single Asset

$$E[R] = \sum_{I=1}^{n} p_i \cdot R_i$$

$$Var[R] = E[(R - E[R])^2]$$
$$= \sum_{I=1}^{n} p_i \cdot (R_i - E[R])^2$$
$$= E[R^2] - (E[R])^2$$

$$SD[R] = \sqrt{Var[R]}$$

Realized Returns

$$R_{t+1} = \text{Capital Gain} + \text{Div Yield}$$
$$= \frac{P_{t+1} - P_t}{P_t} + \frac{D_{t+1}}{P_t}$$

Annual Realized Returns

$$R = (1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})(1 + R_{Q4}) - 1$$

Average Annual Returns

Based on arithmetic average:

$$\bar{R} = \frac{1}{T}\sum_{t=1}^{T} R_t$$

Is used to estimate a stock's expected return over a future horizon based on its past performance.

Compound Annual Returns

Based on geometric average:

$$\bar{R} = [(1 + R_1)(1 + R_2) \ldots (1 + R_T)]^{\frac{1}{T}} - 1$$

Is a better description of the long-run historical performance of a stock.

Variance of Returns

$$Var[R] = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2$$

$$\text{Standard Error} = \frac{SD(R)}{\sqrt{\#\text{ Observations}}}$$

95% confidence interval for expected return

$= \text{Average return} \pm 2 \cdot \text{Standard Error}$

$$= \bar{R} \pm 2 \cdot \frac{SD(R)}{\sqrt{\#\text{ Observations}}}$$

Risk and Return of a Portfolio

$$R_P = x_1R_1 + x_2R_2 + \cdots + x_nR_n$$
$$E[R_P] = x_1E[R_1] + x_2E[R_2] + \cdots + x_nE[R_n]$$
$$x_i = \frac{\text{Value of Investment } i}{\text{Total Portfolio Value}}; \sum x_i = 1$$
$$Cov[R_i, R_j] = E\left[(R_i - E[R_i])(R_j - E[R_j])\right]$$
$$= \frac{1}{T-1}\sum_{i=1}^{T}(R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$
$$= \rho_{i,j} \cdot \sigma_i \cdot \sigma_j$$

For a 2-stock portfolio:

$$R_P = x_1R_1 + x_2R_2$$
$$\sigma_P^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2Cov[R_1, R_2]$$

For a 3-stock portfolio:

$$R_P = x_1R_1 + x_2R_2 + x_3R_3$$
$$\sigma_P^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + x_3^2\sigma_3^2 + 2x_1x_2Cov[R_1, R_2] + 2x_1x_3Cov[R_1, R_3] + 2x_2x_3Cov[R_2, R_3]$$

For an n-stock portfolio:

$$\sigma_P^2 = \sum_{i=1}^{n} x_i \text{Cov}[R_i, R_P]$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \text{Cov}[R_i, R_j]$$

In the covariance matrix, we have:

- $n \times n = n^2$ total elements
- n variance terms
- $n^2 - n$ true covariance terms
- $(n^2 - n)/2$ unique true covariance terms

Note that:

- $\text{Cov}[R_i, R_j] = \text{Cov}[R_j, R_i] = \rho_{i,j} \cdot \sigma_i \cdot \sigma_j$
- $\text{Cov}[R_i, R_i] = \text{Var}[R_i]$
- $\text{Cov}[aR_1 + bR_2, cR_1 + dR_2]$
  $= ac\text{Cov}[R_1, R_1] + ad\text{Cov}[R_1, R_2]$
  $+ bc\text{Cov}[R_2, R_1] + bd\text{Cov}[R_2, R_2]$
  $= ac\text{Var}[R_1] + ad\text{Cov}[R_1, R_2]$
  $+ bc\text{Cov}[R_2, R_1] + bd\text{Var}[R_2]$

Diversification

Systematic risk

- Also known as **common, market, or non-diversifiable risk**.
- Fluctuations in a stock's return that are due to **market-wide** news.

Nonsystematic risk

- Also known as **firm-specific, independent, idiosyncratic, unique, or diversifiable risk**.
- Fluctuations in a stock's return that are due to **firm-specific** news.

Total risk = Systematic risk + Unsystematic risk

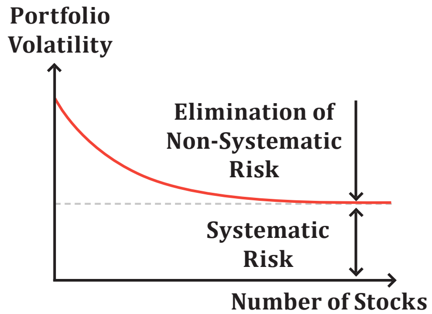
Diversification reduces a portfolio's total risk by averaging out **nonsystematic** fluctuations:

- Investors can eliminate nonsystematic risk for free by diversifying their portfolios. Thus, the risk premium for nonsystematic risk is zero.
- The risk premium of a security is determined by its systematic risk and does not depend on its nonsystematic risk.

For an equally-weighted *n*-stock portfolio:

$$\sigma_P^2 = \frac{1}{n} \cdot \overline{\text{Var}} + \left(1 - \frac{1}{n}\right) \cdot \overline{\text{Cov}}$$

- In a very large portfolio ($n \to \infty$), the covariance among the stocks accounts for the bulk of portfolio risk:
$$\sigma_P^2 = \overline{\text{Cov}}$$
- If the stocks are independent and have identical risks, then $\overline{\text{Cov}} = 0$, and:
$$\sigma_P^2 = \frac{1}{n} \cdot \overline{\text{Var}}$$
As $n \to \infty$, $\sigma_P^2 \to 0$. Thus, a very large portfolio with independent and identical risks will have *zero risk*.



Observations:

- The diversification effect is most significant initially.
- Even with a very large portfolio, we cannot eliminate all risk. The remaining risk is systematic risk that cannot be avoided through diversification.

For a portfolio with n individual stocks with arbitrary weights:

$$\sigma_P = \sum_{i=1}^{n} x_i \cdot \sigma_i \cdot \rho_{i,P}$$

- Each security contributes to the portfolio volatility according to its total risk scaled by its correlation with the portfolio, which adjusts for the fraction of the total risk that is common to the portfolio.
- As long as the correlation is not +1, the volatility of the portfolio is always less than the weighted average volatility of the individual stocks.

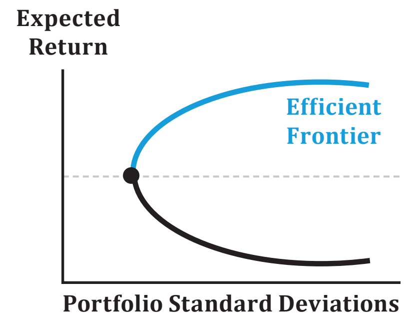
Mean-Variance Portfolio Theory

Assumptions of Mean-Variance Analysis

- All investors are risk-averse.
- The expected returns, variances, and covariances of all assets are known.
- To determine optimal portfolios, investors only need to know the expected returns, variances, and covariances of returns.
- There are no transactions costs or taxes.

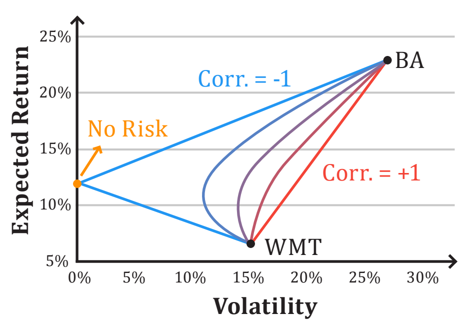
Efficient Frontier

- A portfolio is **efficient** if the portfolio offers the highest level of expected return for a given level of volatility.
- The portfolios that have the greatest expected return for each level of volatility make up the *efficient frontier*.



The Effect of Correlation

- If $\rho_{i,j} = 1$, no diversification. The portfolio's volatility is simply the weighted average volatility of the two risky assets.
- If $\rho_{i,j} < 1$, the portfolio's volatility is reduced due to diversification. It is less than the weighted average volatility of the two risky assets.
- If $\rho_{i,j} = -1$, a zero-risk portfolio can be constructed.



Combining Risky Assets with Risk-free Assets

Suppose we invest a proportion of funds (x) in a risky portfolio and the remainder (1 − x) in a risk-free asset. Then, we have:

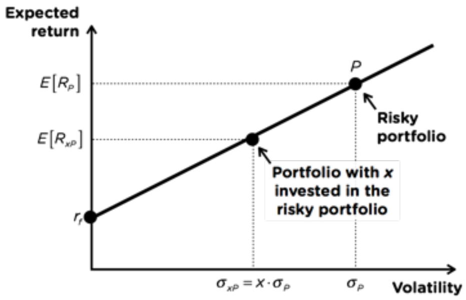
$$\begin{aligned} \mathrm{E}[\mathrm{R}_{\mathrm{xP}}] &= \mathrm{xE}[\mathrm{R}_{\mathrm{P}}] + (1-\mathrm{x})\mathrm{r}_{\mathrm{f}} \\ &= \mathrm{r}_{\mathrm{f}} + \mathrm{x}(\mathrm{E}[\mathrm{R}_{\mathrm{P}}] - \mathrm{r}_{\mathrm{f}}) \end{aligned}$$

$$\sigma_{\mathrm{xP}} = \mathrm{x} \cdot \sigma_{\mathrm{P}}$$

Note: xP means x% is invested in P

Capital Allocation Line (CAL) is a line representing possible combinations from combining a risky portfolio and a risk-free asset:

$$\mathrm{E}[\mathrm{R}_{\mathrm{xP}}] = \mathrm{r}_{\mathrm{f}} + \left(\frac{\mathrm{E}[\mathrm{R}_{\mathrm{P}}] - \mathrm{r}_{\mathrm{f}}}{\sigma_{\mathrm{P}}}\right)\sigma_{\mathrm{xP}}$$



- At the intercept, the portfolio only consists of the **risk-free** asset.
- At point P, the portfolio only consists of **risky** assets.
- The line extending to the right of σ_P represents portfolios that invest more than 100% in the risky portfolio P. This is done by using leverage (borrow money to invest). A portfolio that consists of a short position in the risk-free asset is known as a **levered** portfolio.

Optimal Portfolio Choice

- The optimal risky portfolio to combine with the risk-free asset is the one with the **highest Sharpe ratio**, where the CAL just touches (i.e., tangent to) the efficient frontier of risky investments. The portfolio that generates this **tangent line** is known as the **tangent portfolio**.
- The tangent line will always provide the best risk and return tradeoff available to investors. All portfolios on the tangent line (i.e., all portfolios that are combinations of the risk-free asset and the tangent portfolio) are **efficient** portfolios.
- The tangent portfolio is the optimal risky portfolio that will be selected by a rational investor **regardless of risk preference**.

Capital Market Line (CML)

- Assume investors have **homogeneous** expectations.
  - All investors have the same efficient frontier of risky portfolios, and thus the same optimal risky portfolio and CAL.
  - Every investor will use the same optimal risky portfolio -- the *market portfolio*.
  - When the market portfolio is used as the risky portfolio, the resulting CAL is CML.
- The equation for CML is:

$$\mathrm{E}[\mathrm{R}_{\mathrm{xM}}] = \mathrm{r}_{\mathrm{f}} + \left(\frac{\mathrm{E}[\mathrm{R}_{\mathrm{M}}] - \mathrm{r}_{\mathrm{f}}}{\sigma_{\mathrm{M}}}\right)\sigma_{\mathrm{xM}}$$

- Only efficient portfolios plot on CML. Individual securities plot below this line.

Adding a New Investment

Suppose we have a portfolio, P, with an expected return of $\mathrm{E}[\mathrm{R}_{\mathrm{P}}]$ and a volatility of σ_{P}.

Add the new investment to the portfolio if:

$$\text{Sharpe Ratio}_{\text{new}} > \rho_{\text{new,P}} \cdot \text{Sharpe Ratio}_{\text{P}}$$

$$\frac{\mathrm{E}[\mathrm{R}_{\text{new}}] - \mathrm{r}_{\mathrm{f}}}{\sigma_{\text{new}}} > \rho_{\text{new,P}} \cdot \frac{\mathrm{E}[\mathrm{R}_{\mathrm{P}}] - \mathrm{r}_{\mathrm{f}}}{\sigma_{\mathrm{P}}}$$

ASSET PRICING MODELS

Cost of capital is the rate of return that the providers of capital *require* in order for them to contribute their capital to the firm.

Equity cost of capital / cost of equity / required return on equity: the rate of return that the *equity holders* require in order for them to contribute their capital to the firm.

Debt cost of capital / cost of debt / required return on debt: the rate of return that the *debt holders* require in order for them to contribute their capital to the firm.

Beta

Definitions and Key Facts

- Measures the sensitivity of the asset's return to the market return.
- Is defined as the expected percent change in an asset's return given a 1% change in the market return.
- The beta for a stock, on average, is around 1.
- Cyclical industries (tech, luxury goods) tend to have higher betas.
- Non-cyclical industries (utility, pharmaceutical) tend to have lower betas.

Interpreting Beta

- If $\beta = 1$, then the asset has the **same** systematic risk as the market. The asset will tend to go up and down the **same** percentage as the market.
- If $\beta > 1$, then the asset has **more** systematic risk than the market. The asset will tend to go up and down **more** than the market, on a percentage basis.
- If $\beta < 1$, then the asset has **less** systematic risk than the market. The asset will tend to go up and down **less** than the market, on a percentage basis.
- If $\beta = 0$, then the asset's return is **uncorrelated** with the market return.

Calculating Beta

$$\beta_i = \beta_{i,Mkt} = \frac{Cov[R_i, R_{Mkt}]}{\sigma^2_{Mkt}} = \rho_{i,Mkt} \cdot \frac{\sigma_i}{\sigma_{Mkt}}$$

$$\beta_P = \sum_{i=1}^{n} x_i \beta_i$$

$\rho_{i,Mkt} \cdot \sigma_i$ represents the systematic risk of i.

Beta can be estimated using linear regression:

$R_i - r_f = \alpha_i + \beta_i(R_{Mkt} - r_f) + \epsilon_i$

Capital Asset Pricing Model (CAPM)

$r_i = E[R_i] = r_f + \beta_i[E[R_{Mkt}] - r_f]$

- $E[R_{Mkt}] - r_f$ is the *market risk premium* or the *expected excess return of the market.*
- $E[R_i] - r_f$ or $\beta_i[E[R_{Mkt}] - r_f]$ is the *risk premium for security i* or the *expected excess return of security i.*

Since β_i only captures systematic risk, $E[R_i]$ under the CAPM is not influenced by nonsystematic risk.

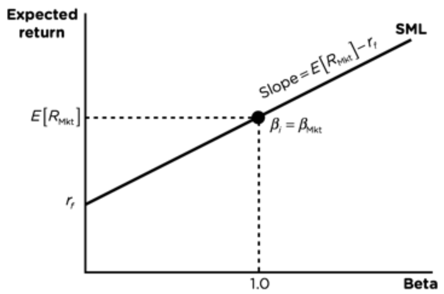
Assumptions of the CAPM

- Investors can buy and sell all securities at competitive market prices. There are no taxes or transaction costs. Investors can borrow and lend at the risk-free interest rate.
- Investors hold only efficient portfolios of traded securities.
- Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities.

The consequence of these assumptions is that the **market portfolio** is the **efficient portfolio**.

Security Market Line (SML)

SML is a graphical representation of CAPM:



CML vs. SML

CML	SML
The x-axis is based on **total** risk (i.e., volatility)	The x-axis is based on **systematic** risk (i.e., beta)
Only holds for **efficient** portfolios (b/c all combinations of the risk-free asset and the market portfolio are efficient portfolios)	Holds for **any** security or combination of securities (b/c the CAPM can be used to calculate the expected return for any security)

Alpha

The difference between a security's expected return and the required return (as predicted by the CAPM) is called *alpha*:

$$\alpha_i = E[R_i] - r_i$$
$$= E[R_i] - (r_f + \beta_i[E[R_{Mkt}] - r_f])$$

If the market portfolio is **efficient**, then all securities are on the SML, and:

$$E[R_i] = r_i \text{ and } \alpha_i = 0$$

If the market portfolio is **not efficient,** then the securities will not all lie on the SML, and:

$$E[R_i] \neq r_i \text{ and } \alpha_i \neq 0$$

Investors can improve the market portfolio by:

- buying stocks whose $E[R_i] > r_i$ (i.e., $\alpha_i > 0$)
- selling stocks whose $E[R_i] < r_i$ (i.e., $\alpha_i < 0$)

Required Return on New Investment

Adding the new investment will increase the Sharpe ratio of portfolio P if its expected return exceeds its **required return**, defined as:

$$r_{New} = r_f + \beta_{New,P}[E[R_P] - r_f]$$

In general, the beta of an asset i with respect to a portfolio P is:

$$\beta_{i,P} = \frac{Cov[R_i, R_P]}{\sigma^2_P} = \rho_{i,P} \cdot \frac{\sigma_i}{\sigma_P}$$

Expected Returns and the Efficient Portfolio

A portfolio is efficient if and only if the expected return of every available asset equals its required return. Thus, we have:

$$E[R_i] = r_i = r_f + \beta_i^{eff}[E[R_{eff}] - r_f]$$

Market Risk Premium

Two methods to estimate the market risk premium:

- **The historical risk premium**: Uses the historical average excess return of the market over the risk-free interest rate.
- **A fundamental approach**: Uses the constant expected growth model to estimate the market portfolio's expected return.

$$P_0 = \frac{Div_1}{E[R_{Mkt}] - g} \Rightarrow E[R_{Mkt}] = \frac{Div_1}{P_0} + g$$

The Debt Cost of Capital

Two methods to estimate debt cost of capital:

- **Adjustment from debt yield:**

  $r_d = y - pL$

  $= \text{Yield to mat.} - \text{Pr(Default)}\ E[\text{Loss rate}]$
- **CAPM using debt betas:**

  $r_d = r_f + \beta_d[E[R_{Mkt}] - r_f]$

Note that:

- It is **difficult** to get the beta estimates for individual debt securities because they are traded less frequently than stocks.
- The average beta for debt tends to be **low**. However, the beta for debt does increase as the credit rating decreases.

Required Return on All-Equity Project

If a project is financed purely with equity, the equity is said to be **unlevered**; otherwise, it is **levered**.

Assuming a project is financed **entirely with equity**, we can estimate the project's cost of capital and beta based on the **asset or unlevered cost of capital** and the beta of comparable firms:

Comparable	All-Equity	Levered
Beta	$\beta_U = \beta_E$	$\beta_U = w_E\beta_E + w_D\beta_D$
Cost of Capital	$r_U = r_E$	$r_U = w_E r_E + w_D r_D$

A firm's **enterprise value** is the risk of the firm's underlying business operations that is separate from its cash holdings. It is the combined market value of the firm's equity and debt, less any excess cash:

$$V = E + D - C$$

To determine the enterprise value, we use the firm's **net debt**:

Net debt = Debt − Excess cash and short-term investments

The beta of the firm's underlying business enterprise is:

$$\beta_U = w_E\beta_E + w_D\beta_D + w_C\beta_C$$

where:

$$w_E = \frac{E}{E+D-C}$$
$$w_D = \frac{D}{E+D-C}$$
$$w_C = \frac{-C}{E+D-C}$$

Required Return on a Leveraged Project

If the project is financed with both equity and debt, then use the **weighted-average cost of capital (WACC)**:

$$\begin{aligned} r_{WACC} &= w_E r_E + w_D r_D(1-\tau_C) \\ &= r_U - w_D r_D \tau_C \end{aligned}$$

where $r_D(1-\tau_C)$ is the **effective after-tax cost of debt**.

Note that:

- WACC is based on the firm's *after-tax* cost of debt while the unlevered cost of capital is based on the firm's *pretax* cost of debt.
- Unlevered cost of capital is also called the *asset cost of capital* or the *pretax WACC*.
- When we say "WACC" with no qualification, we mean "after-tax WACC".

Multi-Factor Model

- If the market portfolio is not efficient, a multi-factor model is an alternative.
- It considers more than one factor when estimating the expected return.
- An efficient portfolio can be constructed from other **well-diversified portfolios**.
- Also known as the Arbitrage Pricing Theory (APT).
- Similar to CAPM, but assumptions are not as restrictive.

Key Equations

Using a collection of N **factor portfolios**:

$$E[R_i] = r_f + \sum_{n=1}^{N} \beta_i^{Fn}(E[R_{Fn}] - r_f)$$

where:

- $\beta_i^{F1}, \ldots, \beta_i^{Fn}$ are the **factor betas** of asset i that measure the sensitivity of the asset to a particular factor, holding other factors constant.
- $E[R_{Fn}] - r_f$ is the **risk premium** or the **expected excess return** for a factor portfolio.

If all factor portfolios are **self-financing**, then we can rewrite the equation as:

$$E[R_i] = r_f + \sum_{n=1}^{N} \beta_i^{Fn}(E[R_{Fn}])$$

For a self-financing portfolio, the portfolio weights sum to zero rather than one.

Fama-French-Carhart (FFC)

This model consists of 4 **self-financing** factor portfolios:

- **Market portfolio.** Accounts for *equity risk*. Take a long position in the market portfolio and finance itself with a short position in the risk-free asset.
- **Small-minus-big (SMB) portfolio.** Accounts for differences in company size based on *market capitalization*. Buy small firms and finance itself by short selling big firms.
- **High-minus-low (HML) portfolio.** Accounts for differences in returns on *value stocks and growth stocks*. Buy high book-to-market stocks (i.e., value stocks) and finance itself by short selling low book-to-market stocks (i.e., growth stocks).
- **Momentum.** Accounts for the tendency of an asset return to be *positively correlated* with the asset return from the previous year. Buy the top 30% stocks and finance itself by short selling the bottom 30% stocks.

The FFC estimates the expected return as:

$$\begin{aligned} E[R_i] = r_f &+ \beta_i^{Mkt}(E[R_{Mkt}] - r_f) \\ &+ \beta_i^{SMB}E[R_{SMB}] \\ &+ \beta_i^{HML}E[R_{HML}] \\ &+ \beta_i^{PR1YR}E[R_{PR1YR}] \end{aligned}$$

where:

SMB = Small-minus-big portfolio
HML = High-minus-low portfolio
PR1YR = Prior 1-year momentum portfolio

MARKET EFFICIENCY AND BEHAVIORAL FINANCE

An *efficient market* is a market in which security prices adjust rapidly to reflect any new information, i.e., security prices reflect all past and present information.

Forms of Market Efficiency

	Market Prices Reflect:		
Form	Past market data	Public info	Private info
Weak	✓		
Semi-strong	✓	✓	
Strong	✓	✓	✓

Weak Form EMH

- It is impossible to consistently attain superior profits by analyzing past returns.

Semi-Strong-Form EMH

- It is impossible to consistently attain superior profits by analyzing public information.
- Prices will adjust immediately upon the release of any public announcements (earnings, mergers, etc.).
- A semi-strong-form efficient market is also weak-form efficient.

Strong-Form EMH

- There are only lucky and unlucky investors. No one (not even company insiders) can consistently attain superior profits.
- Passive strategy is the best.
- A strong-form efficient market is also semi-strong and weak-form efficient.

Important Logic

- If something supports the stronger form of the EMH, then it also supports the weaker forms.
  Example: If something supports the semi-strong form, then it must also support the weak form.
- If something violates the weaker form of the EMH, then it also violates the stronger forms.
  Example: If something violates the semi-strong form, then it must also violate the strong form.
- However, the reverse is not necessarily true.

Empirical Evidence Supporting EMH

Supporting Weak Form of EMH

- Kendall found that prices followed a random walk model, i.e., past stock prices have no bearing on future prices.
- Brealey, Meyers, and Allen created a scatter plot for price changes of four stocks.
  - No distinct pattern in the points, with the concentration of points around the origin. No bias toward any quadrants.
  - Autocorrelation coefficients were close to 0.
- Poterba and Summers found that variance of multi-period change is approximately proportional to number of periods.

Supporting Semi-Strong Form of EMH

- 3 months prior to a takeover announcement, the stock price gradually increased. At the time of announcement, stock price instantaneously jumped. After the announcement, the abnormal returns dropped to zero.
- Abnormal Return = Actual Return – Expected Return where:
  Expected Return $= \alpha + \beta \cdot$ Actual Market Return

Supporting Strong Form of EMH

- Top performing fund managers in one year only have a 50% chance to beat their reference index the following year.
- The performance of actively managed mutual funds from 1971 to 2013 only beat the Wilshire 5000 index 40% of the time.

Empirical Evidence Against EMH

Calendar/Time Anomalies

- **January effect:** Returns have been higher in January (and lower in December) than in other months.
- **Monday effect**: Returns have been lower on Monday (and higher on Friday) than on other days of the week.
- **Time-of-day effect:** Returns are more volatile close to the opening and closing hours for the market. Also, the trading volumes are higher during these times.

Underreaction/Overreaction Anomalies

- **New-Issue/IPO puzzle:** Overreaction to new issues pushes up stock prices initially.
- **Earnings announcement puzzle:** Investors underreacted to the earnings announcement.
- **Momentum effect**: There is a positive serial correlation in stock prices as investors underreact to new information.
- **Reversal effect**: There is a negative serial correlation in stock prices as investors overreact to new information.

Other Anomalies
- **Siamese twins:** Two stocks with claims to a common cash flow should be exposed to identical risks but perform differently.
- **Political cycle effect:** For a given political administration, its first year and last year yield higher returns than the years in between.
- **Stock split effect:** Returns are higher before and after the company announces the stock split.
- **Neglected firm effect:** Lesser-known firms yield abnormally high returns.
- **Super Bowl effect:** Historical data shows in the year after the Super Bowl, the stock market is more likely to do better if an NFC team won and worse if an AFC team won.
- **Size effect:** Small-cap companies have outperformed large-cap companies on a risk-adjusted basis.
- **Value effect:** Value stocks have consistently outperformed growth stocks.

Bubbles also violate market efficiency. It happens when the market value of the asset significantly deviates from its intrinsic value.

The Efficiency of the Market Portfolio
Under the CAPM assumptions, the market portfolio is an efficient portfolio. All investors should hold the market portfolio (combined with risk-free investments). This investment advice **does not depend on the quality of an investor's information or trading skill**.

The assumption of **rational expectations** is less rigid than that of homogeneous expectations. If we assume investors have rational expectations, then all investors correctly interpret and use their own information, along with information from market prices and the trades of others.

The market portfolio can be inefficient (and thus it is possible to beat the market) only if a significant number of investors:
- Do not have rational expectations (thus information is misinterpreted).
- Care about aspects of their portfolio other than expected return and volatility (thus they are willing to hold portfolios that are mean-variance inefficient).

Takeover Offer:
- After the initial jump in the stock price at the time of the announcement, target stocks **do not appear to generate abnormal subsequent returns on average**.
- Stocks that are ultimately acquired tend to appreciate and have positive alphas, while stocks that are not acquired tend to depress and have negative alphas.

Stock Recommendation:
- When a stock recommendation is given at the same time that news about the stock is released, the initial stock price reaction appears correct. The stock price increases in the beginning, then it flattens out.
- When a stock recommendation is given without news, the stock price seems to **overreact**. The stock price surges the following day, then it falls compared to the market

The Performance of Fund Managers:
- The **median** mutual fund actually **destroys** value.
- The mutual fund industry still has positive value added because **skilled managers manage more money** and add value to the whole industry.
- On average, an investor does not profit more from investing in an actively managed mutual fund compared to investing in passive index funds.

  The value added by a fund manager is offset by the mutual fund fees.
- Superior past performance of funds was not a good predictor of future ability to outperform the market

Reasons Why the Market Portfolio Might Not Be Efficient:
- **Proxy Error:** Due to the lack of competitive price data, the market proxy cannot include most of the tradable assets in the economy.
- **Behavioral Biases**: Investors may be subject to *systematic* behavioral biases and therefore hold inefficient portfolios.
- **Alternative Risk Preferences:** Some investors focus on risk characteristics other than the volatility of their portfolio, and they may choose inefficient portfolios as a result.
- **Non-Tradable Wealth:** Investors are exposed to significant risks outside their portfolio. They may choose to invest less in their respective sectors to offset the inherent exposures from their human capital.

The Behavior of Individual Investors
The following behaviors *do not impact* the efficiency of the market and have no effect on market prices or returns.

Underdiversification
Individual investors fail to diversify their portfolios adequately. They invest in stocks of companies that are in the same industry or are geographically close. Explanations:
- Investors suffer from **familiarity bias**, favoring investments in companies they are familiar with.
- Investors have **relative wealth concerns**, caring most about how their portfolio performs relative to their peers.

Excessive Trading and Overconfidence
Individual investors tend to trade very actively. Explanations:
- **Overconfidence bias**. They often overestimate their knowledge or expertise. Men tend to be more overconfident than women.
- Trading activity increases with the number of speeding tickets an individual receives – **sensation seeking**.

Systematic Trading Biases

The following behaviors lead investors to depart from the CAPM in systematic ways and subsequently *impact* the efficiency of the market.

Holding on to Losers and the Disposition Effect

Investors tend to hold on to investments that have lost value and sell investments that have increased in value.

- People tend to prefer avoiding losses more than achieving gains. They refuse to "admit a mistake" by taking the loss.
- Investors are more willing to take on risk in the face of possible losses.
- The disposition effect has negative tax consequences.

Investor Attention, Mood, and Experience

- Individual investors tend to be influenced by **attention-grabbing news or events**. They buy stocks that have recently been in the news.
- Sunshine has a positive effect on **mood** and stock returns tend to be higher on a sunny day at the stock exchange.
- Major sports events have impacts on **mood**. A loss in the World Cup reduces the next day's stock returns in the losing country.
- Investors appear to put inordinate weight on their **experience** compared to empirical evidence. People who grew up during a time of high stock returns are more likely to invest in stocks.

Herd Behavior

Investors actively try to follow each other's behavior. Explanations:

- Investors believe others have superior information, resulting in **information cascade effect**.
- Investors follow others to avoid the risk of underperforming compared to their peers (**relative wealth concerns**).
- Investment managers may risk damaging their reputations if their actions are far different from their peers. If they feel they are going to fail, then they would rather fail with most of their peers than fail while most succeed.

INVESTMENT RISK AND PROJECT ANALYSIS

Measures of Investment Risk

Variance

- The average of the squared deviations above and below the mean:

$$\text{Variance} = \mathrm{E}[(\mathrm{R} - \mathrm{E}[\mathrm{R}])^2]$$

Semi-variance / Downside Semi-variance

- Only cares about downside risk; ignores upside variability.
- The average of the squared deviations below the mean:

$$\text{Semi-variance} = \mathrm{E}[\min(0, \mathrm{R} - \mathrm{E}[\mathrm{R}])^2]$$

- The sample semi-variance is:

$$\text{Semi-variance} = \frac{1}{n}\sum \min(0, \mathrm{R}_i - \mathrm{E}[\mathrm{R}])^2$$

- Semi-variance $\le$ Variance
- For a symmetric distribution:

$$\text{Semi-variance} = \frac{1}{2}\text{Variance}$$

Value-at-Risk (VaR)

- VaR of X at the $100\alpha\%$ confidence level is its $100\alpha^{\text{th}}$ percentile, denoted as $\mathrm{VaR}_\alpha(\mathrm{X})$ or π_α.

$$\Pr[\mathrm{X} \le \pi_\alpha] = \alpha$$

- If X is continuous, then:

$$\mathrm{F_X}(\pi_\alpha) = \alpha \Rightarrow \pi_\alpha = \mathrm{F_X^{-1}}(\alpha)$$

- $\mathrm{VaR}_{0.50}(\mathrm{X})$ is the 50^{th} percentile or the median of X.

Tail Value-at-Risk (TVaR)

- TVaR focuses on what happens in the adverse tail of the probability distribution.
- Also known as the *conditional tail expectation* or *expected shortfall*.
- If X represents **gains**, then the risk we are concerned about comes from the **low end** of the distribution:

$$\mathrm{TVaR}_\alpha = \mathrm{E}[\mathrm{X} \mid \mathrm{X} \le \pi_\alpha] = \frac{1}{\alpha}\int_{-\infty}^{\pi_\alpha} x \cdot f_\mathrm{X}(x)\,dx$$

- If X represents **losses**, then the risk we are concerned about comes from the **high end** of the distribution:

$$\mathrm{TVaR}_\alpha = \mathrm{E}[\mathrm{X} \mid \mathrm{X} > \pi_\alpha] = \frac{1}{1-\alpha}\int_{\pi_\alpha}^{\infty} x \cdot f_\mathrm{X}(x)\,dx$$

- If the risk we are concerned about is unclear, then use the following rule of thumb:
  - If $\alpha < 0.5$, then presumably the risk of concern comes from the low end.
  - If $\alpha > 0.5$, then presumably the risk of concern comes from the high end.
- TVaR will provide a more conservative number than VaR.

Coherent Risk Measures

$g(\mathrm{X})$ is **coherent** if it satisfies (for $c > 0$):

- Translation invariance: $g(\mathrm{X} + c) = g(\mathrm{X}) + c$
- Positive homogeneity: $g(c\mathrm{X}) = c \cdot g(\mathrm{X})$
- Subadditivity: $g(\mathrm{X} + \mathrm{Y}) \le g(\mathrm{X}) + g(\mathrm{Y})$
- Monotonicity: If $\mathrm{X} \le \mathrm{Y}$, then $g(\mathrm{X}) \le g(\mathrm{Y})$

Variance, Semi-Variance, VaR, TVaR:

- Variance and semi-variance do not satisfy any of the 4 characteristics; not coherent.
- VaR is **usually not coherent** since it does not satisfy the subadditivity characteristic. If the distributions are assumed to be *normal*, then VaR can be shown to be coherent.
- TVaR is **always coherent**.

Project Risk Analysis

The *net present value (NPV)* of a project equals the present value of all expected net cash flows from the project. The discount rate for a project is its *cost of capital*.

Breakeven Analysis

- Calculate the value of each parameter so that the project has an NPV of zero.
- The internal rate of return (IRR) is the rate at which the NPV is zero.

Sensitivity Analysis

- Change the input variables **one at a time** to see how sensitive NPV is to each variable. Using this analysis, we can identify the most significant variables by their effect on the NPV.
- The range is the difference between the best-case NPV and the worst-case NPV.

Scenario Analysis

- Change **several input variables at a time**, then calculate the NPV for each scenario. The greater the dispersion in NPV across the given scenarios, the higher the risk of the project.
- The underlying variables are interconnected.

Monte Carlo Simulation

General steps:

1. Build the model of interest, which is a function of several input variables. Assume a specific probability distribution for each input variable.
2. Simulate random draws from the assumed distribution for each input variable.
3. Given the inputs from Step 2, determine the value of the quantity of interest.
4. Repeat Steps 2 and 3 many times.
5. Using the simulated values of the quantity of interest, calculate the mean, variance, and other measures.

Inversion method: Set $F_X(x) = u$.

Real Options

Real options are capital budgeting options that give managers the right, but not the obligation, to make a particular business decision in the future after new information becomes available.

Decision tree is a graphical approach that illustrates alternative decisions and potential outcomes in an uncertain economy.

2 kinds of nodes in the decision tree:

- The square node (■) is the decision node where you have control over the decision.
- The circular node (●) is the information node where you have no control over the outcome.

Solving a Decision Tree:

- Work backwards from the end of the tree.
- At each decision node, determine the optimal choice by comparing the PV of remaining payoffs along each branch.
- At each information node, compute the expected present value of the payoffs from the subsequent branches.
- Discount rate:
  - If the true probability is given, use the cost of capital to discount.
  - If the risk-neutral probability is given, use the risk-free rate to discount.

Value of real option
= NPV(with option) – NPV(without option)

Timing Option (Call Option)

Gives a company the option to delay making an investment with the hope of having better information in the future.

Can be valued using the Black-Scholes formula:

$S =$ current market value of asset
$K =$ initial investment required
$T =$ final decision date
$r_f =$ risk-free rate
$\sigma =$ volatility of asset value
$\text{Div} =$ free cash flow (FCF) lost from delay

Discount FCF at the cost of capital.
Discount K at the risk-free rate.

Factors affecting the timing of investment:

- NPV of the investment
  - Without the timing option, invest today if NPV of investing today is positive.
  - With the timing option, invest today only if NPV of investing today exceeds the value of the option of waiting, assuming the NPV is positive.
- Volatility
  - When huge uncertainty exists regarding the future value of the investment (i.e., high volatility), the option to wait is more valuable.
- Dividends
  - It is better for an investor to wait unless the cost of waiting is greater than the value of waiting.

Sizing Option

- *Growth options* give the company an option to make additional investments when it is optimistic about the future.
- *Abandonment options* give the company an option to abandon the project when it is pessimistic about the future.

CAPITAL STRUCTURE

Equity Financing

Equity Funding for Private Companies

Source of funding for private companies:

- Angel Investors
- Venture Capital Firms
- Private Equity Firms
- Institutional Investors
- Corporate Investors

When a private company first sells equity, it typically issues preferred stock instead of common stock.

A *funding round* occurs when a private company raises money. An initial funding round might start with a "seed round," and then in later funding rounds the securities are named "Series A," "Series B," etc.

Pre-Money and Post-Money Valuation

- The value of the firm *before* a funding round is called the *pre-money valuation*.
- The value of the firm *after* a funding round is called the *post-money valuation*.

Post-money valuation
= Pre-money valuation + Amount invested
= # shares after the funding rounds
× Pre-money price per share

Percentage ownership

$$= \frac{\text{Amount invested}}{\text{Post-money valuation}}$$

$$= \frac{\#\text{ shares owned} \times \begin{matrix}\text{Pre-money}\\ \text{price per share}\end{matrix}}{\text{Post-money valuation}}$$

$$= \frac{\#\text{ shares owned}}{\text{Total }\#\text{ shares}}$$

<u>Venture Capital Financing Terms</u>
Venture capitalists typically hold convertible preferred stock, which differs from common stock due to:
- Liquidity preference

Liquidity preference = Multiplier × Initial inv
- Participation rights
- Seniority
- Anti-dilution protection
- Board membership

There are two ways to exit from a private company:
- Acquisition
- Public offering

<u>Initial Public Offering</u>
An *initial public offering (IPO)* is the first time a company sells its stock to the public.

Advantages of IPO:
- Greater liquidity
- Better access to capital

Disadvantages of IPO:
- Dispersed equity holdings
- Compliance is costly and time-consuming

There are two major types of offerings:
- **Primary offerings**: New shares sold to raise new capital.
- **Secondary offerings**: Existing shares sold by current shareholders.

When issuing an IPO, the company and underwriter must decide on the best mechanism to sell shares:
- **Best-efforts**: Shares will be sold at the best possible price. Usually used in smaller IPOs.
- **Firm commitment**: All shares are guaranteed to be sold at the offer price. Most common.
- **Auction IPOs**: Shares sold through an auction system and directly to the public.

Standard steps to launching a typical IPO:
1. Underwriters typically manage an IPO and they are important because they:
   - Market the IPO.
   - Assist in required filings.
   - Ensure the stock's liquidity after the IPO.
2. Companies must file a *registration statement*, which contains two main parts:
   - Preliminary prospectus/red herring.
   - Final prospectus.
3. A fair valuation of the company is performed by the underwriter through road show and book building.
4. The company will pay the IPO underwriters an *underwriting spread*. After the IPO, underwriters can protect themselves more against losses by using the *over-allotment allocation* or *greenshoe provision*.

4 IPO Puzzles:
- The average IPO seems to be priced too low.
- New issues appear cyclical.
- The transaction costs of an IPO are high.
- Long-run performance after an IPO is poor on average.

Debt Financing
<u>Corporate Debt: Public Debt</u>
Public debt trades on public exchanges. The bond agreement takes the form of an *indenture,* which is a legal agreement between the bond issuer and a trust company.
4 common types of corporate debt:
- Notes (Unsecured)
- Debentures (Unsecured)
- Mortgage bonds (Secured)
- Asset-backed bonds (Secured)

The new debt that has lower seniority than existing debenture issues is called *a subordinated debenture*.

International bonds are classified into four broadly defined categories:
- Domestic bonds – issued by local, bought by foreign
- Foreign bonds – issued by foreign, bought by local
- Eurobonds – issued by local or foreign
- Global bonds

<u>Corporate Debt: Private Debt</u>
Private debt is negotiated directly with a bank or a small group of investors. It is cheaper to issue due to the absence of the cost of registration.

2 main types of private debt:
- Term loan
- Private placement

<u>Other Types of Debt</u>
Government entities issue *sovereign debt* and *municipal bonds* to finance their activities.

Sovereign debt is issued by the national government. In the US, sovereign debt is issued as bonds called "Treasury securities."

There are four types of Treasury securities:
- Treasury bills
- Treasury notes
- Treasury bonds
- Treasury inflation-protected securities (TIPS)

Municipal bond is issued by the state and local governments.

There are also several types of municipal bonds based on the source of funds that back them:
- Revenue bonds
- General obligation bonds

<u>Asset-Backed Securities</u>
An *asset-backed security (ABS)* is a security whose cash flows are backed by the cash flows of its underlying securities.

The biggest sector of the ABS market is the *mortgage-backed security (MBS)* sector. An MBS has its cash flows backed by home mortgages. Because mortgages can be repaid early, the holders of an MBS face *prepayment risk*.

Banks also issue ABS using consumer loans, such as credit card receivables and automobile loans.

A private ABS can be backed by another ABS. This new ABS is known as a *collateralized debt obligation (CDO)*.

Capital Structure Theory: Perfect Capital Markets

Perfect Capital Markets

- Investors and firms can trade the same set of securities at competitive market prices equal to the present value of their future cash flows.
- No taxes, transaction costs, or issuance costs.
- The financing and investment decisions are independent of each other.

MM Proposition I

- The total value of a firm is equal to the market value of the total cash flows generated by its asset.
- The value of a firm is **unaffected** by its choice of capital structure.
- Changing a firm's capital structure merely changes how the value of its assets is divided between debt and equity, but not the firm's total value.
- $V_L = V_U$

Homemade leverage:

- Investors can borrow or lend at no cost on their own to achieve a capital structure different from what the firm has chosen.
- If an investor adds \$x worth of debt to the capital structure, then he must reduce the equity by \$x in order for the total firm's value to remain unchanged. To determine x, set the adjusted current debt-equity ratio to equal the target debt-equity ratio:

$$\frac{D + x}{E - x} = \left(\frac{D}{E}\right)_{\text{Target}}$$

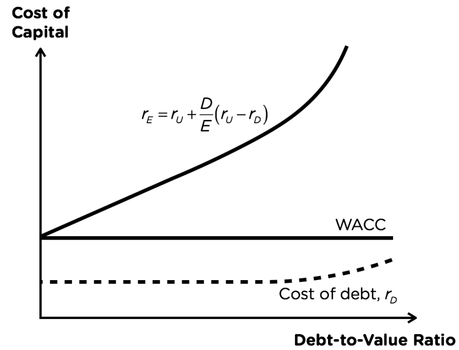
MM Proposition II

- The cost of capital of levered equity increases with the firm's debt-to-equity ratio:

$$r_E = r_U + \frac{D}{E}(r_U - r_D)$$

- Because there are no taxes in a perfect capital market, the firm's WACC and the unlevered cost of capital coincide:

$$r_U = r_{WACC} = \frac{E}{E + D} r_E + \frac{D}{E + D} r_D$$



Note:

- Since debt holders have a priority claim on assets and income above equity holders, debt is less risky than equity, and thus $r_D < r_E$.
- As companies take on more debt, the risk to equity holders increases, and subsequently the cost of equity increases.
- As the amount of debt increases, the chance that the firm will default increases, and subsequently the cost of debt increases.
- Although both cost of debt and cost of equity increase as the company takes on more debt, WACC remains unchanged because more weight is placed on the lower-cost debt.

WACC with Multiple Securities:

$$r_U = r_{WACC} = \sum w_i \cdot r_i$$

Levered and Unlevered Betas:

$$\beta_U = w_E \beta_E + w_D \beta_D$$

$$\beta_E = \beta_U + \frac{D}{E}(\beta_U - \beta_D)$$

Capital Structure Theory: Taxes and Financial Distress Costs

Interest Tax Shield

- The use of debt results in **tax savings** for the firm, which adds to the value of the firm.
- $V_L = V_U + \text{PV(Interest tax shield)}$
  Interest tax shield $=$ Corp. Tax Rate $\times$ Int Pmt

For a firm that borrows debt D and keeps the **debt permanently**, if the firm's marginal tax rate (a.k.a. **effective tax advantage of debt**) is τ_C, then the present value of the interest tax shield is:

$$\text{PV(Interest tax shield)} = \tau_C \cdot D$$

WACC with Taxes

The firm's effective after-tax WACC measures the required return to the firm's investors after taking into account the benefit of the interest tax shield:

$$\begin{aligned} r_{WACC} &= w_E r_E + w_D r_D (1 - \tau_C) \\ &= w_E r_E + w_D r_D - w_D r_D \tau_C \end{aligned}$$

where:

$w_E r_E + w_D r_D = r_U =$ pretax WACC

$w_D r_D \tau_C =$ reduction due to tax shield

Note:

- As debt increases, the reduction due to interest tax shield increases, WACC falls, and thus the value of the firm increases.

Interest Tax Shield with a Target Debt-Equity Ratio

When a firm adjusts its debt over time so that its debt-equity ratio is expected to remain constant, we can value the interest tax shield by:

1. Calculating the value of the unlevered firm, V_U, by discounting cash flows at the unlevered cost of capital (i.e., pre-tax WACC).
2. Calculating the value of the levered firm, V_L, by discounting cash flows at the WACC (i.e., after-tax WACC).
3. $\text{PV(Interest tax shield)} = V_L - V_U$

Financial Distress Costs

A firm that fails to make its required payments to debt holders is said to **default** on its debt.

After the firm defaults, the debt holders have claims to the firm's assets through a legal process called **bankruptcy**.

Two forms of bankruptcies:

- **Chapter 7 liquidation.** A trustee supervises the liquidation of the firm's assets through an auction. The proceeds from the liquidation are used to pay the firm's creditors, and the firm ceases to exist.
- **Chapter 11 reorganization.** The firm's existing management is given the opportunity to propose a reorganization plan. While developing the plan, management continues to operate the business.

The *present value of financial distress costs* has three components:

1. The costs of financial distress and bankruptcy, in the event they occur.

- Direct costs – fees to outside professionals like legal and accounting experts, consultants, appraisers, auctioneers, and investment bankers.
  - Higher for firms with more complicated business operations
  - Typically higher, in percentage terms, for smaller firms
- Indirect costs – loss of customers, loss of suppliers, loss of employees, loss of receivables, fire sale of assets, inefficient liquidation, cost to creditors.
  - Are difficult to measure and are often much larger than direct costs
  - May occur even prior to bankruptcy if the potential perceived threat of future bankruptcy is high
- Companies with marketable tangible assets (e.g., airlines, steel manufacturers) have lower costs of financial distress than companies without these assets (e.g., information technology companies, companies in the service industry) because tangible assets can be sold relatively easily.

Alternatives to bankruptcy designed to save the direct costs:

- **Workout**. The company negotiates directly with creditors and works out an agreement.
- **Prepackaged bankruptcy** (or "prepack"). The company will first create a reorganization plan with the agreement of its primary creditors, and then file Chapter 11 reorganization to implement the plan.

2. The probability of financial distress and bankruptcy occurring

- Companies with a higher debt-to-equity ratio have a higher probability of bankruptcy.
- This probability increases when the volatility of a firm's cash flows and asset values increases.
  - Firms with *steady* cash flows (e.g., utility companies) can use high levels of debt and still have low probability of default.
  - Firms with *volatile* cash flows (e.g., semiconductor firms) must have low levels of debt in order to have low probability of default

3. **The appropriate discount rate for the distress costs**

- The higher the firm's beta:
  - The more likely it will be in distress
  - The more negative the beta of the distress costs
  - The lower the discount rate for the distress costs
  - The higher the PV of distress costs

Who Bears the Financial Distress Costs?

- Debt holders recognize that when the firm defaults, they will not be able to obtain the full value of the assets. As a result, they will pay less (or demand higher yields) for the debt initially.
- It is the equity holders who most directly bears the financial distress costs.

Capital Structure Theory: Agency Cost and Asymmetric Information

<u>The Agency Costs of Leverage</u>

- **Excessive risk-taking and asset substitution.** A company replacing its low-risk assets with high-risk investments. Shareholders may benefit from high-risk projects, even those with negative NPV.
- **Debt overhang or underinvestment.** Shareholders may be unwilling to finance new, positive-NPV projects.
- **Cashing out.** When a firm faces financial distress, shareholders have an incentive to liquidate assets at prices below their market values and distribute the proceeds as dividends.

<u>Estimating the Debt Overhang</u>

Equity holders will benefit from a new investment requiring investment I only if:

$$\frac{NPV}{I} > \frac{\beta_D D}{\beta_E E}$$

<u>Who Bears the Agency Costs?</u>

- When an unlevered firm issues new debt, equity holders will ultimately bear the costs.
- Once a firm has debt already in place, some of the bankruptcy or agency costs from taking on **additional** debt can fall on **existing** debt holders.

The **leverage ratchet effect** explains that once existing debt is in place:

- Equity holders may have an incentive to take on more debt even if it reduces the firm value.
- Equity holders will not have an incentive to decrease leverage by buying back debt even if it will increase the firm value.

<u>Reducing Agency Costs</u>

To mitigate the agency costs of debt, firms and debt holders can:

- Issue short-term debt
- Include debt covenants in bonds that place restrictions on the actions a firm can take

<u>The Agency Benefits of Leverage</u>

Managers have interests that may differ from shareholders' and debt holders' interests:

- **Empire building**. Managers tend to take on investments that increase the size, rather than the profitability, of the firm
- **Managerial entrenchment**. Because managers face little threat of being replaced, managers can run the firm to suit their interests.

Leverage can provide incentives for managers to run a firm more efficiently and effectively due to:

- Increased ownership concentration.
- Reduced wasteful investment.
- Reduced managerial entrenchment and increased commitment.

Free cash flow hypothesis: Wasteful spending is more likely to happen when firms have high levels of cash flow in excess of what is needed.

Costs of Asymmetric Information

When managers have more information about a firm than investors, there is **asymmetric information**.

Lemons principle: When managers have private information about the value of a firm, investors will discount the price they are willing to pay for new equity issue due to adverse selection.

Adverse selection: A seller with private information is likely to sell you worse-than-average goods.

Credibility principle: Claims in one's self-interest are credible only if they are supported by actions that would be too costly to take if the claims were untrue.

Managers consider how their actions will be perceived by investors in selecting financing methods for new investments:

- **Issuing equity** is typically viewed as a **negative** signal as managers tend to issue equity when they believe that the firm's stock is overvalued.
- **Issuing more debt** is typically viewed as a **positive** signal as the company is taking on commitment to make timely interest and principal payments.

Adverse selection has several implications for equity issuance:

- The stock price declines on the announcement of an equity issue.
- The stock price tends to rise prior to the announcement of an equity issue.
- Firms tend to issue equity when information asymmetries are minimized, such as immediately after earnings announcement.

Pecking order hypothesis: Managers prefer to make financing choices that send positive rather than negative signals to outside investors.

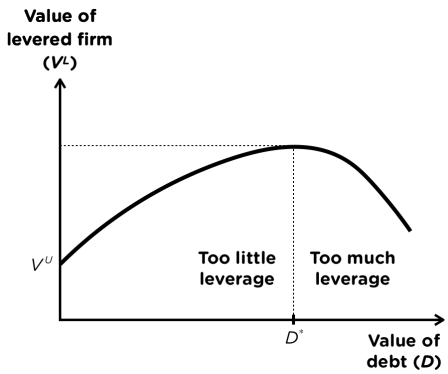
The pecking order (from most favored to least favored financing option):

Internally generated equity (i.e., retained earnings) > Debt > External equity (i.e., newly issued shares)

Trade-Off Theory

Balance the value-enhancing effects of debt on a firm's capital structure with the value-reducing effects.

$$\begin{aligned} &V_L \\ &= V_U + \text{PV(Interest tax shield)} \\ &- \text{PV(Financial distress costs)} \\ &- \text{PV(Agency costs of debt)} \\ &+ \text{PV(Agency benefits of debt)} \end{aligned}$$



The optimal level of debt, D^*, occurs at the point where the firm's value is maximized. It balances the benefits and costs of leverage.

Characteristics of the firm will impact the relative importance of these costs and benefits:

- **R&D-Intensive Firms.** Firms with high research and development costs typically maintain low levels of debt.
- **Low-Growth, Mature Firms.** Mature, low-growth firms with stable cash flows and tangible assets benefit from high debt.

USEFUL FORMULAS FROM EXAMS P & FM

Poisson with mean λ	$\Pr[N = n] = \frac{e^{-\lambda}\lambda^n}{n!}$
Exponential with mean θ	$F_X(x) = 1 - \exp\left(-\frac{x}{\theta}\right)$
Uniform on $[a, b]$	$F_X(x) = \frac{x-a}{b-a}$ $E[X] = \frac{a+b}{2}$ $\text{Var}[X] = \frac{(b-a)^2}{12}$

Interest rate conversion:

$$(1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = e^{rt}$$

Geometric series:

$$\text{Sum} = \frac{\text{First Term} - \text{First Omitted Term}}{1 - \text{Common Ratio}}$$

Infinite geometric series:

$$\text{Sum} = \frac{\text{First Term}}{1 - \text{Common Ratio}}$$

The PV of an annuity:

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$$

$$a_{\overline{\infty|}} = v + v^2 + \cdots = \frac{1}{i}$$

The PV of an n-year annuity immediate with payments of $1, (1+k), (1+k)^2, \ldots, (1+k)^{n-1}$:

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k}$$

The PV of a geometrically increasing perpetuity immediate with payments of $1, (1+k), (1+k)^2, \ldots$:

$$PV = \frac{1}{i-k}$$