Calcul différentiel

Exercice 1 :

$f(x,y) = x^2(x+y)$ $\quad f \in \mathcal{C}^2(\mathbb{R}^2)$ car polynôme, soient $x,y \in \mathbb{R}$,

Dérivée partielle ordre 1 :

$$\frac{\partial f}{\partial x}(x,y) = 2x(x+y) + x^2 = 2x^2+2xy+x^2 = 3x^2+2xy \qquad \frac{\partial f}{\partial y}(x,y) = x^2$$

Dérivées partielles ordre 2 :

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x+2y \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 0 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 2x = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

thm de Schwarz

$f(x,y) = e^{xy}$ $\quad f \in \mathcal{C}^2(\mathbb{R}^2)$, car compo fct expo + polynôme en x,y. Soient $x,y \in \mathbb{R}^2$

Dérivée partielle ordre 1 :

$$\frac{\partial f}{\partial x}(x,y) = ye^{xy} \qquad \frac{\partial f}{\partial y}(x,y) = xe^{xy}$$

Dérivées partielles ordre 2 :

$$\frac{\partial^2 f}{\partial x^2}(x,y) = y^2e^{xy} \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = x^2e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = e^{xy} + xye^{xy} = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

Thm de Schwarz : si $\mathcal{C}^2$ donc der croisées égales

Exercice 2 :

Rappel : Soit $f : \mathbb{R}^n \to \mathbb{R}^p$, $(x_1,\dots,x_n) \mapsto f(x_1,\dots,x_n)$ et $g : \mathbb{R}^p \to \mathbb{R}^q$, $(y_1,\dots,y_n) \mapsto g(y_1,\dots,y_n)$.
Si f est de classe $\mathcal{C}^1(\mathbb{R}^n)$ et g de classe $\mathcal{C}^1(\mathbb{R}^p)$ alors $g \circ f \in \mathcal{C}^1(\mathbb{R}^n)$ et on a :
$\forall (x_1,\dots,x_n) \in \mathbb{R}^n$, $\forall i \in [\![1,n]\!]$, $\forall j \in [\![1,q]\!]$,

$$\frac{\partial (g\circ f)_j}{\partial x_i} = \sum_{k=1}^{p} \frac{\partial g_j}{\partial y_k}(f(x_1,\dots,x_n)) \cdot \frac{\partial f_k}{\partial x_i}(x_1,\dots,x_n)$$

1. $g : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto f(2+2t, t^2) \end{cases}$

La fonction $t \mapsto (2+2t, t^2)$ est de classe $\mathcal{C}^1$ (polynômes), donc g est $\mathcal{C}^1$ par composition.

On va noter $u(t) = 2+2t$ et $v(t) = t^2$

$$g'(t) = u'(t) \cdot \frac{\partial f}{\partial x}(u(t), v(t)) + v'(t) \cdot \frac{\partial f}{\partial y}(u(t), v(t)) = 2 \cdot \frac{\partial f}{\partial x}(2+2t, t^2) + 2t \cdot \frac{\partial f}{\partial y}(2+2t, t^2)$$

2. La fonction $(u,v) \mapsto (uv, u^2+v^2)$ est $\mathcal{C}^1$ (polynômes), donc par composition h est de classe $\mathcal{C}^1$.

$$\frac{\partial h}{\partial u}(u,v) = \frac{\partial f}{\partial x}(uv, u^2+v^2) \cdot \frac{\partial (uv)}{\partial u} + \frac{\partial f}{\partial y}(uv, u^2+v^2) \cdot \frac{\partial (u^2+v^2)}{\partial u} = v \cdot \frac{\partial f}{\partial x}(uv, u^2+v^2) + 2u \cdot \frac{\partial f}{\partial y}(uv, u^2+v^2)$$

$$\frac{\partial h}{\partial v}(u,v) = u \cdot \frac{\partial f}{\partial x}(uv, u^2+v^2) + 2v \cdot \frac{\partial f}{\partial y}(uv, u^2+v^2)$$

Exercice 3 :

1. g est de classe $\mathcal{C}^1$ comme composée de fct de classe $\mathcal{C}^1$.

$$g'(t) = \frac{\partial f}{\partial x}(tx, ty) \cdot \frac{\partial (tx)}{\partial t} + \frac{\partial f}{\partial y}(tx, ty) \cdot \frac{\partial (ty)}{\partial t} = x \cdot \frac{\partial f}{\partial x}(tx, ty) + y \cdot \frac{\partial f}{\partial y}(tx, ty)$$

2. $f(tx, ty) = t f(x,y)$ $\quad \forall (t,x,y) \in \mathbb{R}^3$

a) On a $g(t) = t f(x,y)$ donc $g'(t) = f(x,y)$. Or d'après 1. $g'(t) = x \cdot \frac{\partial f}{\partial x}(tx,ty) + y \cdot \frac{\partial f}{\partial y}(tx,ty)$ $\Rightarrow$ d'où le résultat.

b) En posant $t=0$: $f(x,y) = x \cdot \underbrace{\frac{\partial f}{\partial x}(0,0)}_{\alpha} + y \cdot \underbrace{\frac{\partial f}{\partial y}(0,0)}_{\beta} = \alpha x + \beta y$

Adimensionnement :

Exercice 4 :

$$\begin{cases} \frac{\partial \rho}{\partial t} = D \cdot \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x}\left(\chi \rho \cdot \frac{\partial a}{\partial x}\right) + (\alpha_0 - \beta_0 \rho)\rho \\ \frac{\partial a}{\partial t} = D_a \cdot \frac{\partial^2 a}{\partial x^2} + h\rho - ka \end{cases}$$

Eq de réaction de diffusion avec chimiotactisme

Rappels :

Adimensionner : intro des $q^{tés}$ / valeurs | caractéristiques / de référence qui vont permettre, via des changements de variables / inconnues, de travailler avec des var / inc sans dimension.

But : obtenir un modèle sans dimensions avec des param (sans dim) sur lesquels on va pouvoir faire des hypothèses (temps long, faible diffusion, faible réaction...) et étudier le comportement des solutions dans ce cadre.

Deux phases :

1. Chgt de var et d'inc en introduisant des valeurs caractéristiques (sans trouver de valeurs)
2. Choisir les valeurs des $q^{tés}$ de réf.

Phase 1 :

On introduit L, T, ρ_0 et A tq :

$x = L\tilde{x}$ (où L en m et $\tilde{x}$ ss unité)

$t = T\tilde{t}$ (où T en s et $\tilde{t}$ ss unité)

$\rho(t,x) = \rho_0 \bar{\rho}(t,x) = \rho_0 \bar{\rho}(T\tilde{t}, L\tilde{x}) = \rho_0 \tilde{\rho}(\tilde{t},\tilde{x})$ où $\tilde{\rho}(\tilde{t},\tilde{x}) = \frac{1}{\rho_0}\rho(T\tilde{t}, L\tilde{x})$ (où ρ_0 en m^{-1} et $\tilde{\rho}(\tilde{t},\tilde{x})$ ss unité)

car concentrations

$a(t,x) = A\bar{a}(t,x) = A\bar{a}(T\tilde{t}, L\tilde{x}) = A\tilde{a}(\tilde{t},\tilde{x})$ où $\tilde{a}(\tilde{t},\tilde{x}) = \frac{1}{A}a(T\tilde{t}, L\tilde{x})$ (où A en m^{-1} et $\tilde{a}(\tilde{t},\tilde{x})$ ss unité)

on voit une fct composée

$$\frac{\partial \rho}{\partial t}(t,x) = \frac{\partial}{\partial t}(\rho_0 \tilde{\rho}(\tilde{t},\tilde{x})) = \rho_0 \frac{\partial}{\partial t}(\tilde{\rho}(\tilde{t},\tilde{x})) = \rho_0 \frac{\partial}{\partial t}\left(\tilde{\rho}\left(\frac{t}{T},\frac{x}{L}\right)\right) = \rho_0 \frac{\partial \tilde{\rho}}{\partial \tilde{t}}\left(\frac{t}{T},\frac{x}{L}\right) \cdot \frac{\partial\left(\frac{t}{T}\right)}{\partial t} = \frac{\rho_0}{T} \cdot \frac{\partial \tilde{\rho}}{\partial \tilde{t}}(\tilde{t},\tilde{x})$$

$$\frac{\partial \rho}{\partial x}(t,x) = \frac{\partial}{\partial x}(\rho_0 \tilde{\rho}(\tilde{t},\tilde{x})) = \frac{\rho_0}{L} \cdot \frac{\partial \tilde{\rho}}{\partial \tilde{x}}(\tilde{t},\tilde{x})$$

$$\frac{\partial^2 \rho}{\partial x^2}(t,x) = \frac{\rho_0}{L^2} \cdot \frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2}(\tilde{t},\tilde{x})$$

De même, on a :

$$\frac{\partial a}{\partial t}(t,x) = \frac{A}{T} \cdot \frac{\partial \tilde{a}}{\partial \tilde{t}}(\tilde{t},\tilde{x}) \qquad \frac{\partial^2 a}{\partial x^2}(t,x) = \frac{A}{L^2} \cdot \frac{\partial^2 \tilde{a}}{\partial \tilde{x}^2}(\tilde{t},\tilde{x})$$

$$\chi \rho(t,x) \cdot \frac{\partial a}{\partial x}(t,x) = \frac{\chi \rho_0 A}{L} \cdot \tilde{\rho}(\tilde{t},\tilde{x}) \cdot \frac{\partial \tilde{a}}{\partial \tilde{x}}(\tilde{t},\tilde{x}) \Rightarrow \frac{\partial}{\partial x}\left(\chi \rho(t,x) \cdot \frac{\partial a}{\partial x}(t,x)\right) = \frac{\chi \rho_0 A}{L^2} \cdot \frac{\partial}{\partial \tilde{x}}\left(\tilde{\rho}(\tilde{t},\tilde{x}) \cdot \frac{\partial \tilde{a}}{\partial \tilde{x}}(\tilde{t},\tilde{x})\right)$$

En remplaçant dans le modèle :

$$\begin{cases} \frac{\rho_0}{T} \cdot \frac{\partial \tilde{\rho}}{\partial \tilde{t}}(\tilde{t},\tilde{x}) = D \cdot \frac{\rho_0}{L^2} \cdot \frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2}(\tilde{t},\tilde{x}) - \frac{\chi \rho_0 A}{L^2} \cdot \frac{\partial}{\partial \tilde{x}}\left(\tilde{\rho}(\tilde{t},\tilde{x}) \cdot \frac{\partial \tilde{a}}{\partial \tilde{x}}(\tilde{t},\tilde{x})\right) + (\alpha_0 - \beta_0 \rho_0 \tilde{\rho}(\tilde{t},\tilde{x}))\rho_0 \tilde{\rho}(\tilde{t},\tilde{x}) \\ \frac{A}{T} \cdot \frac{\partial \tilde{a}}{\partial \tilde{t}}(\tilde{t},\tilde{x}) = D_a \cdot \frac{A}{L^2} \cdot \frac{\partial^2 \tilde{a}}{\partial \tilde{x}^2}(\tilde{t},\tilde{x}) + h\rho_0 \tilde{\rho}(\tilde{t},\tilde{x}) - kA\tilde{a}(\tilde{t},\tilde{x}) \end{cases}$$

$$\begin{cases} \dfrac{\rho_0}{T}\dfrac{\partial\tilde\rho}{\partial\tilde t}(\tilde t,\tilde x) = D\cdot\dfrac{\rho_0}{L^2}\cdot\dfrac{\partial^2\tilde\rho}{\partial\tilde x^2}(\tilde t,\tilde x) - \dfrac{\chi\rho_0 A}{L^2}\cdot\dfrac{\partial}{\partial\tilde x}\left(\tilde\rho(\tilde t,\tilde x)\cdot\dfrac{\partial\tilde a}{\partial\tilde x}(\tilde t,\tilde x)\right) + (\alpha_0-\beta_0\rho_0\tilde\rho(\tilde t,\tilde x))\rho_0\tilde\rho(\tilde t,\tilde x) \\[2ex] \dfrac{A}{T}\cdot\dfrac{\partial\tilde a}{\partial\tilde t}(\tilde t,\tilde x) = D_a\cdot\dfrac{A}{L^2}\dfrac{\partial^2\tilde a}{\partial\tilde x^2}(\tilde t,\tilde x) + h\rho_0\tilde\rho(\tilde t,\tilde x) - kA\tilde a(\tilde t,\tilde x)\end{cases}$$

On adimensionne :

- $$\frac{\rho_0}{T}\frac{\partial\tilde\rho}{\partial\tilde t}(\tilde t,\tilde x) = D\cdot\frac{\rho_0}{L^2}\cdot\frac{\partial^2\tilde\rho}{\partial\tilde x^2}(\tilde t,\tilde x) - \frac{\chi\rho_0 A}{L^2}\cdot\frac{\partial}{\partial\tilde x}\left(\tilde\rho(\tilde t,\tilde x)\cdot\frac{\partial\tilde a}{\partial\tilde x}(\tilde t,\tilde x)\right) + (\alpha_0-\beta_0\rho_0\tilde\rho(\tilde t,\tilde x))\rho_0\tilde\rho(\tilde t,\tilde x)$$

$$\Longleftrightarrow \frac{\partial\tilde\rho}{\partial\tilde t}(\tilde t,\tilde x) = \underbrace{\frac{D\cdot T}{L^2}}_{\text{∅ dim}}\frac{\partial^2\tilde\rho}{\partial\tilde x^2}(\tilde t,\tilde x) - \underbrace{\frac{\chi TA}{L^2}}_{\text{∅ dim}}\frac{\partial}{\partial\tilde x}\left(\tilde\rho(\tilde t,\tilde x)\cdot\frac{\partial\tilde a}{\partial\tilde x}(\tilde t,\tilde x)\right) + \underbrace{T(\alpha_0-\beta_0\rho_0}_{\text{∅ dim}}\tilde\rho(\tilde t,\tilde x))\rho_0\tilde\rho(\tilde t,\tilde x)$$

⚠ vérif cohérence dim

$\frac{\partial\rho}{\partial t} \to m^{-1}.s^{-1}$

$\frac{\partial^2\rho}{\partial x^2} \to m^{-1}.m^{-2} = m^{-3}$

$\frac{\partial\rho}{\partial t} \sim D\frac{\partial^2\rho}{\partial x^2} \to m^{-1}.s^{-1} \sim D\,m^{-3}$

$\to D \sim m^2.s^{-1}$

On réécrit la 1ère éq sans les tildes :

$$\frac{\partial\rho}{\partial t} = \frac{DT}{L^2}\frac{\partial^2\rho}{\partial x^2} - \frac{\chi AT}{L^2}\frac{\partial}{\partial x}\left(\rho\frac{\partial a}{\partial x}\right) + T\alpha_0\,\rho - T\beta_0\rho_0\rho^2$$

- $$\frac{A}{T}\frac{\partial a}{\partial t} = \frac{D_a A}{L^2}\frac{\partial^2 a}{\partial x^2} + h\rho_0\rho - kAa$$

$$\Rightarrow \frac{\partial a}{\partial x} = \frac{D_a T}{L^2}\frac{\partial^2 a}{\partial x^2} + \frac{h\rho_0 T}{A}\rho - kTa$$

Un choix naturel pour ρ_0 et A, est de prendre la/les valeur(s) de l'état d'équilibre du système (solution stationnaire et homogène) qui vérifie :

$$\begin{cases}(\alpha_0-\beta_0\rho^*)\rho^* = 0 \\ h\rho^* - ka^* = 0\end{cases} \Longleftrightarrow \begin{cases}\cancel{\rho^* = 0} \\ \cancel{a^* = 0}\end{cases} \text{ et } \begin{cases}\rho^* = \dfrac{\alpha_0}{\beta_0} \\ a^* = \dfrac{h\alpha_0}{k\beta_0}\end{cases}$$

On pose donc : $\rho_0 = \frac{\alpha_0}{\beta_0}$ et $a^* = \frac{h\alpha_0}{k\beta_0}$

On injecte dans le modèle:

- $$\frac{\partial\rho}{\partial t} = \frac{DT}{L^2}\frac{\partial^2\rho}{\partial x^2} - \frac{\chi AT}{L^2}\frac{\partial}{\partial x}\left(\rho\frac{\partial a}{\partial x}\right) + T\alpha_0\rho - T\cancel{\beta_0}\left(\frac{\alpha_0}{\cancel{\beta_0}}\right)\rho^2 \Rightarrow \frac{\partial\rho}{\partial t} = \frac{DT}{L^2}\frac{\partial^2\rho}{\partial x^2} - \frac{\chi AT}{L^2}\frac{\partial}{\partial x}\left(\rho\frac{\partial a}{\partial x}\right) + T\alpha_0(1-\rho)\rho$$

- $$\frac{\partial a}{\partial t} = \frac{D_a T}{L^2} + h\left(\frac{\alpha_0}{\cancel{\beta_0}}\right)T\left(\frac{k\cancel{\beta_0}}{\cancel{h\alpha_0}}\right)\rho - kTa \Rightarrow \frac{\partial a}{\partial t} = \frac{D_a T}{L^2}\frac{\partial^2 a}{\partial x^2} + kT(\rho - a)$$

Deux choix possibles pour T : $T = \frac{1}{\alpha_0}$ ou $T = \frac{1}{k}$

L : $\frac{DT}{L^2} = 1$ ou $\frac{D_a T}{L^2} = 1$

$\Longleftrightarrow L^2 = DT$ ou $L^2 = D_a T$

$\Longleftrightarrow L = \sqrt{DT}$ ou $L = \sqrt{D_a T}$

On obtient alors : $$\frac{\partial\rho}{\partial t} = \frac{\partial\rho^2}{\partial x^2} - \overset{\varepsilon}{\boxed{\frac{\chi A}{D}}}\frac{\partial}{\partial x}\left(\rho\frac{\partial a}{\partial x}\right) + (1-\rho)\rho$$

$$\frac{\partial a}{\partial t} = \overset{r}{\boxed{\frac{D_a}{D}}}\frac{\partial a^2}{\partial x^2} + \overset{\mu}{\boxed{\frac{k}{\alpha_0}}}(\rho - a)$$

Possibilité : étudier les solutions qd $r \ll 1$ ou qd $r \gg 1$

Exercice 5 :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + v \frac{\partial u}{\partial x}(t,x) = f(t,x) & , \forall t>0, \forall x \in \mathbb{R} \\ u(t=0,x) = u_0(x) \end{cases}$$

$$u(t,x) = u_0(x - vt) + \int_0^t f(s, x-(t-s)v)\,ds$$

1. Solution de l'équation de transport avec second membre : $u(t,x) = u_0(x-vt) + \underbrace{\int_0^t f(s, x-(t-s)v)\,ds}_{=g(t,x)}$

Soit $(t,x) \in \mathbb{R}^2$ et $h \in \mathbb{R}$,

$$\frac{1}{h}\left(g(t+h,x) - g(t,x)\right) = \frac{1}{h}\left[\int_0^{t+h} f(s, x-(t+h-s)v)\,ds - \int_0^t f(s,x-(t-s)v)\,ds\right] = \frac{1}{h}\left[\int_0^t \left(f(s, x-(t+h-s)v) - f(s,x-(t-s)v)\right)ds + \int_t^{t+h} f(s, x-(t+h-s)v)\,ds\right]$$

$$= \int_0^t \frac{1}{h}\left(f(s, x-(t+h-s)v) - f(s,x-(t-s)v)\right)ds + I(t,x,h)$$

On note $z : \begin{pmatrix} \mathbb{R}^3 \longrightarrow \mathbb{R} \\ (t,x,s) \mapsto f(s, x-(t-s)v) \end{pmatrix}$

Donc $\frac{1}{h}\left(f(s, x-(t+h-s)v) - f(s,x-(t-s)v)\right) = \frac{1}{h}\left(z(t+h,x,s) - z(t,x,s)\right) = \frac{\partial z}{\partial t}(t,x,s) + o(h)$

Or $\frac{\partial z}{\partial t}(t,x,s) = \frac{\partial f}{\partial x}(s, x-(t-s)v) \cdot \frac{\partial(x-(t-s)v)}{\partial t} = -v \frac{\partial f}{\partial x}(s, x-(t-s)v)$

Ici, on a supposé implicitement que f admet une dérivée partielle par rapport à sa seconde variable en tout point de $\mathbb{R}^2$.

Donc $\frac{1}{h}\left(g(t+h,x) - g(t,x)\right) = \int_0^t -v \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds + o(h) + I(t,x,h)$

$$I(t,x,h) = \frac{1}{h}\int_t^{t+h} f(s, x-(t+h-s)v)\,ds$$

Théorème de la moyenne : si f est continue sur $[a,b]$. Alors $\exists\, \alpha \in [a,b]$ tq $\int_a^b f(t)\,dt = (b-a) f(\alpha)$

On va supposer f continue sur $\mathbb{R}$ par rapport à sa première variable, et on applique le théorème :

$$\exists\, \alpha_h \in [t,t+h], \ \frac{1}{h}\int_t^{t+h} f(s, x-(t+h-s)v)\,ds = \frac{1}{h}\cdot h\, f(\alpha_h, x-(t+h-\alpha_h)v) = f(\alpha_h, x-(t+h-\alpha_h)v)$$

On a $\alpha_h \in [t,t+h]$, donc $\alpha_h \xrightarrow[h\to 0]{} t$ et $x-(t+h-\alpha_h)v \xrightarrow[h\to 0]{} x$

Pour passer à la limite dans la fonction f, on doit supposer f continue au point t et au p^t x, $\forall (t,x) \in \mathbb{R}^2$.

Donc on suppose $f \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$, alors $\lim_{h\to 0} f(\alpha_h, x-(t+h-\alpha_h)v) = f(t,x)$

Au final, on aura donc

$$\frac{1}{h}\left(g(t+h,x) - g(t,x)\right) \xrightarrow[h\to 0]{} -v\int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds + f(t,x)$$

Bilan 1 :

- $\frac{\partial g}{\partial t}(t,x)$ existe et est continue sur $\mathbb{R}$ si $\frac{\partial f}{\partial x}(t,x) \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$ et $f \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$ et alors $\frac{\partial g}{\partial t}(t,x) = -v\int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds + f(t,x)$

- De la même manière, on aura :

$$\frac{1}{h}\left(g(t,x+h) - g(t,x)\right) = \frac{1}{h}\left[\int_0^t f(s,x+h-(t-s)v)\,ds - \int_0^t f(s,x-(t-s)v)\,ds\right] = \int_0^t \frac{1}{h}\left(f(s,x+h-(t-s)v) - f(s,x-(t-s)v)\right)ds = \int_0^t \frac{1}{h}\left(z(t,x+h,s) - z(t,x,s)\right)dz \to \int_0^t \frac{\partial z}{\partial x}(t,x,s)\,ds$$

Or $\frac{\partial z}{\partial x}(t,x,s) = \frac{\partial f}{\partial x}(s, x-(t-s)v) \cdot \frac{\partial (x-(t-s)v)}{\partial x}$

$= \int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds$??

Donc $\frac{1}{h}(g(t,x+h) - g(t,x)) = \int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds + o(h) \xrightarrow[h\to 0]{} \int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds$

Bilan 2 :

$\frac{\partial g}{\partial x}(t,x) \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$ si $\frac{\partial f}{\partial x} \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$ et $\frac{\partial g}{\partial x}(t,x) = \int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds$

Retour à la solution u : $u(t,x) = u_0(x-vt) + g(t,x)$

Au final, on aura donc $u \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R})$ si on a :
1. $u_0 \in \mathcal{C}^1(\mathbb{R})$
2. $f \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$
3. $\frac{\partial f}{\partial x} \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$

Et on a : $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \frac{\partial}{\partial t}(u_0(x-vt)) + v\frac{\partial}{\partial x}(u_0(x-vt)) + \frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x}$

$= -v u_0'(x-vt) + v u_0'(x-vt) - v\int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds + f(t,x) + v\int_0^t \frac{\partial f}{\partial x}(s, x-(t-s)v)\,ds = f(t,x)$

$\Rightarrow$ $\boxed{u \text{ est bien solution}}$

2. Soit $(t,x) \in [0,T]\times\mathbb{R}$,

$|u(t,x)| = |u_0(x-vt) + g(t,x)|$

$\leq |u_0(x-vt)| + |g(t,x)|$

$\leq \|u_0\|_{L^\infty(\mathbb{R})} + |g(t,x)|$

et $|g(t,x)| = \left|\int_0^t f(s, x-(t-s)v)\,ds\right| \leq \int_0^t |f(s, x-(t-s)v)|\,ds$

$\leq \int_0^t \|f\|_{L^\infty(\mathbb{R}\times\mathbb{R})}\,ds$

$\leq \|f\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \cdot t$

$\leq T\|f\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$

Donc $|u(t,x)| \leq \|u_0\|_{L^\infty(\mathbb{R})} + T\|f\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$ d'où $u \in L^\infty([0,T]\times\mathbb{R})$

3. Il suffit de prendre $f = 0$ et $u_0 : x \mapsto \mathbb{1}_{[0,1]}(x)$ (u_0 discontinue en 0 et en 1) ou bien $u_0 : x \mapsto x^2\sin\left(\frac{1}{x}\right)$ (u_0 non dérivable en 0), (cu 1.1)

Exercice 8 :

Méthode des caractéristiques : (éq de transport)

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + v(t,x)\cdot\frac{\partial u}{\partial x}(t,x) = f(t,x,u(t,x)) & \forall (t,x) \in \mathbb{R}_+^*\times\mathbb{R} \\ u(0,x) = u_0(x) \end{cases}$$

On pose $z(t,x) = u(t, X(t,x))$ avec $\boxed{\begin{cases} \frac{\partial X}{\partial t}(t,x) = v(t, X(t,x)) \\ X(0,x) = x_0 \end{cases}}$ (EDO)

On obtient alors l'éq en $z(t,x)$:

$$\frac{\partial z}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,X(t,x)) + \frac{\partial u}{\partial x}(t,X(t,x)) \cdot \overbrace{\frac{\partial X}{\partial t}(t,x)}^{=v(t,X(t,x))}$$

$$= \left(\frac{\partial u}{\partial t} + v\,\frac{\partial u}{\partial x}\right)(t,X(t,x)) = f(t,X(t,x),u(t,X(t,x)))$$

Donc on obtient :

$$\boxed{\begin{cases} \frac{\partial z}{\partial t}(t,x) = f(t,X(t,x),z(t,x)) \\ z(0,x) = u(0,X(0,x)) = u(0,x) = u_0(x) \end{cases}}$$ (EDO)

Plan :

1. Résolution de l'éq en X

2. ——————— en z

3. Inverser la relation entre $X(t,x)$ et x pour obtenir $u(t,x)$

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1. $$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \underbrace{(2x+3)}_{=v(t,x)}\frac{\partial u}{\partial x}(t,x) = \underbrace{u(t,x)-1}_{=f(t,x,u(t,x))}, & \forall t>0,\ \forall x\in\mathbb{R} \\ u(0,x) = \cos(x), & \forall x\in\mathbb{R} \end{cases}$$

On pose $z(t,x) = u(t,X(t,x))$ avec :

$$\begin{cases} \frac{\partial X}{\partial t}(t,x) = 2X(t,x)+3 \\ X(0,x) = x \end{cases}$$

Et alors on obtient : $$\begin{cases} \frac{\partial z}{\partial t}(t,x) = z(t,x)-1 \\ z(0,x) = \cos(x) \end{cases}$$

• Résolution en X :

$$\frac{\partial X}{\partial t}(t,x) - 2X(t,x) = 3 \iff X(t,x) = c(x)e^{2t} - \frac{3}{2}$$

Or $X(0,x) = x$ donc $c(x) - \frac{3}{2} = x \Rightarrow c(x) = x + \frac{3}{2}$. Donc $X'(t,x) = (x+\frac{3}{2})e^{2t} - \frac{3}{2}$

• Résolution en z :

$$\frac{\partial z}{\partial t}(t,x) - z(t,x) = -1 \iff z(t,x) = c(x)e^{t} + 1$$

Or $z(0,x) = \cos(x)$ donc $c(x)+1 = \cos(x) \Rightarrow c(x) = \cos(x) - 1$. Donc $z(t,x) = (\cos(x)-1)e^{t} + 1$

• Expression de $u(t,x)$:

On sait que $z(t,x) = u(t,X(t,x))$.

Donc $u(t,(x+\frac{3}{2})e^{2t} - \frac{3}{2}) = (\cos(x)-1)e^{t}+1$. On pose $y = (x+\frac{3}{2})e^{2t} - \frac{3}{2}$.

Alors on a :

$$(x+\frac{3}{2})e^{2t} = y + \frac{3}{2} \Rightarrow x = (y+\frac{3}{2})e^{-2t} - \frac{3}{2}$$

On trouve alors l'expression de $u(t,y)$:

$$u(t,y) = \left(\cos\left((y+\frac{3}{2})e^{-2t} - \frac{3}{2}\right) - 1\right)e^{t} + 1$$

D'où l'expression de $u(t,x)$:

$$u(t,x) = \left(\cos\left((x+\frac{3}{2})e^{-2t} - \frac{3}{2}\right) - 1\right)e^{t} + 1$$

2. $\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \overbrace{x^2}^{= v(t,x)} \dfrac{\partial u}{\partial x}(t,x) = \underbrace{1 - u(t,x)}_{= f(t,x,u(t,x))} & , \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0,x) = g(x) & , \forall x \in \mathbb{R} \end{cases}$

On pose $z(t,x) = u(t, X(t,x))$ où $\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = X^2(t,x) \\ X(0,x) = x \end{cases}$ et alors : $\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = 1 - z(t,x) \\ z(0,x) = g(x) \end{cases}$

- Résolution en X :

$\dfrac{\partial X}{\partial x}(t,x) = X^2(t,x) \iff \dfrac{\partial X/\partial t\,(t,x)}{X^2(t,x)} = 1$

$\iff \dfrac{\partial}{\partial t}\left(\dfrac{-1}{X(t,x)}\right) = 1$

$\iff \dfrac{-1}{X(t,x)} = t + c(x)$

$\iff X(t,x) = \dfrac{-1}{t + c(x)}$

Or $X(0,x) = x$ donc $-\dfrac{1}{c(x)} = x \Rightarrow c(x) = -\dfrac{1}{x}$

Donc $X(t,x) = \dfrac{x}{1-tx}$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R} \setminus \{(t, \frac{1}{t}), t \in \mathbb{R}_+^*\}$

- Résolution en z :

$\dfrac{\partial z}{\partial t}(t,x) = 1 - z(t,x) \iff z(t,x) = c(x)e^{-t} + 1$

Et $z(0,x) = g(x) \Rightarrow c(x) = g(x) - 1$

Donc $z(t,x) = (g(x)-1)e^{-t} + 1$

- Expression de u :

On obtient $u\left(t, \dfrac{x}{1-tx}\right) = (g(x)-1)e^{-t} + 1$

On pose $y = \dfrac{x}{1-tx} \iff (1+yt)x = y \iff x = \dfrac{y}{1+ty}$

Donc

$u(t,y) = \left(g\left(\dfrac{y}{1+ty}\right) - 1\right)e^{-t} + 1$, $\forall (t,y) \in \mathbb{R}_+ \times \mathbb{R} \setminus \{(t, -\frac{1}{t}), t \in \mathbb{R}_+^*\}$

Exercice 6 :

$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - c^2 \dfrac{\partial^2 u}{\partial x^2}(t,x) = f(t,x) & , \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0,x) = u_0(x) & , \forall x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x) & , \forall x \in \mathbb{R} \end{cases}$

26/03

1. a) $w(t,x) = \dfrac{\partial u}{\partial t}(t,x) + c\dfrac{\partial u}{\partial x}(t,x)$ $\rightarrow$ on veut mq $\dfrac{\partial w}{\partial t}(t,x) - c\dfrac{\partial w}{\partial x}(t,x) = 0$

- $\dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial t^2}(t,x) + c\dfrac{\partial^2 u}{\partial t \partial x}(t,x)$
- $\dfrac{\partial w}{\partial x}(t,x) = \dfrac{\partial^2 u}{\partial x \partial t}(t,x) + c\dfrac{\partial^2 u}{\partial x^2}(t,x)$

Donc $\dfrac{\partial w}{\partial t}(t,x) - c\dfrac{\partial w}{\partial x}(t,x) = \dfrac{\partial^2 u}{\partial t^2}(t,x) + c\dfrac{\partial^2 u}{\partial t \partial x}(t,x) - c\dfrac{\partial^2 u}{\partial x \partial t}(t,x) - c^2\dfrac{\partial^2 u}{\partial x^2}(t,x) = \underbrace{\left(\dfrac{\partial^2 u}{\partial t^2}(t,x) - c^2\dfrac{\partial^2 u}{\partial x^2}(t,x)\right)}_{= 0} + c\left(\dfrac{\partial^2 u}{\partial t \partial x}(t,x) - \dfrac{\partial^2 u}{\partial x \partial t}(t,x)\right)$

On suppose u de classe $\mathcal{C}^2$ sur $\mathbb{R}_+ \times \mathbb{R}$ (u solution n'implique à minima que u de classe $\mathcal{C}^1$ sur $\mathbb{R}_+ \times \mathbb{R}$ et $\dfrac{\partial^2 u}{\partial t^2}(t,x)$ et $\dfrac{\partial^2 u}{\partial x^2}(t,x)$ existent).

Donc d'après le théorème de Schwarz :

$$\frac{\partial^2 u}{\partial t \partial x}(t,x) = \frac{\partial^2 u}{\partial x \partial t}(t,x)$$

D'où

$$\boxed{\frac{\partial w}{\partial t}(t,x) - c\frac{\partial w}{\partial x}(t,x) = 0}$$

b) Condition initiale sur w :

$$w(0,x) = \frac{\partial u}{\partial t}(0,x) + c\frac{\partial u}{\partial x}(0,x)$$
$$= v_0(x) + c\underbrace{\frac{\partial u}{\partial x}(0,x)}_{\overset{?}{=}\frac{\partial}{\partial x}(u(0,x))}$$

Taylor à l'ordre 1 : $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $\forall h \in \mathbb{R}$,

$$u(t,x+h) = u(t,x) + h\frac{\partial u}{\partial x}(t,x) + o(h)$$

Donc

$$u(0,x+h) = u(0,x) + h\frac{\partial u}{\partial x}(0,x) + o(h)$$

$$\Leftrightarrow u_0(x+h) = u_0(x) + h\frac{\partial u}{\partial x}(0,x) + o(h)$$

Donc $\frac{\partial u}{\partial x}(0,x) = \frac{u_0(x+h) - u_0(x)}{h} + o(1)$ d'où qd $h \to 0$, $\frac{\partial u}{\partial x}(0,x) = \lim_{h \to 0}\left(\frac{u_0(x+h) - u_0(x)}{h}\right) = u_0'(x)$

Donc

$$\boxed{w(0,x) = v_0(x) + cu_0'(x)}$$

L'équation à résoudre est donc :

$$\begin{cases} \frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x} = 0 \\ w(0,x) = v_0(x) + cu_0'(x) \end{cases}$$

C'est une équation de transport de vitesse $-c$ constante, donc la solution est donnée par :

$$\boxed{w(t,x) = (v_0 + cu_0')(x+ct)}$$

On note $w_0 = v_0 + cu_0'$

Equation en u :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + c\frac{\partial u}{\partial x}(t,x) = \boxed{w(t,x)} \\ u(0,x) = u_0(x) \end{cases}$$

D'après le cours (ou l'exo 5), on a :

$$u(t,x) = u_0(x-ct) + \int_0^t w(\underbrace{s}_{t}, \underbrace{x-(t-s)c}_{x})\,ds$$
$$= u_0(x-ct) + \int_0^t w_0(x-(t-s)c + cs)\,ds$$
$$\boxed{= u_0(x-ct) + \int_0^t w_0(x-(t-2s)c)\,ds}$$

On note W_0, une primitive de w_0. On a alors :

$$\frac{\partial}{\partial s}\left(W_0(x-(t-2s)c)\right) = w_0(x-(t-2s)c) \times 2c$$
$$= 2c \cdot w_0(x-(t-2s)c)$$

Donc $\displaystyle\int_0^t w_0(x-(t-2s)c)\,ds = \int_0^t \frac{1}{2c}\frac{\partial W_0}{\partial s}(x-(t-2s)c)\,ds = \frac{1}{2c}\Big[W_0(x-(t-2s)c)\Big]_0^t = \frac{1}{2c}\left(W_0(x+ct)-W_0(x-ct)\right)$

Donc

$$u(t,x) = u_0(x-ct) - \frac{1}{2c}W_0(x-ct) + \frac{1}{2c}W_0(x+ct) \iff \boxed{u(t,x) = \left(u_0 - \frac{1}{2c}W_0\right)(x-ct) + \frac{1}{2c}W_0(x+ct)}$$

On retrouve bien $u(t,x) = A(x-ct)+B(x+ct)$ avec $A = u_0 - \frac{1}{2c}W_0$ et $B = \frac{1}{2c}W_0$.

c) $u(t,x) = A(x-ct)+B(x+ct)$.

On a alors,

$$\begin{cases} u(0,x) = u_0(x) = A(x)+B(x) \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x) = \dfrac{\partial}{\partial t}\Big[A(x-ct)+B(x+ct)\Big](0,x) \\ \qquad = A'(x)(-c) + B'(x).c \end{cases}$$

Donc

$$\boxed{\begin{cases} A(x)+B(x) = u_0(x) \\ -cA'(x)+cB'(x) = v_0(x) \end{cases}}$$

$$\Rightarrow \begin{cases} A'(x)+B'(x) = u_0'(x) \\ -cA'(x)+cB'(x) = v_0(x) \end{cases} \quad \underset{L_2 \leftarrow L_2 + cL_1}{\Rightarrow} \begin{cases} A'(x) = u_0'(x) - B'(x) = u_0'(x) - \dfrac{1}{2c}(v_0(x)+cu_0'(x)) \\ 2c\,B'(x) = v_0(x) + cu_0'(x) \end{cases}$$

D'où $\begin{cases} A(x) = \displaystyle\int_0^x \left(\frac{1}{2}u_0'(y) - \frac{1}{2c}v_0(y)\right)dy + k_1 \\ B(x) = \displaystyle\frac{1}{2c}\int_0^x (v_0(y)+cu_0'(y))\,dy + k_2 \end{cases}$ et $A(x)+B(x) = u_0(x) \Rightarrow A(0)+B(0) = k_1+k_2 = u_0(0)$

Donc $\begin{cases} A(x) = \displaystyle\frac{1}{2}\int_0^x u_0'(y)\,dy - \frac{1}{2c}\int_0^x v_0(y)\,dy + k_1 \\ B(x) = \displaystyle\frac{1}{2c}\int_0^x v_0(y)\,dy + \frac{1}{2}\int_0^x u_0'(y)\,dy + u_0(0) - k_1 \end{cases} \iff \begin{cases} A(x) = \displaystyle\frac{1}{2}(u_0(x)-u_0(0)) - \frac{1}{2c}\int_0^x v_0(y)\,dy + k \\ B(x) = \displaystyle\frac{1}{2}(u_0(x)-u_0(0)) + \frac{1}{2c}\int_0^x v_0(y)\,dy + (u_0(0)-k) \end{cases}$

Donc

$$u(t,x) = A(x-ct)+B(x+ct)$$
$$= \frac{1}{2}u_0(x-ct) - \frac{1}{2}\cancel{u_0(0)} - \frac{1}{2c}\int_0^{x-ct} v_0(y)\,dy + \cancel{k} + \frac{1}{2}u_0(x+ct) + \frac{1}{2}\cancel{u_0(0)} + \frac{1}{2c}\int_0^{x+ct} v_0(y)\,dy - \cancel{k}$$
$$= \frac{1}{2}\left(u_0(x-ct)+u_0(x+ct)\right) + \frac{1}{2c}\left[-\int_0^{x-ct} v_0(y)\,dy + \int_0^{x+ct} v_0(y)\,dy\right]$$

2. Cas général : $f \neq 0$ 29/03

a) On suppose $u_0 = v_0 = 0$

Rappel : $w(t,x) = \dfrac{\partial u}{\partial t}(t,x) + c\dfrac{\partial u}{\partial x}(t,x)$

$$\frac{\partial w}{\partial t}(t,x) - c\frac{\partial w}{\partial x}(t,x) \overset{q°1.a}{=} \frac{\partial^2 u}{\partial t^2}(t,x) - c^2\frac{\partial^2 u}{\partial x^2}(t,x) + c\left(\cancel{\frac{\partial^2 u}{\partial t \partial x}(t,x) - \frac{\partial^2 u}{\partial x \partial t}(t,x)}\right) = f(t,x)$$

Schwarz

Donc

$$\boxed{\frac{\partial w}{\partial t}(t,x) - c\frac{\partial w}{\partial x}(t,x) = f(t,x)}$$

Et $w(0,x) = \frac{\partial u}{\partial t}(0,x) + c\frac{\partial u}{\partial x}(0,x) \overset{q°1.b)}{=} v_0(x) + c u_0'(x) \Rightarrow \boxed{w(0,x) = 0}$

Pour finir, $u(0,x) = u_0(x) \Rightarrow \boxed{u_0(x) = 0}$

b) On commence par résoudre le problème en w :

$$\begin{cases} \frac{\partial w}{\partial t}(t,x) - c\frac{\partial w}{\partial x}(t,x) = f(t,x) \\ w(0,x) = 0 \end{cases}$$

(solution exo 5 → signe de c ≠)

Cette solution est connue : (cf. exo 5)

$$\boxed{w(t,x) = \int_0^t f(s, x + (t-s)c)\,ds}$$

Donc le problème en u s'écrit :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + c\frac{\partial u}{\partial x}(t,x) = \int_0^t f(s, x+(t-s)c)\,ds = g(t,x) \\ u(0,x) = 0 \end{cases}$$

La solution est donnée par :

$$u(t,x) = \int_0^t g(y, x - (t-y)c)\,dy$$

$$= \int_0^t \left(\int_0^y f(s, (x-(t-y)c) + (y-s)c\,ds\right)dy$$

$$= \int_0^t\left(\int_0^y f(s, x - ct + cy + cy - cs)\,ds\right)dy$$

$$\Rightarrow \boxed{u(t,x) = \int_0^t\int_0^y f(s, x - (t-2y+s)c\,ds\,dy}$$

3. On note $\tilde{u}(t,x)$ la solution précédente du pb:

$$\begin{cases} \frac{\partial^2 \tilde{u}}{\partial t^2}(t,x) - c^2\frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) = f(t,x) \\ \tilde{u}(0,x) = \frac{\partial \tilde{u}}{\partial x}(0,x) = 0 \end{cases}$$