Post-Lecture Question 1

(a)

$$\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V + \mu\,\mathrm{d}N$$

$$\left(\frac{\partial E}{\partial S}\right)_{V,N} = T \quad \Rightarrow \quad \left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T}$$

$$T\left(\frac{\partial S}{\partial N}\right)_{E,V} + \mu = 0 \quad \Rightarrow \quad \left(\frac{\partial S}{\partial N}\right)_{E,V} = -\frac{\mu}{T}$$

(b)

$$\begin{aligned}
&\ln \Omega_{\text{bath}}(N - N_S, E - E_S) \\
&= \ln \Omega_{\text{bath}}(N, E) - N_S\left(\frac{\partial \ln \Omega_{\text{bath}}}{\partial N}\right)_E (N, E) - E_S\left(\frac{\partial \ln \Omega_{\text{bath}}}{\partial E}\right)_N (N, E) + \mathcal{O}(N_S^n E_S^{2-n}) \\
&= \ln \Omega_{\text{bath}}(N, E) - \frac{N_S}{k_B}\left(\frac{\partial S}{\partial N}\right)_E - \frac{E_S}{k_B}\left(\frac{\partial S}{\partial E}\right)_N \\
&= \ln \Omega_{\text{bath}}(N, E) - \beta E_S + \beta N_S \mu
\end{aligned}$$

(c)

$$\exp(\ln \Omega_{\text{bath}}(N - N_S, E - E_S)) = \Omega_{\text{bath}}(N, E) e^{-\beta E_S} e^{\beta N_S \mu}$$

$$\begin{aligned}
\Rightarrow \quad P(N_S, E_S) &= \frac{\Omega_{\text{bath}}(N, E) e^{-\beta E_S} e^{\beta N_S \mu}}{\sum_i \sum_j \Omega_{\text{bath}}(N, E) e^{-\beta E_j} e^{\beta N_i \mu}} \\
&= \frac{e^{-\beta E_S} e^{\beta N_S \mu}}{\sum_i \sum_j e^{-\beta E_j} e^{\beta N_i \mu}}
\end{aligned}$$

$$\begin{aligned}
P(N_S) &= \frac{\sum_j e^{-\beta E_j} e^{\beta N_S \mu}}{\sum_i \sum_j e^{-\beta E_j} e^{\beta N_i \mu}} \\
&= \frac{Q(N_S, V, T) e^{\beta N_S \mu}}{\sum_{N=0}^{\infty} Q(N, V, T) e^{\beta N \mu}}
\end{aligned}$$