

Problem 1:

Let $f, g \in \mathcal{O}(D_r(c))$, $g(c) = 0$, and $g'(c) \neq 0$.

Without loss of generality, $c = 0$. Now, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Because $g(0) = 0$, we have that $b_0 = 0$. So,

$$\begin{aligned}
 \operatorname{Res}_0 \frac{f}{g} &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{f}{g} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=0}^{\infty} b_n z^n} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=1}^{\infty} b_n z^n} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{\sum_{n=0}^{\infty} a_n z^n}{z \sum_{n=0}^{\infty} b_{n+1} z^n} dz \\
 &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D_r(0)} \frac{a_n z^n}{z \sum_{m=0}^{\infty} b_{m+1} z^m} dz
 \end{aligned}$$

All but the first of those terms vanish; $\frac{z^n a_n}{z b_1 + z^2 b_2 \dots} = \frac{z^n a_n}{z h(z)} = \frac{z^{n-1} a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 if $n \geq 1$ ($h(z)$ is nonzero on a small enough disk, else g is identically zero...and so $g' = 0$. It's also nonzero at 0, because $b_1 \neq 0$ (else $g'(0) = 0$)).

So, using h as above,

$$\begin{aligned}
\operatorname{Res}_0 \frac{f}{g}(c) &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D_r(0)} \frac{a_n z^n}{z \sum_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{a_0}{\sum_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= \frac{a_0}{2\pi i} \int_{\partial D_r(0)} \frac{1}{h(z)} dz \\
&= \frac{a_0}{2\pi i} 2\pi i \frac{1}{h(0)} \\
&= a_0/b_1 \\
&= f(0)/g'(0)
\end{aligned}$$

Yielding our result.

Problem 2:

Let $f \in \mathcal{O}(\dot{D}_r(c))$ with c not an essential singularity. Without loss of generality, $c = 0$.

Consider $\operatorname{Res}_0 \frac{f'}{f}$. Now, let $f(z) = \sum_{n=k}^{\infty} a_n z^n$ with a_k nonzero, so that

$f'(z) = \sum_{n=k}^{\infty} n a_n z^{n-1}$; k will be the order of zero if positive, and -1 times the order of pole if negative, and this is clear. So,

$$\begin{aligned}
\operatorname{Res}_0 \frac{f'}{f} &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{f'}{f} dz \\
&= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{\sum_{n=k}^{\infty} n a_n z^{n-1}}{\sum_{n=k}^{\infty} a_n z^n} dz \\
&= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{\sum_{n=k}^{\infty} n a_n z^{n-1}}{z \sum_{n=k}^{\infty} a_n z^{n-1}} dz \\
&= \frac{1}{2\pi i} \sum_{n=k}^{\infty} \int_{\partial D_r(0)} \frac{n a_n z^{n-1}}{z \sum_{m=k}^{\infty} a_m z^{m-1}} dz
\end{aligned}$$

All but the first of those terms vanish; $\frac{nz^n a_n}{zb_1+z^2b_2\ldots} = \frac{z^n a_n}{zh(z)} = \frac{z^{n-1} a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 ($h(z)$ is nonzero on a small enough disk, else a_k was zero...).

So,

$$\begin{aligned}
\operatorname{Res}_0 \frac{f'}{f} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D_r(0)} \frac{n a_n z^{n-1}}{z \sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
&= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{k a_k z^{k-1}}{z \sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
&= \frac{k a_k}{2\pi i} \int_{\partial D_r(0)} \frac{1}{h(z)} dz \\
&= \frac{a_0}{2\pi i} 2\pi i \frac{1}{h(0)} \\
&= k
\end{aligned}$$

Yielding our result.

Problem 3:

A real-variable analogue of Rouché's Theorem would be:

“Let I be an open interval (a, b) , f, g be differentiable on I , and let J be an open interval containing the closure of I .

If $|f(a)| < |g(a)|$ and $|f(b)| < |g(b)|$, then $g, g - f$ have the same number of zeroes in I .”

The obvious counterexample is $f(x) = 0$ if $x = 0$, $f(x) = \sin(1/x)$ otherwise, and $g(x) = 1$ on the interval $(0, 1/2\pi)$. Now, $f(x) = 0$ at $0, 1/2\pi$, and $g(x) = 1$, so $|f| < |g|$ on the boundary of the interval. But g has no zeroes, and $g - f$ has infinitely many zeroes. So this breaks.

Problem 4:

Consider $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2} dx$, with $a \in \mathbb{R}$ and $a > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iz} + e^{-iz}}{z^2 + a^2} dz \\
&= \int_0^{\infty} \frac{e^{iz} + e^{-iz}}{z^2 + a^2} dz \quad (\text{because the function is even...}) \\
&= \int_0^{\infty} \frac{e^{iz}}{z^2 + a^2} dz + \int_0^{\infty} \frac{e^{-iz}}{z^2 + a^2} dz \\
&= \int_0^{\infty} \frac{e^{iz}}{z^2 + a^2} dz - \int_0^{-\infty} \frac{e^{iz}}{z^2 + a^2} dz \quad (\text{u-substitute } -z) \\
&= \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2 + a^2} dz \\
&= \int_{-\infty}^{\infty} \frac{\sum_{n=0}^{\infty} \frac{(iz)^n}{n!}}{z^2 + a^2} dz \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\frac{(iz)^n}{n!}}{z^2 + a^2} dz \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} \frac{z^n}{z^2 + a^2} dz \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \sum_{c \in \mathbb{C}^+} \text{Res}_c \frac{z^n}{z^2 + a^2} \quad (\text{As discussed in class})
\end{aligned}$$

With the last line being discussed in class, and C^+ being the upper half of the complex plane. Now, $\frac{z^n}{z^2 + a^2}$ can only have poles where $z^2 + a^2 = 0$; that is, where $z = \pm ia$.

So, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx &= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \sum_{c \in \mathbb{C}} \text{Res}_c \frac{z^n}{z^2 + a^2} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\text{Res}_{ia} \frac{z^n}{z^2 + a^2} \right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\text{Res}_0 \frac{(z + ia)^n}{(z + ia)^2 + a^2} \right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\text{Res}_0 \frac{\sum_{m=0}^n \binom{n}{m} z^m (ia)^{n-m}}{z^2 + 2ia} \right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\text{Res}_0 \frac{\sum_{m=0}^n \binom{n}{m} z^m (ia)^{n-m}}{z^2 + 2ia} \right]
\end{aligned}$$

Applying problem 1 to the above, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx &= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_0 \frac{\sum_{m=0}^n \binom{n}{m} z^m (ia)^{n-m}}{z^2 + 2iaz} \right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\frac{(ia)^n}{2ia} \right] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{i^n}{n!} [(ia)^n] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{i^n}{n!} [(ia)^n] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{(a i i)^n}{n!} \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \\
&= \frac{\pi}{a} e^{-a}
\end{aligned}$$

Which is the desired result.

Problem 5:

Consider $\int_{\Gamma_T} z^\alpha R(z) dz$ (with R a rational function, and Γ_T as pictured below.)

Now,

$$\begin{aligned}
\int_{\Gamma_T} z^\alpha R(z) dz &= \int_{\gamma_1} z^\alpha R(z) dz + \int_{\gamma_2} z^\alpha R(z) dz + \int_{\gamma_3} z^\alpha R(z) dz \\
&= \int_T^0 (t - 1/T)^\alpha R(t - 1/T) dt + \int_{\gamma_2} z^\alpha R(z) dz + \int_0^T (t + 1/T)^\alpha R(t + 1/T) dt
\end{aligned}$$

Problem 6:

Consider $e^z = 6z^2 + 1$. This is equivalent to $0 = 6z^2 + 1 - e^z$.

Define $g(z) = 6z^2 + 1$ and $f(z) = e^z$. When $|z| = 2$, $|g| \geq |6z^2| - 1 = 23$ and $|f| \leq e^2 \leq 9$. So $g > f$ when $|z| = 2$.

So Rouché's Theorem applies: $e^z = 6z^2 + 1$ has the same number of solutions as $0 = 6z^2 + 1$ on the disk bounded by $|z| = 2$.

Now, $6z^2 + 1$ has two solutions, by the fundamental theorem of algebra. Moreover, $\pm \frac{i}{\sqrt{6}}$ are solutions, as is readily checked. These solutions are both in that disk. So $6z^2 - 1$ has two zeroes on the disk bounded by $|z| = 2$.

So $e^z = 6z^2 + 1$ has 2 solutions on the disk bounded by $|z| = 2$.

Problem 7:

Consider a polynomial, $f(z) = \sum_{n=0}^N a_n z^n$.

Define $M = 9000N \sum |a_n|$ (Note: M is chosen so that $a_N M^N > \sum_{i=0}^n |a_i M^i|$ for any $n < N$). Define $g_0(z) = a_0$. Now, $|f| > |g_0|$ on the boundary of the disk of radius M centered at 0. So $f - g_0$ and f have the same number of zeroes in this disk.

Define $g_1(z) = a_1 z$. Now, $|f - g_0| > |g_1|$ on the boundary of the disk of radius M centered at 0. So $f - g_0 - g_1$ and $f - g_0$ and f have the same number of zeroes in this disk.

The above process can be iterated: define $g_n(z) = a_n z^n$. Then $\left| f - \sum_{m=0}^{n-1} g_m \right| >$

$|g_n|$. So $f - \sum_{m=0}^n g_m$ and f have the same number of zeroes in that disk.

So f and $a_n z^n$ have the same number of zeroes on the disk of radius M centered at 0. So f has n zeroes.

Note that we can pick M arbitrarily large (that was the point of M) and have this work. Thus, f has n zeroes on \mathbb{C} ; this is the fundamental theorem of algebra.

Problem 8:

Let Ω be “standard” (open, bounded, boundary is finitely many piecewise C^1 Jordan curves). Let $f \in \mathcal{O}(G)$, where $G \supset \bar{\Omega}$, and $f \neq 0$ anywhere on $\partial\Omega$.

Consider $\frac{1}{2\pi i} \int_{\partial\Omega} \frac{z^k f'(z)}{f(z)} dz$, where $k \in \mathbb{N}$.

This is equal to $\sum_{c \in \Omega} \text{Res}_c z^k f'/f$.

Consider any individual singularity, $c \in \Omega$. Without loss of generality, $c = 0$.

Now, let $f(z) = \sum_{n=l}^{\infty} a_n z^n$ with a_l nonzero, so that $f'(z) = \sum_{n=l}^{\infty} n a_n z^{n-1}$; l will be the order of zero. It's positive, because $f \in \mathcal{O}(\Omega)$.

$$\begin{aligned}
 \text{Res}_0 z^k \frac{f'}{f} &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{z^k f'}{f} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{z^k \sum_{n=l}^{\infty} n a_n z^{n-1}}{\sum_{n=l}^{\infty} a_n z^n} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{z^k \sum_{n=l}^{\infty} n a_n z^{n-1}}{z \sum_{n=l}^{\infty} a_n z^{n-1}} dz \\
 &= \frac{1}{2\pi i} \sum_{n=l}^{\infty} \int_{\partial D_r(0)} \frac{n a_n z^{n-1}}{z \sum_{m=l}^{\infty} a_m z^{m-1}} dz
 \end{aligned}$$

All but the first of those terms vanish; $\frac{nz^n a_n}{zb_1+z^2b_2\ldots} = \frac{z^n a_n}{zh(z)} = \frac{z^{n-1} a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 ($h(z)$ is nonzero on a small enough disk, else a_k was zero...).

So,

$$\begin{aligned}
 \operatorname{Res}_0 \frac{f'}{f} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\partial D_r(0)} \frac{na_n z^{n-1}}{z \sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
 &= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{ka_k z^{n-1}}{z \sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
 &= \frac{ka_k}{2\pi i} \int_{\partial D_r(0)} \frac{1}{h(z)} dz \\
 &= \frac{a_0}{2\pi i} 2\pi i \frac{1}{h(0)} \\
 &= k
 \end{aligned}$$