Problem 1:

Let $k \subset K$ and $k \subset L$ be finite field extensions contained in some field.

In the following, I freely use the fact that $\dim(U)\dim(V) = \dim(UV)/\dim(U \cap V)$ for any vector spaces U and V over the same field.

Part a:

First, $[KL : L][L : k] = [KL : k]$.

So this means that

$$\frac{[K : k]}{[KL : L]} = \frac{\dim(K)}{\frac{\dim(KL)}{\dim(L)}} = \frac{\dim(K)\dim(L)}{\dim(KL)}$$

That is, we have reduced this problem to the following one; if $[KL : k] \leq [K : k][L : k]$, then the right hand side is at least 1.

So $[K : k] \geq [KL : L]$ if we manage to solve the following part.

Part b:

A result of linear algebra states that for any two vector spaces over the same field, U and V, we have $\dim(U)\dim(V) = \dim(UV)/\dim(U \cap V)$.

So $[KL : k][K \cap L : k] = [K : k][L : k]$. So $[KL : k] \leq [K : k][L : k]$

Part c:

If we have equality in the above, this means that $[K \cap L : k] = 1$, so that $K \cap L$ has dimension 1. That is, that $K \cap L = k$.

If $K \cap L = k$, then $[K \cap L : k] = 1$, so we have equality in the above.

Problem 2:

Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Then $[K : \mathbb{Q}] = [K : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 * 2 = 4$. (The minimal polynomial of $\sqrt{2}$ in $\mathbb{Q}[x]$ is $x^2 - 2$, a polynomial with $\sqrt{3}$ as a root in $\mathbb{Q}(\sqrt{2})[x]$ is $x^2 - 3$, and $\sqrt{3}$ isn't in $\mathbb{Q}(\sqrt{2})$.

Now, $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$; first, $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subset K$, because $\sqrt{2} + \sqrt{3} \in K$; that is, all of the right hand side's generators are in K, so $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subset K$. Next, we have that

$$\frac{1}{\sqrt{2} + \sqrt{3}} = \frac{-\sqrt{2} + \sqrt{3}}{5}$$

So $\sqrt{2} - \sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Adding (or subtracting) $\sqrt{2} + \sqrt{3}$ to this and dividing by 2 shows us that $\sqrt{2}$ and $\sqrt{3}$ are in $\mathbb{Q}(\sqrt{2}+\sqrt{3})$, so all of K's generators are in $\mathbb{Q}(\sqrt{2} + \sqrt{3})$; $K \subset \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Problem 3:

Let $k \subset K$ be an algebraic field extension.

Then every k-homomorphism $\delta : K \to K$ is a monomorphism; this is immediate, as discussed in class.

Next, let $a \in K$. Then a is the root of some $f \in k[x]$, because the extension is algebraic.

Now, for all $b \in K$, $\delta(f(b)) = f(\delta(b))$; this is because δ fixes all of the coefficients of f.

So if b is a root of f that is in K, then

$$\begin{aligned} \delta(f(b)) &= f(\delta(b)) \\ 0 &= f(\delta(b)) \end{aligned}$$

Note that $\delta(b)$ is also in K, so this means that δ permutes the roots of f that are in K. (There's finitely many of them, and δ is a one-to-one map, so it's a permuation.)

This means that there's a $b \in K$ such that $\delta(b) = a$, for all $a \in K$.

So δ is onto. So δ is a one-to-one and onto k-homomorphism, it is an isomorphism.

Problem 4:

If k is finite, then k^* is cyclic: this is example 5.8 a.

If k^* is cyclic, then there is an $a \in k^*$ such that for all $x \in k^*$ there is an $n \in \mathbb{N}$ such that $x = a^n$. This means that k^* is countable; there is a map $\phi : \mathbb{N} \to k^*$ given by $n \mapsto a^n$ that is onto.

Now, either k is the trivial field (in which case, our result is trivial) or there is an $x \in k^*$ with $x \neq 1$.

Pick such an x. Then $x = a^n$ for some n, and $n \neq 0$. Also, there is an x^{-1} with $x^{-1} = a^m$ for some m. So $1 = xx^{-1} = a^n a^m = a^{n+m}$. So $a^0 = a^{n+m}$. This means k^* is finite; it has at least $n + m$ elements.

So $k^* = k \setminus \{0\}$ is finite, so k is finite.

So k is finite if and only if k^* is cyclic.

Problem 5:

Let k be a field, and let $k(x)$ be the field of rational functions of k.

Assume that $[\overline{k(x)} : k(x)]$ is finite. First, we know that $k(x) \subset \overline{k(x)}$ is an algebraic field extension.