Problem 1:

The map described in class is $f \circ g \circ h$, where $f(z) = \frac{z-1}{z+1}$, $g(z) = \sqrt{z}$, and $h(z) = \frac{z-1}{z+1}$.

Its inverse is thus $h^{-1} \circ g^{-1} \circ f^{-1}$, which is $F : D_1(0) \to \overline{\mathbb{C}} \setminus [-1, 1]$ where $F(z) = \frac{\left(\frac{z+1}{1-z}\right)^2 + 1}{1 - \left(\frac{z+1}{1-z}\right)^2} = \frac{-z^2-1}{2z}$.

Consider the set $\partial D_r(0)$ where $r < 1$. We see that $F(\partial D_r(0)) = \{\frac{-z^2-1}{2z} : |z| = r\}$.

Problem 2:

Problem 3:

Consider $f(z) = f(\phi_{-z}(0))$. Taking derivatives, we get

$$\begin{aligned}
f'(z) &= (f(\phi_{-z}(0)))' \\
&= f'(\phi_{-z}(0))\phi'_{-z}(0) \\
&= f'(\phi_{-z}(0))\frac{1}{1 - |z|^2} \\
|f'(z)| &\leq \frac{1}{1 - |z|^2}
\end{aligned}$$

with the last line being due to the previous problem after an adjustment.

Problem 4:

Consider $\{z \in \mathbb{C} : A\,|z|^2 + 2\text{Re}(Bz^2) + 2\text{Re}(Cz) + D = 0\}$, with $A, D \in \mathbb{R}$, $B, C \in \mathbb{C}$ (A, B, C, D fixed).

This describes a line when $A = B = 0$; If A or B is nonzero, then However, if $A = B = 0$, then the set becomes $\{z \in \mathbb{C} : 2\text{Re}(Cz) = D\}$, which is rather clearly a line.

This describes a circle when

Problem 5:

(Note: I had read this in Complex Made Simple before this was assigned.)

Let $\phi \in \mathrm{Aut}(\overline{\mathbb{C}})$. Say $\mathcal{C}$ is the set of all circles and lines in the complex plane.

Note that $\mathrm{Aut}(\overline{\mathbb{C}})$ is the set of linear-fractional transformations. Further note that the set of linear-fractional transformations is generated, as a group, by the set of maps $z \mapsto az + b$ (with $a, b \in \mathbb{C}$ and the map $z \mapsto 1/z$.

It suffices to show our result for the generating set.

The result is clear for linear maps (for circles, note that they're isometries. For lines, note that they're a dilation followed by a translation followed by a rotation.)

For the map $1/z$, consider a line $\ell = \{z \in \mathbb{C} :\}$.

Problem 6:

Let $\Omega \subset \mathbb{C}$ be open, $f_n \in \mathcal{O}(\Omega)$, $\sup(|f_n(z)|) = L < \infty$, $\xi_j \in \Omega$, $\xi_j \to \xi \in \Omega$, and $f_n(\xi_j) \to \Xi_j$ for some Ξ_j.

Problem 7:

Problem 8: