Problem 1:
Let G be a group, with $H < G$ and $K < G$.
Let $HK < G$.

Now, let $HK = KH$.

Thus, $HK < G$ if and only if $KH = HK$.

Problem 2:
Let G be a group and $H \trianglelefteq G$ and $K \trianglelefteq G$, such that $H \cup K = \{e\}$.
Part a:
Let $h \in H$, $k \in K$.
So $hk = kh$ for all $h \in H$, $k \in K$.

Part b:
From the above, it is clear that $HK = KH$. From this fact and problem 1, it follows that HK is a subgroup of G.
Now, let $\phi : H \times K \to HK$ be given by $\phi((h, k)) = hk$.
We show that ϕ is an isomorphism:
First, ϕ is a homomorphism:

Next, ϕ is one-to-one:

Last ϕ is onto:

Thus, there is an isomorphism from $H \times K$ to HK. That is, $H \times K \cong HK$.

Problem 3:
First, Q_8 is non-Abelian:

$$\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$$
$$\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$$

However, all of Q_8's subgroups are normal.

Because $8 = 2 * 2 * 2$, any subgroup of Q_8 has order 1,2,4, or 8.

Any subgroup of order 1,4, or 8 is trivially normal, from the discussion in class. It only remains to show that the subgroups of order 2 are normal.

Now, there is only one subgroup of order 2 in Q_8; it is $I, -I$. This is clear because there is only one element of order 2 in Q_8, and any subgroup of order 2 has to have exactly one element of order 2 (which is trivial from Cayley's theorem... elements of a group must have order dividing the group, and there can only be one element of order 1 (e). So there has to be an element of an order other than 1...there must be an element of order 2. But e has to be in the subgroup, so there's an element of order 1. And because there's only two elements, one of them is e and the other is the element of order 2).

Problem 4:

Consider $\langle s \rangle < \langle s, r^2 \rangle < D_4$.

Now, $\langle s, r^2 \rangle = \{e, s, r^2, sr^2\}$ has order 4; it is normal in D_4.

Also, $\langle s \rangle$ has order 2; it is normal in $\langle blah \rangle$.

However, $\langle s \rangle$ is not normal in D_4: $rsr^{-1} = sr^3r^{-1} = sr^2$, and $sr^2 \notin \langle s \rangle$.

So $\langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_4$, but $\langle s \rangle$ isn't a normal subgroup of D_4.

Problem 5:

Part a:

Part b (i):

Part b (ii):

Problem 6: