Problem 1:
Consider $u(x, y)$ solving

$$u_y^2 u_{xx} + u u_{xy} + u_x^2 u_{yy} = u^2 + 1$$
$$u(x, 0) = \sin(x), u_y(x, 0) = \cos(x)$$

Then the order 2 (and lower) partials for u at $(0, 0)$ are:

$$\begin{aligned} u(0,0) &= \sin(0) = 0 \\ u_x(0,0) &= \cos(0) = 1 \\ u_y(0,0) &= \cos(0) = 1 \\ u_{xx}(0,0) &= -\sin(0) = 0 \\ u_{xy}(0,0) &= -\sin(0) = 0 \\ u_{yy}(0,0) &= 1 \end{aligned}$$

The first five are obtained by the initial conditions and applying partial derivatives to them, and the last is obtained by plugging this information into the PDE. Thus, the second-order Taylor Approximation of u about the point $(0, 0)$ is

$$u(x, y) \approx x + y + \frac{y^2}{2}$$

Now, some of the order 2 (and lower) partials for u at $(\pi/2, 0)$ are:

$$\begin{aligned} u(0,0) &= \sin(\pi/2) = 1 \\ u_x(0,0) &= \cos(\pi/2) = 0 \\ u_y(0,0) &= \cos(\pi/2) = 0 \\ u_{xx}(0,0) &= -\sin(\pi/2) = -1 \\ u_{xy}(0,0) &= -\sin(\pi/2) = -1 \end{aligned}$$

Plugging this information into the PDE yields $-1 = 2$, which is nonsense; thus, u is inconsistent at $(\pi/2, 0)$.

Problem 2:
Let

$$L[u] = yu_{xx} + (x+y)u_{xy} + xu_{yy} - u_x - u_y$$

Part a:
The equation L is hyperbolic when $\Delta = \left(\frac{x+y}{2}\right)^2 - xy > 0$.
Rewriting this condition, we get:

$$\begin{aligned}\left(\frac{x+y}{2}\right)^2 - xy &> 0\\ \left(\frac{x-y}{2}\right)^2 &> 0\\ (x-y)^2 &> 0\\ x &\neq y\end{aligned}$$

That is, L is hyperbolic except when $x = y$.

Part b:
By the discussion in John, the characteristic curves of this PDE satisfy $\frac{dy}{dx} = \frac{(x+y)/2 \pm (x-y)/2}{y}$. That is, the characteristic curves satisfy either $\frac{dy}{dx} = \frac{x}{y}$ or $\frac{dy}{dx} = \frac{-y}{y}$. Solving the ODEs, we get that the characteristic curves are the hyperbolas given by $y^2 - x^2 = c$ for some constant c, and the lines $y = -x + c$.

Part c:
First, consider the solutions to $y\lambda^2 + (x+y)\lambda + x = 0$; they are $\lambda_1 = -1$ and $\lambda_2 = -x/y$, by the quadratic formula.
We want ξ and η so that $\xi_x = -\lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$. Choosing $\xi = x - y$ and $\eta = y^2 - x^2$ works for this. Also, we can solve for x and y in terms of ξ and η: $x = \frac{\eta - \xi^2}{2\xi}$ and $y = \frac{\eta - \xi^2}{2\xi} - \xi$.

Problem 3:
Part a:

Part b: