Problem 1:
Let $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, and let $u(x, t)$ solve

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

Let $v(x, t) = u(x, t) - g(t)$, and extend v to $\{x < 0\}$ by odd reflection (just call the resulting extension v). Then v solves

$$\begin{cases} v_t - v_{xx} = -g' & \text{in } \mathbb{R}_+ \times (0, \infty) \\ v = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ v = 0 & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

So $v(x, t) = \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} -g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dyds$ by formula 17 in the book.

Now,

$$\begin{aligned}
v(x,t) &= \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} -g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= -\left[\int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \right] \\
&= -\left[\int\limits_{\mathbb{R}} \int\limits_0^t g'(s) \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-|x-y|^2}{4(t-s)}} dsdy \right] \\
&= -\left[\int\limits_{\mathbb{R}} \frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{4\pi(t-s)} g(s) \Big|_0^t - \int\limits_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] dsdy \right]
\end{aligned}$$

Now, observe that $\frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{4\pi(t-s)} g(s) \Big|_0^t$ is 0; evaluating at 0 yields 0, and taking the limit as $s \to t$ of that also yields 0.

So,

$$v(x,t) = -\left[\int_{\mathbb{R}} - \int_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}}\right] dsdy\right]$$
$$= -\left[\int_{\mathbb{R}} - \int_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}}\right] dsdy\right]$$
$$= \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}}\right] dyds$$
$$= \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}}\right] dyds - \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}}\right] dyds$$

Thus, by adding $g(t)$ to $v(x,t)$, we get that $u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s)ds$ as desired.

Problem 2:

Let $g \in C(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, $|g| < M$ for some M. Let u be the bounded solution to

$$\begin{cases} \Delta u - u_t = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Part a:

Then $u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y)dy$. Let $\epsilon > 0$. There is a sequence $h_k \in \mathbb{R}^n$ with compact support such that $|g| \geq |h_k|$ and $|g - h_k| \to 0$ uniformly. Also, let $v_k(x,t)$ be the bounded solution to

$$\begin{cases} \Delta v_k - v_{k,t} = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ v_k(x,0) = h_k(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Now,

$$\left|\frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}}(g(y)-h_k(y))dy\right| \leq \frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}}\left|e^{\frac{-|x-y|^2}{4t}}(g(y)-h_k(y))\right|dy$$
$$\leq \left|\frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}}\epsilon dy\right|$$
$$= \epsilon$$

So $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}^n}|u(x,t)| = \lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}^n}\lim\limits_{k\to\infty}|v_k(x,t)| = \lim\limits_{t\to\infty}\lim\limits_{k\to\infty}\sup\limits_{x\in\mathbb{R}^n}|v_k(x,t)|$. But $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}^n}|v_k(x,t)| = 0$, because $h(x)$ has compact support (I'm fairly sure we touched on this in class). So, $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}^n}|u(x,t)| = 0$, as desired.

Part b:

Consider $v(x,t) = u(x,t) - g(x)$. Then $v(x,t)$ solves

$$\begin{cases} \Delta v - v_t = -\Delta g(x) & \text{for } t>0, x\in\mathbb{R}^n \\ v(x,0) = 0 & \text{for } x\in\mathbb{R}^n \end{cases}$$

Thus, $v(x,t) = \int\limits_0^t\int\limits_{\mathbb{R}^n}\Phi(x-y,t-s)\Delta g(y)dyds$.

So $\int\limits_{\mathbb{R}^n} v(x,t)dx = \int\limits_{\mathbb{R}^n}\int\limits_0^t\int\limits_{\mathbb{R}^n}\Phi(x-y,t-s)\Delta g(y)dydsdx$

Switching the order of integration, we get $\int\limits_0^t\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\Phi(x-y,t-s)\Delta g(y)dxdyds$.

Yet, this is $\int\limits_0^t\int\limits_{\mathbb{R}^n}\Delta g(y)dyds$, because $\int\limits_{\mathbb{R}^n}\Phi(x-y,t-s)dx = 1$.

The integral $\int\limits_{\mathbb{R}^n}\Delta g(y)dy$ is constant with respect to t and x; call it C. Then $\int\limits_{\mathbb{R}^n} v(x,t)dx = ct$.

So $\int\limits_{\mathbb{R}^n} u(x,t)dx = ct + \int\limits_{\mathbb{R}^n} g(x)dx$. Yet, the left hand side does not tend to infinity as $t\to\infty$ ($\left|\int\limits_{\mathbb{R}^n} u(x,t)dx\right| = \left|\int\limits_{\mathbb{R}^n}\frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}}g(y)dydx\right| \leq \left|\int\limits_{\mathbb{R}^n}\frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}}Mdydx\right| = M < \infty$); so c must be 0.

So, $\int\limits_{\mathbb{R}^n} u(x,t)dx = \int\limits_{\mathbb{R}^n} g(x)dx$.

Problem 3:
Part a:
Fix $\alpha \in (0,1)$, $\beta \geq 0$.
Note first that $z^\beta e^{-z} = e^{\beta \ln(z) - z}$. So, the desired result is

$$e^{\beta \ln(z) - z} \leq M e^{-\alpha z}$$

for some M, which is equivalent to

$$\beta \ln(z) - z \leq \ln(M) - \alpha(z)$$

for some M. Now, this is equivalent to

$$-\ln(M) \leq (1-\alpha)z - \beta \ln(z)$$

for some M. By applying basic calculus, the right hand side takes a minimum at $z = \beta/(1-\alpha)$, so taking $M = (1-\alpha)(\beta/(1-\alpha)) - \beta \ln(\beta/(1-\alpha))$ suffices so that $-\ln(M) \leq (1-\alpha)z - \beta\ln(z)$ and, equivalently, $z^\beta e^{-z} \leq M e^{-\alpha z}$ for all $z \geq 0$, as desired.

Part b:
First, consider that $\partial_t \Phi(x,t) = e^{-\frac{|x|^2}{4t}} \left[-\frac{2\pi n}{(4\pi t)^{n/2+1}} + \frac{|x|^2}{4t^2(4\pi t)^{n/2}} \right]$. Simplifying,

$$\partial_t \Phi(x,t) = \Phi(x,t) \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right]$$

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, we get

$$
\begin{aligned}
|\partial_t \Phi(x,t)| &= \left|\Phi(x,t)\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&= \Phi(x,t)\left|\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&= \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\left|\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&\leq \frac{Me^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}(|x|^2/(4t))^{\beta}}\left|\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&= M2^{n/2}\Phi(x,2t)(|x|^2/(4t))^{-\beta}\left|\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&= M_1\Phi(x,2t)(|x|^2/(4t))^{-\beta}\left|\left[\frac{|x|^2}{4t^2} - \frac{n}{2t}\right]\right| \\
&= M_1\Phi(x,2t)(|x|^2/(4t))^{-\beta}\frac{|x|^2}{4t^2} \\
&= M_1\Phi(x,2t)/t
\end{aligned}
$$

As desired.

Next, consider that $\partial_{x_i}\Phi(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\left[-\frac{x_i}{2t}\right]$.

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1/2$, we get

$$
\begin{aligned}
|\partial_{x_i}\Phi(x,t)| &= \left|\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\left[-\frac{x_i}{2t}\right]\right| \\
&\leq \frac{Me^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}}\left|\frac{x_i}{2tz^{\beta}}\right| \\
&\leq M2^{n/2}\phi(x,2t)\left|\frac{|x|}{2t}\frac{2\sqrt{t}}{|x|}\right| \\
&= M_2\phi(x,2t)/\sqrt{t}
\end{aligned}
$$

As desired.

Next, let $i \neq j$. Then $\partial_{x_i x_j}\Phi(x,t) = \frac{-x_i}{2t}\partial_{x_j}\Phi(x,t) = \frac{x_i x_j}{4t^2}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$.
Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, we get

$$\begin{aligned}
\left|\partial_{x_i x_j}\Phi(x,t)\right| &= \left|\frac{x_i x_j}{4t^2}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\right| \\
&\leq \frac{Me^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}}\left|\frac{x_i x_j}{4t^2 z^\beta}\right| \\
&\leq M2^{n/2}\phi(x,2t)\left|\frac{|x|^2}{4t^2}\frac{4t}{|x|^2}\right| \\
&= M_3\phi(x,2t)/t
\end{aligned}$$

And also, if $i = j$, then Then $\partial_{x_i x_j}\Phi(x,t) = -\frac{x_i x_j}{4t^2}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} - \frac{1}{2t}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$
Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, $\beta' = 0$, we get

$$\begin{aligned}
\left|\partial_{x_i x_j}\Phi(x,t)\right| &= \left|-\frac{x_i x_j}{4t^2}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} - \frac{1}{2t}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}\right| \\
&\leq M2^{n/2}\left|-\frac{x_i x_j}{8t^2}\frac{1}{z^\beta} - \frac{1}{2t}\frac{1}{z^{\beta'}}\right| \\
&\leq M_3 2^{n/2}\left|1/t\right| \\
&\leq M_3 2^{n/2}/t
\end{aligned}$$

leaving us with all of the results we wanted for this part.

Part c:
Let $u(x,t) = \int\limits_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy$, where $|g| < M$.