Problem 1 (Problem 2 in book):

Let u be such that $\Delta u = 0$ and let v be such that $v(x) = u(Ox)$ for some orthogonal $n \times n$ matrix O.

Then we have:

$$\begin{aligned}
\Delta v(x) &= \Delta u(x) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(Ox) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(Ox_1e_1 + Ox_2e_2 \ldots + Ox_ne_n) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i x_i}(Ox_je_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i x_i}(x_je_j) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(x) \\
&= 0
\end{aligned}$$

That is, solutions to $\Delta u = 0$ are rotation-invariant.

Problem 2 (Problem 3 in book):

Let u be such that

$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

with dimension $n \geq 3$.

Then define $\phi(r) = \fint\limits_{\partial B(0,r)} u(y)dS(y) - \frac{1}{n(n-2)\alpha(n)} \int\limits_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) \Delta u(x)dx.$

Then

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,r)} \frac{\partial u}{\partial \nu} dS(y) - \frac{\partial}{\partial r}\left(\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx\right) \\
&= \frac{r}{n} \fint_{B(0,r)} \Delta u(y)dy - \frac{\partial}{\partial r}\left(\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx\right) \\
&= \frac{r}{n\alpha(n)r^n} \int_{B(0,r)} \Delta u(y)dy - \frac{\partial}{\partial r}\left(\frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx\right) \\
&= \frac{1}{n\alpha(n)}\left(\frac{1}{r^{n-1}} \int_{B(0,r)} \Delta u(y)dy - \frac{\partial}{\partial r}\left(\frac{1}{(n-2)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx\right)\right) \\
&= \frac{1}{n\alpha(n)r^{n-1}}\left(\int_{B(0,r)} \Delta u(y)dy - \frac{r^{n-1}}{n-2}\frac{\partial}{\partial r}\left(\int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx\right)\right)
\end{aligned}$$

(The next chunk is basically "apply polar coordinates in the same fashion as in the book".)

$$\begin{aligned}
&= \frac{1}{n\alpha(n)r^{n-1}}\left(\int_{B(0,r)} \Delta u(y)dy - \int_{B(0,r)} \Delta u(x)dx\right) \\
&= 0
\end{aligned}$$

So ϕ' is identically zero. So ϕ is constant. So $\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0} \fint_{\partial B(0,r)} u(y)dS(y) - \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\Delta u(x)dx = u(0)$.

Thus, by replacing u with g on the boundary and Δu with $-f$ on the interior, we have:

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f dx$$

Problem 3 (Only homework on page):
Let $u(x)$ be a C^2 solution to

$$\Delta u(x) = |x|^2 \text{ on } \mathbb{R}^n$$

with $n \geq 3$.
Set $m(r) = \fint_{\partial B(0,r)} u(y)dS(y)$.
Then u solves

$$\Delta(u) = |x|^2 \text{ in } B(0,r)$$
$$u = u \text{ on } \partial B(0,r)$$

So by the above problem,

$$u(0) = \fint_{\partial B(0,r)} u(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) |x|^2 \, dx$$
$$= m(r) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) |x|^2 \, dx$$
$$= m(r) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{|x|^2}{|x|^{n-2}} - \frac{|x|^2}{r^{n-2}} \right) dx$$

(I must admit that I'm unsure of the details of this next step.)

$$= m(r) - \frac{r^4}{4(n+2)}$$

which is the result we wanted.

If $n = 1$, then $u(x) = x^4 + Cx + D$ are the only solutions to this, where $C, D \in \mathbb{R}$; this follows from elementary differential equations. It suffices to prove that $m(r) = \frac{r^4}{4(n+2)}$, because the addition of a constant to u leaves $u(0) - m(r)$ invariant (and this can be readily checked.)

(I ran out of time with this, but I would've just explicitly calculated $m(r)$ here.)

(I don't have this for $n = 2$.)