Problem 1:

Let $f_n \to f$ in measure, with an integrable function g such that $|f_n| \leq g$ for all n.

We proceed by proving the fact if the domain of each f_n is $[0, 1]$; this fact can be extended to all of $\mathbb{R}$ by applying the $\epsilon 2^{-n}$ method.

Let $\epsilon > 0$.

Because $f_n \to f$ in measure, there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $m(x : |f_n(x) - f(x)| \geq \epsilon/2) < \epsilon/2$.

Problem 2:

Let f be continuous on $[a, b]$, with one of its derivates everywhere non-negative on (a, b).

If this derivate is D_+, then for all $x \in [a, b]$, $\liminf\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq 0$.

If this derivate is D_-, then for all $x \in [a, b]$, $\liminf\limits_{h \to 0^+} \frac{f(x)-f(x+h)}{h} \geq 0$.

If this derivate is D^+, then for all $x \in [a, b]$, $\limsup\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq 0$.

If this derivate is D^-, then for all $x \in [a, b]$, $\limsup\limits_{h \to 0^+} \frac{f(x)-f(x+h)}{h} \geq 0$.

So in all cases, f is nondecreasing.

Problem 3:

Suppose that $f_n(x) \to f(x)$ at each $x \in [a, b]$.

Problem 4:

Suppose that $f \in BV([a, b])$. Then f' exists, by a theorem in class.

Problem 5:

Let g be an absolutely continuous monotone function on $[0, 1]$, and E be a set of measure 0.

We know that g is the antiderivative of some function, f. That is, $g(x) = \int\limits_0^x f(t)dt + g(0)$. for some f.

Problem 6:

Let f be a nonnegative measurable function on $[0, 1]$.

We know that ln is a concave function on $[0, 1]$ (if this is not clear, it's the inverse of a convex function).

So $-\ln$ is a convex function on $[0, 1]$.

So Jensen's inequality applies:

$$-\ln \int f \leq -\int \ln f$$
$$\ln \int f \geq \int \ln f$$

This satisfies the problem.