Problem 1:

Let R be a UFD and P be a prime ideal.

Let P fail to be principal. Let $a \in P$.

Now, a has a prime factorization, $p_1^{\alpha_1} \dots p_n^{\alpha_n}$.

Then one of the $p_i^{\alpha_i}$ is in P; $a \in P$, so $p_1^{\alpha_1} \in P$ or $p_2^{\alpha_2} \dots p_n^{\alpha_n} \in P$. If $p_2^{\alpha_2} \dots p_n^{\alpha_n} \in P$, then $p_2^{\alpha_2} \in P$ or $p_3^{\alpha_3} \dots p_n^{\alpha_n} \in P$. We can iterate this process, so one of the $p_i^{\alpha_i}$ is in P.

So $p_i \in P$, by applying the same method.

So $(p_i) \subset P$. Because p_i is prime, (p_i) is prime (and nonzero). But it's not P, as P is not principal.

So P has a proper, nonzero prime ideal.

Problem 2:

Let k be a field and $n \geq 2$.

If $\text{char}(k) = 2$, $x_1^2 + x_2^2 \dots x_n^2 - 1$ is equal to $(x_1 + x_2 + \dots + x_n - 1)^2$ (when you multiply it out,every term has a factor of 2 except the x_i^2 and -1 terms) and so $x_1^2 + x_2^2 \dots x_n^2 - 1$ is reducible.

Now, if $\text{char}(k) \neq 2$, then $x_1^2 + x_2^2 - 1$ is irreducible in $k[x_1, x_2]$; $x_1^2 + x_2^2 - 1$ is a is a monic polynomial of degree 2 in $k[x_1][x_2] = k[x_1, x_2]$. So if it factors, it factors into a product of degree 1 polynomials; so it factors into something of the form $(x_2 + s)(x_2 + r)$, with r and s both in $k[x_1]$. But the only way for this to happen is if $s = -r$. That is, $1 - x_1^2$ must be a perfect square. However, its unique prime factorization is $(x_1 + 1)(x_1 - 1)$; it is not a perfect square.

We proceed by induction:

Let $x_1^2 + x_2^2 \dots x_{n-1}^2 - 1$ is irreducible in $k[x_1, x_2 \dots x_{n-1}]$, and set this equal to p. It is clear that $x_n^2 + p$ is a monic polynomial of degree 2 in $k[x_1, x_2 \dots x_{n-1}][x_n] = k[x_1, x_2 \dots x_n]$. So if it factors, it factors into a product of degree 1 polynomials; so it factors into something of the form $(x_n + s)(x_n + r)$, with r and s both in $k[x_1, x_2 \dots x_{n-1}]$. But this would mean that $p = rs$ for some $r, s \in k[x_1, x_2 \dots x_{n-1}]$, so p would be reducible.

So if $x_1^2 + x_2^2 \dots x_{n-1}^2 - 1$ is irreducible in $k[x_1, x_2 \dots x_{n-1}]$, then $x_1^2 + x_2^2 \dots x_n^2 - 1$ is irreducible in $k[x_1, x_2 \dots x_n]$.

So we have our result.

Problem 3:

By the reduction criterion, $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Z}[x]$ if it is irreducible in $\mathbb{Z}/(7)[x]$.

Now, if $x^4 + 3x^3 + 3x^2 - 5 = x^4 + 3x^3 + 3x^2 + 2$ is reducible, it either has a root or it can be written as a product of two monic order 2 polynomials.

But $p(x) = x^4 + 3x^3 + 3x^2 + 2$ has no roots;

$$\begin{aligned} p(0) &= 2 \\ p(1) &= 2 \\ p(2) &= 5 \\ p(3) &= 2 \\ p(4) &= 1 \\ p(5) &= 6 \\ p(6) &= 3 \end{aligned}$$

So if p is reducible, then p can be written as a product of two monic order 2 polynomials. That is,

$$\begin{aligned} x^4 + 3x^3 + 3x^2 + 2 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd \end{aligned}$$

This means that

$$\begin{aligned} a + c &= 3 \\ b + d + ac &= 3 \\ ad + bc &= 0 \\ bd &= 2 \end{aligned}$$

The first equation can be reduced to $c = 3 - a$, which yields

$$\begin{aligned} b + d + 3a - a^2 &= 3 \ (\alpha) \\ ad + 3b - ba &= 0 \ (\beta) \\ bd &= 2 \ (\gamma) \end{aligned}$$

Now, (γ) only 6 solutions; we are working in a field, so for any given b there is a unique solution of that equation for d. Also, $b = 0$ fails.

So we have that the only six valid solutions for b and d are:

$$b = 1, d = 2$$
$$b = 2, d = 1$$
$$b = 3, d = 3$$
$$b = 4, d = 4$$
$$b = 5, d = 6$$
$$b = 6, d = 5$$

The middle two fail, for any value of a; because $ad = ba$, (β) gives us $3b = 0$, which fails for any nonzero b. We are left with

$$b = 1, d = 2$$
$$b = 2, d = 1$$
$$b = 5, d = 6$$
$$b = 6, d = 5$$

A rearrangement of (α) gives us $a(3 - a) = 3 - b - d$.

For the first two cases, $b + d = 3$, so we have that $a = 3$ or $a = 0$. If $a = 0$, then (β) reduces to $3b = 0$, which fails for any nonzero b. If $a = 3$, then (β) reduces to $3d = 0$ which fails for any nonzero d.

For the last two cases, $b + d = 30 = 2$. This means that (α) gives us that $a(3 - a) = 1$. But this is unsatisfiable; if $q = a(3 - a)$, then:

$$q(0) = 0$$
$$q(1) = 2$$
$$q(2) = 2$$
$$q(3) = 0$$
$$q(4) = 3$$
$$q(5) = 4$$
$$q(6) = 3$$

That is, every possible solution for (γ) modulo 7 fails to satisfy the system of equations. That is, we cannot reduce $x^4 + 3x^3 + 3x^2 + 2$ modulo 7.

So by the reduction criterion, $x^4+3x^3+3x^2-5$ is irreducible in $\mathbb{Z}[x]$. So $x^4+3x^3+3x^2-5$ is irreducible in $\mathbb{Q}[x]$.

Problem 4:

Let $R=\mathbb{Z}[\sqrt{-5}]$, and $K=\text{Quot}(R)$.

Consider $3x^2+4x+3$. By the quadratic formula, if this polynomial has roots, they are $\frac{-2}{3} \pm \frac{\sqrt{-5}}{3}$. A factorization of $3x^2+4x+3$ is given by $3(x+\frac{2}{3}+\frac{\sqrt{-5}}{3})(x+\frac{2}{3}-\frac{\sqrt{-5}}{3})$. So the polynomial is reducible in $K[x]$.

Now, in $R[x]$, $3x^2+4x+3$ cannot have a constant factored out of it. As it is a degree 2 polynomial, this means that it factors only as a product of two degree 1 polynomials. So any factorization of that polynomial must be of the form $(rx+r'(2+\sqrt{-5}))(sx+s'(2-\sqrt{-5}))$, with $r', s' \in \mathbb{Z}[\sqrt{-5}]$ and $r=3r'$, $s=3s'$. Yet, this means that the leading coefficient of the polynomial is a multiple of 9, which 3 isn't. So the polynomial is irreducible in $R[x]$.

Problem 5:

Note: I'm playing fast and loose with notation. I recognize this, but feel that it's still clear in context what is meant.

Let R be a UFD and P be a prime ideal of $R[x]$ with $P \cap R = 0$. Define $K = \text{quot}(R)$.

We can view P as a subset of $K[x]$. Consider $(P) \subset K[x]$. We see that (P) is principal, as $K[x]$ is a principal ideal domain. Moreover, (P) is not the entire ring, because $P \cap R = 0$. So $(P) = (p)$ for some $p \in K[x]$, with p having degree at least 1.

We can impose that $p \in P$; if it isn't, then we can multiply p by the least common multiple of the quotients of the coefficients of p to get it in $R[x]$. Also, $R[x] \cap (P) = P$; if $x \in (P) \cap R[x]$ and, then there's a nonzero $r \in R$ such that $rx \in P$, but $r \notin P$, so $x \in P$. (Also, $P \subset R[x]$ and $P \subset (P)$).

Further, we can impose that the leading coefficient of p divides the leading coefficient of any element of P.

This means that p must be prime in $K[x]$; let $r, s \in K[x]$ with $rs = p$. One of r or s must have degree at least one, then. We can factor out a constant, k, with $p = kr's'$ and $r', s' \in R[x]$. That is, $r's' \in (P) \cap R[x]$, so $r's' \in P$. This means that r' or s' is in P. So r' or s' is in (P). So r or s is a multiple of p in $K[x]$. So p is irreducible in $K[x]$, so p is prime.

Thus, p is prime in $R[x]$.

Now, we imposed that $p \in P$, so $(p) \subset P$.

Next, let $q \in P$. Then $q \in (P) \subset K[x]$, so $q \in (p) \subset K[x]$. That means that $p \mid q$ in $K[x]$; there is an $r \in K[x]$ such that $q = pr$. By multiplying by the least common multiple of the quotients of the coefficients of r, we can say that there is a $k \in R$ such that $kq = pr'$ for some $r' \in R[x]$. That is, $kq \in (p)$ for some $k \in R$. But (p) is prime, and $(p) \subset P$. So $k \notin (p)$, so $q \in (p)$.

So $(p) = P$; P is a principal ideal.