Chapter 3:

Problem 4:

Let $m^*(E) = \infty$ for an infinite set and $m^*(E) = |E|$ for a finite set.

It's clear that m^* is defined for all sets of real numbers, is translation invariant, and countably additive. So m^* is a measure; we call it the counting measure.

Problem 7:

If $m^*(E)$ is the Lebesgue Outer Measure, it's somewhat clear that it's translation invariant; we can do this by making an open cover and shifting it.

Problem 8:

If $m^*(A) = 0$, then $m^*(A \cup B) \geq m^*(B)$, by monotonicity.

But also, $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$ by countable subadditivity.

So $m^*(A \cup B) = m^*(B)$.

Problem 11:

Each (a, ∞) is measurable.

We have $\bigcap^{\infty}(n, \infty) = \emptyset$ which has measure 0, but $m((n, \infty)) \to \infty$. So $m(\bigcap^{n} E_i) \not\to m(\bigcap^{\infty} E_n)$

Problem 12:

Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets, and A be a set.
Then $m^*(A \cap \bigcup^n E_i) = \sum^n m^*(A \cap E_i)$.
So $m^*(A \cap \bigcup^\infty E_i) \geq \sum^n m^*(A \cap E_i)$.
But n is arbitrary, so $m^*(A \cap \bigcup^\infty E_i) \geq \sum^\infty m^*(A \cap E_i)$.
Either by employing a similar argument or appealing to countable subadditivity, we get $m^*(A \cap \bigcup^\infty E_i) = \sum^\infty m^*(A \cap E_i)$.

Problem 14:
Part a:
The Cantor set has measure zero; it's usually defined as

$$[0,1] \setminus ((1/3,2/3) \cup ((1/9,2/9) \cup (7/9,8/9)) \cup \ldots)$$

Now, $[0,1]$ has measure 1, and is measurable.
Also, $((1/3,2/3) \cup ((1/9,2/9) \cup (7/9,8/9)) \cup \ldots)$ has measure 1 (consider the geometric series $1/3, 2(1/3)^2 \ldots$. It sums up to 1).
So the measure of the cantor set is $1 - 1 = 0$.

Part b:
If we only remove $\alpha 3^- n$ at each step when we define the cantor set, then we can show that it would still be closed (as a complement in $[0,1]$ of an open set) and by employing the same geometric series argument, it would have measure $1 - \alpha$.

Problem 17:
Part a:
Consider the P_is as defined in this section. We're given that $m[0,1) = \sum m^* P_i = \sum m^* P$, so that the right hand side is either zero or infinite. But if it was zero, then we break countable subadditivity; it must be infinite. So we have an example where $m(\bigcup E_i) \leq \sum m^*(E_i)$.

Part b:
Define $E_0 = [0,1) \setminus P_0$ and $E_n = [0,1) \setminus P_n$ to get the desired result.

Problem 22:
Part a:
If f is measurable, then the restriction of f to any measurable set is measurable. If D_1 isn't measurable, then the interesection of all of the $\{x : f(x) \geq n\}$ isn't measurable, which is bad. Similarly, D_2 must be measurable.

Now, if all of D_1, D_2 and the restriction of f to $D \setminus D_1 \cup D_2$ are measurable, then for each α we get $\{x : f(x) \geq \alpha\}$ the union of D_1 and a measurable set, so we win.

Part b:
Apply the same trick as used earlier this chapter; prove that if f and g measurable, then so is f^2 and $f+g$, and win using $fg = 1/2[(f+g)^2 - f^2 - g^2]$.

Parts c and d are painfully trivial.
Problem 23:
This was a homework problem; just go there.

Problem 28:
I'm not sure how to do this one.

Problem 31:
Not sure how to do this one either. It looks like a very likely qual problem, too...:/

Chapter 4:

Problem 2:

Part a: Let f be a bounded function on $[a, b]$ and let h be the upper envelope of f (that is, $h(x) = \inf_{\delta>0} \sup_{|x-y|<\delta} (f(y)))$

Then $U - \int_a^b f \geq \int_a^b h$; let ϕ be a step function with $\phi \geq f$. Then $\phi \geq h$ except at a finite number of points, because step functions are discontinuous on only finitely many points and the upper envelope is lower than any continuous function above f.

Also, $U - \int_a^b f \leq \int_a^b h$; there's a sequence of step functions converging downwards to h, so by bounded convergence, we have our result.

So $U - \int_a^b f = \int_a^b h$.

Part b:

We get a similar result for the lower envelope. So a bounded function on $[a, b]$ is Riemann-integrable if and only if the integrals of its upper and lower envelopes are equal.

If the upper and lower envelopes are unequal on a set of greater than measure zero, this fails, as the lower envelope is always lower than the upper envelope.

If the upper and lower envelopes are equal except on a set of measure zero, this succeeds, rather obviously.

So a bounded function on $[a, b]$ is Riemann-integrable if and only if the upper and lower envelopes are equal except on a set of measure zero. That is, a bounded function on $[a, b]$ is Riemann-integrable if and only if the function is continuous except on a set of measure zero.

Problem 8:

Let $\langle f_n \rangle$ be a sequence of nonnegative functions on a domain, E. Define $f(x) = \liminf f_n(x)$.

Let $h \leq f$ be any non-negative, simple function with finite measure support on the domain (say it has finite measure support on F.

Then define $h_n = \min(h, f_n)$.

Now, $\int\limits_E h \leq \int\limits_F h = \lim \int\limits_F h_n \leq \liminf \int\limits_E f_n$.

By taking supremums over h, we have our result.

Problem 14:

NOTE: a similar problem was an exam problem. This problem can be generalized, and should be done in the context of L^p spaces.

Part a:

Let $\langle g_n \rangle \to g$ almost everywhere, $\langle f_n \rangle \to f$ almost everywhere, and $|f_n| \leq g_n$, with all of the above functions being measurable, and $\int g = \lim \int g_n$.

Then $|f_n - f| \to 0$ almost everywhere, and $|f_n - f| < g_n + g$. So by applying dominated convergence theorem, we have our result.

Part b:

Let $\langle f_n \rangle$ be a sequence of integrable functions in L^p with $f_n \to f$ almost everywhere.

If $\|f_n\| \not\to \|f\|$, then there's an $\epsilon > 0$ and a subsequence f_{n_k} with $|\|f_{n_k}\| - \|f\|| \geq \epsilon$. But

$$\begin{aligned} \|f_n - f\| &\geq |\|f_n\| - \|f\|| \\ &\not\to 0 \end{aligned}$$

If $\|f_n\| \to \|f\|$, then a modification of part a applies. So we have our result.

So $\|f_n - f\| \to 0$ if and only if $\|f_n\| \to \|f\|$.

Problem 15:

The entire problem is "Apply Littlewood's Three Principles" and the "$2^{-n}\epsilon$ trick". (On $[-1,1]$ there is a (property) function such that $|f-\phi_1| < 2^{-1}\epsilon/2$... similarly, there is such a function on $[-2,-1)$ and $(1,2]$ such that $|f-\phi_2| < 2^{-2}\epsilon/2$...induct, paste everything together, integrate, geometric series, win.)

Problem 16: NOTE: this was an exam problem.

First, note that if we have that this is true for all step functions vanishing except on a vinite interval, then we have our result; if $\lim\limits_{n\to\infty} cos(nx)\phi(x)dx = 0$ for all such step functions ϕ, then because there's such a step function with $\int |f-\phi| < \epsilon$ for all $\epsilon > 0$, we have our result.

So, let ϕ be a step function on $[a,b]$, and let $\epsilon > 0$. Partition $[a,b]$ by $a = x_0 < x_1 \ldots x_l = b$ so that ϕ is constant on each (x_i, x_i+1). Let M be the maximum of $|\phi|$ (which exists, as ϕ takes only finitely many values). Pick n large enough so that $2\pi/n < \epsilon/(lM)$. Integrate over each chunk of the partition; we end up with everything cancelling out except on sets of length less than $2\pi/n$. There's at most l of them, having magnitude at most M; we've won.

Problem 22:

Note: This problem is lol.

Let there be a sequence, $\langle f_n\rangle$, of measurable functions on a set, E, of finte measure, with $f_n \to f$ in measure.

Then every subsequence of f_n converges to f in measure, so every subsequence has a subsequence converging to f in measure.

Now, let there be a sequence, $\langle f_n\rangle$, of measurable functions on a set, E, of finte measure, with every subsequence of f_n having a subsequence converging to f in measure. Then every subsequence of f_n has a subsequence which has every subsequence have a subsequence that converges almost everywhere to f. Thus, every subsequence of f_n has a subsequence that converges almost everywhere to f. So f_n converges to f in measure.

Problem 25:

...Seriously, the hint gives this entire question away. Pretty lame stuff, bro.

Chapter 5:

Problem 4:

Let f be continuous on $[a, b]$ and one of its derivates is everywhere nonnegative on (a, b). Then D^+ or D^- is everywhere nonnegative on (a, b); $D^+ \geq D_+$, and $D^- \geq D_-$.

We proceed by handling this for D^+, and note that the proof for D^- is similar.

First, if $D^+ \geq \epsilon > 0$ on (a, b), then $\limsup\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq \epsilon$ for all x. If for any x there's an h such that $f(x+h) - f(x) < \epsilon$, then we can apply the sup method to find a contradiction (there should be a least possible h so that we have this property, but this breaks down pretty quickly).

Now, consider $g_\epsilon(x) = f(x)+\epsilon x$. Each g_ϵ is increasing, by the above thing. Now, $f = \lim\limits_{\epsilon \to 0} g_\epsilon$. So f is a limit of increasing functions, it's increasing.

Problem 5/8:

5 is a mostly trivial epsilon-delta proof. Note: for part c, it's better to work from the right hand side than the left hand side.

8 is painfully clear.

Problem 10:

Note: I could've sworn one of these wasn't of bounded variation...

Part a:

(Note: we can restrict our attention to $[0, 1]$, rather clearly.)

Consider $f(x) = x^2 \sin(1/x^2)$ on $[0, 1]$ (with $f(0) = 0$).

This is decreasing when $1/x^2 \in [(4n+1)\pi/2, (4n+3)\pi/2]$ for some $n \in \mathbb{N}$.

So it's decreasing when $x^2 \in [2/((4n+3)\pi), 2/((4n+1)\pi)]$ for each $n \in \mathbb{N}$. So its negative variance is $\sum\limits_{i=0}^{\infty} 2/((4n+1)\pi) - 2/((4n+3)\pi) = \sum\limits_{i=0}^{\infty} 4/pi[1/(4n+1)(4n+3)]$. This sum converges, so the negative variance is finite. Similarly, the positive variance is finite. So the total variance is finite, the function is of bounded variation.

Part b:

Consider $f(x) = x^2 \sin(1/x)$ on $[0, 1]$ (with $f(0) = 0$).

This is decreasing when $1/x \in [(4n+1)\pi/2, (4n+3)\pi/2]$ for some $n \in \mathbb{N}$.

So it's decreasing when $x \in [2/((4n+3)\pi), 2/((4n+1)\pi)]$ for each $n \in \mathbb{N}$. So its negative variance is $\sum_{i=0}^{\infty}(2/((4n+1)\pi))^2 - (2/((4n+3)\pi))^2$, which converges. So the total variance is finite, the function is of bounded variation.

Problem 14:

Part a:

Sums and differences of two absolutely continuous functions are also absolutely continous, rather clearly. (If $\epsilon > 0$, use $\delta = \max(de_1, \delta_2)$, with δ_1 working for $\epsilon/2$ for one function and δ_2 working for $\epsilon/2$ with the other function.)

Part b:

Products of two absolutely continuous functions are absolutely continuous; the domain is necessarily a closed interval, continuous functions on closed intervals are bounded. Pick one function; it is absolutely bounded by M. For the other function, for all $\epsilon > 0$ there's a $\delta > 0$ that works for ϵ/M. Use this δ.

Part c:

I'm absolutely stuck on this one.

Problem 16:

Part a:

Let f be a monotone increasing function. Then f' exists almost everywhere; Let f_c be the indefinite integral of f', so that f_c is absolutely continuous. Then $f_s = f - f_c$ has derivative zero almost everywhere. That is, $f = f_s + f_c$ is a decomposition of f into a sum of singular and absolutely continuous functions.

Part b:

Let f be a nondecreasing singular function on $[a, b]$. Let $\epsilon > 0$, $\delta > 0$.

First, f is bounded; it is monotonic on a closed interval. Second, f' is zero almost everywhere. That is, there's a collection of nonoverlapping (open) intervals, call it $\mathcal{I}$, whose total length is less than δ that covers the set of points where f' is nonzero (or undefined).

That is, f is constant on the entire domain except (possibly) those intervals.

So, we have that $\sum\limits_{I\in\mathcal{I}} l(I) < \delta$, and $\sum\limits_{I\in\mathcal{I}} d(f(I)) = f(b)-f(a)$, where $d(f(I))$ is the distance between the endpoints of $f(I)$. (We have the second part, as f is constant except on intervals in $\mathcal{I}$.)

Thus, $\sum\limits_{I\in\mathcal{I}} d(f(I)) = f(b)-f(a)$ converges upwards to $f(b)-f(a)$; there's a finite collection of intervals in $\mathcal{I}$ (call it $\mathcal{F}$) with $\sum\limits_{I\in\mathcal{F}} d(f(I))+\epsilon = f(b)-f(a)$. This satisfies the problem. (We summarize this result as "nondecreasing singular functions have property (S)").

Part c:

Let f be a nondecreasing function on $[a, b]$ with property (S). Then $f = f_s + f_c$ for some singular f_s and absolutely continuous f_c. Moreover, we can take both f_c and f_s nondecreasing.

Now, let $\epsilon > 0$. There's a $\delta > 0$ with the property $\sum |x_i - x_{i+1}| < \delta$ implies that $\sum |f_c(x_i) - f_c(x_{i+1})| < \epsilon$ for any non-overlapping, finite collection of intervals (x_i, x_{i+1}). Also, there's a finite collection of such intervals with the property $\sum |f(x_i) - f(x_{i+1})| + \epsilon > f(b) - f(a)$, for any $/ep > 0$.

Part d:

Let $\langle f_n \rangle$ be a sequence of nondecreasing singular functions on $[a, b]$ with $f = \sum f_n(x)$ everywhere finite.

Then f is singular, by term-by-term differentiation.

Part e:

A series of indicator functions of the form $\chi_{[q,1]}$ with $q \in \mathbb{Q}$ cobbled together with the 2^{-n} trick suffices. (Enumerate the rationals, sum them up.)

Problem 20:

Part a: is a simple epsilon-delta proof.

Part b:

Let f be absolutely continuous.

Then if f fails to satisfy a Lipschitz condition, there is a pair of sequences x_n, y_n that break the Lipschitz condition. As f is absolutely continuous, such a subsequence must exist with $d(x_n, y_n) \to 0$; this is because f is bounded. So we can find a sequence of pairs of points arbitrarily close together whose difference quotient is arbitrarily large; the derivative thus cannot be bounded.

If f satisfies a Lipschitz condition, then the difference quotient is uniformly bounded at all points; so the limits of these difference quotients are uniformly bounded, so the derivative is bounded.

Part c:

If one of the derivates of a function is bounded, say $D+$, then...

Problem 23:

Part a:

Let ϕ be a convex function on a finite interval, $[a, b)$.

Let ϕ not be bounded below. Because ϕ is absolutely continuous on each closed subinterval of $[a, b)$, this means that ϕ is bounded on each closed subinterval of $[a, b)$. Thus, if ϕ is not bounded below, then we have that $\lim_{x \to b} \phi(x) = -\infty$ (else, ϕ can be extended to $[a, b]$, and is thus bounded).

But this goes bad pretty quickly (Pick a point on the curve, draw a chord from $(a, \phi(a))$, make it have arbitrarily negative slope, it'll go below that point eventually.)

Part b:

Part c is trivial

Problem 27:

I've spent way too long on this problem. It's similar to a problem in Zigmund's book, apparently...

Chapter 6:

Problem 2:

Let $f \in L^\infty[0,1]$. Define $\|f\|_\infty = M$. Then for all p, $\|f\|_p/M = \|f/M\|_p$. It suffices to show that $\lim\limits_{p\to\infty} \|f/M\|_p = 1$.

Now, for all $p > 0$, we have that $[\int |f/M|^p]^{1/p} \leq [\int |f/M|]^{1/p} \leq 1^{1/p} = 1$. So the limit as p tends to infinity is bounded above by 1.

Next, let $\epsilon \in (0,1]$. Then because the essential supremum of f/M is 1, there's a set, E, of measure $\delta > 0$ with the property $x \in E$ implies that $f(x) \geq |1 - \epsilon/2|$. Now,

$$\begin{aligned}
[\int |f/M|^p]^{1/p} &\geq [\int_E |f/M|^p]^{1/p} \\
&\geq [\int_E (1-\epsilon/2)^p]^{1/p} \\
&= [(1-\epsilon/2)^p m(E)]^{1/p} \\
&= (1-\epsilon/2) m(E)^{1/p}
\end{aligned}$$

By taking p sufficiently large, we have this greater than $1 - \epsilon$.

So we have our result.

Problem 4:

Let $f \in L^1$, $g \in L^\infty$. Define $M = \|g\|_\infty$.

$$\begin{aligned}
\int |fg/M| &\leq \int |f| \\
&= \|f\|_1
\end{aligned}$$

So,

$$\int |fg| \leq M\|f\|_1 = \|f\|_1 \|g\|_\infty$$

Problem 8:

This was effectively done in your homework.

Problem 10:

Note: This is effectively saying that the norm on L^∞ is the norm induced by the metric of uniform convergence. This is probably given outright in Pugh.

Let $\langle f_n \rangle$ be a sequence of functions in L^∞.

Let there fail to be a set, E, of measure 0 such that f_n converges to f uniformly on E^c. That is, there's a set of nonzero measure such that f_n fails to converge to f uniformly on E. Then there's an $\epsilon > 0$ and a subsequence of f_n with the property $\|f_{n_k} - f\|_\infty \geq \epsilon$ on a set of greater than measure zero. That is, there's a subsequence of f_n such that $\|f_{n_k} - f\|_\infty \geq \epsilon$; f_n does not converge to f in L^∞.

If there's a set, E, of measure 0 such that f_n converges to f uniformly on E^c, then for all $\epsilon > 0$ there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f| < \epsilon$ except on E. That is, $\|f_n - f\|_\infty < \epsilon$. That is, $f_n \to f$ in L^∞.

Problem 11:

This is pulled straight out of Pugh; if a sequence of functions has cauchy uniform convergence, it has pointwise cauchy convergence, so it has pointwise convergence. This convergence is also uniform, somewhat trivially.

Problem 15:

Note: this looks like a good qual problem...

Let c be the space of all convergent sequences of real numbers, given the ℓ^∞ norm.

First, this is a normed linear space; ℓ^∞ norm is a norm, and the space is clearly linear.

Now, let $\langle\langle n_k\rangle\rangle$ be a Cauchy sequence of converging sequences.

Then for all $\epsilon > 0$, there's an N such that $n, m > N$ implies that for all k, $|n_k - m_k| < \epsilon$. That is, for fixed k, $\langle n_k\rangle$ treated as a sequence in n is Cauchy. So each of these converges to something; call it N_k. It is rather clear that $\langle n_k\rangle \to \langle N_k\rangle$ as a sequence using the ℓ^∞ norm from this.

Now, let c_0 be the space of all sequences of real numbers converging to zero. It is also a normed linear space. It's also a closed subset of c; if $\langle\langle n_k\rangle\rangle \to \langle a_k\rangle$ with $\langle n_k\rangle$ each in c_0, then $a_k \to 0$, this is clear by triangle inequality.

So c_0 is a closed subset of a complete normed linear space. It is complete. So c_0 is a Banach space.

Problem 18:

Note: This is very similar to problems given in past quals...

We first note that $f_n g_n \to fg$ almost everywhere, and that this is clear. We proceed by applying the an earlier problem; the result follows if $\|f_n g_n\|_p \to \|fg\|_p$.

Somewhat clearly, if the left hand side converges, then it converges to the right hand side. Moreover, the sequence is Cauchy, so it converges.

Problem 23:

This problem is made of nope and nope.

Problem 24:

Let $g, h \in L^q$ be such that $\int fg = \int fh$ for all $f \in L^p$.

Then $\int f(g - h) = 0$ for all $f \in L^p$. So in particular, this is true for all indicator functions on sets of finite length. We can game this so that $\int f(g-h)$ is always positive; $f = 1$ if $g-h$ is positive, and $f = -1$ if $h-g$ is positive works, and is measurable as h and g are. We know that this means that $f(g-h)$ is zero almost everywhere; thus, we get that $g-h$ is zero almost everywhere, so g must have been almost-everywhere-unique.

Chapter 11:

Problem 2:

Let $(X_i, \mathcal{B}_i, \mu_i)$ b e a collection of measure spaces with each X_i disjoint from the others.

Part a:

The set $\mathcal{B} = \{B : \forall i, B \cap X_i \in \mathcal{B}_i\}$ is a σ-algebra; Let $\langle B_i \rangle$ be a countable collection of sets in $\mathcal{B}$. Then $B = \bigcup B_i$ has that $B \cap X_i = \bigcup B_i \cap X_i$, and since each $B_i \cap X_i \in \mathcal{B}_i$, $\bigcup B_i \cap X_i \in \mathcal{B}_i$.

Part b:

The function $\mu(B) = \sum \mu_i(B \cap X_i)$ is a measure; let $\langle B_i \rangle$ be a sequence of disjoint sets in $\mathcal{B}$. Then $\mu(\bigcup(B)) = \sum \mu_i(\bigcup B \cap X_i) = \sum \sum \mu_i(B_j \cap X_i) = \sum \mu(B_j)$. (There's a subtle change of order of summation, I'm too lazy to type it out.)

Part c:

If there's only a countable number of measures in the collection, and all of them are σ-finite, then the pasted measure space is σ-finite;

If each $X_i = \bigcup\limits_{j \in \mathbb{N}} E_{i_j}$, then $\bigcup X_i = \bigcup\limits_{i \in \mathbb{N}, j \in \mathbb{N}} E_{i_j}$. If each E_{i_j} can be taken to have finite measure, then we've won.

The result I've proved is somewhat clearly equivalent to the problem.

Problem 5:

Part a:

Sums of measures are clearly mesures.

Part b:

Let μ and ν be measures on $\mathcal{B}$ with $\mu \geq \nu$.

Define $\lambda(E) = \mu(E) - \nu(E)$. Then λ is positive, and has the property that $\mu = \nu + \lambda$.

Also, λ is a measure:

First, $\lambda(\emptyset) = 0 - 0 = 0$.

Second, let E_i be a disjoint sequence of measurale sets. Thenb $\lambda(\bigcup E_i) = \mu(\bigcup E_i) - \nu(\bigcup E_i) = \sum \mu(E_i) - \sum \nu(E_i) = \sum(\mu(E_i) - \nu(E_i)) = \sum \lambda(E_i)$.

Part c:

Let ν be σ-finite, and let λ and γ be such that $\mu = \nu + \lambda = \nu + \gamma$.

Let E be a measurable set. Then E can be written as a countable disjoint union of sets of finite (ν) measure, call them E_i. We proceed to prove the result for such sets, and point out that the whole result follows (as if $\lambda = \gamma$ for sets of finite ν measure, we can write a set of infinite ν measure as a union of these sets...and thus, we get that λ and γ must be equal there, as otherwise we could decompose one of them and fail to get a measure having the right properties.)

Now, let E be a set of finite ν measure. Then $\mu(E)$ is infinite or it is finite. If $\mu(E)$ is infinite, then both γ and λ must be infinite. If $\mu(E)$ is finite, then $\gamma = \lambda$ by arithmetic. Done.

Part d:

Let $\mu = \nu = \infty$ for all nonempty E.

Then $\lambda = 0$ and $\gamma = \mu$ have the property $\mu = \nu + \lambda = \nu + \gamma$.

Let $\lambda' = \inf(\lambda : \mu = \nu + \lambda)$. λ' is a measure, and it is indeed the least such measure;

Let E_i be a disjoint sequence of sets. Then

$$\begin{aligned}
\lambda'(\bigcup E_i) &= \inf(\lambda(\bigcup E_i) : \mu(\bigcup E_i) = \nu(\bigcup E_i) + \lambda(\bigcup E_i)) \\
&= \inf(\lambda(\bigcup E_i) : \sum \mu(E_i) = \sum \nu(E_i) + \sum \lambda(E_i)) \\
&= \inf(\sum \lambda(E_i) : \sum \mu(E_i) = \sum \nu(E_i) + \sum \lambda(E_i)) \\
&= \sum \inf(\lambda(E_i) : \mu(E_i) = \nu(E_i) + \lambda(E_i)) \\
&= \sum \lambda'(E_i)
\end{aligned}$$

which gives us our result.

Problem 7:

Let $(X, \mathcal{B}, \mu)$ be a measure space.

Define $\mathcal{B}_0 = \mathcal{B} \cup \{B : \exists B' : (\mu(B') = 0) B \subset B'\}$. This is a σ-algebra:

Let B_i be a sequence of sets in $\mathcal{B}_0$. Then each B_i is either in $\mathcal{B}$ or is a subset of a set of measure zero. Let $\mathcal{B}'$ be the sets in $\mathcal{B}$ and $\mathcal{B}''$ be the subsets of sets of measure zero. Then $\bigcup_{B \in \mathcal{B}'} B \in \mathcal{B}$, and $\bigcup_{B \in \mathcal{B}''} B$ is a subset of a set of measure zero (this is somewhat clear; take $B_i \subset B_{i_0}$, the union of the guys on the left hand side has measure zero, win). So by taking the union of $\bigcup_{B \in \mathcal{B}'} B$ and $\bigcup_{B \in \mathcal{B}''} B$, we have our result.

We clearly have properties $i)$ and $iii)$ from this.

Define $\mu_0(E) = \sup \mu(E')$, where $E' \in \mathcal{B}$ and $E' \subset E$. Then μ_0 is a measure;

Let E_i be a sequence of disjoint sets in $\mathcal{B}$. Then

$$\begin{aligned}
\mu_0(\bigcup(E_i)) &= \sup \mu(E' : E' \in \mathcal{B}, E' \subset \bigcup(E_i)) \\
&= \sum \sup \mu(E' : E' \in \mathcal{B}, E' \subset E_i \\
&= \sum \mu_0(E_i)
\end{aligned}$$

Also, it is clear that $E \in \mathcal{B}$ implies that $\mu_0(E) = \mu(E)$.

Problem 10:

Let f be a nonnegative measurable function.

Then the sets $A_{n,k} = \{x : f(x) \in [k2^{-n}, (k+1)2^{-n})\}$ are all measureable; we can define $\phi_n(x) = k2^{-n}$ if $f(x) \in [k2^{-n}, (k+1)2^{-n})$, each are simple, the sequence of ϕ_ns are increasing, and the limit approaches f.

If the measure is σ-finite, we can write the space as a countable union of disjoint sets of finite measure, and define each ϕ_n as above except by throwing in an indicator function at each step.

Problem 11:

Let μ be a complete measure and f be a measurable function, with $f = g$ almost everywhere.

First, this means that for all α, $B = \{x : f(x) > \alpha\} \in \mathcal{B}$. Also, B', the set where $f \neq g$, is in $\mathcal{B}$. So $C = \{x : f(x) \neq g(x) \text{ and } g(x) > \alpha\} \in \mathcal{B}$. Thus, $\{x : g(x) > \alpha\} = (B \setminus B') \cup C \in \mathcal{B}$, so we have our result.

Now, this may fail if μ is not a complete measure; pick any incomplete measure space. Take the function $f(x) = 1$, and let $g(x)$ equal 1 except on a non-measurable subset of measure zero. Then $f(x) = g(x)$ except on a set of measure zero, but g isn't measurable (as $\{x : g(x) < 1/2\}$ isn't measurable.)

Problem 13:

Part a:

Let $f_n \to f$ in measure.

Pick f_{n_k}, E_k such that f_{n_k} and E_k "work" for $\epsilon = 2^{-k}$ in the definition of convergence to measure. Mor

Then if $x \notin \bigcup\limits_{k=1} E_k$, then $f_{n_k}(x) \to f(x)$. But we could have started the index at any point: so if $x \notin \bigcap\limits_{i=1}^{\infty} \bigcup\limits_{k=i}^{\infty} E_k$, then $f_{n_k}(x) \to f(x)$. But $\mu(A) < \mu(\bigcup_{k=i} E_k) < 2^{-k+1}$ for all k, so $\mu(A) = 0$, giving the desired result.

Part b:

Let $\langle f_n \rangle$ be a sequence of measurable functions each vanishing outside the same A with $\mu(A)$ finite, and $f_n \to f$ pointwise almost everywhere.

Part c:

Problem 21:

Part a:

Let f be integrable. Consider $\{x : f(x) \neq 0\}$. This is the union of the sets $E_n = \{x : |f(x)| \geq 1/n\}$. If any of these has infinite measure, then f is clearly not integrable. So $\{x : f(x) \neq 0\}$ can be written as a countable union of sets of finite measure; it is of σ-finite measure.

Part b:

Note: We did this earlier...

Let f be integrable, $f \geq 0$. We employ the same trick as the last time we did this problem; consider the sets $E_{k,n} = \{x : f(x) \in [k2^{-n}, (k+1)2^{-n})\}$, and for each $n \in \mathbb{N}$, define $\phi_n(x) = k2^{-n}$ if $x \in E_{k,n}$. We now have an increasing sequence of simple functions converging to f; by multiplying these with the indicator functions of the sets in part a, we have our result.

Part c:

Apply part b and the Lebesgue convergence theorem to the positive and negative parts of f, and win.

Problem 24:

Let $\mu_n \uparrow \mu$ setwise. Let E_i be a sequence of disjoint measurable sets. Let $\epsilon > 0$.

For all $i \in \mathbb{N}$ there is an N_i such that for all $n \geq N_i$, $\mu(E_i) - \mu_{N_i}(E_i) < \epsilon/2^i$.

So we have that $\mu(\bigcup^n E_i) - \sum^n \mu_{N_n}(E_i) < \epsilon$, for any arbitrary n (this is true because $\mu_n(E)$ is increasing).

Taking limits as $n \to \infty$ and applying the ϵ principle yields the result.

Problem 25:

Note: we apply the so-called "goatse" trick of taking the line, penetrating it at a point, and spreading the penetration outwards.

Consider $\mu_n(E) = \int_E \chi_{(-n,n)^c}$. For all sets of finite lebesgue measure, $\mu_n(E) \to 0$. It's somewhat clear that $\mu_n \to 0$ for lebesgue finite measure sets and $\mu_n \to \infty$ for sets of infinite lebesgue measure. So by considering $\mu(\bigcup[n, n+1]) \neq \sum 0$, we have our result.

Problem 28:

Let ν be a signed measure, with $\nu + -\nu^- = \nu = \mu^+ - \mu^-$ and ν^+ mutually singular to ν^-.

Let E be a measurable set. Then E has a Hahn decomposition (which is unique up to null sets...thus the measures of the sets are uniquely determined); let $E = A \cup B$, with A a positive set and B a negative set, and A and B disjoint.

Now, it is somewhat clear that $\nu^+(B) = \mu^+(B) = 0$ and $\nu^-(A) = \mu^-(A) = 0$ (if this is not clear...assume not, consider the first case without loss of generality. Then $\nu(B) = \nu^+(B) - \nu^-(B)$ with both $\nu^+(B)$ and $\nu^-(B)$ nonzero. This contradicts the fact that they are mutually singular). Thus, because these are measures, we can take $\nu^+(E) = \mu^+(E) = \nu(A)$ and $\nu^-(E) = \mu^-(E) = \nu(B)$.

Thus, $\mu^+ = \nu^+$ and $\mu^- = \nu^-$; the Jordan decomposition must be unique.

Problem 30:

Let ν, μ be finite signed measures, and α, β be real numbers.

Then $\alpha\nu + \beta\mu$ is a finite signed measure: it clearly never assumes ∞ or $-\infty$, it maps the empty set to 0, and countable additivity can be checked readily.

Now, consider $|\alpha\nu|$, let E be measurable. Then

$$\begin{aligned}|\alpha\nu|(E) &= \left|\alpha\nu^+\right|(E) + \left|\alpha\nu^-\right|(E)\\ &= |\alpha|\left|\nu^+\right|(E) + |\alpha|\left|\nu^-\right|(E)\\ &= |\alpha|\left(\left|\nu^+\right|(E) + \left|\nu^-\right|(E)\right)\\ &= |\alpha|\,|\nu|(E)\end{aligned}$$

Next, consider $|\nu + \mu|$, let E be measurable. Then E has a Hahn Decomposition over $\nu + \mu$, $E = A \cup B$, with A a positive set and B a negative set.

$$\begin{aligned}|\nu+\mu|(E) &= |\nu+\mu|(A \cup B)\\ &= |\nu+\mu|(A) + |\nu+\mu|(B)\\ &= (\nu+\mu)^+(A) + (\nu+\mu)^-(A) + (\nu+\mu)^+(B) + (\nu+\mu)^-(B)\\ &= (\nu+\mu)^+(A) + (\nu+\mu)^-(B)\end{aligned}$$

Now, $(\nu+\mu)^+(A) \leq \nu^+(A) + \mu^(A)$, because

Thus,

$$\begin{aligned}|\nu+\mu|(E) &= (\nu+\mu)^+(A) + (\nu+\mu)^-(B)\\ &\leq \nu^+(A) + \mu^+(A) + \nu^-(B) + \mu^-(B)\\ &= \nu^+(E) + \mu^+(E) + \nu^-(E) + \mu^-(E)\\ &= |\nu|(E) + |\mu|(E)\end{aligned}$$

Problem 33:

Part a:

Seriously, the hint gives it away.

Part b:

Let f, g be such that for all measurable E, $\nu(E) = \int_E f d\mu = \int_E g d\mu$.

Then for all measurable E, $\int_E f - g d\mu = 0$. An earlier problem implies that this means that $f - g = 0$ almost everywhere, so $f = g$ almost everywhere.

Problem 34:

Note: Part c was an exam question. This may show up on the qual.

Let μ, ν, λ be σ-finite.

Part a:

Let f be a nonnegative measurable function.

If $\nu << \mu$, then we know that for all real α and measurable E,

$$\int_E \alpha d\nu = \int_E \alpha \frac{d\nu}{d\mu} d\mu$$

And thus, we have the result for all simple functions, ϕ.

By taking the supremum over all $\phi \leq f$, we have our result.

Part b:

Let $\nu_1 << \mu$, $\nu_2 << \mu$. Let E be a measurable set. Then

$$\begin{aligned}
\int\limits_E \frac{d(\nu_1 + \nu_2)}{d\mu} d\mu &= \nu_1(E) + \nu_2(E) \\
&= \int\limits_E \frac{d\nu_1}{d\mu} d\mu + \int\limits_E \frac{d\nu_2}{d\mu} d\mu \\
&= \int\limits_E \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} d\mu
\end{aligned}$$

Part c:

Let $\nu << \mu << \lambda$. Let E be a measurable set.
Then:

$$\begin{aligned}\int\limits_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}d\lambda &= \int\limits_E \frac{d\nu}{d\mu}d\mu \\ &= \nu(E)\end{aligned}$$

So we have the result.

Part d:
If $\nu << \mu << \nu$, we know that $\frac{d\nu}{d\nu} = 1$. So by part c, we have that $1 = \frac{d\nu}{d\nu} = \frac{d\mu}{d\nu}\frac{d\nu}{d\mu}$. This is equal to our result.

Problem 39:

Consider $\mathcal{X} = [0,1]$, $\mathcal{B}$ the set of lebesgue measurable subsets of $\mathcal{X}$, and ν the Lebesgue measure and μ the counting measure. Then ν is finite, $\nu << \mu$, but if $\nu(E) = \int\limits_E f d\mu$ for all measurable E, then $\int\limits_E f d\mu = 0$ for all single sets E. For the counting measure, this means that f is identically 0. But this obviously fails, as it would imply that $\nu(E)$ is zero for all E.

Problem 41:

Problem 44:

Problem 47:

Chapter 12:

Problem 1:
Let $B \subset E$ be a set with $\overline{\mu}(E) = 0$. Then for all A,

$$\mu^*(A) \geq \mu^*(A \cap B^c) = \mu^*(A \cap B^c + \mu^*(B) \geq \mu^*(A \cap B^c + \mu^*(A \cap B)$$

Which is sufficient to get the result.

Problem 4:

Problem 8: