Problem 1:

Let $k \subset K$ and $k \subset L$ be finite field extensions contained in some field.

In the following, I freely use the fact that $\dim(U)\dim(U) \geq \dim(U+V)$ for any vector space.

Part a:

First, $[KL : L][L : k] = [KL : k]$.

So this means that

$$\frac{[K : k]}{[KL : L]} = \frac{\dim(K)}{\frac{\dim(KL)}{\dim(L)}}$$
$$= \frac{\dim(K)\dim(L)}{\dim(KL)}$$

That is, we have reduced this problem to the following one; if $[KL : k] \leq [K : k][L : k]$, then the right hand side is at least 1.

So $[K : k] \geq [KL : L]$ if we manage to solve the following part.

Part b:

A result of linear algebra states that for any two vector spaces, U and V, we have $\dim(U)\dim(V) = \dim(UV)/\dim(U \cap V)$.

So $[KL : k][K \cap L : k] = [K : k][L : k]$. So $[KL : k] \leq [K : k][L : k]$

Part c:

If we have equality in the above, this means that $[K \cap L : k] = 1$, so that $K \cap L$ has dimension 1. That is, that $K \cap L = k$.

If $K \cap L = k$, then $[K \cap L : k] = 1$, so we have equality in the above.

Problem 2:

Problem 3:

Problem 4:

Problem 5: