(I worked with Sarah Percival and Frankie Chan a little).

Problem 1:

Part a:

Consider the set $A = \{z \in \mathbb{C} : e^z = 0\}$.

If $z = a + bi \in A$ (with $a, b \in \mathbb{R}$), then $e^z = 0$. So $e^a e^{bi} = 0$.

For $a \in \mathbb{R}$, $e^a \neq 0$. So this means that $e^{bi} = 0$. But this never happens either, because $\left|e^{bi}\right| = 1$ for all $b \in \mathbb{R}$ (because $\left|e^{bi}\right| = |\cos(b) + i\sin(b)| = \sqrt{\cos^2(b) + \sin^2(b)} = 1$).

So we have a contradiction. So $A = \emptyset$.

Part b:

Consider the set $B = \{z \in \mathbb{C} : e^z = 1\}$.

If $z = a + bi \in B$ (with $a, b \in \mathbb{R}$), then $e^z = 1$. So $e^z = e^a e^{bi} = 1$.

This means that $\left|e^a e^{bi}\right| = |e^a| \left|e^{bi}\right| = 1$. But $\left|e^{bi}\right| = 1$ for all $b \in \mathbb{R}$. So, $|e^a| = 1$, so $e^a = 1$, so $a = 0$.

So $z = bi$ for some $b \in \mathbb{R}$.

By applying the equivalence of polar and trigonometric forms, this means that $e^{ib} = \cos(b) + isin(b) = 1$. So, $\cos(b) = 1$ and $\sin(b) = 0$. This means that $b = 2k\pi$ for some $k \in \mathbb{Z}$.

So $B \subset \{2k\pi i \in \mathbb{C} : k \in \mathbb{Z}\}$.

Now, if $z = 2k\pi i$ for some $k \in \mathbb{Z}$, then $e^z = \cos(2k\pi) + i\sin(2k\pi) = 1$. So $z \in B$.

So $B = \{2k\pi i \in \mathbb{C} : k \in \mathbb{Z}\}$.

Part c:

Consider the set $C = \{z \in \mathbb{C} : \sin(z) = 0\}$.

Let $z = a + bi \in C$. Then $sin(z) = 0$. So $\frac{e^{iz} - e^{-iz}}{2i} = 0$, so that $e^{iz} = e^{-iz}$.

In other words, $e^{-b}e^{ai} = e^b e^{-ai}$. So, $e^{2b} = e^{2ai}$. Because $|e^{2ai}| = 1$, this means that $e^{2b} = 1$. So, $b = 0$, and $e^{2ai} = 1$. So $2ai = 2k\pi i$ for some $k \in \mathbb{Z}$.

So, $z = k\pi$ for some $k \in \mathbb{Z}$. So $C \subset \{k\pi \in \mathbb{C} : k \in \mathbb{Z}\}$.

Now, if $z = k\pi$ for some $k \in \mathbb{Z}$, then $\sin(z) = 0$, and this is very well known. So $z \in C$.

So $C = \{k\pi \in \mathbb{C} : k \in \mathbb{Z}\}$.

Problem 2:

Let $\Omega \subset \mathbb{C}$ be an open connected set, and $f \in C(\Omega)$ be such that for all closed, piecewise continuous curves, Γ, with $\Gamma \subset \Omega$, $\int\limits_{\Gamma} f(z)dz = 0$.

Pick $z \in \Omega$. Let $p \in \Omega$, and γ be a curve from p to z. We showed in class that $\int\limits_{\gamma} f(\xi)d\xi$ is independent of γ; that is, $\int\limits_{\gamma} f(\xi)d\xi$ only depends on p and z.

So, we can define $g(z) = \int\limits_{\gamma} f$, where γ is a curve from a chosen fixed point,p, to z.

Now, fix $z_0 \in \Omega$. It is clear that $\lim\limits_{z \to z_0} \frac{\int_{z_0}^{z} f(w)dw}{z - z_0} = f(z_0)$, because $\frac{\int_{z_0}^{z} f(w)dw}{z - z_0}$ is the average value of $f(w)$ on the line segment. Now, because $\frac{g(z)-g(z_0)}{z-z_0} = \frac{\int_{z_0}^{z} f(w)dw}{z - z_0}$, this means that $g'(z_0) = \lim\limits_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim\limits_{z \to z_0} \frac{\int_{z_0}^{z} f(w)dw}{z - z_0} = f(z_0)$

That is, $g'(z_0) = f(z_0)$ for all $z_0 \in \Omega$; g is a primitive of f.

Problem 3:

(I must admit to having read this in Complex Made Simple by David C. Ullrich prior to the assignment of this problem, and that I used it as a reference.)

Let $f \in \mathcal{O}(D_1(0))$, with $f = \sum\limits_{n=0}^{\infty} a_n z^n$.

Then consider $f_N = \sum\limits_{n=0}^{N} a_n z^n$.

$$\begin{aligned}
\int_0^{2\pi} \left| f_N(re^{it}) \right|^2 dt &= \int_0^{2\pi} f_N(re^{it})\overline{f_N(re^{it})}dt \\
&= \int_0^{2\pi} \sum_0^N a_n r^n e^{int} \overline{\sum_0^N a_n (r^n e^{int})}dt \\
&= \int_0^{2\pi} \sum_{n,m=0,0}^{N,N} a_n \overline{a_m} r^{2n} e^{i(n-m)t} dt
\end{aligned}$$

It is readily checked that all of the terms in the above, except for those where $n = m$, vanish; this is because $\int\limits_0^{2\pi} e^{int}dt = 0$ when $n \neq 0$. Thus, we have

$$\begin{aligned}\int\limits_0^{2\pi} \left|f_N(re^{it})\right|^2 dt &= \int\limits_0^{2\pi} \sum_{n=0}^{N} a_n\overline{a_n}r^{2n}dt \\ &= \sum_0^N 2\pi \left|a_n\right|^2 r^{2n}\end{aligned}$$

So, for all $N \in \mathbb{N}$, $\int\limits_0^{2\pi} f_N(re^{it})dt = \sum\limits_0^N 2\pi \left|a_n\right|^2 r^{2n}$.

Taking limits as $N \to \infty$, we have $\int\limits_0^{2\pi} f(re^{it})dt = \sum\limits_0^\infty 2\pi \left|a_n\right|^2 r^{2n}$, which is what we wanted.

Problem 4:

Let $\phi, \psi : [a, b] \to \mathbb{R}$ be log-convex.

Then $\ln(\phi)$ and $\ln(\psi)$ are convex.

So for all $x, y \in [a, b]$ with $x \leq y$ and for all $t \in [0, 1]$, $\ln(\psi(tx+(1-t)y)) \leq t \ln \psi(x) + (1-t) \ln \psi(y)$ and $\ln(\phi(tx + (1-t)y)) \leq t \ln \phi(x) + (1-t) \ln \phi(y)$. Note that because e^x is an increasing function, $a < b$ if and only if $e^a < e^b$, so that these are equivalent to $\phi(tx + (1-t)y) \leq \phi(x)^t\phi(y)^{(1-t)}$ and $\psi(tx + (1-t)y) \leq \psi(x)^t\psi(y)^{(1-t)}$.

Consider $\ln(\phi+\psi)$. Note that because e^x is an increasing function, $a < b$ if and only if $e^a < e^b$.

Now, fix $x, y \in [a, b]$ with $x < y$ and fix $t \in [0, 1]$.

$$\begin{aligned}
e^{\ln(\phi+\psi)(tx+(1-t)y)} &= (\phi+\psi)(tx+(1-t)y) \\
&= \phi(tx+(1-t)y) + \psi(tx+(1-t)y) \\
&\leq \phi(x)^t\phi(y)^{(1-t)} + \psi(x)^t\psi(y)^{(1-t)} \\
&\leq e^{t\ln((\phi+\psi)(x))+(1-t)\ln((\phi+\psi)(y))}
\end{aligned}$$

(I must admit to not being sure how to do that last step, but it's clear this is what is needed.)

So $\ln(\phi+\psi)(tx+(1-t)y) \leq \ln(t\ln\phi(x)+(1-t)\ln\phi(y)+t\ln\psi(x)+(1-t)\ln\psi(y))$.

That is, $\phi+\psi$ is log-convex if ϕ and ψ are.

Problem 5:

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $f \in \mathcal{O}(\Omega)$, $f(z) \neq 0$ for any $z \in \Omega$.

We showed in class that $g(z) = \int\limits_p^z \frac{f'(w)}{f(w)}dw + \lambda$ with p chosen arbitrarily in Ω and $e^\lambda = f(p)$ satisfies $f = e^g$, and that $g \in \mathcal{O}(\Omega)$.

Now, let $h \in \mathcal{O}(\Omega)$ be such that $f = e^h$.

Then $\frac{f}{f} = \frac{e^g}{e^h}$, so that $1 = e^{g-h}$. Thus, by problem 1, we have that for all $z \in \mathbb{C}$, $g(z) - h(z) = 2k\pi i$ for some $k \in \mathbb{Z}$. All that remains is to show that k does not depend on z: consider $(g-h)'$. Now, $g' = h' = f'/f$, as was discussed in class. So $(g-h)'$ is zero; $g-h$ is constant. So $g-h$ doesn't depend on z; $g(z)-h(z) = 2k\pi i$ for some fixed k.

That is, any two functions, g and h, satisfying $e^g = e^h = f$ differ only by $2k\pi i$ for some $k \in \mathbb{Z}$.

Problem 6:

(Once again, I used Complex Made Simple as a reference for this.)

Let $\phi \in \mathcal{O}(D_1(0))$. Suppose that ϕ takes its maximum at 0.

Because ϕ is holomorphic on $D_1(0)$, we know that ϕ has a power series representation, $\phi(z) = \sum\limits_0^\infty a_n z^n$, on any disk $\overline{D_r(0)}$ with $r \in (0,1)$.

So, problem 3 applies: $\int\limits_0^{2\pi} \left|\phi(re^{it})\right|^2 dt = \sum\limits_0^{\infty} 2\pi \left|a_n\right|^2 r^{2n}$. Now, $\phi(0) = a_0$. So, $\int\limits_0^{2\pi} \left|\phi(re^{it})\right|^2 dt = \sum\limits_0^{\infty} 2\pi \left|a_n\right|^2 r^{2n} = 2\pi \left|a_0\right|^2 + \sum\limits_1^{\infty} 2\pi \left|a_n\right|^2 r^{2n}$. So $\int\limits_0^{2\pi} \left|\phi(re^{it})\right|^2 dt \geq 2\pi \left|\phi(0)\right|^2$.

Thus, for all r, $\int\limits_0^{2\pi} \left|\phi(re^{it})\right| dt \geq 2\pi \left|\phi(0)\right|^2$. But because $\phi(re^{it}) \leq \phi(0)$ for all $r, t \in \mathbb{C}$, this means that $\phi(re^{it}) = \phi(0)$ for all $r, t \in \mathbb{R}$. That is, $\phi(z) = \phi(0)$ for all $z \in D_1(0)$.

Problem 7:

Suppose that $\phi \in \mathcal{O}(\Omega)$ with Ω a domain, and that there is a $c \in \Omega$ such that $|(|\,\phi(c)) = max(|(|\,\phi))$.

Then ϕ is constant on any disk centered at c, by problem 6 (by expanding and translating appropriately).

Now, Ω is path connected (it is a domain).

Let $z \in \Omega$, and let $\gamma : [0, 1] \to \Omega$ be a path from z to c with $\gamma \subset \Omega$. We can cover the image of the path with a finite number of open disks contained in Ω, because paths are compact. Also ϕ is constant on each of these open balls: if not, then $\sup\{t \in [0, 1] : \phi\gamma(t) \neq c\} = s$ for some $s \in [0, 1]$. But then there's an ϵ-ball around s where $\phi \circ \gamma$ takes the value c somewhere...which means that $\phi(\gamma(s)) = c$, which is a contradiction.

So ϕ is constant along the path: $\phi(z) = \phi(c)$.

So for all $z \in \Omega$, $\phi(z) = \phi(c)$. So ϕ is constant.