

**Problem 1:**

Consider  $\int_0^\infty \frac{1-\cos(x)}{x^2} dx$ .

Now,  $\int_0^T \frac{1-\cos(z)}{z^2} dz = \int_0^T \frac{1-\frac{e^{iz}+e^{-iz}}{2}}{z^2} dz = -\left[ \int_0^T \frac{e^{iz}-1}{2z^2} dz + \int_0^T \frac{e^{-iz}-1}{2z^2} dz \right]$ . Both of the functions under the integrands are holomorphic, except at the origin.

Using a  $u$ -substitution, we get  $\int_0^T \frac{e^{-iz}-1}{2z^2} dz = -\int_0^{-T} \frac{e^{iz}-1}{2z^2} dz$ .

So,

$$\begin{aligned} \int_0^T \frac{1-\cos(z)}{z^2} dz &= -\left[ \int_0^T \frac{e^{iz}-1}{2z^2} dz - \int_0^{-T} \frac{e^{iz}-1}{2z^2} dz \right] \\ &= \frac{1}{2} \left[ \int_0^T \frac{1-e^{iz}}{z^2} dz - \int_0^{-T} \frac{1-e^{iz}}{z^2} dz \right] \\ &= \frac{1}{2} \left[ \int_{-T}^T \frac{1-e^{iz}}{z^2} dz \right] \end{aligned}$$

**Problem 2:**

Let  $\Omega \subset \mathbb{C}$  be open and simply connected,  $f \in \mathcal{O}(\Omega)$ ,  $f$  is always nonzero,  $k \in \mathbb{Z}$ .

There is an  $h \in \mathcal{O}(\Omega)$  such that  $e^h = f$ . Define  $\bar{h} = h/k$ . Then  $e^{\bar{h}k} = f$ .

**Problem 3:**

Consider  $\sqrt[n]{-1} = (-1)^{1/n} = (e)^{\ln(-1)/n} = e^{\ln(-1)/n} = e^{\ln(-1)/n}$ . As discussed in class, the logarithms of  $-1$  are  $(2k+1)\pi i$  for each  $k \in \mathbb{Z}$ . That is, the possible values of  $\sqrt[n]{-1}$  are given by

$$e^{((2k+1)\pi i)/n} = e^{\frac{1}{n}(2k+1)\pi i}$$

for any given  $k, j \in \mathbb{Z}$ .

Yet, this is an intractable mess. Consider that  $e^{\frac{1}{2}((2j+1)\pi i)} = e^{j\pi i + \frac{1}{2}\pi i} = e^{j\pi i} e^{\frac{1}{2}\pi i} = (-1)^j e^{\frac{1}{2}\pi i}$ . Thus, our original expression becomes

$$e^{((2k+1)\pi i)(-1)^j e^{\frac{1}{2}\pi i}}$$

To clean this up even more,  $e^{\frac{1}{2}\pi i} = i$ . So, we have our original expression as

$$e^{((2k+1)\pi i)(-1)^j i} = e^{-((2k+1)\pi)(-1)^j}$$

Now, it is somewhat clear that  $\{e^{-((2k+1)\pi)(-1)^j} : j, k \in \mathbb{Z}\} = \{e^{\pm((2k+1)\pi)} : k \in \mathbb{Z}\} = \{e^{-((2k+1)\pi)} : k \in \mathbb{Z}\}$ .

**Problem 4:**

Let  $\ln(z)$  be the principal branch of the logarithm of  $z$ , and let  $z_1, z_2$  have positive real component.

**Problem 5:**

Consider  $\sin(\frac{1}{z})$ . We know that  $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ . So, where defined,  $\sin(\frac{1}{z}) = \sum_{n=0}^{\infty} (-1)^n \frac{\frac{1}{z^{2n+1}}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+1}(2n+1)!}$ .

That is, we have found a Laurent series for  $\sin(\frac{1}{z})$  about 0. We are done.

**Problem 6:**

**Problem 7:**