Problem 1:

Let $f : X \to Y$ be a function such that there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Then for all $y \in Y$, $f(g(y)) = y$.

So for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$ $(x = f(y))$.

So f is onto if there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Problem 2:

Let $\mathcal{C}$ be an arbitrary collection of sets.

First, we show that $(\bigcup\limits_{C\in\mathcal{C}} C)^c = \bigcap\limits_{C\in\mathcal{C}} C^c$.

Let $x \in (\bigcup\limits_{C\in\mathcal{C}} C)^c$.

Then $x \notin (\bigcup\limits_{C\in\mathcal{C}} C)$.

So x is not in C for any $C \in \mathcal{C}$.

So x is in C^c for all $C \in \mathcal{C}$.

So $x \in \bigcap\limits_{C\in\mathcal{C}} C^c$.

This means that $(\bigcup\limits_{C\in\mathcal{C}} C)^c \subset \bigcap\limits_{C\in\mathcal{C}} C^c$.

Next, let $x \in \bigcap\limits_{C\in\mathcal{C}} C^c$.

So x is in C^c for all $C \in \mathcal{C}$.

So x is not in C for any $C \in \mathcal{C}$.

Then $x \notin (\bigcup\limits_{C\in\mathcal{C}} C)$.

Thus, $x \in (\bigcup\limits_{C\in\mathcal{C}} C)^c$.

This means that $(\bigcup\limits_{C\in\mathcal{C}} C)^c \supset \bigcap\limits_{C\in\mathcal{C}} C^c$.

So $(\bigcup\limits_{C\in\mathcal{C}} C)^c = \bigcap\limits_{C\in\mathcal{C}} C^c$.

Next, we show that $(\bigcap\limits_{C\in\mathcal{C}} C)^c = \bigcup\limits_{C\in\mathcal{C}} C^c$.

Let $x \in (\bigcap\limits_{C\in\mathcal{C}} C)^c$.

Then $x \notin (\bigcap\limits_{C\in\mathcal{C}} C)$.

So x is not in C for all $C \in \mathcal{C}$.

So x is in C^c for some $C \in \mathcal{C}$.

So $x \in \bigcup\limits_{C\in\mathcal{C}} C^c$.

This means that $(\bigcap_{C\in\mathcal{C}} C)^c \subset \bigcup_{C\in\mathcal{C}} C^c$.

Next, let $x \in \bigcup_{C\in\mathcal{C}} C^c$.

So x is in C^c for some $C \in \mathcal{C}$.

So x is not in C for all $C \in \mathcal{C}$.

Then $x \notin (\bigcap_{C\in\mathcal{C}} C)$.

Thus, $x \in (\bigcap_{C\in\mathcal{C}} C)^c$.

This means that $(\bigcap_{C\in\mathcal{C}} C)^c \supset \bigcup_{C\in\mathcal{C}} C^c$.

Thus, $(\bigcap_{C\in\mathcal{C}} C)^c = \bigcup_{C\in\mathcal{C}} C^c$.

Problem 3:

Let $\mathcal{C}$ be a collection of sets, and E be in the σ-algebra generated by $\mathcal{C}$.

Now, let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$ and let $\mathcal{B} = \bigcup_{\mathcal{C}_0\subset\mathcal{C}} \sigma(\mathcal{C}_0)$, where each $\mathcal{C}_0$ is countable and $\sigma(\mathcal{C})$ is the σ-algebra generated by $\mathcal{C}$. We proceed by showing that $\mathcal{A} = \mathcal{B}$.

Now, let $A \in \mathcal{A}$.

Then A is in the smallest σ-algebra containing $\mathcal{C}$, meaning that A is in every σ-algebra contianing $\mathcal{C}$.

We show that $A \in \mathcal{B}$ by showing that $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$.

Let $B \in \mathcal{B}$.

Then B is in the σ-algebra generated by some countable subcollection of $\mathcal{C}$.

So B^c is in the σ-algebra generated by some countable subcollection of $\mathcal{C}$.

So $B^c \in \mathcal{B}$.

Next, let $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$.

Then each B_i is in the σ-algebra generated by some countable subcollection of $\mathcal{C}$.

So every B_i is in the σ-algebra generated by some countable subcollection of $\mathcal{C}$. (The union of all of those countable subcollections in the previous lines is a countable union of countable sets...which is countable.)

So $\bigcup_{i\in\mathbb{N}} \subset \mathcal{B}$.

So $\mathcal{B}$ is a σ-algebra.

Note that $\mathcal{B}$ contains $\mathcal{C}$ because each $C \in \mathcal{C}$ is in a countable subcollection of $\mathcal{C}$. (If this is not clear, try $\{C\}$.)

So $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$.

So $A \in \mathcal{B}$.

So, $\mathcal{A} \subset \mathcal{B}$.

Next, let $B \in \mathcal{B}$.

Then B is in the σ-algebra generated by some countable subcolection of $\mathcal{C}$.

So B is in the σ-algebra generated by $\mathcal{C}$.

So $B \in \mathcal{A}$.

So, $\mathcal{A} \supset \mathcal{B}$.

Thus, $\mathcal{A} = \mathcal{B}$.

Now, this means that any element of the σ-algebra generated by $\mathcal{C}$ is in $\mathcal{B}$. In other words, it is in the σ-algebra generated by some countable subcollection of $\mathcal{C}$.

So for all E there is a countable subcollection, $\mathcal{C}_0$ of $\mathcal{C}$ such that E is in the σ-algebra generated by $\mathcal{C}_0$.

Problem 4:

Let $\langle x_n \rangle$ be a sequence of real numbers.

If $\lim\limits_{n\to\infty} x_n$ exists, then $\langle x_n \rangle$ is Cauchy.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \frac{\epsilon}{2}$, where $L = \lim\limits_{n\to\infty} x_n$.

Now, if $n, m \geq N$, then $|x_n - L| < \frac{\epsilon}{2}$ and $|x_m - L| < \frac{\epsilon}{2}$.

By the triangle inequality,

$$\begin{aligned} |x_n - x_m| &\leq |x_n - L| + |x_m - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $|x_n - x_m| < \epsilon$. That is, $\langle x_n \rangle$ is Cauchy.

Next, if $\langle x_n \rangle$ is Cauchy, then $\lim\limits_{n\to\infty} x_n$ exists.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < \frac{\epsilon}{2}$.

This means that for all $n \geq N$, x_n is within the closed interval $[x_N - \frac{\epsilon}{2}, x_N + \frac{\epsilon}{2}]$.

So $\langle x_n \rangle$ has a converging subsequence. Say that the converging subsequence converges to L.

Now, $L \in [x_N - \frac{\epsilon}{2}, x_N + \frac{\epsilon}{2}]$. (Closed interval is closed.)

This means that for all $n \geq N$, $|x_n - L| < \epsilon$ (both x_n and L are in a closed ball of radius ϵ.)

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$ for some $L \in \mathbb{R}$. That is, $\langle x_n \rangle$ converges to L; $\lim_{n\to\infty} x_n$ exists.

Therefore $\lim_{n\to\infty} x_n$ exists if and only if $\langle x_n \rangle$ is Cauchy.

Problem 5:

Let $\langle x_n \rangle$ be a sequence of real numbers, with $\liminf(x_n)$ and $\limsup(x_n)$ both existing.

Then $\liminf(x_n)$ is the infimum of the set of limit points of $\langle x_n \rangle$.

Also, $\limsup(x_n)$ is the supremum of the set of limit points of $\langle x_n \rangle$.

Because $inf(A) \leq sup(A)$ for all nonempty $A \subset \mathbb{R}$, this means that $\liminf(x_n) \leq \limsup(x_n)$.

Moving on, let $\lim_{n\to\infty} x_n = L$.

Then all subsequences of $\langle x_n \rangle$ converge to L. So the infimum and the supremum of the set of limit points of $\langle x_n \rangle$ are both L (the infimum and supremum of $\{L\}$ is L). So $\limsup(x_n) = \liminf(x_n) = L$.

Next, let $\liminf(x_n) = \limsup(x_n) = L$.

Then if a subsequence of $\langle x_n \rangle$ converges, it converges to L; L is the only limit point of $\langle x_n \rangle$.

Moreover, $\langle x_n \rangle$ is bounded. Else, there is an unbounded subsequence of $\langle x_n \rangle$, and so either $\limsup(x_n)$ or $\liminf(x_n)$ is ∞ or $-\infty$.

Now, let $\epsilon > 0$. There is a last $N \in \mathbb{N}$ such that $|x_N - L| \geq \epsilon$.

Else, there are infinitely many $n \in \mathbb{N}$ such that $|x_N - L| \geq \epsilon$. These create a subsequence of $\langle x_n \rangle$. This subsequence is a sequence in its own right, it is bounded, it has a converging subsequence. This converging subsequence cannot converge to L; each point of that subsequence is at least ϵ away from L. Thus, $\langle x_n \rangle$ has a subsequence converging to something other than L...which means that either $\liminf(x_n) \neq L$ or $\limsup(x_n) \neq L$. Either way, that contradicts our original assumption.

So, for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. That is, $\lim(x_n) = L$.

So $\liminf(x_n) = \limsup(x_n) = L$ if and only if $\lim(x_n) = L$.

Problem 6:

It is a well-known fact that if there is an injection, $f : X \to Y$ and an injection $g : Y \to X$, then there is a bijection $h : X \to Y$. We exploit this fact.

Denote the Cantor set by C. A quick note: C is closed, as it is the complement of an open set. (It is usually defined as the complement of an infinite union of open intervals. This union of open intervals is open...and thus, its complement is closed.)

There is an injection $f : C \to [0, 1]$ given by $f(x) = x$.

Next, there is an injection $g : [0, 1] \to C$ given as follows:

Let $x \in [0, 1]$. We know that x has a binary expansion, $0.a_1a_2\ldots$. (Even in the case of $x = 1$, this expansion can be $0.11111\ldots$.)

Note that this binary expansion need not be unique. We just choose one (which we can do, by the Axiom of Choice).

We next define a sequence of nested closed subsets of C as follows:

If $a_1 = 0$, set $C_1(x) = [0, 1/3]$. Else, set $C_1(x) = [2/3, 1]$.

If $a_2 = 0$, set $C_2(x)$ equal to the lower third of $C_1(x)$. Else, set $C_2(x)$ equal to the upper third of $C_1(x)$.

For all $n > 1$, set $C_n(x)$ equal to the lower third of $C_{n-1}(x)$ if $a_n = 0$, else set $C_n(x)$ equal to the upper third of $C_{n-1}(x)$.

The sequence $\langle C_n \rangle$ is a sequence of nested, closed subsets of C whose diameter approaches 0. So, there is a unique point,y, in the intersection of these closed subsets.

We define $g(x) = y$.

Now, if $a, b \in [0, 1]$ and $a \neq b$, then $g(a) \neq g(b)$:

Because $a \neq b$, a and b have two different binary expansions, $0.a_1a_2\ldots$ and $0.b_1b_2\ldots$.

So there is an index, i, with $a_i \neq b_i$.

This means that $g(a)$ and $g(b)$ are contained in disjoint closed intervals, $C_i(a)$ and $C_i(b)$. So $g(a) \neq g(b)$.

We have an injection $f : C \to [0, 1]$ and an injection $g : [0, 1] \to C$, so there is a bijection $h : C \to [0, 1]$.

Problem 7:

First, note that for any subset, A, of $\mathbb{R}$, the set of accumulation points, A', is a subset of the closure of A. That is $A' \subset \overline{A}$.

The Cantor Set is closed. So, $C' \subset C$.

Now, let $x \in C$.

Then $x \in [0, 1/3]$ or $x \in [2/3, 1]$. Define C_1 to be the interval that x is in.

Similarly, x is in either the lower third or the upper third of C_1. Define C_2 to be the third of C_1 that x is in.

For each $n \in \mathbb{N}$, define C_n to be the third of C_{n-1} that x is in.

Now, for each $n \in \mathbb{N}$, pick an $x_n \in C_n$ such that $x_n \neq x$.

Note that the C_ns are nested, closed subsets of $\mathbb{R}$ whose diameter approaches 0. This means that their intersection has a unique point. Because $x \in C_n$ for all $n \in \mathbb{N}$, this means that x is that unique point.

Moreover, it is clear that $\langle x_n \rangle \to x$. (Should I break out my epsilons, or is it OK to just state this?)

So there's a sequence $\langle x_n \rangle$ in C such that $\lim n \to \infty x_n \to x$ and $x_n \neq x$. So $x \in C'$.

So $C \subset C'$.

So $C = C'$. That is, the set of accumulation points of the Cantor Set is the Cantor Set itself.