Problem 1:

Let $f_n \to f$ in measure, with an integrable function g such that $|f_n| \leq g$ for all n.

Because $f_n \to f$ in measure, there's a subsequence $\langle f_{n_k} \rangle$ such that $f_{n_k} \to f$ almost everywhere.

So the Lebesgue convergence theorem applies to that subsequence: $|f_{n_k} - f| \to 0$ and there's an integrable function g such that $|f_{n_k}| \leq g$ for all k, so we have $\int |f_{n_k} - f| \to 0$.

So if the sequence $\left\langle \int |f_n - f| \right\rangle$ converges, then it converges to 0.

Assume that the above sequence doesn't converge.

That is, there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n > N$ such that $\left| \int |f_n - f| \right| \geq \epsilon$.

Thus, there is a sequence, $\langle f_{n_k} \rangle$, such that $\left| \int |f_{n_k} - f| \right| \geq \epsilon$ for all k.

We know that $f_{n_k} \to f$ in measure. So there's a subsequence, $\left\langle f_{n_{k_j}} \right\rangle$ such that $f_{n_{k_j}} \to f$ almost everywhere. So the Lebesgue convergence theorem applies to that subsequence: $\int \left| f_{n_{k_j}} - f \right| \to 0$. But this is a clear contradiction.

So that sequence converges, and it converges to the right thing.

Problem 2:

Let f be continuous on $[a, b]$, with one of its derivates everywhere nonnegative on (a, b).

First, we will show this for a function g with $D^+(g) \geq \epsilon > 0$. If g is such a function, then $\limsup\limits_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq \epsilon > 0$. This means that g is nondecreasing:

We proceed by contradiction: let there be $x, y \in [a, b]$ (with $x < y$, without loss of generality) be such that $f(x) > f(y)$.

Consider the set $A = \{\alpha \in [x, y) : f(\alpha) > f(y)\}$. This set has a supremum, as it's nonempty. Define $\alpha = \sup(A)$. Either $\alpha = b$ (in which case, the derivate is negative at b, leading to our contradiction), or there is a $\delta > 0$ such that if $t \in [\alpha, \alpha + \delta]$, then $f(t) < f(y)$. Moreover, $f(\alpha) = f(y)$: else, $|f(\alpha) - f(y)| = \epsilon'$ for some $\epsilon' > 0$, so by using continuity (specifically, the intermediate value theorem) we can find an α' between α and y with $f(\alpha')$ between $f(\alpha)$ and $f(y)$, which causes a contradiction. So for any sequence (t_n) decreasing to α, we have $\frac{f(t_n)-f(\alpha)}{t_n - \alpha}$ negative. This means that $D^+(\alpha) \leq 0$. This contradicts our assumption on D^+.

We can mimic this proof to show that if g has $D^-(g) \geq \epsilon > 0$, then g is nondecreasing.

Now, let f have a derivate everywhere nonnegative on (a, b). This means, in particular, that either D^+ or D^- is everywhere nonnegative on (a, b).

Then for every $\epsilon > 0$, $g_\epsilon(x) = f(x) + \epsilon x$ has $D^+(g_\epsilon)$ (or $D^-(g_\epsilon)$) greater than ϵ. So for all $\epsilon > 0$, g_ϵ is nondecreasing. So for all $x, y \in [a, b]$ with $x < y$, $g_\epsilon(x) \leq g_\epsilon(y)$. That is, $f(x) + \epsilon x \leq f(y) + \epsilon y$. Taking limits as $\epsilon \to 0$, this means that $f(x) \leq f(y)$, for all $x, y \in [a, b]$ with $x < y$.

So f is nondecreasing on $[a, b]$ if some derivate is everywhere nonnegative on $[a, b]$.

Problem 3:

Suppose that $f_n(x) \to f(x)$ at each $x \in [a, b]$.

Let $\Delta = \{x_1 < x_2 \ldots < x_k\}$ be a partition of $[a, b]$. Then $t(f) = \sum |f(x_i) - f(x_{i-1})|$, and $t(f_n) = \sum |f_n(x_i) - f_n(x_{i-1})|$.

Now, $f_n \to f$ at all $x \in [a, b]$.

That is, for all $\epsilon > 0$ and $x \in [a, b]$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. So for all $\epsilon > 0$ there's an $N \in \mathbb{N}$ such that for every x_i in our partition and for all $n \geq N$, $|f_n(x_i) - f(x)| < \epsilon/(2k)$.

So we have:

$$\begin{aligned}
t(f) - t(f_n) &= \sum |f(x_i) - f(x_{i-1})| - \sum |f_n(x_i) - f_n(x_{i-1})| \\
&\leq \sum |f(x_i) - f_n(x_i) - (f(x_{i-1}) - f_n(x_{i-1}))| \\
&\leq \sum |f(x_i) - f_n(x_i)| + \sum |f(x_{i-1}) - f_n(x_{i-1})| \\
&< \epsilon
\end{aligned}$$

To rephrase, $t(f) < t(f_n) + \epsilon$ for sufficiently large n.

So $T_a^b(f) \leq \liminf T_a^b(f_n)$, by taking limits as the mesh of $\Delta \to 0$. (Note: A fellow classmate pointed out that there's a subtlety here. I feel like it's clear, but the idea is that we only need a subsequence of the f_ns to make this work, and we can build off of that.)

Problem 4:

Suppose that $f \in BV([a,b])$. Then f' exists almost everywhere, by a theorem in class. Moreover, f is the difference of two monotone functions. That is, $f = f^+ - f^-$ for some monotone functions f^+ and f^-.

So, this means that we have

$$\begin{aligned}\int_a^b |f'| &= \int_a^b \left|(f^+)' - (f^-)'\right| \\ &\le \int_a^b \left|(f^+)'\right| + \left|(f^-)'\right|\end{aligned}$$

Now, we show that $\int_a^b |(f^+)'| \le P_a^b(f)$.

By one of the important theorems that is like the fundamental theorem of calculus, $\int_a^b |(f^+)'| \le f^+(b) - f^+(a)$.

We also know that $f^+(b) - f^+(a) \le P_a^b$;

$$\begin{aligned}P &= N + f(b) - f(a) \\ &= N + f^+(b) - f^-(b) - f^+(a) + f^-(a) \\ &= N + (f^+(b) - f^+(a)) + (f^-(b) - f^-(a)) \\ &\ge N + (f^+(b) - f^+(a)) \\ &\ge (f^+(b) - f^+(a))\end{aligned}$$

Similarly, $\int_a^b |(f^-)'| \le N_a^b(f)$. So, we have

$$\begin{aligned}\int_a^b |f'| &\le \int_a^b \left|(f^+)'\right| + \left|(f^-)'\right| \\ &\le P_a^b + N_a^b \\ &\le T_a^b\end{aligned}$$

as we desired.

Problem 5:

Let g be an absolutely continuous monotone function on $[0,1]$, and E be a set of measure 0. Without loss of generality, we can take g to be increasing. Further, for this problem we can take $g(0) = 0$ without loss of generality, as the measure of a set is translation invariant.

We know that g is the antiderivative of some function, f. That is,

$$g(x) = \int\limits_0^x f(t)dt$$

Moreover, because g is increasing, this means that f is nonnegative almost everywhere.

Now, $[0,1] \setminus E$. It is true that $\int\limits_{[0,x]\setminus E} f(t)dt = g(x)$.

Assume that $g(E)$ has nonzero measure. Then $g(E)$ contains a closed interval, call it $[a,b]$.

Then $\{y : \int\limits_0^x f(t)dt = y \text{ for some } x \in E\}$ contains $[a,b]$.

Let x_1 be a value such that $\int\limits_0^x f(t)dt = a$ and x_2 be a value such that $\int\limits_0^x f(t)dt = b$. Then $[x_1, x_2] \subset E$: this is somewhat clear.

So E must contain a closed interval, and is thus of nonzero measure, contrary to our assumption.

So $g(E)$ must have had zero measure.

Problem 6:

Let f be a nonnegative measurable function on $[0,1]$.

We know that ln is a concave function on $[0,1]$ (if this is not clear, it's the inverse of a convex function).

So $-\ln$ is a convex function on $[0,1]$.

So Jensen's inequality applies:

$$-\ln \int f \leq -\int \ln f$$
$$\ln \int f \geq \int \ln f$$

This satisfies the problem.