Problem 1 (Problem 2 in book):

Let u be such that $\Delta u = 0$ and let v be such that $v(x) = u(Ox)$ for some orthogonal $n \times n$ matrix O.

Then we have:

$$\begin{aligned}
\Delta v(x) &= \Delta u(x) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(Ox) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(Ox_1 e_1 + Ox_2 e_2 \ldots + Ox_n e_n) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_i}(Ox_j e_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_i x_i}(x_j e_j) \\
&= \sum_{i=1}^{n} u_{x_i x_i}(x) \\
&= 0
\end{aligned}$$

That is, solutions to $\Delta u = 0$ are rotation-invariant.

Problem 2 (Problem 3 in book):

Let u be such that

$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

with dimension $n \geq 3$.

Then define $\phi(r) = \fint_{\partial B(0,r)} u(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) u(x)dx$.

Then

$$\begin{aligned}
\phi'(r) &= stuff \\
&= 0
\end{aligned}$$

So ϕ' is identically zero. So ϕ is constant. So $\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0} \fint_{\partial B(0,r)} u(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) u(x)dx = u(0)$.

Thus, by replacing u with f and g on the interior/exterior,

$$u(0) = \fint_{\partial B(0,r)} gdS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) fdx$$

Problem 3 (Only homework on page):
Let $u(x)$ be a C^2 solution to

$$\Delta u(x) = |x|^2 \text{ on } \mathbb{R}^n$$

with $n \geq 3$.
Set $m(r) = \fint_{\partial B(0,r)} u(y)dS(y)$.
Then u solves

$$\begin{aligned} \Delta(u) &= |x|^2 \text{ in } B(0,r) \\ u &= r \text{ on } \partial B(0,r) \end{aligned}$$

So by the above problem,

$$
\begin{aligned}
u(0) &= \fint_{\partial B(0,r)} u(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) |x|^2 \, dx \\
&= m(r) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) |x|^2 \, dx \\
&= m(r) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{|x|^2}{|x|^{n-2}} - \frac{|x|^2}{r^{n-2}} \right) dx \\
&= m(r) - \frac{r^4}{4(n+2)}
\end{aligned}
$$

which is the result we wanted.

If $n = 1$, then $u(x) = x^4 + Cx + D$ are the only solutions to this, where $C, D \in \mathbb{R}$; this follows from elementary differential equations.

If $n = 2$, then ...