Note to self: Replace $Spec(R)$ with $\mathrm{Spec}(R)$.

Problem 1:

Part a:

Let $r \in R^*$.

Then there is an $r^{-1} \in R$ such that $r^{-1}r = 1$. If $r \in M$ for any maximal ideal, M, then $r^{-1}r \in M$, so $1 \in M$, so $M = R$. This means that M is not a maximal ideal. That is,$r \notin \bigcup\limits_{M \in m-Spec(R)} M$, so $r \in R \setminus \bigcup\limits_{M \in m-Spec(R)} M$.

Now, let $r \notin \bigcup\limits_{M \in m-Spec(R)} M$. That is, $r \in R \setminus \bigcup\limits_{M \in m-Spec(R)} M$ isn't in any maximal ideal.

Then r is not in any ideal other than R; all ideals other than R are contained in some maximal ideal.

So $(r) = R$. This means that $1 \in (r)$. So there's an element, $r^{-1} \in R$, such that $r^{-1}r = 1$. So $r \in R^*$.

So $R^* \subset R \setminus \bigcup\limits_{M \in m-Spec(R)} M$ and $R^* \supset R \setminus \bigcup\limits_{M \in m-Spec(R)} M$.

So $R^* = R \setminus \bigcup\limits_{M \in m-Spec(R)} M$.

Part b:

We freely use the fact that $R \setminus R^* = \bigcup\limits_{M \in m-Spec(R)} M$. This follows from the above, and is clear with proper notation:

$$R^* = \left(\bigcup_{M \in m-Spec(R)} M \right)^c$$
$$(R^*)^c = \bigcup_{M \in m-Spec(R)} M$$

If $R \setminus R^* = \bigcup\limits_{M \in m-Spec(R)} M$ is an ideal, then $\bigcup\limits_{M \in m-Spec(R)} M$ is a maximal ideal or R; it contains every maximal ideal, so it contains every ideal other than R. But this ideal is not R, otherwise R^* is empty (and we know that R^* contains 1.) So $\bigcup\limits_{M \in m-Spec(R)} M$ is a maximal ideal that contains every maximal ideal. That is, it is the unique maximal ideal. So R is local.

If R is local, then say that M' is R's unique maximal ideal. Then we have that $R \setminus R^* = \bigcup\limits_{M \in m-Spec(R)} M = M'$ is an ideal.

Problem 2:

Let $P \in Spec(R)$. Consider PR_P.

We can see that PR_P is an R_P-ideal:

Further, if I is an R_P-ideal other than R_P, then $I \subset PR_P$:

So PR_P contains every ideal other than the entire ring; it is the unique maximal ideal, making R_P local.

Problem 3:

Define $rad(R) = \sqrt{0} = \{a \in R : a^n = 0 \text{ for some } n \geq 0\}$.

Then if $r \in rad(R)$, then $r \in \bigcap\limits_{P \in Spec(R)} P$:

We know that $r^n = 0$ for some $n \in \mathbb{N}$. For any prime ideal, P, $0 \in P$. So this means that either r or r^{n-1} is in P. If $r \in P$, we're done.

If $r^{n-1} \in P$, this means that r or r^{n-2} is in P. If $r \in P$, we're done.

We can iterate down to r; the process above ends, so we can get that $r \in P$ for all prime ideals, P. That is, $r \in \bigcap\limits_{P \in Spec(R)} P$.

Next, if $r \in \bigcap\limits_{P \in Spec(R)} P$, then $r \in rad(R)$:

If $r \in \bigcap\limits_{P \in Spec(R)} P$, then $ar \in P$ for all $a \in R$, $P \in Spec(R)$. So in particular, $r^n \in P$ for all $n \in \mathbb{N}$, $P \in Spec(R)$.

Problem 4:

Let $u \in R^*$ and $a \in rad(0)$.

Then $a^{2^n} = 0$ for some $n \in \mathbb{N}$: since $a \in rad(0)$, $a^k = 0$ for some $k \in \mathbb{N}$. So for all $j \geq k$, $a^j = 0$. There's an $n \in \mathbb{N}$ such that $2^n \geq k$, so we have what we want.

Now, consider the product

$$\begin{aligned}(u+a)(u-a)(u^2+a^2)\dots(u^{2^{n-1}}+a^{2^{n-1}}) &= u^{2^n} - a^{2^n} \\ &= u^{2^n}\end{aligned}$$

Now, there is a $u^{-1} \in R$ such that $u^{-1}u = 1$. It is clear also that $(u^{-1})^{2^n} u^{2^n} = 1$. So we have

$$
\begin{aligned}
u^{-2^n}(u+a)(u-a)(u^2+a^2)\dots(u^{2^{n-1}}+a^{2^{n-1}}) &= u^{-2^n}(u^{2^n}-a^{2^n}) \\
&= u^{-2^n}u^{2^n} \\
&= 1
\end{aligned}
$$

So $(u+a)u^{-2^n}(u-a)(u^2+a^2)\dots(u^{2^{n-1}}+a^{2^{n-1}}) = 1$. So $u+a \in R^*$.

Problem 5:

Let R be a PID, and P be a nonzero prime ideal of R.

Then $P = (p)$ for some nonzero $p \in P$. We know that prime elements are irreducible. Ideals generated by a single irreducible element are maximal among proper principal ideals. All ideals are pricipal. So P is a maximal ideal.

Problem 6:

Let R be a domain, and $a, b \in R$.

Let $(a) \cap (b)$ be a principal ideal. Then $(a) \cap (b) = (c)$ for some $c \in R$. It is clear that c is a multiple of both a and b. Further, (c) is the set of all multiples of c (this is clear from the definitions), so this means that every element of $(a) \cap (b)$ is a multiple of c. So $a \mid x$ and $b \mid x$ implies that $c \mid x$. That is, c is the *lcm* of a and b; $lcm(a, b)$ exists.

Let $lcm(a, b)$ exist. Then $(lcm(a, b)) = (a) \cap (b)$: first, as $lcm(a, b)$ is a multiple of a, it is in (a), so $lcm(a, b) \subset (a)$. Similarly, $lcm(a, b) \subset (b)$. So $(lcm(a, b)) \subset (a) \cap (b)$. Next, if $x \in (a) \cap (b)$, then x is a multiple of both a and b. So x is a multiple of $lcm(a, b)$. This means that $x \in (lcm(a, b))$. So $(lcm(a, b)) \supset (a) \cap (b)$, so $(lcm(a, b)) = (a) \cap (b)$. In particular, this means that $(a) \cap (b)$ is a principal ideal.

So $lcm(a, b)$ exists if and only if $(a) \cap (b)$ is a principal ideal, and in this case $(a) \cap (b) = (lcm(a, b))$.