Problem 1a, p20:

Let $f : A \to B$. Let $A_0 \subset A$, $B_0 \subset B$.

Let $a \in A_0$. Then $f(a) \in f(A_0)$. Now, $a \in f^{-1}(f(A_0))$ if there is $a' \in f(A_0)$ with $f(a) = a'$. Because $f(a) \in f(A_0)$, there is $a' \in f(A_0)$ with $f(a) = a'$. So $a \in f^{-1}(f(A_0))$.

So, $A_0 \subset f^{-1}(f(A_0))$.

Now, let f be injective and $a \in f^{-1}(f(A_0))$. Then there is $b \in f(A_0)$ with $f(a) = b$. Now, because $b \in f(A_0)$, we have that $b = f(a')$ for some $a' \in A_0$. Because f is injective, $f(a) = b = f(a')$ implies that $a = a'$. So, because $a' \in A_0$, this means that $a \in A_0$.

So, $A_0 \supset f^{-1}(f(A_0))$ if f is injective. So, $A_0 = f^{-1}(f(A_0))$ if f is injective.

Problem 1b, p20:

Let $f : A \to B$. Let $A_0 \subset A$, $B_0 \subset B$.

Let $b \in f(f^{-1}(B_0))$. Then there is an $a \in f^{-1}(B_0)$ with $f(a) = b$. Now, because $a \in f^{-1}(B_0)$, there is $b' \in B_0$ with $f(a) = b'$. Now, f is a function. So $f(a) = b' = b$. So $b \in B_0$.

So, $f(f^{-1}(B_0)) \subset B_0$.

Now, let f be surjective and $b \in B_0$. Then there is $a \in A$ with $f(a) = b$, as f is surjective. That is, $f^{-1}(B_0)$ is nonempty, and there is $a \in f^{-1}(B_0)$ with $f(a) = b$. Now, $f(a) = b$, so $b \in f(f^{-1}(B_0))$.

So, $f(f^{-1}(B_0)) \supset B_0$ if f is surjective. So, $f(f^{-1}(B_0)) = B_0$ if f is surjective.

Problem 2g, p20:

Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$.

Let $b \in f(A_0 \cap A_1)$. Then there is $a \in A_0 \cap A_1$ with $f(a) = b$. So there is $a \in A_0$ with $f(a) = b$; that is, $b \in f(A_0)$. Also there is $a \in A_1$ with $f(a) = b$; that is, $b \in f(A_1)$. So $b \in f(A_0) \cap f(A_1)$, because $b \in f(A_0)$ and $b \in f(A_1)$.

So, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, let f be injective and $b \in f(A_0) \cap f(A_1)$. Then there is $a_0 \in A_0$ with $f(a_0) = b$. Also, there is $a_1 \in A_0$ with $f(a_1) = b$. Because f is injective, $f(a_0) = b = f(a_1)$ implies that $a_0 = a_1$. So $a_0 \in A_1$. So $a_0 \in A_0 \cap A_1$. So because $f(a_0) = b$, we have that $b = f(a)$ for some $a \in A_0 \cap A_1$. So $b \in f(A_0 \cap A_1)$.

So, $f(A_0 \cap A_1) \supset f(A_0) \cap f(A_1)$ if f is injective. So if f is injective, $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$.

Problem 3, p83:

Let X be a set. Consider the collection $\mathcal{T}_C$, the collection of all subsets U of X such that $X \setminus U$ is countable or all of X.

Then $\emptyset \in \mathcal{T}_C$, as $X \setminus \emptyset = X$, which is all of X. Also, $X \in \mathcal{T}_C$, as $X \setminus X = \emptyset$, which is countable.

That is, $\emptyset \in \mathcal{T}_C$ and $X \in \mathcal{T}_C$.

Next, let $\mathcal{C} \subset \mathcal{T}_C$ be nonempty. If the only element of $\mathcal{C}$ is $\emptyset$, then $X \setminus \bigcup\limits_{U \in \mathcal{C}} U = X$, so that $\bigcup\limits_{U \in \mathcal{C}} U \in \mathcal{T}_C$. Else, pick a nonempty element of $\mathcal{C}$; call it U_0. Then $\bigcup\limits_{U \in \mathcal{C}} U \supset U_0$, so $X \setminus U_0 \supset X \setminus \bigcup\limits_{U \in \mathcal{C}} U$. Because $X \setminus U_0$ is finite (as $U_0 \in \mathcal{T}_C$ so that $X \setminus U_0$ is either finite or all of X, and $X \setminus U_0$ is not all of X, as U_0 is nonempty), this means that $X \setminus \bigcup\limits_{U \in \mathcal{C}} U$ is finite. That is, $X \setminus \bigcup\limits_{U \in \mathcal{C}} U \in \mathcal{T}_C$.

That is, arbitrary unions of elements in $\mathcal{T}_C$ are contained in $\mathcal{T}_C$.

Last, let $\{U_0, U_1, \ldots U_n\} \subset \mathcal{T}_C$. If $\bigcap\limits_{i=0}^{n} U_i = \emptyset$, then $X \setminus \bigcap\limits_{i=0}^{n} U_i = X \setminus \emptyset = X$. Else, recall that $X \setminus \bigcap\limits_{i=0}^{n} U_i = \bigcup\limits_{i=0}^{n} X \setminus U_i$, which is a finite union of finite sets (None of $X \setminus U_i$ are all of X, else $\bigcap\limits_{i=0}^{n} U_i = \emptyset$). So, $X \setminus \bigcap\limits_{i=0}^{n} U_i$ is finite, so that $\bigcap\limits_{i=0}^{n} U_i \in \mathcal{T}_C$.

That is, finite intersections of elements in $\mathcal{T}_C$ are contained in $\mathcal{T}_C$.

So $\mathcal{T}_C$ is a topology.

Problem 5, p83:

Let $\mathcal{A}$ be a basis for a topology, call it $\mathcal{T}$, on X.

Consider $\mathcal{T}' = \bigcap\limits_{\mathcal{T}'' : \mathcal{T}'' \supset \mathcal{A}} \mathcal{T}''$; the intersection of all topologies containing $\mathcal{A}$.

First, $\mathcal{T}' \subset \mathcal{T}$: Let $U \in \mathcal{T}'$. Then U is in the intersection of all topologies containing $\mathcal{A}$. So U is in any topology containing $\mathcal{A}$. Now, $\mathcal{T}$ is a topology containing $\mathcal{A}$. So $\mathcal{A} \in \mathcal{T}$.

Next, $\mathcal{T} \subset \mathcal{T}'$: Let $U \in \mathcal{T}$, let $\mathcal{T}''$ be a topology containing $\mathcal{A}$. Then U is a union of elements in $\mathcal{A}$. So U is a union of elements in $\mathcal{T}''$. So $U \in \mathcal{T}''$. So U is in any topology containing $\mathcal{A}$. So U is in the intersection of all topologies containing $\mathcal{A}$. So $U \in \mathcal{T}'$.

So, $\mathcal{T} = \mathcal{T}'$. So, the topology generated by $\mathcal{A}$ is the intersection of all topologies containing $\mathcal{A}$.

Next, let $\mathcal{A}$ be a subbasis for a topology, call it $\mathcal{T}$, on X.

Consider $\mathcal{T}' = \bigcap_{\mathcal{T}'' : \mathcal{T}'' \supset \mathcal{A}} \mathcal{T}''$; the intersection of all topologies containing $\mathcal{A}$.

First, $\mathcal{T}' \subset \mathcal{T}$: Let $U \in \mathcal{T}'$. Then U is in the intersection of all topologies containing $\mathcal{A}$. So U is in any topology containing $\mathcal{A}$. Now, $\mathcal{T}$ is a topology containing $\mathcal{A}$. So $\mathcal{A} \in \mathcal{T}$.

Next, $\mathcal{T} \subset \mathcal{T}'$: Let $U \in \mathcal{T}$, let $\mathcal{T}''$ be a topology containing $\mathcal{A}$. Then U is a union of finite intersections of elements of $\mathcal{A}$. So U is a union of finite intersections of elements of $\mathcal{T}''$. So $U \in \mathcal{T}''$. So U is in any topology containing $\mathcal{A}$. So U is in the intersection of all topologies containing $\mathcal{A}$. So $U \in \mathcal{T}'$.

So, $\mathcal{T} = \mathcal{T}'$. So, the topology generated by $\mathcal{A}$ is the intersection of all topologies containing $\mathcal{A}$.

Problem 8b, p83:

Consider $\mathcal{C} = \{[a, b) : a < b, a, b \in \mathbb{Q}\}$.

For each $x \in \mathbb{R}$, $[\lfloor x \rfloor, \lceil x + 1 \rceil) \in \mathcal{C}$ contains x.

That is, for each $x \in \mathbb{R}$, there is a $B \in \mathcal{C}$ containing x.

Now, let $B_1, B_2 \in \mathcal{C}$; say that $B_1 = [a, b)$ and $B_2 = [c, d)$ with $a, b, c, d \in \mathbb{Q}$. Let $x \in B_1 \cap B_2$. Define $e = \max(a, c)$ and $f = \min(b, d)$. Then it is clear that $[e_x, f_x) = B_1 \cap B_2$ (so that $[e_x, f_x) \subset B_1 \cap B_2$), and also $[e, f)$ contains x.

That is, for each $B_1, B_2 \in \mathcal{C}$, and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{C}$ such that $x \in B_3 \subset B_1 \cap B_2$.

So, $\mathcal{C}$ is a basis.

Now, consider the lower-limit topology on $\mathbb{R}$, generated by the basis $\mathcal{D} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$. It is clear that $[\sqrt{2}, 3)$ is in the lower-limit topology. However, $[\sqrt{2}, 3)$ is not in the topology generated by $\mathcal{C}$: else, $[\sqrt{2}, 3)$ is the union of some collection, $\mathcal{E}$, of sets in $\mathcal{C}$. This means that $\sqrt{2}$ is in some element, E, of $\mathcal{E}$. Yet, $\sqrt{2}$ is irrational; if $\sqrt{2} \in E$, then there is some $r < \sqrt{2}$ with $r \in E$, because E's endpoints are rational. So, r is in the union of sets of $\mathcal{E}$. Yet, because $r < \sqrt{2}$, this means that the union of sets of $\mathcal{E}$ could not have been $[\sqrt{2}, 3)$.

So there is some set that is open in the lower-limit topology on $\mathbb{R}$ that is not open in this topology.

Problem 1, p91:

Let Y be a subspace of X, and $A \subset Y$. Let $\mathcal{T}$ be the topology A inherits as a subspace of Y, and $\mathcal{T}'$ be the topology A inherits as a subspace of X.

First, $\mathcal{T} \subset \mathcal{T}'$: Let $U \in \mathcal{T}$. Then $U = A \cap U'$ for some open U' in X. Now, $A = A \cap Y$, so $U = A \cap Y \cap U' = A \cap (Y \cap U')$. That is, $U = A \cap U''$ for some U'' open in Y; $U \in \mathcal{T}'$.

Next, $\mathcal{T}' \subset \mathcal{T}$: Let $U \in \mathcal{T}'$. Then $U = A \cap U'$ for some open U' in Y. Now, $U' = Y \cap U''$ for some open U'' in X, so $U = A \cap Y \cap U'' = A \cap U''$. That is, $U = A \cap U''$ for some U'' open in X; $U \in \mathcal{T}$.

So, $\mathcal{T} = \mathcal{T}'$.

Problem 4, p91:

Let X, Y be topological spaces. Consider $\pi_1 : X \times Y \to X$, a projection map.

Let $W \subset X \times Y$ be an open set. Then U is the union of some collection of sets $\mathcal{C}$, with each $W' \in \mathcal{C}$ a product of two open sets. Now $\pi_1(U) = \{x \in X : \exists (x, y) \in U\} = \{x \in X : \exists W' \in \mathcal{C} : x \in W'\} = \bigcup_{W' \in \mathcal{C}} \pi_1(W')$, which is a union of open sets (as $\pi_1(W') = \pi_1(U' \times V') = U'$ for some U', V' open in X, Y for each $W' \in \mathcal{C}$.

So π_1 maps open sets to open sets; π_1 is an open map.

Similarly, π_2 is an open map.

Problem 6, p91:

Consider $\mathcal{B} = \{(a, b) \times (c, d) : a < b, c < d, \{a, b, c, d\} \subset \mathbb{Q}\}$.

For each $(x, y) \in \mathbb{R}^2$, the element $(\lfloor x-1 \rfloor, \lceil x+1 \rceil) \times (\lfloor y-1 \rfloor, \lceil y+1 \rceil) \in \mathcal{B}$ contains (x, y). (Where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the standard floor and ceiling functions.)

That is, for each $(x, y) \in \mathbb{R}^2$, at least one element in $\mathcal{B}$ contains (x, y).

Next, let $B_1, B_2 \in \mathcal{B}$, with $B_1 = (a, b) \times (c, d)$ and $B_2 = (e, f) \times (g, h)$. Define $i = \max(a, e)$, $j = \min(b, f)$, $k = \max(c, g)$, and $l = \min(d, h)$. Then it is clear that $B_1 \cap B_2 = (i, j) \times (k, l) \in \mathcal{B}$. Let $x \in B_1 \cap B_2$. Then $B_3 = (i, j) \times (k, l)$ contains x and $B_3 \subset B_1 \cap B_2$.

That is, if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Thus, $\mathcal{B}$ is a basis.

Problem 9, p91:

Consider the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and the product topology $\mathbb{R}_d \times \mathbb{R}$, with $\mathbb{R}_d$ being $\mathbb{R}$ given the discrete topology.

Let U be open in the dictionary order topology on $\mathbb{R}^2$. Then U is a union of "open intervals" (intervals in the dictionary order) of the form $((a, b), (c, d))$ with $a < c$ or $a = c$ and $b < d$. Now, each such open interval in the dictionary order is equal to $\{a\} \times (b, \infty) \cup (a, c) \times \mathbb{R} \cup \{c\} \times (-\infty, d)$ or $\{a\} \times (b, d)$. So U is equal to a union of sets of the form $\{a\} \times I$ for some open interval I. So U is equal to a union of basis elements of the product topology $\mathbb{R}_d \times \mathbb{R}$ (as the collection of elements of the form $\{a\} : a \in \mathbb{R}$ is a basis for the discrete topology on $\mathbb{R}$ and the collection of open intervals/open rays is a basis for the standard topology on $\mathbb{R}$, so the collection of sets of the form $\{a\} \times I$ with I some open interval/open ray is a basis for the product topology $\mathbb{R}_d \times \mathbb{R}$): U is open in $\mathbb{R}_d \times \mathbb{R}$.

That is, U is open in the product topology on $\mathbb{R}_d \times \mathbb{R}$ if U is open in the dictionary order topology on $\mathbb{R}^2$.

Next, let U be open in the product topology on $\mathbb{R}_d \times \mathbb{R}$. Then U is a union of elements of the form $\{a\} \times I$ with I some open interval/open ray and $a \in \mathbb{R}$, as above. Each such set is open in the dictionary order topology on $\mathbb{R}^2$, as above. So U is a union of open sets in the dictionary order topology; U is open in the dictionary order topology.

That is, U is open in the dictionary order topology on $\mathbb{R}^2$ if U is open in the product topology on $\mathbb{R}_d \times \mathbb{R}$.

That is, those two topologies are equal.