**Problem 1:**

Part a:

The set of points $z$ in the complex plane described by $|z - z_1| = |z - z_2|$ is the set of points equidistant from $z_1$ and $z_2$.

That is, it's described by the line going through the midpoint of $z_1$ and $z_2$ that is perpendicular to the line going through $z_1$ and $z_2$.

Part b:

The set of points $z$ in the complex plane described by $1/z = \overline{z}$ is the set of points with $z\overline{z} = 1$, so that $|z| = \sqrt{z\overline{z}} = 1$.

That is, it's the unit circle centered at the origin.

Part c:

The set of points $z$ in the complex plane described by $\mathrm{Re}(z) = 3$ is the set of points with real component 3.

That is, it's described by a vertical line with x-intercept 3.

Part d:

The set of points $z$ in the complex plane described by $\mathrm{Re}(z) > c$ is the set of points with real component larger than $c$.

That is, it's described by the right half of a plane cut by the vertical line with x-intercept $c$ (excluding the line itself).

**Problem 2:**

Let $z, w \in \mathbb{C}$, with $z \neq w$, be vertices of a square, with $z = (x, y)$ and $w = (x', y')$.

Either $z$ and $w$ are the opposite vertices of a square or they are adjacent vertices of a square.

If they are opposite vertices of a square, then there is a uniquely determined square given these vertices. (And this is geometrically clear.)

In this case, we proceed by finding the center of the square, finding a line perpendicular to the line through $z$ and $w$ through the midpoint, and finding points $a$ and $b$ on this line that make a square.

The center of the square is given by $A = (\frac{x+x'}{2}, \frac{y+y'}{2})$; it is the midpoint of two opposite sides.

Now, consider the vector $p = z - w = (x - x', y - y')$, and the vector $q = (y - y', -(x - x'))$. It is readily checked that $p \cdot q = 0$, so that these vectors are perpendicular.

Now, the points given by $a = A + q/2 = (\frac{x+x'+y-y'}{2}, \frac{y+y'-(x-x')}{2})$ and $b = A - q/2 = (\frac{x+x'-(y-y')}{2}, \frac{y+y'+x-x'}{2})$ are the desired points, as $a, b, z, w$ are four points equidistant from $A$ with the property that $(a - b) \cdot (z - w) = 0$ (that is, we can make opposite sides make a right angle with a point). We've already shown the second part: $p = z - w$ and $q = a - b$. Also, $z$ and $w$ are equidistant from $A$, as $A$ is their midpoint. It is also readily checked that $A$ is $a$ and $b$'s midpoint. Moreover, the length of $p$ and $q$ are the same (by symmetry). So $z = A + p/2$ and $a = A + q/2$ are equidistant from $A$; all of $a, b, w, z$ are the same distance from $A$.

If they are adjacent vertices of a square, then there are exactly two different squares given these vertices. (And this is geometrically clear.)

In this case, we proceed by making a line through $z$ perpendicular to the line through $z$ and $w$, finding points of the appropriate length, and repeating the process with a line through $w$ perpendicular to the line through $z$ and $w$.

Now, consider the vector $p = z - w = (x - x', y - y')$, and the vector $q = (y - y', -(x - x'))$. It is readily checked that $p \cdot q = 0$, so that these vectors are perpendicular.

The points given by $a = z + q$ and $b = w + q$ together with $z$ and $w$ are the vertices of a square; because $p \cdot q = 0$, we have that these are the vertices of some rectangle (by similar work as above). Moreover, the lengths of each side are the same; the lengths of $p$ and $q$ as vectors are the same, and they represent the lengths of the sides of the shape.

Also, the points given by $a' = z - q$ and $b' = w - q$ together with $z$ and $w$ are the vertices of a square, similarly.

To summarize the above:

If $z$ and $w$ are the opposite corners of the square, then $(\frac{x+x'+y-y'}{2}, \frac{y+y'-(x-x')}{2})$ and $(\frac{x+x'-(y-y')}{2}, \frac{y+y'+x-x'}{2})$ are the other corners of the square.

If $z$ and $w$ are adjacent corners of the square, then either $z + q$ and $w + q$ are the other corners of the square or $z - q$ and $w - q$ are, where $q = (y - y', -(x - x'))$.

**Problem 3:**

Let $|a| = |b| = 1$, with $a = (x, y)$ and $b = (x', y')$, with $a \neq b$ and $\overline{a}b \neq 1$. First, note that in this case, $\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a-b|}{|1-\overline{a}b|}$. Now, we have:

$$\begin{aligned}(1-\overline{a}b)\overline{(1-\overline{a}b)} &= (1 - xx' - yy', xy' - x'y)(1 - xx' - yy', -xy' + x'y) \\ &= ((1 - xx' - yy')^2 + (xy' - x'y)^2, 0) \\ &= (1 - 2xx' - 2yy' + x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2, 0)\end{aligned}$$

Also,

$$\begin{aligned}(a-b)\overline{(a-b)} &= (x - x', y - y')(x - x', y' - y) \\ &= ((x - x')^2 + (y - y')^2, 0) \\ &= (2 - 2xx' - 2yy', 0)\end{aligned}$$

Now, because $|a| = |b| = 1$, we have:

$$\begin{aligned}x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2 &= (x^2 + y^2)(x'^2 + y'^2) \\ &= (1)(1) = 1\end{aligned}$$

Thus,

$$\begin{aligned}(1-\overline{a}b)\overline{(1-\overline{a}b)} &= (1 - 2xx' - 2yy' + x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2, 0) \\ &= (2 - 2xx' - 2yy', 0) \\ &= (a-b)\overline{(a-b)}\end{aligned}$$

That is, we have that $(1-\overline{a}b)\overline{(1-\overline{a}b)} = (a-b)\overline{(a-b)}$, so that $|a-b| = |1-\overline{a}b|$, producing the desired result.

**Problem 4:**

Let $f : [a, b] \to \mathbb{C}$ be continuous.

We take for granted that $[a, b]$ is connected.

Now, assume that $f([a,b])$ isn't connected. Then there are nonempty open sets, $U$ and $V$, with $U \cap V = \emptyset$ and $U \cup V \supset f([a,b])$. Now, $f^{-1}(U)$ and $f^{-1}(V)$ are both open, as $f$ is continuous. They are also disjoint; otherwise, there is a point $x$ with $f(x) \in U$ and $f(x) \in V$. Moreover, $f^{-1}(U) \cup f^{-1}(V) = [a,b]$, because $U \cup V = f([a,b])$. So $f^{-1}(U)$ and $f^{-1}(V)$ disconnect $[a,b]$, which is impossible.

So by contradiction, $f([a,b])$ must be connected.