Problem 1:

Consider $g : \{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\} \to \mathbb{C}$ given by $g(z) = e^z$. We know that g is holomorphic. Define $A = \{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\}$.

Also, g is injective: let $g(z) = g(w)$, where $z = a + bi \in A$ and $w = c + di \in A$ (and $a, b, c, d \in \mathbb{R}$). Then:

$$\begin{aligned} e^z &= e^w \\ e^{a+bi} &= e^{c+di} \\ e^a e^{bi} &= e^c e^{di} \end{aligned}$$

So $e^a = e^c$, so $a = c$. Also, $e^{bi} = e^{di}$, so because $b, d \in (0, 2\pi)$, we have that $b = d$.

So $z = w$, as desired.

So g is an injective holomorphism: it is a biholomorphism between A and $g(A)$.

Moreover, $g(\{z \in \mathbb{C} : 0 < \text{Im}(z) < 2\pi\}) = \mathbb{C} \setminus \mathbb{R}^+$: if $z \in \mathbb{C} \setminus \mathbb{R}^+$, then $\ln(|z|) + i\arg(z) \mapsto z$.

Also, if $z \in \mathbb{R}^+$, then $e^{a+bi} = z$ implies that b is a natural number times 2π, which all lie outside our domain.

Problem 2:

(Note: I read a stronger version of this proof in Complex Made Simple prior to this problem being assigned; the extra assumption allows for a lot of stripping away of details.)

Let Ω be a convex open set, $\phi \in \mathcal{O}(\Omega)$, with $\text{Re}(\phi'(z)) > 0$.

We know that ϕ is holomorphic.

Consider $\phi(a) - \phi(b)$. Because Ω is convex, we can calculate this by integrating over the line segment $[a, b]$:

$$|\phi(a) - \phi(b)| = \left| \int_{[a,b]} \phi'(z) dz \right|$$
$$\geq \left| \int_{[a,b]} \mathrm{Re}(\phi'(z)) dz \right|$$

Because $\mathrm{Re}(\phi'(z)) > 0$, the absolute value of the integral is greater than zero if $a \neq b$. So $\phi(a) - \phi(b) \neq 0$ if $a \neq b$. That is, ϕ is injective.

So ϕ is a biholomorphism.

Problem 3:

Let $S_{0,\alpha} = \{z \in \mathbb{C} : 0 < \arg(z) < \alpha\}$ for all $0 < \alpha \leq 2\pi$.

Consider $S_{0,\alpha}$ and $S_{0,\beta}$. The map $\phi : S_{0,\alpha} \to S_{0,\beta}$ given by $\phi(re^{i\theta}) = re^{\frac{\beta}{\alpha} i\theta}$ is a biholomorphism.

First, ϕ is well defined: if $re^{i\theta} \in S_{0,\alpha}$, then $\phi(z) = re^{\frac{\beta}{\alpha} i\theta}$ has $0 < \frac{\beta}{\alpha}\theta < \beta$, so that $\phi(z) \in S_{0,\beta}$. Moreover, because $0 < \alpha < 2\pi$, for each z there is a unique θ with $r > 0$ and $re^{i\theta} = z$.

Next, ϕ is a holomorphism, and this is clear using polar coordinates.

Last, ϕ is injective: let $\phi(z) = \phi(w)$, with $z = ae^{ib}$ and $w = ce^{id}$ with $a, b, c, d \in \mathbb{R}$. Then:

$$ae^{\frac{\beta}{\alpha} ib} = ce^{\frac{\beta}{\alpha} id}$$

So $a = c$, and $e^{\frac{\beta}{\alpha} ib} = e^{\frac{\beta}{\alpha} id}$. Because $\frac{\beta}{\alpha} b, \frac{\beta}{\alpha} d \in (0, 2\pi]$, this means that $b = d$. So $z = w$, as desired.

So we have a biholomorphism, $\phi : S_{0,\alpha} \to S_{0,\beta}$. Thus, $S_{0,\alpha}$ and $S_{0,\beta}$ are conformally equivalent.

Problem 4:

Consider $A = \text{Aut}(\{z : \text{Im}(z) > 0\})$ and B, the set of maps of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$, as well as $C = \{z : \text{Im}(z) > 0\}$.

Let $\phi \in B$.

First, ϕ is injective: let $\phi(z) = \phi(w)$. Then:

$$\begin{aligned}
\frac{az+b}{cz+d} &= \frac{aw+b}{cw+d} \\
(az+b)(cw+d) &= (aw+b)(cz+d) \\
acwz + adz + bcw + bd &= acwz + adw + bcz + bd \\
(ad-bc)z &= (ad-bc)w \\
z &= w
\end{aligned}$$

So where ϕ is defined, ϕ is injective.

Also, ϕ is a holomorphism: except in the cases where $c \neq 0$ and at the point $z = -d/c$, this is clear. But because $-d/c$ is real, we don't have to consider this: it lies outside of our domain.

Thus, ϕ is a biholomorphism into some set.

Now, $\phi(C) = C$:

Let $z \in C$. Then we have that $\frac{az+b}{cz+d} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{(cz+d)\overline{(cz+d)}}$. The sign of the imaginary part is determined solely by $adz + bc\overline{z}$; because this has positive imaginary part ($ad - bc > 0$ and $\text{Im}(z) > 0$), we have that $\phi(z) \in C$.

Also, if $z \in \phi(C)$, then note that ϕ has a holomorphic inverse (it is $\phi^{-1}(z) = \frac{dz-b}{a-cz}$. The same analysis as above verifies that this is holomorphic.)

Thus, ϕ is a biholomorphism from C to C. That is, $\phi \in A$.

Now, let $\phi \in A$. Then ϕ can be extended to a biholomorphism in $\overline{\mathbb{C}}$. That is, $\tilde{\phi} \in \text{Aut}(\overline{\mathbb{C}})$ for some $\tilde{\phi}$. So $\tilde{\phi}$ is linear-fractional: $\tilde{\phi}(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$.

Now, $\tilde{\phi}(C) = C$, because ϕ is a biholomorphism from C to C. So we can require that $a, b, c, d \in \mathbb{R}$: write $a = re^{i\alpha}$, $b = se^{i\beta}$, $c = te^{i\gamma}$, $d = ue^{i\delta}$. Then $\phi(z) = \frac{az+b}{cz+d} = \frac{re^{i\alpha}z + se^{i\beta}}{te^{i\gamma}z + ue^{i\delta}} = e^{i(\alpha-\gamma)} \frac{rz + se^{i(\beta-\alpha)}}{tz + ue^{i(\delta-\gamma)}}$. But we know that $\frac{rz+se^{i(\beta-\alpha)}}{tz+ue^{i(\delta-\gamma)}}$ is a biholomorphism from C to C by the above; multiplication by $e^{i\theta}$ rotates the half-plane by θ, so $\alpha - \gamma$ must have been a multiple of 2π. So we have $\phi(z) = \frac{rz+se^{i(\beta-\alpha)}}{tz+ue^{i(\delta-\gamma)}}$. Similarly, we have that $\phi(z) = \frac{rze^{i(\alpha-\beta)}+s}{tze^{i(\gamma-\delta)}+u}$. A bit of linear algebra shows that this combination implies that $\alpha - \beta$ and $\delta - \gamma$ are multiples of 2π. So $\phi(z) = \frac{rz+s}{tz+u}$.

That is, $\phi \in A$.

So $A = B$, as desired.

Problem 5:

Let $g : D_1(0) \to D_1(0)$ be holomorphic, with $g(0) = g'(0) = \ldots g^{(k)}(0) = 0$.

Then $h(z) = \frac{g(z)}{z^{k+1}}$ is holomorphic; on $D_1(0)$, $h(z) = \frac{\sum_{n=k+1}^{\infty} a_n z^n}{z^{k+1}} = \sum_{n=k+1} a_n z^{n-(k+1)}$. That is, h is represented as a power series, h is holomorphic.

Now, $|h(z)| \leq \max_{\partial D_r(0)} |h|$ for all $z \in D_r(0)$ with $r < 1$. So $|h(z)| \leq \frac{1}{r}$ for all $r < 1$. By taking limits as $r \to 1$, we get that $|h(z)| \leq 1$ for all $z \in D_1(0)$.

So $g(z) \leq |z|^{k+1}$ for $z \in D_1(0)$.

Now, if we have $g(z) = |z|^{k+1}$ for some $z \in D_1(0)$, we get $|h(z)| = 1$. That is, h achieves its maximum. So by the maximum principle, h is constant; say $h = c$. So then we have $g(z) = cz^{k+1}$, for some $c \in \mathbb{C}$ with $|c| = 1$.