Problem 1:

Consider $a, b \in \mathbb{R}$, and consider the set of functions $u \in C^1([0,1])$ such that $u(0) = a, u(1) = b$. Without loss of generality, we can take $a = 0$ and b positive. (We are focused on the derivative's absolute value, so we can add/subtract constants and multiply by -1 as desired.)

If $b = 0$, then the integral $\int\limits_0^1 |u'(x)|^2 \, dx$ is minimized only by $u(x) = 0$, and this is clear.

Consider the set of functions $\mathcal{U}_b$ that minimize the integral $\int\limits_0^1 |u'(x)|^2 \, dx$ with respect to the set of functions $u \in C^1([0,1])$ such that $u(0) = 0$ and $u(1) = b$. This set injects into the set of functions $\mathcal{U}_b - bx = \{u - bx : u \in \mathcal{U}\}$.

Now, functions in $\mathcal{U}_b - bx$ minimize the integral with respect to $u(0) = 0$ and $u(1) = 0$, and I can't figure out why. I think that a good proof of this would probably go through Normal Families somehow.

Either way, the point is that because $\mathcal{U}_b - bx = \{0\}$, the only function that minimizes the integral $\int\limits_0^1 |u'(x)|^2 \, dx$ subject to $u(0) = 0$ and $u(1) = b$ is bx. This yields the result desired.

Problem 2:

Consider the set $A = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \setminus [0, a]$ with $a \in \mathbb{R}^+$.

Define the sets $B = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq a^2\}$, and $C = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$.

First, the map $\phi : A \to B$ given by $z \mapsto z^2$ is a biholomorphism from A to B, and this is clear; the argument that $z \mapsto z^2$ gives a biholomorphism from the half-plane to the slit plane was made in class, and note that the image of $[0, a]$ under this map is $[0, a^2]$: so because the map is a biholomorphism, the half plane exluding $[0, a]$ has the image of the slit plane excluding $[0, a^2]$.

Second, the map $\psi : B \to C$ given by $z \mapsto z - a^2$ is a biholomorphism from B to C, and this is clear (this is a straight translation).

Third, the map $\xi : C \to \{\mathrm{Re}(z) > 0\}$ given by $z \mapsto \sqrt{z}$ (using the branch of $\sqrt{z}$ that is the natural inverse of z^2, of course) is a biholomorphism from C to $\{\mathrm{Re}(z) > 0\}$, and this was discussed in class.

So their composition is a biholomorphism from A to $\{\mathrm{Re}(z) > 0\}$; that is, the map $f(z) = \sqrt{z^2 - a^2}$ is a biholomorphism from the above set to $\{\mathrm{Re}(z) > 0\}$.

Problem 3:

Let Ω be open and symmetric about the $\mathbb{R}$-axis.

Let $f \in C(\Omega)$, and f be holomorphic except perhaps on the $\mathbb{R}$-axis. Note that $f = 0$ on the $\mathbb{R}$-axis.

Our goal is to show that $f \in \mathcal{O}(\Omega)$; we only need to check that f is holomorphic on the $\mathbb{R}$-axis. So, let $z \in \mathbb{R} \cap \Omega$. Then there is an open ball centered at z, call it $D_r(z)$, contained in Ω. This open ball is simply connected. Now, the real part of f, say $u = \mathrm{Re}(f)$, is harmonic on $D_r(z) \setminus \mathbb{R}$. By the reflection principle discussed in class, u is harmonic on all of $D_r(z)$.

Now, u is the real part of some holomorphic function, g, and this holomorphic function is unique up to addition of a constant. So, we can take $g(z) = 0$.

Now, $h = f - g$ is holomorphic except perhaps on the real axis, where it is 0. Moreover, the real part of h is 0; by the Cauchy-Riemann equations, the imaginary part of h must be constant (except perhaps on the real axis). Thus, because the imaginary part of h is 0 on the real axis (and h is continuous), the imaginary part of h is 0. So, $h = 0$; that is, $f = g$.

So, f is holomorphic on $D_r(z)$; in particular, f is holomorphic at z.

Because holomorphy is a local property, this yields the desired result; f is holomorphic on Ω.

Problem 4:

Let $\phi \in \mathrm{Aut}(\overline{\mathbb{C}})$ be such that $\phi(\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}) = D_1(0)$.

One biholomorphism that takes the disk $D_1(0)$ to $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is $\phi_C : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto i\frac{1+z}{1-z}$; this is the Cayley transform. Its inverse is $\psi_C : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto \frac{z-i}{z+i}$. (I pulled these maps from Complex Made Simple; any other biholomorphism would've probably worked).

So, $\psi = \phi_C \circ \phi$ is a biholomorphism of the plane that fixes $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$; by an earlier homework problem, this means that $\phi_C \circ \phi$ is of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$.

Fix $z \in \mathbb{C}$. Let $w = \overline{z}$. Then

$$
\begin{aligned}
\psi(w) &= \frac{aw+b}{cw+d} \\
&= \frac{a\overline{z}+b}{c\overline{z}+d} \\
&= \frac{\overline{az+b}}{\overline{cz+d}} \\
&= \overline{\psi(z)}
\end{aligned}
$$

Now, $\psi_C \circ \psi = \phi$. So,

$$
\begin{aligned}
\phi(w) &= \psi_C(\psi(w)) \\
&= \psi_C(\overline{\psi(z)}) \\
&= \frac{\overline{\psi(z)}-i}{\overline{\psi(z)}+i} \\
&= \frac{\overline{i\frac{1+\phi(z)}{1-\phi(z)}}-i}{\overline{i\frac{1+\phi(z)}{1-\phi(z)}}+i} \\
&= \frac{-i\overline{\frac{1+\phi(z)}{1-\phi(z)}}-i}{-i\overline{\frac{1+\phi(z)}{1-\phi(z)}}+i} \\
&= \frac{\overline{\frac{1+\phi(z)}{1-\phi(z)}}+1}{\overline{\frac{1+\phi(z)}{1-\phi(z)}}-1} \\
&= \frac{\frac{1+\overline{\phi(z)}}{1-\overline{\phi(z)}}+1}{\frac{1+\overline{\phi(z)}}{1-\overline{\phi(z)}}-1} \\
&= \frac{\frac{2}{1-\overline{\phi(z)}}}{\frac{2\overline{\phi(z)}}{1-\overline{\phi(z)}}} \\
&= \frac{1}{\overline{\phi(z)}} \\
&= \frac{\phi(z)}{|\phi(z)|^2}
\end{aligned}
$$

which is the desired result.

Problem 5:

Let $f \in \mathcal{O}(\Omega)$, where Ω is a symmetric domain (with respect to $\mathbb{R}$), and $\mathbb{R} \cap \Omega \neq \emptyset$. Moreover, let $f(\mathbb{R} \cap \Omega) \subset \mathbb{R}$.

Because f is holomorphic on a domain, it can be written as $f = u + iv$, with u and v real-valued harmonic functions.

So, v is a harmonic function with $v = 0$ on $\mathbb{R} \cap \Omega$ (because $f(\mathbb{R} \cap \Omega) \subset \mathbb{R}$). So $v(\overline{z}) = -v(z)$; if not, then v restricted to Ω^+ extends to the harmonic function on Ω that we get via reflection principle (call it v'), and then $v - v'$ vanishes on Ω^+...so $v - v'$ vanishes everywhere by minimum principle, so $v = v'$.

So, $v(\overline{z}) = -v(z)$, for all $z \in \Omega$.

Now, $u(\overline{z}) = u(z)$ by Cauchy Riemann: $u_y(\overline{z}) = -v_x(\overline{z}) = v_x(z) = -u_y(z)$. That is, u_y is an odd function of y, so u is an even function of y (which is another way of saying the first sentence.)

Thus, $f(\overline{z}) = u(\overline{z}) + iv(\overline{z}) = u(z) - iv(z) = \overline{f(z)}$, as desired.

Problem 6:

Consider $\psi(z) = z + \frac{1}{z}$. Fix $a \in [0, 1]$. Consider $U = D_1(0) \setminus ([-1, -a] \cup [a, 1])$.

Then

$$
\begin{aligned}
\psi(U) &= \psi(D_1(0)) \setminus \psi([-1, -a] \cup [a, 1]) \\
&= \mathbb{C} \setminus ([-2, 2] \cup \psi([-1, -a] \cup [a, 1])) \\
&= \mathbb{C} \setminus ([-2, 2] \cup [-a - \frac{1}{a}, -2] \cup [2, a + \frac{1}{a}]) \\
&= \mathbb{C} \setminus [-a - \frac{1}{a}, a + \frac{1}{a}]
\end{aligned}
$$

So we can dilate $\psi(U)$ to yield $\mathbb{C} \setminus [-1, 1]$, by the map α given by $z \mapsto \frac{z}{a + \frac{1}{a}}$. That is, $\alpha \circ \psi(U) = \mathbb{C} \setminus [-1, 1]$.

So by using the biholomorphism discussed in class, $\phi(z) : \mathbb{C} \setminus [-1, 1] \to D_1(0)$ given by $z \mapsto \sqrt{z^2 - 1} - z$, we have that $\phi \circ \alpha \circ \psi$ is a biholomorphism from U to $D_1(0)$.

That is, the map $\beta : U \to D_1(0)$ given by $z \mapsto \sqrt{(\frac{z+\frac{1}{z}}{a+\frac{1}{a}})^2 - 1} - \frac{z+\frac{1}{z}}{a+\frac{1}{a}}$ is a biholomorphism from U to $D_1(0)$, as desired.