

I don't know if we covered the  $\rho$  metric ( $\rho(f, g) = \sup(|f(x) - g(x)| : x \in X)$ ), where  $X$  is the domain of  $f$  and  $g$ ), but I'm using it because it's nice and I like it.

**Problem 1:**

Consider the sequence of functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f_n(x) = x + 1/n$ . It is rather clear that this sequence of functions converges uniformly (to  $x$ ).

However, the sequence of functions  $\langle f_n^2 \rangle$  fails to converge uniformly:

For each  $n \in \mathbb{N}$ ,  $f_n^2(x) = x^2 + 2x/n + 1/n^2$ . It is rather clear that  $\langle f_n^2 \rangle$  converges pointwise to  $x^2$ . So if  $\langle f_n^2 \rangle$  converges uniformly to something, it must converge uniformly to  $x^2$ . However,  $\langle f_n^2 \rangle$  does not converge uniformly to  $x^2$ :

Let  $\epsilon > 0$ , and pick  $n \in \mathbb{N}$ . Consider  $|f_n^2(x) - x^2| = |2x/n + 1/n^2|$ . Pick  $x = n\epsilon$ . Then  $|2x/n + 1/n^2| = 2\epsilon + 1/n^2 \geq \epsilon$ .

So for all  $\epsilon > 0$  and  $n \in \mathbb{N}$  there is an  $x \in \mathbb{R}$  such that  $|f_n^2(x) - x^2| \geq \epsilon$ ;  $f_n^2$  does not converge uniformly to  $x^2$ .

So  $\langle f_n \rangle$  converges uniformly on  $\mathbb{R}$ , but  $\langle f_n^2 \rangle$  doesn't. This satisfies the problem.

**Problem 2:**

Longish note: After finishing this problem, I noticed that this follows immediately from a fragment of the proof of Arzela-Ascoli. I prefer this proof, as it is smoother, but it is important to note that such a thing is possible. Moreover, I used a general metric in this one, because it seems like if we had an appropriate extension of Arzela-Ascoli, then this proof could be extended to other types of metric spaces...Does such a theorem exist?

Proof starts here: Let  $\langle f_n \rangle$  be an equicontinuous sequence of functions on a compact set,  $K$ , with  $\langle f_n \rangle$  converging pointwise to some function, say  $f$ .

By the Arzela-Ascoli theorem, we know that  $\langle f_n \rangle$  has some subsequence that uniformly converges to some function. We know that this function must be  $f$ : if a subsequence of functions converges uniformly to  $f$ , it converges pointwise to  $f$ . If a sequence of functions converges pointwise to a function,  $f$ , then all of its subsequences converge to  $f$ . So if a sequence of functions converges pointwise to  $f$ , then any subsequence of functions that converges uniformly to a function must converge uniformly to  $f$ .

Now, consider such a converging subsequence,  $\langle f_{n_j} \rangle$ .

Let  $\epsilon > 0$ . There is a  $J \in \mathbb{N}$  such that for all  $j \geq J$ ,  $\rho(f_{n_j}, f) < \epsilon/4$ .

In addition, by equicontinuity, there is a  $\delta_1 > 0$  such that for all  $n \in \mathbb{N}$ ,  $x, y \in K$ ,  $d(x, y) < \delta_1$  implies that  $d(f_n(x), f_n(y)) < \epsilon/4$ .

Moreover, we know that converging sequences of continuous functions converge to continuous functions. We also know that continuous functions on a compact domain are uniformly continuous. Thus,  $f$  is uniformly continuous; there is a  $\delta_2 > 0$  such that for all  $n \in \mathbb{N}$ ,  $x, y \in K$ ,  $d(x, y) < \delta_2$  implies that  $d(f(x), f(y)) < \epsilon/4$ .

Define  $\delta = \min(\delta_1, \delta_2)$ , so that we have both of the lines for  $\delta$ .

We know that compact sets are totally bounded. (If this is not clear, consider a career in pastry making.)

So, let  $F$  be a finite collection of points of  $K$  such that for all  $x \in K$ ,  $d(x, y) < \delta$  for some  $y \in F$ .

Now, for each  $y \in F$ , there is an  $N_y \in \mathbb{N}$  such that for all  $n \geq N_y$ ,  $d(f_n(y), f(y)) < \epsilon/4$ .

Define  $N = \max(N_y, n_J)$ .

Now, for all  $n \geq N$ , and for all  $x \in K$ , we have, for some  $y \in F$  (we pick  $y$  with  $d(x, y) < \delta$ ):

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_n(y)) + d(f_n(y), f_{n_J}(y)) + d(f_{n_J}(y), f(y)) + d(f(y), f(x)) \\ &\leq \epsilon \text{ (That's good enough.)} \end{aligned}$$

So for all  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , for all  $x \in K$ ,  $d(f_n(x), f(x)) < \epsilon$ . That is,  $f_n$  converges uniformly to  $f$ .

To summarize, if  $\langle f_n \rangle$  is an equicontinuous sequence of functions on a compact set,  $K$ , with  $\langle f_n \rangle$  converging pointwise, then  $\langle f_n \rangle$  converges uniformly.

### Problem 3:

Let  $\langle f_n \rangle$  be a uniformly bounded sequence of functions that are Riemann-integrable on  $[a, b]$ . Set

$$F_n(x) = \int_a^x f_n(t) dt$$

Moreover, let  $L$  be the absolute value of a lower bound for the  $f_n$ s and let  $U$  be the absolute value of an upper bound for the  $f_n$ s. (I say absolute values here because I don't want to bother with them later.)

Now, the set of  $F_n$ s are equicontinuous:

Let  $\epsilon > 0$ . Define  $\delta = \epsilon / \max(U, L)$ . Then for all  $n \in \mathbb{N}$ ,  $x, y \in [a, b]$  with  $|x - y| < \delta$  (WLOG,  $x \leq y$ ), we have:

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left| \int_a^x f_n(t) dt - \int_a^y f_n(t) dt \right| \\ &= \left| \int_x^y f_n(t) dt \right| \quad \text{I exploit the absolute value here too.} \\ &\leq |(x - y) \max(U, L)| \quad \text{This is some sort of obvious property of integrals we should know.} \\ &< \epsilon \end{aligned}$$

So for all  $\epsilon > 0$  there is a  $\delta > 0$  such that for all  $n \in \mathbb{N}$ ,  $x, y \in [a, b]$ ,  $|x - y| < \delta$  implies that  $|F_n(x) - F_n(y)| < \epsilon$ . That is, the set of  $F_n$ s are equicontinuous.

In addition, the  $F_n$ s are defined on  $[a, b]$ , which is a compact space. By Arzela-Ascoli, there is a subsequence  $\langle F_{n_j} \rangle$  that converges uniformly on  $[a, b]$ .

#### Problem 4:

Let  $\langle f_n \rangle$  be a sequence of increasing functions on  $\mathbb{R}$  with  $0 \leq f_n(x) \leq 1$ .

#### Problem 5:

Let  $\alpha$  be increasing on  $[a, b]$ ,  $g$  continuous, and  $g(x) = G'(x)$  for all  $x \in [a, b]$ .

Then note that both  $\int_a^b \alpha(x)g(x)dx$  and  $\int_a^b Gd\alpha$  exist; the first because  $g$  is continuous, and the second because  $G$  is differentiable, thus continuous, thus  $G \in \mathcal{R}(\alpha)$ .

Let  $\epsilon > 0$ . There is a  $\delta > 0$  such that if the mesh of a partition,  $P$  of  $[a, b]$ , is less than  $\delta$ , then for any set of tags of that partition,  $T$ ,

$$\left| \sum_{i=1}^n g(t_i) \alpha(t_i) \Delta x_i - \int_a^b \alpha(x) g(x) dx \right| < \epsilon/2$$

$$\left| \sum_{i=1}^n G(t_i) \Delta \alpha_i - \int_a^b G(x) d\alpha \right| < \epsilon/2$$

Now, pick a set of tags,  $t_i \in [x_{i-1}, x_i]$  such that  $G(x_i) - G(x_{i-1}) = g(t_i) \Delta x_i$ . We can do this, because of the mean value theorem.

**Problem 6:**

Let  $\alpha$  be an increasing function on  $[a, b]$ , and for  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left( \int_a^b |u|^2 d\alpha \right)^{1/2}.$$

Let  $f, g, h \in \mathcal{R}(\alpha)$ .

Then we have the following:

$$\begin{aligned}
\|f - h\|_2 &= \left( \int_a^b |f - h|^2 d\alpha \right)^{1/2} \\
&= \left( \int_a^b |f - g + g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left( \int_a^b (|f - g| + |g - h|)^2 d\alpha \right)^{1/2} \\
&= \left( \int_a^b |f - g|^2 + 2|f - g||g - h| + |g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left( \int_a^b |f - g|^2 + |g - h|^2 d\alpha \right)^{1/2} \\
&= \|f - g\|_2 + \|g - h\|_2
\end{aligned}$$

Thus, we have the triangle inequality for this ~~norm~~ strange function I have never seen before.