

Problem 1:

Let $g : [0, \infty) \rightarrow \mathbb{R}$ with $g(0) = 0$, and let $u(x, t)$ solve

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

Let $v(x, t) = u(x, t) - g(t)$, and extend v to $\{x < 0\}$ by odd reflection (just call the resulting extension v). Then v solves

$$\begin{cases} v_t - v_{xx} = -g' & \text{in } \mathbb{R}_+ \times (0, \infty) \\ v = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ v = 0 & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

So $v(x, t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} -g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds$ by formula 17 in the book.

Now,

$$\begin{aligned} v(x, t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} -g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds \\ &= - \left[\int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} g'(s) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds \right] \\ &= - \left[\int_{\mathbb{R}} \int_0^t g'(s) \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{-|x-y|^2}{4(t-s)}} ds dy \right] \\ &= - \left[\int_{\mathbb{R}} \frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{4\pi(t-s)} g(s) \Big|_0^t - \int_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] ds dy \right] \end{aligned}$$

Now, observe that $\frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{4\pi(t-s)} g(s) \Big|_0^t$ is 0; evaluating at 0 yields 0, and taking the limit as $s \rightarrow t$ of that also yields 0.

So,

$$\begin{aligned}
v(x, t) &= - \left[\int_{\mathbb{R}} - \int_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] ds dy \right] \\
&= - \left[\int_{\mathbb{R}} - \int_0^t g(s) e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] ds dy \right] \\
&= \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} - \frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] dy ds \\
&= \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{1}{4\sqrt{\pi}(t-s)^{3/2}} \right] dy ds - \int_0^t g(s) \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4(t-s)}} \left[\frac{(x-y)^2}{8\sqrt{\pi}(t-s)^{5/2}} \right] dy ds
\end{aligned}$$

I'm not quite sure how the rest of this plays out, but the idea is that we add $g(t)$ to $v(x, t)$, so we get that $u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s) ds$ as desired.

Problem 2:

Let $g \in C(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, $|g| < M$ for some M . Let u be the bounded solution to

$$\begin{cases} \Delta u - u_t = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Part a:

Then $u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y) dy$. Let $\epsilon > 0$. There is a sequence $h_k \in \mathbb{R}^n$ with compact support such that $|g| \geq |h_k|$ and $|g - h_k| \rightarrow 0$ uniformly. Also, let $v_k(x, t)$ be the bounded solution to

$$\begin{cases} \Delta v_k - v_{k,t} = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ v_k(x, 0) = h_k(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Now,

$$\begin{aligned}
\left| \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} (g(y) - h_k(y)) dy \right| &\leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} \left| e^{\frac{-|x-y|^2}{4t}} (g(y) - h_k(y)) \right| dy \\
&\leq \left| \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} \epsilon dy \right| \\
&= \epsilon
\end{aligned}$$

So $\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^n} |u(x, t)| = \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \lim_{k \rightarrow \infty} |v_k(x, t)| = \lim_{t \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^n} |v_k(x, t)|$.
But $\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^n} |v_k(x, t)| = 0$, because $h(x)$ has compact support (I'm fairly sure we touched on this in class). So, $\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^n} |u(x, t)| = 0$, as desired.

Part b:

Consider $v(x, t) = u(x, t) - g(x)$. Then $v(x, t)$ solves

$$\begin{cases} \Delta v - v_t = -\Delta g(x) & \text{for } t > 0, x \in \mathbb{R}^n \\ v(x, 0) = 0 & \text{for } x \in \mathbb{R}^n \end{cases}$$

Thus, $v(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) \Delta g(y) dy ds$.

So $\int_{\mathbb{R}^n} v(x, t) dx = \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) \Delta g(y) dy ds dx$

Switching the order of integration, we get $\int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x - y, t - s) \Delta g(y) dx dy ds$.

Yet, this is $\int_0^t \int_{\mathbb{R}^n} \Delta g(y) dy ds$, because $\int_{\mathbb{R}^n} \Phi(x - y, t - s) dx = 1$.

The integral $\int_{\mathbb{R}^n} \Delta g(y) dy$ is constant with respect to t and x ; call it C .

Then $\int_{\mathbb{R}^n} v(x, t) dx = ct$.

So $\int_{\mathbb{R}^n} u(x, t) dx = ct + \int_{\mathbb{R}^n} g(x) dx$. Yet, the left hand side does not tend to in-

finiteness as $t \rightarrow \infty$ ($\left| \int_{\mathbb{R}^n} u(x, t) dx \right| = \left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y) dy dx \right| \leq \left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} M dy dx \right| = M < \infty$); so c must be 0.

So, $\int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} g(x) dx$.

Problem 3:

Part a:

Fix $\alpha \in (0, 1)$, $\beta \geq 0$.Note first that $z^\beta e^{-z} = e^{\beta \ln(z) - z}$. So, the desired result is

$$e^{\beta \ln(z) - z} \leq M e^{-\alpha z}$$

for some M , which is equivalent to

$$\beta \ln(z) - z \leq \ln(M) - \alpha z$$

for some M . Now, this is equivalent to

$$-\ln(M) \leq (1 - \alpha)z - \beta \ln(z)$$

for some M . By applying basic calculus, the right hand side takes a minimum at $z = \beta/(1 - \alpha)$, so taking $M = (1 - \alpha)(\beta/(1 - \alpha)) - \beta \ln(\beta/(1 - \alpha))$ suffices so that $-\ln(M) \leq (1 - \alpha)z - \beta \ln(z)$ and, equivalently, $z^\beta e^{-z} \leq M e^{-\alpha z}$ for all $z \geq 0$, as desired.

Part b:

First, consider that $\partial_t \Phi(x, t) = e^{-\frac{|x|^2}{4t}} \left[-\frac{2\pi n}{(4\pi t)^{n/2+1}} + \frac{|x|^2}{4t^2(4\pi t)^{n/2}} \right]$. Simplifying,

$$\partial_t \Phi(x, t) = \Phi(x, t) \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right]$$

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, we get

$$\begin{aligned}
|\partial_t \Phi(x, t)| &= \left| \Phi(x, t) \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= \Phi(x, t) \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&\leq \frac{M e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2} (|x|^2 / (4t))^\beta} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M 2^{n/2} \Phi(x, 2t) (|x|^2 / (4t))^{-\beta} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M_1 \Phi(x, 2t) (|x|^2 / (4t))^{-\beta} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M_1 \Phi(x, 2t) (|x|^2 / (4t))^{-\beta} \frac{|x|^2}{4t^2} \\
&= M_1 \Phi(x, 2t) / t
\end{aligned}$$

As desired.

Next, consider that $\partial_{x_i} \Phi(x, t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left[-\frac{x_i}{2t} \right]$.

Now, using part a above with $\alpha = 1/2$, $z = |x|^2 / (4t)$, and $\beta = 1/2$, we get

$$\begin{aligned}
|\partial_{x_i} \Phi(x, t)| &= \left| \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left[-\frac{x_i}{2t} \right] \right| \\
&\leq \frac{M e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}} \left| \frac{x_i}{2t z^\beta} \right| \\
&\leq M 2^{n/2} \phi(x, 2t) \left| \frac{|x|}{2t} \frac{2\sqrt{t}}{|x|} \right| \\
&= M_2 \phi(x, 2t) / \sqrt{t}
\end{aligned}$$

As desired.

Next, let $i \neq j$. Then $\partial_{x_i x_j} \Phi(x, t) = \frac{-x_i}{2t} \partial_{x_j} \Phi(x, t) = \frac{x_i x_j}{4t^2} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$.

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, we get

$$\begin{aligned} |\partial_{x_i x_j} \Phi(x, t)| &= \left| \frac{x_i x_j}{4t^2} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \right| \\ &\leq \frac{M e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}} \left| \frac{x_i x_j}{4t^2 z^\beta} \right| \\ &\leq M 2^{n/2} \phi(x, 2t) \left| \frac{|x|^2}{4t^2} \frac{4t}{|x|^2} \right| \\ &= M_3 \phi(x, 2t)/t \end{aligned}$$

And also, if $i = j$, then $\partial_{x_i x_j} \Phi(x, t) = -\frac{x_i x_j}{4t^2} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} - \frac{1}{2t} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, $\beta' = 0$, we get

$$\begin{aligned} |\partial_{x_i x_j} \Phi(x, t)| &= \left| -\frac{x_i x_j}{4t^2} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} - \frac{1}{2t} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \right| \\ &\leq M 2^{n/2} \left| -\frac{x_i x_j}{8t^2} \frac{1}{z^\beta} - \frac{1}{2t} \frac{1}{z^{\beta'}} \right| \\ &\leq M_3 2^{n/2} |1/t| \\ &\leq M_3 2^{n/2}/t \end{aligned}$$

leaving us with all of the results we wanted for this part.

Part c:

Let $u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$, where $|g| < M$.

I completely ran out of time to do this.