Problem 1:

Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{1}{1+z^2}$. Then f is holomorphic except at i and $-i$. Thus, f is holomorphic on $\mathbb{R}$: f is locally given by a convergent power series.

(It appears that there is also an argument through Real Analysis using part of the proof of one of the many things called "Taylor's Theorem". This yields an absolute mess, don't try this.)

Yet, the power series at the origin is given by $g(x) = \sum\limits_{n=0}^{\infty} (-1)^n x^{2n}$, which diverges when $|x| \geq 1$.

Problem 2:

Let A and B be open, and $f \in \mathcal{O}(A)$ and $g \in \mathcal{O}(B)$, with $f = g$ on $A \cap B$. The function F given by $F(z) = f(z)$ if $z \in A$ and $F(z) = g(z)$ if $z \in B$ is holomorphic; it is equal to a holomorphic function in A, it is equal to a holomorphic function in B.

The fact that A and B are open is used implicitly in the last line; holomorphy is a local property, and this is because we only speak of holomorphic functions on open sets.

Problem 3:

Consider $f : \overline{D_1(-1)} \to \overline{D_1(-1)}$ given by $f(z) = z$ and $g : \overline{D_1(1)} \to \overline{D_1(1)}$ given by $g(z) = -z$. Then $\overline{D_1(-1)} \cap \overline{D_1(1)} = \{0\}$, f and g agree on the intersection, but no analytic function exists with the desired properties (the function we get out of this isn't differentiable at 0.)

Problem 4:

Let $f \in H(\mathbb{C})$ and $|f(z)| \leq e^{\mathrm{Re}(z)}$ for all z.

Then $\left|\frac{f(z)}{e^{\mathrm{Re}(z)}}\right| = \left|\frac{f(z)}{e^z}\right| \leq 1$ for all z. That is, $\frac{f(z)}{e^z}$ is a bounded entire function: it's constant. So $\frac{f(z)}{e^z} = c$ for some $c \in \mathbb{C}$: $f(z) = ce^z$ for some $c \in \mathbb{C}$.

Problem 5:

Let $f \in H(\mathbb{C})$, $n \in \mathbb{N}$, and $|f(z)| \leq (1 + |z|)^n$ for all z. Then $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for some $\langle c_m \rangle \in \mathbb{C}$. Yet, $|f(z)| \leq (1 + |z|)^n$; if $c_m \neq 0$ for $m > n$, then the estimate breaks down. (Choosing $z \in \mathbb{R}$ incredibly large should get you this result.) Thus, $c_m = 0$ for sufficiently large m; f is a polynomial.

Problem 6:

Let $f, g \in H(D_r(z))$ and $1 \leq m \leq n$ be such that f has a zero of order n at z and g has a zero of order m at z.

Consider f/g. Then

$$\begin{aligned} f/g(w) &= \frac{f(w)}{g(w)} \\ &= \frac{(w-z)^n h(w)}{(w-z)^m j(w)} \\ &= \frac{(w-z)^{n-m} h(w)}{j(w)} \end{aligned}$$

where h and j holomorphic, and nonzero at z. It is clear that this has a limit as $w \to z$ (as h/j has a limit as $w \to z$); the singularity at z is removable.

Problem 7:

Let $f \in H(\mathbb{C})$, and $f(n) = 0$ for all $n \in \mathbb{Z}$. Consider $f(z)/\sin(\pi z)$. Note that $\sin(\pi z)$'s zeroes are all of order 1; this follows readily from the power series expansion. So, by the prior exercise, $f(z)/\sin(\pi z)$ has only removable singularities.

Problem 8:

Let $f \in H(\mathbb{C})$, $f(z+1) = -f(z)$ for all z, $f(0) = 0$, and $|f(z)| \leq e^{\pi|\mathrm{Im}(z)|}$ for all z.

Note that $f(z+2) = f(z)$ for all $z \in \mathbb{C}$; so, $\left|\frac{f(z)}{\sin(\pi z)}\right| \leq \frac{e^{\pi|\mathrm{Im}(z)|}}{\sin(\pi z)}$

Problem 9:

Let V be a connected open set, and $f \in H(V)$.

Pick $a \in V$. For any $b \in D_r(a)$, where $D_r(a) \subset V$, consider that $f(a) - f(b) = \int\limits_{[a,b]} f'(z)dz = 0$. That is, f is constant on a disk around a; f is constant, by the uniqueness theorem or the fact that the derivatives all vanish at a point.

Problem 10:

Pick $V = \{z : \mathrm{Im}(z) \neq 0\}$ and $f(z) = -1$ when $\mathrm{Im}(z) > 0$ and $f(z) = 1$ when $\mathrm{Im}(z) < 0$.

We see that f isn't constant, but f is holomorphic and $f'(z) = 0$ for all $z \in V$.

(The point is that V isn't connected.)

Problem 11:

Problem 12:

Problem 13:

Problem 14:

Problem 15:

Problem 16:

Problem 17:

Problem 18:

Problem 19:

Problem 20: