Problem 1:

Part a:

Consider the set $A = \{z \in \mathbb{C} : e^z = 0\}$.

If $z = a + bi \in A$ (with $a, b \in \mathbb{R}$), then $e^z = 0$. So $e^a e^{bi} = 0$.

For $a \in \mathbb{R}$, $e^a \neq 0$. So this means that $e^{bi} = 0$. But this never happens either.

So we have a contradiction. So $A = \emptyset$.

Part b:

Consider the set $B = \{z \in \mathbb{C} : e^z = 1\}$.

If $z = a + bi \in B$ (with $a, b \in \mathbb{R}$), then $e^z = 1$. So $e^z = e^a e^{bi} = 1$.

This means that $\left|e^a e^{bi}\right| = |e^a| \left|e^{bi}\right| = 1$. But $\left|e^{bi}\right| = 1$ for all $b \in \mathbb{R}$. So, $|e^a| = 1$, so $e^a = 1$, so $a = 0$.

So $z = bi$ for some $b \in \mathbb{R}$.

Part c:

Consider the set $C = \{z \in \mathbb{C} : \sin(z) = 0\}$.

Problem 2:

Let $\Omega \subset \mathbb{C}$ be an open connected set, and $f \in C(\Omega)$ be such that for all closed, piecewise continuous curves, Γ, with $\Gamma \subset \Omega$, $\int\limits_{\Gamma} f(z)dz = 0$.

Pick $z \in \Omega$. Let $p \in \Omega$, and γ be a curve from p to z. We showed in class that $\int\limits_{\gamma} f(\xi)d\xi$ is independent of γ; that is, $\int\limits_{\gamma} f(\xi)d\xi$ only depends on p and z.

Define $g(z) = \int\limits_{\gamma} f$.

Problem 3:

Let $f \in \mathcal{O}(D_1(0))$, with $f = \sum\limits_{n=0}^{\infty} a_n z^n$.

Then consider $\int\limits_0^{2\pi} |f(re^{it})|^2 \, dt$.

Problem 4:

Let $\phi, \psi : [a, b] \to \mathbb{R}$ be log-convex.

Problem 5:

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $f \in \mathcal{O}(\Omega)$, $f(z) \neq 0$ for any $z \in \Omega$.

We showed in class that $g(z) = \int\limits_p^z \frac{f'(w)}{f(w)} dw + \lambda$ with p chosen arbitrarily in Ω and $e^\lambda = f(p)$ satisfies $f = e^g$, and that $g \in \mathcal{O}(\Omega)$.

Now, let $h \in \mathcal{O}(\Omega)$ be such that $f = e^h$.

Then $\frac{f}{f} = \frac{e^g}{e^h}$, so that $1 = e^{g-h}$. Thus, by problem 1, we have that $g - h = 2k\pi i$ for some $k \in \mathbb{Z}$.

That is, any two functions, g and h, satisfying $e^g = e^h = f$ differ only by $2k\pi i$ for some $k \in \mathbb{Z}$.

Problem 6:

Let $\phi \in \mathcal{O}(D_1(0))$. Suppose that ϕ takes its maximum at 0.

Problem 7:

Suppose that $\phi \in \mathcal{O}(\Omega)$ with Ω a domain, and that there is a $c \in \Omega$ such that $|(|\,\phi(c)) = max(|(|\,\phi))$.

Then ϕ is constant on any ball centered at c, by problem 6.

Now, Ω is path connected (it is a domain).

Let $z \in \Omega$, and let γ be a path from z to c.