For reference: in the below, $G(z) = z \prod_{n=1}^{\infty} (1 + \frac{z}{n})e^{-z/n}$. I don't know if this is standard, so it's worth including.

Problem 1:

Consider $\sum_{n=1}^{\infty} \frac{1}{n^s}$, where $\mathrm{Re}(s) > 1$.

The sum converges if and only if the integral $\int_1^{\infty} \frac{1}{n^s} dn$ does. We know that $\int_1^{\infty} \frac{1}{n^s} ds \leq \int_1^{\infty} \left|\frac{1}{n^s}\right| ds = \int_1^{\infty} \frac{1}{|n^s|} ds \leq \int_1^{\infty} \frac{1}{n^{\mathrm{Re}(s)}} ds$, and the last integral converges, so the first one must have as well.

Moreover, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is holomorphic on $\mathrm{Re}(s) > 1$: it's a limit of holomorphic functions.

Problem 2:

Consider $\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$.

The taylor coefficients attached to z^2 and z^4 of $\frac{\sin(\pi z)}{\pi z}$ are $-\pi/6$ and $\pi^2/120$, respectively, because $\frac{\sin(\pi z)}{\pi z} = \frac{\sum_{n=1}^{\infty} \frac{(-1)^n (\pi z)^{2n+1}}{(2n+1)!}}{\pi z} = \sum_{n=1}^{\infty} \frac{(-1)^n (\pi z)^{2n}}{(2n+1)!}$.

The taylor coefficients attached to z^2 and z^4 of $\prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$ are $-\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{i \neq j} \frac{1}{i^2 j^2}$, respectively; these follow by multiplying the product out.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi/6$.

We can rewrite the second one as $\sum_{i,j} \frac{1}{i^2 j^2} - \sum_{n=1}^{\infty} \frac{1}{n^4}$. We evaluate:

$$\begin{aligned}
\sum_{i,j} \frac{1}{i^2 j^2} &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^2} \frac{1}{j^2} \\
&= \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \\
&= \sum_{i=1}^{\infty} \frac{1}{i^2} \frac{\pi}{6} \\
&= \pi^2/36
\end{aligned}$$

Therefore, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \pi/6$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{36} - \frac{\pi^2}{120} = \frac{7\pi^2}{120}$.

Problem 3:
First: note that $\Gamma(n+1) = n\Gamma(n)$, and $\Gamma(1) = 1$; as discussed in class,

$$\begin{aligned}
\Gamma(1) &= \frac{e^{-\gamma}}{G(1)} \\
&= \frac{e^{-\gamma}}{\prod\limits_{n=1}^{\infty} (1+1/n)e^{-1/n}} \\
&= \frac{e^{-\gamma}}{e^{-(\sum\limits_{n=1}^{\infty}(1/n - \ln(1+1/n)))}} \\
&= \frac{e^{-\gamma}}{e^{-(\sum\limits_{n=1}^{\infty}(1/n - \ln(1+1/n)))}} \\
&= \frac{e^{-\gamma}}{e^{-\gamma}} \\
&= 1
\end{aligned}$$

So $\Gamma(n) = (n-1)!$, by a relatively clear induction argument, recreated below so the problem doesn't look too short:

First, $\Gamma(1) = 1!$.

Next, if $\Gamma(n) = (n-1)!$, then $\Gamma(n+1) = n\Gamma(n) = n!$. So by induction, we have our result.

Problem 4:

Consider $\Gamma(z)\Gamma(z-1)$.

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \frac{e^{-\gamma z}e^{-\gamma+\gamma z}}{G(z)G(1-z)} \\
&= \frac{e^{-\gamma}}{G(z)G(1-z)} \\
&= stuff \\
&= \frac{\pi}{\sin(\pi z)}
\end{aligned}$$

Thus, $\Gamma(1/2)\Gamma(1-1/2) = \frac{\pi}{\sin(\pi/2)} = \pi$.

That is, $\Gamma(1/2) = \sqrt{\pi}$. (Note that $\Gamma(z) > 0$ if $z > 0$, so $\Gamma(1/2) \neq -\sqrt{\pi}$).

Problem 5:

Problem 6:

Problem 7:

Define $S_{a,b} = \{z \in \mathbb{C} : a < \arg(z) < b\}$.

The function $f : S_{\alpha,\beta} \to S_{2\alpha,2\beta}$ given by $f(z) = z^2$ is biholomorphic when $\beta - \alpha < \pi$.

First, f is holomorphic, and this is clear.

Next, f is injective: let $a = re^{i\theta}, b = r'e^{i\theta'} \in S_{\alpha,\beta}$ with $f(a) = f(b)$. Then $r^2e^{i2\theta} = r'^2e^{i2\theta'}$. So $e^{i2\theta} = e^{i2\theta'}$, and $r^2 = r'^2$. But because $\beta - \alpha < \pi$, this means that $|\theta - \theta'| < 2\pi$. So This means that $\theta = \theta'$, so we end up with $a = re^{i\theta} = r'e^{i\theta'} = b$, as desired.

So, from the result in class, f is biholomorphic.

Note that if $\beta - \alpha \geq \pi$, there are $z, z' \in S_{\alpha,\beta}$ with $z = re^{i\theta}$ and $z' = re^{i(\theta+\pi)}$, and thus $z^2 = r^2e^{2i\theta} = r^2e^{2i\theta+2\pi} = r^2e^{2i(\theta+\pi)} = z'^2$, so the above condition is the strongest we can get.