Problem 1:

Let G be a group, with $H < G$ and $K < G$.

Let $HK < G$.

Then for all $x, y \in HK$, $xy^{-1} \in HK$.

Let $h \in H$ and $k \in K$. Then $e, h, k, h^{-1}, k^{-1} \in HK$. This means that $h^{-1}k^{-1} \in HK$. So $ekh = kh \in HK$. That is, if $h \in H$ and $k \in K$, $kh \in HK$. That is, any element of KH is contained in HK. Similarly, any element of HK is contained in KH

So $HK = KH$ if HK is a subgroup.

Now, let $HK = KH$.

Then let $x, y \in HK$. There are $h_1, h_2 \in H$, $k_1, k_2 \in K$ such that $h_1k_1 = x$ and $h_2k_2 = y$.

Now, $HK = KH$. So $xy^{-1} = h_1k_1k_2^{-1}h_2^{-1} \in HKKH = HKH = HHK = HK$

So if $x, y \in HK$, then $xy^{-1} \in HK$. This means that HK satisfies the subgroup criterion; HK is a subgroup.

Thus, $HK < G$ if and only if $KH = HK$.

Problem 2:

Let G be a group and $H \trianglelefteq G$ and $K \trianglelefteq G$, such that $H \cup K = \{e\}$.

Part a:

Let $h \in H$, $k \in K$.

Because K is normal, $hkh^{-1} \in K$.

Thus, $hkh^{-1}k^{-1} \in K$.

But because H is normal, $kh^{-1}k^{-1} \in H$.

So $hkh^{-1}k^{-1} \in H$.

So $hkh^{-1}k^{-1} \in H \cup K$, which means that $hkh^{-1}k^{-1} = e$.

So $hkh^{-1} = k$, which means that $hk = kh$.

So $hk = kh$ for all $h \in H$, $k \in K$.

Part b:

From the above, it is clear that $HK = KH$. From this fact and problem 1, it follows that HK is a subgroup of G.

Now, let $\phi : H \times K \to HK$ be given by $\phi((h, k)) = hk$.

We show that ϕ is an isomorphism:

First, ϕ is a homomorphism:

Let $(a,b),(c,d) \in H \times K$.

Then

$$\begin{aligned}\phi((a,b)(c,d)) &= \phi((ac,bd))\\ &= acbd\\ &= abcd\\ &= \phi((a,b))\phi((c,d))\end{aligned}$$

(The third line follows from part a)

To summarize the above, $\phi((a,b)(c,d)) = \phi((a,b))\phi((c,d))$ for all $(a,b),(c,d) \in H \times K$. That is, ϕ is a homomorphism.

Next, ϕ is one-to-one:

Let $(a,b) \in \ker(\phi)$. Then $ab = e$. In other words, $a = b^{-1}$. This implies that $a \in K$, which would mean that $a = e$. This means that $b = e$.

So $\ker(\phi) = \{e\}$. This means that ϕ is one-to-one. (If we don't know this implication yet...If ϕ is not one-to-one, then there are two distinct elements (x,y) that map to the same thing (z) so you can show that there is an element (xy^{-1}) that maps to e...So the kernel would have more than one thing in it. Take the contrapositive, and you get this result.)

Last ϕ is onto:

Let $x \in HK$. Then $x = hk$ for some $h \in H$, $k \in K$. So $x = \phi((h,k))$ for some $h \in H$, $k \in K$.

Thus, there is an isomorphism from $H \times K$ to HK. That is, $H \times K \cong HK$.

Problem 3:

First, Q_8 is non-Abelian:

$$\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}i&0\\0&-i\end{smallmatrix}\right]$$
$$\left[\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}-i&0\\0&i\end{smallmatrix}\right]$$

However, all of Q_8's subgroups are normal.

Because $8 = 2*2*2$, any subgroup of Q_8 has order 1,2,4, or 8.

Any subgroup of order 1,4, or 8 is trivially normal, from the discussion in class. It only remains to show that the subgroups of order 2 are normal.

Now, there is only one subgroup of order 2 in Q_8; it is $\{I, -I\}$. This is clear because there is only one element of order 2 in Q_8, and any subgroup of order 2 has to have exactly one element of order 2 (which is trivial from Cayley's theorem... elements of a group must have order dividing the group, and there can only be one element of order 1 (e). So there has to be an element of an order other than 1...there must be an element of order 2. But e has to be in the subgroup, so there's an element of order 1. And because there's only two elements, one of them is e and the other is the element of order 2).

Now, $\{I, -I\}$ is normal:

Let $A \in Q_8$.

Recall that I and $-I$ commute with every matrix.

$AIA^{-1} = IAA^{-1} = II = I$.

$A(-I)A^{-1} = (-I)AA^{-1} = (-I)I = -I$.

So for all $A \in Q_8$ and $B \in \{I, -I\}$, $ABA^{-1} \in \{I, -I\}$. That is, $\{I, -I\}$ is normal.

So all of Q_8's subgroups of order 1, 2, 4, and 8 are normal. That is, all of Q_8's subgroups are normal even though Q_8 isn't abelian.

Problem 4:

Consider $\langle s \rangle < \langle s, r^2 \rangle < D_4$.

Now, $\langle s, r^2 \rangle = \{e, s, r^2, sr^2\}$ has order 4; it is normal in D_4.

Also, $\langle s \rangle$ has order 2; it is normal in $\langle blah \rangle$.

However, $\langle s \rangle$ is not normal in D_4: $rsr^{-1} = sr^3r^{-1} = sr^2$, and $sr^2 \notin \langle s \rangle$.

So $\langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_4$, but $\langle s \rangle$ isn't a normal subgroup of D_4.

Problem 5:

Part a:

Part b (i):

Part b (ii):

Problem 6: