Note: The theme for a good chunk of these problems is "Analyze the proof, change a component of it, re-execute the proof." In this spirit, a lot of this is ripped from Royden.

Problem 1:

(If $1 \leq p < \infty$, find a representation for the bounded linear functionals on ℓ^p, where ℓ^p consists of sequences $\langle x_n \rangle$ of real numbers such that

$$(\sum |x_n|^p)^{1/p} < \infty)$$

Problem 2:

(Let $f \in L^p$, and let $T_\Delta(f)$ denote the Δ-approximant of f. Prove that

$$\|T_\Delta(f)\|_p \leq \|f\|_p)$$

Let $\Delta = \{x_0, x_1 \ldots x_m\}$ be a partition of $[0, 1]$. Then

$$T_\Delta(f) = \sum_{n=1}^{m} \frac{\chi_{[x_{n-1}, x_n]}}{x_n - x_{n-1}} \int\limits_{x_{n-1}}^{x_n} f(t)dt$$

So we have

$$\begin{aligned}
\|T_\Delta(f)\|_p^p &= \int \left| \sum_{n=1}^{m} \frac{\chi_{[x_{n-1},x_n]}}{x_n - x_{n-1}} \int\limits_{x_{n-1}}^{x_n} f(t)dt \right|^p dx \\
&\leq \int \sum_{n=1}^{m} \left| \frac{\chi_{[x_{n-1},x_n]}}{x_n - x_{n-1}} \int\limits_{x_{n-1}}^{x_n} f(t)dt \right|^p dx \\
&= \sum_{n=1}^{m} \int \left| \frac{\chi_{[x_{n-1},x_n]}}{x_n - x_{n-1}} \int\limits_{x_{n-1}}^{x_n} f(t)dt \right|^p dx \\
&= \sum_{n=1}^{m} \int\limits_{x_{n-1}}^{x_n} \left| \frac{1}{x_n - x_{n-1}} \int\limits_{x_{n-1}}^{x_n} f(t)dt \right|^p dx \\
&= \sum_{n=1}^{m} (x_n - x_{n-1}) \left| \int\limits_{x_{n-1}}^{x_n} \frac{f(t)}{(x_n - x_{n-1})} dt \right|^p \\
&\leq \sum_{n=1}^{m} (x_n - x_{n-1}) \frac{\int\limits_{x_{n-1}}^{x_n} |f(t)|^p \, dt}{(x_n - x_{n-1})} \quad \text{*See note} \\
&= \sum_{n=1}^{m} \int\limits_{x_{n-1}}^{x_n} |f(t)|^p \\
&= \int\limits_0^1 |f(t)|^p \, dt \\
&= \|f\|_p^p
\end{aligned}$$

So we have that $\|T_\Delta(f)\|_p \leq \|f\|_p$.

*Note: I'm convinced that I have written a true fact but can't seem to prove that this step works.

Problem 3:

(Prove that ℓ^p, $1 \leq p < \infty$, and L^∞ are complete.)

(Note: I am completely convinced that there's a way to get this next part straight from Riesz-Fischer (without building up the machinery again), but

I'm not quite good enough to see it. If there is a way, could you write it in the margins?)

First, ℓ^p is complete if $p \in [1, \infty)$; we show this by mimicking the proof of the Riesz-Fischer Theorem, as I am an uncreative n00b. To display the sheer laziness of this approach, I will be using the notation $a(n)$ to denote the nth term of a sequence in ℓ^p for this problem only.

Before doing this, we must prove the Minkowski Inequality for ℓ^p with $p \in [1, \infty)$:

Let f, g be two non-negative sequences in ℓ^p with $p \in [1, \infty)$.

Note that this is trivial if $\|f\|_p$ or $\|g\|_p$ is zero, and we have equality. So, let us assert that this is not the case. Define $f_0 = f/\|f\|_p$ and $g_0 = g/\|g\|_p$. Also, define $\alpha = \|f\|_p$, $\beta = \|g\|_p$, and $\lambda = \alpha/(\alpha + \beta)$.

Note that $\|f_0\|_p = \|g_0\|_p = 1$.

Then we have:

$$
\begin{aligned}
|f+g|^p &= [\alpha f_0 + \beta g_0]^p \\
&= (\alpha+\beta)^p[\lambda f_0 + (1-\lambda)g_0]^p \\
&\leq (\alpha+\beta)^p[\lambda f_0^p + (1-\lambda)g_0^p]
\end{aligned}
$$

With the last step by the concavity of x^p for $p \in (0, 1)$.

Summing up over both sides, we have:

$$
\begin{aligned}
\|f+g\|_p^p &\geq (\alpha+\beta)^p[\lambda\|f_0\|_p^p + (1-\lambda)\|g_0\|_p^p] \\
&= (\alpha+\beta)^p \\
&= (\|f\|_p + \|g\|_p)^p
\end{aligned}
$$

Raising both sides to the $1/p$th power yields the desired inequality.

Now, we can proceed by ripping the proof out of the Royden almost verbatim.

We need only show that each absolutely summable series in ℓ^p is summable in ℓ^p to some element of ℓ^p.

Let $\langle f_n \rangle$ be a sequence in ℓ^p with $\sum_{n=1}^{\infty} \|f_n\| = M < \infty$. Define sequences g_n by setting $g_n(m) = \sum_{k=1}^{n} |f_n(m)|$. From the Minkowski inequality, we have

$$
\|g_n\| \leq \sum_{k=1}^{n} \|f_k\| \leq M
$$

So

$$\sum_{x=0}^{\infty} g_n(x)^p \leq M^p$$

For each $x \in \mathbb{N}$, $\langle g_n(x) \rangle$ is an increasing sequence of (extended) real numbers, and so must converge to an extended real number $g(x)$. By taking limits, we have

$$\sum_{x=0}^{\infty} g(x)^p \leq M^p$$

So $g(x)$ is finite for all $x \in \mathbb{N}$.

Now, for each x, $\sum_{k=1}^{\infty} f_k(x)$ is absolutely summable; this is because $g(x)$ was finite. So $\sum_{k=1}^{\infty} f_k(x)$ is summable; say it sums to $s(x)$. Say that s is the limit of the partial sums $s_n = \sum_{k=1}^{n} f_k$. Now since $|s_n(x)| \leq g(x)$, we have $|s(x)| \leq g(x)$. So $s \in \ell^p$.

Now, $|s_n - s| \to 0$. So $|s_n - s|^p \to 0$; this is because $p \geq 1$. Moreover, we know that $|s_n - s|^p \leq 2^p g(x)^p$.

Recall that $g(x)^p$ had a finite sum. This means that $|s_n - s|^p$ has a finite sum, for all $n \in \mathbb{N}$.

So we have that $\sum_{x=1}^{\infty} |s_n - s|^p \to 0$;

That is, we have that the series $\langle f_n \rangle$ converges to some sum, s. That is, every absolutely summable series is summable, so ℓ^p is complete.

Next, L^∞ is complete;

Let $\langle f_n \rangle$ be a Cauchy sequence in L^∞. So at almost every $t \in [0, 1]$, we have $f_n(t)$ Cauchy in $\mathbb{R}$; because $\|f_n\|_\infty$ is Cauchy, the essential supremum of the f_ns as a sequence is Cauchy, so the supremum of the f_ns is Cauchy except on a countable union of sets of measure zero (that is, almost everywhere), so $f_n(t)$ is Cauchy at almost every $t \in [0, 1]$. So f_n converges pointwise almost everywhere to some f. In fact, it is readily checked (...it's an elementary epsilon-delta proof that you don't want to see) that f_n converges uniformly to this f except on a set of measure zero.

Now, this f is in L^∞: There's an f_n within 1 of f almost everywhere. The essential supremum of f is at most 1 away from the essential supremum of f_n. So the essential supremum of f is finite. So $f \in L^\infty$.

Next, $\|f_n - f\|_\infty \to 0$: We know that $|f_n - f| \to 0$ except on a set of measure zero. Take essential supremums of both sides, then you win. (Note: we really do need uniform convergence here.)

So $\langle f_n \rangle$ converges in the mean to some $f \in L^\infty$ if $\langle f_n \rangle$ is Cauchy; we have our result.

Problem 4:

(Let ℓ^∞ denote the set of all bounded sequences of real numbers. Set $\|(x_n)\|_\infty = \sup |x_n|$. Prove that this is a norm, and ℓ^∞ is a Banach Space.)

First, this is a norm:

Let $\langle x_n \rangle$, $\langle y_n \rangle$ be bounded sequences of real numbers, and $\alpha \in \mathbb{R}$.

First, $\|x_n\|_\infty = 0$ if and only if x_n is identically zero: $\sup(|x_n|) = 0$ if each x_n is zero. Further, if any x_n is nonzero, then $\sup(|x_n|)$ is nonzero.

Next, $\|\alpha x_n\|_\infty = \sup(|\alpha x_n|) = \sup(|\alpha|\,|x_n|) = |\alpha| \sup(|x_n|) = |\alpha|\,\|x_n\|_\infty$. (All of this follows from basic properties of the sup.)

Last:

$$\begin{aligned}
\|x_n + y_n\| &= \sup(|x_n + y_n|) \\
&\leq \sup(|x_n| + |y_n|) \\
&\leq \sup(|x_n|) + \sup(|y_n|) \\
&= \|x_n\| + \|y_n\|
\end{aligned}$$

So this norm is a norm.

Next, ℓ^∞ is complete. We show this by mimicking the proof for L^∞, as I am again an uncreative n00b:

Let $\langle f_n \rangle$ be a Cauchy sequence in ℓ^∞. So at every $t \in [0, 1]$, we have $f_n(t)$ Cauchy in $\mathbb{R}$; because $\|f_n\|_\infty$ is Cauchy, the supremum of the f_ns as a sequence is Cauchy, so the supremum of the f_ns is Cauchy, so $f_n(t)$s must be Cauchy at each $t \in [0, 1]$. So f_n converges pointwise everywhere to some f. In fact, it is readily checked that f_n converges uniformly to this f except on a set of measure zero.

Now, this f is in ℓ^∞: There's an f_n within 1 of f almost everywhere. The supremum of f is at most 1 away from the supremum of f_n. So the supremum of f is finite. So $f \in \ell^\infty$.

Next, $\|f_n - f\|_\infty \to 0$: We know that $|f_n - f| \to 0$. Take essential supremums of both sides, then you win. (Note: we really do need uniform convergence here.)

So $\langle f_n \rangle$ converges in the mean to some $f \in \ell^\infty$ if $\langle f_n \rangle$ is Cauchy; we have our result.

So, ℓ^∞ is a complete normed vector space. It's a Banach space.

Problem 5:

(Prove the Minkowski inequality for $0 < p < 1$.)

We proceed by mimicking the proof of the Minkowski inequality for $p \geq 1$.

Let f, g be two non-negative functions in L^p with $p \in (0, 1)$.

Note that this is trivial if $\|f\|_p$ or $\|g\|_p$ is zero, and we have equality. So, let us assert that this is not the case. Define $f_0 = f/\|f\|_p$ and $g_0 = g/\|g\|_p$. Also, define $\alpha = \|f\|_p$, $\beta = \|g\|_p$, and $\lambda = \alpha/(\alpha + \beta)$.

Note that $\|f_0\|_p = \|g_0\|_p = 1$.

Then we have:

$$
\begin{aligned}
|f + g|^p &= [\alpha f_0 + \beta g_0]^p \\
&= (\alpha + \beta)^p [\lambda f_0 + (1 - \lambda) g_0]^p \\
&\geq (\alpha + \beta)^p [\lambda f_0^p + (1 - \lambda) g_0^p]
\end{aligned}
$$

With the last step by the concavity of x^p for $p \in (0, 1)$.

Integrating both sides, we have:

$$
\begin{aligned}
\|f + g\|_p^p &\geq (\alpha + \beta)^p [\lambda \|f_0\|_p^p + (1 - \lambda) \|g_0\|_p^p] \\
&= (\alpha + \beta)^p \\
&= (\|f\|_p + \|g\|_p)^p
\end{aligned}
$$

Raising both sides to the $1/p$th power yields the result.

Problem 6:

(Young's inequality states that if $a, b \geq 0$, $1 < p < \infty$, and $1/p + 1/q = 1$, then

$$ab \leq a^p/p + b^q/q$$

Prove the Hölder inequality using this.)

First, we should perhaps establish Young's inequality; it was not discussed in class.

I acknowledge that the following is an unintuitive mess; I hope that there is a better means of doing this.

First, consider the right hand side of Young's inequality. Given all of the hypotheses, we have:

$$\begin{aligned} a^p/p + b^q/q &= a^p/p + \frac{b^{p/(p-1)}}{p/(p-1)} \\ &= a^p/p + (p-1)b^{p/(p-1)}/p \\ &= \frac{a^p + (p-1)b^{p/(p-1)}}{p} \end{aligned}$$

Thus, we have that

$$a^p/p + b^q/q - ab = \frac{a^p - pab + (p-1)b^{p/(p-1)}}{p}$$

We have Young's Inequality if the left hand side is greater than or equal to 0. So we have Young's Inequality if the numerator is greater or equal to 0, as we have that $p > 1$, so that the denominator is positive.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^p - pxb + (p-1)b^{p/(p-1)}$$

We know that $f(0) = (p-1)b^{p/(p-1)}$, so that $f(0)$ is positive. Also, it's clear that $\lim_{x\to\infty} f(x) = \infty$. It's also clearly a differentiable function, with

$$f'(x) = px^{p-1} - pb$$

So f has a critical point at $b^{1/(p-1)}$. It is clear that $f'(0)$ is negative and that $f'(b)$ is positive, so f takes its minimum at $b^{1/(p-1)}$.

However, $f(b^{1/(p-1)}) = 0$. So $f \geq 0$.

So $f(a) \geq 0$. So $a^p - pab + (p-1)b^{p/(p-1)} \geq 0$ which yields Young's Inequality, as stated above.

Moving on, we use this to prove the Hölder inequality. Assume that $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, and let $f \in L^p$ and $g \in L^q$.

Without loss of generality, we take f and g positive everywhere, as this has no effect on the norms.

Define $\alpha = \|f\|_p$, $\beta = \|g\|_q$, $f_0 = f/\alpha$, $g_0 = g/\beta$. It's clear that $\|f_0\|_p = \|g_0\|_q = 1$.

Now:

$$\begin{aligned}
\frac{1}{\alpha\beta}\int fg &= \int f_0 g_0 \\
&\leq \int \frac{f_0^p}{p} + \frac{g_0^q}{q} \\
&= \|f_0\|_p^p/p + \|g_0\|_q^q/q \\
&= 1/p + 1/q \\
&= 1
\end{aligned}$$

So $\int fg \leq \alpha\beta$, that is, $\int fg \leq \|f\|_p \|g\|_q$, which is the desired result.

Problem 7:

(Pick up my dry cleaning.)

I cannot pick up your dry cleaning; I don't have a car, as I am too poor to afford one.