Note to professor: I routinely use the notation $X_{d,r}(p)$ for the r-ball around the point p in the metric space (X, d). If the metric is understood, I will suppress the distance function to shorten this to $X_r(p)$.

I know that the suggested notation for the ball was discussed once, but I prefer this notation.

Problem 1:

Let E' be the set of limit points of a set E.

First, E' is closed.

Let $\langle x_n \rangle$ be a converging sequence of points in E'; say $x_n \to x$.

Because each $x_n \in E'$, we know that for all x_n, there is a sequence of points of E converging to x_n; call this sequence $\langle x_{n,j} \rangle$.

Now, let $m \in \mathbb{N}$. There is an N such that $|x_N - x| < 1/2m$. Also, there is a J such that $|x_{N,J} - x_N| < 1/2m$.

So we define $x_m = x_{N,J}$ for each $m \in \mathbb{N}$. It is clear (use the triangle inequality on the above line) that $|x_m - x| < 1/m$ for each $m \in \mathbb{N}$. It is also clear that $x_m \in E$ for all $m \in \mathbb{N}$. It is also clear that $x_m \to x$.

So we have a sequence of points in E, $\langle x_m \rangle$, converging to x. This means that $x \in E'$.

So if $\langle x_n \rangle$ is a converging sequence of points in E', then it converges to a point in E'. That is, E' is closed.

Next, E and $\overline{E}$ have the same limit points.

First, because $E \subset \overline{E}$, $E' \subset \overline{E}'$. (If this is not clear...pick a limit point of E, it has a sequence of distinct points in E converging to it, that sequence of distinct points is also in $\overline{E}$.)

Next, recall that $\overline{E} = E \cup E'$. So $\overline{E}' = (E \cup E')'$. Now; if there is a sequence of points in $(E \cup E')$ converging to a point, x, then either there are infinitely many points in that sequence in E or in E'. In the first case, we have $x \in E'$. In the second case, we have $x \in E''$. However, E' is closed, as above. So $E'' \subset E'$, so $x \in E'$ in either case. That is, if $x \in \overline{E}'$, then $x \in E'$.

Thus, $E' = \overline{E}'$; in other words, E and $\overline{E}$ have the same limit points.

Now, E and E' do not always have the same limit points: consider $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$, but $E'' = \emptyset$.

Problem 2:

The following are metrics on $\mathbb{R}$:

The function $d_2(x, y) = \sqrt{|x - y|}$ is a metric:

First, $d_2(x, y)$ is well-defined: $|x - y| \geq 0$ for all $x, y \in \mathbb{R}$ and so

$\sqrt{|x-y|}$ is defined for all $x, y \in \mathbb{R}$.

Next, d_2 is positive-definite; the square root function is always positive.

Next, $d_2(x, y) = 0$ if and only if $x = y$:

Let $x = y$. Then $d_2(x, y) = \sqrt{|x-y|} = \sqrt{|0|} = \sqrt{0} = 0$.

Next, let $d_2(x, y) = 0$. Then $\sqrt{|x-y|} = 0$. Then $|x - y| = 0$. So $x = y$.

Also, d_2 is symmetric.

Let $x, y \in \mathbb{R}$. Then $d_2(x, y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y, x)$.

Last, d_2 satisfies the triangle inequality:

Let $x, y, z \in \mathbb{R}$.

Then we have:

$$
\begin{aligned}
(d_2(x,y) + d_2(y,z))^2 &= (\sqrt{|x-y|} + \sqrt{|y-z|})^2 \\
&= |x-y| + 2\sqrt{|x-y||y-z|} + |y-z| \\
&\geq |x-y| + |y-z| \\
&\geq |x-z| \\
&= d_2(x,z)^2
\end{aligned}
$$

By taking square roots of both sides of that, we have our result.

The function $d_5(x, y) = |x - y|/(1 + |x - y|)$ is a metric:

First, $d_5(x, y)$ is well-defined: $|x - y| + 1 > 0$ for all $x, y \in \mathbb{R}$, so the above ratio is defined for all $x, y \in \mathbb{R}$.

Next, d_5 is positive-definite; $|x - y| \geq 0$ and $|x - y| + 1 > 0$ for all $x, y \in \mathbb{R}$. So the above ratio is non-negative for all $x, y \in \mathbb{R}$.

Next, $d_5(x, y) = 0$ if and only if $x = y$: $|x - y|/(1 + |x - y|) = 0$ if and only if $|x - y| = 0$, and $|x - y| = 0$ if and only if $x = y$. So $d_5(x, y) = 0$ if and only if $x = y$.

Also, d_5 is symmetric.

Let $x, y \in \mathbb{R}$. Then $d_5(x, y) = |x - y|/(1 + |x - y|) = |y - x|/(1 + |y - x|) = d_5(y, x)$

Last, d_5 satisfies the triangle inequality:

Let $x, y, z \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
d_5(x,y)+d_5(y,z) &= |x-y|/(1+|x-y|)+|y-z|/(1+|y-z|)\\
&= \frac{|x-y|+2|x-y||y-z|+|y-z|}{1+|x-y|+|y-z|+|x-y||y-z|}\\
&\geq \frac{|x-y|+|x-y||y-z|+|y-z|}{1+|x-y|+|y-z|+|x-y||y-z|}\\
&\geq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \text{ (Because it is known that x/(1+x) is increasing.)}\\
&\geq \frac{|x-z|+}{1+|x-z|} \text{ (Use the triangle inequality and the above trick.)}\\
&= d_5(x,z)
\end{aligned}$$

The following are not metrics on $\mathbb{R}$:

The function $d_1(x,y)=(x-y)^2$ is not a metric on $\mathbb{R}$: consider $0,1,2$; $d_1(0,2)=4$, $d_1(0,1)=1$, and $d_1(1,2)=1$. But $4>1+1$, so d_1 fails the triangle ineqality and is therefore not a metric.

The function $d_3(x,y)=|x^2-y^2|$ is not a metric on $\mathbb{R}$: consider $0,1,2$; $d_3(0,2)=4$, $d_3(0,1)=1$, and $d_3(1,2)=1$. But $4>1+1$, so d_3 fails the triangle ineqality and is therefore not a metric.

The function $d_4(x,y)=|x-2y|$ is not a metric on $\mathbb{R}$: $d_4(0,1)=2$, but $d_4(1,0)=1$. So d_4 is not a metric, as it fails symmetry.

Problem 3:

Interiors of connected sets are not always connected.

Closures of connected sets are always connected.

For the first; consider the union of the pair of closed balls in $\mathbb{R}^2$, $\overline{\mathbb{R}^2_1((0,0))}\cup\overline{\mathbb{R}^2_1((0,2))}$. This is a connected set (if this is not clear, recall that path connectivity implies connectivity, and we can draw a path from any point to $(0,1)$ (which will imply path connectivity)).

However, the interior of that union is $\mathbb{R}^2_1((0,0))\cup\mathbb{R}^2_1((0,2))$, which is not connected (It's clear that this is the interior, and pulling out your epsilons and deltas will get it done. If it's not clear that this isn't connected, look at it again.)

Moving on, closures of connected sets are always connected. (I must admit: I have done this problem twice before.)

It is equivalent to show that if the closure of a set, E, is disconnected, then E is also disconnected.

Let X be a metric space, and $E \subset X$ with $\overline{E}$ disconnected.

Then there are U, V both nonempty and open such that $U \cup V = \overline{E}$.

Now, consider $U \cap E$ and $V \cap E$. Both are open in E. Moreover, neither is empty:

We know that U contains a point, x, in $\overline{E}$.

But U is open; for some $r > 0$, $X_r(x) \subset U \cap \overline{E}$.

We know that x is either a limit point of E or is in E. In the latter case, this means that $U \cap E$ is nonempty. In the former case, there is a point $x' \in X_r(x)$ with $x' \in E$, otherwise x is not a limit point of E.

So $U \cap E$ is nonempty. Similarly, $V \cap E$ is nonempty.

So U and V disconnect E; if $\overline{E}$ is disconnected, then so is E. By contrapositive, we have our result.

To summarize: closures of connected sets are always connected, interiors of connected sets aren't always connected.

Problem 4:

Let X be a separable metric space.

Then X has a countable dense subset; call it C.

Now, consider the collection $\mathcal{C} = \{X_q(c) : c \in C, q \in \mathbb{Q}\}$.

First, note that $\mathcal{C}$ is countable; it injects naturally into the set $\mathbb{Q} \times C$, which is a product of countable sets (which is therefore countable). So because $\mathcal{C}$ injects into a countable set, it is countable.

Now, we show that any open set is a union of elements of $\mathcal{C}$:

Let U be an open set. Either U is open (in which case U is the empty union of elements in $\mathcal{C}$), or U has at least one point.

Let $x \in U$. Then there is some $r_x > 0$ such that $X_{r_x}(x) \subset U$. Without loss of generality, we can pick r_x to be rational (if our original r_x was irrational, then we know that there is an r'_x between $r_x/2$ and r_x with r'_x rational. We know that $X_{r'_x}(x) \subset X_{r_x}(x) \subset U$). Consider $X_{r_x/2}(x)$; because C is dense, $C \cap X_{r_x/2}(x) \neq \emptyset$. Let $q_x \in C \cap X_{r_x/2}(x)$; then $x \in X_{r_x/2}(q_x)$. Also, $X_{r_x/2}(q_x) \subset X_{r_x}(x)$ (if this is not clear, the key idea of the proof is the triangle inequality).

The upshot is that $X_{r_x/2}(q_x)$ contains x and $X_{r_x/2}(q_x) \subset X_{r_x}(x) \subset U$.

Now, take the collection $\{X_{r_x/2}(q_x) : x \in U\}$. The union is somewhat clearly U.

But also, because each r_x was rational, this is a collection of sets in $\mathcal{C}$. So U is a union of elements from $\mathcal{C}$.

Thus, we can write any open set, U, as a collection of sets in $\mathcal{C}$; $\mathcal{C}$ is a base of X.

So, we can construct a countable base from a countable dense subset. That is, every separable metric space has a countable base.

Problem 5:

Let X be a compact metric space.

For each rational number, q, consider the open cover $\mathcal{C}_q = \{X_q(x) : x \in X\}$. This open cover has a finite subcover; call it $\mathcal{F}_q$.

Now, consider the collection of open sets $\mathcal{C}' = \bigcup_{q \in \mathbb{Q}} \mathcal{F}_q$. This collection is countable. (If not clear: each $q \in \mathbb{Q}$ is contributes a countable number of open sets...so this set injects naturally into $\mathbb{Q} \times \mathbb{N}$, so it is countable.)

Moving on, we show that $\mathcal{C}'$ is a base;

Let $U \subset X$ be open. If $U = \emptyset$, then U is the empty union of sets in $\mathcal{C}'$.

Else, let $x \in U$. For some $r_x > 0$, $X_{r_x}(x) \subset U$. Without loss of generality, we can pick $r_x \in \mathbb{Q}$ (I make this argument above.) Now, we know that $x \in X_{r_x/2}(p)$ for some $p_x \in X$ with $X_{r_x/2}(p_x) \in \mathcal{C}'$ (for each $q \in \mathbb{Q}$, there is a finite subcover consisting of balls of radius q contained in $\mathcal{C}'$). Also, we know that $X_{r_x/2}(p_x) \subset X_{r_x}(x)$ by the triangle inequality.

The upshot is that $X_{r_x/2}(p_x)$ contains x and $X_{r_x/2}(p_x) \subset X_{r_x}(x) \subset U$.

Now, take the collection $\{X_{r_x/2}(p_x) : x \in U\}$. The union is somewhat clearly U.

But this was also a union of sets in $\mathcal{C}'$. So U is a union of elements of $\mathcal{C}'$.

Thus, we can write any open set, U, as a collection of sets in $\mathcal{C}'$; $\mathcal{C}'$ is a base of X.

Now, X has a countable base. Pick a single point from each element of this base, and call the set of such points D. Note that D is countable. Also, because every open set can be written as a union of elements in the base, this means that D intersects every nonempty open set. So, D is dense. So X has a countable dense subset, X is separable.

To summarize: any compact metric space has a countable base, and is therefore separable.

Problem 6:

Let $f, g : X \to Y$ be continuous maps between metric spaces. Let E be dense in X.

Then $f(E)$ is dense in $f(X)$:

Let $y \in f(X)$. Then there is a point $x \in X$ such that $f(x) = y$. Now, E is dense in X; there is a sequence of points $\langle x_n \rangle$ such that $x_n \to x$ and for all $n \in \mathbb{N}$, $x_n \in E$. Now, f is continuous; this means that $f(x_n) \to f(x) = y$. Now, $f(x_n)$ is a sequence of points in $f(E)$.

So for any $y \in f(X)$ there is a sequence of points in $f(E)$ that converges to y; $f(E)$ is dense in $f(X)$.

Now, let $f(x) = g(x)$ for all $x \in E$. Then $f(x) = g(x)$ for all $x \in X$.

Let $x \in X$. Then because E is dense in X, we know that there is a sequence of points in E that converges to x. Call this sequence $\langle x_n \rangle$. Now, $f(x_n) = g(x_n)$ for all $x \in X$. So because f is continuous, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. But $f(x_n)$ and $g(x_n)$ are the same sequence; they converge to the same thing. So $f(x) = g(x)$.

Problem 7:

Let $f : E \to \mathbb{R}$ be uniformly continuous, with $E \subset \mathbb{R}$ bounded.

Now, because E is bounded, it is contained in some interval, $[a, b]$.

Now, because f is uniformly continuous, there is a $\delta > 0$ such that for all $x, y \in E$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < 1$.

Now, for each $n \in \mathbb{N}$, define x_n by $x_n = a + (n-1)\delta/2$ if $a + n\delta \leq b$, else $x_n = b$. There is a first $N \in \mathbb{N}$ such that $x_N = b$. We now consider the set $\{x_n : n \in \mathbb{N} \text{ and } n \leq N\}$.

We now consider the set of balls $\mathbb{R}_{\delta/2}(x_n)$. Either $E \cap \mathbb{R}_{\delta/2}(x_n)$ is empty or not. If not, pick a point in it; call it y_n.

Now, the set of balls $\mathbb{R}_{\delta}(y_n)$ covers E; this is clear from the triangle inequality. (If this is not clear; let $x \in E$. Then x is within $\delta/2$ of some x_n. Either x_n is within $\delta/2$ of some $y_n \in E$ or $E \cap \mathbb{R}_{\delta/2}(x_n)$ is empty. In the former case, apply the triangle inequality to get that x is within δ of some $y_n \in E$. In the latter case, infer that $x \notin E$.)

So for all $x \in E$, there is a $y_n \in E$ such that $|x - y_n| < \delta$; so $|f(x) - f(y_n)| < 1$ for all $x \in E$.

But the set of all defined y_ns is finite; this means that it has a maximum, M, and a minimum, m. So by using this together with the above line, we have that for all $x \in E$, $f(x) < M + 1$ and $f(x) > m - 1$. So, we can construct bounds for f; f is bounded on E if f is uniformly continuous.

Problem 8:

We proceed by applying the following lemma: "Let X, Y be metric spaces, with Y compact; $f : X \to Y$ is continuous if and only if the graph of f is closed." (Note: I'm pretty sure the condition can be weakened to local compactness for Y. I don't know how much further that condition can be weakened, though.)

So, let $f : X \to Y$ and denote G as the graph of f.

First, let $f : X \to Y$ be continuous.

Let $\langle p_n \rangle$ be a converging sequence of points in G.

For each p_n, there is an x_n such that $p_n = (x_n, f(x_n))$. Now, because $\langle p_n \rangle$ converges, we know that $x_n \to x$ for some $x \in \mathbb{R}$. Because f is continuous, we know that $f(x_n) \to f(x)$. So $p_n \to (x, f(x))$ for some $x \in X$.

So each converging sequence of G converges to a point in G; G is closed.

Now, let the graph of f be closed.

Let $\langle x_n \rangle$ be a sequence in X converging to a point, say x.

Consider the sequence $\langle (x_n, f(x_n)) \rangle$. Take a subsequence of $\langle x_n \rangle$, $\langle x_{n_j} \rangle$, such that $f(x_{n_j})$ converges. Then the sequence $\langle (x_{n_j}, f(x_{n_j})) \rangle$ converges to a point in G. That is, $\langle (x_{n_j}, f(x_{n_j})) \rangle \to (y, f(y))$ for some $y \in X$. Now, we know that $x_{n_j} \to x$. This means that $\langle (x_{n_j}, f(x_{n_j})) \rangle \to (x, f(x))$; so $f(x_{n_j}) \to f(x)$.

So if $f(x_n)$ converges, then it converges to $f(x)$. We proceed by showing that $f(x_n)$ converges.

Assume not. Then there is some $\epsilon > 0$ such that for any N, there is some $n > N$ such that $d(f(x), f(x_n)) \geq \epsilon$.

For each $n \in \mathbb{N}$, pick $j > n$ such that $f(x_j) \geq \epsilon$. This sequence of points has a converging sequence, $f(x_{j_k})$ (say that it converges to z). This converging subsequence does not converge to $f(x)$, as each $f(x_{j_k})$ is at least ϵ away from $f(x)$.

Now, we know that $x_{j_k} \to x$. So the sequence $\langle (x_{j_k}, f(x_{j_k}) \rangle$ is a sequence of G. It also converges, to (x, z). But $z \neq f(x)$; $\langle (x_{j_k}, f(x_{j_k}) \rangle$ is a converging sequence of G that fails to converge to a point in G. So G is not closed, contrary to our assumption.

So $f(x_n)$ converges to $f(x)$.

To summarize; if we have a sequence, $\langle x_n \rangle$, converging to x, then $f(x_n) \to f(x)$. That is, f is continuous.

So $f : X \to Y$, where Y is compact, is continuous if and only if the graph of f is closed.

If E is compact, then a subset of E is closed if and only if it is compact. (Compact sets are always closed, and in compact spaces closed sets are compact).

So if $f : E \to E$, where E is compact, then f is continuous if and only if the graph of f is compact.