Problem 1:
Let $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, and let $u(x, t)$ solve

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

Let $v(x, t) = u(x, t) - g(t)$, and extend v to $\{x < 0\}$ by odd reflection (just call the resulting extension v). Then v solves

$$\begin{cases} v_t - v_{xx} = -g' & \text{in } \mathbb{R}_+ \times (0, \infty) \\ v = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ v = 0 & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

So $v(x, t) = \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} -g'(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds$ by formula 17 in the book.
Now,

$$
\begin{aligned}
v(x,t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} -g'(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= -\left[\int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} g'(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds\right] \\
&= -\left[-\int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} (x-y)g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds\right] \quad \text{(integrate by parts)} \\
&= \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} (x-y)g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \left[x\int_{\mathbb{R}} g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dy - \int_{\mathbb{R}} yg(y)e^{\frac{-|x-y|^2}{4(t-s)}} dy\right] ds \\
&= \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \left[x\int_{\mathbb{R}} g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dy\right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} yg(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dy\right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} yg(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y)e^{\frac{-|x-y|^2}{4(t-s)}} dy\right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} yg(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y)e^{\frac{-(x^2-2xy+y^2)}{4(t-s)}} dy\right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} yg(y)e^{\frac{-(x^2-2xy+y^2)}{4(t-s)}} \\
&= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y)e^{\frac{-(-2xy+y^2)}{4(t-s)}} dy\right] ds - \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} yg(y)e^{\frac{-(-2xy+y^2)}{4(t-s)}} dyd \\
&= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s)ds - g(t)
\end{aligned}
$$

I must admit that I'm not sure how the last line is supposed to go, but that's the idea.

Thus, by adding $g(t)$ to $v(x,t)$, we get that $u(x,t) = \frac{x}{\sqrt{4\pi}} \int\limits_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s)ds$ as desired.

Problem 2:

Let $g \in C(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, $|g| < M$ for some M. Let u be the bounded solution to

$$\begin{cases} \Delta u - u_t = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Part a:

Then $u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y)dy$. Let $\epsilon > 0$. There is an $h \in \mathbb{R}^n$ with compact support such that $|g| \geq |h|$, $\left| \int\limits_{\mathbb{R}^n} g - h \right| < \epsilon/2$ and $|g - h| < \epsilon/(2A)$ where A is the volume of a ball, call it B, that h vanishes outside of.

So for all $\epsilon > 0$ there's an N such that for all $t > N$, $|u(x,t)| < \epsilon$. That is, $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} |u(x,t)| = 0$, which is the desired result.

Part b:

Consider $v(x,t) = u(x,t) - g(x)$. Then $v(x,t)$ solves

$$\begin{cases} \Delta v - v_t = -\Delta g(x) & \text{for } t > 0, x \in \mathbb{R}^n \\ v(x,0) = 0 & \text{for } x \in \mathbb{R}^n \end{cases}$$

Thus, $v(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(x-y, t-s)\Delta g(y)dyds$.

So $\int\limits_{\mathbb{R}^n} v(x,t)dx = \int\limits_{\mathbb{R}^n} \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(x-y, t-s)\Delta g(y)dydsdx$

Switching the order of integration, we get $\int\limits_0^t \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \Phi(x-y, t-s)\Delta g(y)dxdyds$.

Yet, this is $\int\limits_0^t \int\limits_{\mathbb{R}^n} \Delta g(y)dyds$, because $\int\limits_{\mathbb{R}^n} \Phi(x-y, t-s)dx = 1$.

The integral vanishes, by Green's Theorem; $\int\limits_{\mathbb{R}^n} \Delta g(y)dy = \lim\limits_{r\to\infty} \int\limits_{\partial B(0,r)} \frac{\partial g(y)}{\partial \nu} dS$, which must vanish as g is in $L^1(\mathbb{R})$.

So, $\int\limits_{\mathbb{R}^n} v(x,t)dx = 0$; $\int\limits_{\mathbb{R}^n} u(x,t) - g(x)dx = 0$, which yields the desired result of $\int\limits_{\mathbb{R}^n} u(x,t)dx = \int\limits_{\mathbb{R}^n} g(x)dx$.

Problem 3:

Part a:

Fix $\alpha \in (0,1)$, $\beta \geq 0$.

Note first that $z^\beta e^{-z} = e^{\beta \ln(z) - z}$. So, the desired result is

$$e^{\beta \ln(z) - z} \leq Me^{-\alpha z}$$

for some M, which is equivalent to

$$\beta \ln(z) - z \leq \ln(M) - \alpha(z)$$

for some M. Now, this is equivalent to

$$-\ln(M) \leq (1-\alpha)z - \beta \ln(z)$$

for some M. By applying basic calculus, the right hand side takes a minimum at $z = \beta/(1-\alpha)$, so taking $M = (1-\alpha)(\beta/(1-\alpha)) - \beta \ln(\beta/(1-\alpha))$ suffices so that $-\ln(M) \leq (1-\alpha)z - \beta \ln(z)$ and, equivalently, $z^\beta e^{-z} \leq Me^{-\alpha z}$ for all $z \geq 0$, as desired.

Part b:

Part c: