Note to grader: In the below, I freely use the standard notation $\overline{x}$ to denote the equivalence class of x in the quotient space $X/\sim$.

Problem 4a, p127:

Consider the function $h(t) : \mathbb{R} \to \mathbb{R}^\omega$ given by $h(t) = (t, t/2, t/3, \ldots)$.

First, h is not continuous in the box topology; consider the open set $U = \prod_{n=1}^{\infty}(-1/n^2, 1/n^2)$. Then $h^{-1}(U) = \bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is not open.

Next, h is continuous if $\mathbb{R}^\omega$ is given the product topology; let U be a basic open set in $\mathbb{R}^\omega$. Then $U = \prod_{i=1}^{\infty} U_i$, with U_i an open set in $\mathbb{R}$, and only finitely many $U_i \neq \mathbb{R}$. Now, $h^{-1}(U) = U_1 \cap 2U_2 \cap 3U_3 \ldots$ (with $iU_i = \{ix \in \mathbb{R} : x \in U_i\}$.), and this is clear. Now, there is an N such that for all $n \geq N$, $U_n = \mathbb{R}$. So there is an N such that for all $n \geq N$, $nU_n = \mathbb{R}$. So $h^{-1}(U) = U_1 \cap 2U_2 \cap 3U_3 \ldots U_N \cap \mathbb{R} \cap \mathbb{R} \ldots = \bigcap_{i=1}^{N} iU_i$, which is open, as it is an intersection of open sets (it is clear that each nU_n is open, as multiplication by a constant is well known to be a homeomorphism (this is a basic fact from analysis)).

That is, if U is a basic open set in $\mathbb{R}^\omega$ given the product topology, then $h^{-1}(U)$ is open. Let U be an open set in $\mathbb{R}^\omega$; then $h^{-1}(U) = h^{-1}(\bigcup_{\alpha \in A} U_\alpha)$ for some index set A, with each U_α basic. Now, $h^{-1}(U) = h^{-1}(\bigcup_{\alpha \in A} U_\alpha) = \bigcup_{\alpha \in A} h^{-1}(U_\alpha)$, which is open as it is a union of open sets (they are the preimages of basic open sets, so they are open, from the above); that is, if U is open in the product topology, then $h^{-1}(U)$ is open. That is, h is continuous as a function from $\mathbb{R}$ to $\mathbb{R}^\omega$, with $\mathbb{R}^\omega$ given the product topology.

I don't know how to handle the uniform topology, sorry. Moreover, this is the problem I ran out of time on; I apologize if this is sloppy, but I still feel like every step is clear.

Problem 4b, p127:

Consider $\langle x_n \rangle$ as described in the text.

Now, $\langle x_n \rangle \not\to 0$ in the box topology: Let $U = \prod_{n=1}^{\infty}(-1/2^n, 1/2^n)$. Then for all $n \in \mathbb{N}$, $x_n \notin U$; this is because $\pi_n(x_n) = 1/n$, and $1/n > 1/2^n$ for all $n \geq 1$ (this is somewhat obvious analysis). So $\langle x_n \rangle$ doesn't converge to 0 in the box topology; by the logic above, this means that $\langle x_n \rangle$ doesn't converge in the box topology.

Consider $\langle y_n \rangle$ as described in the text.

Now, $\langle y_n \rangle \not\to 0$ in the box topology: Let $U = \prod_{n=1}^{\infty}(-1/2^n, 1/2^n)$. Then for all $n \in \mathbb{N}$, $y_n \notin U$; this is because $\pi_n(y_n) = 1/n$, and $1/n > 1/2^n$ for all $n \geq 1$ (this is somewhat obvious analysis). So $\langle y_n \rangle$ doesn't converge to 0 in the box topology; by the logic above, this means that $\langle y_n \rangle$ doesn't converge in the box topology.

Consider $\langle z_n \rangle$ as described in the text. This sequence converges to 0 in the box topology; let U be a basic open neighborhood of 0 in the box topology. Then $U = \prod U_n$ with each U_n an open neighborhood containing 0. Consider U_1 and U_2; each contains a basic neighborhood $\mathbb{R}_{\epsilon_1}(0)$ and $\mathbb{R}_{\epsilon_2}(0)$ respectively, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ (by the definition on page 78, example 2 on p120, and the definition of the metric topology). Now, by the archimedean principle, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $1/n < \epsilon_1$ and $1/n < \epsilon_2$, so that $\pi_1(z_n) \in U_1$ and $\pi_2(z_n) \in U_2$ for all $n \geq N$. So $z_n \in U$ for all $n \geq N$, because $z_n = (1/n, 1/n, 0, 0, 0 \ldots)$ so that $\pi_m(z_n) \in U_m$ for all $m > 2$ because U_m is a neighborhood of 0 (as U was a neighborhood of 0).

So for any basic open neighborhood, U, of 0, there is an N such that for all $n \geq N$, $z_n \in U$. So for any open neighborhood U of 0, there is an N such that for all $n \geq N$, $z_n \in U$, (by the definition on p78). So, $\langle z_n \rangle \to 0$ in the box topology.

Problem 6b, p127:

Consider the sequence $y = (x_1, x_2 + \epsilon/2, x_3 + 2\epsilon/3, x_4 + 3\epsilon/4, \ldots)$.

Then $y \in U(x, \epsilon)$. So, if $U(x, \epsilon)$ is open, we have that there is a basic open set, $\mathbb{R}^{\omega}_{\epsilon'}(y)$, with $y \in U \subset U(x, \epsilon)$ and $\epsilon' > 0$ (from the definition of basis on p78). Yet, if so, then the point $z = (y_1 + \epsilon'/2, y_2 + \epsilon'/2, \ldots)$ is in $U(x, \epsilon)$. But this is nonsense; there is an n such that $\frac{n\epsilon}{n+1} + \epsilon'/2 > \epsilon$ (this follows from the fact that $\frac{n}{n+1}$ increases to 1, so that $\frac{n\epsilon}{n+1}$ increases to ϵ, so that there's an n with $\epsilon - \frac{n\epsilon}{n+1} < \epsilon'/2$). Thus, there is an n with $z_n = \epsilon'/2 + y_n = \epsilon'/2 + \frac{n\epsilon}{n+1} x_n$, and so $z_n - x_n = \frac{n\epsilon}{n+1} + \epsilon'/2 > \epsilon$; that is, $z_n \notin U(x, \epsilon)$.

So, there is not a basic open neighborhood, $\mathbb{R}^{\omega}_{\epsilon}(y)$, with $y \in \mathbb{R}^{\omega}_{\epsilon}(y) \subset U(x, \epsilon)$; So, $U(x, \epsilon)$ is not open (from the definition of basis on p78).

Problem A:

This problem is Theorem Q.2, which Dr. McClure said we can use on all future homework.

Joking aside...

Let $f : X/\sim \to Y$ be continuous. From Lemma Q.1, we know that the quotient map $q : X \to X/\sim$ is continuous. So, by Theorem 18.2, the composite map $f \circ q$ is continuous, which was what we wanted.

Now, let the composite map $f \circ q$ be continuous. Let U be an open set in Y. Then $(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U))$ is open. So $f^{-1}(U)$ is open, by the definition of the quotient topology.

That is, $f : X/\sim \to Y$ is continuous if and only if the composite map $f \circ q$ is continuous.

Problem B:

Let X, Y be topological spaces.

Let $p : X \to Y$ be surjective, and let p be such that for all $U \subset Y$, U is open if and only if $p^{-1}(U)$ is open in X. (That is, let p be a quotient map as in Munkres, p137.) Let q be the relevant quotient map from $X \to X/\sim$. Then consider $\overline{p} : X/\sim \to Y$, built from p as in Theorem Q.3.

First, $\overline{p}$ is injective; let $\overline{p}(\overline{x}) = \overline{p}(\overline{y})$, for some $x, y \in X$. Then $p(x) = p(y)$, by definition. But, again by definition, this means that $x \sim y$, so that $\overline{x} = \overline{y}$. So, we have that $\overline{p}(\overline{x}) = \overline{p}(\overline{y})$ implies that $\overline{x} = \overline{y}$, so that $\overline{p}$ is injective.

Also, $\overline{p}$ is surjective; if $y \in Y$, note that there is an $x \in X$ with $p(x) = y$. So, $\overline{p}(\overline{x}) = y$, by the definition of $\overline{p}$; that is, for all $y \in Y$, there is an $\overline{x} \in X/\sim$ with $\overline{p}(\overline{x}) = y$, so $\overline{p}$ is surjective.

Now, $\overline{p}$ is continuous: p is continuous, because if U is open in Y, then $p^{-1}(U)$ is open in X, by the hypotheses on p. So by theorem Q.2, $\overline{p}$ is continuous.

Last, $\overline{p}$ has an inverse, $\overline{p}^{-1}$. This inverse is continuous; let U be open in $X/$. Consider $(\overline{p}^{-1})^{-1}(U) = \{y \in Y : \overline{p}^{-1}(\overline{y}) = x \text{ for some } \overline{x} \in U\} = \{y \in Y : y = \overline{p}(\overline{x}) \text{ for some } \overline{x} \in U\} = \{y \in Y : y = \overline{p}(\overline{x}) \text{ for some } x \in q^{-1}(U)\} = \{y \in Y : y = \overline{p}(q(x)) \text{ for some } x \in q^{-1}(U)\} = \overline{p}(q(q^{-1}(U))) = p(q^{-1}(U))$; this set is open; $q^{-1}(U)$ is open, by the definition of the topology on the quotient space, so we get that $p(q^{-1}(U)) = (\overline{p}^{-1})^{-1}(U)$ is open by the hypotheses. Thus, the inverse is continuous.

So, $\overline{p}$ is a homeomorphism; it is a bijection with both the $\overline{p}$ and $\overline{p}^{-1}$ continuous.

Now, let $p : X \to Y$ be such that $\overline{p}$ is a homeomorphism.

Then p is surjective; let $y \in Y$. Then there is $x \in X$ such that $\overline{p}(\overline{x}) = y$, because $\overline{p}$ is a homeomorphism (and thus surjective). So, there is $x \in X$ such that $p(x) = y$, by definition.

Now, let $U \subset Y$ be open. Then note that $\overline{p}$ is continuous, because $\overline{p}$ is a homeomorphism. So, by theorem Q.2, p is continuous. So, $p^{-1}(U)$ is open.

Now, let $p^{-1}(U) \subset X$ be open. Then $p^{-1}(U) = q^{-1}(\overline{p}^{-1}(U))$; I claim that this is an obvious set theory fact, that follows from the fact that $p = \overline{p} \circ q$ (if this is not obvious, then consider that $p^{-1}(U) = \{x \in X : p(x) = y$ for some $y \in U\} = \{x \in X : q(\overline{p}(x)) = y$ for some $y \in U\} = \{x \in X : q(\overline{x}) = y$ for some $\overline{x} \in p^{-1}(U)\} = q^{-1}(\overline{p}^{-1}(U)))$. Now, $p^{-1}(U) = q^{-1}(\overline{p}(U))$ is open; so, by the definition of the quotient space, $\overline{p}(U)$ is open. So, because $\overline{p}$ is a homeomorphism, U is open.

That is, if p is a "Munkres quotient map", then p is a surjective map with $U \subset Y$ open if and only if $p^{-1}(U)$ is open in X.

So, p is a "Munkres quotient map" if and only if p is a surjective map with $U \subset Y$ open if and only if $p^{-1}(U)$ is open in X.

Problem C:

Let $p : X \to Y$ be a Munkres quotient map.

Let $f : Y \to Z$ be continuous. Note that $p : X \to Y$ is continuous; this is a throwaway comment on p137. So by Theorem 18.2, the composite map $f \circ p$ is continuous, which was what we wanted.

Now, let $f \circ p$ be continuous. Let U be an open set in Z. Then $p^{-1}(f^{-1}(U))$ is open in X. So $f^{-1}(U)$ is open in Y, by the definition on p137. That is, for every U open in Z, $f^{-1}(U)$ is open in Y; that is, f is continuous.

So, $f : Y \to Z$ is continuous if and only if $f \circ p$ is continuous.

Problem D:

Let $\sim$ be the equivalence relation on $[-1, 1]$ defined by $x \sim y$ if and only if $x = y$ or $x \sim -y$ with $y \in (-1, 1)$. Let $q : [-1, 1] \to [-1, 1]/\sim$ be the quotient map.

Then $\overline{1} \neq \overline{-1}$ in $Q = [-1, 1]/\sim$. Let U be a neighborhood of $\overline{1}$ in Q, and V be a neighborhood of $\overline{-1}$ in Q. Then $U' = q^{-1}(U)$ and $V' = q^{-1}(V)$ are open in $[-1, 1]$, and also $1 \in U'$ and $-1 \in V'$. Moreover, $U' \cap V' = \emptyset$ (from the definitions). Now, there is an $\epsilon > 0$ with $[-1, 1]_\epsilon(1) \subset U'$ and an $\epsilon' > 0$

with $[-1,1]_\epsilon(-1) \subset V'$, by (obvious fact about metric spaces). So, there is an $n \in \mathbb{N}$ with $-1+1/n \in [-1,1]_\epsilon(-1)$ and $1-1/n \in [-1,1]_\epsilon(1)$, by an application of the archimedean principle. So $1-1/n \in U'$ and $-1+1/n \in V'$. So $\overline{1-1/n} \in U$ and $\overline{-1+1/n} = \overline{1-1/n} \in V$. That is, $U \cap V$ is nonempty.

That is, for any two neighborhoods of $\overline{1}$ and $\overline{-1}$ intersect; $[-1,1]/\sim$ is not Hausdorff.

Problem E:

Let X be a topological space with an equivalence relation $\sim$. Suppose $X/\sim$ is Hausdorff.

Consider $S = \{(x,y) \in X \times X : x \sim y\}$. Now, recall that $\Delta/\sim = \{(\overline{x}, \overline{x}) : \overline{x} \in X/\sim\}$ is closed, by p101, 13.

Let p be the projection map $p : X \to X/\sim$. Then p is continuous, by definition of the quotient topology.

So the set $p^{-1}(\Delta/\sim)$ is closed in X, by theorem 18.1; that is, the set $\{(x,y) \in X \times X : (x,y) = (\overline{x'}, \overline{x'})$ for some $x' \in X\}$ is closed. But this set is just $\{(x,y) \in X \times X : x \sim y\} = S$, because $\sim$ is an equivalence relation. So, S is closed in $X \times X$ if the quotient space $X/\sim$ is Hausdorff.

Problem F i:

Let X be a topological space. Let U be open in X, let $A \subset U$. give U the subspace topology. Let $i : U/A \to X/A$ be given by $\overline{x} \mapsto \overline{x}$. Let $q : X \to X/A$ and $q' : U \to U/A$ be the relevant quotient maps.

Let V be open in X/A. Then $q^{-1}(V)$ is open in X, by theorem Q.1 (q is continuous). So $q^{-1}(V) \cap U = \{x \in X : x \in U$ and $\overline{x} \in V\}$ is open in X, because both of those are open sets in X.

Consider, now, $i^{-1}(V) = \{\overline{x} \in U/A : \overline{x} \in V\}$. Then $q'{-}1(i^{-1}(V)) = \{x \in U : \overline{x} \in V\} = \{x \in X : x \in U$ and $\overline{x} \in V\}$. But this is the same as the set $q^{-1}(V) \cap U$, which is open in X (and also in U, by the definition of subspace topology). So by the definition of the Quotient topology, $i^{-1}(V)$ is open in U/A.

That is, if V is open in X/A, then $i^{-1}(V)$ is open in U/A; that is, i is continuous.

Problem F ii:

Let X be a topological space. Let U be open in X, let $A \subset U$. give U the subspace topology. Let $i : U/A \to X/A$ be given by $\overline{x} \mapsto \overline{x}$. Let $q : X \to X/A$ and $q' : U \to U/A$ be the relevant quotient maps.

Let V be open in U/A. Then $q'^{-1}(V)$ is open in U, by theorem Q. (q' is continuous). So $q'^{-1}(V) = \{x \in X : \overline{x} \in V\}$ is open in X, by lemma 16.2.

Consider, now, $i(V) = \{\overline{x} \in X/A : \overline{x} \in V\}$. Then $q^{-1}(i(V)) = \{x \in X : \overline{x} \in V\}$. But this is the same as $q'^{-1}(V)$, which is open in X. So, by the definition of the quotient topology, $i(V)$ is open in X/A.

That is, if V is open in U/A, then $i(V)$ is open in X/A; that is i is an open map.

Problem G:

Let X be a topological space with a countable basis at each point in X. Let $A \subset X$, and let $x \in \overline{A}$.

Choose a countable basis, $\{U_n\}_{n\in\mathbb{N}}$, around x, so that each U_n is a neighborhood of x. Define $B_N = \bigcap\limits_{n=1}^{N} U_n$. Each B_N intersects A, because each B_N is open (it is a finite intersection of open sets) and because of Theorem 17.5 ($x \in \overline{A}$ if and only if every neighborhood of x intersects A). So, for each B_N, there is an $x_N \in B_N \cap A$.

Consider the sequence $\langle x_N \rangle$. Note that $x_N \in A$ for all N, because $x_N \in B_N \cap A$ for each N. Now, pick a neighborhood, U, of x; then U contains at least one of the sets U_n, by definition. So, because $B_N = \bigcap\limits_{n=1}^{N} U_n \subset U_N$, we have that $B_N \subset U_N$ for each B_N; that is, U contains at least one of B_N. In fact, note that $B_{N+1} = \bigcap\limits_{n=1}^{N+1} U_n \subset \bigcap\limits_{n=1}^{N} U_n = B_N$, so that $B_{N+1} \subset B_N$ for each N; by induction, $B_M \subset B_N$ if $M \geq N$. So, because U contains at least one of B_N, we have that for some N, we have U containing $B_{N'}$ for all $N' \geq N$.

So, for some N, we have that U contains $x_{N'} \in B_{N'}$ for all $N' \geq N$. That is, for any neighborhood, U, of x, there is an N such that U contains $x_{N'} \in B_{N'}$ for all $N' \geq N$.

That is, $\langle x_n \rangle \to x$. That is, $x \in \overline{A}$ implies that there is a sequence of A that converges to x.