Problem 1:

Let $f_n \to f$ in measure, with an integrable function g such that $|f_n| \leq g$ for all n.

We proceed by proving the fact if the domain of each f_n is $[0, 1]$, then extending it to all of $\mathbb{R}$ by applying the $\epsilon 2^{-n}$ method.

Let $\epsilon > 0$.

Because $f_n \to f$ in measure, there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $m(x : |f_n(x) - f(x)| \geq \epsilon/2) < \epsilon/2$.

Problem 2:

Let f be continuous on $[a, b]$, with one of its derivates everywhere non-negative on (a, b).

First, we will show this for a function g with $D^+(g) \geq \epsilon > 0$. If g is such a function, then $\limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq \epsilon > 0$. This means that g is nondecreasing:

We proceed by contradiction: let there be $x, y \in [a, b]$ (with $x < y$, without loss of generality) be such that $f(x) > f(y)$.

Consider the set $A = \{\alpha \in [x, y] : f(\alpha) > f(y)\}$. This set has a supremum, as it's nonempty. Define $\alpha = \sup(A)$. There is a $\delta > 0$ such that if $t \in [\alpha, \alpha + \delta]$, then $f(t) < f(y)$: (Is pretty trivial). Moreover, $f(\alpha) = f(y)$: (Proof is clear, follows from continuity). So for any sequence (t_n) decreasing to α, we have $\frac{f(t_n)-f(\alpha)}{t_n-\alpha}$ negative. This means that $D^+(\alpha) \leq 0$. This contradicts our assumption on D^+.

We can mimic this proof to show that if g has $D^-(g) \geq \epsilon > 0$, then g is nondecreasing.

Now, let f have a derivate everywhere nonnegative on (a, b). This means, in particular, that either D^+ or D^- is everywhere nonnegative on (a, b).

Then for every $\epsilon > 0$, $g_\epsilon(x) = f(x) + \epsilon x$ has $D^+(g_\epsilon)$ (or $D^-(g_\epsilon)$) greater than ϵ. So for all $\epsilon > 0$, g_ϵ is nondecreasing. So for all $x, y \in [a, b]$ with $x < y$, $g_\epsilon(x) \leq g_\epsilon(y)$. That is, $f(x) + \epsilon x \leq f(y) + \epsilon y$. Taking limits as $\epsilon \to 0$, this means that $f(x) \leq f(y)$, for all $x, y \in [a, b]$ with $x < y$.

So f is nondecreasing on $[a, b]$ if some derivate is everywhere nonnegative on $[a, b]$.

Problem 3:
Suppose that $f_n(x) \to f(x)$ at each $x \in [a,b]$.

Problem 4:
Suppose that $f \in BV([a,b])$. Then f' exists almost everywhere, by a theorem in class. Moreover, f is the difference of two monotone functions. That is, $f = f^+ - f^-$ for some monotone functions f^+ and f^-.

So, this means that we have

$$\int_a^b |f'| = \int_a^b \left|(f^+)' - (f^-)'\right|$$
$$\leq \int_a^b \left|(f^+)'\right| + \left|(f^-)'\right|$$

Now, we show that $\int_a^b |(f^+)'| \leq P_a^b(f)$.

By a theorem we have, $\int_a^b |(f^+)'| \leq f^+(b) - f^-(a)$

Similarly, $\int_a^b |(f^-)'| \leq N_a^b(f)$. So, we have

$$\int_a^b |f'| \leq \int_a^b \left|(f^+)'\right| + \left|(f^-)'\right|$$
$$\leq P_a^b + N_a^b$$
$$\leq T_a^b$$

as we desired.

Problem 5:
Let g be an absolutely continuous monotone function on $[0,1]$, and E be a set of measure 0.

Problem 6:

Let f be a nonnegative measurable function on $[0, 1]$.

We know that ln is a concave function on $[0, 1]$ (if this is not clear, it's the inverse of a convex function).

So $-\ln$ is a convex function on $[0, 1]$.

So Jensen's inequality applies:

$$-\ln \int f \leq -\int \ln f$$
$$\ln \int f \geq \int \ln f$$

This satisfies the problem.