Problem 1:
Consider the Cauchy problem

$$xu_y - yu_x = xyu$$
$$\Gamma = \{(x, y) : x = y\}$$
$$u(x, y) = xy \text{ for } (x, y) \in \Gamma$$

We apply the method outlined in Zachmanoglou;

Consider the system of equations $\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{xyz}$. This is solved by $u_1 = \frac{x^2}{2} + \frac{y^2}{2}$ and $u_2 = \frac{y^2}{2} - \ln(z)$. If we parameterize the curve Γ by $x = y = t$, we get that $z = t^2$ on this curve, so that $u_1 = t^2$ and $u_2 = \frac{t^2}{2} - \ln(t^2)$ on this curve. Solving for t, we get $u_2 + \ln(u_1) - \frac{u_1}{2} = 0$. That is, $\frac{y^2}{2} - \ln(z) + \ln(\frac{x^2}{2} + \frac{y^2}{2}) - \frac{\frac{x^2}{2} + \frac{y^2}{2}}{2} = 0$.

Solving for z, we get $\ln(z) = \frac{y^2 - x^2}{4} + \ln(\frac{x^2}{2} + \frac{y^2}{2})$. That is, our solution is

$$u(x, y) = \frac{1}{2}(x^2 + y^2)e^{\frac{y^2 - x^2}{4}}$$

as is readily checked. The solution is unique: Γ is noncharacteristic for this PDE except at $(0, 0)$, so we have uniqueness/existence except at $(0, 0)$ (that is, including $(1, 1)$. In other words, we can solve this around any $(x, x) \in \Gamma$ except perhaps $(0, 0)$. However, note that our solution is valid at $(0, 0)$, so we can solve this at any point on Γ.

Problem 2:
Let $k(x, y) = \frac{1}{2}e^{-|x-y|}$. Fix $f \in C_c^2(\mathbb{R})$. Set $w(x) = \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-|x-y|}f(y)dy = \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-|y|}f(x - y)dy$.

Then $w''(x) = \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-|y|}f''(x - y)dy$, because differentiation under the integral works (as f has compact support). So

$$
\begin{aligned}
w(x) - w''(x) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f''(x-y)dy \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy - \frac{1}{2}\left[e^{-|y|} f'(x-y)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (e^{-|y|})' f'(x-y)dy \right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\left[\int_{-\infty}^{\infty} (e^{-|y|})' f'(x-y)dy \right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\left[\int_{-\infty}^{0} (e^{y})' f'(x-y)dy + \int_{0}^{\infty} (e^{-y})' f'(x-y)dy \right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\left[\int_{-\infty}^{0} e^{y} f'(x-y)dy - \int_{0}^{\infty} e^{-y} f'(x-y)dy \right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\Big[e^{y} f(x-y)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{y} f(x-y)dy \\
&\quad - e^{-y} f(x-y)\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-y} f(x-y)dy \Big] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\left[f(x) - \int_{-\infty}^{0} e^{y} f(x-y)dy + f(x) - \int_{0}^{\infty} e^{-y} f(x-y)dy \right] \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy + \frac{1}{2}\left[2f(x) - \int_{-\infty}^{\infty} e^{-|y|} f(x-y)dy \right] \\
&= f(x)
\end{aligned}
$$

as desired. (All steps are are straightforward integration by parts or simply observing that something cancels or is zero.)

Problem 3:
Let $u(x,t)$ solve

$$\begin{aligned} u_t t - \Delta u = 0 \text{ for } x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = \chi_{B(0,1)}(x) \end{aligned}$$

That is, $u(x,t) = ⨍_{\partial B(x,t)} t\chi_{B(0,1)}(y)dS(y) = \frac{1}{4\pi t}\int_{\partial B(x,t)} \chi_{B(0,1)}(y)dS(y)$.

Part i:

Now, define $\Omega(t) = \{x : u(x,t) \neq 0\}$. Note that $u(x,t) \neq 0$ if $\partial B(x,t) \cap B(0,1)$ is nonempty (this isn't exactly "obvious", but if the intersection has at least one point, then it's open in the right topology...and so it contains some area, which means the integral above is nonzero).

Thus, $u(x,t)$ is nonzero when $|y-x| = t$ for some y with $|y| < 1$. That is, $u(x,t)$ is nonzero when $|x| \in (t-1, t+1)$. (This is geometrically clear.)

So, $\Omega(t) = \{x : |x| \in (t-1, t+1)\}$.

Part ii:

Observe that $\int_{\partial B(x,t)} \chi_{B(0,1)}(y)dS(y) \leq 4\pi$; this is because the surface area of a portion of a sphere embedded in the unit ball must have surface area no greater than the surface area of the sphere. Thus, $u(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} \chi_{B(0,1)}(y)dS(y) \leq \frac{1}{t} < \frac{2}{t+1}$ if $t \geq 1$.

Thus, $\sup\limits_{x \in \mathbb{R}^3} u(x,t) \leq \frac{2}{t+1}$.

Next, notice that because $t \geq 1$, we can pick $x \in \mathbb{R}^3$ so that $\partial B(x,t)$ contains an equator of $\partial B(0,1)$. The surface area of the segment of $\partial B(x,t)$ that intersects $B(0,1)$ is at least π, because the surface of least area containing any equator is a circle. Thus, we can choose x so that $u(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} \chi_{B(0,1)}(y)dS(y) \geq \frac{1}{4\pi t}\pi = \frac{1}{4t} < \frac{1}{t+1}$ if $t \geq 1$.

Thus, $\sup\limits_{x \in \mathbb{R}^3} u(x,t) \geq \frac{1}{t+1}$. Combining this with the above,

$$\frac{1}{t+1} \leq \sup_{x \in \mathbb{R}^3} u(x,t) \leq \frac{2}{t+1}$$

as desired.

Problem 4:

Let T be any rectangle in $\mathbb{R}^2$ with sides parallel to the x and y axes, as pictured in the test. Let $u \in C^2(T) \cap C(\overline{T})$ satisfy $u_{xy} = 0$.

Part a:

Then $u_x = f(x)$ for some differentiable function f. Thus, by the fundamental theorem of calculus, $\int\limits_{[B,A]} u_x = u(A) - u(B)$ and $\int\limits_{[D,C]} u_x = u(C) - u(D)$

However, $u_x = f(x)$ does not depend on y; this means that $\int\limits_{[D,C]} u_x = \int\limits_{[B,A]} u_x$. That is, $u(A) - u(B) = u(C) - u(D)$.

Rewriting that, we have $u(A) - u(B) - u(C) + u(D) = 0$, as desired.

Part b:

Consider the Dirichlet problem,

$$u_{xy} = 0 \text{ in } T$$
$$u(x,y) = g(x,y) \text{ on } \partial T$$

Now, a solution to the Dirichlet problem must have that $u \in C^2(T) \cap C(\overline{T})$. So, $u(A) - u(B) - u(C) + u(D) = 0$. However, we can choose g smooth so that this condition is necessarily violated: say, by taking $g(x,y) = xy$ and the points $A = (1,1), B = (0,1), C = (1,0), D = (0,0)$ so that $u(A) - u(B) - u(C) + u(D) = 1 + 0 + 0 + 0 = 1 \neq 0$.

Thus, there are $g \in C(\partial T)$ so that the Dirichlet problem has no solution.

Part c:

Consider the Dirchlet problem for this PDE on $B(0,1)$, that is,
Consider the Dirichlet problem,

$$u_{xy} = 0 \text{ in } B(0,1)$$
$$u(x,y) = g(x,y) \text{ on } \partial B(0,1)$$

with $g(x,y) = (x + 1/\sqrt{2})(y + 1/\sqrt{2})$. Let u solve the PDE on the domain $B(0,1)$; then u also solves the PDE on the rectangle with vertices $A = (1/\sqrt{2}, 1/\sqrt{2}), B = (-1/\sqrt{2}, 1/\sqrt{2}), C = (1/\sqrt{2}, -1/\sqrt{2}), D =$

$(-1/\sqrt{2}, -1/\sqrt{2})$. So $u(A) - u(B) - u(C) + u(D) = 0$, by part a. However, we can calculate using the boundary data that $u(A) - u(B) - u(C) + u(D) = (2/\sqrt{2})^2 = 2 \neq 0$, which is a contradiction: there is no solution to the problem.

That is, the Dirchlet problem is ill-posed on this domain.