Note to grader: the standard notation for "the ball of radius r around the point x in the metric space X is $B_r(x)$. This is terrible, especially if we are working with more than one metric space at a time; I use the notation $X_r(x)$ to denote "the ball of radius r around the point x in the metric space X, as it is better.

Second note: I write $0 = (0, 0, 0, \ldots)$ in the following. I'm led to believe this is standard.

Problem 7a, p111:

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous from the right. Consider $f : \mathbb{R}_\ell \to \mathbb{R}$. Then, by definition, $\forall a \in \mathbb{R} \forall \epsilon > 0 \exists \delta > 0 : |x - a| < \delta$ and $a > x \implies |f(x) - f(a)| < \epsilon$.

Now, let W be open in $\mathbb{R}$. Then for each $a \in W$, there is an open interval $I \subset W$ containing a. Consider $f^{-1}(I)$; let $a' \in f^{-1}(I)$. Then $f(a') \in I$. Choose $\epsilon > 0 : \mathbb{R}_\epsilon(f(a')) \subset I$ (this is possible because I is open and so there's a basic (read: "basic using the standard basis") neighborhood of $f(a')$ contained in I). Then there is $\delta > 0$ with $|x - a|$ and $x > a$ implying that $|f(x) - f(a)| < \epsilon$. That is, $f([a', a' + \delta)) \subset \mathbb{R}_\epsilon(f(a')) \subset I$; so, $[a', a' + \delta) \subset f^{-1}(I)$. So, for each $a' \in f^{-1}(I)$, there is a set, U open in $\mathbb{R}_\ell$ with $a' \in U \subset f^{-1}(I)$; each $f^{-1}(I)$ is open in $\mathbb{R}_\ell$.

So, because U is a union of open intervals, we have that $f^{-1}(U)$ is a union of open sets. Thus, $f^{-1}(W)$ is open.

That is, for all W open in $\mathbb{R}$, $f^{-1}(W)$ is open in $\mathbb{R}_\ell$; so $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous if f is continuous from the right.

Problem 13, p112:

Let $A \subset X$, let $f : A \to Y$ be continuous, let Y be Hausdorff, let there be a continuous function $g : \overline{A} \to Y$ with $f(x) = g(x)$ for all $x \in A$.

Let there be two distinct such functions, g and h. Then there exists $a \in \overline{A}$ with $g(a) \neq h(a)$. Then, as Y is Hausdorff, there are U and V open in Y with $g(a) \in U$, $h(a) \in V$, and $U \cap V = \emptyset$. Note that because $g(a) \in U$ and $h(a) \in V$, we have that $g^{-1}(U)$ and $h^{-1}(V)$ both contain a. Moreover, both $g^{-1}(U)$ and $h^{-1}(V)$ are open in X, as g and h were both continuous. Consider $g^{-1}(U) \cap h^{-1}(V)$; this set is open in X, as it is the intersection of two open sets in X. Also, it contains $a \in \overline{A}$, so that every open neighborhood of a contains a point $a' \in A$(by theorem 17.5). So, there is an $a' \in A$ with $a' \in g^{-1}(U) \cap h^{-1}(V)$. Because $a' \in A$, we have $f(a') = g(a') = h(a')$. So, because $a' \in g^{-1}(U)$, we have $f(a') = g(a') \in U$. Also, because $a' \in h^{-1}(V)$, we have $f(a') = h(a') \in V$. So $f(a') \in U \cap V$, contradicting the assumption that $U \cap V$ was empty.

So, g and h must be equal at each point; that is, g is uniquely determined by f.

Problem 2, p118:

Let A_α be a subspace of X_α, for each $\alpha \in J$.

Let $\prod X_\alpha$ be given the box topology. Say that $A = \prod A_\alpha$ given the box topology, and $A' = \prod A_\alpha$ given the subspace topology on $\prod X_\alpha$.

Let U be a basic open set ("basic" being "an element of the basis used to define the box topology") in A. Then $U = \prod U_\alpha$, with each U_α open in A_α. That is, each $U_\alpha = U'_\alpha \cap A_\alpha$ for some U'_α open in X_α. So,$\prod U'_\alpha$ is open in $\prod X_a l$. Also, $\prod U_\alpha = \prod U'_\alpha \cap \prod A_\alpha$ (this is basic set theory, and I will use this sort of logic freely in the below). So, U_α is open in A'.

Now, let U be a basic open set in A'. Then $U = U' \cap \prod A_\alpha$ for some U' open in $\prod X_\alpha$. That is, $U = \prod U'_\alpha \cap \prod A_\alpha$ for some U'_α each open in X_α. So, $U = \prod U'_\alpha \cap A_\alpha = U_\alpha$ with each $U_\alpha = U'_\alpha \cap A_\alpha$; as each U_α is open in A_α, we have that U is a product of open sets in A_α; that is, U is open in A.

So each basic open set of A is open in A' and vice versa; that is, $A = A'$; $\prod A_\alpha$ given the box topology is the same as $\prod A_\alpha$ given the subspace topology it inherits from $\prod X_\alpha$ given the box topology.

Now, let $\prod X_\alpha$ be given the product topology. Say that $A = \prod A_\alpha$ given the product topology, and $A' = \prod A_\alpha$ given the subspace topology on $\prod X_\alpha$.

Let U be a basic open set ("basic" being "an element of the basis used to define the product topology") in A. Then $U = \prod U_\alpha$, with each U_α open in A_α and $U_\alpha \neq A_\alpha$ for only finitely many U_α. That is, each $U_\alpha = U'_\alpha \cap A_\alpha$ for some U'_α open in X_α and $U_\alpha \neq A_\alpha$ for only finitely many U_α. So, $\prod U'_\alpha$ is open in $\prod X_a l$. Also, $\prod U_\alpha = \prod U'_\alpha \cap \prod A_\alpha$. So, U_α is open in A'.

Now, let U be a basic open set ("basic" being "an element of the basis used to define the product topology") in A'. Then $U = U' \cap \prod A_\alpha$ for some U' open in $\prod X_\alpha$ and $U_\alpha \neq A_\alpha$ for only finitely many U_α. That is, $U = \prod U'_\alpha \cap \prod A_\alpha$ for some U'_α each open in X_α and $U_\alpha \neq A_\alpha$ for only finitely many U_α. So, $U = \prod U'_\alpha \cap A_\alpha = U_\alpha$ with each $U_\alpha = U'_\alpha \cap A_\alpha$; as each U_α is open in A_α and $U_\alpha \neq A_\alpha$ for only finitely many U_α, we have that U is a product of open sets in A_α; that is, U is open in A.

So each basic open set of A is open in A' and vice versa. So A contains a basis for A', so that $A \supset A'$ and A' contains a basis for A, so that $A' \supset A$; that is, $A = A'$; $\prod A_\alpha$ given the product topology is the same as $\prod A_\alpha$ given the subspace topology it inherits from $\prod X_\alpha$ given the product topology.

Problem 3, p118:

Let each X_α be a Hausdorff space.

First, note that if $X = \prod X_\alpha$ is a Hausdorff space given the product topology, then $X' = \prod X_\alpha$ is a Hausdorff space given the box topology; if $x \neq y$, U open in X is a neighborhood of x, V open in X is a neighborhood of y, and $U \cap V = \emptyset$, then U is open in X' and V is open in X' (as the product topology is coarser than the box topology; this is an offhand remark made after Theorem 19.1, and is taken as "clear"). That is, we have U a neighborhood of x, V a neighborhood of Y, and $U \cap V = \emptyset$; X' is Hausdorff if X is.

Next, X (as above, this is $\prod X_\alpha$ given the product topology) is a Hausdorff space if each X_α is; let $x, y \in X$ with $x \neq y$. Then there is α' such that $\pi_{\alpha'}(x) \neq \pi_{\alpha'}(y)$ (else, $\pi_\alpha(x) = \pi_\alpha(y)$ for all α, so that $x(\alpha) = y(\alpha)$ for all α, so that $x = y$...and we're assuming $x \neq y$). So, because $X_{\alpha'}$ is Hausdorff, there are open (in $X_{\alpha'}$) sets $U_{\alpha'}$ and $V_{\alpha'}$ with $\pi_{\alpha'}(x) \in U_{\alpha'}$ and $\pi_{\alpha'}(y) \in V_{\alpha'}$ and $U_{\alpha'} \cap V_{\alpha'} = \emptyset$.

Now, $U = \prod\limits_{\alpha \neq \alpha'} X_\alpha \times U_{\alpha'}$ and $V = \prod\limits_{\alpha \neq \alpha'} X_\alpha \times V_{\alpha'}$ are open in the product topology on $X = \prod X_\alpha$, as they are products of open sets (with only finitely many of those open sets not equal to X_α). Also, $U \cap V = \prod\limits_{\alpha \neq \alpha'} (X_\alpha \cap X_\alpha) \times (U_{\alpha'} \cap V_{\alpha'}) = \prod\limits_{\alpha \neq \alpha'} (X_\alpha) \times (\emptyset) = \emptyset$. Also, U contains x and V contains y; this is clear.

So, if each X_α is a Hausdorff space, then X given the product topology is a Hausdorff space. From the above discussion, this means that X given the box topology is a Hausdorff space, too.

Problem 6, p118:

Let $\langle x_n \rangle$ be a sequence of points of the product space $\prod X_\alpha$.

Let $\langle x_n \rangle \to x$. Fix β. Consider the sequence $\langle \pi_\beta(x_n) \rangle$; this converges to $\pi_\beta(x)$; let U_β be a neighborhood of $\pi_\beta(x)$; there is an N such that for all $n \geq N$, $x_n \in \prod X_\alpha \times U_\beta$; thus, $\pi_\beta(x_n) \in U$ (because $\pi_\beta(x_n) \in \pi_\beta(\prod X_\alpha U_\beta) = U_\beta$), for all $n \geq N$.

That is, for all open neighborhoods U of $\pi_\beta(x)$, there is an N such that for all $n \geq N$, $\pi_\beta(x_n) \in U$; $\langle \pi_b e(x_n) \rangle$ converges to $\pi_\beta(x)$, for each β.

Now, let $\langle \pi_\alpha(x_n) \rangle$ converge to $\pi_\alpha(x)$ for each α. Consider $\langle x_n \rangle$; we show that this converges to x.

Let $U = \prod U_\alpha$ be a basic open set containing x. Then for each α, $\pi_\alpha(U)$ is open in X_α, and also $\pi_\alpha^(U) \neq X_\alpha$ for only finitely many α (because $U = \prod U_\alpha$ with $U_\alpha \neq X_\alpha$ for only finitely many α, because U was a basic open set). So for each α, there is an N_α such that for all $n \geq N_\alpha$, $\pi_\alpha(x_n) \in \pi_\alpha(U)$. Note that for α with $U_\alpha = X_\alpha$, we can take $N_\alpha = 1$, because for all $n \geq 1$, $\pi_\alpha(x_n) \in X_\alpha$ because $\pi_\alpha(x_n) \in X_\alpha$ for all x_n.

Now, take $N = \max\limits_\alpha N_\alpha$; this exists because there are only finitely many N_α not equal to 1. Now, for all $n \geq N$, we have that $\pi_\alpha(x_n) \in \pi_\alpha(U)$ for all α. That is, for all $n \geq N$, $x_n \in U$.

That is, for all basic open neighborhoods U of x, there is an N such that for all $n \geq N$, $x_n \in U$. Now, each neighborhood, U, of x contains a basic open neighborhood of x (by definition). That is, for all open neighborhoods U of x, there is an N such that for all $n \geq N$, $x_n \in U$. That is, $\langle x_n \rangle \to x$.

So $\langle x_n \rangle \to x$ if and only if $\langle \pi_\alpha(x_n) \rangle$ converges to $\pi_\alpha(x)$ for each α.

This is not true in the box topology; consider the space $A = \{0, 1\}$ given the discrete topology. Consider $\prod_{n=1}^{\infty} A$ given the box topology. Consider the sequence $\langle x_m \rangle$ given by $(1, 1, 1, 1, 1 \ldots)$, $(0, 1, 1, 1, 1 \ldots)$, $(0, 0, 1, 1, 1 \ldots)$ $\ldots$. For each $n \in \mathbb{N}$, $\pi_n(x_m)$ is eventually zero, and thus converges to zero. Yet, for no $m \in \mathbb{N}$ is x_m in the open set $\{0\} \times \{0\} \times \{0\} \ldots$ (an open set containing only the point $(0, 0, 0, 0, 0, \ldots)$).

Problem 7, p118:

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero.

The closure of $\mathbb{R}^\infty$ in the box topology is $\mathbb{R}^\infty$; consider the set $A = \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Then A is open; let $x \in A$. Then x is not eventually zero; that is, for all N there is an $n \geq N$ with $\pi_n(x) \neq 0$. So, there is a subsequence $\pi_{n_k}(x)$ with $\pi_{n_k}(x) \neq 0$ for all n_k (we know this from introductory analysis courses.). Define $A_n = \mathbb{R}$ if $n \neq n_k$ for any k. Define $A_n = (0, x + 1)$ if $n = n_k$ for some k and $\pi_{n_k}(x) > 0$. Define $A_n = (x - 1, 0)$ if $n = n_k$ for some k and $\pi_{n_k}(x) < 0$. Then the product $A' = \prod\limits_{n=1}^{\infty} A_n$ is an open set in the box topology (as it is the product of open sets) and A' contains x; $\pi_n(x) \in A_n$ for all $n \in \mathbb{N}$, so $x \in \prod A_n = A'$. Now, $A' \subset A$; it suffices to show that $a \in A'$ implies that $a \notin \mathbb{R}^\infty$. Yet, if $a \in A'$ and $a \in \mathbb{R}^\infty$, then a is eventually zero, so that $\pi_{n_k}(x)$ is eventually zero. But this means that $\pi_{n_k}(x) = 0$ for infinitely many n_k, when A_{n_k} excludes zero, which contradicts that $x \in A'$. So, $A' \subset A$.

So for all $x \in A$, there is an open (in the box topology) neighborhood of x completely contained in A; A is open, by Lemma C. That is, $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is open, so that $\mathbb{R}^\infty$ is closed.

So, $\mathbb{R}^\infty$ is closed in the box topology; so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$ (the closure of a closed set is itself, this is a throwaway comment on p95).

The closure of $\mathbb{R}^\infty$ in the product topology is $\mathbb{R}^\omega$; let $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$. Let U be a basic open neighborhood of x. Then $U = \prod_{n=1}^\infty U_n$ with $U_n \neq \mathbb{R}$ for only finitely many n. That is, there is an N with $U_n = \mathbb{R}$ for all $n \geq N$. So, the point $(x_1, x_2, \ldots x_N, 0, 0, 0 \ldots) \in \mathbb{R}^\infty$ (by definition) and $(x_1, x_2 \ldots x_N, 0, 0, 0 \ldots) \in U$, because So, every basic neighborhood of $x \in \mathbb{R}^\omega$ intersects $\mathbb{R}^\infty$. So every neighborhood of $x \in \mathbb{R}^\omega$ intersects $\mathbb{R}^\infty$. So $x \in \overline{\mathbb{R}^\infty}$, by theorem 17.5. So $\mathbb{R}^\omega \subset \overline{\mathbb{R}^\infty}$. Because $\mathbb{R}^\infty \subset \mathbb{R}^\omega$, (by definition), we have that $\overline{\mathbb{R}^\infty} \subset \overline{\mathbb{R}^\omega}$ (this is Lemma C). Now, $\overline{\mathbb{R}^\omega} = \mathbb{R}^\omega$, because $\mathbb{R}^\omega$ is the entire space (and thus, is closed) (closure of a closed set is itself, this is a throwaway comment on p95). This means that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$ in the product topology.

Problem 3b, p126:

I've burned an hour and a half on this problem and made zero progress. This problem's not getting done in time.

Problem 4b, p126:

Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.

Consider $\langle w_n \rangle$ as described in the text. This sequence converges to 0 in the product topology: let U be a basic neighborhood of 0. Then $U = \prod U_n$ with U_n each containing 0 and $U_n = \mathbb{R}$ for all $n \geq N$, for some N. So for all $n \geq N$, we have that $w_n \in U$;

That is, for every basic neighborhood U of 0, there is an N such that for all $n \geq N$, $w_n \in U$. So for all neighborhoods U of 0, there is an N such that for all $n \geq N$, $w_n \in U$ (this is clear via the definition of "basis" on p78). So, $\langle w_n \rangle \to 0$ in the product topology.

Yet, $\langle w_n \rangle$ fails to converge in the uniform topology; first, note that if $\langle w_n \rangle$ converged, it would converge to 0 and no other point; this is because the uniform and product topologies are both Hausdorff (problem 3;p 118,20.4, and the fact that topologies finer than Hausdorff spaces are Hausdorff), so sequences only converge to one point (17.10). Yet if $\langle w_n \rangle$ converged to a point other than 0 in the uniform topology, it would converge to a point other than 0 in the product topology; if $\langle w_n \rangle \to x \neq 0$ in the uniform topology, then for all open neighborhoods of the uniform topology, U, of x, there is an N such that for all $n \geq N$, $w_n \in U$. Because all open neighborhoods of the product topology are open in the uniform topology, we have that this means that for all open neighborhoods of the product topology, U, of x, there is an N such that for all $n \geq N$, $w_n \in U$, so that $\langle w_n \rangle \to x \neq 0$ in the product topology.

Now, $\langle w_n \rangle$ fails to converge to 0 in the uniform topology; define $U = \prod_{n=1}^{\infty}(-1,1)$; this is the ball of radius 1 about 0, and is thus open in the uniform topology. Yet, for all $n \geq 2$, $w_n \notin U$. So $\langle w_n \rangle \not\to 0$.

This means that $\langle w_n \rangle$ fails to converge in the box topology; because $\langle w_n \rangle$ fails to converge in the uniform topology, the box topology is finer than the box topology (Theorem 20.4), and we have that there is an open neighborhood, U, of 0 in the uniform topology that for all $N \in \mathbb{N}$ there is an $n \geq N$ with $w_n \notin U$, so that we get that there is an open neighborhood, U, of 0 in the box topology so that for all $N \in \mathbb{N}$ there is an $n \geq N$ with $w_n \notin U$, so that $\langle w_n \rangle$ doesn't converge to 0. Yet, 0 is the only point that $\langle w_n \rangle$ could converge to, by the same reasoning as last time.

So $\langle w_n \rangle$ does not converge in the box topology.

Consider $\langle x_n \rangle$ as described in the text.

Now, $\langle x_n \rangle \to 0$ in the uniform topology; let U be a basic neighborhood of 0. Then $U = \prod\limits_{n=1}^{\infty}(-a,a)$ for some $a \in \mathbb{R}$. By the archimedean property, we have that there is an N such that for all $n \geq N$, $1/n < a$. That is, for all $n \geq N$, $\pi_m(x_n) \in (-a,a)$ for all m. So for all $n \geq N$, $x_n \in U$. So we have that for all basic neighborhoods, U, of 0, we have that for some N, for all $n \geq N$, $x_n \in U$. So we have that for all open neighborhoods, U, of 0, we have that for some N, for all $n \geq N$, $x_n \in U$. That is, $\langle x_n \rangle \to 0$.

By the logic above, this means that $\langle x_n \rangle \to 0$ in the product topology as well.

However, $\langle x_n \rangle \not\to 0$ in the box topology: Let $U = \prod\limits_{n=1}^{\infty}(-1/2^n, 1/2^n)$. Then for all $n \in \mathbb{N}$, $x_n \notin U$; this is because $\pi_n(x_n) = 1/n$, and $1/n > 1/2^n$ for all $n \geq 1$ (this is somewhat obvious analysis). So $\langle x_n \rangle$ doesn't converge to 0 in the box topology; by the logic above, this means that $\langle x_n \rangle$ doesn't converge in the box topology.

Consider $\langle y_n \rangle$ as described in the text.

Now, $\langle y_n \rangle \to 0$ in the uniform topology; let U be a basic neighborhood of 0. Then $U = \prod\limits_{n=1}^{\infty}(-a,a)$ for some $a \in \mathbb{R}$. By the archimedean property, we have that there is an N such that for all $n \geq N$, $1/n < a$. That is, for all $n \geq N$, $\pi_m(y_n) \in (-a, a)$ for all m. So for all $n \geq N$, $y_n \in U$. So we have that for all basic neighborhoods, U, of 0, we have that for some N, for all $n \geq N$, $y_n \in U$. So we have that for all open neighborhoods, U, of 0, we have that for some N, for all $n \geq N$, $y_n \in U$. That is, $\langle y_n \rangle \to 0$.

By the logic above, this means that $\langle y_n \rangle \to 0$ in the product topology as well.

However, $\langle y_n \rangle \not\to 0$ in the box topology: Let $U = \prod_{n=1}^{\infty}(-1/2^n, 1/2^n)$. Then for all $n \in \mathbb{N}$, $y_n \notin U$; this is because $\pi_n(y_n) = 1/n$, and $1/n > 1/2^n$ for all $n \geq 1$ (this is somewhat obvious analysis). So $\langle y_n \rangle$ doesn't converge to 0 in the box topology; by the logic above, this means that $\langle y_n \rangle$ doesn't converge in the box topology.

Consider $\langle z_n \rangle$ as described in the text. This sequence converges to 0 in the box topology; let U be a basic open neighborhood of 0 in the box topology. Then $U = \prod U_n$ with each U_n an open neighborhood containing 0. Consider U_1 and U_2; each contains a basic neighborhood $\mathbb{R}_{\epsilon_1}(0)$ and $\mathbb{R}_{\epsilon_2}(0)$ respectively, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ (by the definition on page 78, example 2 on p120, and the definition of the metric topology). Now, by the archimedean principle, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $1/n < \epsilon_1$ and $1/n < \epsilon_2$, so that $\pi_1(z_n) \in U_1$ and $\pi_2(z_n) \in U_2$ for all $n \geq N$. So $z_n \in U$ for all $n \geq N$, because $z_n = (1/n, 1/n, 0, 0, 0 \ldots)$ so that $\pi_m(z_n) \in U_m$ for all $m > 2$ because U_m is a neighborhood of 0 (as U was a neighborhood of 0).

So for any basic open neighborhood, U, of 0, there is an N such that for all $n \geq N$, $z_n \in U$. So for any open neighborhood U of 0, there is an N such that for all $n \geq N$, $z_n \in U$, (by the definition on p78). So, $\langle z_n \rangle \to 0$ in the box topology.

Thus, $\langle z_n \rangle \to 0$ in the product and uniform topologies; by theorem 20.4, both of these topologies are coarser than the box topology. For any open neighborhood, U, of 0 in the box topology, there is an N such that for all $n \geq N$, $z_n \in U$; so for all open neighborhoods of 0 in the product and uniform topologies, there is an N such that for all $n \geq N$, $z_n \in U$; so $\langle z_n \rangle \to 0$ in the product and uniform topologies.

Problem A:

Let X be a metric space, and let A be a countable subset of X with $\overline{A} = X$.

Consider the collection $\mathcal{C} = \{X_r(x) : x \in A, r \in \mathbb{Q}\}$. Then $\mathcal{C}$ is countable;

it's a countable union of countable sets.

Next, note that $\bigcup_{C \in \mathcal{C}} C = X$; it is clear that $\bigcup_{C \in \mathcal{C}} C \subset X$, as the left hand side is a union of subsets of X. Now, let $x \in X$. Then $x \in \overline{A}$. Consider $X_1(x)$; then there is $a \in A$ with $a \in X_1(x)$, because every open neighborhood of x intersects A by theorem 17.5. Now, note that because $d_X(a, x) < 1$, we have $x \in X_1(a)$. Because $1 \in \mathbb{Q}$ and $a \in A$, we have that $x \in C$ for some $C \in \mathcal{C}$. That is, $x \in \bigcup_{C \in \mathcal{C}} C$. So, $X \subset \bigcup_{C \in \mathcal{C}} C$. So $X = \bigcup_{C \in \mathcal{C}} C$.

Now, let $x \in C_1 \cap C_2$ for some $C_1, C_2 \in \mathcal{C}$. Then consider the set $C_1 \cap C_2$; this set is open, as it is the intersection of two open sets (each set in $\mathcal{C}$ is open, as each set in $\mathcal{C}$ is an open ball). So, there is an open ball $X_r(x) \subset C_1 \cap C_2$. Choose $q \in \mathbb{Q}$ with $0 < q < r/2$ (we can do this by Archimedean property). Consider $X_q(x)$. Then there is an $a \in A$ with $A \in X_q(x)$, by theorem 17.5 as above. Now, as above, we have that $x \in X_q(a)$. Moreover, $X_q(a) \subset C_1 \cap C_2$; if $b \in X_q(a)$, then $d(a, b) < q$, and we know that $d(x, a) < q$, so $d(b, x) \leq 2q < r$ by triangle inequality, so that $b \in X_r(x) \subset C_1 \cap C_2$. Now, note that $X_q(a) \in \mathcal{C}$, as q is rational and $a \in A$.

So, for all $x \in C_1 \cap C_2$, there is a $C_3 \in \mathcal{C}$ with $x \in C_3 \subset C_1 \cap C_2$.

So $\bigcup_{C \in \mathcal{C}} C = X$ and if $x \in C_1 \cap C_2$ for any C_1, $C_2 \in \mathcal{C}$, there is a $C_3 \in \mathcal{C}$ with $x \in C_3 \subset C_1 \cap C_2$; so $\mathcal{C}$ is a basis for X.

So $\mathcal{C}$ is a countable basis for X; a metric space, X, has a countable basis if there is a countable subset A with $\overline{A} = X$.

Problem B:

Let Y be an ordered set, (a, b) and (c, d) be disjoint open intervals, and let there exist $x \in (a, b)$ and $y \in (c, d)$ with $x < y$.

Let there exist x', y' with $x' \in (a, b)$, $y' \in (c, d)$, and $x' \geq y'$. It is clear that $x' \neq y'$, else (a, b) and (c, d) were not disjoint. So, $x' > y'$. Now, $y' > c$ and $x' < b$, as $x' \in (a, b)$ and $y' \in (c, d)$. So, we have that $c < y' < x' < b$. That is, $c < b$. So, $(a, b) \cap (c, d) = (c, b)$, which is nonempty (as y' and x' are in (c, b). This contradicts our assumption that this set was empty.

So, if Y is an ordered set, (a, b) and (c, d) are disjoint open intervals, and there exist $x \in (a, b)$ and $y \in (c, d)$ with $x < y$, then $x' < y'$ for all $x' \in (a, b)$, $y' \in (c, d)$.

Problem C, part i:

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $f(\{1\}) = \{1\}$, so that $f^{-1}(f(\{1\})) = \{-1, 1\}$, so that $f^{-1}(f(\{1\})) \neq \{1\}$.

That is, $f^{-1}(f(A)) = A$ isn't always true.

Problem C, part ii:

Define an equivalence relation on S by $s\ s'$ if and only if $f(s) = f(s')$.

Then $f^{-1}(f(A)) = A$ if and only if $a\ a'$ and $a \in A$ implies that $a' \in A$.

First, we know that $f^{-1}(f(A)) \supset A$, from elementary set theory.

Now, let $f^{-1}(f(A)) = A$. Let $a \in A$, and let $a'\ a$. Then $f(a') = f(a)$. So $a' \in f^{-1}(f(A))$. So $a' \in A$. That is, $a'\ a$ and $a \in A$ implies that $a' \in A$.

Next, let $a'\ a$ and $a \in A$ imply that $a'\ inA$. Let $a \in f^{-1}(f(a))$. Then there is an $a' \in A$ with $f(a) = f(a')$. So $a\ a'$, so $a \in A$. That is, $f^{-1}(f(A)) \subset A$.

So $f^{-1}(f(A)) \subset A$ if and only if $a'\ a$ and $a \in A$ implies that $a' \in A$. So $f^{-1}(f(A)) = A$ if and only if $a'\ a$ and $a \in A$ implies that $a' \in A$.