Problem 1 (23 in book):

Let S denote the square in $\mathbb{R}\times(0,\infty)$ with corners $(0,1),(1,2),(0,3),(-1,2)$. Define

$$f(x,t)=\begin{cases} -1 & \text{for } (x,t)\in S\cap\{t>x+2\} \\ -1 & \text{for } (x,t)\in S\cap\{t<x+2\} \\ 0 & \text{else} \end{cases}$$

Let u solve

$$\begin{cases} u_{tt}-u_{xx}=f & \text{when } t>0 \\ u=0, u_t=0 & \text{when } t=0 \end{cases}$$

Consider u when $t>3$. Then we have

$$u(x,t)=\int_0^t u(x,t;s)ds$$

where $u(x,t;s)=\frac{1}{2}\int_{x-t}^{x+t}f(y,s)dy$ (we get this by Duhamel's principle and the solution of the wave equation in one dimension). In other words,

$$u(x,t)=\int_0^t \frac{1}{2}\int_{x-t}^{x+t} f(y,s)dyds$$

For the sake of sanity, let us write f as $-\chi_A+\chi_B$, with $A=S\cap\{t>x+2\}$ and $B=S\cap\{t<x+2\}$. Then

$$\begin{aligned} u(x,t) &= \int_0^t \frac{1}{2}\int_{x-t}^{x+t} \chi_B-\chi_A dyds \\ &= \frac{1}{2}\left[\int_0^t\int_{x-t}^{x+t}\chi_B dyds - \int_0^t\int_{x-t}^{x+t}\chi_A dyds\right] \end{aligned}$$

Now, if $x-t>1$ or $x+t<-1$, both of those integrals vanish. That is, for fixed $t>3$, $u(x,t)=0$ if $x>1+t$ or $x<-1-t$. Moreover, if $1/2-t<$

$x < -1/2 + t$, then $\int_0^t \int_{x-t}^{x+t} \chi_B dyds = 1$. Similarly, if $t - 1/2 > x > 1/2 - t$, then $\int_0^t \int_{x-t}^{x+t} \chi_A dyds = 1$.

At this point, we can see that $u(x,t)$ vanishes except possibly when $x \in [1-t, 1/2-t] \cup [t-1/2, t+1]$.

Problem 2 (24 in book):

Let u solve the intial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{when } t > 0 \\ u = g, u_t = h & \text{when } t = 0 \end{cases}$$

Let g, h have compact support. Consider $k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)dx$ and $p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t)dx$.

Part a:

Consider $k(t) + p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t) + u_t^2(x,t)dx$.

We know that $u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)dy$

So, we have:

$$u_x(x,t) = \frac{g'(x+t) + g'(x-t)}{2} + \left[\frac{1}{2} \int_{x-t}^{x+t} h(y)dy \right]_x$$
$$u_t(x,t) = \frac{g'(x+t) - g'(x-t)}{2} + \left[\frac{1}{2} \int_{x-t}^{x+t} h(y)dy \right]_t$$

Part b:

Problem 3 (on page):