

Problem 1:

Let G be a finite abelian group, with $n \in \mathbb{N}$ and $n \mid |G|$.

We know that for each $n \in \mathbb{N}$, n has a unique prime factorization; that is, $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ for some p_1, p_2, \dots, p_k each prime, and a_1, a_2, \dots, a_k each positive and nonzero.

Proceed as follows:

For each p_i , there is an element with order p_i in G :

Thus, there is a subgroup, H_1 , with order p_1 in G . This subgroup is normal, because G is abelian.

Consider the new group $G_1 = G/H_1$, along with $n_1 = n/p_1$. Note that $|G_1| = |G|/p_1$; this follows from the theorem that says $|G/H| = |G|/|H|$ if H is normal.

We know that G_1 is abelian:

So by similar logic as above, if $a_1 > 1$, then there is a subgroup of order p_1 in G_1 . Otherwise, we know that there is a subgroup of order p_2 in G_1 .

Either way, there is a subgroup, H'_2 , with order p_1 (or p_2) in G_1 , which is normal.

Consider the group $G_2 = G_1/H'_2$. Note that $G_2 \cong G/H_2$, by the third isomorphism theorem. Moreover, $|G_2| = |G|/p_1^2$ (or $|G_2| = |G|/p_1 p_2$).

We can proceed in the above manner for a_i times for each p_i . We end up with a group, $G_{a_1+a_2+\dots+a_k}$.

Consider $G_{a_1+a_2+\dots+a_k}$; it has order $|G|/n$. It is isomorphic to $G/H_{a_1+a_2+\dots+a_k}$ for some $H \trianglelefteq G$. This means that $|H| = n$ (because $|G/H| = |G|/|H|$...thus, $|H| = |G|/|G/H|$, or in this case, $|H| = |G|/(|G|/n) = n$.)

So G has a normal subgroup of order n if G is a finite abelian group with $n \mid |G|$.

Problem 2:

Let $H < G$ with $[G : H]$ finite.

Problem 3:

Let G be a group acting transitively on a finite set, S , with $|S| > 1$.

Now, the action has only one orbit; for all $x \in S$, $\bar{x} = S$. In other words, for every $x, y \in S$ there is a $g \in G$ such that $g.x = y$.

Assume that for all $g \in G$, there is an $x \in S$ such that $g.x = x$.

Let x, y be distinct points of S .

There is a $g \in G$ such that $g.x = y$.

There is a $z \in S$ such that $g.z = z$.

Consider the set $\{g^n.x : n \in \mathbb{N}\}$.

Either g has finite order or g has infinite order. If g has finite order, then

So g has infinite order. But

Problem 4:

Let G be a group such that $G/Z(G)$ is cyclic.

Then $G/Z(G) = \langle \bar{a} \rangle$ for some $a \in G$.

Note that this fails if $G/Z(G)$ is only abelian:

Consider $D_8 = \langle r, s \rangle$. We note that the center of D_8 is $\{e, r^2\}$;

Also, $D_8 / \langle r^2 \rangle$ is abelian:

But we know from an earlier homework that D_8 is not abelian. So in general, $G/Z(G)$ being abelian does not imply that G is abelian.

Problem 5:

Let p be prime, and let G be a group of order p^2 .

We know that G has an element of order p ; any element has order 1, p , or p^2 . There are at least two elements in G (because 1 is not prime...). Pick an element other than e : call it a . We know that a has order p or p^2 . If it has order p^2 , then consider a^p . We know that $(a^p)^p = a^{p^2}$. So a^p is an element of order p .

The upshot is that there is an element, x , of order p in G . So there is a subgroup, $H = \langle x \rangle$, of order p in G . Moreover, H is cyclic (I think we proved this in class...any group of order p is cyclic. If we haven't proved it, then

Also, H is normal in G :

Now, consider G/H ; this is a group of order p , it is cyclic.

Problem 6:

Let p be prime, and let G be a group of order p^n for some $n \in \mathbb{N}$. Let $H \leq G$ with $H \neq \{e\}$.