Problem 7:

Let $u \in C^0(\Omega)$.

Let u be subharmonic in Ω. Then u satisfies the mean value properties for any ball compactly contained in Ω; it is clear that this implies that u satisfies the mean value properties locally.

Now, let u satisfy the mean value properties locally. Fix $y \in \Omega$. Without loss of generality, we can set $u(y) = 0$. Then there is a $\delta > 0$ such that $u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} uds$ for all $R \leq \delta$.

So, on $B_R(y)$, we have:

$$\begin{aligned}
u(y) &\leq \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} uds \\
0 &\leq \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} uds \\
0 &\leq \int\limits_{\partial B_R(y)} uds
\end{aligned}$$

As the above holds for all $R < \delta$, we get that $\int\limits_{B_R(y)} udx \geq 0$ for all $R < \delta$. Moreover, we get that $\int\limits_{B_R(y)} udx \geq 0$ is increasing as a function of R.

That is,

$$\begin{aligned}
\frac{d}{dR} \int\limits_{B_R(y)} udx &\geq 0 \\
\int\limits_{\partial B_R(y)} \frac{\partial u}{\partial \nu} dx &\geq 0 \\
\int\limits_{B_R(y)} \Delta u dx &\geq 0
\end{aligned}$$

So because the above holds for all $R < \delta$, we have that $\Delta u(y) \geq 0$. That is, u is subharmonic at y.

Problem 9:

Let $u \in C^2(\Omega)$.

First, $\Delta u \geq 0$ in Ω, if and only if u is subharmonic in Ω, by definition.

Next: let u be subharmonic in Ω. Then $\Delta u \geq 0$ in Ω. Let $\phi \geq 0$ be a function in $C_c^2(\Omega)$. So

$$\begin{aligned}\int_\Omega u\Delta\phi dx &= \int_\Omega \phi\Delta u dx + \int_{\partial\Omega} u\partial_\nu\phi - \phi\partial_\nu u dS \\ &= \int_\Omega \phi\Delta u dx \text{ (Note: } \phi \text{ vanishes on } \partial\Omega\text{, as } \phi \text{ has compact support.)} \\ &\geq 0\end{aligned}$$

That is, u is weakly subharmonic if u is subharmonic.

Now, let u be weakly subharmonic. Then for any $\phi \geq 0$ with $\phi \in C_c^2(\Omega)$, we have

$$\begin{aligned}\int_\Omega u\Delta\phi dx &= \int_\Omega \phi\Delta u dx + \int_{\partial\Omega} u\partial_\nu\phi - \phi\partial_\nu u dS \\ &= \int_\Omega \phi\Delta u dx \\ &\geq 0\end{aligned}$$

Thus, $\Delta u \geq 0$ (Else, there's a point y with $\Delta u(y) < 0$, so there's a neighborhood around y with $\Delta u(y) < 0$, so picking $\phi > 0$ on that neighborhood and $\phi = 0$ outside that neighborhood will yield a contradiction.). So, u is subharmonic.

That is, u is subharmonic if u is weakly subharmonic.

That is, u is subharmonic if and only if u is weakly subharmonic.

Thus, the three conditions given are equivalent.