Problem 1:

Part a:

Consider the set $A = \{z \in \mathbb{C} : e^z = 0\}$.

If $z = a + bi \in A$ (with $a, b \in \mathbb{R}$), then $e^z = 0$. So $e^a e^{bi} = 0$.

For $a \in \mathbb{R}$, $e^a \neq 0$. So this means that $e^{bi} = 0$. But this never happens either, because $\left|e^{bi}\right| = 1$ for all $b \in \mathbb{R}$ (because $\left|e^{bi}\right| = |\cos(b) + i\sin(b)| = \sqrt{\cos^2(b) + \sin^2(b)} = 1$).

So we have a contradiction. So $A = \emptyset$.

Part b:

Consider the set $B = \{z \in \mathbb{C} : e^z = 1\}$.

If $z = a + bi \in B$ (with $a, b \in \mathbb{R}$), then $e^z = 1$. So $e^z = e^a e^{bi} = 1$.

This means that $\left|e^a e^{bi}\right| = |e^a| \left|e^{bi}\right| = 1$. But $\left|e^{bi}\right| = 1$ for all $b \in \mathbb{R}$. So, $|e^a| = 1$, so $e^a = 1$, so $a = 0$.

So $z = bi$ for some $b \in \mathbb{R}$.

By applying the equivalence of polar and trigonometric forms, this means that $e^{ib} = \cos(b) + isin(b) = 1$. So, $\cos(b) = 1$ and $\sin(b) = 0$. This means that $b = 2k\pi$ for some $k \in \mathbb{Z}$.

So $B \subset \{2k\pi i \in \mathbb{C} : k \in \mathbb{Z}\}$.

Now, if $z = 2k\pi i$ for some $k \in \mathbb{Z}$, then $e^z = \cos(2k\pi) + i\sin(2k\pi) = 1$. So $z \in B$.

So $B = \{2k\pi i \in \mathbb{C} : k \in \mathbb{Z}\}$.

Part c:

Consider the set $C = \{z \in \mathbb{C} : \sin(z) = 0\}$.

Let $z = a + bi \in C$. Then $sin(z) = 0$. So $\frac{e^{iz} - e^{-iz}}{2i} = 0$, so that $e^{iz} = e^{-iz}$.

In other words, $e^{-b}e^{ai} = e^b e^{-ai}$. So, $e^{2b} = e^{2ai}$. Because $|e^{2ai}| = 1$, this means that $e^{2b} = 1$. So, $b = 0$, and $e^{2ai} = 1$. So $2ai = 2k\pi i$ for some $k \in \mathbb{Z}$.

So, $z = k\pi$ for some $k \in \mathbb{Z}$. So $C \subset \{k\pi \in \mathbb{C} : k \in \mathbb{Z}\}$.

Now, if $z = k\pi$ for some $k \in \mathbb{Z}$, then $\sin(z) = 0$, and this is very well known. So $z \in C$.

So $C = \{k\pi \in \mathbb{C} : k \in \mathbb{Z}\}$.

Problem 2:

Let $\Omega \subset \mathbb{C}$ be an open connected set, and $f \in C(\Omega)$ be such that for all closed, piecewise continuous curves, Γ, with $\Gamma \subset \Omega$, $\int_\Gamma f(z)dz = 0$.

Pick $z \in \Omega$. Let $p \in \Omega$, and γ be a curve from p to z. We showed in class that $\int_\gamma f(\xi)d\xi$ is independent of γ; that is, $\int_\gamma f(\xi)d\xi$ only depends on p and z.

So, we can define $g(z) = \int_\gamma f$, where γ is a curve from a chosen fixed point,p, to z.

Now, fix $z_0 \in \Omega$. It is clear that $\lim_{z \to z_0} \frac{\int_{z_0}^{z} f(w)dw}{z - z_0} = f(z_0)$, because $\frac{\int_{z_0}^{z} f(w)dw}{z - z_0}$ is the average value of $f(w)$ on the line segment. Now, because $\frac{g(z)-g(z_0)}{z-z_0} = \frac{\int_{z_0}^{z} f(w)dw}{z-z_0}$, this means that $g'(z_0) = \lim_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{\int_{z_0}^{z} f(w)dw}{z-z_0} = f(z_0)$

That is, $g'(z_0) = f(z_0)$ for all $z_0 \in \Omega$; g is a primitive of f.

Problem 3:

(I must admit to having read this in Complex Made Simple by David C. Ullrich prior to the assignment of this problem.)

Let $f \in \mathcal{O}(D_1(0))$, with $f = \sum_{n=0}^{\infty} a_n z^n$.

Then consider $\int_0^{2\pi} |f(re^{it})|^2 \, dt$.

Problem 4:

Let $\phi, \psi : [a, b] \to \mathbb{R}$ be log-convex.

Then $\ln(\phi)$ and $\ln(\psi)$ are convex.

So for all $x, y \in [a, b]$ with $x \leq y$ and for all $t \in [0, 1]$, $\ln(\psi(tx+(1-t)y)) \leq t \ln \psi(x) + (1 - t) \ln \psi(y)$ and $\ln(\phi(tx + (1 - t)y)) \leq t \ln \phi(x) + (1 - t) \ln \phi(y)$.

Problem 5:

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $f \in \mathcal{O}(\Omega)$, $f(z) \neq 0$ for any $z \in \Omega$.

We showed in class that $g(z) = \int\limits_p^z \frac{f'(w)}{f(w)} dw + \lambda$ with p chosen arbitrarily in Ω and $e^\lambda = f(p)$ satisfies $f = e^g$, and that $g \in \mathcal{O}(\Omega)$.

Now, let $h \in \mathcal{O}(\Omega)$ be such that $f = e^h$.

Then $\frac{f}{f} = \frac{e^g}{e^h}$, so that $1 = e^{g-h}$. Thus, by problem 1, we have that $g - h = 2k\pi i$ for some $k \in \mathbb{Z}$.

That is, any two functions, g and h, satisfying $e^g = e^h = f$ differ only by $2k\pi i$ for some $k \in \mathbb{Z}$.

Problem 6:

(I must admit to having read this in Complex Made Simple by David C. Ullrich prior to the assignment of this problem.)

Let $\phi \in \mathcal{O}(D_1(0))$. Suppose that ϕ takes its maximum at 0.

Problem 7:

Suppose that $\phi \in \mathcal{O}(\Omega)$ with Ω a domain, and that there is a $c \in \Omega$ such that $|(|\,\phi(c)) = max(|(|\,\phi))$.

Then ϕ is constant on any ball centered at c, by problem 6.

Now, Ω is path connected (it is a domain).

Let $z \in \Omega$, and let γ be a path from z to c with $\gamma \subset \Omega$. We can cover the path with open balls, as Ω is open. ϕ is constant on each of these open balls, as ϕ takes its maximum (or minimum) on these open balls. So ϕ is constant along the path: $\phi(z) = \phi(c)$.

So for all $z \in \Omega$, $\phi(z) = \phi(c)$. So ϕ is constant.