Problem 1:

Let $f \in \mathcal{O}(\mathbb{C})$ be such that $f(1/\nu) = (-1)^\nu/\nu$ for all $\nu \in \mathbb{N}$.

Consider the sequences $\langle a_n \rangle = \frac{1}{2n}$ and $\langle b_n \rangle = \frac{1}{2n+1}$. We know $f(a_n)$ and $f(b_n)$, and both sequences converge to 0. So by the uniqueness theorem, f is uniquely determined by either one of these sequences. Yet, the holomorphic function $g(z) = z$ matches $f(a_n)$ at all points, and $h(z) = -z$ matches $f(b_n)$ at all points; this contradicts the uniqueness theorem.

Problem 2:

Consider $z^7 - 2z^5 + 6z^3 - z + 1$.

First, on $\partial D_1(0)$, $|z^7 - 2z^5 + 6z^3| \geq 3$ and $|-z+1| \leq 2$. So by Rouche's Theorem, $z^7 - 2z^5 + 6z^3$ and $z^7 - 2z^5 + 6z^3 - z + 1$ have the same number of zeroes on $D_1(0)$.

Now, $z^7 - 2z^5 + 6z^3$ has three zeroes (up to multiplicity) on $D_1(0)$; $z^7 - 2z^5 + 6z^3 = z^3(z^4 - 2z^2 + 6) = z^3(z^2 - 1 + i\sqrt{20})(z^2 - 1 - i\sqrt{20})$. The zeroes are 0 (Having multiplicity 3) and four other complex numbers whose values have absolute value greater than 1 (this is clear by inspection.)

So $z^7 - 2z^5 + 6z^3 - z + 1$ has three zeroes on $D_1(0)$.

Problem 3:

Let u be harmonic on $A_{r,R}$, the open annulus with inner radius r and outer radius R.

Consider u_z; this is holomorphic on the annulus. So u_z has a local primitive on the annulus.

So there is a g such that $g' - \log|z| = u_z$. So $(g + \overline{g} - u - c\log|z|)_z = 0$.

Define $v = g + \overline{g} - u - c\log|z|$. Then $v_x - iv_y = 0$. But v_x and v_y are both real. So $v_x = v_y = 0$. So v is constant, D, on the annulus.

So $0 = g + \overline{g} - u - c\log|z| - D$. That is, $u = g + \overline{g} - c\log|z| = 2\mathrm{Re}(g) - c\log|z|$, which is equivalent to the desired result.

Problem 4:

We proceed by induction. (I have a feeling that there's a proof directly from some deep theorem of Algebra, but I don't know any Algebra. :( )

Let Q be a polynomial of degree 2. Note that in this case, we have the desired result if $\sum_{z_j} \frac{1}{Q'(z_j)} = 0$, where $\{z_j\}$ is the set of zeroes of Q. Now, say that $Q(z) = \sum_{k=0}^{2} a_k z^k$, so that $Q'(z) = 2a_2 z + a_1$. Then we have:

$$\begin{aligned}
\sum_{z_j} \frac{1}{Q'(z_j)} &= \frac{1}{2a_2 z_1 + a_1} + \frac{1}{2a_2 z_2 + a_1} \\
&= \frac{(2a_2 z_1 + a_1) + (2a_2 z_2 + a_1)}{(2a_2 z_1 + a_1)(2a_2 z_2 + a_1)} \\
&= \frac{2(a_2(z_1 + z_2) + a_1)}{(2a_2 z_1 + a_1)(2a_2 z_2 + a_1)}
\end{aligned}$$

The numerator vanishes, because the zeroes of Q are given by

$$z_j = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$$

So that

$$a_2(z_1 + z_2) + a_1 = 0$$

Next, let it be that $\sum_{z_j} \frac{z_j^\alpha}{Q'(z_j)} = 0$ when the degree of Q is less than N (where $\{z_j\}$ is the set of zeroes of Q and with $\alpha \leq \deg(Q) - 2$, and where Q only has simple zeroes). Let Q be a polynomial of degree N with only simple zeroes. Then there is a polynomial q with $Q = q(z - z_N)$. Also, $Q' = q + q'(z - z_N)$ by the product rule. Also note that $q(z_j) = 0$ if z_j is any one of Q's zeroes except for z_N. So,

$$\begin{aligned}
\sum_{j=1}^{N} \frac{z_j^\alpha}{Q'(z_j)} &= \sum_{j=1}^{N} \frac{z_j^\alpha}{q(z_j) + (z_j - z_N)q'(z_j)} \\
&= \sum_{j=1}^{N-1} \frac{z_j^\alpha}{(z_j - z_N)q'(z_j)} + \frac{z_N^\alpha}{q(z_N)} \\
&= \sum_{j=1}^{N-1} \frac{(z_j - z_N)^\alpha + P(z_j)}{(z_j - z_N)q'(z_j)} + \frac{z_N^\alpha}{q(z_N)} \\
&= \sum_{j=1}^{N-1} \frac{(z_j - z_N)^{\alpha-1}}{q'(z_j)} + \sum_{j=1}^{N-1} \frac{P(z_j)}{(z_j - z_N)q'(z_j)} + \frac{z_N^\alpha}{q(z_N)} \\
&= \sum_{j=1}^{N-1} \frac{P(z_j)}{(z_j - z_N)q'(z_j)} + \frac{z_N^\alpha}{q(z_N)} \\
&= 0 \text{ (I'm not sure how the last step works out, but this is the idea.)}
\end{aligned}$$

Where $P(z)$ is some polynomial depending on z_N.

So, by induction, $\sum_{z_j} \frac{z_j^\alpha}{Q'(z_j)} = 0$ (where $\{z_j\}$ is the set of zeroes of Q and with $\alpha \leq \deg(Q) - 2$, and where Q only has simple zeroes), which is equivalent to our result.