I don't know if we covered the $\rho$ metric ($\rho(f,g) = \sup(|f(x) - g(x)| : x \in X)$, where $X$ is the domain of $f$ and $g$), but I'm using it because it's nice and I like it.

**Problem 1:**

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = x + 1/n$.

It is rather clear that this sequence of functions converges uniformly (to $x$).

However, the sequence of functions $\langle f_n^2 \rangle$ fails to converve uniformly:

For each $n \in \mathbb{N}$, $f_n^2(x) = x^2 + 2x/n + 1/n^2$. It is rather clear that $\langle f_n^2 \rangle$ converges pointwise to $x^2$. So if $\langle f_n^2 \rangle$ converges uniformly to something, it must converge uniformly to $x^2$. However, $\langle f_n^2 \rangle$ does not converge uniformly to $x^2$:

Let $\epsilon > 0$, and pick $n \in \mathbb{N}$. Consider $|f_n^2(x) - x^2| = |2x/n + 1/n^2|$. Pick $x = n\epsilon$. Then $|2x/n + 1/n^2| = 2\epsilon + 1/n^2 \geq \epsilon$.

So for all $\epsilon > 0$ and $n \in \mathbb{N}$ there is an $x \in \mathbb{R}$ such that $|f_n^2(x) - x^2| \geq \epsilon$; $f_n^2$ does not converge uniformly to $x^2$.

So $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$, but $\langle f_n^2 \rangle$ doesn't. This satisfies the problem.

**Problem 2:**

Longish note: After finishing this problem, I noticed that this follows immediately from a fragment of the proof of Arzela-Ascoli. I prefer this proof, as it is smoother, but it is important to note that such a thing is possible. Moreover, I used a general metric in this one, because it seems like if we had an appropriate extension of Arzela-Ascoli, then this proof could be extended to other types of metric spaces...Does such a theorem exist?

Proof starts here: Let $\langle f_n \rangle$ be an equicontinuous sequence of functions on a compact set, $K$, with $\langle f_n \rangle$ converging pointwise to some function, say $f$.

By the Arzela-Ascoli theorem, we know that $\langle f_n \rangle$ has some subsequence that uniformly converges to some function. We know that this function must be $f$: if a subsequence of functions converges uniformly to $f$, it converges pointwise to $f$. If a sequence of functions converges pointwise to a function, $f$, then all of its subsequences converge to $f$. So if a sequence of functions converges pointwise to $f$, then any subsequence of functions that converges uniformly to a function must converge uniformly to $f$.

Now, consider such a converging subsequence, $\langle f_{n_j} \rangle$.

Let $\epsilon > 0$. There is a $J \in \mathbb{N}$ such that for all $j \geq J$, $\rho(f_{n_j}, f) < \epsilon/4$.

In addition, by equicontinuity, there is a $\delta_1 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x, y) < \delta_1$ implies that $d(f_n(x), f_n(y)) < \epsilon/4$.

Moreover, we know that converging sequences of continuous functions converge to continuous functions. We also know that continuous functions on a compact domain are uniformly continuous. Thus, $f$ is uniformly continuous; there is a $\delta_2 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x, y) < \delta_2$ implies that $d(f(x), f(y)) < \epsilon/4$.

Define $\delta = \min(\delta_1, \delta_2)$, so that we have both of the lines for $\delta$.

We know that compact sets are totally bounded. (If this is not clear, consider a career in pastry making.)

So, let $F$ be a finite collection of points of $K$ such that for all $x \in K$, $d(x, y) < \delta$ for some $y \in F$.

Now, for each $y \in F$, there is an $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $d(f_n(y), f(y)) < \epsilon/4$.

Define $N = \max(N_y, n_J)$.

Now, for all $n \geq N$, and for all $x \in K$, we have, for some $y \in F$ (we pick $y$ with $d(x, y) < \delta$:

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_n(y)) + d(f_n(y), f_{n_j}(y)) + d(f_{n_j}(y), f(y)) + d(f(y), f(x)) \\ &\leq \epsilon \text{ (That's good enough.)} \end{aligned}$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in K$, $d(f_n(x), f(x)) < \epsilon$. That is, $f_n$ converges uniformly to $f$.

To summarize, if $\langle f_n \rangle$ is an equicontinuous sequence of functions on a compact set, $K$, with $\langle f_n \rangle$ converging pointwise, then $\langle f_n \rangle$ converges uniformly.

**Problem 3:**

Let $\langle f_n \rangle$ be a uniformly bounded sequence of functions that are Riemann-integrable on $[a, b]$. Set

$$F_n(x) = \int_a^x f_n(t)dt$$

Moreover, let $L$ be the absolute value of a lower bound for the $f_n$s and let $U$ be the absolute value of an upper bound for the $f_n$s. (I say absolute values here because I don't want to bother with them later.)

Now, the set of $F_n$s are equicontinuous:

Let $\epsilon > 0$. Define $\delta = \epsilon / \max(U, L)$. Then for all $n \in \mathbb{N}$, $x, y \in [a, b]$ with $|x - y| < \delta$ (WLOG, $x \leq y$), we have:

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left| \int_a^x f_n(t)dt - \int_a^y f_n(t)dt \right| \\ &= \left| \int_x^y f_n(t)dt \right| \text{ I exploit the absolute value here too.} \\ &\leq |(x - y)\max(U, L)| \text{ This is some sort of obvious property of integrals we shou} \\ &< \epsilon \end{aligned}$$

So for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$, $x, y \in [a, b]$, $|x - y| < \delta$ implies that $|F_n(x) - F_n(y)| < \epsilon$. That is, the set of $F_n$s are equicontinuous.

In addition, the $F_n$s are defined on $[a, b]$, which is a compact space. By Arzela-Ascoli, there is a subsequence $\langle F_{n_j} \rangle$ that converges uniformly on $[a, b]$.

**Problem 4:**

Let $\langle f_n \rangle$ be a sequence of increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$.

Some subsequence, $\langle f_{n_k} \rangle$, converges at all rational points, to some function $f$ on the rationals.

Now, define $f(x) = \sup(f(r) : r \leq x)$.

Then $f_{n_k}(x) \to f(x)$ at all points of continuity, $x$.

Now, there are countably many points of discontinuity of $f$; $f$ is a limit of increasing functions, it must also be increasing, and thus $f$ has countably many points of discontinuity.

Thus, there is a subsequence of $\langle f_{n_k} \rangle$ that converges to $f$ at every point of discontinuity of $f$;

So, this subsequence converges pointwise to $f$ at every point of $\mathbb{R}$, satisfying the problem.

**Problem 5:**

Let $\alpha$ be increasing on $[a, b]$, $g$ continous, and $g(x) = G'(x)$ for all $x \in [a, b]$.

Then note that both $\int_a^b \alpha(x)g(x)dx$ and $\int_a^b G d\alpha$ exist; the first because it's a product of Riemann-Integrable functions, and the second because $G$ is differentiable, thus continuous, thus $G \in \mathcal{R}(\alpha)$.

Let $\epsilon > 0$. There is a $\delta > 0$ such that if the mesh of a partition, $P$ of $[a, b]$, is less than $\delta$, then for any set of tags of that partition, $T$,

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i - \sum_{i=1}^{n} g(t_i)\alpha(t_i)\Delta x_i \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(t_i)\Delta x_i - \int_a^b \alpha(x)g(x)dx \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} G(t_i)\Delta\alpha_i - \int_a^b G(x)d\alpha \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i - \int_a^b \alpha(x)g(x)dx \right| < 2\epsilon/3$$

The first is because $\alpha$ is increasing; it has only countably many discontinuities, all of which are jump discontinuities. So, the diference of each $\alpha(x_i)$ and $\alpha(t_i)$ can be shrunk by making the difference of $x_i$ and $t_i$ small...which yields the result. The second and third are because either it is the definition of the integral or it is a theorem we should know about integrals. (It depends on the approach, and I'm not sure which one we're taking.) The last is a combination of the first two.

Now, pick a set of tags, $t_i \in [x_{i-1}, x_i]$ such that $G(x_i) - G(x_{i-1}) = g(t_i)\Delta x_i$. We can do this, because of the mean value theorem.

Note that we have the following:

$$\begin{aligned}\sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i + \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i &= \sum_{i=1}^{n}(G(x_i) - G(x_{i-1}))\alpha(x_i) + G(x_{i-1})\Delta\alpha_i \\ &= \sum_{i=1}^{n}(G(x_i) - G(x_{i-1}))\alpha(x_i) + G(x_{i-1})(\alpha(x_i) - \alpha(x_{i-1})) \\ &= \sum_{i=1}^{n} G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_i) \\ &\qquad + G(x_{i-1})\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1})) \\ &= \sum_{i=1}^{n} G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1}) \\ &= G(b)\alpha(b) - G(a)\alpha(a) \text{ (That's a telescoping sum.)}\end{aligned}$$

Next,

$$\begin{aligned}&\left|G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)d\alpha - \int_a^b \alpha(x)g(x)dx\right| \\ &= \left|\sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i + \sum_{i=1}^{n} G(t_i)\Delta\alpha_i - \int_a^b G(x)d\alpha - \int_a^b \alpha(x)g(x)dx\right| \\ &< \epsilon\end{aligned}$$

So for all $\epsilon > 0$, $\left|G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)d\alpha - \int_a^b \alpha(x)g(x)dx\right| < \epsilon$. That is, $G(b)\alpha(b) - G(a)\alpha(a) = \int_a^b G(x)d\alpha + \int_a^b \alpha(x)g(x)dx$, which is the result.

**Problem 6:**
Let $\alpha$ be an increasing function on $[a, b]$, and for $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, d\alpha\right)^{1/2}.$$

First, we show that $\int_a^b |fg| \leq \int_a^b |f| \int_a^b |g|$.

Let $f, g, h \in \mathcal{R}(\alpha)$.
Then we have the following:

$$\begin{aligned}
\int_a^b |f-h|^2 \, d\alpha &= \int_a^b |f-g+g-h|^2 \, d\alpha \\
&= \int_a^b |f-g+g-h| \, |f-g+g-h| \, d\alpha \\
&\leq \int_a^b |f-g|^2 + 2\,|f-g|\,|g-h| + |g-h|^2 \, d\alpha \\
&= \left( \int_a^b |f-g| \, d\alpha + \int_a^b |g-h| \, d\alpha \right)^2
\end{aligned}$$

Taking square roots of both sides yields the triangle inequality for this ~~norm~~ strange function I have never seen before.