Problem 1:

Consider $h(z) = p(z)$ if $z \in \mathbb{C}$, and $h(z) = \infty$ if $z = \infty$, with $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ a nonconstant polynomial of degree n.

Then define $k(z) = 1/p(z)$ if $p(z) \in \mathbb{C}$, and $k(z) = 0$ if $z = \infty$, and define $g(z) = k(1/z)$ when $z \neq 0$ and $g(z) = 0$ when $z = 0$.

Now, when $|z| < \infty$, h is complex-differentiable at z because it is a polynomial.

If $z = \infty$, then $h(z) = \infty$. First, note that h has exclusively non-infinite values except at ∞, so on any neighborhood of ∞, h takes noninfinite values on that neighborhood. Next, we want to show that k is complex differentiable at ∞. So, we want g to be complex differentiable at 0; consider g at 0; $g(0) = 0$. Elsewhere, $g(\zeta) = k(1/\zeta) = 1/p(1/\zeta) = 1/(a_0 + a_1 z^{-1} + a_2 z^{-2} \ldots a_n z^{-n})$.

Except at 0, this means that $g(\zeta) = \frac{z^n}{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}$. This is a holomorphic function except possibly at 0 (where it wasn't explicitly defined). However, the singularity here is removable, and this is clear (the limit as $z \to 0$ of that is just 0...so it's bounded around 0, so the singularity's removable). So, g is complex-differentiable at 0, so k is complex-differentiable at ∞, so h is complex-differentiable at ∞.

...So h is complex-differentiable everywhere.

Problem 2:

Let $\Omega \subset \mathbb{C}$ be bounded, $f_n \in C(\overline{\Omega})$, f_n all holomorphic on Ω, and $f_n \to f$ uniformly on $\partial\Omega$ where f is holomorphic on Ω.

Let $\epsilon > 0$. Pick N such that for all $n \geq N$, $|f_n - f| < \epsilon$ on $\partial\Omega$.

Now, $f_n - f$ is holomorphic on a domain; the maximum principles apply. So, $|f_n - f| < \epsilon$ for all $z \in \Omega$ (And also, for all $z \in \overline{\Omega}$) if $n \geq N$.

So for all $\epsilon > 0$ there is an N such that for all $n \geq N$, $|f_n - f| < \epsilon$ on $\overline{\Omega}$. That is, $f_n \to f$ uniformly on $\overline{\Omega}$, as desired.

Problem 3:

Consider e^{z^2}.

First, the 99th derivative of this at 0 is $\frac{99!}{2\pi i} \int\limits_{|z|=1} \frac{e^{z^2}}{z^{100}} dz$. Now, $\frac{e^{z^2}}{z^{100}}$ is holomorphic except at 0; we can apply the residue theorem;

$$\frac{99!}{2\pi i}\int\limits_{|z|=1}\frac{e^{z^2}}{z^{100}}dz = 99!\mathrm{Res}_0\frac{e^{z^2}}{z^{100}}$$

An expansion of $\frac{e^{z^2}}{z^{100}}$ is $\sum\limits_{n=0}^{\infty}\frac{z^{2n-100}}{n!}$. This has no $1/z$ term; thus, the residue of it is 0. That is,

$$\begin{aligned}\frac{99!}{2\pi i}\int\limits_{|z|=1}\frac{e^{z^2}}{z^{100}}dz &= 99!\mathrm{Res}_0\frac{e^{z^2}}{z^{100}}\\ &= 0\end{aligned}$$

So the 99th derivative of e^{z^2} at 0 is 0.

Next, the 100th derivative of e^{z^2} at 0 is $\frac{100!}{2\pi i}\int\limits_{|z|=1}\frac{e^{z^2}}{z^{101}}dz$. Now, $\frac{e^{z^2}}{z^{101}}$ is holomorphic except at 0; we can apply the residue theorem;

$$\frac{100!}{2\pi i}\int\limits_{|z|=1}\frac{e^{z^2}}{z^{101}}dz = 100!\mathrm{Res}_0\frac{e^{z^2}}{z^{101}}$$

An expansion of $\frac{e^{z^2}}{z^{101}}$ is $\sum\limits_{n=0}^{\infty}\frac{z^{2n-101}}{n!}$. The coefficient attached to the $1/z$ term is $1/(51!)$; thus, the residue of it is $1/(51!)$. That is,

$$\begin{aligned}\frac{100!}{2\pi i}\int\limits_{|z|=1}\frac{e^{z^2}}{z^{101}}dz &= 100!\mathrm{Res}_0\frac{e^{z^2}}{z^{101}}\\ &= 100!/51!\end{aligned}$$

So the 100th derivative of e^{z^2} at 0 is $100!/51!$.

Problem 4:

(Note: I wikipedia'd this to make sure I got the right formula. I got it right the first time; hopefully walking through the logic is convincing enough that I'm not just copying from wikipedia.)

Let $P = (p_1, p_2, p_3) \in S^2$ be a point on the sphere in $\mathbb{R}^3$.

We define the stereographic projection of P onto $\mathbb{C}$ by $SP : S^2 \to \mathbb{C}$ using the following logic:

First, we choose an argument in $[0, 2\pi)$ to be the same as the argument of the complex number (p_1, p_2) in $[0, 2\pi)$; this is because the line we use to define the stereographic projection goes in the same direction as (p_1, p_2).

Next, we define a magnitude using similar triangles: we say that $r = \frac{\sqrt{p_1^2+p_2^2}}{1-p_3}$. This works in both the case where $p_3 \geq 0$ and $p_3 < 0$.

So, $SP(P) = \frac{\sqrt{p_1^2+p_2^2}}{1-p_3} e^{i\arg(p_1,p_2)}$.

That is, $SP(P) = \frac{\sqrt{p_1^2+p_2^2}}{1-p_3} \frac{p_1}{\sqrt{p_1^2+p_2^2}} + i \frac{\sqrt{p_1^2+p_2^2}}{1-p_3} \frac{p_2}{\sqrt{p_1^2+p_2^2}}$

Simplifying, $SP(P) = \frac{p_1}{1-p_3} + i\frac{p_2}{1-p_3}$.

Problem 5:

Let $h : \mathbb{C} \to \mathbb{C}$ be a real-differentiable function.

$$\begin{aligned}
\overline{h_z} &= \frac{1}{2}\overline{h_x - ih_y} \\
&= \frac{1}{2}\overline{h_x} - \overline{ih_y} \\
&= \frac{1}{2}\overline{h_x} + i\overline{h_y} \\
&= \overline{h}_{\overline{z}}
\end{aligned}$$

As desired.

Problem 6:

Let z_1, z_2 be fixed on the unit circle in $\mathbb{R}^2$. Define $\alpha(z) : D_1(0) \to \mathbb{R}$ to be the angle between the line segments $\overline{zz_1}$ and $\overline{zz_2}$. That is, $\alpha(z) = \arg(z_1 - z) - \arg(z_2 - z)$ Because α is continuous and is never zero, this means that α is either strictly positive or strictly negative. Without loss of generality, we can take α strictly positive, by relabeling points.

We know that $\arg(z)$ is harmonic (where it matters) from the discussion in class*. So because sums/differences of harmonic functions are harmonic, the result is clear.

*(We know that $\log(z) = \log(|z|) + i\arg(z)$ is holomorphic except at 0...because $D_1(0)$ is simply connected, there's a branch of log that $\arg(z)$ is harmonic on...which is good enough.)

Problem 7:

Let Ω be an annulus, $u(z) = \log(|z|)$ for all $z \in \Omega$.

Let f be a holomorphic function on Ω with $u = \mathrm{Re}(f)$.

Then $u_x = \frac{x}{x^2+y^2}$ and $u_y = \frac{y}{x^2+y^2}$. But this means that $v_y = \frac{x}{x^2+y^2}$ and $v_x = -\frac{y}{x^2+y^2}$.

However, that system of equations is inconsistent; integrating v_y with respect to y yields $\arctan(y/x) + g(x)$ for some g, and differentiating this with respect to x yields $g'(x) + y/x\frac{1}{y^2+x^2}$, which is necessarily inconsistent with our given v_x.

So u cannot have been the real part of f. That is, u isn't the real part of any holomorphic function.

Problem 8:

Let u and u^2 be harmonic. Define $v = u^2 - u$.

Then:

$$\begin{aligned}
v_{xx} &= 2(u_x)^2 + u_{xx}(2u-1) \\
v_{yy} &= 2(u_y)^2 + u_{yy}(2u-1) \\
v_{xx} + v_{yy} &= 0 \\
&= 2(u_x)^2 + u_{xx}(2u-1) + 2(u_y)^2 + u_{yy}(2u-1) \\
&= 2(u_x)^2 + 2(u_y)^2 \text{ (Cancellation because u is harmonic.)}
\end{aligned}$$

Because $(u_x)^2$ and $(u_y)^2$ must be nonnegative, this means that u_x and u_y are both identically zero.

So, u is constant.

Problem 9:

Let $u \geq 0$ be harmonic on $D_R(0)$ with $R > 1$ and $u(0) = 1$. Let $r < 1$.

Problem 10:

Consider $\cos(\sin(z))$.

By laboriously taking derivatives (that is, by applying wolframalpha rigorously), we get

$$\begin{aligned}
a_0 &= \cos(\sin(0)) = 1 \\
a_1 &= \sin(\sin(0))(-\cos(0)) = 0 \\
a_2 &= \frac{1}{2}(\sin(0)\sin(\sin(0)) - \cos^2(0)\cos(\sin(0))) = \frac{-1}{2} \\
a_3 &= \frac{1}{6}(\sin(\sin(0))\cos^3(0) + 3\sin(0)\cos(0)\cos(\sin(0)) + \sin(\sin(0))\cos(0)) = 0 \\
a_4 &= \frac{5}{24}
\end{aligned}$$

Problem 11:

Let Ω be a "standard" open set, let $a_0, a_1, \dots a_n \in \Omega$. Let $h \in \mathcal{O}(G)$, with G containing $\overline{\Omega}$. Define $w(z) = \prod_{j=0}^{n}(z - a_j)$.

Consider $P(\xi) = \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)}{w(z)}\frac{w(z)-w(\xi)}{z-\xi}dz$

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)}{w(z)}\frac{w(z)-w(\xi)}{z-\xi}dz &= \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)}{z-\xi} - \frac{h(z)w(\xi)}{w(z)(z-\xi)}dz \\
&= h(\xi) - \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)w(\xi)}{w(z)(z-\xi)}dz \\
&= h(\xi) - w(\xi)\frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)}{w(z)(z-\xi)}dz \\
&= h(\xi) - w(\xi)\frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{h(z)}{w(z)(z-\xi)}dz
\end{aligned}$$

Note that at each a_j, w vanishes; so, $P(a_j) = h(a_j)$ for all j.

Next, note that $P(\xi)$ is a polynomial of degree at most n; it is the sum of a constant term and a polynomial of degree n multiplied by some constant.