Problem 1:

Let R be a UFD and P be a prime ideal.

Let P fail to be principal. Let $a \in P$.

Now, a has a prime factorization, $p_1^{\alpha_1} \dots p_n^{\alpha_n}$.

Then one of the $p_i^{\alpha_i}$ is in P; $a \in P$, so $p_1^{\alpha_1} \in P$ or $p_2^{\alpha_2} \dots p_n^{\alpha_n} \in P$. If $p_2^{\alpha_2} \dots p_n^{\alpha_n} \in P$, then $p_2^{\alpha_2} \in P$ or $p_3^{\alpha_3} \dots p_n^{\alpha_n} \in P$. We can iterate this process, so one of the $p_i^{\alpha_i}$ is in P.

So $p_i \in P$, by applying the same method.

So $(p_i) \subset P$. Because p_i is prime, (p_i) is prime (and nonzero). But it's not P, as P is not principal.

So P has a proper, nonzero prime ideal.

Problem 2:

Let k be a field and $n \geq 2$.

If $\text{char}(k) = 2$, $x_1^2 + x_2^2 \dots x_n^2 - 1$ is equal to $(x_1 + x_2 + \ldots + x_n - 1)^2$ (when you multiply it out,every term has a factor of 2 except the x_i^2 and -1 terms) and so $x_1^2 + x_2^2 \dots x_n^2 - 1$ is reducible.

Problem 3:

By the reduction criterion, $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Z}[x]$ if it is irreducible in $\mathbb{Z}/(11)[x]$.

By Eisenstein's criterion, $x^4 + 3x^3 + 3x^2 - 5 = x^4 + 3x^3 + 3x^2 + 6$ is irreducible, using the prime 3.

So $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Z}[x]$. So $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Q}[x]$.

Problem 4:

Let $R = \mathbb{Z}[\sqrt{-5}]$, and $K = \text{Quot}(R)$.

Consider $3x^2 + 4x + 3$. By the quadratic formula, if this polynomial has roots, they are $\frac{-2}{3} \pm \frac{\sqrt{-5}}{3}$. A factorization of $3x^2 + 4x + 3$ is given by $3(x + \frac{2}{3} + \frac{\sqrt{-5}}{3})(x + \frac{2}{3} - \frac{\sqrt{-5}}{3})$. So the polynomial is reducible in $K[x]$.

Now, in $R[x]$, $3x^2+4x+3$ cannot have a constant factored out of it. As it is a degree 2 polynomial, this means that it factors only as a product of two degree 1 polynomials. So any factorization of that polynomial must be of the form $(rx+r'(2+\sqrt{-5}))(sx+s'(2-\sqrt{-5}))$, with $r', s' \in \mathbb{Z}[\sqrt{-5}]$ and $r = 3r'$, $s = 3s'$. Yet, this means that the leading coefficient of the polynomial is a multiple of 9, which 3 isn't. So the polynomial is irreducible in $R[x]$.

Problem 5:

Let R be a UFD and P be a prime ideal of $R[x]$ with $P \cap R = 0$.

Let P fail to be principal. Then there are $p, q \in P$ with $p \nmid q$ and $q \nmid p$. We can pick p to be of minimal degree, and among the q that satisfy these conditions, we can also choose q minimal (note that the degree of p will be less than or equal to the degree of q). Moreover, we can choose p, q to have the leading coefficient of q be a multiple of the leading coefficient of p.

Define $r = \gcd(p, q)$.

Now, r must have a lower degree than p. If not, then r must have the same degree as p (it clearly canot have higher degree). So $rs = p$ for some $r \in R$. Also, $r \mid q$ and $p \nmid q$.