Problem 1:

(Prove: The real 2x2 matrix blah represents a complex-linear map iff a=d, c=-b)

Let the real matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ represent a complex-linear map. Then we have:

$$\begin{aligned} i \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] &= i \left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} -d \\ b \end{smallmatrix}\right] \end{aligned}$$

And also

$$\begin{aligned} i \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] &= \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} -a \\ -c \end{smallmatrix}\right] \end{aligned}$$

So $\left[\begin{smallmatrix} -d \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} -a \\ -c \end{smallmatrix}\right]$; that is, $a = d$ and $b = -c$.

Now, let the real matrix A have that $a = d$ and $c = -b$. Write $A = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$, and let T be the linear map associated with A. Also, let $\vec{z}$ be the vector associated with z, for any $z \in \mathbb{C}$.

Let $z = x + iy, w = x' + iy' \in \mathbb{C}$. Then:

$$\begin{aligned} T(wz) &= A\overrightarrow{wz} \\ &= \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \left[\begin{smallmatrix} xx'-yy' \\ xy'+yx' \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} a(xx'-yy')+b(xy'+yx') \\ a(xy'+yx')-b(xx'-yy') \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right] \times \left[\begin{smallmatrix} ax+by \\ ax-by \end{smallmatrix}\right] \\ &= \vec{w} \times T(\vec{z}) \end{aligned}$$

(with $\times$ being complex multiplication).

That is, A's associated linear map is $\mathbb{C}$-linear.

So, we have that the real matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ represents a complex-linear map if and only if $a = d$ and $b = -c$.

Problem 2:

Consider $\int\limits_{|z|=R} \overline{z}^n dz$. Define $\alpha : [0, 2\pi] \to \mathbb{C}$ by $\alpha(t) = R(\cos(t) + i\sin(t))$.

If $n \neq 1$, we have

$$\begin{aligned}
\int\limits_{|z|=R} \overline{z}^n dz &= \int\limits_{\alpha} \overline{z}^n dz \\
&= \int\limits_0^{2\pi} [R(\cos(t) - i\sin(t))]^n [R(-\sin(t) + i\cos(t))] dt \\
&= R^{n+1} \int\limits_0^{2\pi} \frac{-\sin(t) + i\cos(t)}{(\cos(t) + i\sin(t))^n} dt \\
&= R^{n+1} \frac{1}{n-1} [((\cos(2\pi) + i\sin(2\pi)))^{1-n} - (\cos(0) + i\sin(0))^{1-n} \\
&= R^{n+1} \frac{1}{n-1} [1-1] \\
&= 0
\end{aligned}$$

(Here, we're using freely the fact that $\overline{z} = 1/z$ if $|z| = 1$, and we gloss over the u-substitution with $u = \cos(t) + i\sin(t)$.)

And if $n = 1$, we have

$$\begin{aligned}
\int\limits_{|z|=R} \overline{z}^n dz &= \int\limits_{\alpha} \overline{z} dz \\
&= \int\limits_0^{2\pi} [R(\cos(t) - i\sin(t))][R(-\sin(t) + i\cos(t))] dt \\
&= R^2 \int\limits_0^{2\pi} \frac{-\sin(t) + i\cos(t)}{\cos(t) + i\sin(t)} dt \\
&= R^2 \int\limits_0^{2\pi} i \frac{\cos(t) + i\sin(t)}{\cos(t) + i\sin(t)} dt \\
&= R^2 \int\limits_0^{2\pi} i dt \\
&= R^2 2\pi i
\end{aligned}$$

Problem 3:

Let $D \subset \mathbb{C}$ be open, and let D^* be D's reflection about the x-axis. Let $f \in \mathcal{O}(D)$, and define $g(z) = \overline{f(\overline{z})}$. As an intermediate step, define $h(z) = f(\overline{z})$.

Let $f = u + iv$. Then $h((x, y)) = f((x, -y))$; say that $h = u` + iv`$. Also, $u_x(z) = v_y(z)$ for all $z \in \mathbb{C}$.

Moreover, $u_y(z) = -v_x(z)$ for all $z \in \mathbb{C}$. So

(I remember someone saying something about more problems but I must be missing one...:/)