Problem 1:
Consider $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$. Define $s_n=(-1)^n\frac{z^{2n+1}}{(2n+1)!}$.
Fix $z\in\mathbb{C}$. Then:

$$\begin{aligned}\left|\frac{s_{n+1}}{s_n}\right| &= \left|\frac{(-1)^{n+1}\frac{z^{2n+3}}{(2n+3)!}}{(-1)^n\frac{z^{2n+1}}{(2n+1)!}}\right| \\ &= \left|\frac{(-1)z^2}{(2n+3)(2n+2)}\right| \\ &= \left|\frac{z^2}{4n^2+10n+6}\right|\end{aligned}$$

And it is clear that as $n\to\infty$, $\left|\frac{z^2}{4n^2+10n+6}\right|\to 0$ for all $z\in\mathbb{C}$. So by the ratio test, $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ absolutely converges (and thus, converges) for all $z\in\mathbb{C}$. That is, $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ has infinite radius of convergence.

Similarly (by defining $t_n=(-1)^n\frac{z^{2n}}{(2n)!}$ and applying the ratio test), $\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$ has infinite radius of convergence.

Typically, we define $\sin(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$ and $\cos(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$

Problem 2:
Consider $\sum_{k=1}^{\infty}\frac{z^k}{1-z^k}$.
Define $s_k=\frac{z^k}{1-z^k}$. Fix $z\in D_1(0)$. Then:

$$\begin{aligned}
\left|\frac{s_{k+1}}{s_k}\right| &= \left|\frac{\frac{z^{k+1}}{1-z^{k+1}}}{\frac{z^k}{1-z^k}}\right| \\
&= \left|z\frac{1-z^k}{1-z^{k+1}}\right| \\
&\leq |z|\left|\frac{1-z^k}{1-z^{k+1}}\right| \text{ (By Cauchy-Schwarz)} \\
&\leq |z| \text{ (Because } |z|<1 \text{ and } \left|z^k\right| \text{ is a decreasing sequence when } |z|<1) \\
&< 1
\end{aligned}$$

So, by the ratio test, $\sum\limits_{k=1}^{\infty} \frac{z^k}{1-z^k}$ converges on the unit disk.

Moreover, that sum converges uniformly;

So, because each term is holomorphic on the unit disk, we have that each partial sum is holomorphic on the unit disk, and thus the limit of the partial sums is holomorphic on the unit disk, by Weierstrauss.

Problem 3:

Consider $e^{\overline{z}}$, with $z \in \mathbb{C}$.

$$\begin{aligned}
e^{\overline{z}} &= \sum \frac{\overline{z}^n}{n!} \\
&= \sum \frac{\overline{z^n}}{n!} \\
&= \sum \overline{\frac{z^n}{n!}} \\
&= \overline{\sum \frac{z^n}{n!}} \\
&= \overline{e^z}
\end{aligned}$$

So $e^{\overline{z}} = \overline{e^z}$, for all $z \in \mathbb{C}$.

Problem 4:

(In the following, $\dotplus$ is used for curve concatenation.)

Let $\Omega \subset \mathbb{C}$ be open and connected. Fix $w \in \Omega$, define Ω_1 to be the set of all points that can be joined to w by a curve. Define Ω_2 to be the set of all points that cannot.

Then it is clear that $\Omega = \Omega_1 \cup \Omega_2$, that $\Omega_1 \cap \Omega_2 = \emptyset$, and that $w \in \Omega_1$.

Now, Ω_1 is open:

Let $x \in \Omega_1$. There is an $\epsilon > 0$ such that $D_\epsilon(x) \subset \Omega$. There is a curve, Γ, connecting w to x. For all $y \in D_\epsilon(x)$, the line segment $[x, y] \subset D_\epsilon(x) \subset \Omega$ (because disks are convex); we can parameterize a curve ϕ whose image is $[x, y]$. It is clear that $\Gamma \dotplus \phi$ is a curve connecting w and y. So for all $y \in D_\epsilon(x)$, $y \in \Omega_1$; $D_\epsilon(x) \subset \Omega_1$.

So for all $x \in \Omega_1$, there's an open disk that contains x that is a subset of Ω_1. So Ω_1 is open.

Also, Ω_2 is open:

Let $x \in \Omega_2$. There is an $\epsilon > 0$ such that $D_\epsilon(x) \subset \Omega$. There is no curve connecting w to x. For all $y \in D_\epsilon(x)$, the line segment $[y, x] \subset D_\epsilon(x) \subset \Omega$ (because disks are convex); we can parameterize a curve ϕ whose image is $[y, x]$.

If for any $y \in D_\epsilon(x)$ there is a curve, Γ, connecting y and w, then $\Gamma \dotplus \phi$ is a curve connecting w and x. This would contradict the fact that $x \in \Omega_2$. So for all $y \in D_\epsilon(x)$, $y \in \Omega_2$; $D_\epsilon(x) \subset \Omega_2$.

So for all $x \in \Omega_2$, there's an open disk that contains x that is a subset of Ω_2. So Ω_2 is open.

So if Ω_2 was nonempty, we would have two nonempty open sets whose union was Ω. That is, we would be able to disconnect Ω, which is contrary to our assumption.

Thus, Ω_2 is empty, so $\Omega_1 = \Omega$. That is, an open, connected subset of $\mathbb{C}$ is path-connected.