

**Problem 1:**

Let  $f_n \rightarrow f$  in measure, with an integrable function  $g$  such that  $|f_n| \leq g$  for all  $n$ .

We proceed by proving the fact if the domain of each  $f_n$  is  $[0, 1]$ , then extending it to all of  $\mathbb{R}$  by applying the  $\epsilon 2^{-n}$  method.

Let  $\epsilon > 0$ .

Because  $f_n \rightarrow f$  in measure, there's an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $m(\{x : |f_n(x) - f(x)| \geq \epsilon/2\}) < \epsilon/2$ .

**Problem 2:**

Let  $f$  be continuous on  $[a, b]$ , with one of its derivatives everywhere non-negative on  $(a, b)$ .

First, we will show this for a function  $g$  with  $D^+(g) \geq \epsilon > 0$ . If  $g$  is such a function, then  $\limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \geq \epsilon > 0$ . This means that  $g$  is nondecreasing:

We proceed by contradiction: let there be  $x, y \in [a, b]$  (with  $x < y$ , without loss of generality) be such that  $f(x) > f(y)$ .

Consider the set  $A = \{\alpha \in [x, y] : f(\alpha) > f(y)\}$ . This set has a supremum, as it's nonempty. Define  $\alpha = \sup(A)$ . Either  $\alpha = b$  (in which case, the derivative is negative at  $b$ , leading to our contradiction), or there is a  $\delta > 0$  such that if  $t \in [\alpha, \alpha + \delta]$ , then  $f(t) < f(y)$ . Moreover,  $f(\alpha) = f(y)$ : else,  $|f(\alpha) - f(y)| = \epsilon'$  for some  $\epsilon' > 0$ , so by using continuity (specifically, the intermediate value theorem) we can find an  $\alpha'$  between  $\alpha$  and  $y$  with  $f(\alpha')$  between  $f(\alpha)$  and  $f(y)$ , which causes a contradiction. So for any sequence  $(t_n)$  decreasing to  $\alpha$ , we have  $\frac{f(t_n) - f(\alpha)}{t_n - \alpha}$  negative. This means that  $D^+(\alpha) \leq 0$ . This contradicts our assumption on  $D^+$ .

We can mimic this proof to show that if  $g$  has  $D^-(g) \geq \epsilon > 0$ , then  $g$  is nondecreasing.

Now, let  $f$  have a derivative everywhere nonnegative on  $(a, b)$ . This means, in particular, that either  $D^+$  or  $D^-$  is everywhere nonnegative on  $(a, b)$ .

Then for every  $\epsilon > 0$ ,  $g_\epsilon(x) = f(x) + \epsilon x$  has  $D^+(g_\epsilon)$  (or  $D^-(g_\epsilon)$ ) greater than  $\epsilon$ . So for all  $\epsilon > 0$ ,  $g_\epsilon$  is nondecreasing. So for all  $x, y \in [a, b]$  with  $x < y$ ,  $g_\epsilon(x) \leq g_\epsilon(y)$ . That is,  $f(x) + \epsilon x \leq f(y) + \epsilon y$ . Taking limits as  $\epsilon \rightarrow 0$ , this means that  $f(x) \leq f(y)$ , for all  $x, y \in [a, b]$  with  $x < y$ .

So  $f$  is nondecreasing on  $[a, b]$  if some derivative is everywhere nonnegative on  $[a, b]$ .

**Problem 3:**

Suppose that  $f_n(x) \rightarrow f(x)$  at each  $x \in [a, b]$ .

**Problem 4:**

Suppose that  $f \in BV([a, b])$ . Then  $f'$  exists almost everywhere, by a theorem in class. Moreover,  $f$  is the difference of two monotone functions. That is,  $f = f^+ - f^-$  for some monotone functions  $f^+$  and  $f^-$ .

So, this means that we have

$$\begin{aligned} \int_a^b |f'| &= \int_a^b |(f^+)' - (f^-)'| \\ &\leq \int_a^b |(f^+)'| + |(f^-)'| \end{aligned}$$

Now, we show that  $\int_a^b |(f^+)'| \leq P_a^b(f)$ .

By one of the important theorems that is like the fundamental theorem of calculus,  $\int_a^b |(f^+)'| \leq f^+(b) - f^+(a)$ .

We also know that  $f^+(b) - f^+(a) \leq P_a^b(f)$ ;

$$\begin{aligned} P &= N + f(b) - f(a) \\ &= N + f^+(b) - f^-(b) - f^+(a) + f^-(a) \\ &= N + (f^+(b) - f^+(a)) + (f^-(b) - f^-(a)) \\ &\geq N + (f^+(b) - f^+(a)) \\ &\geq (f^+(b) - f^+(a)) \end{aligned}$$

Similarly,  $\int_a^b |(f^-)'| \leq N_a^b(f)$ . So, we have

$$\begin{aligned}
\int_a^b |f'| &\leq \int_a^b |(f^+)'| + |(f^-)'| \\
&\leq P_a^b + N_a^b \\
&\leq T_a^b
\end{aligned}$$

as we desired.

**Problem 5:**

Let  $g$  be an absolutely continuous monotone function on  $[0, 1]$ , and  $E$  be a set of measure 0.

Let  $\epsilon > 0$ .

There's a  $\delta > 0$  such that for any finite set of intervals,  $\{(x_i, x'_i)\}_{i \in \mathcal{F}}$ , then  $\sum_{\mathcal{F}} |x'_i - x_i| < \delta$  implies that  $\sum_{\mathcal{F}} |f(x'_i) - f(x_i)| < \epsilon$ .

Because  $E$  has measure 0, there's a set of intervals covering  $E$ , call it  $\mathcal{I}$ , with the sum of their lengths less than  $\delta$ .

So  $g(\bigcup_{I \in \mathcal{I}} I)$  covers  $g(E)$  and has the sum of the lengths of the component intervals less than  $\epsilon$ , as desired.

**Problem 6:**

Let  $f$  be a nonnegative measurable function on  $[0, 1]$ .

We know that  $\ln$  is a concave function on  $[0, 1]$  (if this is not clear, it's the inverse of a convex function).

So  $-\ln$  is a convex function on  $[0, 1]$ .

So Jensen's inequality applies:

$$\begin{aligned}
-\ln \int f &\leq -\int \ln f \\
\ln \int f &\geq \int \ln f
\end{aligned}$$

This satisfies the problem.