Problem 7:

Let $u \in C^0(\Omega)$.

Let u be subharmonic in Ω. Then u satisfies the mean value properties for any ball compactly contained in Ω; it is clear that this implies that u satisfies the mean value properties locally.

Now, let u satisfy the mean value properties locally. Fix $y \in \Omega$. Then there is a $\delta > 0$ such that $u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} uds$ for all $R \leq \delta$.

So, on $B_R(y)$, we have:

$$u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(y)} uds$$

Problem 9:

Let $u \in C^2(\Omega)$.

First, $\Delta u \geq 0$ in Ω, if and only if u is subharmonic in Ω, by definition.

Next: let u be subharmonic in Ω. Then $\Delta u \geq 0$ in Ω. Let $\phi \geq 0$ be a function in $C_c^2(\Omega)$. So

$$\begin{aligned}\int\limits_\Omega u\Delta\phi dx &= \int\limits_\Omega \phi\Delta u dx + \int\limits_{\partial\Omega} u\partial_\nu phi - \phi\partial_\nu u dS \\ &= \int\limits_\Omega \phi\Delta u dx \text{ (Note: } \phi \text{ vanishes on } \partial\Omega\text{, as } \phi \text{ has compact support.)} \\ &\geq 0\end{aligned}$$

That is, u is weakly subharmonic if u is subharmonic.

Now, let u be weakly subharmonic. Then for any $\phi \geq 0$ with $\phi \in C_c^2(\Omega)$, we have

$$\begin{aligned}\int\limits_{\Omega} u\Delta\phi dx &= \int\limits_{\Omega} \phi\Delta u dx + \int\limits_{\partial\Omega} u\partial_{\nu} phi - \phi\partial_{\nu} u dS \\ &= \int\limits_{\Omega} \phi\Delta u dx\end{aligned}$$

Thus, the three conditions given are equivalent.