Problem 1 (23 in book):

Let S denote the square in $\mathbb{R}\times(0,\infty)$ with corners $(0,1),(1,2),(0,3),(-1,2)$. Define

$$f(x,t) = \begin{cases} -1 & \text{for } (x,t) \in S \cap \{t > x+2\} \\ -1 & \text{for } (x,t) \in S \cap \{t < x+2\} \\ 0 & \text{else} \end{cases}$$

Let u solve

$$\begin{cases} u_{tt} - u_{xx} = f & \text{when } t > 0 \\ u = 0, u_t = 0 & \text{when } t = 0 \end{cases}$$

Consider u when $t > 3$. Then we have

$$u(x,t) = \int_0^t u(x,t;s)ds$$

where $u(x,t;s) = \frac{1}{2}\int_{x-t}^{x+t} f(y,s)dy$ (we get this by Duhamel's principle and the solution of the wave equation in one dimension). In other words,

$$u(x,t) = \int_0^t \frac{1}{2}\int_{x-t}^{x+t} f(y,s)dyds$$

For the sake of sanity, let us write f as $-\chi_A + \chi_B$, with $A = S\cap\{t > x+2\}$ and $B = S \cap \{t < x+2\}$. Then

$$\begin{aligned} u(x,t) &= \int_0^t \frac{1}{2}\int_{x-t}^{x+t} \chi_B - \chi_A dyds \\ &= \frac{1}{2}\left[\int_0^t \int_{x-t}^{x+t} \chi_B dyds - \int_0^t \int_{x-t}^{x+t} \chi_A dyds\right] \end{aligned}$$

Now, if $x - t > 1$ or $x + t < -1$, both of those integrals vanish. That is, for fixed $t > 3$, $u(x,t) = 0$ if $x > 1 + t$ or $x < -1 - t$. Moreover, if $1/2 - t <$

$x < -1/2 + t$, then $\int\limits_0^t \int\limits_{x-t}^{x+t} \chi_B dyds = 1$. Similarly, if $t - 1/2 > x > 1/2 - t$, then $\int\limits_0^t \int\limits_{x-t}^{x+t} \chi_A dyds = 1$.

At this point, we can see that $u(x,t)$ vanishes except possibly when $x \in [1-t, 1/2-t] \cup [t-1/2, t+1]$.

Problem 2 (24 in book):

Let u solve the intial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{when } t > 0 \\ u = g, u_t = h & \text{when } t = 0 \end{cases}$$

Let g, h have compact support. Consider $k(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_t^2(x,t)dx$ and $p(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_x^2(x,t)dx$.

Part a:

Consider $k(t) + p(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_x^2(x,t) + u_t^2(x,t)dx$.

We know that $u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy$

So, we have:

$$\begin{aligned} u_x(x,t) &= \frac{g'(x+t) + g'(x-t)}{2} + \left[\frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy \right]_x \\ &= \frac{g'(x+t) + g'(x-t)}{2} + \frac{1}{2} [h(x+t) - h(x-t)] \\ u_t(x,t) &= \frac{g'(x+t) - g'(x-t)}{2} + \left[\frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy \right]_t \\ &= \frac{g'(x+t) - g'(x-t)}{2} + \frac{1}{2} [h(x+t) + h(x-t)] \end{aligned}$$

This means that

$$\begin{aligned}
u_x^2 + u_t^2 &= \left(\frac{g'(x+t) + g'(x-t)}{2} + \frac{1}{2}\left[h(x+t) - h(x-t)\right] \right)^2 \\
&\quad + \left(\frac{g'(x+t) - g'(x-t)}{2} + \frac{1}{2}\left[h(x+t) + h(x-t)\right] \right)^2 \\
&= (\frac{g'(x+t) + g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t) + g'(x-t)}{2}\left[h(x+t) - h(x-t)\right] \\
&\quad + (\frac{1}{2}\left[h(x+t) - h(x-t)\right])^2 \\
&\quad + (\frac{g'(x+t) - g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t) - g'(x-t)}{2}\left[h(x+t) + h(x-t)\right] \\
&\quad + (\frac{1}{2}\left[h(x+t) + h(x-t)\right])^2 \\
&= \frac{1}{4}g'(x+t)^2 + \frac{1}{4}g'(x-t)^2 + \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad + \frac{1}{2}\left[h(x+t)g'(x+t) - h(x-t)g'(x+t) + h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad + \frac{1}{4}h(x+t)^2 + \frac{1}{4}h(x-t)^2 - \frac{1}{2}h(x+t)h(x-t) \\
&\quad + \frac{1}{4}g'(x+t)^2 + \frac{1}{4}g'(x-t)^2 - \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad + \frac{1}{2}\left[h(x+t)g'(x+t) + h(x-t)g'(x+t) - h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad + \frac{1}{4}h(x+t)^2 + \frac{1}{4}h(x-t)^2 + \frac{1}{2}h(x+t)h(x-t) \\
&= \frac{1}{2}g'(x+t)^2 + \frac{1}{2}g'(x-t)^2 \\
&\quad + \left[h(x+t)g'(x+t) - h(x-t)g'(x-t)\right] \\
&\quad + \frac{1}{2}h(x+t)^2 + \frac{1}{2}h(x-t)^2
\end{aligned}$$

Now, we integrate:

$$
\begin{aligned}
\int_{\mathbb{R}} u_x^2 + u_t^2 dx = \int_{\mathbb{R}} & \frac{1}{2}g'(x+t)^2 + \frac{1}{2}g'(x-t)^2 \\
& + [h(x+t)g'(x+t) - h(x-t)g'(x-t)] \\
& + \frac{1}{2}h(x+t)^2 + \frac{1}{2}h(x-t)^2 dx
\end{aligned}
$$

The above is constant (with respect to t), and this is clear by applying appropriate substitutions to each term.

Part b:
Using the above, consider that

$$
\begin{aligned}
u_x^2 - u_t^2 &= \left(\frac{g'(x+t)+g'(x-t)}{2} + \frac{1}{2}\left[h(x+t)-h(x-t)\right]\right)^2 \\
&\quad - \left(\frac{g'(x+t)-g'(x-t)}{2} + \frac{1}{2}\left[h(x+t)+h(x-t)\right]\right)^2 \\
&= (\frac{g'(x+t)+g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t)+g'(x-t)}{2}\left[h(x+t)-h(x-t)\right] \\
&\quad + (\frac{1}{2}\left[h(x+t)-h(x-t)\right])^2 \\
&\quad - (\frac{g'(x+t)-g'(x-t)}{2})^2 \\
&\quad - \frac{g'(x+t)-g'(x-t)}{2}\left[h(x+t)+h(x-t)\right] \\
&\quad - (\frac{1}{2}\left[h(x+t)+h(x-t)\right])^2 \\
&= \frac{1}{4}g'(x+t)^2 + \frac{1}{4}g'(x-t)^2 + \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad + \frac{1}{2}\left[h(x+t)g'(x+t) - h(x-t)g'(x+t) + h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad + \frac{1}{4}h(x+t)^2 + \frac{1}{4}h(x-t)^2 - \frac{1}{2}h(x+t)h(x-t) \\
&\quad - \frac{1}{4}g'(x+t)^2 - \frac{1}{4}g'(x-t)^2 + \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad - \frac{1}{2}\left[h(x+t)g'(x+t) + h(x-t)g'(x+t) - h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad - \frac{1}{4}h(x+t)^2 - \frac{1}{4}h(x-t)^2 - \frac{1}{2}h(x+t)h(x-t) \\
&= g'(x+t)g'(x-t) \\
&\quad + \left[-h(x-t)g'(x+t) + h(x+t)g'(x-t)\right] \\
&\quad + h(x+t)h(x-t)
\end{aligned}
$$

Integrating, we get

$$\int_{\mathbb{R}} u_x^2 - u_t^2 dx = \int_{\mathbb{R}} g'(x+t)g'(x-t) + [-h(x-t)g'(x+t) + h(x+t)g'(x-t)] + h(x+t)h(x-t)dx$$

Because g and h have compact support, there's a t large enough that all of the above products vanish for all x. (Taking t to be twice the diameter of the larger of the sets g and h have support on suffices.)

Thus, the above integral vanishes for some sufficiently large t, yielding our result.

Problem 3 (on page):

Assume $f(x,t) = 1$ if $|x| \leq 1$ and $0 \leq t \leq 1$, and $f(x,t) = 0$ otherwise. Let u solve

$$\begin{cases} u_{tt} - \Delta u = f & \text{when } t > 0 \\ u = 0, u_t = 0 & \text{when } t = 0 \end{cases}$$

Consider $u(0,t)$ when $t > 2$.

If $n = 1$, then...

If $n = 2$, then...

If $n = 3$, then...