Problem 1:

Let u be such that $\Delta u = 0$ in $\mathbb{R}^2$, $u = 0$, $\frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1)$ when $x_2 = 0$.

We try to find u with $u(x_1, x_2) = v(x_1)w(x_2)$ for some v, w.

If this holds, then we get $v''(x_1)w(x_2) - v(x_1)w''(x_2) = 0$, which we can rearrange to get $\frac{v''(x_1)}{v(x_1)} = -\frac{w''(x_2)}{w(x_2)} = \mu$.

Also, we get that $v(x_1)w'(0) = \frac{1}{n}\sin(nx_1)$. Thus, we determine that either $v(x_1) = \frac{c}{n}\sin(nx_1)$, where $c = \frac{1}{w'(0)}$. (Note: $w'(0) \neq 0$, $\frac{1}{n}\sin(nx_1)$ is identically zero...)

Because of the situation, we can arbitrarily choose $w'(0) = 1$, by rescaling v and w appropriately. This yields $v(x_1) = \frac{1}{n}\sin(nx_1)$.

So, $v''(x_1) = -n\sin(nx_1)$, so we get $\mu = -n^2$ in the above.

So $\frac{w''(x_2)}{w(x_2)} = n^2$. Using techniques of ODE (note that it's necessary to use that $w(0) = 0$ and $w'(0) = 1$ for this...it's just a second order linear ODE with constant coefficients), we find that $w(x_2) = \frac{1}{n}\sinh(nx_2)$. Thus,

$$\begin{aligned} u(x_1, x_2) &= v(x_1)w(x_2) \\ &= \frac{1}{n^2}\sin(nx_1)\sinh(nx_2) \end{aligned}$$

as desired.

Note that as $n \to \infty$, $u(x_1, x_2)$ vanishes along $x_1 = 0$ and $x_2 = 0$. However, in all other cases,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^2}\sin(nx_1)\sinh(nx_2) &= \lim_{n\to\infty} \frac{n\left[\cos(nx_1)\sinh(nx_2) + \sin(nx_1)\cosh(nx_2)\right]}{2n} \\ &= \lim_{n\to\infty} \frac{\left[\cos(nx_1)\sinh(nx_2) + \sin(nx_1)\cosh(nx_2)\right]}{2} \end{aligned}$$

This limit does not exist; $\frac{[\cos(nx_1)\sinh(nx_2)+\sin(nx_1)\cosh(nx_2)]}{2}$ oscillates wildly. However, as $n \to \infty$, the Cauchy data goes to 0. Thus, we can say that the Cauchy Problem for Laplace's Equation isn't well-posed.

Problem 2:

Consider the line $\{t = 0\}$ and the heat equation $u_t - u_{xx} = 0$. Then for this PDE, we have that $-1\nu^{(0,2)} = 0$ on the line given (because $\nu = (0, 1)$).

So the line is everywhere noncharacteristic for the PDE: it is characteristic.

Now, assume that there's an analytic solution, u, of the heat equation in $\mathbb{R} \times \mathbb{R}$ with $u = \frac{1}{1+x^2}$ on $\{t = 0\}$. That is, $u = \int\limits_{\mathbb{R}^2} \frac{1}{y^2+1} \frac{e^{\frac{-(x-y)^2}{4t}}}{4\pi t} dy = \sum \alpha^{\infty} a_\alpha (x,t)^\alpha$.

Consider $u(0,t) = \frac{1}{4\pi t} \int\limits_{\mathbb{R}^2} \frac{e^{\frac{-y^2}{4t}}}{1+y^2} dy = \frac{ce^{-1/4t}}{t}$.

Note that $u(0,t)$'s second derivative explodes at 0; thus, u cannot have been analytic.