Let g be an absolutely continuous monotone function on $[0,1]$, and E be a set of measure 0. Without loss of generality, we can take g to be increasing. Further, for this problem we can take $g(0)=0$ without loss of generality, as the measure of a set is translation invariant.

We know that g is the antiderivative of some function, f. That is,

$$g(x) = \int\limits_0^x f(t)dt$$

Moreover, because g is increasing, this means that f is nonnegative almost everywhere.

Now, $[0,1] \setminus E$. It is true that $\int\limits_{[0,x]\setminus E} f(t)dt = g(x)$.

Assume that $g(E)$ has nonzero measure. Then $g(E)$ contains a closed interval, call it $[a,b]$.

Then $\{y : \int\limits_0^x f(t)dt = y \text{ for some } x \in E\}$ contains $[a,b]$.

Let x_1 be a value such that $\int\limits_0^{x_1} f(t)dt = a$ and x_2 be a value such that $\int\limits_0^{x_2} f(t)dt = b$. Then $[x_1, x_2] \subset E$: this is somewhat clear.

So E must contain a closed interval, and is thus of nonzero measure, contrary to our assumption.

So $g(E)$ must have had zero measure.