Problem 1:

Let $f : X \to Y$ be a function such that there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Then for all $y \in Y$, $f(g(y)) = y$.

So for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$ ($x = f(y)$).

So f is onto if there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Problem 2:

Let $\mathcal{C}$ be an arbitrary collection of sets.

First, we show that $(\bigcup\limits_{C\in\mathcal{C}} C)^c = \bigcap\limits_{C\in\mathcal{C}} C^c$.

Let $x \in (\bigcup\limits_{C\in\mathcal{C}} C)^c$.

Then $x \notin (\bigcup\limits_{C\in\mathcal{C}} C)$.

So x is not in C for any $C \in \mathcal{C}$.

So x is in C^c for all $C \in \mathcal{C}$.

So $x \in \bigcap\limits_{C\in\mathcal{C}} C^c$.

This means that $(\bigcup\limits_{C\in\mathcal{C}} C)^c \subset \bigcap\limits_{C\in\mathcal{C}} C^c$.

Next, let $x \in \bigcap\limits_{C\in\mathcal{C}} C^c$.

So x is in C^c for all $C \in \mathcal{C}$.

So x is not in C for any $C \in \mathcal{C}$.

Then $x \notin (\bigcup\limits_{C\in\mathcal{C}} C)$.

Thus, $x \in (\bigcup\limits_{C\in\mathcal{C}} C)^c$.

This means that $(\bigcup\limits_{C\in\mathcal{C}} C)^c \supset \bigcap\limits_{C\in\mathcal{C}} C^c$.

So $(\bigcup\limits_{C\in\mathcal{C}} C)^c = \bigcap\limits_{C\in\mathcal{C}} C^c$.

Next, we show that $(\bigcap\limits_{C\in\mathcal{C}} C)^c = \bigcup\limits_{C\in\mathcal{C}} C^c$.

Let $x \in (\bigcap\limits_{C\in\mathcal{C}} C)^c$.

Next, let $x \in \bigcup\limits_{C\in\mathcal{C}} C^c$.

Problem 3:

This is not true.

Let $\mathcal{C}$ be the collection of singleton sets in $[0, 1]$. Then $[0, 1]$ is an element of the σ-algebra generated by $\mathcal{C}$. Yet, $[0, 1]$ is not countable; it cannot be the countable union of a collection of singleton sets!

There can be no countable subcollection $\mathcal{C}_0$ of $\mathcal{C}$ such that $[0, 1]$ is contained in the σ-algebra generated by $\mathcal{C}$!

Problem 4:

Let $\langle x_n \rangle$ be a sequence of real numbers.

If $\lim_{n\to\infty} x_n$ exists, then $\langle x_n \rangle$ is Cauchy.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \frac{\epsilon}{2}$, where $L = \lim_{n\to\infty} x_n$.

Now, if $n, m \geq N$, then $|x_n - L| < \frac{\epsilon}{2}$ and $|x_m - L| < \frac{\epsilon}{2}$.

By the triangle inequality,

$$\begin{aligned} |x_n - x_m| &\leq |x_n - L| + |x_m - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $|x_n - x_m| < \epsilon$. That is, $\langle x_n \rangle$ is Cauchy.

Next, if $\langle x_n \rangle$ is Cauchy, then $\lim_{n\to\infty} x_n$ exists.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < \frac{\epsilon}{2}$.

This means that for all $n \geq N$, x_n is within the closed interval $[x_N - \frac{\epsilon}{2}, x_N + \frac{\epsilon}{2}]$.

So $\langle x_n \rangle$ has a converging subsequence. Say that the converging subsequence converges to L.

Now, $L \in [x_N - \frac{\epsilon}{2}, x_N + \frac{\epsilon}{2}]$. (Closed interval is closed.)

This means that for all $n \geq N$, $|x_n - L| < \epsilon$ (both x_n and L are in a closed ball of radius ϵ.)

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$ for some $L \in \mathbb{R}$. That is, $\langle x_n \rangle$ converges to L; $\lim_{n\to\infty} x_n$ exists.

Therefore $\lim_{n\to\infty} x_n$ exists if and only if $\langle x_n \rangle$ is Cauchy.

Problem 5:

Let $\langle x_n \rangle$ be a sequence of real numbers, with $\liminf(x_n)$ and $\limsup(x_n)$ both existing.

Then $\liminf(x_n)$ is the infimum of the set of blah.

Also, $\limsup(x_n)$ is the supremum of the set of blah.

Because $inf(A) \leq sup(A)$ for all nonempty $A \subset \mathbb{R}$, this means that $\liminf(x_n) \leq \limsup(x_n)$.

Moving on, let $\lim_{n \to \infty} x_n = L$.

Next, let $\liminf(x_n) = \limsup(x_n) = L$.

Problem 6:

It is a well-known fact that if there is an injection, $f : X \to Y$ and an injection $g : Y \to X$, then there is a bijection $h : X \to Y$. We exploit this fact.

Denote the Cantor set by C. A quick note: C is closed, as it is the complement of an open set. (It is usually defined as the complement of an infinite union of open intervals. This union of open intervals is open...and thus, its complement is closed.)

There is an injection $f : C \to [0, 1]$ given by $f(x) = x$.

Next, there is an injection $g : [0, 1] \to C$ given as follows:

Let $x \in [0, 1]$. We know that x has a binary expansion, $0.a_1 a_2 \ldots$. (Even in the case of $x = 1$, this expansion can be $0.11111\ldots$.)

Note that this binary expansion need not be unique. We just choose one (which we can do, by the Axiom of Choice).

We next define a sequence of nested closed subsets of C as follows:

If $a_1 = 0$, set $C_1(x) = [0, 1/3]$. Else, set $C_1(x) = [2/3, 1]$.

If $a_2 = 0$, set $C_2(x)$ equal to the lower third of $C_1(x)$. Else, set $C_2(x)$ equal to the upper third of $C_1(x)$.

For all $n > 1$, set $C_n(x)$ equal to the lower third of $C_{n-1}(x)$ if $a_n = 0$, else set $C_n(x)$ equal to the upper third of $C_{n-1}(x)$.

The sequence $\langle C_n \rangle$ is a sequence of nested, closed subsets of C whose diameter approaches 0. So, there is a unique point,y, in the intersection of these closed subsets.

We define $g(x) = y$.

Now, if $a, b \in [0, 1]$ and $a \neq b$, then $g(a) \neq g(b)$:

Because $a \neq b$, a and b have two different binary expansions, $0.a_1a_2\ldots$ and $0.b_1b_2\ldots$.

So there is an index, i, with $a_i \neq b_i$.

This means that $g(a)$ and $g(b)$ are contained in disjoint closed intervals, $C_i(a)$ and $C_i(b)$. So $g(a) \neq g(b)$.

We have an injection $f : C \to [0,1]$ and an injection $g : [0,1] \to C$, so there is a bijection $h : C \to [0,1]$.

Problem 7:

First, note that for any subset, A, of $\mathbb{R}$, the set of accumulation points, A', is a subset of the closure of A. That is $A' \subset \overline{A}$.

The Cantor Set is closed. So, $C' \subset C$.

Now, let $x \in C$.

Then $x \in [0, 1/3]$ or $x \in [2/3, 1]$. Define C_1 to be the interval that x is in.

Similarly, x is in either the lower third or the upper third of C_1. Define C_2 to be the third of C_1 that x is in.

For each $n \in \mathbb{N}$, define C_n to be the third of C_{n-1} that x is in.

Now, for each $n \in \mathbb{N}$, pick an $x_n \in C_n$ such that $x_n \neq x$.

Note that the C_ns are nested, closed subsets of $\mathbb{R}$ whose diameter approaches 0. This means that their intersection has a unique point. Because $x \in C_n$ for all $n \in \mathbb{N}$, this means that x is that unique point.

Moreover, it is clear that $\langle x_n \rangle \to x$. (Should I break out my epsilons, or is it OK to just state this?)

So there's a sequence $\langle x_n \rangle$ in C such that $\lim n \to \infty x_n \to x$ and $x_n \neq x$. So $x \in C'$.

So $C \subset C'$.

So $C = C'$. That is, the set of accumulation points of the Cantor Set is the Cantor Set itself.