Note: I am accustomed to writing "The element $g \in G$ acting on the element $x \in S$" as $g.x$ instead of gx. I use the stated notation, as I feel it is clearer.

Problem 1:

Let G be a finite abelian group, with $n \in \mathbb{N}$ and $n \mid |G|$.

We know that for each $n \in \mathbb{N}$, n has a unique prime factorization; that is, $n = p_1^{a_1} p_2^{a_2} p_3 a_3 \dots p_k^{a_k}$ for some $p_1, p_2 \dots p_k$ each prime, and $a_1, a_2 \dots a_k$ each positive and nonzero.

Proceed as follows:

For each p_i, there is an element with order p_i in G:

Thus, there is a subgroup, H_1, with order p_1 in G. This subgroup is normal, because G is abelian.

Consider the new group $G_1 = G/H_1$, along with $n_1 = n/p_1$. Note that $|G_1| = |G|/p_1$; this follows from the theorem that says $|G/H| = |G|/|H|$ if H is normal.

We know that G_1 is abelian:

So by similar logic as above, if $a_1 > 1$, then there is a subgroup of order p_1 in G. Otherwise, we know that there is a subgroup of order p_2 in G.

Either way, there is a subgroup, H_2', with order p_1 (or p_2) in G_1, which is normal.

Consider the group $G_2 = G_1/H_2$. Note that $G_2 \cong G/H_2$, by the third isomorphism theorem. Moreover, $|G_2| = |G|/p_1^2$ (or $|G_2| = |G|/p_1p_2$).

We can proceed in the above manner for a_i times for each p_i. We end up with a group, $G_{a_1+a_2 \dots +a_k}$.

Consider $G_{a_1+a_2 \dots +a_k}$; it has order $|G|/n$. It is isomorphic to $G/H_{a_1+a_2 \dots +a_k}$ for some $H \trianglelefteq G$. This means that $|H| = n$ (because $|G/H| = |G|/|H|$...thus, $|H| = |G|/|G/H|$, or in this case, $|H| = |G|/(|G|/n) = n$.)

So G has a normal subgroup of order n if G is a finite abelian group with $n \mid |G|$.

Problem 2:

Let $H < G$ with $[G : H]$ finite.

Problem 3:

Let G be a group acting transitively on a finite set, S, with $|S| > 1$.

Now, the action has only one orbit; for all $x \in S$, $\overline{x} = S$. In other words, for every $x, y \in S$ there is a $g \in G$ such that $g.x = y$.

Before proceeding, I wish to point out that I use the following freely:

If $g.x = x$, then $g^{-1}.x = x$: This is clear by applying g^{-1} to both sides of the equation.

If $g.x = y$, then $g^{-1}.y = x$: This is clear by applying g^{-1} to both sides of the equation.

Assume that for all $g \in G$, there is an $x \in S$ such that $g.x = x$. We proceed by constructing an infinite set of points in S, by induction.

Because $|S| > 1$, there are at least two distinct points of S: call them x_0 and x_1.

There is an element, g_2, such that $g_2.x_0 = x_1$, by transitivity of the action.

There is an x_2 such that $g_2.x_2 = x_2$, by the assumption we made earlier.

Now, $x_2 \neq x_0$, else:

$$\begin{aligned} g_2.x_0 &= g_2.x_2 \\ x_1 &= x_2 = x_0 \end{aligned}$$

which is a contradiction.

Also, $x_2 \neq x_1$, else:

$$\begin{aligned} g_2^{-1}.x_1 &= g_2^{-1}.x_2 \\ x_0 &= x_2 = x_1 \end{aligned}$$

which is also a contradiction.

So x_2 is distinct from x_0 and x_1.

Now, assume that we have the following: we have defined x_n for each $n \in \mathbb{N}$ such that $n < N$, and g_n for each $n \in \mathbb{N}$ such that $n < N - 1$ and $n \geq 2$, with the following properties: $g_n.x_n = x_n$ and $g_n.x_0 = x_{n-1}$.

Then there is a g_N such that $g_N.x_0 = x_{N-1}$, because the action is transitive.

Also, there is an x_N such that $g_N.x_N = x_N$, by the assumption we made earlier.

Now, $x_N \neq x_0$, else:

$$\begin{aligned} g_N.x_0 &= g_N.x_N \\ x_{N-1} &= x_N = x_0 \end{aligned}$$

which is a contradiction.
Also, $x_N \neq x_{N-1}$, else:

$$g_N^{-1}.x_{N-1} = g_N^{-1}.x_N$$
$$x_0 = x_N = x_{N-1}$$

which is also a contradiction.
Further, $x_N \neq x_i$ for any i between 0 and $N-1$ (exclusive), else:

$$g_N g_i^{-1}.x_i = g_N.x_i$$
$$g_n x_0 = g_N.x_N$$
$$x_{N-1} = x_N$$

which is also a contradiction.
So x_N is distinct from each x_i with $i < N$.
So we have two distinct points, and if we have n distinct points in S, we can make $n+1$ distinct points in S; we can make infinitely many distinct points, thus S is infinite.
So, if for all $g \in G$, g has a fixed point, then S is infinite.
Or, in other words, because S is finite, there is a $g \in G$ that has no fixed point.

Problem 4:
Let G be a group such that $G/Z(G)$ is cyclic.
Then $G/Z(G) = \langle \overline{a} \rangle$ for some $a \in G$.
Note that this fails if $G/Z(G)$ is only abelian:
Consider $D_8 = \langle r, s \rangle$. We note that the center of D_8 is $\{e, r^2\}$;

Also, $D_8/\left\langle r^2 \right\rangle$ is abelian:

But we know from an earlier homework that D_8 is not abelian. So in general, $G/Z(G)$ being abelian does not imply that G is abelian.

Problem 5:

Let p be prime, and let G be a group of order p^2.

We know that G has an element of order p; any element has order 1, p, or p^2. There are at least two elements in G (because 1 is not prime...). Pick an element other than e: call it a. We know that a has order p or p^2. If it has order p^2, then consider a^p. We know that $(a^p)^p = a^{p^2}$. So a^p is an element of order p.

The upshot is that there is an element, x, of order p in G. So there is a subgroup, $H = \langle x \rangle$, of order p in G. Moreover, H is cyclic (I think we proved this in class...any group of order p is cyclic. If we haven't proved it, then

Also, H is normal in G:

Now, consider G/H; this is a group of order p, it is cyclic.

Problem 6:

Let p be prime, and let G be a group of order p^n for some $n \in \mathbb{N}$. Let $H \trianglelefteq G$ with $H \neq \{e\}$.