Problem 1:

Part a:

The set of points z in the complex plane described by $|z - z_1| = |z - z_2|$ is the set of points equidistant from z_1 and z_2.

That is, it's described by the line going through the midpoint of z_1 and z_2 that is perpendicular to the line going through z_1 and z_2.

Part b:

The set of points z in the complex plane described by $1/z = \overline{z}$ is the set of points with $z\overline{z} = 1$, so that $|z| = \sqrt{z\overline{z}} = 1$.

That is, it's the unit circle centered at the origin.

Part c:

The set of points z in the complex plane described by $\text{Re}(z) = 3$ is the set of points with real component 3.

That is, it's described by a vertical line with x-intercept 3.

Part d:

The set of points z in the complex plane described by $\text{Re}(z) > c$ is the set of points with real component larger than c.

That is, it's described by the right half of a plane cut by the vertical line with x-intercept c (excluding the line itself).

Problem 2:

Let $z, w \in \mathbb{C}$, with $z \neq w$, be vertices of a square, with $z = (x, y)$ and $w = (x', y')$.

Either z and w are the opposite vertices of a square or they are adjacent vertices of a square.

If they are opposite vertices of a square, then there is a uniquely determined square given these vertices. (And this is geometrically clear.)

In this case, we proceed by finding the center of the square, finding a line perpendicular to the line through z and w through the midpoint, and finding points a and b on this line that make a square.

The center of the square is given by $A = (\frac{x+x'}{2}, \frac{y+y'}{2})$; it is the midpoint of two opposite sides.

Now, consider the vector $p = z - w = (x - x', y - y')$, and the vector $q = (y - y', -(x - x'))$. It is readily checked that $p \cdot q = 0$, so that these vectors are perpendicular.

Now, the points given by $a = A + q/2 = (\frac{x+x'+y-y'}{2}, \frac{y+y'-(x-x')}{2})$ and $b = A - q/2 = (\frac{x+x'-(y-y')}{2}, \frac{y+y'+x-x'}{2})$ are the desired points, as a, b, z, w are four points equidistant from A with the property that $(a - b) \cdot (z - w) = 0$ (that is, we can make opposite sides make a right angle with a point). We've already shown the second part: $p = z - w$ and $q = a - b$. Also, z and w are equidistant from A, as A is their midpoint. It is also readily checked that A is a and b's midpoint. Moreover, the length of p and q are the same (by symmetry). So $z = A + p/2$ and $a = A + q/2$ are equidistant from A; all of a, b, w, z are the same distance from A.

If they are adjacent vertices of a square, then there are exactly two different squares given these vertices. (And this is geometrically clear.)

In this case, we proceed by making a line through z perpendicular to the line through z and w, finding points of the appropriate length, and repeating the process with a line through w perpendicular to the line through z and w.

Now, consider the vector $p = z - w = (x - x', y - y')$, and the vector $q = (y - y', -(x - x'))$. It is readily checked that $p \cdot q = 0$, so that these vectors are perpendicular.

The points given by $a = z + q$ and $b = w + q$ together with z and w are the vertices of a square; because $p \cdot q = 0$, we have that these are the vertices of some rectangle (by similar work as above). Moreover, the lengths of each side are the same; the lengths of p and q as vectors are the same, and they represent the lengths of the sides of the shape.

Also, the points given by $a' = z - q$ and $b' = w - q$ together with z and w are the vertices of a square, similarly.

To summarize the above:

If z and w are the opposite corners of the square, then $(\frac{x+x'+y-y'}{2}, \frac{y+y'-(x-x')}{2})$ and $(\frac{x+x'-(y-y')}{2}, \frac{y+y'+x-x'}{2})$ are the other corners of the square.

If z and w are adjacent corners of the square, then either $z + q$ and $w + q$ are the other corners of the square or $z - q$ and $w - q$ are, where $q = (y - y', -(x - x'))$.

Problem 3:

Let $|a| = |b| = 1$, with $a = (x, y)$ and $b = (x', y')$, with $a \neq b$ and $\overline{a}b \neq 1$. First, note that in this case, $\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a-b|}{|1-\overline{a}b|}$. Now, we have:

$$
\begin{aligned}
(1 - \overline{a}b)\overline{(1 - \overline{a}b)} &= (1 - xx' - yy', xy' - x'y)(1 - xx' - yy', -xy' + x'y) \\
&= ((1 - xx' - yy')^2 + (xy' - x'y)^2, 0) \\
&= (1 - 2xx' - 2yy' + x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2, 0)
\end{aligned}
$$

Also,

$$
\begin{aligned}
(a - b)\overline{(a - b)} &= (x - x', y - y')(x - x', y' - y) \\
&= ((x - x')^2 + (y - y')^2, 0) \\
&= (2 - 2xx' - 2yy', 0)
\end{aligned}
$$

Now, because $|a| = |b| = 1$, we have:

$$
\begin{aligned}
x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2 &= (x^2 + y^2)(x'^2 + y'^2) \\
&= (1)(1) = 1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(1 - \overline{a}b)\overline{(1 - \overline{a}b)} &= (1 - 2xx' - 2yy' + x^2x'^2 + y^2y'^2 + x^2y'^2 + x'^2y^2, 0) \\
&= (2 - 2xx' - 2yy', 0) \\
&= (a - b)\overline{(a - b)}
\end{aligned}
$$

That is, we have that $(1 - \overline{a}b)\overline{(1 - \overline{a}b)} = (a - b)\overline{(a - b)}$, so that $|a - b| = |1 - \overline{a}b|$, producing the desired result.

Problem 4:

Let $f : [a, b] \to \mathbb{C}$ be continuous.
We take for granted that $[a, b]$ is connected.

Now, assume that $f([a,b])$ isn't connected. Then there are nonempty open sets, U and V, with $U \cap V = \emptyset$ and $U \cup V \supset f([a,b])$. Now, $f^{-1}(U)$ and $f^{-1}(V)$ are both open, as f is continuous. They are also disjoint; otherwise, there is a point x with $f(x) \in U$ and $f(x) \in V$. Moreover, $f^{-1}(U) \cup f^{-1}(V) = [a,b]$, because $U \cup V = f([a,b])$. So $f^{-1}(U)$ and $f^{-1}(V)$ disconnect $[a,b]$, which is impossible.

So by contradiction, $f([a,b])$ must be connected.