Chapter 3:

Problem 4:

Let $m^*(E) = \infty$ for an infinite set and $m^*(E) = |E|$ for a finite set.

It's clear that m^* is defined for all sets of real numbers, is translation invariant, and countably additive. So m^* is a measure; we call it the counting measure.

Problem 7:

If $m^*(E)$ is the Lebesgue Outer Measure, it's somewhat clear that it's translation invariant; we can do this by making an open cover and shifting it.

Problem 8:

If $m^*(A) = 0$, then $m^*(A \cup B) \geq m^*(B)$, by monotonicity.

But also, $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$ by countable subadditivity.

So $m^*(A \cup B) = m^*(B)$.

Problem 11:

Each (a, ∞) is measurable.

We have $\bigcap^{\infty}(n, \infty) = \emptyset$ which has measure 0, but $m((n, \infty)) \to \infty$. So $m(\bigcap^{n} E_i) \not\to m(\bigcap^{\infty} E_n)$

Problem 12:

Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets, and A be a set.

Then $m^*(A \cap \bigcup^n E_i) = \sum^n m^*(A \cap E_i)$.

So $m^*(A \cap \bigcup^\infty E_i) \geq \sum^n m^*(A \cap E_i)$.

But n is arbitrary, so $m^*(A \cap \bigcup^\infty E_i) \geq \sum^\infty m^*(A \cap E_i)$.

Either by employing a similar argument or appealing to countable subadditivity, we get $m^*(A \cap \bigcup^\infty E_i) = \sum^\infty m^*(A \cap E_i)$.

Problem 14:

Part a:

The Cantor set has measure zero; it's usually defined as

$$[0,1] \setminus ((1/3, 2/3) \cup ((1/9, 2/9) \cup (7/9, 8/9)) \cup \ldots)$$

Now, $[0,1]$ has measure 1, and is measurable.

Also, $((1/3, 2/3) \cup ((1/9, 2/9) \cup (7/9, 8/9)) \cup \ldots)$ has measure 1 (consider the geometric series $1/3, 2(1/3)^2 \ldots$. It sums up to 1).

So the measure of the cantor set is $1 - 1 = 0$.

Part b:

If we only remove $\alpha 3^- n$ at each step when we define the cantor set, then we can show that it would still be closed (as a complement in $[0,1]$ of an open set) and by employing the same geometric series argument, it would have measure $1 - \alpha$.

Problem 17:

Part a:

Consider the P_is as defined in this section. We're given that $m[0,1) = \sum m^* P_i = \sum m^* P$, so that the right hand side is either zero or infinite. But if it was zero, then we break countable subadditivity; it must be infinite. So we have an example where $m(\bigcup E_i) \leq \sum m^*(E_i)$.

Part b:
Define $E_0 = [0,1) \setminus P_0$ and $E_n = [0,1) \setminus P_n$ to get the desired result.

Problem 22:
Part a:
If f is measurable, then the restriction of f to any measurable set is measurable. If D_1 isn't measurable, then the interesection of all of the $\{x : f(x) \geq n\}$ isn't measurable, which is bad. Similarly, D_2 must be measurable.

Now, if all of D_1, D_2 and the restriction of f to $D \setminus D_1 \cup D_2$ are measurable, then for each α we get $\{x : f(x) \geq \alpha\}$ the union of D_1 and a measurable set, so we win.

Part b:
Apply the same trick as used earlier this chapter; prove that if f and g measurable, then so is f^2 and $f+g$, and win using $fg = 1/2[(f+g)^2 - f^2 - g^2]$.

Parts c and d are painfully trivial.
Problem 23:
This was a homework problem; just go there.

Problem 28:
I'm not sure how to do this one.

Problem 31:
Not sure how to do this one either. It looks like a very likely qual problem, too...:/

Chapter 4:

Problem 2:

Part a: Let f be a bounded function on $[a, b]$ and let h be the upper envelope of f (that is, $h(x) = \inf\limits_{\delta>0} \sup\limits_{|x-y|<\delta} (f(y)))$

Then $U - \int\limits_a^b f \geq \int\limits_a^b h$; let ϕ be a step function with $\phi \geq f$. Then $\phi \geq h$ except at a finite number of points, because step functions are discontinuous on only finitely many points and the upper envelope is lower than any continuous function above f.

Also, $U - \int\limits_a^b f \leq \int\limits_a^b h$; there's a sequence of step functions converging downwards to h, so by bounded convergence, we have our result.

So $U - \int\limits_a^b f = \int\limits_a^b h$.

Part b:

We get a similar result for the lower envelope. So a bounded function on $[a, b]$ is Riemann-integrable if and only if the integrals of its upper and lower envelopes are equal.

If the upper and lower envelopes are unequal on a set of greater than measure zero, this fails, as the lower envelope is always lower than the upper envelope.

If the upper and lower envelopes are equal except on a set of measure zero, this succeeds, rather obviously.

So a bounded function on $[a, b]$ is Riemann-integrable if and only if the upper and lower envelopes are equal except on a set of measure zero. That is, a bounded function on $[a, b]$ is Riemann-integrable if and only if the function is continuous except on a set of measure zero.

Problem 8:

Let $\langle f_n \rangle$ be a sequence of nonnegative functions on a domain, E. Define $f(x) = \liminf f_n(x)$.

Let $h \leq f$ be any non-negative, simple function with finite measure support on the domain (say it has finite measure support on F.

Then define $h_n = \min(h, f_n)$.

Now, $\int\limits_E h \leq \int\limits_F h = \lim \int\limits_F h_n \leq \liminf \int\limits_E f_n$.

By taking supremums over h, we have our result.

Problem 14:

NOTE: a similar problem was an exam problem. This problem can be generalized, and should be done in the context of L^p spaces.

Part a:

Let $\langle g_n \rangle \to g$ almost everywhere, $\langle f_n \rangle \to f$ almost everywhere, and $|f_n| \leq g_n$, with all of the above functions being measurable, and $\int g = \lim \int g_n$.

Then $|f_n - f| \to 0$ almost everywhere, and $|f_n - f| < g_n + g$. So by applying dominated convergence theorem, we have our result.

Part b:

Let $\langle f_n \rangle$ be a sequence of integrable functions in L^p with $f_n \to f$ almost everywhere.

If $\|f_n\| \not\to \|f\|$, then there's an $\epsilon > 0$ and a subsequence f_{n_k} with $|\|f_{n_k}\| - \|f\|| \geq \epsilon$. But

$$\begin{aligned} \|f_n - f\| &\geq |\|f_n\| - \|f\|| \\ &\not\to 0 \end{aligned}$$

If $\|f_n\| \to \|f\|$, then a modification of part a applies. So we have our result.

So $\|f_n - f\| \to 0$ if and only if $\|f_n\| \to \|f\|$.

Problem 15:

The entire problem is "Apply Littlewood's Three Principles" and the "$2^{-n}\epsilon$ trick". (On $[-1,1]$ there is a (property) function such that $|f-\phi_1| < 2^{-1}\epsilon/2$... similarly, there is such a function on $[-2,-1)$ and $(1,2]$ such that $|f-\phi_2| < 2^{-2}\epsilon/2$...induct, paste everything together, integrate, geometric series, win.)

Problem 16: NOTE: this was an exam problem.

First, note that if we have that this is true for all step functions vanishing except on a vinite interval, then we have our result; if $\lim_{n\to\infty} cos(nx)\phi(x)dx = 0$ for all such step functions ϕ, then because there's such a step function with $\int |f-\phi| < \epsilon$ for all $\epsilon > 0$, we have our result.

So, let ϕ be a step function on $[a,b]$, and let $\epsilon > 0$. Partition $[a,b]$ by $a = x_0 < x_1 \ldots x_l = b$ so that ϕ is constant on each (x_i, x_i+1). Let M be the maximum of $|\phi|$ (which exists, as ϕ takes only finitely many values). Pick n large enough so that $2\pi/n < \epsilon/(lM)$. Integrate over each chunk of the partition; we end up with everything cancelling out except on sets of length less than $2\pi/n$. There's at most l of them, having magnitude at most M; we've won.

Problem 22:

Note: This problem is lol.

Let there be a sequence, $\langle f_n \rangle$, of measurable functions on a set, E, of finte measure, with $f_n \to f$ in measure.

Then every subsequence of f_n converges to f in measure, so every subsequence has a subsequence converging to f in measure.

Now, let there be a sequence, $\langle f_n \rangle$, of measurable functions on a set, E, of finte measure, with every subsequence of f_n having a subsequence converging to f in measure. Then every subsequence of f_n has a subsequence which has every subsequence have a subsequence that converges almost everywhere to f. Thus, every subsequence of f_n has a subsequence that converges almost everywhere to f. So f_n converges to f in measure.

Problem 25:

...Seriously, the hint gives this entire question away. Pretty lame stuff, bro.

Chapter 5:

Problem 4:

Let f be continuous on $[a, b]$ and one of its derivates is everywhere nonnegative on (a, b). Then D^+ or D^- is everywhere nonnegative on (a, b); $D^+ \geq D_+$, and $D^- \geq D_-$.

We proceed by handling this for D^+, and note that the proof for D^- is similar.

First, if $D^+ \geq \epsilon > 0$ on (a, b), then $\limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \geq \epsilon$ for all x. If for any x there's an h such that $f(x+h) - f(x) < \epsilon$, then we can apply the sup method to find a contradiction (there should be a least possible h so that we have this property, but this breaks down pretty quickly).

Now, consider $g_\epsilon(x) = f(x) + \epsilon x$. Each g_ϵ is increasing, by the above thing. Now, $f = \lim_{\epsilon \to 0} g_\epsilon$. So f is a limit of increasing functions, it's increasing.

Problem 5/8:

5 is a mostly trivial epsilon-delta proof. Note: for part c, it's better to work from the right hand side than the left hand side.

8 is painfully clear.

Problem 10:

Note: I could've sworn one of these wasn't of bounded variation...

Part a:

(Note: we can restrict our attention to $[0, 1]$, rather clearly.)

Consider $f(x) = x^2 \sin(1/x^2)$ on $[0, 1]$ (with $f(0) = 0$).

This is decreasing when $1/x^2 \in [(4n+1)\pi/2, (4n+3)\pi/2]$ for some $n \in \mathbb{N}$.

So it's decreasing when $x^2 \in [2/((4n+3)\pi), 2/((4n+1)\pi)]$ for each $n \in \mathbb{N}$. So its negative variance is $\sum_{i=0}^{\infty} 2/((4n+1)\pi) - 2/((4n+3)\pi) = \sum_{i=0}^{\infty} 4/pi[1/(4n+1)(4n+3)]$. This sum converges, so the negative variance is finite. Similarly, the positive variance is finite. So the total variance is finite, the function is of bounded variation.

Part b:

Consider $f(x) = x^2 \sin(1/x)$ on $[0, 1]$ (with $f(0) = 0$).

This is decreasing when $1/x \in [(4n+1)\pi/2, (4n+3)\pi/2]$ for some $n \in \mathbb{N}$.

So it's decreasing when $x \in [2/((4n+3)\pi), 2/((4n+1)\pi)]$ for each $n \in \mathbb{N}$. So its negative variance is $\sum_{i=0}^{\infty}(2/((4n+1)\pi))^2 - (2/((4n+3)\pi))^2$, which converges. So the total variance is finite, the function is of bounded variation.

Problem 14:

Part a:

Sums and differences of two absolutely continuous functions are also absolutely continous, rather clearly. (If $\epsilon > 0$, use $\delta = \max(de_1, \delta_2)$, with δ_1 working for $\epsilon/2$ for one function and δ_2 working for $\epsilon/2$ with the other function.)

Part b:

Products of two absolutely continuous functions are absolutely continuous; the domain is necessarily a closed interval, continuous functions on closed intervals are bounded. Pick one function; it is absolutely bounded by M. For the other function, for all $\epsilon > 0$ there's a $\delta > 0$ that works for ϵ/M. Use this δ.

Part c:

I'm absolutely stuck on this one.

Problem 16:

Part a:

Let f be a monotone increasing function. Then f' exists almost everywhere; Let f_c be the indefinite integral of f', so that f_c is absolutely continuous. Then $f_s = f - f_c$ has derivative zero almost everywhere. That is, $f = f_s + f_c$ is a decomposition of f into a sum of singular and absolutely continuous functions.

Part b:

Let f be a nondecreasing singular function on $[a, b]$. Let $\epsilon > 0$, $\delta > 0$.

First, f is bounded; it is monotonic on a closed interval. Second, f' is zero almost everywhere. That is, there's a collection of nonoverlapping (open) intervals, call it $\mathcal{I}$, whose total length is less than δ that covers the set of points where f' is nonzero (or undefined).

That is, f is constant on the entire domain except (possibly) those intervals.

So, we have that $\sum\limits_{I\in\mathcal{I}} l(I) < \delta$, and $\sum\limits_{I\in\mathcal{I}} d(f(I)) = f(b) - f(a)$, where $d(f(I))$ is the distance between the endpoints of $f(I)$. (We have the second part, as f is constant except on intervals in $\mathcal{I}$.)

Thus, $\sum\limits_{I\in\mathcal{I}} d(f(I)) = f(b) - f(a)$ converges upwards to $f(b) - f(a)$; there's a finite collection of intervals in $\mathcal{I}$ (call it $\mathcal{F}$) with $\sum\limits_{I\in\mathcal{F}} d(f(I)) + \epsilon = f(b) - f(a)$. This satisfies the problem. (We summarize this result as "nondecreasing singular functions have property (S)").

Part c:

Let f be a nondecreasing function on $[a, b]$ with property (S). Then $f = f_s + f_c$ for some singular f_s and absolutely continuous f_c. Moreover, we can take both f_c and f_s nondecreasing.

Now, let $\epsilon > 0$. There's a $\delta > 0$ with the property $\sum |x_i - x_{i+1}| < \delta$ implies that $\sum |f_c(x_i) - f_c(x_{i+1})| < \epsilon$ for any non-overlapping, finite collection of intervals (x_i, x_{i+1}). Also, there's a finite collection of such intervals with the property $\sum |f(x_i) - f(x_{i+1})| + \epsilon > f(b) - f(a)$, for any $/ep > 0$.

Part d:

Let $\langle f_n \rangle$ be a sequence of nondecreasing singular functions on $[a, b]$ with $f = \sum f_n(x)$ everywhere finite.

Then f is singular, by term-by-term differentiation.

Part e:

A series of indicator functions of the form $\chi_{[q,1]}$ with $q \in \mathbb{Q}$ cobbled together with the 2^{-n} trick suffices. (Enumerate the rationals, sum them up.)

Problem 20:

Part a: is a simple epsilon-delta proof.

Part b:

Let f be absolutely continuous.

Then if f fails to satisfy a Lipschitz condition, there is a pair of sequences x_n, y_n that break the Lipschitz condition. As f is absolutely continuous, such a subsequence must exist with $d(x_n, y_n) \to 0$; this is because f is bounded. So we can find a sequence of pairs of points arbitrarily close together whose difference quotient is arbitrarily large; the derivative thus cannot be bounded.

If f satisfies a Lipschitz condition, then the difference quotient is uniformly bounded at all points; so the limits of these difference quotients are uniformly bounded, so the derivative is bounded.

Part c:

If one of the derivates of a function is bounded, say $D+$, then...

Problem 23:

Part a:

Let ϕ be a convex function on a finite interval, $[a, b)$.

Let ϕ not be bounded below. Because ϕ is absolutely continuous on each closed subinterval of $[a, b)$, this means that ϕ is bounded on each closed subinterval of $[a, b)$. Thus, if ϕ is not bounded below, then we have that $\lim_{x \to b} \phi(x) = -\infty$ (else, ϕ can be extended to $[a, b]$, and is thus bounded).

But this goes bad pretty quickly (Pick a point on the curve, draw a chord from $(a, \phi(a))$, make it have arbitrarily negative slope, it'll go below that point eventually.)

Part b:

Part c is trivial

Problem 27:

I've spent way too long on this problem. It's similar to a problem in Zigmund's book, apparently...

Chapter 6:

Problem 2:

Let $f \in L^\infty[0,1]$. Define $\|f\|_\infty = M$. Then for all p, $\|f\|_p/M = \|f/M\|_p$. It suffices to show that $\lim_{p\to\infty} \|f/M\|_p = 1$.

Now, for all $p > 0$, we have that $[\int |f/M|^p]^{1/p} \leq [\int |f/M|]^{1/p} \leq 1^{1/p} = 1$. So the limit as p tends to infinity is bounded above by 1.

Next, let $\epsilon \in (0,1]$. Then because the essential supremum of f/M is 1, there's a set, E, of measure $\delta > 0$ with the property $x \in E$ implies that $f(x) \geq |1 - \epsilon/2|$. Now,

$$\begin{aligned}
[\int |f/M|^p]^{1/p} &\geq [\int_E |f/M|^p]^{1/p} \\
&\geq [\int_E (1-\epsilon/2)^p]^{1/p} \\
&= [(1-\epsilon/2)^p m(E)]^{1/p} \\
&= (1-\epsilon/2) m(E)^{1/p}
\end{aligned}$$

By taking p sufficiently large, we have this greater than $1 - \epsilon$.

So we have our result.

Problem 4:

Let $f \in L^1$, $g \in L^\infty$. Define $M = \|g\|_\infty$.

$$\begin{aligned}
\int |fg/M| &\leq \int |f| \\
&= \|f\|_1
\end{aligned}$$

So,

$$\int |fg| \leq M\|f\|_1 = \|f\|_1 \|g\|_\infty$$

Problem 8:

This was effectively done in your homework.

Problem 10:

Note: This is effectively saying that the norm on L^∞ is the norm induced by the metric of uniform convergence. This is probably given outright in Pugh.

Let $\langle f_n \rangle$ be a sequence of functions in L^∞.

Let there fail to be a set, E, of measure 0 such that f_n converges to f uniformly on E^c. That is, there's a set of nonzero measure such that f_n fails to converge to f uniformly on E. Then there's an $\epsilon > 0$ and a subsequence of f_n with the property $\|f_{n_k} - f\|_\infty \geq \epsilon$ on a set of greater than measure zero. That is, there's a subsequence of f_n such that $\|f_{n_k} - f\|_\infty \geq \epsilon$; f_n does not converge to f in L^∞.

If there's a set, E, of measure 0 such that f_n converges to f uniformly on E^c, then for all $\epsilon > 0$ there's an $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n - f| < \epsilon$ except on E. That is, $\|f_n - f\|_\infty < \epsilon$. That is, $f_n \to f$ in L^∞.

Problem 11:

This is pulled straight out of Pugh; if a sequence of functions has cauchy uniform convergence, it has pointwise cauchy convergence, so it has pointwise convergence. This convergence is also uniform, somewhat trivially.

Problem 15:

Note: this looks like a good qual problem...

Let c be the space of all convergent sequences of real numbers, given the ℓ^∞ norm.

First, this is a normed linear space; ℓ^∞ norm is a norm, and the space is clearly linear.

Now, let $\langle\langle n_k\rangle\rangle$ be a Cauchy sequence of converging sequences.

Then for all $\epsilon > 0$, there's an N such that $n, m > N$ implies that for all k, $|n_k - m_k| < \epsilon$. That is, for fixed k, $\langle n_k\rangle$ treated as a sequence in n is Cauchy. So each of these converges to something; call it N_k. It is rather clear that $\langle n_k\rangle \to \langle N_k\rangle$ as a sequence using the ℓ^∞ norm from this.

Now, let c_0 be the space of all sequences of real numbers converging to zero. It is also a normed linear space. It's also a closed subset of c; if $\langle\langle n_k\rangle\rangle \to \langle a_k\rangle$ with $\langle n_k\rangle$ each in c_0, then $a_k \to 0$, this is clear by triangle inequality.

So c_0 is a closed subset of a complete normed linear space. It is complete. So c_0 is a Banach space.

Problem 18:

Note: This is very similar to problems given in past quals...

We first note that $f_n g_n \to fg$ almost everywhere, and that this is clear. We proceed by applying the an earlier problem; the result follows if $\|f_n g_n\|_p \to \|fg\|_p$.

Problem 23:

Problem 24:

Let $g, h \in L^q$ be such that $\int fg = \int fh$ for all $f \in L^p$.

Then $\int f(g-h) = 0$ for all $f \in L^p$. So in particular, this is true for all indicator functions on sets of finite length. We can game this so that $\int f(g-h)$ is always positive; $f = 1$ if $g-h$ is positive, and $f = -1$ if $h-g$ is positive works, and is measurable as h and g are. We know that this means that $f(g-h)$ is zero almost everywhere; thus, we get that $g-h$ is zero almost everywhere, so g must have been almost-everywhere-unique.