Problem 3, p100:

Let A be closed in X and B be closed in Y.

Then $X \setminus A$ is open in X and $Y \setminus B$ is open in Y.

So $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$ are open in $X \times Y$; these are products of open sets.

So , $X \times Y \setminus (X \setminus A) \times Y$ and $X \times Y \setminus X \times (Y \setminus B)$ are closed in $X \times Y$ by definition.

Now, $X \times Y \setminus (X \setminus A) \times Y = A \times Y$, and $X \times Y \setminus X \times (Y \setminus B) = X \times B$; $X \times Y \setminus (X \setminus A) \times Y = \{(x, y) \in X \times Y : (x, y) \notin (X \setminus A) \times Y\} = \{(x, y) \in X \times Y : x \notin (X \setminus A)\} = \{(x, y) \in X \times Y : x \in A\} = A \times Y$. Similarly, $X \times Y \setminus X \times (Y \setminus B) = \{(x, y) \in X \times Y : (x, y) \notin X \times (Y \setminus B)\} = \{(x, y) \in X \times Y : y \notin (Y \setminus B)\} = \{(x, y) \in X \times Y : y \in B\} = X \times B$.

So, $A \times Y \cap X \times B$ is closed. But $A \times Y \cap X \times B = A \times B$. So $A \times B$ is closed, as desired.

Problem 6b, p100:

Let A, B be subsets of a space, X.

First: if $A \subset B$, then $\overline{A} \subset \overline{B}$, because if $x \in \overline{A}$, then every neighborhood of x intersects A, so every neighborhood of x intersects B, so $x \in \overline{B}$.

Next, note that $\overline{A} \cup \overline{B}$ is closed; it's a union of closed sets. Moreover, $A \cup B \subset \overline{A} \cup \overline{B}$, because $A \subset \overline{A}$ and $B \subset \overline{B}$. Now, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, because $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, and $\overline{A \cup B}$ is the intersection of all closed sets containing $A \cup B$.

Next, let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$. So $x \in \overline{A \cup B}$, because $A \subset A \cup B$. That is, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

So, $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Problem 6c, p100:

Let A_α be a collection of subsets of a space, X.

Let $x \in \bigcup \overline{A_\alpha}$. Then $x \in \overline{A_\alpha}$ for some α. So every neighborhood of x intersects A_α for some α, by theorem 17.5. So every neighborhood of x intersects $\bigcup A_\alpha$. So $x \in \overline{\bigcup A_\alpha}$.

That is, $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$.

Equality fails; consider $\mathbb{R}$ with the standard topology. Then $(1/n, 1]$ has closure $[1/n, 1]$ for each $n \in \mathbb{N}$ (as in Example 6 on p96). Now, $0 \notin$

$\bigcup_{n=2}^{\infty} \overline{(1/n, 1]} = \bigcup_{n=2}^{\infty} [1/n, 1]$, else $0 \in [1/n, 1]$ for some $n \in \mathbb{N}$, which is obvious nonsense. But $0 \in \overline{\bigcup_{n=2}^{\infty} (1/n, 1]}$; every neighborhood of zero contains $2/n$ for some $n \in \mathbb{N}$ larger than 2 (this is a basic fact about the real numbers). So every neighborhood of zero intersects $(1/n, 1]$ for some $n \in \mathbb{N}$. So every neighborhood of zero intersects $\bigcup_{n=2}^{\infty} (1/n, 1]$, so by theorem 17.5, $0 \in \overline{\bigcup_{n=2}^{\infty} (1/n, 1]}$.

Problem 7, p100:

It fails here: "...U must intersect some A_α, so that x must belong to the closure of some A_α." We need a little more power than we're given: We have that every neighborhood intersects some A_α, which may depend on U. However, we need the A_α to be fixed with respect to U to apply theorem 17.5.

This is the line where I would make a joke about the looseness of the word "criticize" in the problem statement, but I am too unfunny to pull this off.

Problem 9, p100:

Let $A \subset X$ and $B \subset Y$.

First, note that $\overline{A}$ and $\overline{B}$ are closed, as they are the closures of some set. Now, $\overline{A} \times \overline{B}$ is closed, by exercise 3 (done above, in this homework set). Moreover, note that $\overline{A} \times \overline{B}$ contains $A \times B$, as $A \subset \overline{A}$ and $B \subset \overline{B}$. So, $\overline{A} \times \overline{B}$ is a closed set containing $A \times B$; $\overline{A \times B} \subset \overline{A} \times \overline{B}$, because $\overline{A \times B}$ is the intersection of all closed sets containing $A \times B$.

Let $(x, y) \in \overline{A} \times \overline{B}$. Then $x \in \overline{A}$ and $y \in \overline{B}$. So every neighborhood of x intersects A and every neighborhood of y intersects B, by theorem 17.5. So every neighborhood of (x, y) intersects $A \times Y$ and $X \times B$. So every neighborhood of (x, y) intersects $A \times B$, because $A \times Y \cap X \times B = A \times B$. So $(x, y) \in \overline{A \times B}$, by theorem 17.5.

Problem 10, p100:

Let X be an ordered set, and give X the order topology.

Let $a, b \in X$, with $a \neq b$. Without loss of generality, say that $a < b$.

Either a is the smallest element of X or not.

Either b is the largest element of X or not.

Either there is a $c \in X$ with $a < c < b$ or not.

If a is not the smallest element of X, b is not the largest element of X, and there is not $c \in X$ with $a < c < b$, then there are A and B with $A < a$ and $b < B$. The sets (A, b) and (a, B) have $a \in (A, b)$ (as $A < a < b$) and $b \in (a, B)$ (as $a < b < B$). Also, $(A, b) \cap (a, B) = (a, b) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is the smallest element of X, b is not the largest element of X, and there is not $c \in X$ with $a < c < b$, then there is B with $b < B$. The sets $[a, b)$ and (a, B) have $a \in [a, b)$ and $b \in (a, B)$ (as $a < b < B$). Also, $[a, b) \cap (a, B) = (a, b) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is not the smallest element of X, b is the largest element of X, and there is not $c \in X$ with $a < c < b$, then there is A with $A < a$. The sets (A, b) and $(a, b]$ have $a \in (A, b)$ (as $A < a < b$) and $b \in (a, b]$. Also, $(A, b) \cap (a, b] = (a, b) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is the smallest element of X, b is the largest element of X, and there is not $c \in X$ with $a < c < b$, then the sets $[a, b)$ and $(a, b]$ have $a \in [a, b)$ and $b \in (a, b]$. Also, $[a, b) \cap (a, b] = (a, b) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is not the smallest element of X, b is not the largest element of X, and there is $c \in X$ with $a < c < b$, then pick some such c. Now, there are A and B with $A < a$ and $b < B$. The sets (A, c) and (c, B) have $a \in (A, c)$ (as $A < a < c$) and $b \in (c, B)$ (as $c < b < B$). Also, $(A, c) \cap (c, B) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is the smallest element of X, b is not the largest element of X, and there is $c \in X$ with $a < c < b$, then pick some such c. Now, there is B with $b < B$. The sets $[a, c)$ and (c, B) have $a \in [a, c)$ and $b \in (c, B)$ (as $c < b < B$). Also, $[a, c) \cap (c, B) = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is not the smallest element of X, b is the largest element of X, and there is $c \in X$ with $a < c < b$, then pick some such c. Now, there is A with $A < a$. The sets (A, c) and $(c, b]$ have $a \in (A, c)$ (as $A < a < c$) and $b \in (c, b]$. Also, $(A, c) \cap (c, b] = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

If a is the smallest element of X, b is the largest element of X, and there is $c \in X$ with $a < c < b$, then pick some such c. Now, the sets $[a, c)$ and $(c, b]$ have $a \in [a, c)$ and $b \in (c, b]$. Also, $[a, c) \cap (c, b] = \emptyset$; that is, the points a and b are separated by open sets in the order topology.

So in all cases, the points a and b are separated by open sets in the order topology; the order topology is Hausdorff.

(Question: Would appealing to Theorem 17.11 have gotten me points for this?)

Problem 13, p100:

Let X be a Hausdorff space. Let $(a, b) \in X \times X \setminus \Delta$, with $\Delta = \{(x, x) : x \in X\}$. Then there are U, V open in X with $a \in U$, $b \notin U$, $a \notin V$, $b \in V$, and $U \cap V = \emptyset$. Note that $U \times V \cap \Delta = \emptyset$; else, there is $(x, x) \in U \times V$ for some $x \in X$, so that there is an $x \in U \cap V$, which contradicts the fact that $U \cap V$ is empty. So, for each $(a, b) \in X \times X \setminus \Delta$, there's a neighborhood of (a, b) contained in $X \times X \setminus \Delta$. That is $X \times X \setminus \Delta$ is open in $X \times X$; so Δ is closed in $X \times X$.

So if X is a Hausdorff space, then Δ is closed in $X \times X$.

Let $\Delta = \{(x, x) : x \in X\}$ be closed in $X \times X$. Pick $a, b \in X$ with $a \neq b$. Then consider $(a, b) \in X \times X$; because $(a, b) \notin \Delta$, $(a, b) \in X \times X \setminus \Delta$. Now, $X \times X \setminus \Delta$ is open, because Δ is closed. So there are U and V each open in X such that $(x, y) \in U \times V$ and $U \times V \cap \Delta = \emptyset$, because products of open sets are a basis for the product topology. Because $U \times V \cap \Delta = \emptyset$, the points (a, a) and (b, b) are not in $U \times V$. Now, U contains a and V contains b, because $(a, b) \in U \times V$. Also, $b \notin U$, else $(b, b) \in U \times V$. Also, $a \notin V$, else $(a, a) \in U \times V$. So U is an open set in X containing a and not b, and V is an open set in X containing b and not a.

So if Δ is closed in $X \times X$, then any two points can be separated by open sets in X; that is, X is Hausdorff.

Problem 4, p111:

Fix $x_0 \in X$, $y_o \in Y$, with X and Y topological spaces. Consider $f : X \to X \times Y$ and $g : Y \to X \times Y$ given by $f(x) = (x, y_0)$ and $g(y) = (x_0, y)$.

First, f and g are injective: let $f(a) = f(b)$. Then $f(a) = (a, y_0) = f(b) = (b, y_0)$, so that $a = b$. Similarly, if $g(a) = g(b)$,then $g(a) = (x_0, a) = g(b) = (x_0, b)$, so that $a = b$

Next, f and g are continuous: let W be an open set in $X \times Y$. Then $f^{-1}(W) = \{x \in X : (x, y_0) \in W\}$. Now, for each $x \in f^{-1}(W)$, there is a pair of open sets $U \subset X$ and $V \subset Y$ with $(x, y_0) \in U \times V$ and $U \times V \subset W$. So, there is a $U \subset X$ with $x \in U$ and $U \subset f^{-1}(W)$. So $f^{-1}(W)$ is open in X. So $f^{-1}(W)$ is open in X for all W open in $X \times Y$; f is continuous. Similarly, let W be an open set in $X \times Y$. Then $g^{-1}(W) = \{y \in Y : (x_0, y) \in W\}$. Now, for each $y \in g^{-1}(W)$, there is a pair of open sets $U \subset X$ and $V \subset Y$ with $(x_0, y) \in U \times V$ and $U \times V \subset W$. So, there is a $V \subset Y$ with $y \in V$ and $V \subset g^{-1}(W)$. So $g^{-1}(W)$ is open in Y. So $g^{-1}(W)$ is open in Y for all W open in $X \times Y$; g is continuous.

Next, f and g map onto $X \times \{y_0\}$ and $\{x_0\} \times Y$, respectively; this is clear.

We can readily construct an inverse to f and g; the maps $f^{-1} : X \times \{y_0\} \to X$ and $g^{-1} : \{x_0\} \times Y \to Y$ given by $f^{-1}(x, y_0) = x$ and $g^{-1}(x_0, y) = y$ work, and this is clear.

These inverses are continuous; let W be open in X. Then consider $f^{-1}(W) = W \times \{y_0\}$; this is open in $X \times \{y_0\}$, as $W \times \{y_0\} = W \times Y \cap X \times \{y_0\}$, which is the intersection of an open set in the space $X \times Y$ and the subspace $X \times \{y_0\}$. That is, $f^{-1}(W)$ is open if W is; f is continuous. Similarly, let W be open in Y. Then consider $g^{-1}(W) = \{x_0\} \times W$; this is open in $\{x_0\} \times Y$, as $\{x_0\} \times W = X \times W \cap \{x_0\} \times Y$, which is the intersection of an open set in the space $X \times Y$ and the subspace $\{x_0\} \times Y$. That is, $g^{-1}(W)$ is open if W is; g is continuous.

So, f and g are injective, continuous, and have continuous inverses on their image sets; f and g are imbeddings.

Problem 8a, p111:

Let Y be an ordered set, given the order topology. Let f, g be continuous maps from X to Y.

Consider $A = \{x : f(x) \leq g(x)\}$.

Let $x \in \overline{A}$. Then every neighborhood, U, of x intersects A. Let U be a basic neighborhood (read: "interval") of $f(x)$ and V be a basic neighborhood of $g(x)$ with the property that $U \cap V = \emptyset$ (we can do this, as Y is Hausdorff; this is problem 10 on page 100. So we can choose open sets with these properties, and so we can choose a basis element with these properties by simply choosing any basis element contained in U (or V) that contains $f(x)$ (or $g(x)$).).

Now, $f^{-1}(U)$ and $g^{-1}(V)$ are both open, because f and g are continuous. Moreover, they are both neighborhoods of x, as U contained $f(x)$ and V

contained $g(x)$. Now, consider $B = f^{-1}(U) \cap g^{-1}(V)$. Then $x \in B$, and B is open, as it's an intersection of two open sets; that is, B is a neighborhood of x. So, $B \cap A$ is nonempty, by theorem 17.5. So there is some $a \in B$ with $f(a) \leq g(a)$. So there is some $a \in f^{-1}(U) \cap g^{-1}(V)$ with $f(a) \leq g(a)$.

This means that for all $u \in U$, $v \in V$, $u < v$ (see Appendix A).

So, $f(x) \leq g(x)$, because $f(x) \in U$ and $g(x) \in V$. So, $x \in A$.

So, $\overline{A} \subset A$. So because $\overline{A} \supset A$ (this is clear from Theorem 17.6), we have that $\overline{A} = A$. This means that A is closed; this is mentioned on page 95 of Munkres, right in the middle.

Problem 8b, p111:

Let Y be an ordered set, given the order topology. Let f, g be continuous maps from X to Y.

Let $h : X \to Y$ be the function $h(x) = \min f(x), g(x)$.

Then consider $A = \{x : f(x) \leq g(x)$ and $B = \{x : g(x) \leq f(x)\}$. From problem 8a, both A and B are closed. Moreover, $X = A \cup B$, (as for all $x \in X$, $f(x) \leq g(x)$ or $g(x) \leq f(x)$).

Now, consider $f_A : A \to Y$ given by $f_A(x) = f(x)$ and $g_B : B \to Y$ given by $g_B(x) = g(x)$. Then note that $f_A(x) = g_B(x)$ on $A \cap B$, because on $A \cap B$, we have $f(x) \leq g(x)$ and $g(x) \leq f(x)$ so that $f(x) = g(x) = f_A(x) = g_B(x)$.

So, $h : X \to Y$ given by $h(x) = f_A(x)$ on A and $h(x) = g_B(x)$ on B is continuous. That is, $h(x) = \min f(x), g(x)$ is continuous.

Problem A:

Let X be a topological space with open sets U_i for $i = 1, 2, 3 \ldots n$, with $\overline{U_i} = X$ for all i.

Then consider $A = \overline{\bigcap_{i=1}^{n} U_i}$.

First, $A \subset X$, and this is clear.

Next: let $x \in X$. Then for any open neighborhood of x, say, U, the intersection $U \cap U_i$ is nonempty for all $i \leq n$; this is because $\overline{U_i} = X$. The intersection $\bigcap_{i=1}^{n} U \cap U_i$ is open, as it is a finite intersection of open sets. This set equals $U \cap \bigcap_{i=1}^{n} U_i$, by known set theory. It is also nonempty, and we prove this by induction: we know that U_1 intersects every open set in X (else, there is some neighborhood of some point that U_1 fails to intersect, so that $\overline{U_1} \neq X$, which is a contradiction of our original assumptions on U_i). If $\bigcap_{i=1}^{m} U_i$ intersects every open set for $m < n$, then consider $\bigcap_{i=1}^{m+1} U_i = U_{m+1} \cap \bigcap_{i=1}^{m} U_i$. Now, let U be an open set. Then $U \cap \bigcap_{i=1}^{m} U_i$ is nonempty and open. So because U_{m+1} intersects every open set (as above), then $U \cap U_{m+1} \cap \bigcap_{i=1}^{m} U_i$ is nonempty; that is, $\bigcap_{i=1}^{m+1} U_i$ intersects every open set. By induction, $\bigcap_{i=1}^{n} U_i$ intersects every open set, so that $U \cap \bigcap_{i=1}^{n} U_i$ is nonempty when U is open.

To summarize, for any open neighborhood of any $x \in X$, $\bigcap_{i=1}^{m+1} U_i$ intersects said neighborhood. That is, $x \in \overline{\bigcap_{i=1}^{n} U_i}$, for all $x \in X$. So $X \subset A$.

So $X = A$.

Appendix A:

Let Y be an ordered set, (a, b) and (c, d) be disjoint open intervals, and let there exist $x \in (a, b)$ and $y \in (c, d)$ with $x < y$.

Let there exist x', y' with $x' \in (a, b)$, $y' \in (c, d)$, and $x' \geq y'$. It is clear that $x' \neq y'$, else (a, b) and (c, d) were not disjoint. So, $x' > y'$. Now, $y' > c$ and $x' < b$, as $x' \in (a, b)$ and $y' \in (c, d)$. So, we have that $c < y' < x' < b$. That is, $c < b$. So, $(a, b) \cap (c, d) = (c, b)$, which is nonempty (as y' and x' are in (c, b). This contradicts our assumption that this set was empty.

So, if Y is an ordered set, (a, b) and (c, d) are disjoint open intervals, and there exist $x \in (a, b)$ and $y \in (c, d)$ with $x < y$, then $x' < y'$ for all $x' \in (a, b)$, $y' \in (c, d)$.