Problem 1:

Consider $\int\limits_0^\infty \frac{1-\cos(x)}{x^2}dx$. Note that this must be positive. (If this is not clear, we can estimate the integral from below by an alternating series that converges to something positive.)

Now, $\int\limits_0^T \frac{1-\cos(z)}{z^2}dz = \int\limits_0^T \frac{1-\frac{e^{iz}+e^{-iz}}{2}}{z^2}dz = -\left[\int\limits_0^T \frac{e^{iz}-1}{2z^2}dz + \int\limits_0^T \frac{e^{-iz}-1}{2z^2}dz\right]$. Both of the functions under the integrands are holomorphic, except at the origin. Using a u-substitution, we get $\int\limits_0^T \frac{e^{-iz}-1}{2z^2}dz = -\int\limits_0^{-T} \frac{e^{iz}-1}{2z^2}dz$.

So,

$$\begin{aligned}
\int\limits_0^T \frac{1-\cos(z)}{z^2}dz &= -\left[\int\limits_0^T \frac{e^{iz}-1}{2z^2}dz - \int\limits_0^{-T} \frac{e^{iz}-1}{2z^2}dzdz\right] \\
&= \frac{1}{2}\left[\int\limits_0^T \frac{1-e^{iz}}{z^2}dz - \int\limits_0^{-T} \frac{1-e^{iz}}{z^2}dz\right] \\
&= \frac{1}{2}\left[\int\limits_{-T}^T \frac{1-e^{iz}}{z^2}dz\right]
\end{aligned}$$

Now, we know that $\int\limits_{-T}^T \frac{1}{z^2}dz = \infty$. The remaining term is more difficult to handle.

First, by Cauchy's Theorem, we can integrate the remaining term along the path γ, pictured below:

So,

$$\frac{1}{2}\left[\int_{-T}^{T}\frac{1-e^{iz}}{z^2}dz\right] = \frac{1}{2}\left[\int_{-T}^{-T+i\sqrt{T}}\frac{1-e^{iz}}{z^2}dz\right] + \frac{1}{2}\left[\int_{-T+i\sqrt{T}}^{T+i\sqrt{T}}\frac{1-e^{iz}}{z^2}dz\right] + \frac{1}{2}\left[\int_{T+i\sqrt{T}}^{T}\frac{1-e^{iz}}{z^2}dz\right]$$

Using the ML-inequality/trivial estimate, we can estimate the first term: $\int\limits_{-T}^{-T+i\sqrt{T}} \frac{1-e^{iz}}{z^2}dz \leq \sqrt{T}\sup(\left|\frac{1-\cos(z)}{z^2}\right|) = \frac{1}{\sqrt{T}}$

Similarly, we can estimate the last term: $\int\limits_{T+i\sqrt{T}}^{T} \frac{1-e^{iz}}{z^2}dz \leq \sqrt{T}\sup(\left|\frac{1-\cos(z)}{z^2}\right|) = \frac{1}{\sqrt{T}}$

And we can estimate the middle term: $\int\limits_{-T+i\sqrt{T}}^{T+i\sqrt{T}} \frac{1-e^{iz}}{z^2}dz \leq 2T\sup(\left|\frac{1-\cos(z)}{z^2}\right|) = 2/T$.

So as $T \to \infty$, all of these terms vanish. So $\frac{1}{2}\left[\int\limits_{-T}^{T} \frac{1-e^{iz}}{z^2}dz\right] = \frac{1}{2}\left[\int\limits_{-T}^{-T+i\sqrt{T}} \frac{1-e^{iz}}{z^2}dz\right]$ vanishes as $T \to \infty$.

So $\int\limits_{0}^{\infty} \frac{1-\cos(x)}{x^2}dx = 0$.

Problem 2:

Let $\Omega \subset \mathbb{C}$ be open and simply connected, $f \in \mathcal{O}(\Omega)$, f is always nonzero, $k \in \mathbb{Z}^+$.

There is an $h \in \mathcal{O}(\Omega)$ such that $e^h = f$. Define $\tilde{h} = h/k$. Then:

$$
\begin{aligned}
e^{\tilde{h}k} &= f \\
e^{\tilde{h}+\tilde{h}+\tilde{h}\ldots+\tilde{h}} &= f \\
e^{\tilde{h}}e^{\tilde{h}}e^{\tilde{h}}\ldots e^{\tilde{h}} &= f \\
(e^{\tilde{h}})^k &= f
\end{aligned}
$$

So, if $\Omega \subset \mathbb{C}$ be open and simply connected, $f \in \mathcal{O}(\Omega)$, f is always nonzero, $k \in \mathbb{Z}^+$, then there's a $g \in \mathcal{O}(\Omega)$ with $g^k = f$.

Now, if $k \in \mathbb{Z}^-$, then find h with $h^{-k} = f$. Next, define $g = 1/h$. Then we have that $g^k = \frac{1}{h}^k = e^{\ln(1/h)k} = e^{-ln(h)k} = h^{-k} = f$, which yields our result.

Problem 3:

Consider $\sqrt[\sqrt{-1}]{-1} = (-1)^{sqrt-1} = (e)^{\ln(-1)\sqrt{-1}} = e^{\ln -1 e^{\frac{1}{2}\ln(-1)}}$. As discussed in class, the logarithms of -1 are $(2k+1)\pi i$ for each $k \in \mathbb{Z}$. That is, the possible values of $\sqrt[\sqrt{-1}]{-1}$ are given by

$$e^{((2k+1)\pi i)e^{\frac{1}{2}((2j+1)\pi i)}}$$

for any given $k, j \in \mathbb{Z}$.

Yet, this is an intractible mess. Consider that $e^{\frac{1}{2}((2j+1)\pi i)} = e^{j\pi i + \frac{1}{2}\pi i} = e^{j\pi i}e^{\frac{1}{2}\pi i} = (-1)^j e^{\frac{1}{2}\pi i}$. Thus, our original expression becomes

$$e^{((2k+1)\pi i)(-1)^j e^{\frac{1}{2}\pi i}}$$

To clean this up even more, $e^{\frac{1}{2}\pi i} = i$. So, we have our original expression as

$$e^{((2k+1)\pi i)(-1)^j i} = e^{-((2k+1)\pi)(-1)^j}$$

Now, it is somewhat clear that $\{e^{-((2k+1)\pi)(-1)^j} : j, k \in \mathbb{Z}\} = \{e^{\pm((2k+1)\pi)} : k \in \mathbb{Z}\} = \{e^{-((2k+1)\pi)} : k \in \mathbb{Z}\}$.

So, the set of values $\sqrt[\sqrt{-1}]{-1}$ are $\{e^{-((2k+1)\pi)} : k \in \mathbb{Z}\}$.
And yes, taking $k = -1$ yields a value of e^{π}, which is "about 23".

Problem 4:
Let $\ln(z)$ be the principal branch of the logarithm of z, and let z_1, z_2 have positive real component.

Then $e^{\ln(z_1)+\ln(z_2)} = e^{\ln(z_1)}e^{\ln(z_2)} = z_1 z_2 = e^{\ln(z_1 z_2)}$.

Now, e^{a+bi} is one-to-one given that $a \in \mathbb{R}$ and $b \in (-\pi, \pi)$. Because we're working in the principal branch and the real components of z_1 and z_2 are (strictly) positive, $z_1 z_2 = e^{a+bi}$ has $a \in \mathbb{R}$ and $b \in (-\pi, \pi)$. For the same reason, $\ln(z_1) + \ln(z_2) = e^{a'+b'i}$ has $a' \in \mathbb{R}$ and $b' \in (-\pi, \pi)$. So e^z is one-to-one for a domain containing both $\ln(z_1) + \ln(z_2)$ and $\ln(z_1 z_2)$. Thus, $\ln(z_1) + \ln(z_2) = \ln(z_1 z_2)$.

Problem 5:
Consider $\sin(\frac{1}{z})$. We know that $\sin(z) = \sum\limits_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$. So, where defined, $\sin(\frac{1}{z}) = \sum\limits_{n=0}^{\infty}(-1)^n \frac{\frac{1}{z}^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n+1}(2n+1)!}$.

That is, we have found a Laurent series for $\sin(\frac{1}{z})$ about 0. We are done.

Problem 6:
Consider $\frac{\sin(z)}{1-z}$. Because $\frac{1}{1-z} = \sum\limits_{n=0}^{\infty} z^n$ (when $z \in D_1(0)$, which we are working on because of the singularity at 1) and $\sin(z) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, we have $\frac{\sin(z)}{1-z} = \sum\limits_{n=0}^{\infty} z^n \sum\limits_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} a_n z^n$.

The first seven coefficients of this expansion (that is, those with $n \leq 6$), are as follows (this follows trivially by computation, which I will invariably screw up.)

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= 1 \\
a_3 &= 5/6 \\
a_4 &= 5/6 \\
a_5 &= 5/6 + 1/60 \\
a_6 &= 5/6 + 1/60
\end{aligned}
$$

Problem 7:

(Preliminary note: these integrals land in the real numbers...there's no ambiguity about what ln is...).

Let $f \in \mathcal{O}(D_R(0))$. Consider $\ln(\int\limits_0^{2\pi} |f(e^{s+it})|^2 \, dt)$ as a function of s.

We can apply Parseval's Formula (one of the earlier homeworks): let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$.

Then $\ln(\int\limits_0^{2\pi} |f(e^{s+it})|^2 \, dt) = \ln(2\pi \sum\limits_{n=0}^{\infty} |a_n|^2 e^{2sn})$. Moreover, we have $\ln(2\pi \sum\limits_{n=0}^{\infty} |a_n|^2 e^{2sn}) = \ln(2\pi \lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} |a_n|^2 e^{2sn})$.

By the earlier homework, the sums of log-convex functions are log-convex. By a trivial induction argument, all finite sums of log-convex functions are log-convex. Moreover, $\ln(2\pi |a_n|^2 e^{2sn})$ is convex; $\ln(2\pi |a_n|^2 e^{2sn}) = 2sn \ln([2\pi |a_n|^2]^{-2sn}) = 2snc$ for some $c \in \mathbb{R}$, which is clearly convex as a function of s.

So we have that $2\pi \sum\limits_{n=0}^{N} |a_n|^2 e^{2sn}$ is log-convex, for all $N \in \mathbb{N}$; in other words, $\ln(2\pi \sum\limits_{n=0}^{N} |a_n|^2 e^{2sn})$ is convex for all N.

Now, the limit of a sequence of log-convex functions is log-convex: let $x, y \in \mathbb{R}$, and $t \in [0, 1]$, and let $\phi_N \to \phi$ be a sequence of log-convex functions. Then:

$$t\ln(\phi_N(x)) + (1-t)\ln(\phi_N(y)) \leq \ln(\phi_N(tx+(1-t)y))$$
$$t\ln(\phi(x)) + (1-t)\ln(\phi(y)) \leq \ln(\phi(tx+(1-t)y))$$

because inequality is preserved over limits, because ln is continuous.

Thus, $2\pi \lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} |a_n|^2 e^{2sn}$ is log-convex: So, $\ln(2\pi \lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} |a_n|^2 e^{2sn}) = \ln(\int\limits_0^{2\pi} |f(e^{s+it})|^2 \, dt)$ is convex; this is the result we wanted.

Problem 8:

Let $\psi, \phi \in \mathcal{O}(\mathbb{C})$, and $|\psi| \leq |\phi|$ on $\mathbb{C}$.

First, $\phi = 0$, $\psi = 0$ trivially by the assumption.

Next, $|\psi| / |\phi| \leq 1$ on $\mathbb{C}$, except where $\phi = 0$. Thus, $\left|\frac{\psi}{\phi}\right| \leq 1$ on $\mathbb{C}$, except where $\phi = 0$. Because $\frac{\psi}{\phi}$ is bounded, all of its singularities are removable; we can define ξ holomorphic and equal to $\frac{\psi}{\phi}$ except where $\phi = 0$.

Now, ξ is a bounded, entire function; it is constant, by Liouville.

So $\frac{\psi}{\phi} = c$ on $\mathbb{C}$, except where $\phi = 0$, for some $c \in \mathbb{C}$. Also, $\phi = \psi = c\phi$ where $\phi = 0$. Thus, $\psi = c\phi$.

Problem 9:

(Without loss of generality, let $c = 0$.)

Let f have an essential singularity at 0, and let $f = \sum\limits_{-\infty}^{\infty} a_n z^n$ on some disk of radius r. Now, $a_n \neq 0$ for infinitely many negative $n \in \mathbb{Z}$. Now, let $1/f = \sum\limits_{-\infty}^{\infty} b_n z^n$.

Now, $f1/f = 1 = \sum\limits_{-\infty}^{\infty} a_n z^n \sum\limits_{-\infty}^{\infty} b_n z^n$. So...