Problem 1:

Consider $\prod_{n=1}^{\infty} \cos(\frac{z}{2^n})$.

To evaluate this, we apply the telescoping trick; so, note that for each $N \in \mathbb{N}$, $\sin(\frac{z}{2^N}) \prod_{n=1}^{N} \cos(\frac{z}{2^n}) = \frac{1}{2^N}\sin(z)$.

So $\prod_{n=1}^{N} \cos(\frac{z}{2^n}) = \frac{\sin(z)}{2^N \sin(\frac{z}{2^N})}$, when $\sin(\frac{z}{2^N}) \neq 0$. By taking limits as $N \to \infty$, we get that $\prod_{n=1}^{\infty} \cos(\frac{z}{2^n}) = \lim_{N\to\infty} \frac{-\ln(2)2^{-N}\sin(z)}{-\ln(2)2^{-N}\cos(z2^{-N})} = \sin(z)$ when $\sin(\frac{z}{2^N}) \neq 0$ for any $N \in \mathbb{N}$.

When $\sin(\frac{z}{2^N}) = 0$ for some $N \in \mathbb{N}$, we have that $\sin(z) = 0$. Also, we have that $\cos(\frac{z}{2^{N+k}}) = 0$ for some $k \geq 0$ if $z \neq 0$ (as z is a power of 2 times π times an odd number...we divide by enough 2s to reduce it to $\pi/2$ times an odd number); the above formula holds true when $z \neq 0$, then.

Yet when $z = 0$, the product above is trivially 1.

So:

$$\prod_{n=1}^{\infty} \cos(\frac{z}{2^n}) = \begin{cases} \sin(z) & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

Problem 2:

Consider that $\pi^2 \csc^2(\pi z) = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$.

By taking derivatives, we have that $-2\pi^3 \csc^2(\pi z)\cot(\pi z) = -2\sum_{-\infty}^{\infty} \frac{1}{(z-n)^3}$.

Dividing by our original identity,

$$
\begin{aligned}
-2\pi\cot(\pi z) &= \frac{-2\pi^3\csc^2(\pi z)\cot(\pi z)}{\pi^2\csc^2(\pi z)} \\
&= \frac{(\pi^2\csc^2(\pi z))'}{\pi^2\csc^2(\pi z)} \\
&= \sum_{-\infty}^{\infty} \frac{\frac{-2}{(z-n)^3}}{\frac{1}{(z-n)^2}} \text{ (Because } f'/f = \sum (f_n'/f)\text{).} \\
&= \sum_{-\infty}^{\infty} \frac{-2}{z-n} \\
&= -2\left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n}\right] \\
&= -2\left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}\right]
\end{aligned}
$$

Which is equivalent to our desired result.