Problem 1:

Consider $a, b \in \mathbb{R}$, and consider the set of functions $u \in C^1([0,1])$ such that $u(0) = a, u(1) = b$. Call the set of such functions $\mathcal{U}$.

Let u and v both minimize the integral $\int_0^1 |f'(x)|^2 \, dx$ among functions in $\mathcal{U}$. Then $u - v$ minimizes the same integral subject to $f(0) = f(1) = 0$:

The only function that minimizes the integral subject to $f(0) = f(1) = 0$ is the zero function; this is because the integral $\int_0^1 |f'|^2$ is nonzero when $f \in C^1([0,1])$ is nonzero. Thus, $u = v$; there is only one function that minimizes the integral $\int_0^1 |f'(x)|^2 \, dx$ in $\mathcal{U}$.

Next: the linear function minimizes the integral:

Problem 2:

Consider the set $A = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \setminus [0, a]$ with $a \in \mathbb{R}^+$.

Define the sets $B = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq a^2\}$, and $C = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$.

First, the map $\phi : A \to B$ given by $z \mapsto z^2$ is a biholomorphism from A to B, and this is clear.

Second, the map $\psi : B \to C$ given by $z \mapsto z - a^2$ is a biholomorphism from B to C, and this is clear.

Third, the map $\xi : C \to \{\mathrm{Re}(z) > 0\}$ given by $z \mapsto \sqrt{z}$ is a biholomorphism from C to $\{\mathrm{Re}(z) > 0\}$, and this is clear.

So their composition is a biholomorphism from A to $\{\mathrm{Re}(z) > 0\}$; that is, the map $f(z) = \sqrt{z^2 - a^2}$ is a biholomorphism from the above set to $\{\mathrm{Re}(z) > 0\}$.

Problem 3:

Let Ω be open and symmetric about the $\mathbb{R}$-axis.

Let $f \in C(\Omega)$, and f be holomorphic except perhaps on the $\mathbb{R}$-axis.

Our goal is to show that $f \in \mathcal{O}(\Omega)$; we only need to check that f is holomorphic on the $\mathbb{R}$-axis. So, let $z \in \mathbb{R} \cap \Omega$. Then there is an open ball centered at z, call it $D_r(z)$, contained in Ω. This open ball is simply connected. Now, the real part of f, say $u = \mathrm{Re}(f)$, is harmonic on $D_r(z) \setminus \mathbb{R}$.

By the reflection principle discussed in class, u is harmonic on all of $D_r(z)$.

Now, u is the real part of some holomorphic function, g, and this holomorphic function is unique up to addition of a constant. So, we can take $g(z) = 0$.

Problem 4:

Let $\phi \in \mathrm{Aut}(\overline{\mathbb{C}})$ be such that $\phi(\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}) = D_1(0)$. That is, ϕ is given by $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$.

Let $w = \overline{z}$. Then:

$$\begin{aligned}
\frac{\phi(z)}{|\phi(z)|^2} &= \frac{\frac{az+b}{cz+d}}{\left|\frac{az+b}{cz+d}\right|^2} \\
&= \frac{\frac{az+b}{cz+d}}{\frac{az+b}{cz+d}\overline{\left(\frac{az+b}{cz+d}\right)}} \\
&= \frac{\frac{az+b}{cz+d}}{\frac{az+b}{cz+d}\frac{\overline{az+b}}{\overline{cz+d}}} \\
&= \frac{\overline{cz+d}}{\overline{az+b}}
\end{aligned}$$

Problem 5: