Problem 1:

Let f be defined in a neighborhood around z.

If f is complex-differentiable at z, then $\frac{f(z+h)-f(z)}{h} \to a$ as $h \to 0$. Thus, $|f(z+h) - f(z) - ah| \left|\frac{1}{h-z}\right| \to 0$. Thus, $f(z+h) - f(z) - ah + o(h) = 0$ as $h \to 0$.

So $f(z+h) = f(z) + ah + o(h)$ as $h \to 0$, which is our result.

Now, let there be a such that $f(z+h) = f(z) + ah + o(h)$ as $h \to 0$.

Then $\frac{f(z+h)-f(z)}{h} = \frac{ah-o(h)}{h} = a$ as $h \to 0$. That is, f is complex-differentiable at z.

Problem 2:

Let f be complex-differentiable at z. Then f is real-differentiable at z. So f is continuous at z.

Problem 3:

Let $T : \mathbb{C} \to \mathbb{C}$ be $\mathbb{R}$-linear.

If T is $\mathbb{C}$-linear, then $T(iz) = iT(z)$ for all z, by definition.

If $T(iz) = iT(z)$ for all z, then let $x, z \in \mathbb{C}$, with $x = a + bi$. Then we have

$$\begin{aligned} T(xz) &= T((a+bi)z) \\ &= T(az) + T(biz) \\ &= T(az) + iT(bz) & &= (a+ib)T(z) \\ &= xT(z) \end{aligned}$$

yielding the desired result; T is $\mathbb{C}$-linear.

So T is $\mathbb{C}$-linear if and only if T is $\mathbb{R}$-linear and $iT(z) = T(iz)$ for all $z \in \mathbb{C}$.

Problem 4:

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the $\mathbb{R}$-linear mapping given by the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

If $a = d$ and $b = -c$, then for all $z = x + iy \in \mathbb{C}$, we have

$$\begin{aligned} T(iz) &= \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] i \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \end{aligned}$$

Problem 5:

Somehow, the "correct" way to do this is to search for a basic calculus book and copy the proof contained within.

Suppose that f and g are complex-differentiable at z.

Part i:

Consider $f + g$. Then $\frac{(f+g)(z+h)-(f+g)(z)}{h} = \frac{f(z+h)-f(z)}{h} + \frac{g(z+h)-g(z)}{h} \to f'(z) + g'(z)$.

Consider fg. Then $\frac{(fg)(z+h)-(fg)(z)}{h} = \frac{f(z+h)g(z+h)-f(z)g(z)}{h}$.

Part ii:

Part iii:

Part iv:

Problem 6:

Problem 7:

Problem 8: