Problem 1:

Find a representation for the linear functionals on ℓ^p, where ℓ^p consists of sequences $\langle x_n \rangle$ of real numbers such that

$$(\sum |x_n|^p)^{1/p} < \infty$$

Problem 2:

Let $f \in L^p$, and let $T_\Delta(f)$ denote the Δ-approximant of f. Prove that

$$\|T_\Delta(f)\|_p \leq \|f\|_p$$

Problem 3:

(Prove that ℓ^p, $1 \leq p < \infty$, and L^∞ are complete.)

First, ℓ^p is complete if $p \in [1, \infty)$; we show this by mimicking the proof of the Riesz-Fischer Theorem, as I am an uncreative n00b. To display the sheer laziness of this approach, I will be using the notation $a(n)$ to denote the nth term of a sequence in ℓ^p for this problem only.

Before doing this, we must prove the Minkowski Inequality for ℓ^p with $p \in [1, \infty)$:

Now, we can proceed by ripping the proof out of the Royden almost verbatim.

We need only show that each absolutely summable series in ℓ^p is summable in ℓ^p to some element of ℓ^p.

Let $\langle f_n \rangle$ be a sequence in ℓ^p with $\sum\limits_{n=1}^{\infty} \|f_n\| = M < \infty$. Define sequences g_n by setting $g_n(m) = \sum\limits_{k=1}^{n} |f_n(m)|$. From the Minkowski inequality

Next, L^∞ is complete;

Let $\langle f_n \rangle$ be a Cauchy sequence in L^∞. So at almost every $t \in [0, 1]$, we have $f_n(t)$ Cauchy in $\mathbb{R}$. So $f_n(t)$ converges pointwise almost everywhere to some f.

Now, this f is in L^∞: There's an f_n within 1 of f almost everywhere. The essential supremum of f is at most 1 away from the essential supremum of f_n. So the essential supremum of f is finite. So $f \in L^\infty$.

Next, $\|f_n - f\| \to 0$: $|f_n - f| \to 0$ almost everywhere. Define $g_n = |f_n - f|$ where $f_n \to f$, and $g_n = 0$ elsewhere. Then $\text{esssup}(g_n) = 0$, and $g_n = f_n - f$ almost everywhere. So $\text{esssup}(|f_n - f|) = 0$.

So $\langle f_n \rangle$ converges in the mean to some $f \in L^\infty$ if $\langle f_n \rangle$ is Cauchy; we have our result.

Problem 4:

(Let ℓ^∞ denote the set of all bounded sequences of real numbers. Set $\|(x_n)\|_\infty = \sup |x_n|$. Prove that this is a norm, and ℓ^∞ is a Banach Space.)

First, this is a norm:

Let $\langle x_n \rangle$, $\langle y_n \rangle$ be bounded sequences of real numbers, and $\alpha \in \mathbb{R}$.

First, $\|x_n\| = 0$ if and only if x_n is identically zero: $\sup(|x_n|) = 0$ if each x_n is zero. Further, if any x_n is nonzero, then $\sup(|x_n|)$ is nonzero.

Next, $\|\alpha x_n\| = \sup(|\alpha x_n|) = \sup(|\alpha|\,|x_n|) = |\alpha| \sup(|x_n|) = |\alpha|\,\|x_n\|$. (All of this follows from basic properties of the sup.)

Last:

$$\begin{aligned}
\|x_n + y_n\| &= \sup(|x_n + y_n|) \\
&\leq \sup(|x_n| + |y_n|) \\
&\leq \sup(|x_n|) + \sup(|y_n|) \\
&= \|x_n\| + \|y_n\|
\end{aligned}$$

So this norm is a norm.

Next, ℓ^∞ is complete. We show this by mimicking the proof for L^∞, as I am again an uncreative n00b:

Let $\langle f_n \rangle$ be a Cauchy sequence in ℓ^∞. So at almost every $t \in [0, 1]$, we have $f_n(t)$ Cauchy in $\mathbb{R}$. So $f_n(t)$ converges pointwise almost everywhere to some f.

Now, this f is in L^∞: There's an f_n within 1 of f almost everywhere. The essential supremum of f is at most 1 away from the essential supremum of f_n. So the essential supremum of f is finite. So $f \in L^\infty$.

Next, $\|f_n - f\| \to 0$: $|f_n - f| \to 0$ almost everywhere. Define $g_n = |f_n - f|$ where $f_n \to f$, and $g_n = 0$ elsewhere. Then $\text{esssup}(g_n) = 0$, and $g_n = f_n - f$ almost everywhere. So $\text{esssup}(|f_n - f|) = 0$.

So $\langle f_n \rangle$ converges in the mean to some $f \in L^\infty$ if $\langle f_n \rangle$ is Cauchy; we have our result.

So, ℓ^∞ is a complete normed vector space. It's a Banach space.

Problem 5:

(Prove the Minkowski inequality for $0 < p < 1$.)

We proceed by mimicking the proof of the Minkowski inequality for $p \geq 1$.

Let f, g be two non-negative functions in L^p with $p \in (0, 1)$.

Note that this is trivial if $\|f\|_p$ or $\|g\|_p$ is zero, and we have equality. So, let us assert that this is not the case. Define $f_0 = f/\|f\|_p$ and $g_0 = g/\|g\|_p$. Also, define $\alpha = \|f\|_p$, $\beta = \|g\|_p$, and $\lambda = \alpha/(\alpha + \beta)$.

Note that $\|f_0\|_p = \|g_0\|_p = 1$.

Then we have:

$$\begin{aligned}
|f + g|^p &= [\alpha f_0 + \beta g_0]^p \\
&= (\alpha + \beta)^p [\lambda f_0 + (1 - \lambda) g_0]^p \\
&\geq (\alpha + \beta)^p [\lambda f_0^p + (1 - \lambda) g_0^p]
\end{aligned}$$

With the last step by the concavity of x^p for $p \in (0, 1)$.

Integrating both sides, we have:

$$\begin{aligned}
\|f + g\|_p^p &\geq (\alpha + \beta)^p [\lambda \|f_0\|_p^p + (1 - \lambda) \|g_0\|_p^p] \\
&= (\alpha + \beta)^p \\
&= (\|f\|_p + \|g\|_p)^p
\end{aligned}$$

Raising both sides to the $1/p$th power yields the result.

Problem 6:

(Young's inequality states that if $a, b \geq 0$, $1 < p < \infty$, and $1/p + 1/q = 1$, then

$$ab \leq a^p/p + b^q/q$$

Prove the Hölder inequality using this.)

First, we should perhaps establish Young's inequality; it was not discussed in class.

I acknowledge that the following is an unintuitive mess; I hope that there is a better means of doing this.

First, consider the right hand side of Young's inequality. Given all of the hypotheses, we have:

$$\begin{aligned}
a^p/p + b^q/q &= a^p/p + \frac{b^{(p-1)/p}}{(p-1)/p} \\
&= a^p/p + pb^{(p-1)/p}/(p-1) \\
&= \frac{(p-1)a^p + pb^{(p-1)/p}}{p(p-1)}
\end{aligned}$$

Thus, we have that

$$a^p/p + b^q/q - ab = \frac{(p-1)a^p + pb^{(p-1)/p} - abp(p-1)}{p(p-1)}$$

We have Young's Inequality if the left hand side is greater than or equal to 0. So we have Young's Inequality if the numerator is greater or equal to 0, as we have that $p > 1$, so that the denominator is positive.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = (p-1)x^p - xbp(p-1) + pb^{(p-1)/p}$$

We know that $f(0) = pb^{(p-1)/p}$, so that $f(0)$ is positive. Also, it's clear that $\lim_{x\to\infty} f(x) = \infty$. It's also clearly a differentiable function, with

$$f'(x) = p(p-1)x^{p-1} - p(p-1)b$$

So f has a critical point at $b^{1/(p-1)}$. It is clear that $f'(0)$ is negative and that $f'(b)$ is positive, so f takes its minimum at $b^{1/(p-1)}$.

However, $f(b^{1/(p-1)}) = 0$. So $f \geq 0$.

So $f(a) \geq 0$. So $(p-1)a^p + pb^{(p-1)/p} - abp(p-1) \geq 0$ which yields Young's Inequality, as stated above.

Moving on, we use this to prove the Hölder inequality. Assume that $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, and let $f \in L^p$ and $g \in L^q$.

Problem 7:

I cannot pick up your dry cleaning.

I don't have a car, as I am too poor to afford one.

I will gladly pick up your dry cleaning if you are willing to pay me enough to purchase a car and gas.