I don't know if we covered the ρ metric ($\rho(f, g) = \sup(|f(x) - g(x)| : x \in X)$, where X is the domain of f and g), but I'm using it because it's nice and I like it.

Problem 1:

Consider the sequence of functions $f_n : (-\pi/2, \pi/2) \to \mathbb{R}$ given by $f_n(x) = \sec(x) + 1/n$.

It is rather clear that this sequence of functions converges uniformly (to $\sec(x)$).

However, the sequence of functions $\langle f_n^2 \rangle$ fails to converve uniformly:

For each $n \in \mathbb{N}$, $f_n^2(x) = \sec(x)^2 + 2\sec(x)/n + 1/n^2$. It is rather clear that $\langle f_n^2 \rangle$ converges pointwise to $\sec(x)^2$. However, $\langle f_n^2 \rangle$ does not converge uniformly:

So $\langle f_n \rangle$ converges uniformly on $(-\pi/2, \pi/2)$, but $\langle f_n^2 \rangle$ doesn't. This satisfies the problem.

Problem 2:

Note: After finishing this problem, I noticed that this follows immediately from a fragment of the proof of Arzela-Ascoli. I prefer this proof, as it is smoother, but it is important to note that such a thing is possible.

Let $\langle f_n \rangle$ be an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise to some function, say f.

By the Arzela-Ascoli theorem, we know that $\langle f_n \rangle$ has some subsequence that uniformly converges to some function. We know that this function must be f: if a subsequence of functions converges uniformly to f, it converges pointwise to f. If a sequence of functions converges pointwise to a function, f, then all of its subsequences converge to f. So if a sequence of functions converges pointwise to f, then any subsequence of functions that converges uniformly to a function must converge uniformly to f.

Now, consider such a converging subsequence, $\langle f_{n_j} \rangle$.

Let $\epsilon > 0$. There is a $J \in \mathbb{N}$ such that for all $j \geq J$, $\rho(f_{n_j}, f) < \epsilon/3$.

In addition, by equicontinuity, there is a $\delta > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x, y) < \delta$ implies that $d(f_n(x), f_n(y)) < \epsilon/3$.

We know that compact sets are totally bounded. (If this is not clear, consider a career in pastry making.)

So, let F be a finite collection of points of K such that for all $x \in K$, $d(x, y) < \delta$ for some $y \in F$.

Now, for each $y \in F$, there is an $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $d(f_n(y), f(y)) < \epsilon/3$.

Define $N = \max(N_y, n_J)$.

Now, for all $n \geq N$, and for all $x \in K$, we have:

$$The proof$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in K$, $d(f_n(x), f(x)) < \epsilon$. That is, f_n converges uniformly to f.

To summarize, if $\langle f_n \rangle$ is an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise, then $\langle f_n \rangle$ converges uniformly.

Problem 3:

Let $\langle f_n \rangle$ be a uniformly bounded sequence of functions that are Riemann-integrable on $[a, b]$. Set

$$F_n(x) = \int_a^x f_n(t)dt$$

Now, the set of F_ns are equicontinuous:

In addition, the F_ns are defined on $[a, b]$, which is a compact space. By Arzela-Ascoli, there is a subsequence $\left\langle F_{n_j} \right\rangle$ that converges uniformly on $[a, b]$.

Problem 4:

Problem 5:

Let α be increasing on $[a, b]$, g continous, and $g(x) = G'(x)$ for all $x \in [a, b]$.

Then note that both $\int_a^b \alpha(x)g(x)dx$ and $\int_a^b G d\alpha$ exist, because both α and g are Riemann integrable.

Problem 6:
Let α be an increasing function on $[a, b]$, and for $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 d\alpha \right)^{1/2}.$$

Let $f, g, h \in \mathcal{R}(\alpha)$.
Then we have the following:

$$\begin{aligned}
\|f - h\|_2 &= \left(\int_a^b |f - h|^2 d\alpha \right)^{1/2} \\
&= \left(\int_a^b |f - g + g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left(\int_a^b (|f - g| + |g - h|)^2 d\alpha \right)^{1/2} \\
&= \left(\int_a^b |f - g|^2 + 2|f - g||g - h| + |g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left(\int_a^b |f - g|^2 + |g - h|^2 d\alpha \right)^{1/2} \\
&= \|f - g\|_2 + \|g - h\|_2
\end{aligned}$$

Thus, we have the triangle inequality for this ~~norm~~ strange function I have never seen before.