Problem 1:

Consider $\int\limits_0^\infty \frac{1-\cos(x)}{x^2}dx$.

Now, $\int\limits_0^T \frac{1-\cos(z)}{z^2}dz = \int\limits_0^T \frac{1-\frac{e^{iz}+e^{-iz}}{2}}{z^2}dz = -\left[\int\limits_0^T \frac{e^{iz}-1}{2z^2}dz + \int\limits_0^T \frac{e^{-iz}-1}{2z^2}dz\right]$. Both of the functions under the integrands are holomorphic, except at the origin. Using a u-substitution, we get $\int\limits_0^T \frac{e^{-iz}-1}{2z^2}dz = -\int\limits_0^{-T} \frac{e^{iz}-1}{2z^2}dz$.

So,

$$\begin{aligned}
\int\limits_0^T \frac{1-\cos(z)}{z^2}dz &= -\left[\int\limits_0^T \frac{e^{iz}-1}{2z^2}dz - \int\limits_0^{-T} \frac{e^{iz}-1}{2z^2}dzdz\right] \\
&= \frac{1}{2}\left[\int\limits_0^T \frac{1-e^{iz}}{z^2}dz - \int\limits_0^{-T} \frac{1-e^{iz}}{z^2}dz\right] \\
&= \frac{1}{2}\left[\int\limits_{-T}^T \frac{1-e^{iz}}{z^2}dz\right]
\end{aligned}$$

Problem 2:

Let $\Omega \subset \mathbb{C}$ be open and simply connected, $f \in \mathcal{O}(\Omega)$, f is always nonzero, $k \in \mathbb{Z}$.

There is an $h \in \mathcal{O}(\Omega)$ such that $e^h = f$. Define $\overline{h} = h/k$. Then $e^{\overline{h}k} = f$.

Problem 3:

Consider $\sqrt[\sqrt{-1}]{-1} = (-1)^{sqrt-1} = (e)^{\ln(-1)\sqrt{-1}} = e^{\ln -1 e^{\frac{1}{2}\ln(-1)}}$. As discussed in class, the logarithms of -1 are $(2k+1)\pi i$ for each $k \in \mathbb{Z}$. That is, the possible values of $\sqrt[\sqrt{-1}]{-1}$ are given by

$$e^{((2k+1)\pi i)e^{\frac{1}{2}((2j+1)\pi i)}}$$

for any given $k, j \in \mathbb{Z}$.

Yet, this is an intractible mess. Consider that $e^{\frac{1}{2}((2j+1)\pi i)} = e^{j\pi i+\frac{1}{2}\pi i} = e^{j\pi i}e^{\frac{1}{2}\pi i} = (-1)^j e^{\frac{1}{2}\pi i}$. Thus, our original expression becomes

$$e^{((2k+1)\pi i)(-1)^j e^{\frac{1}{2}\pi i}}$$

To clean this up even more,

Problem 4:

Let $\ln(z)$ be the principal branch of the logarithm of z, and let z_1, z_2 have positive real component.

Problem 5:

Consider $\sin(\frac{1}{z})$. We know that $\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$. So, where defined, $\sin(\frac{1}{z}) = \sum_{n=0}^{\infty}(-1)^n \frac{\frac{1}{z}^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n+1}(2n+1)!}$

Problem 6: