

Problem 1:

Note: Seriously, who says that $D_6 = \langle a, b \rangle$? Hungerford's the first time I've seen that, and doing that completely strips away all hope of understanding it intuitively. Here, $D_6 = \langle r, s \rangle$, with r being "rotation" and s being "reflection".

First, we point out that D_6 and $\{e\}$ are normal subgroups of D_6 . Also, $\langle r \rangle$ is normal, as it is a subgroup of index 2 (We know that subgroups of index 2 are normal, by a theorem in class.) Also, $\langle r^3 \rangle$ is normal; it is the center of D_6 (as was discussed in class), and is thus normal. In addition, $\langle sr^3 \rangle$ is normal: if $a = r^n$, then $asr^3a^{-1} = r^n sr^3 r^{-n} = r^n r^{n-3} s = r^{-3} s = sr^3$. Also, if $a = sr^n$, then $asr^3a^{-1} = sr^n sr^3 r^{-n} s = sr^n r^{n-3} s s = sr^{-3} = sr^3$. (Note that this doesn't work for other elements of the form sr^n as we have used the fact that $r^3 s = sr^3$.) In addition, $\langle r^2 \rangle$ is normal:

stuff

We can exclude conjugation by elements of the form r^n in the above, because such elements commute with r^2 and r^4 , and thus conjugation of either by such an element leaves r^2 and r^4 fixed.

However, this is all; let H be a normal subgroup.

If $sr^n \in H$ where $n \neq 3$, then...

If $r^2 \in H$, then...

Problem 2:

Let G, H, K be finite abelian groups, with $G \oplus H \cong G \oplus K$.

Now, because $G \oplus H$ is a finite abelian group, it is isomorphic to a direct sum of the form $\bigoplus_{i=1}^m \mathbb{Z}/(p_i^{\alpha_i} \mathbb{Z})$, with each p_i prime. Moreover, $G \oplus K$ is isomorphic to the same direct sum, by a theorem discussed in class.

Moreover, there's an injective homomorphism from G to $G \oplus H$ given by $\phi : G \rightarrow G \oplus H$ where $\phi(g) = (g, 0)$. Thus, G is isomorphic to a subgroup of $G \oplus H$. Thus, G is isomorphic to a direct sum of the form $\bigoplus_{k=1}^n \mathbb{Z}/(p_{i_k}^{\alpha_{i_k}} \mathbb{Z})$ (as any subgroup of $\bigoplus_{i=1}^m \mathbb{Z}/(p_i^{\alpha_i} \mathbb{Z})$ is of this form...we probably discussed this some time.)

Now, by rearranging terms, we can see that $G \oplus H \cong \bigoplus_{k=1}^n \mathbb{Z}/(p_{i_k}^{\alpha_{i_k}} \mathbb{Z}) \oplus$

$\bigoplus_{\{i:i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i} \mathbb{Z})$. It is clear (if it isn't, then projection maps will get you there...) from this that $H \cong \bigoplus_{\{i:i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i} \mathbb{Z})$. Similar to the above, $K \cong \bigoplus_{\{i:i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i} \mathbb{Z})$. So $K \cong H$.

Problem 3:

Let G be a finite abelian group.

First, if G is cyclic, then for each $n \in \mathbb{N}$ with $n \neq 1$, then either n is a multiple of $|G|$, in which case there are $|G|$ elements, a , satisfying $na = 0$, or n is not a multiple of $|G|$, in which case there are none (because every element in G has the same order, except for 0). And of course, if $n = 1$, then there's only 1 element satisfying $1a = 0$, that is, 0.

So if G is cyclic, then for each $n \in \mathbb{N}$ there are at most n elements $a \in G$ satisfying $na = 0$.

Now, let G not be cyclic. Then $G \cong \bigoplus_{i=1}^n \mathbb{Z}/(m_i^{\alpha_i} \mathbb{Z})$, with each m_i dividing m_{i+1} , each $m_i > 1$, and $n \geq 2$ (else, G is obviously cyclic...). Consider m_n . Every element, a , of G has the property $m_n a = 0$; let $a \in G$. Then $\phi(a) = (a_1, a_2 \dots a_n) \in \bigoplus_{i=1}^n \mathbb{Z}/(m_i^{\alpha_i} \mathbb{Z})$, with an isomorphism, ϕ determined by the FTFA. Moving on $m_n(a_1, a_2 \dots a_n) = (m_n a_1, m_n a_2 \dots m_n a_n) = (0, 0, \dots 0)$ (the last of these is because each of the component groups has order dividing m_n). Thus, $a^{m_n} = 0$, for all $a \in G$. However, $m_n < |G|$, because $|G| = \prod m_i \geq m_{n-1} m_n > m_n$.

So if G is not cyclic, then there is an $n \in \mathbb{N}$ with more than n elements $a \in G$ satisfying $na = 0$.

So G is cyclic if and only if for each $n \in \mathbb{N}$ there are at most n elements $a \in G$ satisfying $na = 0$.