Problem 1 (23 in book):

Let S denote the square in $\mathbb{R}\times(0,\infty)$ with corners $(0,1),(1,2),(0,3),(-1,2)$. Define

$$f(x,t)=\begin{cases}-1 & \text{for } (x,t)\in S\cap\{t>x+2\}\\ -1 & \text{for } (x,t)\in S\cap\{t<x+2\}\\ 0 & \text{else}\end{cases}$$

Let u solve

$$\begin{cases}u_{tt}-u_{xx}=f & \text{when } t>0\\ u=0, u_t=0 & \text{when } t=0\end{cases}$$

Consider u when $t>3$. Then we have

$$u(x,t)=\int_0^t u(x,t;s)ds$$

where $u(x,t;s)=\frac{1}{2}\int_{x-t}^{x+t} f(y,s)dy$ (we get this by Duhamel's principle and the solution of the wave equation in one dimension). In other words,

$$u(x,t)=\int_0^t \frac{1}{2}\int_{x-t}^{x+t} f(y,s)dyds$$

I appear to be pressed on time; I present the below diagram without argument, and hope that the nature of the argument follows from it. I will note that each segment of the curve is a segment of a parabola, and this is somewhat clear given the geometry of the desired shape.

Problem 2 (24 in book):

Let u solve the intial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{when } t > 0 \\ u = g, u_t = h & \text{when } t = 0 \end{cases}$$

Let g, h have compact support. Consider $k(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_t^2(x,t)dx$ and $p(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_x^2(x,t)dx$.

Part a:

Consider $k(t) + p(t) = \frac{1}{2} \int\limits_{\mathbb{R}} u_x^2(x,t) + u_t^2(x,t)dx$.

We know that $u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy$

So, we have:

$$\begin{aligned} u_x(x,t) &= \frac{g'(x+t) + g'(x-t)}{2} + \left[\frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy \right]_x \\ &= \frac{g'(x+t) + g'(x-t)}{2} + \frac{1}{2} \left[h(x+t) - h(x-t) \right] \\ u_t(x,t) &= \frac{g'(x+t) - g'(x-t)}{2} + \left[\frac{1}{2} \int\limits_{x-t}^{x+t} h(y)dy \right]_t \\ &= \frac{g'(x+t) - g'(x-t)}{2} + \frac{1}{2} \left[h(x+t) + h(x-t) \right] \end{aligned}$$

This means that

$$
\begin{aligned}
u_x^2 + u_t^2 &= \left(\frac{g'(x+t) + g'(x-t)}{2} + \frac{1}{2} \left[h(x+t) - h(x-t) \right] \right)^2 \\
&\quad + \left(\frac{g'(x+t) - g'(x-t)}{2} + \frac{1}{2} \left[h(x+t) + h(x-t) \right] \right)^2 \\
&= (\frac{g'(x+t) + g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t) + g'(x-t)}{2} \left[h(x+t) - h(x-t) \right] \\
&\quad + (\frac{1}{2} \left[h(x+t) - h(x-t) \right])^2 \\
&\quad + (\frac{g'(x+t) - g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t) - g'(x-t)}{2} \left[h(x+t) + h(x-t) \right] \\
&\quad + (\frac{1}{2} \left[h(x+t) + h(x-t) \right])^2 \\
&= \frac{1}{4} g'(x+t)^2 + \frac{1}{4} g'(x-t)^2 + \frac{1}{2} g'(x+t) g'(x-t) \\
&\quad + \frac{1}{2} \left[h(x+t) g'(x+t) - h(x-t) g'(x+t) + h(x+t) g'(x-t) - h(x-t) g'(x-t) \right] \\
&\quad + \frac{1}{4} h(x+t)^2 + \frac{1}{4} h(x-t)^2 - \frac{1}{2} h(x+t) h(x-t) \\
&\quad + \frac{1}{4} g'(x+t)^2 + \frac{1}{4} g'(x-t)^2 - \frac{1}{2} g'(x+t) g'(x-t) \\
&\quad + \frac{1}{2} \left[h(x+t) g'(x+t) + h(x-t) g'(x+t) - h(x+t) g'(x-t) - h(x-t) g'(x-t) \right] \\
&\quad + \frac{1}{4} h(x+t)^2 + \frac{1}{4} h(x-t)^2 + \frac{1}{2} h(x+t) h(x-t) \\
&= \frac{1}{2} g'(x+t)^2 + \frac{1}{2} g'(x-t)^2 \\
&\quad + \left[h(x+t) g'(x+t) - h(x-t) g'(x-t) \right] \\
&\quad + \frac{1}{2} h(x+t)^2 + \frac{1}{2} h(x-t)^2
\end{aligned}
$$

Now, we integrate:

$$\int_{\mathbb{R}} u_x^2 + u_t^2 dx = \int_{\mathbb{R}} \frac{1}{2}g'(x+t)^2 + \frac{1}{2}g'(x-t)^2$$
$$+ [h(x+t)g'(x+t) - h(x-t)g'(x-t)]$$
$$+ \frac{1}{2}h(x+t)^2 + \frac{1}{2}h(x-t)^2 dx$$

The above is constant (with respect to t), and this is clear by applying appropriate substitutions to each term.

Part b:
Using the above, consider that

$$\begin{aligned}
u_x^2 - u_t^2 &= \left(\frac{g'(x+t)+g'(x-t)}{2} + \frac{1}{2}\left[h(x+t)-h(x-t)\right]\right)^2 \\
&\quad - \left(\frac{g'(x+t)-g'(x-t)}{2} + \frac{1}{2}\left[h(x+t)+h(x-t)\right]\right)^2 \\
&= (\frac{g'(x+t)+g'(x-t)}{2})^2 \\
&\quad + \frac{g'(x+t)+g'(x-t)}{2}\left[h(x+t)-h(x-t)\right] \\
&\quad + (\frac{1}{2}\left[h(x+t)-h(x-t)\right])^2 \\
&\quad - (\frac{g'(x+t)-g'(x-t)}{2})^2 \\
&\quad - \frac{g'(x+t)-g'(x-t)}{2}\left[h(x+t)+h(x-t)\right] \\
&\quad - (\frac{1}{2}\left[h(x+t)+h(x-t)\right])^2 \\
&= \frac{1}{4}g'(x+t)^2 + \frac{1}{4}g'(x-t)^2 + \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad + \frac{1}{2}\left[h(x+t)g'(x+t) - h(x-t)g'(x+t) + h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad + \frac{1}{4}h(x+t)^2 + \frac{1}{4}h(x-t)^2 - \frac{1}{2}h(x+t)h(x-t) \\
&\quad - \frac{1}{4}g'(x+t)^2 - \frac{1}{4}g'(x-t)^2 + \frac{1}{2}g'(x+t)g'(x-t) \\
&\quad - \frac{1}{2}\left[h(x+t)g'(x+t) + h(x-t)g'(x+t) - h(x+t)g'(x-t) - h(x-t)g'(x-t)\right] \\
&\quad - \frac{1}{4}h(x+t)^2 - \frac{1}{4}h(x-t)^2 - \frac{1}{2}h(x+t)h(x-t) \\
&= g'(x+t)g'(x-t) \\
&\quad + \left[-h(x-t)g'(x+t) + h(x+t)g'(x-t)\right] \\
&\quad + h(x+t)h(x-t)
\end{aligned}$$

Integrating, we get

$$\int_{\mathbb{R}} u_x^2 - u_t^2 dx = \int_{\mathbb{R}} g'(x+t)g'(x-t) \\ + [-h(x-t)g'(x+t) + h(x+t)g'(x-t)] \\ + h(x+t)h(x-t)dx$$

Because g and h have compact support, there's a t large enough that all of the above products vanish for all x. (Taking t to be twice the diameter of the larger of the sets g and h have support on suffices.)

Thus, the above integral vanishes for some sufficiently large t, yielding our result.

Problem 3 (on page):

Assume $f(x,t) = 1$ if $|x| \leq 1$ and $0 \leq t \leq 1$, and $f(x,t) = 0$ otherwise. Let u solve

$$\begin{cases} u_{tt} - \Delta u = f & \text{when } t > 0 \\ u = 0, u_t = 0 & \text{when } t = 0 \end{cases}$$

Consider $u(0,t)$ when $t > 2$. Then $u(0,t) = \int\limits_0^t f(0,t;s)ds$ where $f(x,t;s)$ solves the IVP at time s.

If $n = 1$, then this means that $u(0,t) = \frac{1}{2}\int\limits_0^t \int\limits_{-t}^t f(y,s)dyds = \frac{1}{2}\int\limits_0^1 \int\limits_{-1}^1 1dyds =$ 1.

So if $n = 1$, then $u(0,t) = 1$ when $t > 2$.

If $n = 2$, then:

$$
\begin{aligned}
u(0,t) &= \frac{1}{2}\int_0^t \fint_{B(0,t)} \frac{t^2 f(y,s)}{\sqrt{t^2-|y|^2}}dyds \\
&= \frac{t^2}{2}\int_0^1 \frac{1}{\pi t^2}\int_{B(0,1)} \frac{1}{\sqrt{t^2-|y|^2}}dyds \\
&= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{t^2-|r|^2}}drd\theta ds \\
&= \int_0^1\int_0^1 \frac{r}{\sqrt{t^2-|r|^2}}drds \\
&= \int_0^1 \frac{r}{\sqrt{t^2-|r|^2}}dr \\
&= \sqrt{t^2}-\sqrt{t^2-1} \\
&= t-\sqrt{t^2-1}
\end{aligned}
$$

So if $n=2$, then $u(0,t)=t-\sqrt{t^2-1}$ when $t>2$.

If $n=3$, then this means that $u(0,t)=\int_0^t \fint_{\partial B(0,t)} tf(y,s)dS(y)ds=0$ (it vanishes because $f(y,s)$ vanishes on the shells we're working on.)

So if $n=3$, then $u(0,t)=0$ when $t>2$.