Problem 1:

Consider $\operatorname{Ind}(\partial D(a,r), a)$. Define $\theta : [0,1] \to \mathbb{R}$ by $x \to 2\pi x$. Note that θ satisfies $\gamma(t) - a = |\gamma(t) - a|\, e^{i\theta(t)}$, where $\gamma(t) = re^{2\pi t} + a$. From the definitions, we have

$$\begin{aligned}\operatorname{Ind}(\partial D(a,r), a) &= \frac{1}{2\pi}(\theta(1) - \theta(0)) \\ &= \frac{1}{2\pi}(\theta(1) - \theta(0)) \\ &= 1\end{aligned}$$

Problem 2:

Consider $\operatorname{Ind}(\partial D(a,r), z)$. Because $\partial D(a,r)$ is a smooth, closed curve, we have that $\operatorname{Ind}(\partial D(a,r), z) = \frac{1}{2\pi i} \int\limits_{\partial D(a,r)} \frac{dw}{w-z}$.

So,

$$\begin{aligned}\operatorname{Ind}(\partial D(a,r), z) &= \frac{1}{2\pi i} \int\limits_{\partial D(a,r)} \frac{dw}{w-z} \\ &= \frac{1}{2\pi i} \int\limits_{\partial D(z,r)} \frac{dw}{w-z} \\ &= \frac{1}{2\pi i} 2\pi i \\ &= 1\end{aligned}$$

if $z \in D(a,r)$.

And

$$\begin{aligned}\operatorname{Ind}(\partial D(a,r), z) &= \frac{1}{2\pi i} \int\limits_{\partial D(a,r)} \frac{dw}{w-z} \\ &= \frac{1}{2\pi i} \int\limits_{\partial D(z,r)} \frac{dw}{w-z} \\ &= 0\end{aligned}$$

if $z \notin D(a, r)$ (as the function is holomorphic on that disk.)

Note that in our definitions, the index isn't defined when $a \in \gamma^*$. That is, $\mathrm{Ind}(\partial D(a, r), z)$ isn't defined when $z \in \partial D(a, r)$.

Problem 3:

Let V be an open subset of the plane and Γ be a cycle in V. Also, let Γ have the property that $\mathrm{Ind}(\Gamma, a) = 0$ for all $a \in \mathbb{C} \setminus V$.

Let $f \in H(V)$.

Then define $g(w) = \frac{f(w)}{w-z}$. We know that g is differentiable on $V \setminus \{z\}$. Referring to a figure much like Figure 2.1 and using Theorem 4.10 to show that the "outer" chunks of the integral vanish, we get $\int\limits_{\Gamma} g(w)dw = \int\limits_{\partial D(z,r)} g(w)dw$ for arbitrarily small $r > 0$.

Problem 4:

Problem 5:

Problem 6:

Problem 7:

Problem 8:

Problem 9:

Consider f, holomorphic on some disk, Ω, centered at z. Consider $g(w) = \frac{f(w)}{w-z}$; then we have that $\int\limits_{\partial Om} g(w)dw = 2\pi i \text{Res}_z g$. (Note that g's only singularity is at z.) Moreover, note that $g(w) = \frac{\sum\limits_{n=0}^{\infty} a_n(w-z)^n}{w-z}$. Thus, by the residue theorem, $\int\limits_{\partial\Omega} \frac{f(w)}{w-z}dw = f(z)$.

Problem 10:

Problem 11:

Problem 12:

Problem 13:

Problem 14:

Problem 15:

Problem 16:

Problem 17:

Problem 18:

Problem 19:

Problem 20:

Problem 21:

Problem 22:

Problem 23: