

Problem 1: Problem 6 in textbook:

Let U be a bounded, open subset of \mathbb{R}^n .

We freely use the preceding problem's result that if $-\Delta v \leq 0$, then $\max_{\bar{U}} v = \max_{\partial U} v$, and also the hint given; $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$.

Define $\lambda = \max_{\bar{U}} |f|$. Define $M = \max(1, \frac{r^2}{2n})$ where r is an upper bound on the distance of a point in U from 0.

So we have:

$$\begin{aligned}
 \max_{\bar{U}} u &\leq \max_{\bar{U}} (u + \frac{|x|^2}{2n}\lambda) \\
 &= \max_{\partial U} (u + \frac{|x|^2}{2n}\lambda) \\
 &\leq \max_{\partial U} (u) + \max_{\partial U} \frac{|x|^2}{2n}\lambda \\
 &\leq \max_{\partial U} (u) + M\lambda \\
 &= \max_{\partial U} (g) + M\lambda \\
 &\leq \max_{\partial U} (|g|) + M \max_{\bar{U}} (|f|) \\
 &\leq M(\max_{\partial U} (|g|) + \max_{\bar{U}} (|f|))
 \end{aligned}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Noting that we get the same result for $-u$, we have our result.

Problem 2: Problem 9 in textbook:

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n \\ u = g & \text{on } \partial \mathbb{R}_+^n \end{cases}$$

Assume g is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}_+^n$ with $|x| \leq 1$.

Then $u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{|y|}{|x-y|^n} dy$, by Poisson's formula.

Consider $\frac{u(\lambda e_n) - u(0)}{\lambda}$ (with $1 > \lambda > 0$). We can see that:

$$\begin{aligned} \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2\lambda}{\lambda n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{|y|}{|\lambda e_n - y|^n} dy \\ &= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{|y|}{|\lambda e_n - y|^n} dy \\ &= C \int_{\partial\mathbb{R}_+^n} \frac{|y|}{|\lambda e_n - y|^n} dy \end{aligned}$$

As $\lambda \rightarrow 0$, that integral approaches ∞ (because the integrand approaches $\frac{1}{|y|^{n-1}}$, and the integral of this explodes.) So, because the derivatives are continuous, this means that Du is unbounded around 0.

Problem 3: Problem 10 in textbook:

Part a:

Let U^+ be the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C^2(\overline{U^+})$ is harmonic in U^+ , with $u = 0$ on $\partial U^+ \cap \{x : x_n = 0\}$. Now, set

$$v(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for x in the open unit ball, U .

First, $v \in C^2(U \setminus \{x : x_n = 0\})$, and this is clear.

Next, v is continuous on $\{x : x_n = 0\}$, and this is clear. Also, v 's derivatives on $\{x : x_n = 0\}$ are continuous:

Last, v 's second derivatives on $\{x : x_n = 0\}$ are continuous, because (reasons).

So, $v \in C^2(U)$, and $\Delta v = 0$ except perhaps when $x_n = 0$. Thus, $\Delta v = 0$ even on the line, by continuity of the second partials. So v is harmonic on U .

Part b:

Let $u \in C^2(U^+) \cap C(\overline{U^+})$, and define v as above.

First, $v \in C^2(U \setminus \{x : x_n = 0\})$, and this is clear.

Next, v is continuous on $\{x : x_n = 0\}$, and this is clear. Also, v 's derivatives on $\{x : x_n = 0\}$ are continuous:

Last, v 's second derivatives on $\{x : x_n = 0\}$ are continuous, because (reasons).

So, $v \in C^2(U)$, and $\Delta v = 0$ except perhaps when $x_n = 0$. Thus, $\Delta v = 0$ even on the line, by continuity of the second partials. So v is harmonic on U .

Problem 4: Only problem on sheet:

Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.

By appealing to either Theorem 14 or the fact that the question bashes us over the head with it, there's at least one function, u , with:

- $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n) \cap C(\overline{\mathbb{R}_+^n})$
- $\Delta u = 0$ in \mathbb{R}_+^n
- $u(x', 0) = g(x')$ on \mathbb{R}^{n-1} .

Let u and v be such functions. Then there's a function \tilde{u} and \tilde{v} with

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

and

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x_n \geq 0 \\ -v(x_1, x_2, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Now, consider $w = \tilde{u} - \tilde{v}$. Then w is harmonic on the entire space: it's the sum of two harmonic functions, as explained in the previous problem.

Moreover, w is bounded: both u and v are bounded, so \tilde{u} and \tilde{v} are bounded, so their difference is bounded.

So, by Liouville, w must be constant. However, \tilde{u} and \tilde{v} are the same at a point ($\tilde{u}(0) = u(0) = g(0) = v(0) = \tilde{v}(0)$). So $w = 0$ at a point. So $\tilde{u} = \tilde{v}$. So $u = v$.