Problem 1:

Let $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, and let $u(x,t)$ solve

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0,\infty) \end{cases}$$

Let $v(x,t) = u(x,t) - g(t)$, and extend v to $\{x < 0\}$ by odd reflection (just call the resulting extension v). Then v solves

$$\begin{cases} v_t - v_{xx} = -g' & \text{in } \mathbb{R}_+ \times (0,\infty) \\ v = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ v = 0 & \text{on } \{x = 0\} \times [0,\infty) \end{cases}$$

So $v(x,t) = \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} -g'(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds$ by formula 17 in the book.

Now,

$$\begin{aligned}
v(x,t) &= \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} -g'(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= -\left[\int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} g'(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \right] \\
&= -\left[-\int\limits_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int\limits_{\mathbb{R}} (x-y) g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \right] \quad \text{(integrate by parts)} \\
&= \int\limits_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int\limits_{\mathbb{R}} (x-y) g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dyds \\
&= -\int\limits_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int\limits_{\mathbb{R}} (y) g(x-y) e^{\frac{-|y|^2}{4(t-s)}} dyds
\end{aligned}$$

$$= \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \left[x \int_{\mathbb{R}} g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy - \int_{\mathbb{R}} y g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy \right] ds$$

$$= \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \left[x \int_{\mathbb{R}} g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy \right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} y g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds$$

$$= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy \right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} y g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds$$

$$= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy \right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} y g(y) e^{\frac{-|x-y|^2}{4(t-s)}} dy ds$$

$$= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y) e^{\frac{-(x^2-2xy+y^2)}{4(t-s)}} dy \right] ds - \int_0^t \frac{1}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} y g(y) e^{\frac{-(x^2-2xy+y^2)}{4(t-s)}} dy ds$$

$$= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{2(t-s)^{3/2}} \left[\int_{\mathbb{R}} g(y) e^{\frac{-(-2xy+y^2)}{4(t-s)}} dy \right] ds - \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{2\sqrt{4\pi}(t-s)^{3/2}} \int_{\mathbb{R}} y g(y) e^{\frac{-(-2xy+y^2)}{4(t-s)}} dy ds$$

$$= \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s) ds - g(t)$$

I must admit that I'm not sure how the last line is supposed to go, but that's the idea.

Thus, by adding $g(t)$ to $v(x,t)$, we get that $u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} g(s) ds$ as desired.

Problem 2:

Let $g \in C(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, $|g| < M$ for some M. Let u be the bounded solution to

$$\begin{cases} \Delta u - u_t = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Part a:

Then $u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y)dy$. Let $\epsilon > 0$. There is a sequence $h_k \in \mathbb{R}^n$ with compact support such that $|g| \geq |h_k|$ and $|g - h_k| \to 0$ uniformly. Also, let $v_k(x,t)$ be the bounded solution to

$$\begin{cases} \Delta v_k - v_{k,t} = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ v_k(x,0) = h_k(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Now,

$$\begin{aligned} \left| \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} (g(y) - h_k(y))dy \right| &\leq \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} \left| e^{\frac{-|x-y|^2}{4t}} (g(y) - h_k(y)) \right| dy \\ &\leq \left| \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} \epsilon dy \right| \\ &= \epsilon \end{aligned}$$

So $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} |u(x,t)| = \lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} \lim\limits_{k\to\infty} |v_k(x,t)| = \lim\limits_{t\to\infty} \lim\limits_{k\to\infty} \sup\limits_{x\in\mathbb{R}^n} |v_k(x,t)|$. But $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} |v_k(x,t)| = 0$, because $h(x)$ has compact support (I'm fairly sure we touched on this in class). So, $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} |u(x,t)| = 0$, as desired.

Part b:

Consider $v(x,t) = u(x,t) - g(x)$. Then $v(x,t)$ solves

$$\begin{cases} \Delta v - v_t = -\Delta g(x) & \text{for } t > 0, x \in \mathbb{R}^n \\ v(x,0) = 0 & \text{for } x \in \mathbb{R}^n \end{cases}$$

Thus, $v(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(x-y, t-s)\Delta g(y)dyds$.

So $\int\limits_{\mathbb{R}^n} v(x,t)dx = \int\limits_{\mathbb{R}^n} \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(x-y,t-s)\Delta g(y)dydsdx$

Switching the order of integration, we get $\int\limits_0^t \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \Phi(x-y,t-s)\Delta g(y)dxdyds$.

Yet, this is $\int\limits_0^t \int\limits_{\mathbb{R}^n} \Delta g(y)dyds$, because $\int\limits_{\mathbb{R}^n} \Phi(x-y,t-s)dx = 1$.

The integral vanishes, by Green's Theorem; $\int\limits_{\mathbb{R}^n} \Delta g(y)dy = \lim\limits_{r\to\infty} \int\limits_{\partial B(0,r)} \frac{\partial g(y)}{\partial \nu} dS$, which must vanish, because g is in $L^1(\mathbb{R})$.

So, $\int\limits_{\mathbb{R}^n} v(x,t)dx = 0$; $\int\limits_{\mathbb{R}^n} u(x,t) - g(x)dx = 0$, which yields the desired result of $\int\limits_{\mathbb{R}^n} u(x,t)dx = \int\limits_{\mathbb{R}^n} g(x)dx$.

Problem 3:

Part a:

Fix $\alpha \in (0,1)$, $\beta \geq 0$.

Note first that $z^\beta e^{-z} = e^{\beta \ln(z) - z}$. So, the desired result is

$$e^{\beta \ln(z) - z} \leq Me^{-\alpha z}$$

for some M, which is equivalent to

$$\beta \ln(z) - z \leq \ln(M) - \alpha(z)$$

for some M. Now, this is equivalent to

$$-\ln(M) \leq (1-\alpha)z - \beta \ln(z)$$

for some M. By applying basic calculus, the right hand side takes a minimum at $z = \beta/(1-\alpha)$, so taking $M = (1-\alpha)(\beta/(1-\alpha)) - \beta \ln(\beta/(1-\alpha))$ suffices so that $-\ln(M) \leq (1-\alpha)z - \beta\ln(z)$ and, equivalently, $z^\beta e^{-z} \leq Me^{-\alpha z}$ for all $z \geq 0$, as desired.

Part b:

First, consider that $\partial_t \Phi(x,t) = e^{-\frac{|x|^2}{4t}} \left[-\frac{2\pi n}{(4\pi t)^{n/2+1}} + \frac{|x|^2}{4t^2 (4\pi t)^{n/2}} \right]$. Simplifying,

$$\partial_t \Phi(x,t) = \Phi(x,t) \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right]$$

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1$, we get

$$\begin{aligned}
|\partial_t \Phi(x,t)| &= \left| \Phi(x,t) \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= \Phi(x,t) \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&\leq \frac{M e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2} (|x|^2/(4t))^{\beta}} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M 2^{n/2} \Phi(x,2t) (|x|^2/(4t))^{-\beta} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M_1 \Phi(x,2t) (|x|^2/(4t))^{-\beta} \left| \left[\frac{|x|^2}{4t^2} - \frac{n}{2t} \right] \right| \\
&= M_1 \Phi(x,2t) (|x|^2/(4t))^{-\beta} \frac{|x|^2}{4t^2} \\
&= M_1 \Phi(x,2t)/t
\end{aligned}$$

As desired.

Next, consider that $\partial_{x_i} \Phi(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left[-\frac{x_i}{2t} \right]$.

Now, using part a above with $\alpha = 1/2$, $z = |x|^2/(4t)$, and $\beta = 1/2$, we get

$$\begin{aligned}
|\partial_{x_i} \Phi(x,t)| &= \left| \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \left[-\frac{x_i}{2t} \right] \right| \\
&\leq \frac{M e^{-\frac{|x|^2}{8t}}}{(4\pi t)^{n/2}} \left| \frac{x_i}{2t z^{\beta}} \right| \\
&\leq M 2^{n/2} \phi(x,2t) \left| \frac{|x|}{2t} \frac{2\sqrt{t}}{|x|} \right| \\
&= M_2 \phi(x,2t)/\sqrt{t}
\end{aligned}$$

As desired.

Part c: