Note: in the below, we adopt the notation $\phi_a : D_1(0) \to D_1(0)$ to be given by $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. This is the same as the f_a given in class, but that notation lends itself to issues in this homework.

I should've said this in the other homework as well, but I use $\overline{\mathbb{C}}$ to denote the Riemann Sphere because I can't figure out how to get $\mathbb{C}$ with a hat over it.

Problem 1:

The map described in class is $f \circ g \circ h$, where $f(z) = \frac{z-1}{z+1}$, $g(z) = \sqrt{z}$, and $h(z) = \frac{z-1}{z+1}$.

Its inverse is thus $h^{-1} \circ g^{-1} \circ f^{-1}$, which is $F : D_1(0) \to \overline{\mathbb{C}} \setminus [-1, 1]$ where $F(z) = \frac{\left(\frac{z+1}{1-z}\right)^2+1}{1-\left(\frac{z+1}{1-z}\right)^2} = \frac{-z^2-1}{2z}$.

Consider the set $\partial D_r(0)$ where $r < 1$. We see that $F(\partial D_r(0)) = \{\frac{-z^2-1}{2z} : |z| = r\}$....

I have recognized that this is horribly broken (1/2 maps to 3/4, which is on the line segment we excluded), and I have no earthly clue how to fix this.

Problem 2:

Consider $\phi_{f(0)}(f(z))$. Taking a derivative, we get:

$$
\begin{aligned}
(\phi_{f(0)}(f(z)))' &= \phi'_{f(0)}(f(z))f'(z) \\
f'(z) &= \frac{(\phi_{f(0)}(f(z)))'}{\phi'_{f(0)}(f(z))} \\
|f'(z)| &= \left|\frac{(\phi_{f(0)}(f(z)))'}{\phi'_{f(0)}(f(z))}\right| \\
|f'(0)| &= \left|\frac{(\phi_{f(0)}(f(0)))'}{\phi'_{f(0)}(f(0))}\right| \\
|f'(0)| &= (1 - |f(0)|^2) \left|(\phi_{f(0)}(f(0)))'\right|
\end{aligned}
$$

with the last line being because $|\phi'_a(a)| = \frac{1}{1-|a|^2}$, which was discussed in class.

Moreover, $\left|(\phi_{f(0)}(f(0)))'\right| \leq 1$, by Schwarz's lemma. (Note that $\phi_{f(0)}(f(z))$ is a holomorphic map fixing the origin, so its derivative at the origin is at most 1.)

Thus, we have $|f'(z)| \leq (1 - |f(0)|^2)$.

Problem 3:

Fix $z \in D_1(0)$. Consider $f(\phi_{-z}(w))$ as a function of w. Taking a derivative, we get:

$$\begin{aligned}
(f(\phi_{-z}(w)))' &= f'(\phi_{-z}(w))\phi'_{-z}(w) \\
(f(\phi_{-z}(0)))' &= f'(\phi_{-z}(0))\phi'_{-z}(0) \\
\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)} &= f'(\phi_{-z}(0)) \\
\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)} &= f'(z) \\
\left|\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)}\right| &= |f'(z)| \\
|f'(z)| &= |f(\phi_{-z}(0)))'| \frac{1}{1 - |-z|^2} \\
|f'(z)| &\leq \frac{1}{1 - |z|^2}
\end{aligned}$$

With the last line being by Schwarz's lemma, as above.

Problem 4:

Consider $\{z \in \mathbb{C} : A|z|^2 + 2\mathrm{Re}(Bz^2) + 2\mathrm{Re}(Cz) + D = 0\}$, with $A, D \in \mathbb{R}$, $B, C \in \mathbb{C}$ (A, B, C, D fixed).

This describes a line when $A = B = 0$; If A or B is nonzero, then something. However, if $A = B = 0$, then the set becomes $\{z \in \mathbb{C} : 2\mathrm{Re}(Cz) = D\}$, which is rather clearly a line.

This describes a circle when

Problem 5:

(Note: I had read this in Complex Made Simple before this was assigned.)

Let $\phi \in \text{Aut}(\overline{\mathbb{C}})$. Say $\mathcal{C}$ is the set of all circles and lines in the complex plane.

Note that $\text{Aut}(\overline{\mathbb{C}})$ is the set of linear-fractional transformations. Further note that the set of linear-fractional transformations is generated, as a group, by the maps of the form $z \mapsto az + b$ (with $a, b \in \mathbb{C}$) and the map $z \mapsto 1/z$.

It suffices to show our result for the generating set.

The result is clear for linear maps (note that they're a dilation followed by a translation followed by a rotation.)

For the map $f(z) = 1/z$, let ℓ be a line through the origin: that is, $\ell = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = \epsilon r\}$ for some fixed $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$. Then $f(\ell)$ is another line: $f(\ell) = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = 1/(\epsilon r)\}$; note that $|1/\epsilon| = 1$ and $1/r$ is an automorphism of $\overline{\mathbb{R}}$.

If ℓ is a line that misses the origin: that is, $\ell = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = \epsilon r + c\}$ for some fixed $\epsilon \in \mathbb{C}$ and $c \in \mathbb{C}$ with $|\epsilon| = 1$. Then $f(\ell)$ is a circle: $f(\ell) = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = 1/(\epsilon r + c)\}$, which is a circle. (I am somewhat certain we discussed this in class.)

Let Γ be a circle centered at the origin. Then

Let Γ be a circle not centered at the origin. Then

So in all cases, $f(z) = 1/z$ maps $\mathcal{C}$ to itself.

So we have the desired result.

Problem 6:

Let $\Omega \subset \mathbb{C}$ be open, $f_n \in \mathcal{O}(\Omega)$, $\sup(|f_n(z)|) = L < \infty$, $\xi_j \in \Omega$, (with each ξ_j distinct), $\xi_j \to \xi \in \Omega$, and $f_n(\xi_j) \to \Xi_j$ for some Ξ_j.

By Vitali-Montel, there's a subsequence of f_n, call it f_{n_k}, that converges locally uniformly to some holomorphic function, f.

Consider $\mathcal{F}$, the set of functions f such that f_{n_k} converges to f for some subsequence f_{n_k}.

So f_n converges to f, and f_n has a subsequence that converges locally uniformly to f.

Problem 7:

Consider $\text{Aut}(\mathbb{C} \setminus \{0\})$.

Let $\phi \in \mathrm{Aut}(\mathbb{C} \setminus \{0\})$. Then ϕ is an injective holomorphism with singularities at 0 and ∞. By the exam problem, ϕ has removable singularities or (first order) poles at 0 and ∞.

If ϕ has a removable singularity at 0, then ϕ is extended naturally to an automorphism of $\mathbb{C}$. Thus, ϕ is given by $z \mapsto az + b$ for some $a, b \in \mathbb{C}$. Note that $b = 0$ in this case, otherwise $\phi(-b/a) = 0$, so that ϕ is no longer well defined. Moreover, $a \neq 0$, else ϕ isn't injective.

So if ϕ has a removable singularity at 0, then ϕ is given by $z \mapsto az$ for some $a \in \mathbb{C}$, $a \neq 0$.

Next, let ϕ have a pole at 0. Then

Problem 8: