Problem 1:

Let $f, g \in \mathcal{O}(D_r(c))$, $g(c) = 0$, and $g'(c) \neq 0$.

Without loss of generality, $c = 0$. Now, let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$. Because $g(0) = 0$, we have that $b_0 = 0$. So,

$$\begin{aligned}
\mathrm{Res}_0 \frac{f}{g} &= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{f}{g} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{\sum\limits_{n=0}^{\infty} b_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{\sum\limits_{n=1}^{\infty} b_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{z \sum\limits_{n=0}^{\infty} b_{n+1} z^n} dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{\partial D_r(0)} \frac{a_n z^n}{z \sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz
\end{aligned}$$

All but the first of those terms vanish; $\frac{z^n a_n}{zb_1 + z^2 b_2 \ldots} = \frac{z^n a_n}{zh(z)} = \frac{z^{n-1} a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 if $n \geq 1$ ($h(z)$ is nonzero on a small enough disk, else g is identically zero...and so $g' = 0$. It's also nonzero at 0, because $b_1 \neq 0$ (else $g'(0) = 0$)).

So, using h as above,

$$\begin{aligned}
\mathrm{Res}_0 \frac{f}{g}(c) &= \frac{1}{2\pi i} \sum_{n=k}^{\infty} \int_{\partial D_r(0)} \frac{a_n z^n}{z \sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= \frac{1}{2\pi i} \int_{\partial D_r(0)} \frac{a_0}{z \sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= \frac{a_0}{2\pi i} \int_{\partial\partial D_r(0)} \frac{1}{z h(z)} dz \\
&= \frac{a_0}{2\pi i} 2\pi i \frac{1}{h(0)} \\
&= a_0 / b_1 \\
&= f(0)/g'(0)
\end{aligned}$$

Yielding our result.

Problem 2:

Let $f \in \mathcal{O}(\dot{D}_r(c))$ with c not an essential singularity. Without loss of generality, $c = 0$.

Consider $\mathrm{Res}_0 \frac{f'}{f}$. Now, let $f(z) = \sum\limits_{n=k}^{\infty} a_n z^n$ with a_k nonzero, so that $f'(z) = \sum\limits_{n=k}^{\infty} n a_n z^{n-1}$; k will be the order of zero if positive, and -1 times the order of pole if negative, and this is clear. So,

$$
\begin{aligned}
\mathrm{Res}_0 \frac{f'}{f} &= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{f'}{f} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{\sum\limits_{n=k}^{\infty} n a_n z^{n-1}}{\sum\limits_{n=k}^{\infty} a_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{\sum\limits_{n=k}^{\infty} n a_n z^{n-1}}{z \sum\limits_{n=k}^{\infty} a_n z^{n-1}} dz \\
&= \frac{1}{2\pi i} \sum_{n=k}^{\infty} \int\limits_{\partial D_r(0)} \frac{n a_n z^{n-1}}{z \sum\limits_{m=k}^{\infty} a_m z^{m-1}} dz
\end{aligned}
$$

All but the first of those terms vanish; $\frac{z^n a_n}{z(a_k z^k + a_{k+1} z^{k+1} + \ldots)} = \frac{z^n a_n}{z z^k h(z)} = \frac{z^{n-k} a_n}{z h(z)}$ is holomorphic on a sufficiently small disk around 0 if $n > k$ ($h(z)$ is nonzero on a small enough disk, else a_k was zero...).

So,

$$
\begin{aligned}
\mathrm{Res}_0 \frac{f'}{f} &= \frac{1}{2\pi i} \sum_{n=-k}^{\infty} \int\limits_{\partial D_r(0)} \frac{n a_n z^{n-1}}{z \sum\limits_{m=-k}^{\infty} a_m z^{m-1}} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{k a_k z^{k-1}}{z \sum\limits_{m=-k}^{\infty} a_m z^{m-1}} dz \\
&= \frac{k a_k}{2\pi i} \int\limits_{\partial D_r(0)} \frac{1}{z h(z)} dz \\
&= \frac{k a_k}{2\pi i} 2\pi i \frac{1}{h(0)} \\
&= k
\end{aligned}
$$

Yielding our result.

Problem 3:

A real-variable analogue of Rouche's Theorem would be:

"Let I be an open interval (a, b), f, g be differentiable on I, and let J be an open interval containing the closure of I.

If $|f(a)| < |g(a)|$ and $|f(b)| < |g(b)|$, then g, $g - f$ have the same number of zeroes in I."

The obvious counterexample is $f(x) = 0$ if $x = 0$, $f(x) = \sin(1/x)$ otherwise, and $g(x) = 1$ on the interval $(0, 1/2\pi)$. Now, $f(x) = 0$ at $0, 1/2\pi$, and $g(x) = 1$, so $|f| < |g|$ on the boundary of the interval. But g has no zeroes, and $g - f$ has infinitely many zeroes. So this breaks.

Problem 4:

Consider $\int\limits_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2} dx$, with $a \in \mathbb{R}$ and $a > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iz}+e^{-iz}}{z^2+a^2}dz \\
&= \int_{0}^{\infty} \frac{e^{iz}+e^{-iz}}{z^2+a^2}dz \text{ (because the function is even...)} \\
&= \int_{0}^{\infty} \frac{e^{iz}}{z^2+a^2}dz + \int_{0}^{\infty} \frac{e^{-iz}}{z^2+a^2}dz \\
&= \int_{0}^{\infty} \frac{e^{iz}}{z^2+a^2}dz - \int_{0}^{-\infty} \frac{e^{iz}}{z^2+a^2}dz \text{ (u-substitute -z)} \\
&= \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+a^2}dz \\
&= \int_{-\infty}^{\infty} \frac{\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{(iz)^n}{n!}}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty} \frac{z^n}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}2\pi i\sum_{c\in\mathbb{C}^+}\operatorname{Res}_c\frac{z^n}{z^2+a^2} \text{ (As discussed in class)}
\end{aligned}$$

With the last line being discussed in class, and C^+ being the upper half of the complex plane. Now, $\frac{z^n}{z^2+a^2}$ can only have poles where $z^2+a^2=0$; that is, where $z=\pm ia$.

So, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}dx &= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \sum_{c\in\mathbb{C}} \operatorname{Res}_c \frac{z^n}{z^2+a^2} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_{ia} \frac{z^n}{z^2+a^2}\right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_0 \frac{(z+ia)^n}{(z+ia)^2+a^2}\right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_0 \frac{\sum_{m=0}^{n} \binom{n}{m} z^m (ia)^{n-m}}{z^2+2ia}\right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_0 \frac{\sum_{m=0}^{n} \binom{n}{m} z^m (ia)^{n-m}}{z^2+2iaz}\right]
\end{aligned}
$$

Applying problem 1 to the above, we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}dx &= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\operatorname{Res}_0 \frac{\sum_{m=0}^{n} \binom{n}{m} z^m (ia)^{n-m}}{z^2+2iaz} \right] \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} 2\pi i \left[\frac{(ia)^n}{2ia} \right] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[(ia)^n \right] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[(ia)^n \right] \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{(aii)^n}{n!} \\
&= \frac{\pi}{a} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \\
&= \frac{\pi}{a} e^{-a}
\end{aligned}
$$

Which is the desired result.

Problem 5:

Consider $\int\limits_{\Gamma_T} z^\alpha R(z)dz$ with $R(z) = P(z)/Q(z)$ (with R a rational function, P and Q polynomials, and Γ_T as pictured below.)

For this problem, we can take T large enough that the above closed curve fails to enclose any complex zeroes of Q, but encloses all real zeroes of Q.

$$
\int_{\Gamma_T} z^\alpha R(z)dz = \int_{\gamma_1} z^\alpha R(z)dz - \int_{\gamma_2} z^\alpha R(z)dz
$$

Now, $\int_{\gamma_1} z^\alpha R(z)dz = \sum_{z\in\mathbb{R}} \text{Res}_z z^\alpha R(z)$.
Consider $\int_{\gamma_2} z^\alpha R(z)dz$.

$$\int_{\gamma_2} z^\alpha R(z)dz = \int_{-1}^{1} (T + it/T)^\alpha R(T + it/T)(i/T)dt$$

The above being readily computed if R is known. So,

$$\begin{aligned}\int_{\Gamma_T} z^\alpha R(z)dz &= \int_{\gamma_1} z^\alpha R(z)dz - \int_{\gamma_2} z^\alpha R(z)dz \\ &= \sum_{z\in\mathbb{R}} \text{Res}_z z^\alpha R(z) - \int_{-1}^{1} (T + it/T)^\alpha R(T + it/T)(i/T)dt\end{aligned}$$

Although the above expression appears disgusting, it suffices for the desired purpose.

Now, let $T \to \infty$. The integral $\int_{-1}^{1} (T+it/T)^\alpha R(T+it/T)(i/T)dt$ vanishes, which is clear by applying the ML-inequality/trivial estimate.

The limit above represents the integral $\int_{\infty}^{0} x^\alpha R(x)dx + \int_{0}^{\infty} x^\alpha R(x)dx$. Intuition demands that this integral vanish, but weird things happen at infinity.

Problem 6:

Consider $e^z = 6z^2 + 1$. This is equivalent to $0 = 6z^2 + 1 - e^z$.

Define $g(z) = 6z^2 + 1$ and $f(z) = e^z$. When $|z| = 2$, $|g| \geq |6z^2| - 1 = 23$ and $|f| \leq e^2 \leq 9$. So $g > f$ when $|z| = 2$.

So Rouche's Theorem applies: $e^z = 6z^2 + 1$ has the same number of solutions as $0 = 6z^2 + 1$ on the disk bounded by $|z| = 2$.

Now, $6z^2 + 1$ has two solutions, by the fundamental theorem of algebra. Moreover, $\pm\frac{i}{\sqrt{6}}$ are solutions, as is readily checked. These solutions are both in that disk. So $6z^2 - 1$ has two zeroes on the disk bounded by $|z| = 2$.

So $e^z = 6z^2 + 1$ has 2 solutions on the disk bounded by $|z| = 2$.

Problem 7:

Consider a polynomial, $f(z) = \sum_{n=0}^{N} a_n z^n$.

Define $M = 9000N \sum |a_n|$ (Note: M is chosen so that $a_N M^N > \sum_{i=0}^{n} |a_i M^i|$ for any $n < N$). Define $g_0(z) = a_0$. Now, $|f| > |g_0|$ on the boundary of the disk of radius M centered at 0. So $f - g_0$ and f have the same number of zeroes in this disk.

Define $g_1(z) = a_1 z$. Now, $|f - g_0| > |g_1|$ on the boundary of the disk of radius M centered at 0. So $f - g_0 - g_1$ and $f - g_0$ and f have the same number of zeroes in this disk.

The above process can be iterated: define $g_n(z) = a_n z^n$. Then $\left| f - \sum_{m=0}^{n-1} g_m \right| > |g_n|$. So $f - \sum_{m=0}^{n} g_m$ and f have the same number of zeroes in that disk.

So f and $a_n z^n$ have the same number of zeroes on the disk of radius M centered at 0. So f has n zeroes.

Note that we can pick M arbitrarily large (that was the point of M) and have this work. Thus, f has n zeroes on $\mathbb{C}$; this is the fundamental theorem of algebra.

Problem 8:

Let Ω be "standard" (open, bounded, boundary is finitely many piecewise C^1 Jordan curves). Let $f \in \mathcal{O}(G)$, where $G \supset \overline{\Omega}$, and $f \neq 0$ anywhere on $\partial\Omega$.

Consider $\frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{z^k f'(z)}{f(z)} dz$, where $k \in \mathbb{N}$.

This is equal to $\sum_{c \in \Omega} \mathrm{Res}_c z^k f'/f$.

Consider any individual singularity, $c \in \Omega$. Without loss of generality, $c = 0$.

Now, let $f(z) = \sum_{n=l}^{\infty} a_n z^n$ with a_l nonzero, so that $f'(z) = \sum_{n=l}^{\infty} n a_n z^{n-1}$; l will be the order of zero. It's positive, because $f \in \mathcal{O}(\Omega)$.

$$\begin{aligned}
\mathrm{Res}_0 z^k \frac{f'}{f} &= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{z^k f'}{f} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{z^k \sum\limits_{n=l}^{\infty} n a_n z^{n-1}}{\sum\limits_{n=l}^{\infty} a_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{z^k \sum\limits_{n=l}^{\infty} n a_n z^{n-1}}{z \sum\limits_{n=l}^{\infty} a_n z^{n-1}} dz \\
&= \frac{1}{2\pi i} \sum_{n=l}^{\infty} \int\limits_{\partial D_r(0)} \frac{z^k n a_n z^{n-1}}{z \sum\limits_{m=l}^{\infty} a_m z^{m-1}} dz
\end{aligned}$$

All but the $l-k$th of those terms vanish; $\frac{z^k z^n a_n}{z(a_l z^l + a_{l+1} z^{l+1} + \ldots)} = \frac{z^k z^n a_n}{z z^l h(z)} = \frac{z^{n+k-l} a_n}{z h(z)}$ is holomorphic on a sufficiently small disk around 0 if $n+k-l > 0$ ($h(z)$ is nonzero on a small enough disk, else a_l was zero...).

So,

$$\begin{aligned}
\mathrm{Res}_0 z^k \frac{f'}{f} &= \frac{1}{2\pi i} \sum_{n=l}^{\infty} \int\limits_{\partial D_r(0)} \frac{z^k n a_n z^{n-1}}{z \sum\limits_{m=l}^{\infty} a_m z^{m-1}} dz \\
&= \frac{1}{2\pi i} \int\limits_{\partial D_r(0)} \frac{(l-k) a_{l-k} z^{l-k-1}}{z \sum\limits_{m=l}^{\infty} a_m z^{m-1}} dz
\end{aligned}$$

This vanishes if $k > l$, because $a_{l-k} = 0$ then. Else,

$$\begin{aligned}
\mathrm{Res}_0 z^k \frac{f'}{f} &= \frac{(l-k) a_{l-k}}{2\pi i} \int\limits_{\partial D_r(0)} \frac{1}{z h(z)} dz \\
&= \frac{a_{l-k}}{2\pi i} 2\pi i \frac{1}{h(0)} \\
&= l - k
\end{aligned}$$

So, back to our original problem; $\frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{z^k f'(z)}{f(z)} dz = \sum\limits_{c\in\Omega} \text{Res}_c z^k f'/f = \sum\limits_{c\in\Omega} \max(l_c - k, 0)$, where l is the order of zero at c.

In words, $\frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{z^k f'(z)}{f(z)} dz$ is equal to the sum of the orders of zero at points with order of zero at least k, minus the number of such zeroes times k.