Note: in the below, we adopt the notation $\phi_a : D_1(0) \to D_1(0)$ to be given by $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. This is the same as the f_a given in class, but that notation lends itself to issues in this homework.

I should've said this in the other homework as well, but I use $\overline{\mathbb{C}}$ to denote the Riemann Sphere because I can't figure out how to get $\mathbb{C}$ with a hat over it.

Problem 1:

(I googled facts about ellipses in order to get the last equation.)

The map described in class is $f \circ g \circ h$, where $f(z) = \frac{z-1}{z+1}$, $g(z) = \sqrt{z}$, and $h(z) = \frac{z-1}{z+1}$.

Its inverse is thus $h^{-1} \circ g^{-1} \circ f^{-1}$, which is $F : D_1(0) \to \overline{\mathbb{C}} \setminus [-1, 1]$ where $F(z) = \frac{\left(\frac{z+1}{1-z}\right)^2+1}{1-\left(\frac{z+1}{1-z}\right)^2} = \frac{-z^2-1}{2z}$.

Consider the set $\partial D_r(0)$ where $r < 1$. We see that $F(\partial D_r(0)) = \{\frac{-z^2-1}{2z} : |z| = r\}$.

Now, let $p \in F(\partial D_r(0))$, so that for some $z \in D_1(0)$, we have $\frac{-z^2-1}{2z} = p$. Say that $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Then we have

$$\begin{aligned}
|p+1| + |p-1| &= \left|\frac{-z^2-1}{2z} + 1\right| + \left|\frac{-z^2-1}{2z} - 1\right| \\
&= \left|\frac{-z^2+2z-1}{2z}\right| + \left|\frac{-z^2-2z-1}{2z}\right| \\
&= \frac{1}{2r}\left[\left|-z^2+2z-1\right| + \left|-z^2-2z-1\right|\right] \\
&= \frac{1}{2r}\left[|z-1|^2 + |z+1|^2\right] \\
&= \frac{1}{2r}\left[(x-1)^2 + y^2 + (x+1)^2 + y^2\right] \\
&= \frac{1}{2r}\left[2y^2 + 2x^2 + 2\right] \\
&= \frac{r^2+1}{r}
\end{aligned}$$

So the sum of the distance from p to 1 and from p to -1 does not depend on p; this set is an ellipse with foci at 1 and -1; the length of the semimajor axis is given by half the sum of the distances; that is, it is $\frac{r^2+1}{2r}$.

To be more explicit, the set is given by the equation $1 = \frac{x}{(\frac{r^2+1}{2r})^2} + \frac{y}{1-(\frac{r^2+1}{2r})^2}$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Problem 2:
Consider $\phi_{f(0)}(f(z))$. Taking a derivative, we get:

$$\begin{aligned}
(\phi_{f(0)}(f(z)))' &= \phi'_{f(0)}(f(z))f'(z) \\
f'(z) &= \frac{(\phi_{f(0)}(f(z)))'}{\phi'_{f(0)}(f(z))} \\
|f'(z)| &= \left|\frac{(\phi_{f(0)}(f(z)))'}{\phi'_{f(0)}(f(z))}\right| \\
|f'(0)| &= \left|\frac{(\phi_{f(0)}(f(0)))'}{\phi'_{f(0)}(f(0))}\right| \\
|f'(0)| &= (1 - |f(0)|^2)\left|(\phi_{f(0)}(f(0)))'\right|
\end{aligned}$$

with the last line being because $|\phi'_a(a)| = \frac{1}{1-|a|^2}$, which was discussed in class.

Moreover, $\left|(\phi_{f(0)}(f(0)))'\right| \leq 1$, by Schwarz's lemma. (Note that $\phi_{f(0)}(f(z))$ is a holomorphic map fixing the origin, so its derivative at the origin is at most 1.)

Thus, we have $|f'(z)| \leq (1 - |f(0)|^2)$.

Problem 3:
Fix $z \in D_1(0)$. Consider $f(\phi_{-z}(w))$ as a function of w. Taking a derivative, we get:

$$\begin{aligned}
(f(\phi_{-z}(w)))' &= f'(\phi_{-z}(w))\phi'_{-z}(w) \\
(f(\phi_{-z}(0)))' &= f'(\phi_{-z}(0))\phi'_{-z}(0) \\
\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)} &= f'(\phi_{-z}(0)) \\
\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)} &= f'(z) \\
\left|\frac{(f(\phi_{-z}(0)))'}{\phi'_{-z}(0)}\right| &= |f'(z)| \\
|f'(z)| &= |f(\phi_{-z}(0)))'| \frac{1}{1-|-z|^2} \\
|f'(z)| &\leq \frac{1}{1-|z|^2}
\end{aligned}$$

With the last line being by Schwarz's lemma, as above.

Problem 4:

Consider $\{z \in \mathbb{C} : A|z|^2 + 2\mathrm{Re}(Bz^2) + 2\mathrm{Re}(Cz) + D = 0\}$, with $A, D \in \mathbb{R}$, $B, C \in \mathbb{C}$ (A, B, C, D fixed). We are given that this represents a conic section.

This describes a line when $A = B = 0$; If A or B is nonzero, then consider any two points z, z' in the set where $z \neq z'$. Then $\frac{z+z'}{2}$ is not in the set;

$$\begin{aligned}
&A\left|\frac{z+z'}{2}\right|^2 + 2\mathrm{Re}(B(\frac{z+z'}{2})^2) + 2\mathrm{Re}(C\frac{z+z'}{2}) + D \\
&= A\left|\frac{z+z'}{2}\right|^2 + \frac{1}{2}\mathrm{Re}(B(z+z')^2) + \mathrm{Re}(C(z+z')) + D \\
&= \frac{A}{4}|z+z'|^2 + \frac{1}{2}\mathrm{Re}(Bz^2) + \frac{1}{2}\mathrm{Re}(Bz'^2) + \mathrm{Re}(Bzz') + \mathrm{Re}(Cz) + \mathrm{Re}(Cz') + D \\
&= \frac{A}{4}|z+z'|^2 - \frac{A}{2}|z|^2 - \frac{A}{2}|z'|^2 - \frac{1}{2}\mathrm{Re}(Bz^2) - \frac{1}{2}\mathrm{Re}(Bz'^2) + \mathrm{Re}(Bzz') \\
&= \frac{A}{4}\left[|z+z'|^2 - 2|z|^2 - 2|z'|^2\right] + \mathrm{Re}(B(zz' - \frac{z^2+z'^2}{2}))
\end{aligned}$$

Both $\left[|z+z'|^2 - 2|z|^2 - 2|z'|^2\right]$ and $(zz' - \frac{z^2+z'^2}{2})$ vanish only when $z = z'$, and this is somewhat clear.

Thus, this could not have described a line; the midpoint of two points on it fails to be in the set.

However, if $A = B = 0$, then the set becomes $\{z \in \mathbb{C} : 2\mathrm{Re}(Cz) = D\}$, which is rather clearly a line (if this is not clear, repeating the same analysis as above yields that the midpoint of any two points is in the set...and the only conic section satisfying this is the line.).

This describes a circle when

Problem 5:

(Note: I had read this in Complex Made Simple before this was assigned.)

Let $\phi \in \mathrm{Aut}(\overline{\mathbb{C}})$. Say $\mathcal{C}$ is the set of all circles and lines in the complex plane.

Note that $\mathrm{Aut}(\overline{\mathbb{C}})$ is the set of linear-fractional transformations. Further note that the set of linear-fractional transformations is generated, as a group, by the maps of the form $z \mapsto az + b$ (with $a, b \in \mathbb{C}$) and the map $z \mapsto 1/z$.

It suffices to show our result for the generating set.

The result is clear for linear maps (note that they're a dilation followed by a translation followed by a rotation.)

For the map $f(z) = 1/z$, let ℓ be a line through the origin: that is, $\ell = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = \epsilon r\}$ for some fixed $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$. Then $f(\ell)$ is another line: $f(\ell) = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = 1/(\epsilon r)\}$; note that $|1/\epsilon| = 1$ and $1/r$ is an automorphism of $\overline{\mathbb{R}}$.

If ℓ is a line that misses the origin: that is, $\ell = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = \epsilon r + c\}$ for some fixed $\epsilon \in \mathbb{C}$ and $c \in \mathbb{C}$ with $|\epsilon| = 1$. Then $f(\ell)$ is a circle: $f(\ell) = \{z \in \mathbb{C} : \exists r \in \overline{\mathbb{R}} : z = 1/(\epsilon r + c)\}$, which is a circle. (I am somewhat certain we discussed this in class.)

Let Γ be a circle centered at the origin. Then

Let Γ be a circle not centered at the origin. Then

So in all cases, $f(z) = 1/z$ maps $\mathcal{C}$ to itself.

So we have the desired result.

Problem 6:

Let $\Omega \subset \mathbb{C}$ be open, $f_n \in \mathcal{O}(\Omega)$, $\sup(|f_n(z)|) = L < \infty$, $\xi_j \in \Omega$, (with each ξ_j distinct), $\xi_j \to \xi \in \Omega$, and $f_n(\xi_j) \to \Xi_j$ for some Ξ_j.

By Vitali-Montel, there's a subsequence of f_n, call it f_{n_k}, that converges locally uniformly to some holomorphic function, f.

Consider $\mathcal{F}$, the set of functions f such that f_{n_k} converges to f for some subsequence f_{n_k}.

So f_n converges to f, and f_n has a subsequence that converges locally uniformly to f.

Problem 7:

Consider $\mathrm{Aut}(\mathbb{C} \setminus \{0\})$.

Let $\phi \in \mathrm{Aut}(\mathbb{C} \setminus \{0\})$. Then ϕ is an injective holomorphism with singularities at 0 and ∞. By the exam problem, ϕ has removable singularities or (first order) poles at 0 and ∞.

If ϕ has a removable singularity at 0, then ϕ is extended naturally to an automorphism of $\mathbb{C}$. Thus, ϕ is given by $z \mapsto az + b$ for some $a, b \in \mathbb{C}$. Note that $b = 0$ in this case, otherwise $\phi(-b/a) = 0$, so that ϕ is no longer well defined. Moreover, $a \neq 0$, else ϕ isn't injective.

So if ϕ has a removable singularity at 0, then ϕ is given by $z \mapsto az$ for some $a \in \mathbb{C}$, $a \neq 0$.

Next, let ϕ have a pole at 0. Then

Problem 8: