Problem 1: Problem 6 in textbook:

Let U be a bounded, open subset of $\mathbb{R}^n$.

We freely use the preceding problem's result that if $-\Delta v \leq 0$, then $\max\limits_{\overline{U}} v = \max\limits_{\partial U} v$.

Define $\lambda = \max\limits_{\overline{U}} |f|$. Define $M = \max(1, \frac{r^2}{2n})$ where r is an upper bound on the distance of a point in U from 0.

So we have:

$$\begin{aligned}
\max_{\overline{U}} u &\leq \max_{\overline{U}}(u + \frac{|x|^2}{2n}\lambda) \\
&= \max_{\partial U}(u + \frac{|x|^2}{2n}\lambda) \\
&\leq \max_{\partial U}(u) + \max_{\partial U} \frac{|x|^2}{2n}\lambda) \\
&\leq \max_{\partial U}(u) + M\lambda \\
&= \max_{\partial U}(g) + M\lambda \\
&\leq \max_{\partial U}(|g|) + M \max_{\overline{U}}(|f|) \\
&\leq M(\max_{\partial U}(|g|) + \max_{\overline{U}}(|f|))
\end{aligned}$$

Problem 2: Problem 9 in textbook:

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$ with $|x| \leq 1$.

Then $u(x) = -\int\limits_{\partial U} |x| \, (\frac{\partial G}{\partial \nu}(x, y) dS(y))$, where G is the Green's function for the half-space, $G(x, y) = \Phi(y - x) - \Phi(y - \overline{x})$.

Consider $\frac{u(\lambda e_n) - u(0)}{\lambda}$ (with $\lambda > 0$). We can see that:

$$\begin{aligned}
\frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{\int\limits_{\partial U} |\lambda e_n| \left[-\frac{\partial G}{\partial \nu}(\lambda e_n, y)\right] dS(y)}{\lambda} \\
&= \frac{\int\limits_{\partial U} |\lambda e_n| \left[-\frac{\partial \Phi}{\partial \nu}(y - \lambda e_n) + \frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n})\right)\right] dS(y)}{\lambda} \\
&= \frac{\int\limits_{\partial U} |\lambda e_n| \left[\frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n}) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)\right)\right] dS(y)}{\lambda} \\
&= \int\limits_{\partial U} \left[\frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n}) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)\right)\right] dS(y) \\
&= \int\limits_{\partial U} \left[\frac{\partial \Phi}{\partial \nu}(y + \lambda e_n) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)\right)\right] dS(y)
\end{aligned}$$

Problem 3: Problem 10 in textbook:

Part a:

Let U^+ be the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x : x_n = 0\}$. Now, set

$$v(x) = \begin{cases} u & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Part b:

Problem 4: Only problem on sheet: