Problem 1:
Consider $u(x, y)$ solving

$$\begin{aligned} u_y^2 u_{xx} + u u_{xy} + u_x^2 u_{yy} &= u^2 + 1 \\ u(x, 0) = \sin(x), u_y(x, 0) &= \cos(x) \end{aligned}$$

Then the order 2 (and lower) partials for u at $(0, 0)$ are:

$$\begin{aligned} u(0, 0) &= \sin(0) = 0 \\ u_x(0, 0) &= \cos(0) = 1 \\ u_y(0, 0) &= \cos(0) = 1 \\ u_{xx}(0, 0) &= -\sin(0) = 0 \\ u_{xy}(0, 0) &= -\sin(0) = 0 \\ u_{yy}(0, 0) &= 1 \end{aligned}$$

The first five are obtained by the initial conditions and applying partial derivatives to them, and the last is obtained by plugging this information into the PDE. Thus, the second-order Taylor Approximation of u about the point $(0, 0)$ is

$$u(x, y) \approx x + y + \frac{y^2}{2}$$

Now, some of the order 2 (and lower) partials for u at $(\pi/2, 0)$ are:

$$\begin{aligned} u(\pi/2, 0) &= \sin(\pi/2) = 1 \\ u_x(\pi/2, 0) &= \cos(\pi/2) = 0 \\ u_y(\pi/2, 0) &= \cos(\pi/2) = 0 \\ u_{xx}(\pi/2, 0) &= -\sin(\pi/2) = -1 \\ u_{xy}(\pi/2, 0) &= -\sin(\pi/2) = -1 \end{aligned}$$

Plugging this information into the PDE yields $-1 = 2$, which is nonsense; thus, u is inconsistent at $(\pi/2, 0)$.

Problem 2:
Let

$$L[u] = yu_{xx} + (x+y)u_{xy} + xu_{yy} - u_x - u_y$$

Part a:
The equation L is hyperbolic when $\Delta = \left(\frac{x+y}{2}\right)^2 - xy > 0$.
Rewriting this condition, we get:

$$\begin{aligned}
\left(\frac{x+y}{2}\right)^2 - xy &> 0 \\
\left(\frac{x-y}{2}\right)^2 &> 0 \\
(x-y)^2 &> 0 \\
x &\neq y
\end{aligned}$$

That is, L is hyperbolic except when $x = y$.

Part b:
By the discussion in John, the characteristic curves of this PDE satisfy $\frac{dy}{dx} = \frac{(x+y)/2 \pm (x-y)/2}{y}$. That is, the characteristic curves satisfy either $\frac{dy}{dx} = \frac{x}{y}$ or $\frac{dy}{dx} = \frac{y}{y}$. Solving the ODEs, we get that the characteristic curves are the hyperbolas given by $y^2 - x^2 = c$ for some constant c, and the lines $y = x + c$.

Part c:
First, consider the solutions to $y\lambda^2 + (x+y)\lambda + x = 0$; they are $\lambda_1 = -1$ and $\lambda_2 = -x/y$, by the quadratic formula.

We want ξ and η so that $\xi_x = -\lambda_2 \xi_y$ and $\eta_x = \lambda_1 \eta_y$. Choosing $\eta = y - x$ and $\xi = x^2 - y^2$ works for this.

Now, say that $u(x,y)$ solves the PDE, and define $v(\xi, \eta) = v(x^2 - y^2, y - x) = u(x,y)$

Using the Chain Rule, we have:

$$
\begin{aligned}
u_x &= -2yv_\xi + v_\eta \\
u_y &= 2xv_\xi - v_\eta \\
u_{xx} &= 2v_\xi + 4x^2 v_{\xi\xi} - 4xv_{\xi\eta} + v_{\eta\eta} \\
u_{yy} &= -2v_\xi + 4y^2 v_{\xi\xi} - 4yv_{\xi\eta} + v_{\eta\eta} \\
u_{xy} &= -4xyv_{\xi\xi} + 2xv_{\xi\eta} + 2yv_{\xi\eta} + v_{\eta\eta}
\end{aligned}
$$

Thus, the PDE reduces to the canonical form, $4v_\xi(y-x)+v_{\xi\eta}(2x^2-4xy-2y^2) = 4\eta v_\xi + 2\eta^2 v_{\xi\eta} = 0$. We can apply techniques of ODE to determine that $v_\xi = f(\xi)/\eta^2$, and thus $v(\xi,\eta) = F(\xi)/\eta^2 + G(\eta)$ for some F, G.

Returning to x and y variables, this means that $u(x,y) = v(x^2-y^2, y-x) = \frac{F(x^2-y^2)}{(y-x)^2} + G(y-x)$ is the general solution of the PDE given.

Using the initial data, we get that $x^4 - x = \frac{F(x^2)}{x^2} + G(-x)$ and $1 = \frac{2F(x^2)}{x^3} + G'(-x)$. Thus, we have

$$
\begin{aligned}
x/2 &= \frac{F(x^2)}{x^2} + G'(-x)(x/2) \\
x^4 - \frac{3}{2}x &= G(-x) - \frac{1}{2}xG'(-x) \\
x^4 + \frac{3}{2}x &= G(x) + \frac{1}{2}xG'(x) \\
x^5 + 3x^2 &= 2xG(x) + x^2G'(x) \\
x^5 + 3x^2 &= (x^2G(x))' \\
\frac{x^6}{6} + x^3 + C &= x^2G(x) \\
\frac{\frac{x^6}{6} + x^3 + C}{x^2} &= G(x)
\end{aligned}
$$

Where C is some unknown constant. Now, plugging this into the other bit of data, we get

$$
\begin{aligned}
x^4 - x &= \frac{F(x^2)}{x^2} + G(-x) \\
x^4 - x &= \frac{F(x^2)}{x^2} + \frac{\frac{x^6}{6} - x^3 + C}{x^2} \\
x^6 - x^3 &= F(x^2) + \frac{x^6}{6} - x^3 + C \\
\frac{5}{6}x^6 - 2x^3 - C &= F(x^2) \\
\frac{5}{6}x^3 - 2x^{3/2} - C &= F(x)
\end{aligned}
$$

So, to summarize, $u(x,y) = \frac{\frac{5}{6}(x^2-y^2)^3 - 2(x^2-y^2)^{3/2} - C}{(y-x)^2} + \frac{\frac{(y-x)^6}{6} + (y-x)^3 + C}{(y-x)^2}$. After some slight cleaning up, that is

$$
u(x,y) = \frac{\frac{5}{6}(x^2-y^2)^3 - 2(x^2-y^2)^{3/2} + \frac{(y-x)^6}{6} + (y-x)^3}{(y-x)^2}
$$

Problem 3:
Consider $u(x,y)$ solving

$$
\begin{aligned}
2yu_x + u_y &= u^2 \\
u(x,0) = g(x) &= \frac{1}{x^2+1} \text{ for } x \in \mathbb{R}
\end{aligned}
$$

Part a:
We begin by applying the method of characteristics, as in the textbook.

We get that $\dot{x} = 2y$, $\dot{y} = 1$, $\dot{z} = z^2$. Given the parameterization of the curve our initial condition is given on, we get $x(s) = s^2 + x_0$, $y(s) = s$, and $z(s) = \frac{g(x_0)}{1 - sg(x_0)}$.

Now, fix (x,y) near $y = 0$. Then select $s > 0$ and x_0 so that $(x,y) = (x(s), y(s)) = (s^2 + x_0, s)$. In other words, $s = y$ and $x_0 = x - y^2$. Then $u(x,y) = \frac{\frac{1}{1+(x-y^2)^2}}{1 - y\frac{1}{1+(x-y^2)^2}} = \frac{1}{1+(x-y^2)^2 - y}$.

Part b:
I'm not sure how to handle this problem.