I don't know if we covered the ρ metric ($\rho(f,g) = \sup(|f(x) - g(x)| : x \in X)$, where X is the domain of f and g), but I'm using it because it's nice and I like it.

Problem 1:

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = x + 1/n$.

It is rather clear that this sequence of functions converges uniformly (to x).

However, the sequence of functions $\langle f_n^2 \rangle$ fails to converve uniformly:

For each $n \in \mathbb{N}$, $f_n^2(x) = x^2 + 2x/n + 1/n^2$. It is rather clear that $\langle f_n^2 \rangle$ converges pointwise to x^2. So if $\langle f_n^2 \rangle$ converges uniformly to something, it must converge uniformly to x^2. However, $\langle f_n^2 \rangle$ does not converge uniformly to x^2:

Let $\epsilon > 0$, and pick $n \in \mathbb{N}$. Consider $|f_n^2(x) - x^2| = |2x/n + 1/n^2|$. Pick $x = n\epsilon$. Then $|2x/n + 1/n^2| = 2\epsilon + 1/n^2 \geq \epsilon$.

So for all $\epsilon > 0$ and $n \in \mathbb{N}$ there is an $x \in \mathbb{R}$ such that $|f_n^2(x) - x^2| \geq \epsilon$; f_n^2 does not converge uniformly to x^2.

So $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$, but $\langle f_n^2 \rangle$ doesn't. This satisfies the problem.

Problem 2:

Longish note: After finishing this problem, I noticed that this follows immediately from a fragment of the proof of Arzela-Ascoli. I prefer this proof, as it is smoother, but it is important to note that such a thing is possible. Moreover, I used a general metric in this one, because it seems like if we had an appropriate extension of Arzela-Ascoli, then this proof could be extended to other types of metric spaces...Does such a theorem exist?

Proof starts here: Let $\langle f_n \rangle$ be an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise to some function, say f.

By the Arzela-Ascoli theorem, we know that $\langle f_n \rangle$ has some subsequence that uniformly converges to some function. We know that this function must be f: if a subsequence of functions converges uniformly to f, it converges pointwise to f. If a sequence of functions converges pointwise to a function, f, then all of its subsequences converge to f. So if a sequence of functions converges pointwise to f, then any subsequence of functions that converges uniformly to a function must converge uniformly to f.

Now, consider such a converging subsequence, $\langle f_{n_j} \rangle$.

Let $\epsilon > 0$. There is a $J \in \mathbb{N}$ such that for all $j \geq J$, $\rho(f_{n_j}, f) < \epsilon/4$.

In addition, by equicontinuity, there is a $\delta_1 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x, y) < \delta_1$ implies that $d(f_n(x), f_n(y)) < \epsilon/4$.

Moreover, we know that converging sequences of continuous functions converge to continuous functions. We also know that continuous functions on a compact domain are uniformly continuous. Thus, f is uniformly continuous; there is a $\delta_2 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x, y) < \delta_2$ implies that $d(f(x), f(y)) < \epsilon/4$.

Define $\delta = \min(\delta_1, \delta_2)$, so that we have both of the lines for δ.

We know that compact sets are totally bounded. (If this is not clear, consider a career in pastry making.)

So, let F be a finite collection of points of K such that for all $x \in K$, $d(x, y) < \delta$ for some $y \in F$.

Now, for each $y \in F$, there is an $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $d(f_n(y), f(y)) < \epsilon/4$.

Define $N = \max(N_y, n_J)$.

Now, for all $n \geq N$, and for all $x \in K$, we have, for some $y \in F$ (we pick y with $d(x, y) < \delta$:

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_n(y)) + d(f_n(y), f_{n_j}(y)) + d(f_{n_j}(y), f(y)) + d(f(y), f(x)) \\ &\leq \epsilon \text{ (That's good enough.)} \end{aligned}$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in K$, $d(f_n(x), f(x)) < \epsilon$. That is, f_n converges uniformly to f.

To summarize, if $\langle f_n \rangle$ is an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise, then $\langle f_n \rangle$ converges uniformly.

Problem 3:

Let $\langle f_n \rangle$ be a uniformly bounded sequence of functions that are Riemann-integrable on $[a, b]$. Set

$$F_n(x) = \int_a^x f_n(t)dt$$

Moreover, let L be the absolute value of a lower bound for the f_ns and let U be the absolute value of an upper bound for the f_ns. (I say absolute values here because I don't want to bother with them later.)

Now, the set of F_ns are equicontinuous:

Let $\epsilon > 0$. Define $\delta = \epsilon / \max(U, L)$. Then for all $n \in \mathbb{N}$, $x, y \in [a, b]$ with $|x - y| < \delta$ (WLOG, $x \leq y$), we have:

$$
\begin{aligned}
|F_n(x) - F_n(y)| &= \left| \int_a^x f_n(t)dt - \int_a^y f_n(t)dt \right| \\
&= \left| \int_x^y f_n(t)dt \right| \text{ I exploit the absolute value here too.} \\
&\leq |(x - y) \max(U, L)| \text{ This is some sort of obvious property of integrals we shou} \\
&< \epsilon
\end{aligned}
$$

So for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$, $x, y \in [a, b]$, $|x - y| < \delta$ implies that $|F_n(x) - F_n(y)| < \epsilon$. That is, the set of F_ns are equicontinuous.

In addition, the F_ns are defined on $[a, b]$, which is a compact space. By Arzela-Ascoli, there is a subsequence $\langle F_{n_j} \rangle$ that converges uniformly on $[a, b]$.

Problem 4:

Let $\langle f_n \rangle$ be a sequence of increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$.

Some subsequence, $\langle f_{n_k} \rangle$, converges at all rational points, to some function f on the rationals:

Enumerate the rationals, by $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$. By Heine-Borel, there is a subsequence of $\langle f_n \rangle$, call it $\langle f_{n_{r_1}} \rangle$, such that $f_{n_{r_1}}(r_1) \to s_1$ for some $s_1 \in [0, 1]$.

Pick f_{n_1} to be the first function in this subsequence.

Next, there is a subsequence of $\langle f_{n_{r_1}} \rangle$, call it $\langle f_{n_{r_2}} \rangle$, such that $f_{n_{r_2}}(r_2) \to s_2$ for some $s_2 \in [0, 1]$. Moreover, $f_{n_{r_2}}(r_1) \to s_1$, as $\langle f_{n_{r_2}} \rangle$ is a subsequence of $\langle f_{n_{r_1}} \rangle$.

Pick f_{n_2} to be the second function in this subsequence.

We carry out the above process recursively, and produce a subsequence of $\langle f_n \rangle$, called $\langle f_{n_k} \rangle$, that rather clearly converges at all rational points.

Define $f : \mathbb{Q} \to [0, 1]$ by $f(r_n) = s_n$ for each $n \in \mathbb{N}$. Then $\langle f_{n_k} \rangle$ converges pointwise to f on $\mathbb{Q}$, as desired.

Now, define $f(x) = \sup(f(r) : r \leq x)$.

Then $f_{n_k}(x) \to f(x)$ at all points of continuity, x;

Consider a sequence of rationals, $\langle q_n \rangle$, converging to x. We know that $f(q_n) \to f(x)$, by continuity. Moreover, $f_{n_k}(q_n) \to f(q_n)$ at each rational point. Now, there are only finitely many of the f_{n_k}s that are discontinuous at x, otherwise f would be discontinuous at x. Thus, we can pick n_k large enough that $f_{n_k}(q_n) \to f_{n_k}(x)$, for all $n_k > n_K$. From the above, the triangle inequality yields the result.

Now, there are countably many points of discontinuity of f; f is an increasing function by the definition, and thus f has countably many points of discontinuity.

Thus, there is a subsequence of $\langle f_{n_k} \rangle$ that converges to f at every point of discontinuity of f;

Enumerate the ~~rationals~~ discontinuity set, by $\{r_i\}_{i=1}^{\infty}$. By Heine-Borel, there is a subsequence of $\cancel{\langle f_n \rangle}$ $\langle f_{n_k} \rangle$, call it $\cancel{\langle f_{n_{r_1}} \rangle}$ $\left\langle f_{n_{k_{r_1}}} \right\rangle$, such that $f_{n_{k_{r_1}}}(r_1) \to s_1$ for some $s_1 \in [0, 1]$. (That's enough of that joke, I guess.)

Further, $s_1 = f(r_1)$;

Any discontinuity of f must occur at a rational point; let x be irrational, and let $x_n \to x$. Then $f(x_n) = \sup(f(q) : q \in \mathbb{Q})$. This means that for each $n \in \mathbb{N}$, there is a sequence of rationals, $\langle q_n \rangle$, below x_n with the property $q_n \to x_n$. We can build a sequence out of these, $\langle q_{n_m} \rangle$, and it is clear that it converges to x and it is also clear that $f(q_{n_m}) \to f(x)$ and also that this implies that f is continuous at x. (I'm tired of this problem, and this was the last line I wrote of it.)

So, since the discontinuity occurs at a rational point, we know that f_{n_k} converges to f at that point, as this is how we constructed f_{n_k}.

Pick $f_{n_{k_1}}$ to be the first function in this subsequence.

Next, there is a subsequence of $\left\langle f_{n_{k_{r_1}}} \right\rangle$, call it $\left\langle f_{n_{k_{r_2}}} \right\rangle$, such that $f_{n_{k_{r_2}}}(r_2) \to s_2$ for some $s_2 \in [0, 1]$. Moreover, $f_{n_{k_{r_2}}}(r_1) \to s_1$, as $\left\langle f_{n_{k_{r_2}}} \right\rangle$ is a subsequence of $\left\langle f_{n_{k_{r_1}}} \right\rangle$. And also, similar to the argument above, $s_2 = f(r_2)$.

Pick f_{n_2} to be the second function in this subsequence.

We carry out the above process recursively, and produce a subsequence of $\langle f_{n_k} \rangle$, called $\left\langle f_{n_{k_l}} \right\rangle$, that rather clearly converges to f at all of f's points of discontinuity.

So, this subsequence converges pointwise to f at every point of $\mathbb{R}$, satisfying the problem.

Problem 5:

Let α be increasing on $[a, b]$, g continous, and $g(x) = G'(x)$ for all $x \in [a, b]$.

Then note that both $\int_a^b \alpha(x)g(x)dx$ and $\int_a^b G d\alpha$ exist; the first because it's a product of Riemann-Integrable functions, and the second because G is differentiable, thus continuous, thus $G \in \mathcal{R}(\alpha)$.

Let $\epsilon > 0$. There is a $\delta > 0$ such that if the mesh of a partition, P of $[a, b]$, is less than δ, then for any set of tags of that partition, T,

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i - \sum_{i=1}^{n} g(t_i)\alpha(t_i)\Delta x_i \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(t_i)\Delta x_i - \int_a^b \alpha(x)g(x)dx \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} G(t_i)\Delta \alpha_i - \int_a^b G(x)d\alpha \right| < \epsilon/3$$

$$\left| \sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i - \int_a^b \alpha(x)g(x)dx \right| < 2\epsilon/3$$

The first is because α is increasing; it has only countably many discontinuities, all of which are jump discontinuities. So, the diference of each $\alpha(x_i)$ and $\alpha(t_i)$ can be shrunk by making the difference of x_i and t_i small...which yields the result. The second and third are because either it is the definition of the integral or it is a theorem we should know about integrals. (It depends on the approach, and I'm not sure which one we're taking.) The last is a combination of the first two.

Now, pick a set of tags, $t_i \in [x_{i-1}, x_i]$ such that $G(x_i) - G(x_{i-1}) = g(t_i)\Delta x_i$. We can do this, because of the mean value theorem.

Note that we have the following:

$$\begin{aligned}
\sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i + \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i &= \sum_{i=1}^{n}(G(x_i) - G(x_{i-1}))\alpha(x_i) + G(x_{i-1})\Delta\alpha_i \\
&= \sum_{i=1}^{n}(G(x_i) - G(x_{i-1}))\alpha(x_i) + G(x_{i-1})(\alpha(x_i) - \alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_i) \\
&\qquad + G(x_{i-1})\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} G(x_i)\alpha(x_i) - G(x_{i-1})\alpha(x_{i-1}) \\
&= G(b)\alpha(b) - G(a)\alpha(a) \text{ (That's a telescoping sum.)}
\end{aligned}$$

Next,

$$\begin{aligned}
&\left| G(b)\alpha(b) - G(a)\alpha(a) - \int\limits_a^b G(x)d\alpha - \int\limits_a^b \alpha(x)g(x)dx \right| \\
&= \left| \sum_{i=1}^{n} g(t_i)\alpha(x_i)\Delta x_i + \sum_{i=1}^{n} G(t_i)\Delta\alpha_i - \int\limits_a^b G(x)d\alpha - \int\limits_a^b \alpha(x)g(x)dx \right| \\
&< \epsilon
\end{aligned}$$

So for all $\epsilon > 0$, $\left| G(b)\alpha(b) - G(a)\alpha(a) - \int\limits_a^b G(x)d\alpha - \int\limits_a^b \alpha(x)g(x)dx \right| < \epsilon$.
That is, $G(b)\alpha(b) - G(a)\alpha(a) = \int\limits_a^b G(x)d\alpha + \int\limits_a^b \alpha(x)g(x)dx$, which is the result.

Problem 6:
Let α be an increasing function on $[a, b]$, and for $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int\limits_a^b |u|^2 \, d\alpha \right)^{1/2}.$$

First, we prove the Schwarz inequality: $\left(\int_a^b |fg|\right)^2 \leq \int_a^b |f|^2 \int_a^b |g|^2$.

Put $A = \int_a^b |f|^2$, $B = \int_a^b |g|^2$, $C = \left(\int_a^b |fg|\right)$.

If A or B is 0, then f (or g) is 0 almost everywhere. Hence, the Schwarz inequality is reduced to $0 = 0$.

So, let $A, B > 0$. Then:

$$
\begin{aligned}
\int_a^b |Bf - Cg|^2 \, d\alpha &= \int_a^b |(Bf - Cg)| \, |(Bf - Cg)| \, d\alpha \\
&= \int_a^b B^2 \left|f^2\right| - 2BC \, |fg| + C^2 \left|g^2\right| d\alpha \\
&= B^2 \int_a^b \left|f^2\right| d\alpha - 2BC \int_a^b |fg| \, d\alpha + C^2 \int \left|g^2\right| d\alpha \\
&= B^2 A - BC^2 \\
&= B(AB - C^2)
\end{aligned}
$$

As the left hand side is positive, $B(AB - C^2) \geq 0$. As $B > 0$, we have $AB - C^2 \geq 0$, or $C^2 \leq AB$, which is the Schwarz inequality.

Now, let $f, g, h \in \mathcal{R}(\alpha)$.

Then we have the following:

$$
\begin{aligned}
\int_a^b |f - h|^2 \, d\alpha &= \int_a^b |f - g + g - h|^2 \, d\alpha \\
&= \int_a^b |f - g + g - h| \, |f - g + g - h| \, d\alpha \\
&\leq \int_a^b |f - g|^2 + 2 \, |f - g| \, |g - h| + |g - h|^2 \, d\alpha \\
&= \left(\int_a^b |f - g| \, d\alpha + \int_a^b |g - h| \, d\alpha\right)^2
\end{aligned}
$$

Taking square roots of both sides yields the triangle inequality for this ~~norm~~ strange function I have never seen before.