Problem 1:

Problem 2:

Problem 3:

By the reduction criterion, $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Z}[x]$ if it is irreducible in $\mathbb{Z}/(11)[x]$.

By Eisenstein's criterion, $x^4 + 3x^3 + 3x^2 - 5 = x^4 + 3x^3 + 3x^2 + 6$ is irreducible, using the prime 3.

So $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Z}[x]$. So $x^4 + 3x^3 + 3x^2 - 5$ is irreducible in $\mathbb{Q}[x]$.

Problem 4:

Let $R = \mathbb{Z}[\sqrt{-5}]$, and $K = \text{Quot}(R)$.

Consider $3x^2 + 4x + 3$. By the quadratic formula, if this polynomial has roots, they are $\frac{-2}{3} \pm \frac{\sqrt{-5}}{3}$. A factorization of $3x^2 + 4x + 3$ is given by $3(x + \frac{2}{3} + \frac{\sqrt{-5}}{3})(x + \frac{2}{3} - \frac{\sqrt{-5}}{3})$. So the polynomial is reducible in $K[x]$.

Now, in $R[x]$, $3x^2+4x+3$ cannot have a constant factored out of it. As it is a degree 2 polynomial, this means that it factors only as a product of two degree 1 polynomials. So any factorization of that polynomial must be of the form $(rx + r'(2 + \sqrt{-5}))(sx + s'(2 - \sqrt{-5})$, with $r', s' \in \mathbb{Z}[\sqrt{-5}]$ and $r = 3r'$, $s = 3s'$. Yet, this means that the leading coefficient of the polynomial is a multiple of 9, which 3 isn't. So the polynomial is irreducible in $R[x]$.

Problem 5:

Let R be a UFD and P be a prime ideal of $R[x]$ with $P \cap R = 0$.

Let P fail to be principal. Then there are $p, q \in P$ with $p \nmid q$ and $q \nmid p$. We can pick p to be of minimal degree, and among the q that satisfy these conditions, we can also choose q minimal.

Define $r = \gcd(p, q)$.
Now, $\gcd(p, q)$ must have a lower degree than either p or q;
Also, there is a polynomial, s, such that $rs = p$.