Problem 1:

Let $f \in \mathcal{O}(\mathbb{C})$. Then consider $\int\limits_{|z|=2} \frac{f(z)}{z-1} dz$.

Note that $\{z : |z| = 2\}$ is the boundary of the open disc of radius 2, and that 1 is a point in this disc. Thus, Cauchy's formula applies; $f(1) = 1/(2\pi i) \int\limits_{|z|=2} \frac{f(z)}{z-1} dz$, so $\int\limits_{|z|=2} \frac{f(z)}{z-1} dz = 2\pi i f(1)$.

Problem 2:

Let $f \in \mathcal{O}(\mathbb{C})$. Then consider $\int\limits_{|z|=2} \frac{f(z)}{z^2-1} dz$.

Note that $\int\limits_{|z|=2} \frac{f(z)}{z^2-1} dz = \int\limits_{|z|=2} \frac{f(z)}{(z+1)(z-1)} dz$. Now, this function is holomorphic except at 1 and -1. Thus, Cauchy's Theorem applies: the integral of $\frac{f(z)}{z^2-1}$ over a closed loop not containing 1 or -1 is zero. Thus:

$$\int\limits_{|z|=2} \frac{f(z)}{(z+1)(z-1)} dz = \int\limits_{|z-1|=1} \frac{f(z)}{(z+1)(z-1)} dz + \int\limits_{|z+1|=1} \frac{f(z)}{(z+1)(z-1)} dz$$

(This becomes clear given the following picture. Moreover, I freely use this sort of trick in future problems:)

Now, $f/(z+1)$ is holomorphic except at -1, and $f/(z-1)$ is holomorphic except at 1. So, we can apply Cauchy's Formula to the two integrals;

$$\begin{aligned}\int\limits_{|z|=2} \frac{f(z)}{(z+1)(z-1)} dz &= \int\limits_{|z-1|=1} \frac{f(z)}{(z+1)(z-1)} dz + \int\limits_{|z+1|=1} \frac{f(z)}{(z+1)(z-1)} dz \\ &= 2\pi i f(1)/(1+1) + 2\pi i f(-1)/(-1-1) \\ &= \pi i (f(1) - f(-1))\end{aligned}$$

Problem 3:

If Ω is an open set, f is holomorphic on some open set containing Ω's closure, and $w \notin \Omega$, then $\frac{f(z)}{z-w}$ is a product of two holomorphic functions and is thus holomorphic, so $\int\limits_{\partial\Omega} \frac{f(z)}{z-w} dz$ vanishes by Cauchy's Theorem.

Problem 4:
Let f be a holomorphic function on some open set, Ω.
Let $c \in \Omega$. Then

$$\begin{aligned}
\frac{df}{dz}(c) &= \frac{1}{2}(\frac{df}{dx}(c) - i\frac{df}{dy}(c)) \\
\frac{d}{d\bar{z}}\frac{df}{dz}(c) &= \frac{1}{2}\frac{d}{dx}[\frac{1}{2}(\frac{df}{dx}(c) - i\frac{df}{dy}(c))] + i\frac{d}{dy}[\frac{1}{2}(\frac{df}{dx}(c) - i\frac{df}{dy}(c))] \\
&= \frac{1}{4}[\frac{d^2}{dx^2}(c) - i\frac{d^2}{dxdy}(c) + i\frac{d^2}{dxdy}(c) + \frac{d^2}{dy^2}(c)] \\
&= \frac{1}{4}\Delta f
\end{aligned}$$

To summarize, $\frac{d^2 f}{dz d\bar{z}} = \frac{1}{4}\Delta f$.
(Note that we have freely used the symmetry of the second partial derivatives here.)

Problem 5:
Consider $\int\limits_{|z|=2} z^n(z-1)^m dz$ with $n, m \in \mathbb{Z}$.
First, if $n \geq 0$ and $m \geq 0$, $z^n(z-1)^m$ is holomorphic; the integral is 0.

Next, if $n \geq 0$ and $m < 0$, then $f(z) = z^n$ is holomorphic. So, by Cauchy's Formula,

$$
\begin{aligned}
f^{-(m+1)}(1) &= \frac{(-(m+1))!}{2\pi i} \int\limits_{|z|=2} \frac{f(z)}{(z-1)^{-(m+1)+1}} dz \\
&= \frac{(-(m+1))!}{2\pi i} \int\limits_{|z|=2} z^n (z-1)^m dz \\
\frac{f^{-(m+1)}(1)}{(-(m+1))!} 2\pi i &= \int\limits_{|z|=2} z^n (z-1)^m dz \\
\frac{n!}{(-(m+1))!(n+m+1)!} 2\pi i &= \int\limits_{|z|=2} z^n (z-1)^m dz
\end{aligned}
$$

Next, if $n < 0$ and $m \geq 0$, then $f(z) = (z-1)^m$ is holomorphic. So, by Cauchy's Formula,

$$
\begin{aligned}
f^{-(n+1)}(0) &= \frac{(-(n+1))!}{2\pi i} \int\limits_{|z|=2} \frac{f(z)}{z^{-(n+1)+1}} dz \\
&= \frac{(-(n+1))!}{2\pi i} \int\limits_{|z|=2} z^n (z-1)^m dz \\
\frac{f^{-(n+1)}(0)}{(-(n+1))!} 2\pi i &= \int\limits_{|z|=2} z^n (z-1)^m dz \\
\frac{(-1)^{m-(n+1)} m!}{(-(n+1))!(n+m+1)!} 2\pi i &= \int\limits_{|z|=2} z^n (z-1)^m dz
\end{aligned}
$$

Last, if $n < 0$ and $m < 0$, then $z^n(z-1)^m$ is holomorphic except at 0 and 1. So, $\int\limits_{|z|=2} z^n(z-1)^m dz = \int\limits_{|z|=1/2} z^n(z-1)^m dz + \int\limits_{|z-1|=1/2} z^n(z-1)^m dz = \int\limits_{|z|=1/2} z^n(z-1)^m dz + \int\limits_{|z|=1/2} (z+1)^n z^m dz$.

By this and Cauchy's Formula, we have, by letting $f(z) = (z+1)^n$ and $g(z) = (z-1)^m$,

$$
\begin{aligned}
\int\limits_{|z|=2} z^n(z-1)^m dz &= \int\limits_{|z|=1/2} z^n(z-1)^m dz + \int\limits_{|z|=1/2} (z+1)^n z^m dz \\
&= \int\limits_{|z|=1/2} \frac{g(z)}{z^{-n}} dz + \int\limits_{|z|=1/2} \frac{f(z)}{z^{-m}} dz \\
&= 2\pi i \left[\frac{g^{(-(n+1))}(0)}{(-(n+1))!} + \frac{f^{(-(m+1))}(0)}{(-(m+1))!} \right] \\
&= 2\pi i \left[\frac{(-1)^m \frac{(-(m+n+1))!}{(-(m+1))!}}{(-(n+1))!} + \frac{(-1)^{-(n+1)} \frac{(-(m+n+1))!}{(-(n+1))!}}{(-(m+1))!} \right]
\end{aligned}
$$

So in each case, we have a formula for the integral; the problem is satisfied.

Problem 6:
Let $g(z) = \overline{z}$ for all $z : |z| = 1$.
Consider

$$
\begin{aligned}
\int\limits_{|z|=1} g(z) dz &= \int\limits_{|z|=1} \overline{z} dz \\
&= \int_0^{2\pi} (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t)) dt \\
&= \int_0^{2\pi} \sin(t)\cos(t) - \sin(t)\cos(t) + i[\sin^2(t) + \cos^2(t)] dt \\
&= \int_0^{2\pi} i dt \\
&= 2\pi i
\end{aligned}
$$

That is, Cauchy's Theorem would fail if g was holomorphic; g cannot be holomorphic on any open disk containing the set $\{z : |z| = 1\}$, let alone $\mathbb{C}$!

Problem 7:

Let $f \in \mathcal{O}(\mathbb{C})$, with $|f(z)| \leq A + B\,|z|^n$ for some fixed $A, B \in \mathbb{R}$, $n \in \mathbb{N}$, and for all $z \in \mathbb{C}$.

Then for all $z \in \mathbb{C}$, $r > 0 \in \mathbb{R}$, $k \in \mathbb{N}$, we have $\left|f^{(k)}(z)\right| \leq k!\frac{\sup(f(B(z,r)))}{r^k}$.

So $\left|f^{(k)}(z)\right| \leq k!\frac{A+B(|z|+r)^n}{r^k}$.

So $\left|f^{(n+1)}(z)\right| \leq (n+1)!\frac{A+B(|z|+r)^n}{r^{n+1}}$.

By taking a limit as $r \to \infty$, we see that $\left|f^{(n+1)}(z)\right| = 0$ for all $z \in \mathbb{C}$. That is, the $n+1$th derivative of f is identically 0; f is a polynomial of degree at most $n+1$.

Problem 8:

Let $S \subset \mathbb{C}$ be an arbitrary set, $U \subset \mathbb{C}$ be open, and $K \in C(S \times U)$ be such that for all $s \in S$, $f_s(w) = K(s, w)$ is holomorphic on U.

Then $\frac{\partial K(s,w)}{\partial w} = f_s'(w)$. Because f_s is holomorphic on U, $f_s'(w)$ is continuous (if this isn't clear, consider that Cauchy's Formula implies that f is infinitely differentiable...and thus, each derivative is differentiable, and thus continuous). So, $\frac{\partial K(s,w)}{\partial w}$ is continuous for fixed s.

Next, let $s_n \to s$, with each $s_n, s \in S$. Then f_{s_n} converges to f_s uniformly; By Weierstrauss' theorem, this means that f_{s_n}' converges to $f'(s)$ uniformly. That is, $\frac{\partial K(s,w)}{\partial w}$ is continuous for fixed w.

So $\frac{\partial K(s,w)}{\partial w}$ is continuous with either variable fixed; it is continuous.