Problem 1: Problem 6 in textbook:

Let U be a bounded, open subset of $\mathbb{R}^n$.

We freely use the preceding problem's result that if $-\Delta v \leq 0$, then $\max_{\overline{U}} v = \max_{\partial U} v$, and also the hint given; $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$.

Define $\lambda = \max_{\overline{U}} |f|$. Define $M = \max(1, \frac{r^2}{2n})$ where r is an upper bound on the distance of a point in U from 0.

So we have:

$$\begin{aligned}
\max_{\overline{U}} u &\leq \max_{\overline{U}}(u + \frac{|x|^2}{2n}\lambda) \\
&= \max_{\partial U}(u + \frac{|x|^2}{2n}\lambda) \\
&\leq \max_{\partial U}(u) + \max_{\partial U} \frac{|x|^2}{2n}\lambda \\
&\leq \max_{\partial U}(u) + M\lambda \\
&= \max_{\partial U}(g) + M\lambda \\
&\leq \max_{\partial U}(|g|) + M \max_{\overline{U}}(|f|) \\
&\leq M(\max_{\partial U}(|g|) + \max_{\overline{U}}(|f|))
\end{aligned}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Noting that we get the same result for $-u$, we have our result.

Problem 2: Problem 9 in textbook:

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$ with $|x| \leq 1$.

Then $u(x) = -\int\limits_{\partial U} |x| \left(\frac{\partial G}{\partial \nu}(x,y) dS(y) \right)$, where G is the Green's function for the half-space, $G(x,y) = \Phi(y-x) - \Phi(y-\overline{x})$.

Consider $\frac{u(\lambda e_n)-u(0)}{\lambda}$ (with $\lambda > 0$). We can see that:

$$
\begin{aligned}
\frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{\int\limits_{\partial U} |\lambda e_n| \left[-\frac{\partial G}{\partial \nu}(\lambda e_n, y) \right] dS(y)}{\lambda} \\
&= \frac{\int\limits_{\partial U} |\lambda e_n| \left[-\frac{\partial \Phi}{\partial \nu}(y - \lambda e_n) + \frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n})) \right] dS(y)}{\lambda} \\
&= \frac{\int\limits_{\partial U} |\lambda e_n| \left[\frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n}) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)) \right] dS(y)}{\lambda} \\
&= \int\limits_{\partial U} \left[\frac{\partial \Phi}{\partial \nu}(y - \overline{\lambda e_n}) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)) \right] dS(y) \\
&= \int\limits_{\partial U} \left[\frac{\partial \Phi}{\partial \nu}(y + \lambda e_n) - \frac{\partial \Phi}{\partial \nu}(y - \lambda e_n)) \right] dS(y) \\
&= \int\limits_{U} \left[\Delta \Phi(x + \lambda e_n) - \Delta \Phi(x - \lambda e_n) \right] dx \\
&= \int\limits_{U} \left[\Delta (\Phi(x + \lambda e_n) - \Phi(x - \lambda e_n)) \right] dx
\end{aligned}
$$

It was pointed out that the limit as $\lambda \to 0$ of this is ∞. Thus, the derivative in the e_n direction blows up around 0; the Du is unbounded around 0.

Problem 3: Problem 10 in textbook:

Part a:

Let U^+ be the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x : x_n = 0\}$. Now, set

$$
v(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}
$$

for x in the open unit ball, U.

First, $v \in C^2(U \setminus \{x : x_n = 0\})$, and this is clear.

Next, v is continuous on $\{x : x_n = 0\}$, and this is clear. Also, v's derivatives on $\{x : x_n = 0\}$ are continuous:

Last, v's second derivatives on $\{x : x_n = 0\}$ are continuous, because (reasons).

So, $v \in C^2(U)$, and $\Delta v = 0$ except perhaps when $x_n = 0$. Thus, $\Delta v = 0$ even on the line, by continuity of the second partials. So v is harmonic on U.

Part b:

Let $u \in C^2(U^+) \cap C(\overline{U^+})$, and define v as above.

First, $v \in C^2(U \setminus \{x : x_n = 0\})$, and this is clear.

Next, v is continuous on $\{x : x_n = 0\}$, and this is clear. Also, v's derivatives on $\{x : x_n = 0\}$ are continuous:

Last, v's second derivatives on $\{x : x_n = 0\}$ are continuous, because (reasons).

So, $v \in C^2(U)$, and $\Delta v = 0$ except perhaps when $x_n = 0$. Thus, $\Delta v = 0$ even on the line, by continuity of the second partials. So v is harmonic on U.

Problem 4: Only problem on sheet:

Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.

By appealing to either Theorem 14 or the fact that the question bashes us over the head with it, there's at least one function, u, with:

- $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$
- $\Delta u = 0$ in $\mathbb{R}^n_+$
- $u(x', 0) = g(x')$ on $\mathbb{R}^{n-1}$.

Let u and v be such functions. Then there's a function $\tilde{u}$ and $\tilde{v}$ with

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

and

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x_n \geq 0 \\ -v(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Now, consider $w = \tilde{u} - \tilde{v}$. Then w is harmonic on the entire space: it's the sum of two harmonic functions, as explained in the previous problem.

Moreover, w is bounded: both u and v are bounded, so $\tilde{u}$ and $\tilde{v}$ are bounded, so their difference is bounded.

So, by Liouville, w must be constant. However, $\tilde{u}$ and $\tilde{v}$ are the same at a point ($\tilde{u}(0) = u(0) = g(0) = v(0) = \tilde{v}(0)$). So $w = 0$ at a point. So $\tilde{u} = \tilde{v}$. So $u = v$.