Problem 1:

I absolutely can't figure this one out, and I've spent about 3 hours on it. I tried bashing it out with the definitions (there's a q such that $pq = 0$, rip off all of the nonzero coefficients of q, and multiply them), but that doesn't get me something that's necessarily nonzero...so I'm sunk. :/

Problem 2:

Part a:

First, let $p = a_k x^k + \ldots + a_0 \in \mathrm{rad}(R[x])$.

Then $p^n = 0$ for some $n \in \mathbb{N}$. So $a_0^n = 0$.

Now, if $a_j^m = 0$ for all $j < N$, then we have:

Next, let $p \in \{a_n x^n + \ldots + a_0 : a_i \in \mathrm{rad}(R)\}$.

Then for each i, there's an n_i such that $a_i^{n_i} = 0$. Define n to be the product of all of the n_is.

Then we have

$$p^n = stuff$$

Part b:

Let $p \in (R[x])^*$.

Then stuff.

Next, let $p \in \{a_n x^n + \ldots + a_0 : a_i \in \mathrm{rad}(R) \text{ if and only if } i > 0\}$.

Problem 3:

Problem 4:

Let $n = 4$ or $n = 2^i p^j$ for some odd prime p, $i = 0$ or $i = 1$, and $j \geq 0$.

If $n = 4$, then $(\mathbb{Z}/n)^* \cong \mathbb{Z}/2$, which is cyclic.

If $n = p^j$ for some odd prime p, then $(\mathbb{Z}/n)^*$ is generated, as a group, by 2, and is thus cyclic.

If $n = 2p^j$ for some odd prime p, then $(\mathbb{Z}/n)^*$ is generated, as a group, by 3, and is thus cyclic.

Next, let $(\mathbb{Z}/n)^*$ be cyclic.

Assume that $n \neq 4$ and $n \neq 2^i p^j$ for any odd prime p, $i = 0$ or $i = 1$, and $j \geq 0$.

This means that n must be at least 8. We have that n has at least one odd prime in its prime factorization or it is a power of two.

If n is a power of two greater than 8, then $(\mathbb{Z}/n)^*$ is not cyclic;

For the types of n we have described, if n has at least one odd prime in its prime factorization, then we know that either n has two odd primes in its prime factorization or n has a power of two greater than 8 in its prime factorization.

If n has two odd primes in its prime factorization, then $(\mathbb{Z}/n)^*$ is not cyclic; it's isomorphic to a product of more than one nontrivial group, by the Chinese Remainder Theorem. (Note: we needed an odd prime for this because the Chinese Remainder Theorem burns anything that looks like $\mathbb{Z}/2$).

If n has a power of two greater than 8 in its prime factorization, then there's a homomorphism, ϕ, from $\mathbb{Z}/n$ to $\mathbb{Z}/2^k$ for some k at least 3, having the property that ϕ preserves units. We know that $(\mathbb{Z}/2^k)^*$ is not cyclic, and so neither can $(\mathbb{Z}/n)^*$ be. (If $(\mathbb{Z}/n)^*$ was cyclic, we could take its generator, a, and have $\phi(a)$ generate $(\mathbb{Z}/2^k)^*$).

So $(\mathbb{Z}/n)^*$ is cyclic if and only if $n = 4$ or $n = 2^i p^j$ for some odd prime p, $i = 0$ or $i = 1$, and $j \geq 0$.