Problem 1:

Let $f, g \in \mathcal{O}(D_r(c))$, $g(c) = 0$, and $g'(c) \neq 0$.

Without loss of generality, $c = 0$. Now, let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$. Because $g(0) = 0$, we have that $b_0 = 0$. So,

$$\begin{aligned}
\mathrm{Res}_0 \frac{f}{g}(c) &= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{f}{g} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{\sum\limits_{n=0}^{\infty} b_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{\sum\limits_{n=1}^{\infty} b_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{\sum\limits_{n=0}^{\infty} a_n z^n}{z \sum\limits_{n=0}^{\infty} b_{n+1} z^n} dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{D_r(0)} \frac{a_n z^n}{z \sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz
\end{aligned}$$

All but the first of those terms vanish; $\frac{z^n a_n}{z b_1 + z^2 b_2 \ldots} = \frac{z^n a_n}{z h(z)} = \frac{z^{n-1} a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 ($h(z)$ is nonzero on a small enough disk, else g is identically zero...and so $g' = 0$).

So,

$$
\begin{aligned}
\mathrm{Res}_0 \frac{f}{g}(c) &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{D_r(0)} \frac{a_n z^n}{z \sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{a_1}{\sum\limits_{m=0}^{\infty} b_{m+1} z^m} dz \\
&= a_1/b_0 \\
&= f'(c)/g(c)
\end{aligned}
$$

Yielding our result.

Problem 2:

Let $f \in \mathcal{O}(\dot{D}_r(c))$ with c not an essential singularity. Without loss of generality, $c = 0$.

Consider $\mathrm{Res}_0 \frac{f'}{f}$. Now, let $f(z) = \sum\limits_{n=k}^{\infty} a_n z^n$ with a_k nonzero, so that $f'(z) = \sum\limits_{n=k}^{\infty} n a_n z^{n-1}$; k will be the order of zero if positive, and the order of pole if negative, and this is clear. So,

$$
\begin{aligned}
\mathrm{Res}_0 \frac{f'}{f}(c) &= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{f'}{f} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{\sum\limits_{n=k}^{\infty} n a_n z^{n-1}}{\sum\limits_{n=k}^{\infty} a_n z^n} dz \\
&= \frac{1}{2\pi i} \int\limits_{D_r(0)} \frac{\sum\limits_{n=0}^{\infty} n a_n z^{n-1}}{z \sum\limits_{n=0}^{\infty} a_n z^{n-1}} dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{D_r(0)} \frac{n a_n z^{n-1}}{z \sum\limits_{m=0}^{\infty} a_m z^{m-1}} dz
\end{aligned}
$$

All but the first of those terms vanish; $\frac{nz^n a_n}{zb_1+z^2b_2\ldots} = \frac{z^n a_n}{zh(z)} = \frac{z^{n-1}a_n}{h(z)}$ is holomorphic on a sufficiently small disk around 0 ($h(z)$ is nonzero on a small enough disk, else a_k was zero...).

So,

$$\begin{aligned}
\mathrm{Res}_0 \frac{f'}{f}(c) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \int_{D_r(0)} \frac{na_n z^{n-1}}{z\sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
&= \frac{1}{2\pi i}\int_{D_r(0)} \frac{ka_k z^{n-1}}{z\sum_{m=0}^{\infty} a_m z^{m-1}} dz \\
&= k
\end{aligned}$$

Yielding our result.

Problem 3:

A real-variable analogue of Rouche's Theorem would be:

"Let I be an open interval (a, b), f, g be differentiable on I, and let J be an open interval containing the closure of I.

If $|f(a)| < |g(a)|$ and $|f(b)| < |g(b)|$, then g, $g - f$ have the same number of zeroes in I."

The obvious counterexample is $f(x) = 0$ if $x = 0$, $f(x) = \sin(1/x)$ otherwise, and $g(x) = 1$ on the interval $(0, 1/2\pi)$. Now, $f(x) = 0$ at $0, 1/2\pi$, and $g(x) = 1$, so $|f| < |g|$ on the boundary of the interval. But g has no zeroes, and $g - f$ has infinitely many zeroes. So this breaks.

Problem 4:

Consider $\int\limits_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2} dx$, with $a \in \mathbb{R}$ and $a > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iz}+e^{-iz}}{z^2+a^2}dz \\
&= \int_{0}^{\infty} \frac{e^{iz}+e^{-iz}}{z^2+a^2}dz \text{ (because the function is even...)} \\
&= \int_{0}^{\infty} \frac{e^{iz}}{z^2+a^2}dz + \int_{0}^{\infty} \frac{e^{-iz}}{z^2+a^2}dz \\
&= \int_{0}^{\infty} \frac{e^{iz}}{z^2+a^2}dz - \int_{0}^{-\infty} \frac{e^{iz}}{z^2+a^2}dz \text{ (u-substitute -z)} \\
&= \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+a^2}dz \\
&= \int_{-\infty}^{\infty} \frac{\sum\limits_{n=0}^{\infty}\frac{(iz)^n}{n!}}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{(iz)^n}{n!}}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty} \frac{z^n}{z^2+a^2}dz \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}2\pi i\sum_{c\in\mathbb{C}}\text{Res}_c\frac{z^n}{z^2+a^2} \text{ (As discussed in class)}
\end{aligned}$$

Problem 5:

Consider $\int\limits_{\Gamma_T} z^\alpha R(z)dz$ (with R a rational function, and Γ_T as pictured below.)

Problem 6:

Consider $e^z = 6z^2 + 1$. This is equivalent to $0 = 6z^2 + 1 - e^z$.

Define $g(z) = 6z^2 + 1$ and $f(z) = e^z$. When $|z| = 2$, $|g| \geq |6z^2| - 1 = 23$ and $|f| \leq e^2 \leq 9$. So $g > f$ when $|z| = 2$.

So Rouche's Theorem applies: $e^z = 6z^2 + 1$ has the same number of solutions as $0 = 6z^2 + 1$ on the disk bounded by $|z| = 2$.

Now, $6z^2 + 1$ has two solutions, by the fundamental theorem of algebra. Moreover, $\frac{i}{\sqrt{6}}$ are solutions, as is readily checked. These solutions are both in that disk. So $6z^2 - 1$ has two zeroes on the disk bounded by $|z| = 2$.

So $e^z = 6z^2 + 1$ has 2 solutions on the disk bounded by $|z| = 2$.

Problem 7:

Problem 8: