Problem 1:

Let $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, and let $u(x, t)$ solve

$$\begin{cases} u_t - uxx = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

Let $v(x, t) = u(x, t) - g(t)$, and extend v to $\{x < 0\}$ by odd reflection (just call the resulting extension v). Then v solves

$$\begin{cases} v_t - vxx = g'(t) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ v = -g & \text{in } \mathbb{R}_+ \times \{t = 0\} \\ v = 0 & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

So $v(x, t) = \int\limits_{\mathbb{R}} -g(y)e^{\frac{-|x-y|^2}{4t}} dy + \int\limits_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{\mathbb{R}} g'(y)e^{\frac{-|x-y|^2}{4(t-s)}} dyds$ by formula 17 in the book.

Problem 2:

Let $g \in C(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, $|g| < M$ for some M. Let u be the bounded solution to

$$\begin{cases} \Delta u - u_t = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Part a:

Then $u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}} e^{\frac{-|x-y|^2}{4t}} g(y)dy$. Let $\epsilon > 0$. Choose N greater than *something*. Then for all $t > N$, $resultdesired$.

That is, $\lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}^n} |u(x, t)| = 0$, which is the desired result.

Part b:

Consider $v(x, t) = u(x, t) - g(x)$. Then $v(x, t)$ solves

$$\begin{cases} \Delta v - v_t = -\Delta g(x) & \text{for } t > 0, x \in \mathbb{R}^n \\ v(x, 0) = 0 & \text{for } x \in \mathbb{R}^n \end{cases}$$

Thus, $v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\Delta g(y)dyds$.

So $\int_{\mathbb{R}^n} v(x,t)dx = \int_{\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\Delta g(y)dydsdx$

Switching the order of integration, we get $\int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x-y,t-s)\Delta g(y)dxdyds$.

Yet, this is $\int_0^t \int_{\mathbb{R}^n} \Delta g(y)dyds$, because $\int_{\mathbb{R}^n} \Phi(x-y,t-s)dx = 1$.

The integral vanishes, for reasons I haven't figured out yet.

So, $\int_{\mathbb{R}^n} v(x,t)dx = 0$; $\int_{\mathbb{R}^n} u(x,t) - g(x)dx = 0$, which yields the desired result of $\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} g(x)dx$.

Problem 3:

Part a:

Fix $\alpha \in (0,1)$, $\beta \geq 0$.

Note first that $z^\beta e^{-z} = e^{\beta \ln(z) - z}$. So, the desired result is

$$e^{\beta \ln(z) - z} \leq M e^{-\alpha z}$$

for some M, which is equivalent to

$$\beta \ln(z) - z \leq \ln(M) - \alpha(z)$$

for some M. Now, this is equivalent to

$$-\ln(M) \leq (1-\alpha)z - \beta \ln(z)$$

for some M. By applying basic calculus, the right hand side takes a minimum at $z = \beta/(1-\alpha)$, so taking $M = (1-\alpha)(\beta/(1-\alpha)) - \beta \ln(\beta/(1-\alpha))$ suffices.

Part b:

Part c: