Note: I am accustomed to writing "The element $g \in G$ acting on the element $x \in S$" as $g.x$ instead of $gx$. I use the stated notation, as I feel it is clearer.

**Problem 1:**

Let $G$ be a finite abelian group, with $n \in \mathbb{N}$ and $n \mid |G|$.

We know that for each $n \in \mathbb{N}$, $n$ has a unique prime factorization; that is, $n = p_1^{a_1} p_2^{a_2} p_3 a_3 \ldots p_k^{a_k}$ for some $p_1, p_2 \ldots p_k$ each prime, and $a_1, a_2 \ldots a_k$ each positive and nonzero.

Proceed as follows:

For each $p_i$, there is an element with order $p_i$ in $G$:

We proceed by induction. If $|G| = 1$, then $|G|$ has no prime divisors, so there is vacuously an element of order $p$ in $G$ if $p \mid |G|$.

Now, assume that there is an element of order $p$ (with $p$ prime) in $G$ if $p \mid |G|$ for all $G$ with order less than $N$. Let $G$ be a group of order $N$, and let $p \mid N$ be a prime number. Pick an element of $G$, call it $g$, other than the identity. It has some order, $m > 1$. Either $p \mid m$ or not. If so, then take $g^{m/p}$; this element clearly has order $p$. If not, then consider $G/\langle a \rangle$ (this is well defined; $G$ is abelian, so $\langle a \rangle$ is normal, because all subgroups are normal in abelian groups). We know that $p \mid |G/\langle a \rangle|$; $p \mid |G| = |G/\langle a \rangle||a|$, so because $p$ is prime, we know that $p \mid |G/\langle a \rangle|$. So there's an element, $b$, of order $p$ in $G/\langle a \rangle$. So there is an element with $\overline{c} = b$. Now, $c^p \in \langle a \rangle$;

$$\begin{aligned} Proof &= Goes \\ &= Here \end{aligned}$$

So, $c^p = a^l$ for some $l \in \mathbb{N}$. We know that $a^l$ has a finite order, $|a^l|$. So clearly $c^{|}a^l|$ has order $p$.

So in either case, there is an element of order $p$ in $G$ if $p$ is prime and $p \mid |G|$.

Thus, there is a subgroup, $H_1$, with order $p_1$ in $G$. This subgroup is normal, because $G$ is abelian.

Consider the new group $G_1 = G/H_1$, along with $n_1 = n/p_1$. Note that $|G_1| = |G|/p_1$; this follows from the theorem that says $|G/H| = |G|/|H|$ if $H$ is normal.

We know that $G_1$ is abelian:

So by similar logic as above, if $a_1 > 1$, then there is a subgroup of order $p_1$ in $G$. Otherwise, we know that there is a subgroup of order $p_2$ in $G$.

Either way, there is a subgroup, $H_2'$, with order $p_1$ (or $p_2$) in $G_1$, which is normal.

Consider the group $G_2 = G_1/H_2$. Note that $G_2 \cong G/H_2$, by the third isomorphism theorem. Moreover, $|G_2| = |G|/p_1^2$ (or $|G_2| = |G|/p_1p_2$).

We can proceed in the above manner for $a_i$ times for each $p_i$. We end up with a group, $G_{a_1+a_2...+a_k}$.

Consider $G_{a_1+a_2...+a_k}$; it has order $|G|/n$. It is isomorphic to $G/H_{a_1+a_2...+a_k}$ for some $H \trianglelefteq G$. This means that $|H| = n$ (because $|G/H| = |G|/|H|$...thus, $|H| = |G|/|G/H|$, or in this case, $|H| = |G|/(|G|/n) = n$.)

So $G$ has a normal subgroup of order $n$ if $G$ is a finite abelian group with $n \mid |G|$.

**Problem 2:**

Let $H < G$ with $[G : H]$ finite.

**Problem 3:**

Let $G$ be a group acting transitively on a finite set, $S$, with $|S| > 1$.

Now, the action has only one orbit; for all $x \in S$, $\overline{x} = S$. In other words, for every $x, y \in S$ there is a $g \in G$ such that $g.x = y$.

Before proceeding, I wish to point out that I use the following freely:

If $g.x = x$, then $g^{-1}.x = x$: This is clear by applying $g^{-1}$ to both sides of the equation.

If $g.x = y$, then $g^{-1}.y = x$: This is clear by applying $g^{-1}$ to both sides of the equation.

Assume that for all $g \in G$, there is an $x \in S$ such that $g.x = x$. We proceed by constructing an infinite set of points in $S$, by induction.

Because $|S| > 1$, there are at least two distinct points of $S$: call them $x_0$ and $x_1$.

There is an element, $g_2$, such that $g_2.x_0 = x_1$, by transitivity of the action.

There is an $x_2$ such that $g_2.x_2 = x_2$, by the assumption we made earlier.

Now, $x_2 \neq x_0$, else:

$$\begin{aligned} g_2.x_0 &= g_2.x_2 \\ x_1 &= x_2 = x_0 \end{aligned}$$

which is a contradiction.

Also, $x_2 \neq x_1$, else:

$$\begin{aligned} g_2^{-1}.x_1 &= g_2^{-1}.x_2 \\ x_0 &= x_2 = x_1 \end{aligned}$$

which is also a contradiction.

So $x_2$ is distinct from $x_0$ and $x_1$.

Now, assume that we have the following: we have defined $x_n$ for each $n \in \mathbb{N}$ such that $n < N$, and $g_n$ for each $n \in \mathbb{N}$ such that $n < N - 1$ and $n \geq 2$, with the following properties: $g_n.x_n = x_n$ and $g_n.x_0 = x_{n-1}$.

Then there is a $g_N$ such that $g_N.x_0 = x_{N-1}$, because the action is transitive.

Also, there is an $x_N$ such that $g_N.x_N = x_N$, by the assumption we made earlier.

Now, $x_N \neq x_0$, else:

$$\begin{aligned} g_N.x_0 &= g_N.x_N \\ x_{N-1} &= x_N = x_0 \end{aligned}$$

which is a contradiction.

Also, $x_N \neq x_{N-1}$, else:

$$\begin{aligned} g_N^{-1}.x_{N-1} &= g_N^{-1}.x_N \\ x_0 &= x_N = x_{N-1} \end{aligned}$$

which is also a contradiction.

Further, $x_N \neq x_i$ for any $i$ between 0 and $N-1$ (exclusive), else:

$$\begin{aligned} g_N g_i^{-1}.x_i &= g_N.x_i \\ g_n x_0 &= g_N.x_N \\ x_{N-1} = x_N & \end{aligned}$$

which is also a contradiction.

So $x_N$ is distinct from each $x_i$ with $i < N$.

So we have two distinct points, and if we have $n$ distinct points in $S$, we can make $n+1$ distinct points in $S$; we can make infinitely many distinct points, thus $S$ is infinite.

So, if for all $g \in G$, $g$ has a fixed point, then $S$ is infinite.

Or, in other words, because $S$ is finite, there is a $g \in G$ that has no fixed point.

**Problem 4:**

Let $G$ be a group such that $G/Z(G)$ is cyclic.

Then there is an $a \in G$ such that for all $\overline{x} \in G/Z(G)$, $\overline{x} = \overline{a}^n$ for some $n \in \mathbb{N}$.

So for all $y \in G$, there is an $n \in \mathbb{N}$ and $b \in Z(G)$ such that $y = a^n b$; the left cosets of $Z(G)$ partition $G$, and the set of these cosets is $\{a^n Z(G) : n \in \mathbb{N}\}$. So for all $y \in G$, $y \in a^n Z(G)$ for some $n \in \mathbb{N}$.

So for all $y, z \in G$, we have $y = a^n b$ and $z = a^m c$ for some $n, m \in \mathbb{N}$ and $b, c \in Z(G)$.

Now we have:

$$\begin{aligned} yz &= a^n b a^m c \\ &= a^n a^m bc \\ &= a^m a^n bc \\ &= a^m a^n cb \\ &= a^m c a^n b \\ &= zy \end{aligned}$$

Note that this fails if $G/Z(G)$ is only abelian:

Consider $D_8 = \langle r, s \rangle$. We note that the center of $D_8$ is $\{e, r^2\}$.

(We freely use the identity $sr = r^3 s$ in the following).

We know that $e$ commutes with every element, trivially.

Now, $r^2$ commutes with $e$, $r$,$r^2$, and $r^3$ trivially;

Also, $r^2$ commutes with $s$, $sr$, $sr^2$, and $sr^3$:

$$\begin{aligned} r^2 s &= rsr^3 = sr^3 r^3 = sr^2 \\ r^2 sr &= r^2 r^3 s = rs = sr^3 = srr^2 \\ r^2 sr^2 &= r^2 sr^2 \\ r^2 sr^3 &= r^2 rs = sr = sr^5 = sr^3 r^2 \end{aligned}$$

However, $r$ and $s$ do not commute with each other, and $r^3$ and $s$ do not commute with each other:

$$rs = sr^3$$

Also, $sr$, $sr^2$, and $sr^3$ do not commute with $r$:

$$\begin{aligned} rsr &= sr^3r = s \neq sr^2 \\ rsr^2 &= sr^3r^2 = sr \neq sr^3 \\ rsr^3 &= sr^3r^3 = sr^2 \neq sr^3r = s \end{aligned}$$

So every element other than $e$ and $r^2$ fails to commute with something. So

Also, $D_8/\langle r^2 \rangle$ is abelian:

Observe that this group must have order 4, and that $\{\overline{e}, \overline{r}, \overline{s}, \overline{sr}\}$ are all distinct elements of this group (and thus this represents all elements in the quotient group).

Clearly, $\overline{e}$ commutes with everything. We proceed by exhaustion:

$$\begin{aligned} \overline{rs} &= \overline{sr} = \overline{r^3s} = \overline{rs} = \overline{rs} \\ \overline{rsr} &= \overline{rsr} = \overline{sr^3r} = \overline{s} = \overline{sr^2} = \overline{srr} \\ \overline{ssr} &= \overline{s^2r} = \overline{r} = \overline{rss} = \overline{sr^3s} = \overline{srs} = \overline{srs} \end{aligned}$$

But we know from an earlier homework that $D_8$ is not abelian. So in general, $G/Z(G)$ being abelian does not imply that $G$ is abelian.

**Problem 5:**

Let $p$ be prime, and let $G$ be a group of order $p^2$. We know that $Z(G)$ is a subgroup of $G$; so $Z(G)$ has order 1, $p$, or $p^2$.

We know that $Z(G)$ does not have order 1, from the discussion in class.

If $Z(G)$ has order $p$, then $G/Z(G)$ is cyclic (as it is a group of order $p^2/p = p$); we know that $G$ is abelian, from problem 4.

If $Z(G)$ has order $p^2$, then the center is the entire group; that is, $G$ is abelian.

So in all cases, $G$ is abelian.

**Problem 6:**

Let $p$ be prime, and let $G$ be a group of order $p^n$ for some $n \in \mathbb{N}$. Let $H \trianglelefteq G$ with $H \neq \{e\}$.

Now, $G$ acts on $H$ by conjugation. By the class equation,

$$\begin{aligned} |H| &= |\{h \in H : \forall g \in G, ghg^{-1} = h\}| + \sum |\overline{x_i}| \\ &= |H \cap Z(G)| + \sum |[G : G_{x_i}]| \text{ Because the order of the orbit is the index of the stabilizer} \end{aligned}$$

We know that $p \mid |H|$, because $H$ is a nontrivial subgroup of a $p$-group. Also, $p \mid [G : G_{x_i}]$ because each $G_{x_i}$ is a subgroup of $G$, and none are trivial (because none of those orbits contain only one element; if there was an element whose orbit had only one element, then it would be in the set $\{h \in H : \forall g \in G, ghg^{-1} = h\}$).

So we know that $p \mid |H \cap Z(G)| + \sum |[G : G_{x_i}]|$. So because $p \mid \sum |[G : G_{x_i}]|$, we know that $p \mid |H \cap Z(G)|$.

So because $p \mid |H \cap Z(G)|$, we know that $|H \cap Z(G)| \neq 1$; that is $H \cap Z(G)$ is not the trivial subgroup.