Problem 1: Problem 6 in textbook:

Let U be a bounded, open subset of $\mathbb{R}^n$.

We freely use the result that if $-\Delta v \leq 0$, then $\max_{\overline{U}} v = \max_{\partial U} v$, and also the hint given; $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$.

Define $\lambda = \max_{\overline{U}} |f|$. Define $M = \max(1, \frac{r^2}{2n})$ where r is an upper bound on the distance of a point in U from 0.

So we have:

$$\begin{aligned}
\max_{\overline{U}} u &\leq \max_{\overline{U}}(u + \frac{|x|^2}{2n}\lambda) \\
&= \max_{\partial U}(u + \frac{|x|^2}{2n}\lambda) \\
&\leq \max_{\partial U}(u) + \max_{\partial U} \frac{|x|^2}{2n}\lambda \\
&\leq \max_{\partial U}(u) + M\lambda \\
&= \max_{\partial U}(g) + M\lambda \\
&\leq \max_{\partial U}(|g|) + M \max_{\overline{U}}(|f|) \\
&\leq M(\max_{\partial U}(|g|) + \max_{\overline{U}}(|f|))
\end{aligned}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Noting that we get the same result for $-u$, we have our result.

Problem 2: Problem 9 in textbook:

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

Assume g is bounded by M and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$ with $|x| \leq 1$.

Then $u(x) = \frac{2x_n}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy$, by Poisson's formula.

Consider $\frac{u(\lambda e_n) - u(0)}{\lambda}$ (with $1 > \lambda > 0$). We can see that:

$$\begin{aligned}
\left| \frac{u(\lambda e_n) - u(0)}{\lambda} \right| &= \left| \frac{2\lambda}{\lambda n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+} \frac{g(y)}{|\lambda e_n - y|^n} dy \right| \\
&= \left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+} \frac{g(y)}{|\lambda e_n - y|^n} dy \right| \\
&\geq \left| \left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+ \cap B(0,1)} \frac{|y|}{|\lambda e_n - y|^n} dy \right| - \left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+ \setminus B(0,1)} \frac{g(y)}{|\lambda e_n - y|^n} dy \right| \right|
\end{aligned}$$

Now, we know that:

$$\left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+ \setminus B(0,1)} \frac{g(y)}{|\lambda e_n - y|^n} dy \right| \leq \left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+ \setminus B(0,1)} \frac{M}{|\lambda e_n - y|^n} dy \right|$$

which converges when $n > 1$. (When $n = 1$, the problem's statement is unsatisfiable because all harmonic functions are linear there...so we get our result vacuously.)

As $\lambda \to 0$, $\left| \frac{2}{n\alpha(n)} \int\limits_{\partial \mathbb{R}^n_+ \cap B(0,1)} \frac{|y|}{|\lambda e_n - y|^n} dy \right|$ approaches ∞ (because the integrand approaches $\frac{1}{|y|^{n-1}}$, and it is well known that the integral of this explodes on balls containing 0.) So, $\left| \frac{u(\lambda e_n) - u(0)}{\lambda} \right| \to \infty$ as $\lambda \to 0$. So, because the derivatives are continuous, this means that Du is unbounded around 0.

Problem 3: Problem 10 in textbook:

Part a:

Let U^+ be the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x : x_n = 0\}$. Now, set

$$v(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \dots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for x in the open unit ball, U.

First, $v \in C^2(U \setminus \{x : x_n = 0\})$, and this is clear.

Next, v is continuous on $\{x : x_n = 0\}$, and this is clear. Also, v's partial derivatives on $\{x : x_n = 0\}$ are continuous: for the first $n-1$ partials, $v_{x_i}(x) = u_{x_i}(x)$, and this is clear. For the nth partial, $\lim\limits_{\lambda \to 0^+} \frac{v(x+\lambda e_n) - v(x)}{\lambda} = \lim\limits_{\lambda \to 0^+} \frac{u(x+\lambda e_n) - u(x)}{\lambda} = u_{x_n}(x)$ and $\lim\limits_{\lambda \to 0^-} \frac{v(x+\lambda e_n) - v(x)}{\lambda} = \lim\limits_{\lambda \to 0^-} \frac{-u(x+-\lambda e_n) - u(x)}{\lambda} = u_{x_n}(x)$. So, v's derivative exists, and is continuous.

Last, v's second derivatives on $\{x : x_n = 0\}$ are continuous: for the first $i \in [1, n], j \in [1, n-1]$, $v_{x_i, x_j}(x) = u_{x_i, x_j}(x)$ and this is clear (from the above). For $i \in [1, n], j = n$, $\lim\limits_{\lambda \to 0^+} \frac{v_{x_i}(x+\lambda e_n) - v_{x_i}(x)}{\lambda} = \lim\limits_{\lambda \to 0^+} \frac{u_{x_i}(x+\lambda e_n) - u_{x_i}(x)}{\lambda} = u_{x_i, x_n}(x)$ and $\lim\limits_{\lambda \to 0^-} \frac{v_{x_i}(x+\lambda e_n) - v_{x_i}(x)}{\lambda} = \lim\limits_{\lambda \to 0^-} \frac{-u_{x_i}(x+-\lambda e_n) - u_{x_i}(x)}{\lambda} = u_{x_i, x_n}(x)$. So, v's second derivative exists, and is continuous.

So, $v \in C^2(U)$, and $\Delta v = 0$ except perhaps when $x_n = 0$. Thus, $\Delta v = 0$ even on the line, by continuity of the second partials. So v is harmonic on U.

Part b:

Let $u \in C^2(U^+) \cap C(\overline{U^+})$, and define v as above.

Define $w(x) = \frac{1-|x|^2}{n\alpha(n)} \int\limits_{\partial B(0,1)} \frac{v(y)}{|x-y|^n} dS(y)$. Then $\Delta w = 0$, by theorem 15.

Now, consider

$$\begin{aligned} w(x) &= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{v(y)}{|x-y|^n} dS(y) \\ &= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial U^+} \frac{v(y)}{|x-y|^n} dS(y) + \frac{1-|x|^2}{n\alpha(n)} \int_{U^-} \frac{v(y)}{|x-y|^n} dS(y) \\ &= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial U^+} \frac{u(y)}{|x-y|^n} dS(y) + \frac{1-|x|^2}{n\alpha(n)} \int_{U^-} \frac{-u(\overline{y})}{|x-y|^n} dS(y) \end{aligned}$$

Where $\overline{(x_1, x_2, \ldots x_n)} = (x_1, x_2, \ldots - x_n)$. If $x_n = 0$, this vanishes. So $w = u$ on the boundary of U^+ and is harmonic; by uniqueness, this means that $w = u = v$ on U^+. Also, $w(x) = -u(\overline{x})$ and is harmonic; by uniqueness, this means that $w(x) = -u(\overline{x}) = v(x)$ on U^-. Thus, $w = v$ on $U^+ \cup U^- = U$.

Thus, $\Delta v = 0$, which is what we wanted.

Problem 4: Only problem on sheet:

Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.

By appealing to either Theorem 14 or the fact that the question bashes us over the head with it, there's at least one function, u, with:

- $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$
- $\Delta u = 0$ in $\mathbb{R}^n_+$
- $u(x', 0) = g(x')$ on $\mathbb{R}^{n-1}$.

Let u and v be such functions. Then there's a function $\tilde{u}$ and $\tilde{v}$ with

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

and

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x_n \geq 0 \\ -v(x_1, x_2, \ldots x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

Now, consider $w = \tilde{u} - \tilde{v}$. Then w is harmonic on the entire space: it's the sum of two harmonic functions; the previous problem gives us this the reflection principle for unit balls, but this can be expanded to arbitrarily large balls, and thus the entire half space.

Moreover, w is bounded: both u and v are bounded, so $\tilde{u}$ and $\tilde{v}$ are bounded, so their difference is bounded.

So, by Liouville, w must be constant. However, $\tilde{u}$ and $\tilde{v}$ are the same at a point ($\tilde{u}(0) = u(0) = g(0) = v(0) = \tilde{v}(0)$). So $w = 0$ at a point; w is identically 0. So $\tilde{u} = \tilde{v}$. So $u = v$.