Chapter 3:

Problem 4:

Let $m^*(E) = \infty$ for an infinite set and $m^*(E) = |E|$ for a finite set.

It's clear that m^* is defined for all sets of real numbers, is translation invariant, and countably additive. So m^* is a measure; we call it the counting measure.

Problem 7:

If $m^*(E)$ is the Lebesgue Outer Measure, it's somewhat clear that it's translation invariant; we can do this by making an open cover and shifting it.

Problem 8:

If $m^*(A) = 0$, then $m^*(A \cup B) \geq m^*(B)$, by monotonicity.

But also, $m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B)$ by countable subadditivity.

So $m^*(A \cup B) = m^*(B)$.

Problem 11:

Each (a, ∞) is measurable.

We have $\bigcap^{\infty}(n, \infty) = \emptyset$ which has measure 0, but $m((n, \infty)) \to \infty$. So $m(\bigcap^{n} E_i) \not\to m(\bigcap^{\infty} E_n)$

Problem 12:

Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets, and A be a set.
Then $m^*(A \cap \bigcup^n E_i) = \sum^n m^*(A \cap E_i)$.
So $m^*(A \cap \bigcup^\infty E_i) \geq \sum^n m^*(A \cap E_i)$.
But n is arbitrary, so $m^*(A \cap \bigcup^\infty E_i) \geq \sum^\infty m^*(A \cap E_i)$.
Either by employing a similar argument or appealing to countable subadditivity, we get $m^*(A \cap \bigcup^\infty E_i) = \sum^\infty m^*(A \cap E_i)$.

Problem 14:
Part a:
The Cantor set has measure zero; it's usually defined as

$$[0, 1] \setminus ((1/3, 2/3) \cup ((1/9, 2/9) \cup (7/9, 8/9)) \cup \ldots)$$

Now, $[0, 1]$ has measure 1, and is measurable.
Also, $((1/3, 2/3) \cup ((1/9, 2/9) \cup (7/9, 8/9)) \cup \ldots)$ has measure 1 (consider the geometric series $1/3, 2(1/3)^2 \ldots$. It sums up to 1).
So the measure of the cantor set is $1 - 1 = 0$.

Part b:
If we only remove $\alpha 3^- n$ at each step when we define the cantor set, then we can show that it would still be closed (as a complement in $[0, 1]$ of an open set) and by employing the same geometric series argument, it would have measure $1 - \alpha$.

Problem 17:
Part a:
Consider the P_is as defined in this section. We're given that $m[0, 1) = \sum m^* P_i = \sum m^* P$, so that the right hand side is either zero or infinite. But if it was zero, then we break countable subadditivity; it must be infinite. So we have an example where $m(\bigcup E_i) \leq \sum m^*(E_i)$.

Part b:
Define $E_0 = [0,1) \setminus P_0$ and $E_n = [0,1) \setminus P_n$ to get the desired result.

Problem 22:
Part a:
If f is measurable, then the restriction of f to any measurable set is measurable. If D_1 isn't measurable, then the interesection of all of the $\{x : f(x) \geq n\}$ isn't measurable, which is bad. Similarly, D_2 must be measurable.

Now, if all of D_1, D_2 and the restriction of f to $D \setminus D_1 \cup D_2$ are measurable, then for each α we get $\{x : f(x) \geq \alpha\}$ the union of D_1 and a measurable set, so we win.

Part b:
Apply the same trick as used earlier this chapter; prove that if f and g measurable, then so is f^2 and $f+g$, and win using $fg = 1/2[(f+g)^2 - f^2 - g^2]$.

Parts c and d are painfully trivial.
Problem 23:
This was a homework problem; just go there.

Problem 28:
I'm not sure how to do this one.

Problem 31:
Not sure how to do this one either. It looks like a very likely qual problem, too...:/

Chapter 4:

Problem 2:

Part a: Let f be a bounded function on $[a,b]$ and let h be the upper envelope of f (that is, $h(x) = \inf\limits_{\delta>0} \sup\limits_{|x-y|<\delta} (f(y)))$

Then $U - \int\limits_a^b f \geq \int\limits_a^b h$; let ϕ be a step function with $\phi \geq f$. Then $\phi \geq h$ except at a finite number of points, because step functions are discontinuous on only finitely many points and the upper envelope is lower than any continuous function above f.

Also, $U - \int\limits_a^b f \leq \int\limits_a^b h$; there's a sequence of step functions converging downwards to h, so by bounded convergence, we have our result.

So $U - \int\limits_a^b f = \int\limits_a^b h$.

Part b:

We get a similar result for the lower envelope. So a bounded function on $[a,b]$ is Riemann-integrable if and only if the integrals of its upper and lower envelopes are equal.

If the upper and lower envelopes are unequal on a set of greater than measure zero, this fails, as the lower envelope is always lower than the upper envelope.

If the upper and lower envelopes are equal except on a set of measure zero, this succeeds, rather obviously.

So a bounded function on $[a,b]$ is Riemann-integrable if and only if the upper and lower envelopes are equal except on a set of measure zero. That is, a bounded function on $[a,b]$ is Riemann-integrable if and only if the function is continuous except on a set of measure zero.

Problem 8:

Let $\langle f_n \rangle$ be a sequence of nonnegative functions on a domain, E. Define $f(x) = \liminf f_n(x)$.

Let $h \leq f$ be any non-negative, simple function with finite measure support on the domain (say it has finite measure support on F.

Then define $h_n = \min(h, f_n)$.

Now, $\int\limits_E h \leq \int\limits_F h = \lim \int\limits_F h_n \leq \liminf \int\limits_E f_n$.

By taking supremums over h, we have our result.

Problem 14:

Part a:

Let $\langle g_n \rangle \to g$ almost everywhere, $\langle f_n \rangle \to f$ almost everywhere, and $|f_n| \leq g_n$, with all of the above functions being measurable, and $\int g = \lim \int g_n$.

Then $\int |f_n - f| \leq \left| \int f_n - f \right| = \left| \int f_n - \int f \right| \to 0$.

Part b: NOTE: a similar problem was an exam problem. This problem can be generalized, and should be done in the context of L^p spaces.

Let $\langle f_n \rangle$ be a sequence of integrable functions in L^p with $f_n \to f$ almost everywhere.

If $\|f_n\| \not\to \|f\|$, then there's an $\epsilon > 0$ and a subsequence f_{n_k} with $|\|f_{n_k}\| - \|f\|| \geq \epsilon$. But

$$
\begin{aligned}
2\|f_n - f\| &\geq |\|f_n\| - \|f\|| \\
&\not\to 0
\end{aligned}
$$

If $\|f_n\| \to \|f\|$, then:

$$
\begin{aligned}
\|f_n - f\| &\leq |\|f_n\| - \|f\|| \text{ (By reverse triangle inequality.)} \\
&\to 0
\end{aligned}
$$

So $\|f_n - f\| \to 0$ if and only if $\|f_n\| \to \|f\|$.

Problem 15:

The entire problem is "Apply Littlewood's Three Principles" and the "$2^{-n}\epsilon$ trick". (On $[-1,1]$ there is a (property) function such that $|f-\phi_1| < 2^{-1}\epsilon/2$... similarly, there is such a function on $[-2,-1)$ and $(1,2]$ such that $|f-\phi_2| < 2^{-2}\epsilon/2$...induct, paste everything together, integrate, geometric series, win.)

Problem 16: NOTE: this was an exam problem.

First, note that if we have that this is true for all step functions vanishing except on a vinite interval, then we have our result; if $\lim_{n\to\infty} cos(nx)\phi(x)dx = 0$ for all such step functions ϕ, then because there's such a step function with $\int |f-\phi| < \epsilon$ for all $\epsilon > 0$, we have our result.

So, let ϕ be a step function on $[a,b]$, and let $\epsilon > 0$. Partition $[a,b]$ by $a = x_0 < x_1 \ldots x_l = b$ so that ϕ is constant on each (x_i, x_i+1). Let M be the maximum of $|\phi|$ (which exists, as ϕ takes only finitely many values). Pick n large enough so that $2\pi/n < \epsilon/(lM)$. Integrate over each chunk of the partition; we end up with everything cancelling out except on sets of length less than $2\pi/n$. There's at most l of them, having magnitude at most M; we've won.

Problem 22:

Note: This problem is lol.

Let there be a sequence, $\langle f_n \rangle$, of measurable functions on a set, E, of finte measure, with $f_n \to f$ in measure.

Then every subsequence of f_n converges to f in measure, so every subsequence has a subsequence converging to f in measure.

Now, let there be a sequence, $\langle f_n \rangle$, of measurable functions on a set, E, of finte measure, with every subsequence of f_n having a subsequence converging to f in measure. Then every subsequence of f_n has a subsequence which has every subsequence have a subsequence that converges almost everywhere to f. Thus, every subsequence of f_n has a subsequence that converges almost everywhere to f. So f_n converges to f in measure.

Problem 25:

...Seriously, the hint gives this entire question away. Pretty lame stuff, bro.

Chapter 5:

Problem 4:

Problem 5:

Problem 8:

Problem 10:

Problem 14:

Problem 16:

Problem 20:

Problem 23:

Problem 24: