Problem 1:

Problem 2:

Let G be a group and $H \subset G$ be nonempty and finite.

If $H < G$, then if $a \in H$ and $b \in H$, then $ab \in H$. Else, H does not inherit the group operation of G.

Next, assume that $a, b \in H$ implies that $ab \in H$.

Then H inherits the group operation of G.

Because the group operation of G was associative, the inherited group operation on H is also associative.

Further, $e \in H$:

Next, for all $a \in H$, $a^{-1} \in H$:

Thus, H is a group under the inherited group operation of G: $H < G$.

So $H < G$ if and only if for all $a, b \in H$, $ab \in H$.

Problem 3:

Let G be a group such that for all $a, b \in G$ and for three given consectuve integers i, $(ab)^i = a^i b^i$.

Then there is an $n \in \mathbb{Z}$ such that for all $a, b \in G$:

$$\begin{aligned}
&(ab)^n = a^n b^n \\
&(ab)^{n-1} = a^{n-1} b^{n-1} = a^{-1}(ab)^n b^{-1} \\
&(ab)^{n+1} = a^{n+1} b^{n+1} = a(ab)^n b
\end{aligned}$$

Now, for all $a, b \in G$,

$$\begin{aligned}
ab &= (ab)^n((ab)^{n-1})^{-1} \\
&= a^n b^n b((ab)^n)^{-1}a \\
&= a^n b^n b((ab)^n)^{-1}a \\
&= a^{-1}a^{n+1}b^{n+1}((ab)^n)^{-1}a \\
&= a^{-1}(ab)^{n+1}(ab)^{-n}a \\
&= a^{-1}(ab)a \\
&= a^{-1}aba \\
&= ba
\end{aligned}$$

To summarize, for all $a, b \in G$, $ab = ba$. That is, G is abelian.

Problem 4:
Let $G = \langle \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \rangle < GL_2(\mathbb{R})$.
G has 8 elements; they are:

$$\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right],$$
$$\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right].$$

Now, consider $D_4 = \langle r, s \rangle$, where r is a rotation by 90 degrees and s is a reflection.

Let $\phi : G \to D_4$ be as follows:

$$\begin{aligned}
&\phi(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]) = r, \\
&\phi(\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]) = r^2, \\
&\phi(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]) = r^3, \\
&\phi(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) = e, \\
&\phi(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]) = s, \\
&\phi(\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) = sr, \\
&\phi(\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]) = sr^2, \\
&\phi(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]) = sr^3.
\end{aligned}$$

Then it is readily checked (where "readily" means in a series of not more than 64 matrix multiplications that I will not list here) that ϕ is an isomorphism.

Problem 5:

Problem 6: