Problem 1:

Note: Seriously, who says that $D_6 = \langle a, b\rangle$? Hungerford's the first time I've seen that, and doing that completely strips away all hope of understanding it intuitively. Here, $D_6 = \langle r, s\rangle$, with r being "rotation" and s being "reflection".

First, we point out that D_6 and $\{e\}$ are normal subgroups of D_6. Also, $\langle r\rangle$, $\langle s, r^2\rangle$, and $\langle sr, r^2\rangle$ are normal, as it they are subgroups of index 2 (We know that subgroups of index 2 are normal, by a theorem in class.) Also, $\langle r^3\rangle$ is normal; it is the center of D_6 (as was discussed in class), and is thus normal. In addition, $\langle r^2\rangle$ is normal:

$$\begin{aligned}
srr^2(sr)^{-1} &= sr^2s = ssr^4 = r^4\\
sr^2r^2(sr^2)^{-1} &= sr^2s = ssr^4 = r^4\\
sr^3r^2(sr^3)^{-1} &= sr^2s = ssr^4 = r^4\\
sr^4r^2(sr^4)^{-1} &= sr^2s = ssr^4 = r^4\\
sr^5r^2(sr^5)^{-1} &= sr^2s = ssr^4 = r^4\\
sr^2(s)^{-1} &= ssr^4 = r^4\\
srr^4(sr)^{-1} &= sr^4s = ssr^2 = r^2\\
sr^2r^4(sr^2)^{-1} &= sr^4s = ssr^2 = r^2\\
sr^3r^4(sr^3)^{-1} &= sr^4s = ssr^2 = r^2\\
sr^4r^4(sr^4)^{-1} &= sr^4s = ssr^2 = r^2\\
sr^5r^4(sr^5)^{-1} &= sr^4s = ssr^2 = r^2\\
sr^4(s)^{-1} &= ssr^2 = r^2
\end{aligned}$$

We can exclude conjugation by elements of the form r^n in the above, because such elements commute with r^2 and r^4, and thus conjugation of either by such an element leaves r^2 and r^4 fixed.

However, this is all; let H be a normal subgroup. We have shown that every subgroup containing only powers of r is normal in H.

If $sr^n \in H$ where $n \neq 3$, then $rsr^nr^{-1} = rsr^{n-1} = sr^5r^{n-1} = sr^{n+4}$. So $sr^{n+4} \in H$.

Because H is a subgroup, this means that $sr^nsr^{n+4} = r^{-n}ssr^{n+4} = r^{-n}r^{n+4} = r^4 \in H$. By taking an inverse, we also get that $r^2 \in H$.

In other words, if H is normal, then H contains both r^2 and sr^n for some $n \in \mathbb{N}$. If n is even, we can multiply sr^n on the right a number of times to get the result s. Else, we can do the same to get the result sr. So either

s or sr is in H. Thus, H is one of the subgroups already determined to be normal ($\langle r^2, sr\rangle$, $\langle r^2, s\rangle$, or D_6.)

So, we can determine that we have "got 'em all".

Problem 2:

Let G, H, K be finite abelian groups, with $G \oplus H \cong G \oplus K$.

Now, because $G \oplus H$ is a finite abelian group, it is isomorphic to a direct sum of the form $\bigoplus_{i=1}^{m} \mathbb{Z}/(p_i^{\alpha_i}\mathbb{Z})$, with each p_i prime. Moreover, $G \oplus K$ is isomorphic to the same direct sum, by a theorem discussed in class.

Moreover, there's an injective homomorphism from G to $G \oplus H$ given by $\phi : G \to G \oplus H$ where $\phi(g) = (g, 0)$. Thus, G is isomorphic to a subgroup of $G \oplus H$. Thus, G is isomorphic to a direct sum of the form $\bigoplus_{k=1}^{n} \mathbb{Z}/(p_{i_k}^{\alpha_{i_k}}\mathbb{Z})$ (as any subgroup of $\bigoplus_{i=1}^{m} \mathbb{Z}/(p_i^{\alpha_i}\mathbb{Z})$ is of this form...we probably discussed this some time.)

Now, by rearranging terms, we can see that $G \oplus H \cong \bigoplus_{k=1}^{n} \mathbb{Z}/(p_{i_k}^{\alpha_{i_k}}\mathbb{Z}) \oplus \bigoplus_{\{i : i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i}\mathbb{Z})$. Now, $H \cong \bigoplus_{\{i : i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i}\mathbb{Z})$; else, $G \oplus H$ can be "factored" as a product of two distinct sums of the form $\bigoplus_{k=1}^{n} \mathbb{Z}/(p_{i_k}^{\alpha_{i_k}}\mathbb{Z})$. Similarly, $K \cong \bigoplus_{\{i : i \neq i_k \text{ for any } k \in \mathbb{N}\}}^{n-m} \mathbb{Z}/(p_i^{\alpha_i}\mathbb{Z})$. So $K \cong H$.

Problem 3:

Let G be a finite abelian group.

First, if G is cyclic, then for each $n \in \mathbb{N}$ with $n \neq 1$, then either n is a multiple of $|G|$, in which case there are $|G|$ elements, a, satisfying $na = 0$, or n is not a multiple of $|G|$, in which case there are none (because every element in G has the same order, except for 0). And of course, if $n = 1$, then there's only 1 element satisfying $1a = 0$, that is, 0.

So if G is cyclic, then for each $n \in \mathbb{N}$ there are at most n elements $a \in G$ satisfying $na = 0$.

Now, let G not be cyclic. Then $G \cong \bigoplus_{i=1}^{n} \mathbb{Z}/(m_i^{\alpha_i}\mathbb{Z})$, with each m_i dividing m_{i+1}, each $m_i > 1$, and $n \geq 2$ (else, G is obviously cyclic...). Consider m_n. Every element, a, of G has the property $m_n a = 0$; let $a \in G$. Then $\phi(a) = (a_1, a_2 \ldots a_n) \in \bigoplus_{i=1}^{n} \mathbb{Z}/(m_i^{\alpha_i}\mathbb{Z})$, with an isomorphism, ϕ determined by the FTFAG. Moving on $m_n(a_1, a_2 \ldots a_n) = (m_n a_1, m_n a_2 \ldots m_n a_n) = (0, 0, \ldots 0)$ (the last of these is because each of the component groups has order dividing m_n). Thus, $a^{m_n} = 0$, for all $a \in G$. However, $m_n < |G|$, because $|G| = \prod m_i \geq m_{n-1}m_n > m_n$.

So if G is not cyclic, then there is an $n \in \mathbb{N}$ with more than n elements $a \in G$ satisfying $na = 0$.

So G is cyclic if and only if for each $n \in \mathbb{N}$ there are at most n elements $a \in G$ satisfying $na = 0$.