I don't know if we covered the ρ metric ($\rho(f, g) = \sup(|f(x) - g(x)| : x \in X)$, where X is the domain of f and g), but I'm using it because it's nice and I like it.

Problem 1:

Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = x + 1/n$.

It is rather clear that this sequence of functions converges uniformly (to x).

However, the sequence of functions $\langle f_n^2 \rangle$ fails to converve uniformly:

For each $n \in \mathbb{N}$, $f_n^2(x) = x^2 + 2x/n + 1/n^2$. It is rather clear that $\langle f_n^2 \rangle$ converges pointwise to x^2. So if $\langle f_n^2 \rangle$ converges uniformly to something, it must converge uniformly to x^2. However, $\langle f_n^2 \rangle$ does not converge uniformly to x^2:

Let $\epsilon > 0$, and pick $n \in \mathbb{N}$. Consider $|f_n^2(x) - x^2| = |2x/n + 1/n^2|$. Pick $x = n\epsilon$. Then $|2x/n + 1/n^2| = 2\epsilon + 1/n^2 \geq \epsilon$.

So for all $\epsilon > 0$ and $n \in \mathbb{N}$ there is an $x \in \mathbb{R}$ such that $|f_n^2(x) - x^2| \geq \epsilon$; f_n^2 does not converge uniformly to x^2.

So $\langle f_n \rangle$ converges uniformly on $\mathbb{R}$, but $\langle f_n^2 \rangle$ doesn't. This satisfies the problem.

Problem 2:

Longish note: After finishing this problem, I noticed that this follows immediately from a fragment of the proof of Arzela-Ascoli. I prefer this proof, as it is smoother, but it is important to note that such a thing is possible. Moreover, I used a general metric in this one, because it seems like if we had an appropriate extension of Arzela-Ascoli, then this proof could be extended to other types of metric spaces...Does such a theorem exist?

Proof starts here: Let $\langle f_n \rangle$ be an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise to some function, say f.

By the Arzela-Ascoli theorem, we know that $\langle f_n \rangle$ has some subsequence that uniformly converges to some function. We know that this function must be f: if a subsequence of functions converges uniformly to f, it converges pointwise to f. If a sequence of functions converges pointwise to a function, f, then all of its subsequences converge to f. So if a sequence of functions converges pointwise to f, then any subsequence of functions that converges uniformly to a function must converge uniformly to f.

Now, consider such a converging subsequence, $\langle f_{n_j} \rangle$.

Let $\epsilon > 0$. There is a $J \in \mathbb{N}$ such that for all $j \geq J$, $\rho(f_{n_j}, f) < \epsilon/4$.

In addition, by equicontinuity, there is a $\delta_1 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x,y) < \delta_1$ implies that $d(f_n(x), f_n(y)) < \epsilon/4$.

Moreover, we know that converging sequences of continuous functions converge to continuous functions. We also know that continuous functions on a compact domain are uniformly continuous. Thus, f is uniformly continuous; there is a $\delta_2 > 0$ such that for all $n \in \mathbb{N}$, $x, y \in K$, $d(x,y) < \delta_2$ implies that $d(f(x), f(y)) < \epsilon/4$.

Define $\delta = \min(\delta_1, \delta_2)$, so that we have both of the lines for δ.

We know that compact sets are totally bounded. (If this is not clear, consider a career in pastry making.)

So, let F be a finite collection of points of K such that for all $x \in K$, $d(x,y) < \delta$ for some $y \in F$.

Now, for each $y \in F$, there is an $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $d(f_n(y), f(y)) < \epsilon/4$.

Define $N = \max(N_y, n_J)$.

Now, for all $n \geq N$, and for all $x \in K$, we have, for some $y \in F$ (we pick y with $d(x,y) < \delta$:

$$\begin{aligned} d(f_n(x), f(x)) &\leq d(f_n(x), f_n(y)) + d(f_n(y), f_{n_j}(y)) + d(f_{n_j}(y), f(y)) + d(f(y), f(x)) \\ &\leq \epsilon \text{ (That's good enough.)} \end{aligned}$$

So for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in K$, $d(f_n(x), f(x)) < \epsilon$. That is, f_n converges uniformly to f.

To summarize, if $\langle f_n \rangle$ is an equicontinuous sequence of functions on a compact set, K, with $\langle f_n \rangle$ converging pointwise, then $\langle f_n \rangle$ converges uniformly.

Problem 3:

Let $\langle f_n \rangle$ be a uniformly bounded sequence of functions that are Riemann-integrable on $[a,b]$. Set

$$F_n(x) = \int_a^x f_n(t)dt$$

Moreover, let L be the absolute value of a lower bound for the f_ns and let U be the absolute value of an upper bound for the f_ns. (I say absolute values here because I don't want to bother with them later.)

Now, the set of F_ns are equicontinuous:

Let $\epsilon > 0$. Define $\delta = \epsilon / \max(U, L)$. Then for all $n \in \mathbb{N}$, $x, y \in [a, b]$ with $|x - y| < \delta$ (WLOG, $x \leq y$), we have:

$$
\begin{aligned}
|F_n(x) - F_n(y)| &= |\int_a^x f_n(t)dt - \int_a^y f_n(t)dt| \\
&= |\int_x^y f_n(t)dt| \text{ I exploit the absolute value here too.} \\
&\leq |(x - y)\max(U, L)| \text{ This is some sort of obvious property of integrals we shou} \\
&< \epsilon
\end{aligned}
$$

So for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$, $x, y \in [a, b]$, $|x - y| < \delta$ implies that $|F_n(x) - F_n(y)| < \epsilon$. That is, the set of F_ns are equicontinuous.

In addition, the F_ns are defined on $[a, b]$, which is a compact space. By Arzela-Ascoli, there is a subsequence $\langle F_{n_j} \rangle$ that converges uniformly on $[a, b]$.

Problem 4:

Let $\langle f_n \rangle$ be a sequence of increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$.

Problem 5:

Let α be increasing on $[a, b]$, g continous, and $g(x) = G'(x)$ for all $x \in [a, b]$.

Then note that both $\int_a^b \alpha(x)g(x)dx$ and $\int_a^b G d\alpha$ exist; the first because reasons, and the second because G is differentiable, thus continuous, thus $G \in \mathcal{R}(\alpha)$.

Problem 6:

Let α be an increasing function on $[a, b]$, and for $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 d\alpha \right)^{1/2}.$$

Let $f, g, h \in \mathcal{R}(\alpha)$.
Then we have the following:

$$\begin{aligned}
\|f - h\|_2 &= \left(\int_a^b |f - h|^2 d\alpha \right)^{1/2} \\
&= \left(\int_a^b |f - g + g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left(\int_a^b (|f - g| + |g - h|)^2 d\alpha \right)^{1/2} \\
&= \left(\int_a^b |f - g|^2 + 2|f - g||g - h| + |g - h|^2 d\alpha \right)^{1/2} \\
&\leq \left(\int_a^b |f - g|^2 + |g - h|^2 d\alpha \right)^{1/2} \\
&= \|f - g\|_2 + \|g - h\|_2
\end{aligned}$$

Thus, we have the triangle inequality for this ~~norm~~ strange function I have never seen before.