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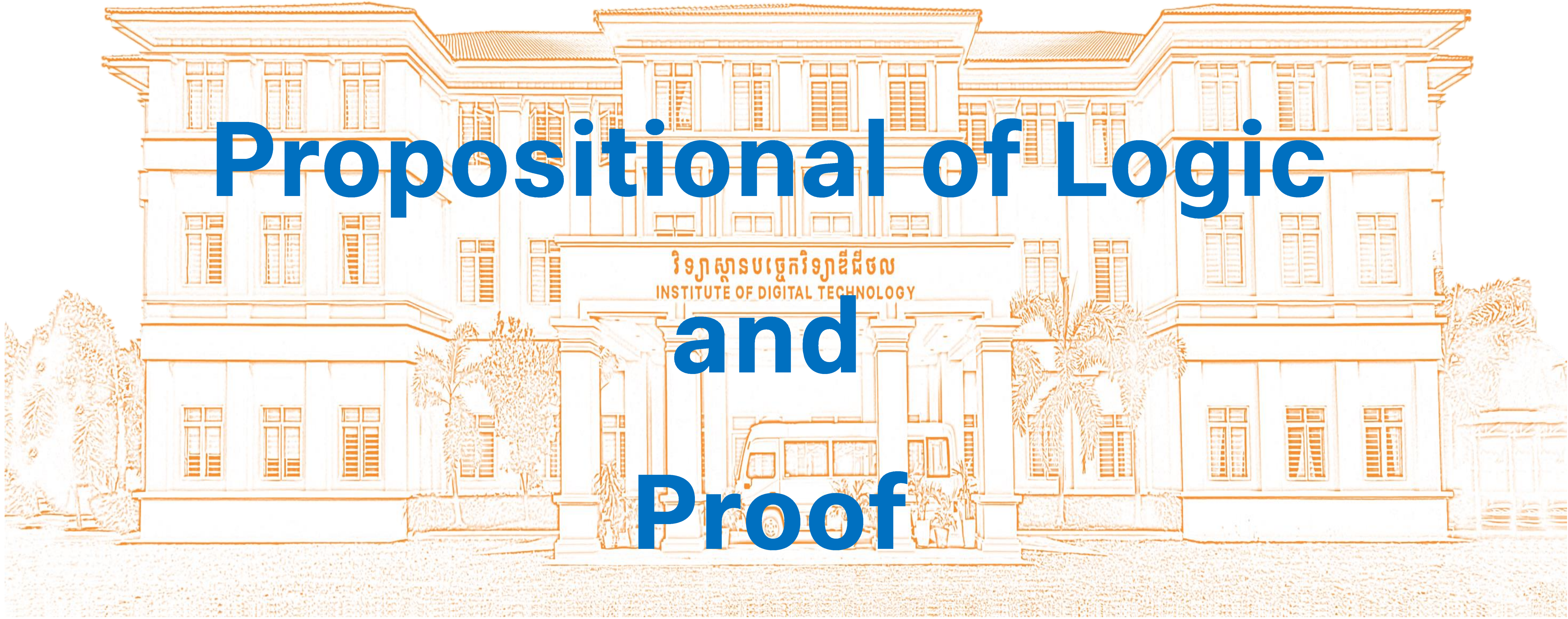

Propositional of Logic and Proof

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In this point we will :

- ❖Learn how to create and use truth table
- ❖Learn some basic operations
- ❖Explain the operations in propositions
- ❖Learn about tautologies and how to show them
- ❖Proof

Propositional of Logic

Basic Definition :

Logic is the study of the criteria used in evaluating inference or arguments.

What is an inference?

An inference is a process of reasoning in which a new belief is formed on the basic of or in virtue of evidence of proof supposedly provided by other beliefs.

What is an argument?

An argument is a collection of statements or propositions, some of which are intended to provide support or evidence in favor of one of the others.

What is a statement or proposition?

A statement or proposition is something that can either be true or false. We usually think of a statement as a declarative sentence, or part of a sentence.

Symbols of connective in Logic

Symbol	Name	Read as	Explanation	Example
$\neg$ *or* $-$	Negation	not	The statement $\neg A$ is true if and only if A is false	$\neg(\neg A) \equiv A$ $x \neq y \;\Leftrightarrow \neg(x = y)$
$\vee$ *or* $+$	Disjunction	or	The statement $A \vee B$ is true if A or B (or both) are true	$n \geq 7 \vee n \leq 5 \;\Leftrightarrow n \neq 6$ $when\ n\ is\ a\ natural\ number$
$\wedge$ *or* $\cdot$	Conjunction	and	The statement $A \wedge B$ is true if A and B are true. otherwise, it is false	$n < 7 \vee n > 5 \;\Leftrightarrow n = 6$ $when\ n\ is\ a\ natural\ number$
$\Rightarrow$ *or* $\rightarrow$	Implication	Implies or If .... then	$A \Rightarrow B$ is false when A is true and B is false, otherwise is true	$x = 1 \Rightarrow x^2 = 1$ is true, but $x^2 = 1 \Rightarrow x = 1$ is false in general $(\,x\ can\ be\ \pm 1\,)$
$\Leftrightarrow$ *or* $\leftrightarrow$ *or* $\equiv$	equivalence	If and only if, iff, or the same as	$A \Leftrightarrow B$ is true if and only if A and B are the same values	$x^2 + 2x - 3 = 0 \Leftrightarrow x = 1\ and\ x = -3$

Truth Values

Truth values of logic it is depend on to the statements or propositional logic

if we have three statements p , q and r so we get the truth values denote by 2^n

The values of p is equal to $2^3 = 8$ we can write $(1,1,1,1,0,0,0,0)$

The values of q is equal to $2^2 = 4$ we can write $(1,1,0,0)$

The values of r is equal to $2^1 = 2$ we can write $(1,0)$

Notice : True values let by number 1 or letter T

False values let by number 0 or letter F

How to write values in truth table

P	q
1	1
1	0
0	1
0	0

p	$\overline{p}$
1	0
0	1

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Truth table

p	q	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	1	1	1	1
1	0	1	0	0	0
0	1	1	0	1	0
0	0	0	0	1	1

Truth Tables

- Useful for determining equivalences and also for shoeing simple true and false values for variables gone through various operation.
- Two values occupy the spaces in a truth table :
  - “ True “ value : Either “ 1 “ or “ T “
  - “False “ value : Either “ 0 “ or “ F “

Notice : In our country we will be using the “ 0 “ and “ 1 “ values

True we use “ 1 “

False we use “ 0 “

Truth Tables

Sample truth table

p is our statement and $\neg p = \bar{p}$ is an operation

p	¬p
0	1
1	0

Truth Tables

- ❖ When creating a truth table, we start off by listing all the statements in one separate column
- ❖ Following the statements, we fill in the operations we need to perform to the columns on the left of the statements
- ❖ In this case for multiple statements, we list all possible combinations of true and false values for the statements and that will determine the amount of rows we have

- The number will be 2^n, where n is the number of statement

- If we have one statement so we get the value are $2^1 = 2$ (" 1 " and " 0 ")

Sample truth table :

Statement : p and q so n=2 and we expect to have $2^2 = 4$ rows

Operation : $p \wedge q$ (these operations will be explained in the next few slides)

p	q	p ^ q
0	0	0
0	1	0
1	0	0
1	1	1

(Note that the first two columns are our statements and the last column is the operation we performed)

Basic Operations

The NOT operation : $\neg$

The AND operation : $\wedge$

The OR operation : $\vee$

The Implication operation : $\longrightarrow$

The equivalence operation : $\longleftrightarrow , \Leftrightarrow$ ***or*** $\equiv$

NOT Operation

- The NOT operation is quite self-explanatory and is what one would think it to be, negation.
- This operation can be only perform to one statement.
- Often denoted by using the symbol " $\neg$ or $\overline{\ }$ "

Example: $\neg p$, p' or $\bar{p}$

Truth Table:

p	¬p or p'
0	1
1	0

(Note that the “true” value for ¬p is when p is false)

NOT Operation

In plain text, we can describe the operation to be the opposite of what the original value is

Example : Let p = “ cat “

then $\neg p$ will be “ not a cat “

Example : Let p = “ tired “

then $\neg p$ = “ not tired “

AND operation

- For the AND Operation, two or more statements are required and will only return " true " if both statements are also " true "
- Often denoted by symbol " $\wedge$ or " · "

Truth Table:

p	q	p ^ q
0	0	0
0	1	0
1	0	0
1	1	1

AND plain text,

Example: let p is “tired” and q is “ hungry ”

then $p \wedge q$ is “ tired and hungry “

also $\neg p \wedge q$ is “ not tired and hungry “

or $p \wedge \neg q$ is “ tired and not hungry “

OR Operation

❖ For the OR Operation, two or more statement are required and will return " true " if one or more of statements are true

❖ Often denoted by the symbol " ∨ " or " + "

Truth Table:

p	q	p ∨ q
0	0	0
0	1	1
1	0	1
1	1	1

OR Operation

In plain text,

Example : let p is " tired " and q is " hungry "

if " tired and not hungry " , $p \vee q$ is " true "

if " not tired and hungry " ,$p \vee q$ is " true "

it " not tired and not hungry " $p \vee q$ is " false "

Implication Operation

- For Implication Operation, it is slightly more complicated and requires two statement. It will return “true “if the initial condition is false, regardless of the value of the second statement, and when both statements ae true.
- Denoted by the symbol “ $\rightarrow$ *or* $\Rightarrow$ “

Truth Table:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Implication Operation

❖ In plain text, we can use it in the form " if p, then q "

Example : let p is hungry and q is eat

$p \rightarrow q$ will be " if I am hungry, I eat "

Example : " you must be 21 to drink "

If p is " John is 21 " and q is " John drink
"

then $q \rightarrow p$ satisfies the statement

Implication Operation

❖ $p \rightarrow q$ is also equivalent (the same as) $\neg p \vee q$

We can verify this using truth tables:

p	q	¬p V q
0	0	1
0	1	1
1	0	0
1	1	1

(Note we receive the same values as the previous table)

Example : Using the operations we have just learned, we will apply them using truth tables :

p	q	p ^ q	¬(p ^ q)	¬p	¬q	¬p ^ ¬q	p V q	¬(p V q)	¬p ^ ¬q
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Equivalence

❖ Equivalence requires two statements and will return “true” if both statements have the same value

❖ Denoted by the symbol " $\Leftrightarrow$ *or* $\equiv$ “

Truth Table:

p	q	p ↔ q
0	0	1
0	1	0
1	0	0
1	1	1

Tautology

- Tautology is very similar to logical equivalence
- When all values are “ true “ that is a tautology

Example : $p \equiv q$ if and only if $p \leftrightarrow q$ is tautology

Example : $p \equiv \neg\neg p$ is tautology

Example: Show that p ^ (q V r) ≡ (p ^ q) V (p ^ r) ← statement

p	q	r	q V r	p ^ (q V r)	p ^ q	p ^ r	(p ^ q) V (p ^ r)	Statement
0	0	0	0	0	0	0	0	1
0	0	1	1	0	0	0	0	1
0	1	0	1	0	0	0	0	1
0	1	1	1	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	0	1	1	1	0	1	1	1
1	1	0	1	1	1	0	1	1
1	1	1	1	1	1	1	1	1

Example

Show that $(p \wedge q) \rightarrow p$ is a tautology:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

All values are true therefore it is a tautology.

Type of Proof in Logic

- ❖ Direct Proof
- ❖ Proof by Contrapositive
- ❖ Proof by Contradiction
- ❖ Proof by Cases (Bi-condition)
- ❖ Proof by counter (Example)

What is Direct Proof ?

The simplest (from a logic perspective) style of proof is a direct proof. Often all that is required to prove something is a systematic explanation of what everything means. Direct proofs are especially useful when proving implications. The general format to prove $p \rightarrow q$ is this :

Assume p explain, explain, ..., explain. Therefore q.

Often we want to prove universal statements, perhaps of the form $\forall x(p(x) \rightarrow q(x))$. Again we will want to assume $p(x)$ is true and deduce $q(x)$. But what about the x? We want this to work for all x. We accomplish this by fixing x to be an arbitrary element (of the sort we are interested in)

Here a few example. First, we will set up the proof structure for direct proof, then fill the details

Example : Prove for all integers n, if n even then n^2 is even

Solution

The format of the proof will be this : let n be an arbitrary integer. Assume that n is even.

Explain, explain ... therefore n^2 is even.

To fill in the details, we will basically just explain what it means for n to be even,

and then see what that means for n^2. Here is a complete proof.

let n be an arbitrary integer. Suppose n is even. Then $n = 2k$ for some integer k.

Now $n^2 = (2k)^2 = 4k^2 = 2(2k^2$. Since $2k^2$ is an integer, n^2 is even

Example : Prove: For all integers $a, b,$ and c, if $a|b$ *and* $b|c$ *then* $a|c$. (Here $x|y$ read "x *divides* y" means that y is a multiple of x, i.e. that x will divide into y without remainder).

Solution

Even before we know what the divides symbol means, we can set up a direct proof for this statement. It will go something like this: Let $a, b,$ *and* c be arbitrary integers. Assume that $a|b$ *and* $b|c$. Dot dot dot. Therefore $a|c$.

How do we connect the dots? We say what our hypothesis $(a|b \text{ and } b|c)$ really means and why this gives us what the conclusion $(a|c)$ really means. Another way to say that $a|b$ is to say that $b = ka$ for some integer k (that is, that b is a multiple of a). What are we going for? That $c = la$, for some integer l (because we want c to be a multiple of a). Here is the complete proof.

Let a, b *and* c be integers. Assume that $a|b$ *and* $b|c$. In other words, b is a multiple of a *and* c is a multiple of b. So there are integers k *and* j such that $b = ka$ *and* $c = jb$. Combining these (through substitution) we get that $c = jka$. But jk is an integer, so this says that c is a multiple of a. Therefore $a|c$.

What is the proof by contrapositive?

Recall that an implication $p \to q$ is logically equivalent to its contrapositive $\neg q \to \neg p$. There are plenty of examples of statements which are hard to prove directly, but whose contrapositive can easily be proved directly. This is all that ***proof by contrapositive*** does. It gives a direct proof of the contrapositive of the implication. This is enough because the contrapositive is logically equivalent to the original implication.

The skeleton of the proof of $p \to q$ by contrapositive will always look roughly like this:

Assume $\neg q$. Explain, explain, … explain. Therefore $\neg p$.

As before, if there are variables and quantifiers, we set them to be arbitrary elements of our domain. Here are two examples:

Example : Is the statement “for all integers n if n^2 is even, then n is even” true?

Solution

This is the converse of the statement we proved above using a direct proof. From trying a few examples, this statement definitely appears to be true. So let's prove it.

A direct proof of this statement would require fixing an arbitrary n and assuming that n^2 is even. But it is not at all clear how this would allow us to conclude anything about n. Just because $\mathrm{n}^2 = 2k$ does not in itself suggest how we could write n as a multiple of 2.

Try something else: write the contrapositive of the statement. We get, for all integers n, if nn is odd then n^2 is odd. This looks much more promising. Our proof will look something like this:

Let n be an arbitrary integer. Suppose that n is not even. This means that …. In other words …. But this is the same as saying …. Therefore n^2 is not even.

Now we fill in the details:

We will prove the contrapositive. Let n be an arbitrary integer. Suppose that n is not even, and thus odd. Then $n = 2k + 1$ for some integer k. Now $\mathrm{n}^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Since $2k^2 + 2k$ is an integer, we see that n^2 is odd and therefore not even.

Example : Consider the following statement: for every prime number p, either $p = 2$ or p is odd. We can rephrase this: for every prime number p, if p≠2, then p is odd. Now try to prove it.

Solution

Let p be an arbitrary prime number. Assume p is not odd. So p is divisible by 2. Since p is prime, it must have exactly two divisors, and it has 2 as a divisor, so p must be divisible by only 1 and 2. Therefore p=2. This completes the proof (by contrapositive).

What is the proof by contradiction?

There might be statements which really cannot be rephrased as implications. For example, "$\sqrt{2}$ is irrational." In this case, it is hard to know where to start. What can we assume? Well, say we want to prove the statement p. What if we could prove that $\neg p \rightarrow q$ where q was false? If this implication is true, and q is false, what can we say about $\neg p$? It must be false as well, which makes p true!

This is why ***proof by contradiction*** works. If we can prove that $\neg p$ leads to a contradiction, then the only conclusion is that $\neg p$ is false, so p is true. That's what we wanted to prove. In other words, if it is impossible for p to be false, p must be true.

Example : Prove that $\sqrt{2}$ is irrational.

Solution

Suppose not. Then $\sqrt{2}$ is equal to a fraction $\frac{a}{b}$. Without loss of generality, assume $\frac{a}{b}$ is in lowest terms (otherwise reduce the fraction). So,

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

Thus a^2 is even, and as such a is even. So $a = 2k$ for some integer k, and $a^2 = 4k^2$. We then have,

$$2b^2 = 4k^2$$

$$b^2 = 2k^2$$

Thus b^2 is even, and as such b is even. Since a is also even, we see that $\frac{a}{b}$ is not in lowest terms, a contradiction.

Thus $\sqrt{2}$ is irrational.

Example : Prove there are no integers x and y such that $x^2 = 4y + 2$.

Solution

We proceed by contradiction. So suppose there *are* integers $x \text{ and } y$ such that $x^2 = 4y + 2 = 2(2y + 1)$. So x^2 is even. We have seen that this implies that x is even. So $x = 2k$ for some integer k. Then $x^2 = 4k^2$. This in turn gives $2k^2 = (2y + 1)$. But $2k^2$ is even, and $2y + 1$ is odd, so these cannot be equal. Thus we have a contradiction, so there must not be any integers $x \text{ and } y$ such that $x^2 = 4y + 2$.

What is the proof by case?

We could go on and on and on about different proof styles (we haven't even mentioned induction or combinatorial proofs here), but instead we will end with one final useful technique: proof by cases. The idea is to prove that p is true by proving that $q \to p \; and \; \neg q \to p$ for some statement q. So no matter what, whether or not q is true, we know that p is true. In fact, we could generalize this. Suppose we want to prove p. We know that at least one of the statements $q_1, q_2, \ldots, q_n$ is true. If we can show that $q_1 \dashrightarrow p$

and $q_2 \to p$ and so on all the way to $q_n \to p$, then we can conclude p. The key thing is that we want to be sure that one of our cases (the q_i 's) must be true no matter what.

Example : Prove For any integer n, the number $(n^3 - n)$ is even.

Solution

We consider two cases: if n is even or if n is odd.

Case 1: n is even. Then $n = 2k$ for some integer k. This give

$n^3 - n = 8k^3 - 2k = 2(4k^3 - k)$, and since $4k^3 - k$ is an integer, this says that $n^3 - n$ is even.

Case 2: n is odd. Then $n = 2k + 1$ for some integer k. This gives

$n^3 - n = (2k+1)^3 - (2k+1) = 8k^3 + 6k^2 + 6k + 1 - 2k - 1 = 2(4k^3 + 3k^2 + 2k)$, and since $4k^3 + 3k^2 + 2k$ is an integer, we see that $n^3 - n$ is even again.

Since $n^3 - n$ is even in both exhaustive cases, we see that $n^3 - n$ is indeed always even.

Proof by (counter) Example

It is almost NEVER okay to prove a statement with just an example.

Certainly none of the statements proved above can be proved through an example. This is because in each of those cases we are trying to prove that something holds of all integers.

Example : Above we proved, “for all integers a and b, if $a + b$ is odd, then a is odd or b is odd.” Is the converse true?

Solution

Now we know what to do. To prove that the converse is false we need to find two integers a and b so that a is odd or b is odd, but $a + b$ is not odd (so even). That's easy: 1 and 3. (remember, “or” means one or the other or both). Both of these are odd, but $1 + 3 = 4$ is not odd.

Thank You for your Attending this class