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Pell's Equation

DISCRETE MATHEMATICS

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PELL S EQUATION

What is Pell’s equation?

How can we solve this x^2−dy^2=1 equation ?

What is Pell's equation?

Pell's equation is Linear Diophantine equation which have form

$x^2 - dy^2 = 1$ where $d \in \mathbb{Z}$ and d non-square integer , $x, y \in \mathbb{Z}^+$

Example : $x^2 - 3y^2 = 1$, $x^2 - 5y^2 = 1$ are

Pell's equation

but $x^2 - 4y^2 = 1$, $x^2 - 9y^2 = 1$ are

not Pell's equation

because 4 and 9 are square integer

PELL S EQUATION

How can we solve this $x^2 - dy^2 = 1$ equation ?

If we have (x_0, y_0) is a solution of $x^2 - dy^2 = 1$ we have all positive solution is (x_n, y_n)

where $x_n + y_n\sqrt{d} = \left(x_0 + y_0\sqrt{d}\right)^n \ (1)$

$x_n - y_n\sqrt{d} = \left(x_0 - y_0\sqrt{d}\right)^n \ (2)$

$(1) + (2) : 2x_n = \left(x_0 + y_0\sqrt{d}\right)^n + \left(x_0 - y_0\sqrt{d}\right)^n$ then $x_n = \frac{\left(x_0+y_0\sqrt{d}\right)^n + \left(x_0-y_0\sqrt{d}\right)^n}{2}$

$(1) - (2) : 2y_n\sqrt{d} = \left(x_0 + y_0\sqrt{d}\right)^n - \left(x_0 - y_0\sqrt{d}\right)^n$ then $y_n = \frac{\left(x_0+y_0\sqrt{d}\right)^n - \left(x_0-y_0\sqrt{d}\right)^n}{2\sqrt{d}}$

so the general solution of equation is

$$\begin{cases} x_n = \dfrac{\left(x_0+y_0\sqrt{d}\right)^n + \left(x_0-y_0\sqrt{d}\right)^n}{2} \\ y_n = \dfrac{\left(x_0+y_0\sqrt{d}\right)^n - \left(x_0-y_0\sqrt{d}\right)^n}{2\sqrt{d}} \end{cases} , \ x, y \in \mathbb{Z}^+, d \in \mathbb{Z}^+, d \ non - square \ integer$$

PELL S EQUATION

How to solve this $x^2 - dy^2 = 1$ equation ?

1. Find initial solution (x_0 , y_0)

2. Write equation

$$x_n + y_n\sqrt{d} = \left(x_0 + y_0\sqrt{d}\,\right)^n \; (1)$$

$$x_n - y_n\sqrt{d} = \left(x_0 - y_0\sqrt{d}\,\right)^n \; (2)$$

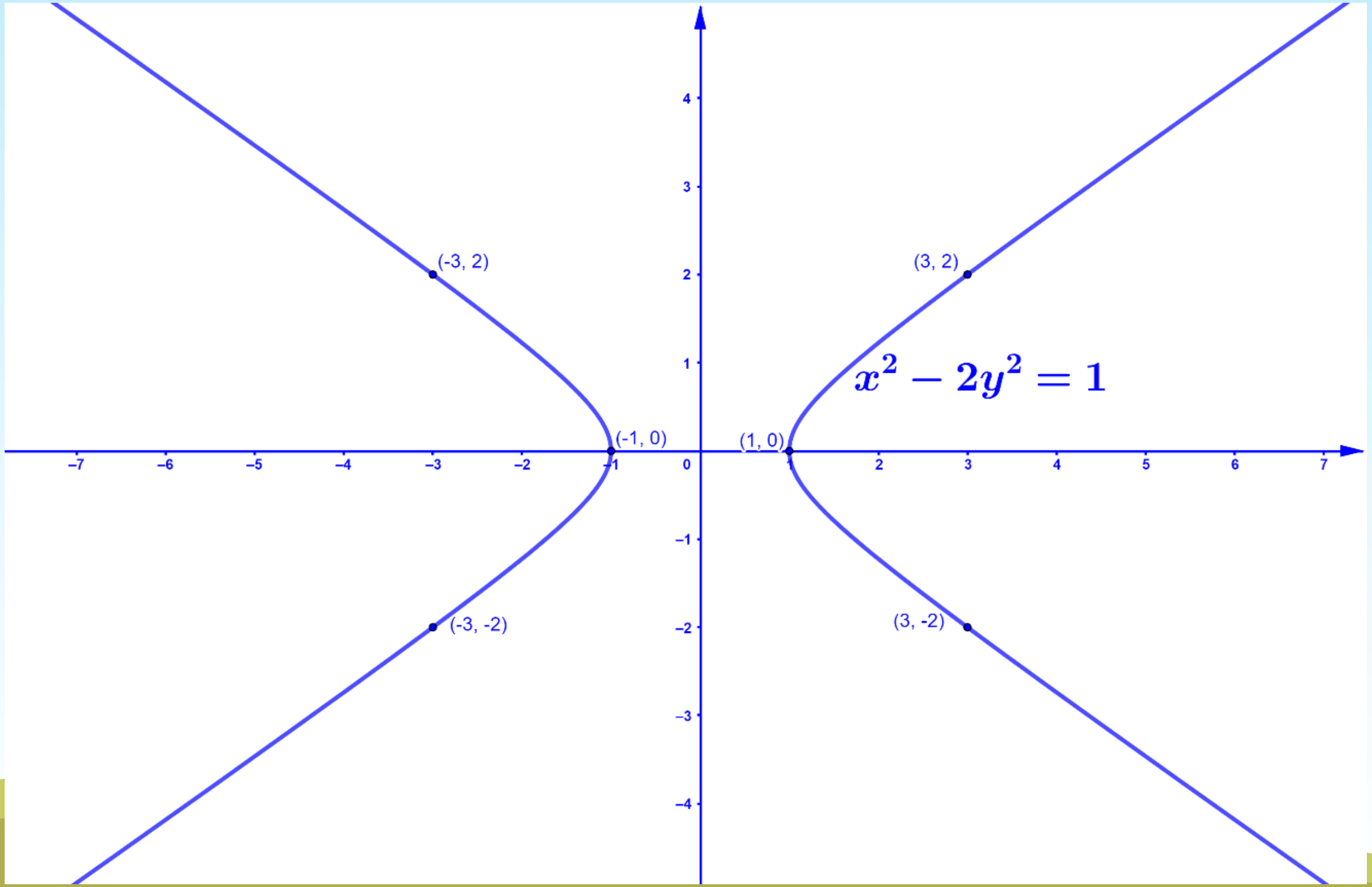
3. General solution

$$\begin{cases} x_n = \dfrac{\left(x_0 + y_0\sqrt{d}\,\right)^n + \left(x_0 - y_0\sqrt{d}\,\right)^n}{2} \\ y_n = \dfrac{\left(x_0 + y_0\sqrt{d}\,\right)^n - \left(x_0 - y_0\sqrt{d}\,\right)^n}{2\sqrt{d}} \end{cases} , n \in \mathbb{N}, x, y \in \mathbb{Z}^+, d \in \mathbb{Z}^+, d\ non-square\ integer$$

PELL S EQUATION

Consider this $x^2 - 2y^2 = 1$

Graph of this equation is look like



PELL S EQUATION

Solve this $x^2 - 2y^2 = 1$ in $\mathbb{Z}^+$

PELL S EQUATION

Solve this $x^2 - 5y^2 = 1$ in $\mathbb{Z}^+$

PELL S EQUATION

Solve this $x^2 - 7y^2 = 1$ in $\mathbb{Z}^+$

PELL S EQUATION

Solve this $x^2 - 17y^2 = 1$ in $\mathbb{Z}^+$

PELL S EQUATION

Solve this $x^2 - 21y^2 = 1$ in $\mathbb{Z}^+$

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PELL S EQUATION

Given $n \in \mathbb{N}$ and equation $2 + 2\sqrt{28n^2 + 1}$ is natural number

Show that $2 + 2\sqrt{28n^2 + 1}$ is square integer of number

We n is natural number , $2 + 2\sqrt{28n^2 + 1}$ is natural number

we let $2 + 2\sqrt{28n^2 + 1} = 2 + 2m\,, m \in \mathbb{N}$

so we get $\quad \mathrm{m} = \sqrt{28n^2 + 1}$

$$m^2 = 28n^2 + 1$$

$$m^2 - 28n^2 = 1 \quad (\,Pell's\ equation\,)$$

$$\underbrace{m^2 = 28n^2 + 1}_{Square\ integer\ by\ given}$$

PELL S EQUATION

$m^2 - 28n^2 = 1$ has $(m_0 , n_0) = (127 , 24)$

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