Alireza Aghaei   Problem 3

(1) It is a sine wave → because there is no friction force in our model, so the Pendulum will swing back and forth.

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(2) $z = \phi - \pi \longrightarrow \phi = z + \pi \longrightarrow \begin{cases} \dot{z} = \dot{\phi} = v \\ \dot{v} = \dfrac{-g}{\ell} \underbrace{\sin(\pi + z)}_{-z} + \dfrac{u}{m\ell^2} \end{cases}$

$$\Longrightarrow \begin{cases} \dot{z} = \dot{\phi} = v \\ \dot{v} = \dfrac{g}{\ell} z + \dfrac{u}{m\ell^2} \end{cases} \Longrightarrow dS = AS + BI$$

$$dS = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} [u]$$

$$\hookrightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

(4) $A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \longrightarrow \det(A - \lambda I) = 0 \longrightarrow \left| \begin{bmatrix} -\lambda & 1 \\ \frac{g}{\ell} & -\lambda \end{bmatrix} \right| = 0$

$$\longrightarrow \lambda^2 - \frac{g}{\ell} = 0 \longrightarrow \lambda = \pm\sqrt{\frac{g}{\ell}} \longrightarrow \begin{cases} \lambda_1 = \sqrt{\frac{g}{\ell}} \\ \lambda_2 = -\sqrt{\frac{g}{\ell}} \end{cases}$$

$\longrightarrow$ because $g > 0$ and $\ell > 0 \longrightarrow \lambda_1 > 0 \longrightarrow$ So the system is unstable.

(5) $dS = AS + BI = AS + B(K(R - S)) =$

$= (A - BK)S + BKR \longrightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$

$K = [k_1 \;\; k_2]$

$$A - BK = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} [k_1 \;\; k_2] =$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{k_1}{m\ell^2} & \frac{k_2}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} - \frac{k_1}{m\ell^2} & -\frac{k_2}{m\ell^2} \end{bmatrix} \Longrightarrow$$

(5) $\Rightarrow \det((A-Bk)-\lambda I)=\emptyset \longrightarrow \left|\begin{bmatrix} -\lambda & 1 \\ \frac{g}{\ell}-\frac{k_1}{m\ell^2} & \frac{-k_2}{m\ell^2}-\lambda \end{bmatrix}\right| = \emptyset$

$\longrightarrow \lambda^2 + \frac{k_2}{m\ell^2}\lambda - \left(\frac{g}{\ell}-\frac{k_1}{m\ell^2}\right) = \emptyset$

$\hookrightarrow \begin{cases} -(\lambda_1+\lambda_2) = \frac{1}{m\ell^2} k_2 \\ \lambda_1\lambda_2 = \frac{-g}{\ell} + \frac{k_1}{m\ell^2} \end{cases} \longrightarrow$ we want to have $\lambda_1=-1$, $\lambda_2=-2$

$\Rightarrow \begin{matrix} 3 = \frac{1}{m\ell^2} k_2 \\ 2 = \frac{-g}{\ell} + \frac{k_1}{m\ell^2} \end{matrix} \Rightarrow \begin{matrix} k_2 = 3m\ell^2 \\ k_1 = m\ell^2\left(2+\frac{g}{\ell}\right) \end{matrix} \longrightarrow$ we know that $m=\ell=1$

$\longrightarrow \begin{matrix} k_2 = 3 \\ k_1 = 11.81 \end{matrix} \longrightarrow \boxed{K = [11.81 \quad 3]}$ gain matrix

(9) $dS = AS + BI \longrightarrow u = k_P e_\phi + k_D \dot{e}_\phi$

$e_\phi = R - S = \pi - \phi = -z$

$\dot{e}_\phi = (R-S)' = (\pi-\phi)' = -\dot{\phi} = -v$

$\hookrightarrow u = [k_P \quad k_D] - S$

$u = -[k_P \quad k_D] S$

$\Rightarrow dS = AS + B(-[k_P \quad k_D]S) = AS - B[k_P \quad k_D]S$

$= (A - B[k_P \quad k_D])S$

$\Rightarrow A - B[k_P \quad k_D] = \begin{bmatrix} \emptyset & 1 \\ \frac{g}{\ell} & \emptyset \end{bmatrix} - \begin{bmatrix} \emptyset & \emptyset \\ \frac{k_P}{m\ell^2} & \frac{k_D}{m\ell^2} \end{bmatrix} =$

$= \begin{bmatrix} \emptyset & 1 \\ \frac{g}{\ell} - \frac{k_P}{m\ell^2} & \frac{-k_D}{m\ell^2} \end{bmatrix}$

(9) $$\begin{vmatrix} -\lambda & 1 \\ \frac{g}{\ell} - \frac{k_P}{m\ell^2} & -\lambda - \frac{k_D}{m\ell^2} \end{vmatrix} = \emptyset \rightarrow \lambda^2 + \frac{k_D}{m\ell^2}\lambda - \left(\frac{g}{\ell} - \frac{k_P}{m\ell^2}\right) = \emptyset$$

$$\rightarrow \begin{cases} -(\lambda_1+\lambda_2) = \frac{k_D}{m\ell^2} \\ \lambda_1\lambda_2 = -\left(\frac{g}{\ell} - \frac{k_P}{m\ell^2}\right) \end{cases}$$

$\rightarrow$ we want the system to be stable $\rightarrow$ $\lambda_1 \leq \emptyset$, $\lambda_2 \leq \emptyset$

$\hookrightarrow$ $\lambda_1 + \lambda_2 \leq \emptyset$, $\lambda_1\lambda_2 \geq \emptyset$ $\Longrightarrow$

$$\Longrightarrow \begin{cases} \frac{k_D}{m\ell^2} \geq \emptyset \longrightarrow m \geq \emptyset, \ell^2 \geq \emptyset \rightarrow \boxed{k_D \geq \emptyset} \\ \frac{g}{\ell} - \frac{k_P}{m\ell^2} \leq \emptyset \longrightarrow \boxed{k_P \geq m\ell g} \end{cases}$$

Stability Condition

When we have $m = \ell = 1 \longrightarrow k_D \geq \emptyset, \; k_P \geq g$