Universality for random permutations

Séminaire MEGA

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Universality

GUE :

	GUE
Biggest particle	T.W
Edge	Soft edge (Airy)
Global convergence	Semi circular
Fluctuations	Gaussian
Bulk	Sine process

Universality

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Bulk	Sine process

Random matrices, OP ensembles, DPPs etc.

- Global (semi circular law): Wigner (1958); Pastur (1972).
- Local convergence (sine process): Lubinsky (2008); Erdos, Péché, Ramírez, Schlein, and Yau (2010).
- Edge (Airy ensemble and Tracy-Widom fluctuations): Soshnikov (1999); Tao and Vu (2011).

Universality

	GUE + Wigner (with a good control on moments)
Biggest particle	T.W
Edge	Soft edge (Airy)
Global convergence	Semi circular
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Universality

	GUE + Wigner (with a good control on moments)	Uniform permutation
Biggest particle	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)
Global convergence	Semi circular	VKLS
Fluctuations	Gaussian	Gaussian
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Universality

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- Universality for random permutations ?
- Independence?
- Moments?

Plan

1. Longest increasing subsequence and Ulam–Hammersley problem.
2. The first arrows of random Young tableaux (edge)
3. The Vershik-Kerov-Logan-Shepp shape

Longest increasing subsequence

- $\mathfrak{S}_n$: symmetric group, (the group of permutations of $\{1, \ldots, n\}$).
- $(\sigma(i_1), \ldots, \sigma(i_k))$ increasing subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$.
- $\ell(\sigma)$: the length of the longest increasing subsequence of σ.
- For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & \color{red}{1} & \color{red}{2} & \color{red}{4} & 7 & \color{red}{5} & \color{red}{6} \end{pmatrix}.$$

$\ell(\sigma) = 5$.

Conjecture (Ulam (1961))

If $\sigma_n \sim U_{\mathfrak{S}_n}$, then

$$\lim_{n \to \infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = c.$$

Longest increasing subsequence

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\ell(\sigma_n))}{\sqrt{n}} = 2$$

and

$$\frac{\ell(\sigma_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 2.$$

Theorem (Baik, Deift, and Johansson (1999))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s).$$

F_2: CDF of the GUE Tracy-Widom distribution.

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Longest increasing subsequence

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* n, σ_n *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

Then for all $s \in \mathbb{R}$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

Ewens' case

Definition (Ewens distribution)

Let $\theta \geq 0$. If $\sigma_n \sim Ew(\theta)$ then

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{k=1}^{n-1}(\theta + k)}.$$

- $\theta = 1$: uniform distribution.
- $\theta = 0$: uniform distribution on permutations with a unique cycle.
- $\mathbb{E}(\#(\sigma_n)) = 1 + \sum_{k=1}^{n-1} \frac{\theta}{\theta+k} \sim \theta \log(n)$.

Ewens' case

Corollary

Assume that $\sigma_n \sim Ew(\theta_n)$. *If*

$$\lim_{n\to\infty} \frac{\theta_n \log(n)}{n^{\frac{1}{6}}} = 0. \tag{H'2}$$

Then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s). \tag{TW}$$

Other applications: Ewens-Pitman, virtual permutations (Kingman), etc.

Proof

By comparision with the uniform setting.
We denote by:

- $A_\sigma := \begin{cases} \{\rho \in \mathfrak{S}_n, \rho = \sigma \circ (i,j) \text{ and } \#(\rho) = \#(\sigma) - 1\} & \text{if } \#(\sigma) > 1 \\ \{\sigma\} & \text{if } \#(\sigma) = 1 \end{cases}.$
- T: Markov operator associated to $\left[\dfrac{\mathbf{1}_{A_{\sigma_1}}(\sigma_2)}{card(A_{\sigma_1})} \right]_{\sigma_1, \sigma_2 \in \mathfrak{S}_n}.$

Proof

- If σ_n is invariant under conjugation, $T(\sigma_n)$ is also invariant under conjugation.
- $\#(T(\sigma_n)) \overset{a.s}{=} \max(\#(\sigma_n) - 1, 1)$.

Consequently, if σ_n is invariant under conjugation, then

- If σ_n is invariant under conjugation, $T^{n-1}(\sigma_n)$ is also invariant under conjugation.
- Almost surely, $\#\left(T^{n-1}(\sigma_n)\right) = 1$.

Proof

Lemma

$\forall \sigma \in \mathfrak{S}_n$, $\forall \tau = (i,j)$ a transposition,

$$|\ell(\sigma \circ \tau) - \ell(\sigma)| \leq 2.$$

Lemma

If

- *σ_n is invariant under conjugation.*
- *Almost surely, $\#(\sigma_n) = 1$.*

Then $\sigma_n \sim Ew(0)$.

Proof

If σ_n is invariant under conjugation then

- $T^{n-1}(\sigma_n) \sim Ew(0)$.
- Almost surely,

$$|\ell(T^{n-1}(\sigma_n)) - \ell(\sigma_n)| = |\ell(T^{\#(\sigma_n)-1}(\sigma_n)) - \ell(\sigma_n)| \leq 2(\#(\sigma_n) - 1).$$

Assume that $\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$. We obtain $\frac{\ell(T^{n-1}(\sigma_n))-\ell(\sigma_n)}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$.

(TW): Uniform $\Rightarrow$ $Ew(0)$ $\Rightarrow$ Invariant under conjugation + $\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$

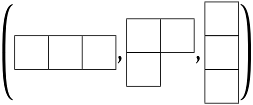
Plan

Young diagram

Definition (Young diagram)

$\lambda = (\lambda_i)_{i\geq 1} \in \mathbb{N}^{\mathbb{N}^*}$ is a Young diagram of size n if

- $\forall i \geq 1,\ \lambda_{i+1} \leq \lambda_i$,
- $\sum_{i=1}^{\infty} \lambda_i = n$.

Example: Young diagrams of size 3 are
$\mathbb{Y}_3 = (3, \underline{0}), (2, 1, \underline{0}), (1, 1, 1, \underline{0})$
or .

Young tableau

Definition (Young tableau)

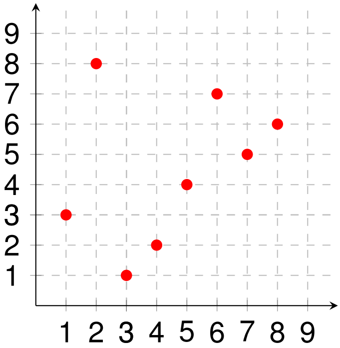
A Young tableau of shape λ is a filling of the boxes of λ using the entries $\{1, 2, \ldots, n\}$ and the entries in each row and each column are increasing.

- Example: Young tableaux of shape $\begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ are

  $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$
- $dim(\lambda) = \#$ Young tableaux of shape λ.
- Example: $dim\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\right) = 3.$
- $dim(\lambda) =$ dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by λ.
- $\sum_{\lambda \in \mathbb{Y}_n} dim(\lambda)^2 = \#(\mathfrak{S}_n) = n!$.

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



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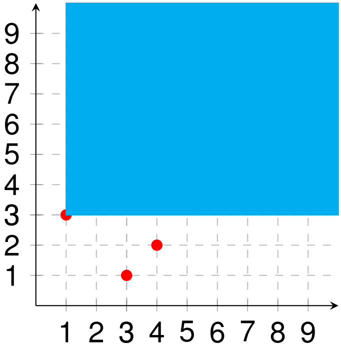

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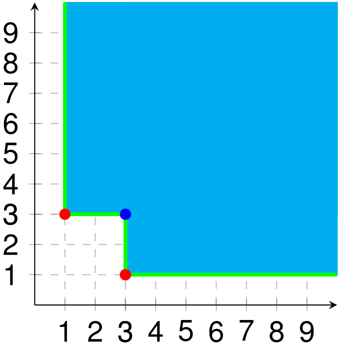

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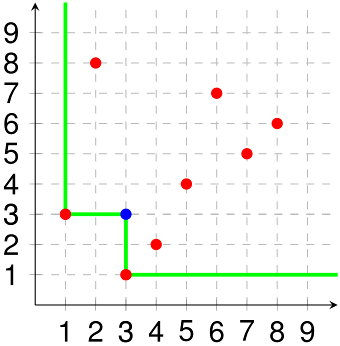

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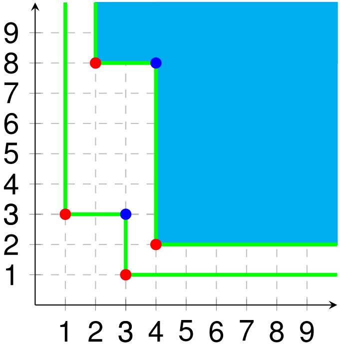

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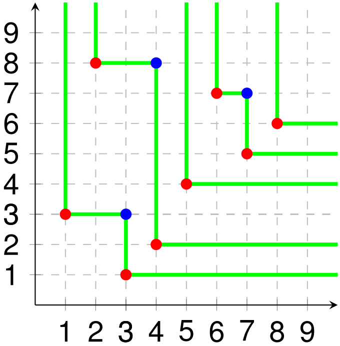
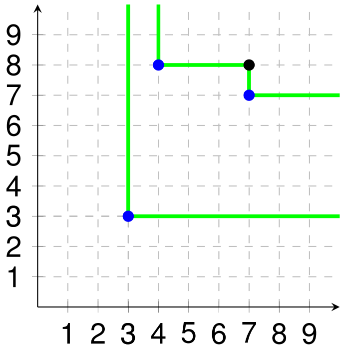


$\boxed{1}\boxed{2}\boxed{4}\boxed{5}\boxed{6}, \boxed{1}\boxed{2}\boxed{5}\boxed{6}\boxed{8}$

Viennot's geometric construction

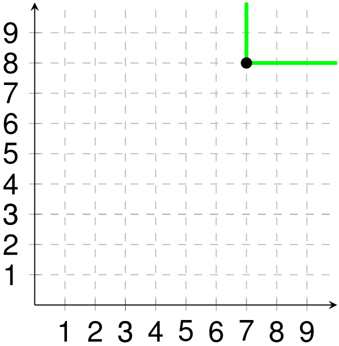
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline 3 & 7 \\ \cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 8 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$

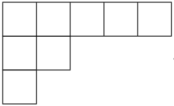


$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline 3 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 8 \\ \hline 3 & 4 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

Robinson-Schensted correspondence

- One-to-one correspondence between permutations and pairs of standard Young tableaux of the same shape.
- We denote by $\lambda(\sigma) := (\lambda_i(\sigma))_{i\geq 1}$ the shape of the image of σ by this correspondence. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \quad \text{then} \quad \lambda(\sigma) =$$



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- $\ell(\sigma) = \lambda_1(\sigma)$.

Plancherel measure

- If $\sigma_n \sim U_{\mathfrak{S}_n}$ then $\lambda(\sigma_n) \sim PL_n$. For any $\mu \in \mathbb{Y}_n$,

$$\mathbb{P}(\lambda(\sigma_n) = \mu) = \frac{\#\{\text{pairs of Young tableaux of shape } \mu\}}{C}$$
$$= \frac{dim(\mu)^2}{n!}.$$

- The poissonized version. If $\lambda \sim PL^\theta$ then for any $\mu \in \cup_{n\geq 1}\mathbb{Y}_n$,

$$\mathbb{P}(\lambda_\theta = \mu) = e^{-\theta}\frac{\theta^{|\mu|}dim(\mu)^2}{|\mu|!^2}.$$

- If $\lambda \sim PL^\theta$ then $\{\lambda_i - i\}_{i\geq 1}$ is determinantal with discrete Bessel kernel.

Edge: Plancherel case

Theorem (Borodin, Okounkov, and Olshanski (2000))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then* $\forall k \geq 1$, $\forall s_1, s_2, \ldots, s_k \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

$\{\xi_1 \geq \xi_2 \geq \cdots \geq \xi_k \geq \ldots\}$: *Airy ensemble.*

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Edge: generalization

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* n, σ_n *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} > \varepsilon\right) = 0. \tag{H2}$$

Then for all $s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\forall i \leq k,\ \frac{\lambda_i(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s_i\right) = \mathbb{P}(\forall i \leq k,\ \xi_i \leq s_i).$$

We denote by

$$\mathfrak{I}_1(\sigma) := \{s \subset \{1,2,\ldots,n\};\ \forall i,j \in s,\ (i-j)(\sigma(i)-\sigma(j)) \geq 0\},$$
$$\mathfrak{I}_{k+1}(\sigma) := \{s \cup s',\ s \in \mathfrak{I}_k,\ s' \in \mathfrak{I}_1\}.$$

We have then

Lemma (Greene (1974))

For any permutation $\sigma \in \mathfrak{S}_n$,

$$\max_{s \in \mathfrak{I}_i(\sigma)} |s| = \sum_{k=1}^{i} \lambda_k(\sigma).$$

In particular,

$$\max_{s \in \mathfrak{I}_1(\sigma)} |s| = \lambda_1(\sigma) = \ell(\sigma).$$

Proof

Lemma

For any permutation σ and transposition τ,

$$\left|\sum_{k=1}^{i} \lambda_k(\sigma) - \lambda_k(\sigma \circ \tau)\right| \leq 2$$

Consequently,

$$\left|\lambda_i(\sigma) - \lambda_i(\sigma \circ \tau)\right| \leq 4.$$

Corollary

$$\left|\lambda_i(\sigma_n) - \lambda_i\left(T^{n-1}(\sigma_n)\right)\right| \leq 4(\#(\sigma_n) - 1). \tag{1}$$

Plan

Russian notations

- Rotate the diagram by $\frac{3\pi}{4}$.
- Complete the high function by $x \to |x|$.
- We denote by L_λ the resulting function.

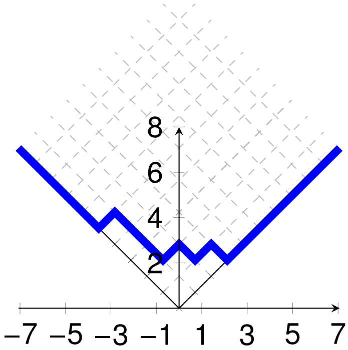

Figure: $L_{(5,2,1,\underline{0})}$

Vershik-Kerov-Logan-Shepp shape

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$, *then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon\right) = 1,$$

where

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s\arcsin(s) + \sqrt{1-s^2}\right) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1 \end{cases}.$$

Vershik-Kerov-Logan-Shepp shape

Ω is strongly related to the semi-circular law.
We denote by

$$\omega(s) := \frac{\Omega(2s) - |2s|}{2}.$$

We have

$$\exp\left(\int_{\mathbb{R}} \frac{d\omega(u)}{u - \frac{1}{x}}\right) = \int_{\mathbb{R}} \frac{d\mu_{sc}(u)}{1 - ux}$$

with

$$d\mu_{sc}(u) := \frac{\sqrt{4 - u^2}}{2\pi} du.$$

Vershik-Kerov-Logan-Shepp shape

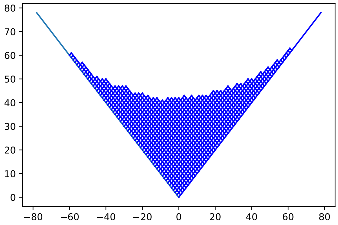

Figure: Typical Young diagram under the Plancherel distribution

	GUE + Wigner (with a good control on moments)	Uniform permutation
Biggest particle	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)
Global convergence	Semi circular	VKLS
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Limit shape

Theorem (K (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ *satisfies:*

- *For all positive integer* n, σ_n *is invariant under conjugation i.e.* $\forall \sigma, \rho \in \mathfrak{S}_n$,
$$\mathbb{P}(\sigma_n = \sigma) = \mathbb{P}(\sigma_n = \rho^{-1}\sigma\rho). \tag{H1}$$
- *The number of cycles is such that: For all* $\varepsilon > 0$,
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\#(\sigma_n)}{n} > \varepsilon\right) = 0. \tag{H3}$$

Then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon\right) = 1.$$

Proof

Lemma

Let $n, m \in \mathbb{N}^*$, $\lambda = (\lambda_i)_{i \geq 1} \in \mathbb{Y}_n$, $\mu = (\mu_i)_{i \geq 1} \in \mathbb{Y}_m$. *Then,*

$$||L_\lambda - L_\mu||_\infty^2 \leq 4 \max_{i \geq 1} \left| \sum_{k=1}^{i} (\lambda_k - \mu_k) \right|.$$

Consequently,

$$\sup_{s \in \mathbb{R}} \frac{1}{\sqrt{2n}} \left| L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - L_{\lambda(T^{n-1}(\sigma_n))}\left(s\sqrt{2n}\right) \right| \leq 2\sqrt{\frac{\#(\sigma_n) - 1}{n}}.$$

Fluctuations : Kerov's central limit theorem

Theorem (Ivanov and Olshanski (2002))

$$L_{\lambda(\sigma_n)}(s\sqrt{n}) - \sqrt{n}\Omega(s) \sim \Delta(s) = \Delta(cos(\theta)) = \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{\xi_k}{\sqrt{k}}sin(k\theta)$$

$\xi_k \sim \mathcal{N}(0,1)$ *i.i.d*

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Bulk

Theorem (Borodin, Okounkov, and Olshanski (2000))

For any $|\alpha| < 2$, under Plancherel measure,

$$\{\lambda_i - i - \alpha\sqrt{n}\}_{i\geq 1} \to Sin_\alpha.$$

Sin_α D.P.P with kernel

$$K_\alpha(x,y) = \begin{cases} \frac{sin(arccos(\frac{\alpha}{2})(x-y))}{\pi(x-y)} & if\, x \neq y \\ \frac{arcos(\frac{\alpha}{2})}{\pi} & if\, x = y \end{cases}$$

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Application: Longest commun subsequence

- $(\sigma(i_1),\ldots,\sigma(i_k))$ subsequence of σ of length k if $i_1 < i_2 <,\ldots,< i_k$.
- $LCS(\sigma_1,\sigma_2)$ the length of the longest common subsequence of two permutations.

Conjecture (Bukh and Zhou (2016))

For any integer $n \geq 1$, for any $\sigma_{1,n}$ and $\sigma_{2,n}$ independent and identically distributed random permutations,

$$\mathbb{E}(LCS(\sigma_{1,n},\sigma_{2,n})) \geq \sqrt{n}.$$

Application: Longest common subsequence

Theorem

For any $0 \leq \alpha < 2$*, there exists* $n_\alpha \geq 1$ *such that for any* $n > n_\alpha$*, for any* $\sigma_{1,n}$ *and* $\sigma_{2,n}$ *independent and identically distributed random permutations with distribution invariant under conjugation.*

$$\mathbb{E}(LCS(\sigma_{1,n}, \sigma_{2,n})) \geq \alpha\sqrt{n}.$$

Conclusion

	GUE + Wigner (with a good control on moments)	Plancherel	Random permutations invariant under conjugation (with a good control on cycles' number)
Biggest particle	T.W	T.W	T.W
Edge	Soft edge (Airy)	Soft edge (Airy)	Soft edge (Airy)
Global convergence	Semi circular	VKLS	VKLS
Fluctuations	Gaussian	Gaussian	??
Bulk	Sine process	Discrete sine process	??

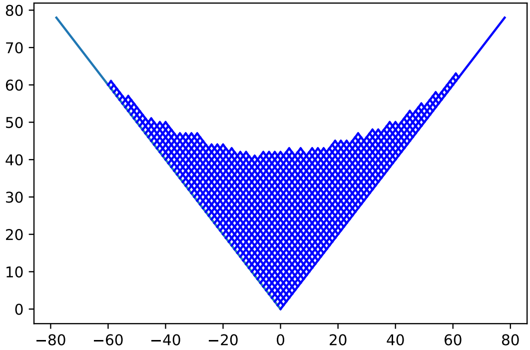
- Independance ~ invarience under conjugation ?
- Moments ~ cycles' structure ?

Conjectures

- We need only $\frac{\#(\sigma_n)}{n^{\frac{1}{2}}} \xrightarrow{\mathbb{P}} 0$ to obtain Tracy-Widom fluctuations.
- For any sequence of random permutations invariant under conjugation, for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{2\sqrt{n}} L_{\lambda(\sigma_n)}\left(2s\sqrt{n - fix(\sigma_n)}\right) - \sqrt{1 - \frac{fix(\sigma_n)}{n}}\,\Omega(s) \right| < \varepsilon \right) = 1.$$

- Under a good control on cycles we have discrete sine process (Bulk).



Thank you for your attention

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