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Asymptotic Symmetries and Soft Photon Theorem

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Abstract

Theories with local symmetries like gravity and abelian gauge theories admit genuine symmetries coming from the set of large gauge transformations of the theory. These symmetry transformations typically lead to constraints on the observables such as $\mathcal{S}$- matrix elements of the corresponding theory. In this project we have learned about a subset of these developments. In particular we review in detail the asymptotic symmetries of Maxwell theories coupled to matter in four dimensional flat Minkowski spacetime. Their consequences to the $\mathcal{S}$-matrix are seen to be the corresponding soft photon theorem.

Contents

Chapter 1

Introduction

The role of asymptotic symmetries are very important in understanding the theory of quantum gravity. In 1962 Bondi, Metzner and Sachs in [1,2] showed that the symmetry group of asymptotically flat spacetime at null infinity was not the usual Poincare group as one would expect since it is an isometry group of Minkowski spacetime but instead an infinite dimensional extension of it which came to be known as BMS group. The BMS group was shown to be the semi-direct product of Lorentz group and abelian group of supertranslations (infinite dimesnional vector space of smooth angle dependent functions defined on S^2 at null infinity) with Poincare group as its subgroup. This was the first example of the sort of symmetry enhancement where the Lie algebra of a killing vector fields defined in the bulk differed with the symmetry algebra on the boundary. After two decades of this discovery Brown and Henneaux in [3] showed that asymptotic symmetry algebra of 3-D gravity with negative cosmological constant is enhanced from $SO(2,2)$ which is the Killing isometry group of ADS_3 to infinite dimensional conformal algebra in two dimension (two copies of Witt algebra) and the charges associated with these asymptotic symmetries obey Virasaro algebra with a central extension to it. The results of the paper [3] hinted at a way of understanding quantum theory of gravity with the tools developed to analyse the infinite dimensional conformal field theory and was one of the first examples of a holographic principle or duality which found a much more concrete footing after Maldacena in [4] proposed AdS/CFT correspondence which conjectured that M/string theory on various Anti-deSitter spacetimes plus boundary conditions is dual to various conformal field theories.

The principle of holography of which AdS/CFT correspondence is the prime example states that information contained inside a spacetime region can be described by data located at the boundary of that region [13]. However the full implications of this principle have yet to be realised but the holographic duality provides a nice set up for asymptotic symmetries of any gravitational theory which may be shown equivalent to the global symmetries of some dual theory living on boundary of manifold. One expect this to hold for the case of flat space holography [5] which currently is an active area of research. From the year 2014 in a series of papers [6, 7] geared towards developing flat space holography Strominger et al showed the equivalence of the Ward identity associated with the symmetry of quantum gravity S- matrix under supertranslations of BMS group with Weinberg's soft graviton theorem [10,11] that relates any S-matrix element in any quantum theory including gravity to a second S-matrix element which differs only by the addition of a graviton in the limit where its four-momentum is taken to zero. Also they proposed that the soft gravitons enter as boundary modes and are manifestly the Goldstone bosons of spontaneously broken supertranslation invariance [5]. Although much simpler relative to the gravitational field the role of asymptotic symmetries in electromagnetic fields were discovered recently in [8,12] where the Ward identity associated with the *large* gauge transformations was shown to be equivalent to Weinberg soft photon theorem for massless scalar QED. In [9] this equivalence was later extended to massive scalar QED.

The aim of this project report is to provide a pedagogical review on the connection between leading soft photon theorems in scalar QED with Ward identity associated with large gauge transformations. The report attempts at studying tools and techniques developed in [8, 9]. Although the number of papers is growing fast in this field with lot of possible future endeavors, with this project we hope to gain a firm understanding with the concept of asymptotic symmetries in scalar QED which would help to carry out further research in this direction.

Asymptotic symmetries of scalar QED can be formally defined as residual gauge transformations which do not die down at infinity and act non trivially on the free data at the boundary. Hence they are called 'large' gauge transformations. Generally gauge transformations are defined as the invariance of Lagrangian of QED under the transformation,

$$A'_\mu = A_\mu + \partial_\mu \lambda. \tag{1.1}$$

λ is the gauge parameter that depends on the coordinates of spacetime. These transformations are not the true symmetry of the Lagrangian in the sense that one cannot derive a Noether charge for these gauge transformations, therefore they are treated as redundancies of physical systems. To rectify this we use the method of gauge fixing. The idea of gauge fixing is to pick out a representative from each equivalence class of vector potential by imposing a condition $F[A] = 0$, where $F[A]$ is some functional of A. Some of the common gauge fixing conditions are Lorentz gauge $\partial_\mu A^\mu = 0$ and radial gauge $A_r = 0$. Only those gauge fields are considered that obey this gauge fixing condition. While studying QED at the boundary of Minkowski spacetime one can define appropriate boundary conditions and fall-off rate of the fields near $\mathscr{I}^+$ to show that under specific choice of gauge fixing conditions there still remain an infinite number of unfixed residual gauge transformations.

These symmetry transformation give infinite number of non zero finite conserved charges at the boundary and the Lie algebra of the charges is shown to be abelian. The Ward identity associated with these symmetries can then be shown equivalent to the soft photon theorem of quantum field theory. The method of defining the boundary of Minkowski spacetime using conformal infinity is reviewed in chapter (2). In chapter (3) the fall-off rate of Maxwell fields are discussed and the conserved charges which generate asymptotic symmetries at the boundary is derived using covariant phase space formulation for massless scalar QED. The calculations that establish the equivalence between Ward identity associated with the large gauge transformations and Weinberg soft photon theorem is reproduced as given in [8]. In chapter (4) this equivalence is reviewed for massive QED which contains additional subtlety that is treated in [9]. These subtleties arise from the fact that massive particles do not reach null infinity and hence a different approach is required to define charges for these fields at the boundary. In chapter (5) the future directions that one can go after familiarizing with the concepts reviewed in this report are discussed. In appendix (A) the saddle point approximation for various fields near null infinity and time-like infinity are derived. In appendix (B) various field commutators using the language of symplectic product is derived. Appendix (C) contains the derivation of soft photon theorem and in appendix (D) the BMS group in $3 + 1$ dimension is reviewed without using on-shell conditions.

Chapter 2

Conformal Infinity

In order to study the electromagnetic radiation of an isolated system near the boundary of Minkowski spacetime, a precise framework and definition of boundary is required for describing mathematical entities as one takes the limit of Maxwell fields to infinity. The construction of conformal infinity of Minkowski spacetime denoted by $\mathcal{M}$ can be done by simply rescaling the metric of $\mathcal{M}$ by a conformal factor Ω^2 and replacing original physical metric by a new unphysical metric $\hat{ds}$ which is conformally related to it [35, 36],

$$\hat{ds}^2 = \Omega^2 \, ds^2. \tag{2.1}$$

Ω is a function defined on $\mathcal{M}$ which is smooth and positive everywhere. The metric tensor and its inverse are scaled appropriately. For a suitable choice of Ω we can add more points to the manifold $\mathcal{M}$ such that the unphysical metric $\hat{ds}^2$ is smoothly defined everywhere, where even ds^2 cannot be extended. To achieve this we demand that the conformal factor goes to zero at these new points so that when our physical metric is infinite at these points then the product of these two is finite (as shown below) and as a result we have a smoothly defined unphysical metric everywhere. In this procedure there is no actual transformation of points on the manifold, one metric is simply replaced by another. For our purposes and discussion of null geodesics this technique is suitable because many physical concepts like causality, massless Klein Gordon field equation and Maxwell field equation in $3+1$ dimensions are invariant under conformal rescaling. In other words we have brought infinity to a finite distance and all the tensor analysis can be done properly without taking any complicated limiting procedure.

In spherical coordinates the metric of Minkowski spacetime is given by

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \tag{2.2}$$

Introducing retarded time u,

$$u = t - r \tag{2.3}$$

The metric then becomes,

$$ds^2 = -du^2 - 2dudr + r^2 \, d\omega_2^2. \tag{2.4}$$

u is a null coordinate which is constant along a null ray and $d\omega_2^2$ is the unit metric on the sphere S^2 . To define the notion of boundary for Minkowski spacetime one needs to make sense of the limit $r \to \infty$ where the metric becomes infinite. Doing a change of coordinates $\Omega = \frac{1}{r}$, the metric becomes

$$ds^2 = -du^2 + \frac{2dud\Omega}{\Omega^2} + \frac{1}{\Omega^2} d\omega_2^2 \tag{2.5}$$

Next we define $d\hat{s}^2$ as,

$$d\hat{s}^2 = \Omega^2 ds^2, \tag{2.6}$$

where $d\hat{s}^2$ is an unphysical metric which is given by multiplying the physical metric ds^2 with a conformal factor Ω^2. The resulting metric is smooth on the original Minkowski manifold and is given by

$$d\hat{s}^2 = -\Omega^2 du^2 + 2dud\Omega + d\omega_2^2 \tag{2.7}$$

This metric is finite in the limit $\Omega \to 0$ and becomes

$$d\hat{s}^2 = 0.du^2 + d\omega_2^2. \tag{2.8}$$

This is a finite metric which is defined on the boundary of Minkowski spacetime for the light rays going into the future and is called $\mathscr{I}^+$ and has the topology $S^2 \times R$. This metric is degenerate because distance between any two points in the u direction is zero. This surface $\mathscr{I}^+$ which is called future null infinity is a null surface which can be shown by calculating the norm of the normal to $\Omega = constant$ surface in the limit $\Omega \to 0$. The normal vector to this surface is,

$$n^\mu = \hat{g}^{\mu\nu}\nabla_\nu\Omega = \hat{g}^{u\Omega}\nabla_\Omega\Omega = (1, 0, 0, 0). \tag{2.9}$$

The norm in the limit $\Omega \to 0$ is given by,

$$n_\mu n^\mu = \hat{g}_{uu}n^u n^u = -\Omega^2 \to 0 \tag{2.10}$$

Since the normal to surface is a null vector therefore $\mathscr{I}^+$ is a null surface where the null geodesic ends. The metric on the sphere in its explicit form is written as

$$d\omega_2^2 = \gamma_{AB}dx^A dx^B, \tag{2.11}$$

where A and B are the coordinates on the sphere (for example θ, ϕ and $z, \bar{z}$). We adopt z and $\bar{z}$ as the coordinates on S^2 which are stereographic coordinates and is related to angular coordinates by $z = e^{i\phi}\tan\frac{\theta}{2}$. The components of a vector on the unit sphere S^2 at null infinity in terms of $z, \bar{z}$ is,

$$\hat{x} = \frac{z + \bar{z}}{1 + z\bar{z}}, \qquad \hat{y} = \frac{-i(z - \bar{z})}{1 + z\bar{z}}, \qquad \hat{z} = \frac{1 - z\bar{z}}{1 + z\bar{z}} \tag{2.12}$$

In these coordinates the metric becomes

$$d\omega_2^2 = 2\gamma_{z\bar{z}}dzd\bar{z} \tag{2.13}$$

where $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. Note that throughout the report both the metric equation (2.11) and equation (2.13) will be used interchangeably. $\mathscr{I}^+$ has boundaries at $u = \pm\infty$ which is denoted as $\mathscr{I}^+_\pm$. This whole exercise can be repeated to get the boundary of Minkowski spacetime in past from where the null rays originate with one change, instead of u we introduce an 'advanced time' defined by,

$$v = t + r \tag{2.14}$$

The metric in Minkowski spacetime then becomes,

$$ds^2 = -dv^2 + 2dvdr + r^2 d\omega_2^2. \tag{2.15}$$

Keeping v constant one can take the limit $r \to \infty$ and reach the past null infinity denoted by $\mathscr{I}^-$. The exact same procedure as mentioned in the previous paragraph can be carried out to see the structure of degenerate metric at $\mathscr{I}^-$ which is same as $\mathscr{I}^+$ except it has an advanced time v instead of retarded time u.

One thing to note is that since null geodesics always travel in straight line at 45 degrees the points on S^2 at future null infinity and past null infinity are antipodally identified since the null geodesic coming out of a point on the sphere at $\mathscr{I}^-$ hits the sphere on $\mathscr{I}^+$ at exactly the opposite point. Using the relation of stereographic coordinates with angular coordinates we see that z at $\mathscr{I}^+$ becomes $-\frac{1}{\bar{z}}$ at $\mathscr{I}^-$ [5].

Chapter 3

Massless QED

The action for massless complex scalar field ϕ coupled with Maxwell field $\mathcal{A}_\mu$ is given by,

$$\mathcal{S} = \int d^4x\sqrt{g}[-g^{\mu\nu}D_\mu\phi(x)(D_\nu\phi(x))^* - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}], \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ and $D_\mu\phi = \partial_\mu - ie\mathcal{A}_\mu\phi$ is the gauge covariant derivative. In this report unless stated otherwise fields are considered in Minkowski spacetime therefore at all times unless stated otherwise $g_{\mu\nu} = \eta_{\mu\nu}$. The equation of motion for Maxwell field by varying the above action is,

$$\nabla^\nu\mathcal{F}_{\nu\mu} = \mathcal{J}^M_\mu. \tag{3.2}$$

$\mathcal{J}^M_\mu = ie[\phi^* D_\mu\phi - \phi(D_\mu\phi)^*]$ is the conserved matter current due to massless charged scalar field and obeys continuity equation $\nabla^\mu\mathcal{J}^M_\mu = 0$. The behavior of fields are examined near $\mathscr{I}^+$ and therefore suitable coordinate system to analyze fields fall-off is $(u, r, z, \bar{z})$. Throughout the report $(z, \bar{z})$ are collectively used interchangeably with $\hat{x}$ which is the short hand notation to denote the coordinates on S^2. The equation of motion has a gauge symmetry under transformation,

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu\hat{\epsilon}(u, r, \hat{x}). \tag{3.3}$$

For gauge fixing of the fields Lorentz gauge is used which imposes the condition,

$$\nabla^\mu\mathcal{A}_\mu = 0. \tag{3.4}$$

Before looking at the fall-off of various fields near $\mathscr{I}^+$, the Christoffel symbols associated with metric $-du^2 - 2dudr + r^2\gamma_{z\bar{z}}dzd\bar{z}$ are [12],

$$\Gamma^z_{rz} = \frac{1}{r}, \quad \Gamma^z_{zz} = \frac{\partial_z\gamma_{z\bar{z}}}{\gamma_{z\bar{z}}}, \quad \Gamma^u_{z\bar{z}} = r\gamma_{z\bar{z}}, \quad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}}. \tag{3.5}$$

3.1 Fall-off of Maxwell Fields near $\mathscr{I}^+$

The expansion of fields near $\mathscr{I}^+$ is given as [12],

$$\begin{aligned}
\lim_{r\to\infty}\mathcal{A}_A &= \sum_{n=0}^{\infty}\frac{A^{(n)}_A(u,\hat{x})}{r^n}\\
\lim_{r\to\infty}\mathcal{A}_u &= \sum_{n=1}^{\infty}\frac{A^{(n)}_u(u,\hat{x})}{r^n}\\
\lim_{r\to\infty}\mathcal{A}_r &= \sum_{n=2}^{\infty}\frac{A^{(n)}_r(u,\hat{x})}{r^n}
\end{aligned} \tag{3.6}$$

To derive these fall-off conditions one can evaluate the saddle point approximation of the Fourier mode expansion in the limit $r \to \infty$ as shown in Appendix (A.2). This gives the fall-off of $\mathcal{A}_u, \mathcal{A}_A$. Once the fall-off of these fields are known then the Lorentz gauge condition gives the fall-off of $\mathcal{A}_r$. Alternatively the long range electric field in radial direction is given by $\mathcal{F}_{ru}$ which should fall-off as

$\mathcal{O}(r^{-2})$ to give finite charges near the boundary using Gauss law. Therefore one can deduce from this that $\mathcal{A}_r$ should fall off as $\mathcal{O}(r^{-2})$ to leading order. The coefficient of fields at various orders near $\mathscr{I}^+$ can be function of other coordinates than r as shown in equation (3.6).

Now it will be shown that the fields $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ are the only free data in the phase space of solutions at the boundary and all the other components of Maxwell fields can be determined from it. Using equation (3.2) and equation (3.5) we can write down the equation for each component of the field in $(u, r, z, \bar{z})$ coordinates as

$$\nabla^r \mathcal{F}_{ru} + \nabla^z \mathcal{F}_{zu} + \nabla^{\bar{z}} \mathcal{F}_{\bar{z}u} = \mathcal{J}_u^M$$

$$(\partial_u - \partial_r)(r^2 \mathcal{F}_{ru}) + \gamma^{z\bar{z}}(\partial_z \mathcal{F}_{u\bar{z}} + \partial_{\bar{z}} \mathcal{F}_{uz}) = r^2 \mathcal{J}_u^M \tag{3.7}$$

$$\nabla^u \mathcal{F}_{ur} + \nabla^z \mathcal{F}_{zr} + \nabla^{\bar{z}} \mathcal{F}_{\bar{z}r} = 0$$

$$-\partial_r(r^2 \mathcal{F}_{ur}) + \gamma^{z\bar{z}}(\partial_{\bar{z}} \mathcal{F}_{zr} + \partial_z \mathcal{F}_{\bar{z}r}) = 0 \tag{3.8}$$

$$\nabla^u \mathcal{F}_{uz} + \nabla^r \mathcal{F}_{rz} + \nabla^{\bar{z}} \mathcal{F}_{\bar{z}z} = 0$$

$$r^2(\partial_u - \partial_r)\mathcal{F}_{rz} + r^2 \partial_r \mathcal{F}_{uz} + \partial_z(\gamma^{z\bar{z}} \mathcal{F}_{z\bar{z}}) = 0 \tag{3.9}$$

For simplicity we have taken $\mathcal{J}_r^M = \mathcal{J}_z^M = 0$ [12]. There will be one more equation of motion for $\mu = \bar{z}$ which we get by replacing $z \leftrightarrow \bar{z}$. The Lorentz gauge condition is given by,

$$\partial_r(r^2 \mathcal{A}_r) - \partial_r(r^2 \mathcal{A}_u) - \partial_u(r^2 \mathcal{A}_r) + \gamma^{z\bar{z}}[\partial_{\bar{z}} \mathcal{A}_z + \partial_z \mathcal{A}_{\bar{z}}] = 0 \tag{3.10}$$

Substituting equation (3.10) in the above equation of motions we get,

$$\partial_r(r^2 \partial_r \mathcal{A}_u) + 2\gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} \mathcal{A}_u - 2r \partial_r(r \partial_u \mathcal{A}_u) = r^2 \mathcal{J}_u^M \tag{3.11}$$

$$-2\partial_r(r \mathcal{A}_u) - 2\partial_u \partial_r(r^2 \mathcal{A}_r) + \partial_r^2(r^2 \mathcal{A}_r) + 2\gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} \mathcal{A}_r = 0 \tag{3.12}$$

$$-2r^2 \partial_u \partial_r \mathcal{A}_z + r^2 \partial_r^2 \mathcal{A}_z + 2r \partial_z(\mathcal{A}_r - \mathcal{A}_u) + 2\partial_z(\gamma^{z\bar{z}} \partial_{\bar{z}} \mathcal{A}_z) = 0 \tag{3.13}$$

Here we can assume that $\mathcal{J}_u^M$ fall-off as $\mathcal{O}(r^{-2})$ so that $r^2 \mathcal{J}_u^M$ is finite. Next we substitute in these equations the fall-off conditions of the field from equation (3.6) and look at the relationship between terms of various order. The Lorentz gauge condition equation (3.10) becomes,

$$\begin{aligned} \partial_u A_r^{(2)} &= \gamma^{z\bar{z}}[\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}] \\ \partial_u A_r^{(3)} &= \gamma^{z\bar{z}}[\partial_{\bar{z}} A_z^{(1)} + \partial_z A_{\bar{z}}^{(1)}] \end{aligned} \tag{3.14}$$

Given $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ one can determine $A_r^{(2)}$ from leading order equation in equation (3.14). To find $A_u^{(1)}$ we use equation equation (3.13),

$$\begin{aligned} \partial_z A_u^{(1)} &= \partial_z(\gamma^{z\bar{z}} \partial_{\bar{z}} A_z^{(0)}) \\ A_u^{(1)} &= \gamma^{z\bar{z}} \partial_{\bar{z}} A_z^{(0)} + C, \end{aligned} \tag{3.15}$$

where C is some integration constant which is function of u. The same thing can be done for the $\bar{z}$ component which is obtained by replacing $z \leftrightarrow \bar{z}$ in the above equation. Finally adding these two equation and differentiating w.r.t to u we get,

$$\partial_u A_u^{(1)} = \gamma^{z\bar{z}} \partial_u[\partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}] \tag{3.16}$$

upto an integration constant. Using this equation we can find leading coefficient A_u in terms of $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$. The higher order coefficient in equation (3.13) are related to each other as,

$$\partial_u A_z^{(1)} - \partial_z A_u^{(1)} + \partial_z(\gamma^{z\bar{z}} \partial_{\bar{z}} A_z^{(0)}) = 0 \tag{3.17}$$

The equation (3.17) is used to find the sub-leading coefficient $A_z^{(1)}$ in terms of $A_u^{(1)}$ and $A_z^{(0)}$ which were already determined before. This $A_z^{(1)}$ in turn determines the sub-leading coefficient $A_r^{(3)}$. The whole point of this exercise is to show that one can use Maxwell equations to derive the relationship between coefficients of various order of fields and that all the components of gauge field $\mathcal{A}_\mu$ can be computed in terms of $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ upto integration constants that follows constraints as dictated by Maxwell equations. $A_z^{(0)}$ and $A_{\bar{z}}^{(0)}$ therefore are the free data on the boundary.

3.2 Large gauge transformations and charge

From the gauge transformations $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \tilde{\lambda}$ there seem to be no specific choice of $\tilde{\lambda}$ which in general can be the function of all coordinates $\tilde{\lambda} = \tilde{\lambda}\,(r, u, z, \bar{z})$ of Minkowski spacetime but this is not the case near $\mathscr{I}^+$. The gauge transformations should preserve fall-off of the field components at the boundary $\mathscr{I}^+$. For $\mu = A$ (coordinates on S^2) we have,

$$\mathcal{A}'_A = \mathcal{A}_A + \partial_A \tilde{\lambda}$$

Substituting the fall-off of $\mathcal{A}_A$ the leading term in the expansion of $\tilde{\lambda}$ should be independent of r but the sub-leading term can be of the order of $\frac{1}{r}$. Therefore one can write $\tilde{\lambda}(u, r, \hat{x}) = \lambda(u, \hat{x}) + \mathcal{O}(1/r)$. For $\mu = u$ we have ,

$$\mathcal{A}'_u = \mathcal{A}_u + \partial_u \tilde{\lambda}$$

From the fall-off condition of the field $\mathcal{A}_u$, one can see that the leading term of $\partial_u \tilde{\lambda}$ should go as $\mathcal{O}(1/r)$ which is only possible if λ is not the function of u but the coefficient of $\mathcal{O}(1/r)$ is u dependent. Similar conclusions along the same line for $\mu = r$ lead us to the following fall off of $\tilde{\lambda}$,

$$\tilde{\lambda}(u, r, \hat{x}) = \lambda(\hat{x}) + \mathcal{O}(1/r) \tag{3.18}$$

Therefore upto the leading order the action of gauge transformation near $\mathscr{I}^+$ take place only for the free fields $A_A^{(0)}$ and is given by,

$$\delta_\lambda A_A(u, z, \bar{z}) = \partial_A \lambda(z, \bar{z}) \tag{3.19}$$

From now $A_A^{(0)}$ will be denoted by A_A. These transformations as mentioned before are unfixed residual gauge transformations which are generated by gauge parameter $\lambda(z, \bar{z})$ defined on the sphere S^2 at the boundary $\mathscr{I}^+$. These transformations are called large gauge transformations because they do not die down at $\mathscr{I}^+$ and give rise to non equivalent field contribution and therefore are real symmetries of the system unlike the gauge transformations defined in Minkowski spacetime. These are what constitutes as the asymptotic symmetries of massless QED. Similar considerations can be done at past null infinity $\mathscr{I}^-$ with the same conclusion. If $\lambda^+(z, \bar{z})$ and $\lambda^-(z, \bar{z})$ are the gauge parameter at $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively then because of antipodal matching condition [5]

$$\lambda^+(z, \bar{z}) = \lambda^-(z, \bar{z}), \tag{3.20}$$

where the two functions are antipodally identified. In the next section we will construct finite non zero infinite number of conserved charges at the boundary for these symmetries.

3.3 Calculation of charges

This section reviews on how to get charges using covariant phase space formulation for a large gauge transformations. In no way this is a complete review. The essential definitions and tools are picked up from the classic paper [14] by Lee and Wald. Many terms will be defined along the way and their meaning in the context of this report will be explained. One starts by writing the action involving general fields,

$$S = \int d^4x\, \mathcal{L}(\phi^a, \partial_\mu \phi^a) \tag{3.21}$$

$\mathcal{L}(\phi^a, \partial_\mu \phi^a)$ is the Lagrangian density for the collection of fields ϕ^a where a label different fields . Varying this action as follows,

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \delta \partial_\mu \phi^a \right]$$

$$\delta S = \int d^4x \left[E(\phi^a) \delta \phi^a + \partial_\mu \Theta^\mu(\delta \phi^a) \right] \tag{3.22}$$

where we have used the technique of integration by parts. One ends up with two terms where $E(\phi^a) = \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right)$ is the Euler-Lagrange(E-L) derivatives which when set equal to zero give E-L equations and,

$$\Theta^\mu(\delta \phi^a) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \delta \phi^a$$

is the symplectic potential current density. For the case of massless complex scalar field coupled to Maxwell field as given in equation (3.1) we get,

$$\begin{aligned}\Theta^\mu(\delta\phi,\delta\phi^*,\delta\mathcal{A}_\mu) =& \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{A}_\nu)}\delta\mathcal{A}_\nu \\ \Theta^\mu(\delta\phi,\delta\phi^*,\delta\mathcal{A}_\mu) =& -\sqrt{g}g^{\mu\nu}(\partial_\nu\phi^*\,\delta\phi + \partial_\nu\phi\,\delta\phi^* + \mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)\end{aligned} \tag{3.23}$$

Using $\Theta^\mu(\delta\phi^a)$ one can construct another quantity called symplectic current density w^μ as,

$$w^\mu(\delta\phi^a,\delta'\phi^a;\phi^a) = \delta\Theta^\mu(\delta'\phi^a) - \delta'\Theta^\mu(\delta\phi^a) \tag{3.24}$$

$\delta\phi$ and $\delta'\phi$ are linearized perturbations and ϕ^a is the background field. If ϕ^a is the solution of field equations and $\delta\phi$ and $\delta'\phi$ are solutions of linearized field equation then w^u is conserved,

$$\partial_\mu w^\mu = 0 \tag{3.25}$$

Using w^μ one can define covariant phase space symplectic product Ω_Σ which is the real valued functional of field variables.

$$\Omega_\Sigma(\delta\phi^a,\delta'\phi^a;\phi) = \int_\Sigma dS_\mu w^\mu, \tag{3.26}$$

where Σ is a Cauchy surface and dS^μ is induced metric times unit normal vector to Σ. Cauchy surface is a surface which is intersected by every causal curve in the spacetime exactly once. If the initial value of field is given on this surface we can exactly determine its future evolution. Substituting equation (3.23) in equation (3.24) we get,

$$w^\mu = -\sqrt{g}[\delta(D^\mu\phi)^*\delta'\phi + \delta(D^\mu\phi)\delta'\phi^* + \delta\mathcal{F}^{\mu\nu}\delta'\mathcal{A}_\nu - \delta\leftrightarrow\delta'] \tag{3.27}$$

The symplectic product for a $t =$ constant Cauchy surface is given by

$$\Omega(\delta,\delta') = \lim_{t\to\infty}\int_{\Sigma_t} d\Sigma_t\, w^t, \tag{3.28}$$

where $dS_\mu = d\Sigma_t n_\mu$ with $n_\mu = \delta^t_\mu$ and $d\Sigma_t$, the invariant volume element of $t =$ constant surface.

The symplectic current density w^t depends on how field are parameterized at t=constant Cauchy surface. Since the fields involved describe massless particles therefore all the action is happening at $\mathscr{I}^+$ so that if one keeps $u = t - r$ constant in the limit $r \to \infty$ then one reaches future null infinity. Since the coordinate system near $\mathscr{I}^+$ is $(u, r, \hat{x})$, therefore using vector transformation law one can get the contribution to w^t from the symplectic current density near the boundary given by,

$$w^t = \frac{\partial t}{\partial r}w^r + \frac{\partial t}{\partial u}w^u = w^r + w^u \tag{3.29}$$

w^r and w^u are components of current density in r and u direction respectively. From equation (3.27) we calculate w^r as,

$$\begin{aligned}w^r =& -r^2\sqrt{\gamma}(\delta(D^r\phi)^*\delta'\phi + \delta(D^r\phi)\delta'\phi^* + \delta\mathcal{F}^{r\nu}\delta'\mathcal{A}_\nu - \delta\leftrightarrow\delta') \\ =& -r^2\sqrt{\gamma}\Big[g^{rr}[\delta(D_r\phi)^*\delta'\phi + \delta(D_r\phi)\delta'\phi^*] + g^{ru}[\delta(D_u\phi)^*\delta'\phi + \delta(D_u\phi)\delta'\phi^*] \\ &+ \delta\mathcal{F}^{r\nu}\delta' A_\nu - \delta\leftrightarrow\delta'\Big]\end{aligned} \tag{3.30}$$

The determiant of metric $g_{\mu\nu}$ is calculated using equation (??) and equation (2.11) and is given by $g = r^4\gamma$, where γ is the determinant of γ_{AB}.

Since we require w^r to be finite as $r \to \infty$ therefore it should not have any dependence on r. From appendix (A.2) we know that massless scalar goes as $\phi = \frac{\phi^b}{r} + \mathcal{O}(1/r^2)$ near $\mathscr{I}^+$. Therefore terms involving $\delta(D_r\phi)$ will be $\mathcal{O}(1/r^2) + \mathcal{O}(1/r^3)$, where the fall-off conditions for gauge field was substituted in the covariant derivative . Therefore any term of the type $\delta\phi^*\delta^{'}(D_r\phi)$ will be proportional to $\mathcal{O}(1/r^3) + \mathcal{O}(1/r^4)$ and the contribution of such terms to w^r will vanish in the limit $r \to \infty$. Similarly the term involving $\delta(D_u\phi)$ will be of $\mathcal{O}(1/r) + \mathcal{O}(1/r^2)$. Therefore terms of the type $\delta(D_u\phi)\delta'\phi^*$ will be of the $\mathcal{O}(1/r^2)$ and $\mathcal{O}(1/r^3)$. The $\mathcal{O}(1/r^2)$ contribution to w^r near $\mathscr{I}^+$ comes only from terms $\delta\phi^b\partial_u\delta'\phi^b$. Following arguments along the similar lines one can verify that the contribution of scalar field to w^u will vanish as $r \to \infty$. The contribution of Maxwell field to w^r is given by,

$$\begin{aligned}
w^r &= -r^2\sqrt{\gamma}(\delta\mathcal{F}^{ru}\delta'\mathcal{A}_u - \delta\mathcal{F}^{rA}\delta'\mathcal{A}_A - \delta \leftrightarrow \delta') \\
w^r &= -r^2\sqrt{\gamma}[g^{ru}g^{ru}(\partial_u\delta\mathcal{A}_r - \partial_r\delta\mathcal{A}_u)\delta'\mathcal{A}_u + g^{ru}g^{AB}(\partial_u\delta\mathcal{A}_B - \partial_B\delta\mathcal{A}_u)\delta'\mathcal{A}_A \\
&\quad - \delta \leftrightarrow \delta']
\end{aligned} \tag{3.31}$$

Substituting the fall-off conditions of gauge fields we see that the first term in the above relation is proportional to $\mathcal{O}(r^{-1})$ which vanishes in the limit $r \to \infty$. Lets look at the second term,

$$w^r = -r^2\sqrt{\gamma}[-\frac{\gamma^{AB}}{r^2}(\partial_u\delta\mathcal{A}_B - \partial_B\delta\mathcal{A}_u)\delta'\mathcal{A}_A) - \delta \leftrightarrow \delta'] \tag{3.32}$$

After substituting fall-off condition we have,

$$w^r = \sqrt{\gamma}\gamma^{AB}\partial_u\delta A_B\delta' A_A - \sqrt{\gamma}\gamma^{AB}\partial_u\delta' A_B\delta A_A \tag{3.33}$$

The term proportional to $\mathcal{A}_u$ drops out. For w^u the contribution of Maxwell field is given by,

$$\begin{aligned}
w^u &= r^2\sqrt{\gamma}(\delta\mathcal{F}^{ur}\delta'\mathcal{A}_r + \delta\mathcal{F}^{uA}\delta'\mathcal{A}_A - \delta \leftrightarrow \delta') \\
w^u &= r^2\sqrt{\gamma}[(g^{ru}g^{ur}\delta\mathcal{F}_{ru}\delta'\mathcal{A}_r) + g^{ur}g^{AB}\delta\mathcal{F}_{rB}\delta' A_A]
\end{aligned} \tag{3.34}$$

Substituting $g^{AB} = \frac{\gamma^{AB}}{r^2}$ and fall off conditions we see that first and second term are $\mathcal{O}(r^{-1})$ and $\mathcal{O}(r^{-2})$ respectively which means that w^u vanishes in the limit $r \to \infty$. Hence the final form for symplectic product as $t \to \infty$ is given by,

$$\Omega_A(\delta, \delta') = \int_{\mathscr{I}^+} \sqrt{\gamma}du(\delta A_A\partial_u\delta' A^A + \partial_u\delta\phi^{b*}\delta'\phi^b + \partial_u\delta\phi^b\delta'\phi^{b*} - \delta \leftrightarrow \delta') \tag{3.35}$$

Choosing $z, \bar{z}$ as the coordinates on the sphere we have [9],

$$\begin{aligned}
\Omega_A(\delta, \delta') = \int_{\mathscr{I}^+} d^2z\, du\gamma_{z\bar{z}}(&\gamma^{z\bar{z}}\delta A_z\partial_u\delta' A_{\bar{z}} + \gamma^{z\bar{z}}\delta A_{\bar{z}}\partial_u\delta' A_z + \partial_u\delta\phi^{b*}\delta'\phi^b \\
&+ \partial_u\delta\phi^b\delta'\phi^{b*} - \delta \leftrightarrow \delta').
\end{aligned} \tag{3.36}$$

The determinant γ in $(z, \bar{z})$ coordinates is given by $\gamma_{z\bar{z}}^2$. To find the conserved charge using the symplectic product we consider one of the variations of the field to be infinitesimal local symmetry variation at the boundary. For large gauge transformation the field variations at $\mathscr{I}^+$ are given by,

$$\delta_\lambda A_A = \partial_A\lambda(z, \bar{z}) \tag{3.37}$$

$$\delta_\lambda\phi = ie\lambda(z, \bar{z})\phi. \tag{3.38}$$

Substituting this in equation (3.36) we have,

$$\begin{aligned}
\delta Q^+ = \Omega_A(\delta, \delta_\lambda) = \int_{\mathscr{I}^+} & d^2z\, du\gamma_{z\bar{z}}[\gamma^{z\bar{z}}\delta A_z\partial_u\partial_{\bar{z}}\lambda(z, \bar{z}) + \gamma^{z\bar{z}}\delta A_{\bar{z}}\partial_u\partial_z\lambda(z, \bar{z}) \\
&+ \partial_u\delta\phi^{b*}ie\lambda(z, \bar{z})\phi^b - \partial_u\delta\phi^b ie\lambda(z, \bar{z})\phi^{b*} - \gamma^{z\bar{z}}\partial_z\lambda(z, \bar{z})\partial_u\delta A_{\bar{z}} \\
&- \gamma^{z\bar{z}}\partial_{\bar{z}}\lambda(z, \bar{z})\partial_u\delta A_z + \partial_u(ie\lambda(z, \bar{z})\phi^{b*})\delta\phi^b - \partial_u(ie\lambda(z, \bar{z})\phi^b\delta\phi^{b*})] \\
= \delta \int_{\mathscr{I}^+} & d^2z\, du[-\partial_z\lambda(z, \bar{z})\partial_u A_{\bar{z}} - \partial_{\bar{z}}\lambda(z, \bar{z})\partial_u A_z + \lambda(z, \bar{z})\gamma_{z\bar{z}}j_u],
\end{aligned} \tag{3.39}$$

where $j_u = ie[\phi^b\partial_u\phi^{b*} - \phi^{b*}\partial_u\phi^b]$ is the matter current passing through $\mathscr{I}^+$. The expression is written as the total variation of some terms in the square bracket. This square bracket gives the expression of total charge at $\mathscr{I}^+$ Integrating by parts and shifting ∂_z on $A_{\bar{z}}$, we get the charge at $\mathscr{I}^+$ in final form as,

$$Q^+ = \int_{\mathscr{I}^+} d^2z\, du\lambda(z, \bar{z})[\partial_z\partial_u A_{\bar{z}} + \partial_{\bar{z}}\partial_u A_z + \gamma_{z\bar{z}}j_u]. \tag{3.40}$$

Equation (3.40) give the infinite number of charges that generate large gauge transformations at $\mathscr{I}^+$. It consists of two parts, a *soft* charge and a hard charge. Hard charge is the matter charge at null infinity which generate large gauge transformations on the massless complex scalar field. The soft charge is the term which is linear in the field. The reason it is called soft charge because the first two terms in equation (3.40) has a piece $\int_{-\infty}^{+\infty} du\, \partial_u A_{\bar{z}}$ which is $E_p \to 0$ limit of,

$$\int_{-\infty}^{+\infty} du\, \partial_u A_z e^{iE_p u} \tag{3.41}$$

This is the Fourier transformation of field and when the theory is quantized it becomes a quantum operator that creates and annihilates photon of specific helicity at $\mathscr{I}^+$. Since the limit $E_p \to 0$, photons that are created or annihilated at $\mathscr{I}^+$ approaches to zero energy and are also called soft photon. Defining ,

$$\int_{-\infty}^{+\infty} du\, \partial_u A_z = N_z = A_z|_{\mathscr{I}^+_+} - A_z|_{\mathscr{I}^+_-} = [A_z] \tag{3.42}$$

N_z is a soft photon mode which is given by the difference of the gauge fields at the boundary of future null infinity. We assume that there are no magnetic charges at the boundary and no long range magnetic fields and therefore we can set $F_{z\bar{z}}|_{\mathscr{I}^+_\pm} = 0$. One can write $N_z = \partial_z N$ and $N_{\bar{z}} = \partial_{\bar{z}} N$ since ,

$$\begin{aligned} \partial_{\bar{z}} N_z - \partial_z N_{\bar{z}} &= \int_{-\infty}^{+\infty} du\, (\partial_{\bar{z}}\partial_u A_z - \partial_z \partial_u A_{\bar{z}}) \\ &= \int_{-\infty}^{+\infty} du\, \partial_u F_{\bar{z}z} = F_{\bar{z}z}|_{\mathscr{I}^+_+} - F_{\bar{z}z}|_{\mathscr{I}^+_-} = 0 \end{aligned} \tag{3.43}$$

so rewriting first two term of the equation (3.40) as,

$$Q^+_{Soft} = 2\int_{\mathscr{I}^+} d^2z\, \lambda(z,\bar{z})\partial_z\partial_{\bar{z}} N \tag{3.44}$$

Therefore total charge becomes,

$$Q^+ = Q^+_{soft} + Q^+_{hard} \tag{3.45}$$

Once the charge is calculated then using the symplectic product at null infinty one can compute various fields commutator. We consider the symplectic product associated only with the soft sector,

$$\Omega_A(\delta,\delta') = \int_{\mathscr{I}^+} d^2z\, du(\delta A_z \partial_u \delta' A_{\bar{z}} + \delta A_{\bar{z}} \partial_u \delta' A_z - \delta' A_z \partial_u \delta A_{\bar{z}} - \delta' A_{\bar{z}} \partial_u \delta A_z) \tag{3.46}$$

Since the boundary fields at $u = \pm\infty$ plays an important role in the theory of gauge transformations, therefore in equation (3.46) one can separate A_z in u dependent term and a u independent term at the boundary of $\mathscr{I}^+$ as follows [5],

$$A_z(u,z,\bar{z}) = \hat{A}_z(u,z,\bar{z}) + \partial_z \Phi(z,\bar{z}), \tag{3.47}$$

where $\partial_z\Phi(z,\bar{z}) = \frac{1}{2}\left[A_z|_{\mathscr{I}^+_+} + A_z|_{\mathscr{I}^+_-}\right]$ gets equal contribution from the boundary terms at $u = \pm\infty$. This term is a pure gauge term hence do not affect equation of motions. Substituting equation (3.47) in equation (3.46) we get,

$$\Omega_A(\delta,\delta') = \Omega_{radiative} + \Omega_{soft} \tag{3.48}$$

$$\Omega_{radiative} = -2\int_{\mathscr{I}^+} d^2z\, du(\delta\partial_u \hat{A}_z \delta' \hat{A}_{\bar{z}} - \delta' \partial_u \hat{A}_z \delta \hat{A}_{\bar{z}})$$

$$\Omega_{soft} = 2\int_{\mathscr{I}^+} d^2z\, (\partial_z \delta\Phi \delta' \partial_{\bar{z}} N - \partial_z \delta' \Phi \delta \partial_{\bar{z}} N) \tag{3.49}$$

Ω_{soft} is the symplectic product with soft photon modes and fields at the boundary and $\Omega_{radiative}$ contains the radiative components $\hat{A}_z$. To obtain commutators from the symplectic form we have to invert it. Since $\Omega_A(\delta,\delta')$ is the sum of symplectic form involving different fields they can be inverted separately. Here results of the various commutator are mentioned. The computation is carried out in Appendix (B).

$$[\partial_u \hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{w}}(u',w,\bar{w})] = -\frac{i}{2}\delta(u-u')\delta^2(z-w) \tag{3.50}$$

$$[\hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{w}}(u',w,\bar{w})] = -\frac{i}{4}\Theta(u-u')\delta^2(z-w) \tag{3.51}$$

$$[\partial_z \Phi(z,\bar{z}), \partial_{\bar{w}} N(w,\bar{w})] = \frac{i}{2}\delta^2(z-w) \tag{3.52}$$

$$[Q^+, A_z] = i\partial_z\lambda(z,\bar{z}), \qquad [Q^+, N(z,\bar{z})] = 0, \qquad [Q^+_\lambda, \hat{A}_z(u,z,\bar{z})] = 0 \tag{3.53}$$

$$[Q^+, \Phi(z,\bar{z})] = i\partial_z\lambda(z,\bar{z}) \qquad [Q^+_{\lambda'}, Q^+_\lambda] = 0 \tag{3.54}$$

$$\tag{3.55}$$

From these commutation relations one can infer that charges Q^+ do indeed generate gauge transformations on the Maxwell fields living at the boundary $\mathscr{I}^+$. These conserved charges are associated with infinite number of symmetries as the parameter $\lambda(z,\bar{z})$ is a function on the sphere S^2 at $\mathscr{I}^+$. As mentioned earlier they are large gauge transformations that do not die off at boundary of Minkowski spacetime. Also the charges are abelian since the commutator of any two charges vanish.

The scalar field like Maxwell fields also gives contribution to the symplectic product at $\mathscr{I}^+$ which is given by equation (3.36). To get the commutators of the field we follow the same procedure as mentioned in the appendix (B) and get,

$$\left[\, \partial_u \phi^{b*},\, \phi^b \,\right] = \frac{i}{2}\gamma^{z\bar{z}}\delta(u-u')\delta^2(z-z') \tag{3.56}$$

$$\left[\, \partial_u \phi^{b},\, \phi^{*b} \,\right] = \frac{i}{2}\gamma^{z\bar{z}}\delta(u-u')\delta^2(z-z') \tag{3.57}$$

To look at the action of Q^+_{hard} on the asymptotic states it is better to work in position space so we can write the general state as,

$$|\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\rangle = \phi^b_1(\vec{x}_1)\phi^b_2(\vec{x}_2), \phi^{b*}_3(\vec{x}_3)..., \phi^b_n(\vec{x}_n)|0\rangle \tag{3.58}$$

Equation (3.58) is our definition of asymptotic state of n particles/antiparticles on the boundary. $\phi^b(\vec{x}_1)$ creates an antiparticle at position $\vec{x}_1$ and $\phi^{b*}(\vec{x}_3)$ creates a particle at position $\vec{x}_3$. The $\vec{x}$ denotes the coordinates $(u,z,\bar{z})$ on the boundary. The total current of n particles j_u passing through $\mathscr{I}^+$ is given by,

$$j_u = \sum_{l=1}^{n} iQ_l[\phi^b_l \partial_u \phi^{b*}_l - \phi^{b*}_l \partial_u \phi^b_l]. \tag{3.59}$$

Therefore the commutator $[\, Q^+_{hard},\, \phi^b_k \,]$ is evaluated as,

$$\begin{aligned}
[\, Q^+_{hard},\, \phi^b_k \,] &= \int_{\mathscr{I}^+} d^2z du \lambda(z,\bar{z})\gamma_{z\bar{z}} \sum_{l=1}^{n} 2iQ_l[\, \phi^b_l \partial_u \phi^{b*},\, \phi^b_k \,] \\
[\, Q^+_{hard},\, \phi^b_k \,] &= -\, \lambda(z,\bar{z})Q_k \phi^b_k(u,z,\bar{z}) = i\delta_\lambda \phi^b_k
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
[\, Q^+_{hard},\, \phi^{*b}_k \,] &= -\int_{\mathscr{I}^+} d^2z du \lambda(z,\bar{z})\gamma_{z\bar{z}} \sum_{l=1}^{n} 2iQ_l[\, \phi^{*b}_l \partial_u \phi^{b},\, \phi^{*b}_k \,] \\
[\, Q^+_{hard},\, \phi^{*b}_k \,] &= \lambda(z,\bar{z})Q_k \phi^b_k(u,z,\bar{z}) = -i\delta_\lambda \phi^b_k
\end{aligned} \tag{3.61}$$

These commutation relations also confirm that Q^+_{hard} charge at the boundary generate large gauge transformations on complex scalar field. Therefore it is shown that Q^+ are the generators of large gauge transformations on $\mathscr{I}^+$ for both complex scalar and Maxwell field.

3.4 Asymptotic structure at $\mathscr{I}^-$

Much of the structure of Maxwell field remains same near $\mathscr{I}^-$. The procedure to reach $\mathscr{I}^-$ is already defined in section (2). All the fall-off of fields are identical except now free field at the boundary is function of v (advanced time) instead of u(retarded time). The points on the sphere at $\mathscr{I}^-$ are identified antipodally with the points on $\mathscr{I}^+$. The associated charge at $\mathscr{I}^-$ is given by,

$$Q^- = \int_{\mathscr{I}^-} d^2z\, dv \lambda(z,\bar{z})[\partial_z \partial_v A_{\bar{z}} + \partial_{\bar{z}} \partial_v A_z + \gamma_{z\bar{z}} j_v] \tag{3.62}$$

The boundary field is given by $\partial_z N^-$ and therefore the soft charge becomes,

$$Q^-_{Soft} = 2\int_{\mathscr{I}^-} d^2z\, \lambda(z,\bar{z}) \partial_z \partial_{\bar{z}} N^-. \tag{3.63}$$

To make sense of the scattering problem there should be some kind of relation between charges defined at $\mathscr{I}^+$ and $\mathscr{I}^-$. Just as in equation (3.20), the relation between Q^+ and Q^- is given by,

$$Q^+ = Q^- \tag{3.64}$$

The charge at $\mathscr{I}^+$ is equal to charge at $\mathscr{I}^-$ and they generate symmetry transformation on $\mathcal{S}-$ matrix.

3.5 Ward Identity

Given a conserved charge in a quantum system one can derive something similar to conservation law called Ward identity where the insertion of the Noether current (associated with the symmetry) in the correlation function give rise to contact terms. Here Ward identity is derived as the symmetry of $\mathcal{S}$- matrix. Since conserved charges commute with the Hamiltonian, it should also commute with the $\mathcal{S}$- matrix which is proportional to the $exp(iHT)$. A general scattering amplitude in quantum field theory is defined as,

$$\langle out|\mathcal{S}|in\rangle \tag{3.65}$$

$\mathcal{S}$ is an operator defined as $\lim_{T\to\infty(1-i\epsilon)} \mathcal{T}\left[exp^{-i\int_{-T}^{T} H_I(t)}\right]$, where $\mathcal{T}$ is the time ordered product. Under finite transformations charge conservation implies,

$$\langle out|\mathcal{S}'|in\rangle = \left\langle out|e^{i\lambda Q^+}\mathcal{S}e^{i\lambda Q^-}|in\right\rangle = \langle out|\mathcal{S}|in\rangle \tag{3.66}$$

For infinitesimal transformation we expand the exponential to first order in λ,

$$\left\langle out|Q^+\mathcal{S} - \mathcal{S}Q^-|in\right\rangle = 0 \tag{3.67}$$

Writing the above expression in terms of soft and hard charges we get,

$$\left\langle out|Q^+_{soft}\mathcal{S}|in\right\rangle - \left\langle out|\mathcal{S}Q^-_{soft}|in\right\rangle = \langle out|\mathcal{S}Q^-_{hard}|in\rangle - \langle out|Q^+_{hard}\mathcal{S}|in\rangle \tag{3.68}$$

From equation (3.44) we substitute the value of soft charge to get,

$$\begin{aligned} -2\int d^2z \partial_{\bar{z}}\lambda \left\langle out|\partial_z N^+\mathcal{S}|in\right\rangle + 2\int d^2z \partial_{\bar{z}}\lambda \left\langle out|\mathcal{S}\partial_z N^-|in\right\rangle \\ = \left\langle out|\mathcal{S}Q^-_{hard} - Q^+_{hard}\mathcal{S}|in\right\rangle \end{aligned} \tag{3.69}$$

Now choosing $\lambda(w,\bar{w}) = \frac{1}{z-w}$ and using the identity $\partial_{\bar{z}} \frac{1}{z-w} = 2\pi\delta^2(z-w)$ the LHS term after carrying out an integration over all z becomes,

$$-4\pi \left\langle out|(\partial_w N^+\mathcal{S} - \mathcal{S}\partial_w N^-)|in\right\rangle \tag{3.70}$$

This is just the expectation value of soft charge that has picked out a special direction w on the asymptotic S^2 i.e the soft photon is either created or annihilated at this fixed angle. After relabeling w as z the final form of equation (3.69) is,

$$-4\pi \left\langle out|\partial_z N^+\mathcal{S} - \mathcal{S}\partial_z N^-|in\right\rangle = \left\langle out|\mathcal{S}Q^-_{hard} - Q^+_{hard}\mathcal{S}|in\right\rangle \tag{3.71}$$

From equation (3.41) and equation (3.42) we define the soft mode field for energy greater than zero as,

$$\partial_z N = \lim_{E_p \to 0} \frac{1}{2} \int_{-\infty}^{+\infty} du (e^{iE_p u} + e^{-iE_p u}) \partial_u A_z \tag{3.72}$$

Just a reminder, here $A_z = A_z^{(0)}$ is the free field at the boundary and leading term of the expansion of field $\mathcal{A}_z$ near $\mathscr{I}^+$. In appendix (A.1) A_z is calculated using saddle point method [5] and is given by,

$$A_z = \frac{-i\sqrt{2}}{8\pi^2(1+z\bar{z})} \int_0^\infty d\omega \left[a_+(\omega\hat{x}) e^{-iu\omega} - a_-^\dagger(\omega\hat{x}) e^{iu\omega} \right] \tag{3.73}$$

Substituting this in equation (3.72) we get,

$$\partial_z N^{+/-} = \lim_{E_p \to 0} - \frac{\sqrt{2}}{8\pi(1+z\bar{z})} [E_p a_+^{out/in}(E_p\hat{x}) + E_p a_-^{\dagger\, out/in}(E_p\hat{x}) \tag{3.74}$$

Upon quantisation the soft mode field is written in terms of creation and annihilation operator that create or annihilate soft photons at asymptotic sphere S^2 at the boundary. Substituting equation (3.74) in equation (3.71) L.H.S becomes,

$$\begin{aligned} \frac{1}{\sqrt{2}(1+z\bar{z})} \Big[&\Big\langle out | E_p a_+^{out}(E_p\hat{x}) \mathcal{S} + E_p a_-^{\dagger\, out}(E_p\hat{x}) \mathcal{S} | in \Big\rangle \\ &- \Big\langle out | \mathcal{S} E_p a_+^{in}(E_p\hat{x}) + S E_p a_-^{\dagger\, in}(E_p, \hat{x}) | in \Big\rangle \Big] \end{aligned} \tag{3.75}$$

$a_-^{\dagger\, out}(E_p, \hat{x})$ acting on $\langle out|$ and $a_+^{in}(E_p\hat{x})$ acting on $|in\rangle$ will annihilate the state and therefore LHS becomes,

$$\frac{1}{\sqrt{2}(1+z\bar{z})} \lim_{E_p \to 0} \Big\langle out | (E_p a_+^{out}(E_p\hat{x}) \mathcal{S} - \mathcal{S} E_p a_-^{\dagger\, in}(E_p, \hat{x})) | in \Big\rangle \tag{3.76}$$

From the crossing symmetry which relates the amplitude of soft photon going out of the external leg and the amplitude of soft photon going in the external leg one gets,

$$\lim_{E_p \to 0} \langle out | E_p a_+^{out}(E_p\hat{x}) \mathcal{S} | in \rangle = \lim_{E_p \to 0} - \langle out | \mathcal{S} E_p a_-^{\dagger\, in}(E_p, \hat{x}) | in \rangle. \tag{3.77}$$

Using above equation LHS finally becomes,

$$\frac{\sqrt{2}}{(1+z\bar{z})} \lim_{E_p \to 0} \langle out | E_p a_+^{out}(E_p\hat{x}) \mathcal{S} | in \rangle \tag{3.78}$$

This says that the final state will have a positive helicity outgoing photon towards a fixed angle on S^2 at $\mathscr{I}^+$.

Now in the scattering of m incoming particles or antiparticles coming from $\mathscr{I}^-$ and n particles or antiparticles going towards $\mathscr{I}^+$ we see in RHS of equation (3.71) that Q^-_{hard} which is charge defined at $\mathscr{I}^-$ acts on the $|in\rangle$ state as follows,

$$Q^-_{hard} |in\rangle = \sum_{k=1}^{m} \lambda(z_k^{in}, \bar{z}_k^{in}) Q_k^{in} |in\rangle \tag{3.79}$$

The action is determined using equation (3.61) and Q_k^{in} is positive for particles and negative for antiparticles. Similarly Q^+_{hard} which is charge defined at $\mathscr{I}^+$ acts on $\langle out|$ state as

$$\langle out | Q^+_{hard} = \langle out | \sum_{k=1}^{n} \lambda(z_k^{out}, \bar{z}_k^{out}) Q_k^{out} \tag{3.80}$$

Q_k^{out} is positive for particles and negative for antiparticles. z_k^{in} labels the point on the asymptotic sphere at past null infinity from where the particles are incoming. Similarly z_k^{out} labels the point on the sphere at $\mathscr{I}^+$ where the outgoing particle will end up. Collecting all the results together equation (3.71) can be written as,

$$\lim_{E_p\to\infty}\langle out|E_p a_+^{out}(E_p\hat{x})\mathcal{S}|in\rangle = \frac{(1+z\bar{z})}{\sqrt{2}}\left[\sum_{k=1}^{m}\lambda(z_k^{in},\bar{z}_k^{in})Q_k^{in} - \lambda(z_k^{out},\bar{z}_k^{out})Q_k^{out}\right] \langle out|\mathcal{S}|in\rangle \tag{3.81}$$

Now we substitute $\lambda(z_k,\bar{z}_k) = \frac{1}{z_k - z}$ and get final expression of Ward identity,

$$\lim_{E_p\to\infty}\left[E_p\langle out|a_+^{out}(E_p\hat{x})\mathcal{S}|in\rangle\right] = \frac{(1+z\bar{z})}{\sqrt{2}}\left[\sum_{k=1}^{n}\frac{Q_k^{out}}{z-z_k^{out}} - \sum_{k=1}^{m}\frac{Q_k^{in}}{z-z_k^{in}}\right]\langle out|\mathcal{S}|in\rangle \tag{3.82}$$

This is the expression of Ward identity associated with the large gauge transformations at the boundary $\mathscr{I}^+$ [8].

3.6 Soft Photon Theorem

The soft photon theorem plays a very important role in removing the infrared divergences from the QED theories at cross section level. The addition of amplitude of emission of soft photon in the final state during a scattering event exactly cancels the infrared divergences caused by the virtual photons in QED scattering event [10]. Soft photon theorem as given by Weinberg relates the amplitude of m incoming particle and n outgoing particle with the amplitude of m incoming particle and n outgoing particle with addition of one photon in the final state whose energy tends to zero. The formula for soft photon theorem is given by ,

$$M^{\mu}_{\beta\alpha}(q) \to M_{\beta\alpha}\sum_n \frac{\eta_n e_n p_n^{\mu}}{p_n.q - i\eta_n\epsilon}, \tag{3.83}$$

This formula is derived in appendix (C). The index α represent the event of m incoming particles and β represent the event of n outgoing particles. $M^{\mu}_{\beta\alpha}(q)$ is the S-matrix for emitting an extra single photon in final state with four momentum q and polarization index μ in the process $\alpha \to \beta$ in the limit when $q \to 0$. η_n is a sign factor with with value $+1$ for particles in final state β and -1 for particles in initial state α. For outgoing photon of positive helicity we contract with the polarisation tensor of positive helicity given by ϵ_μ^+ defined in equation (A.7). Therefore the expression becomes,

$$M^{+}_{\beta\alpha}(q) \to M_{\beta\alpha}\sum_n \frac{\eta_n e_n p_n.\epsilon^+}{p_n.q} \tag{3.84}$$

To recast in a more familiar form we make following changes,

$$M^{+}_{\beta\alpha}(q) = \langle out|a_+^{out}(\vec{q})\mathcal{S}|in\rangle \tag{3.85}$$

$$M_{\beta\alpha} = \langle out|\mathcal{S}|in\rangle. \tag{3.86}$$

Putting labels on the outgoing and incoming four vector momentum of the particles and multiplying both the sides by E_p which is the energy carried by the photon the expression in equation (3.84) in the limit $E_p \to 0$ becomes,

$$\lim_{E_p\to 0}\left[E_p\langle out|a_+^{out}(\vec{q})\mathcal{S}|in\rangle\right] = \left[\sum_{k=1}^{n}E_p\frac{Q_k^{out}p_k^{out}.\epsilon^+}{p_k^{out}.q} - \sum_{k=1}^{m}E_p\frac{Q_k^{in}p_k^{in}.\epsilon^+}{p_k^{in}.q}\right]\langle out|\mathcal{S}|in\rangle \tag{3.87}$$

From saddle point conditions discussed in equation (A.2) we know that in late time limit and $r \to \infty$ the momenta of the massless particle asymptotes to the sphere S^2 at null infinity. The unit vector on the sphere in terms of stereographic coordinates $(z, \bar{z})$ is given in equation (2.12) and therefore using this parameterization we have the following definition of the momenta,

$$
\begin{aligned}
(p_k^{in})^\mu &= E_k^{in}\big(1, \hat{x}_k^{in}\big) && (3.88)\\
(p_k^{out})^\mu &= E_k^{out}\big(1, \hat{x}_k^{out}\big)\\
q^\mu &= E_p\big(1, \hat{x}\big)\\
\epsilon_\mu^+(q) &= \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}) && (3.89)
\end{aligned}
$$

The polarization vector should obey $q^\mu \epsilon_\mu^+ = 0$. Substituting above expressions in the formula of soft photon theorem we see that,

$$
\lim_{E_p \to 0}\Big[E_p\langle out|a_+^{out}(E_p\hat{x})\mathcal{S}|in\rangle\Big] = \frac{(1+z\bar{z})}{\sqrt{2}}\Bigg[\sum_{k=1}^{n}\frac{Q_k^{out}}{z - z_k^{out}} - \sum_{k=1}^{m}\frac{Q_k^{in}}{z - z_k^{in}}\Bigg]\langle out|\mathcal{S}|in\rangle
$$

This equation is same as equation (3.82). Therefore we have shown that Ward identity associated with large gauge transformations at the boundary is equivalent to soft photon theorem.

Chapter 4

Massive QED

The action of free massive charged scalar field is given by [9],

$$\mathcal{S} = \int d^4x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi^*(x) - m^2\phi(x)\phi^*(x)] \tag{4.1}$$

Varying this action one get the equation of motion for $\phi(x)$,

$$-g^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi(x) = 0 \tag{4.2}$$

The solution of this equation is,

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}(b(\vec{p})e^{ip.x} + d^*(\vec{p})e^{-ip.x}) \tag{4.3}$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2}$ and $p.x = -E_p t + \vec{p}.\vec{x}$. This is the classical field solution and in quantum field theory $b(\vec{p})$, $d^*(\vec{p})$ are interpreted as annihilation operators for particles and creation operators for antiparticles respectively.

The solutions are given in terms of $b(\vec{p})$, $d(\vec{p})$ which are the free data in the space of solutions and hence taken as coordinates of the asymptotic phase space defined at some late time asymptote of free complex scalar field below. If the particles were massless then the late time asymptotic of the field is defined at null infinity. As in the case of massless QED there should be a boundary like $\mathscr{I}^+$, a manifold structure where the phase space of asymptotic massive scalar field can be defined. A particle of non zero mass moves along timelike geodesic and end its trajectory at timelike infinity i^+ but since i^+ is a point , a effective manifold structure like $\mathscr{I}^+$ cannot be defined there. This poses a problem which can be bypassed with a coordinate system adopted for this purpose. The trajectory of massive particle is restricted to the inside of light cone and hence obeys the equation

$$-t^2 + r^2 < 0,$$

where $r = \sqrt{|\vec{x}|^2 + |\vec{y}|^2| + |\vec{z}|^2}$. One consider the slicing of Minkowski space by introducing the coordinate τ such that

$$-t^2 + r^2 = -\tau^2, \tag{4.4}$$

where $\tau^2 > 0$. Each constant τ describes a hyperbolic slice and therefore Minkowski space inside the light cone is foliated by these hyperbolic slices. Introducing another coordinate ρ which is defined as

$$t = \sqrt{1+\rho^2}\,\tau, \qquad r = \tau\rho \tag{4.5}$$

The coordinates $x^i = r\hat{x}^i$, where $\hat{x}^i$ are the components of vector defined on the sphere S^2. The Minkowski metric in terms of these coordinates is given by,

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2\gamma_{AB}dx^A dx^B \\ ds^2 &= -d\tau^2 + \tau^2\left(\frac{d\rho^2}{1+\rho^2} + \rho^2\gamma_{AB}dx^A dx^B\right) \end{aligned}$$

where A, B are indices on S^2. As mentioned earlier they can be stereographic coordinates like $(z, \bar{z})$ or angular coordinates like (θ, ϕ). This metric describes the Minkowski spacetime inside the light cone with τ describing the hyperbolic slice and $\frac{d\rho^2}{1+\rho^2} + \rho^2\gamma_{AB}dx^A dx^B$ is the induced metric on the unit hyperboloid $\mathcal{H}^+$ which is spacelike. The hyperboloid $\mathcal{H}^+$ is parameterised by the coordinates ρ and x_A. The $\mathcal{H}^+$ is a suitable candidate for a manifold that describes the asymptotic phase space of the massive field since the time like infinity i^+ can be represented by a manifold $\mathcal{H}^+$ in the limit $\tau \to \infty$.

Before proceeding forward one should look at what the late time limit $t \to \infty$ means for the massive particle. The trajectory of the massive particle with constant momentum is given by $\vec{r} = \frac{\vec{p}}{E}t + \vec{r_0}$ where $\vec{r_0}$ is the initial position vector and $\vec{p}, E$ are the components of four momentum vector $p^\mu = mu^\mu$ with u being the four velocity given by $u = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z)$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$. The new coordinates as $t \to \infty$ are given as [5]

$$\rho = \frac{r}{\sqrt{t^2 - r^2}} = \frac{\sqrt{(\frac{\vec{p}}{E}t + \vec{r_0})^2}}{\sqrt{t^2 - (\frac{\vec{p}}{E}t + \vec{r_0})^2}} \xrightarrow{t\to\infty} \frac{\sqrt{(\frac{\vec{p}}{E})^2}}{\frac{\sqrt{E^2-\vec{p}^2}}{E}} = \frac{|\vec{p}|}{m} \tag{4.6}$$

$$\tau = \sqrt{t^2 - r^2} = \sqrt{t^2 - \left(\frac{\vec{p}t}{E} + \vec{r_0}\right)^2} \xrightarrow{t\to\infty} t\sqrt{\left(\frac{E^2 - \vec{p}^2}{E^2}\right)} = t\frac{m}{E} \tag{4.7}$$

$$\hat{x} = \frac{\vec{r}}{r} = \frac{\frac{\vec{p}t}{E} + \vec{r_0}}{|\frac{\vec{p}t}{E} + \vec{r_0}|} = \frac{\vec{p}t + E\vec{r_0}}{\sqrt{(\vec{p}t + E\vec{r_0})^2}} = \frac{\vec{p} + \frac{E\vec{r_0}}{t}}{\sqrt{(\vec{p} + \frac{E\vec{r_0}}{t})^2}} \xrightarrow{t\to\infty} \frac{\vec{p}}{|\vec{p}|} \tag{4.8}$$

As can be seen the late time asymptotic of the field is taken as $\tau \to \infty$ and in this limit ρ asymptote to $|\vec{p}|$ which is constant for a free particle. Also the unit vector on S^2 $\hat{x}$ asymptote to a fixed unit vector on S^2 in momentum space $\hat{p}$. This means that as $t \to \infty$ the particle reaches to the fixed position on the unit hyperboloid $\mathcal{H}_+$ which is proportional to its constant momentum.

As mentioned before the late time asymptote of scalar field is calculated in the limit $\tau \to \infty$. One can evaluate the field in this limit using saddle point approximation as shown in equation (A.1). The final result is ,

$$\phi(x) = \frac{e^{-3i\pi/4}\sqrt{m}}{2(2\pi\tau)^{3/2}}(b(m\rho\hat{x})e^{-i\tau m} + id^*(m\rho\hat{x})e^{i\tau m}) + \mathcal{O}(\tau^{-5/2}). \tag{4.9}$$

Given the asymptotic behavior of the field in equation (4.9), the next step is to write down the covariant phase space symplectic product on a Cauchy surface in Minkowski spacetime.

4.1 Covariant phase space formulation

For free complex scalar field theory the w^μ is evaluated as follows,

$$\begin{aligned} w^\mu =& -\sqrt{g}g^{\mu\nu}[\delta(\partial_\nu\phi^*\,\delta'\phi + \partial_\nu\phi\,\delta'\phi^*) - \delta'(\partial_\nu\phi^*\,\delta\phi + \partial_\nu\phi\,\delta\phi^*)] \\ =& -\sqrt{g}g^{\mu\nu}[\partial_\nu\delta\phi^*\delta'\phi + \partial_\nu\delta\phi\delta'\phi^* - \partial_\nu\delta'\phi^*\delta\phi - \partial_\nu\delta'\phi\delta\phi^*] \end{aligned} \tag{4.10}$$

It is assumed that ∂_ν and δ commute and also $[\delta, \delta'] = 0$. Since one is interested in evaluation of symplectic product Ω_Σ for τ=constant in the limit $\tau \to \infty$ it is required that the integral is defined on the Cauchy surface Σ but $\tau = constant$ hyperboloid is not a Cauchy surface since the light like causal curves will reach null infinity $\mathscr{I}^+$ without ever crossing it once. Therefore one needs to define proper Cauchy surface on which symplectic product can be evaluated. For this purpose the coordinate u can be written in terms of ρ and τ. Therefore,

$$u = \tau\sqrt{1+\rho^2} - \rho\tau \tag{4.11}$$

where the definition of retarded coordinate $u = t - r$ is used. At constant τ surface one sees that taking the limit $\rho \to \infty$ is equal to $u = 0$, therefore one can conclude that the boundary of any constant hyperboloid intersect with the $u = 0$ point at $\mathscr{I}^+$. Hence to define Cauchy surface one has to take note of the light like causal curves which for $u > 0$ will have to pass through the hyperboloid $\mathcal{H}^+$ but for $u < 0$ it can reach null infinity without ever crossing $\mathcal{H}^+$ therefore in order to completely describe the Cauchy surface one needs to include $u < 0$ portion of $\mathscr{I}^+$. The resultant Cauchy surface Σ_τ is then defined as [9]

$$\Sigma_\tau := t^2 - r^2 = \tau^2 \quad \text{for} \quad t \geq r \tag{4.12}$$

$$\Sigma_\tau := \mathscr{I}^+ \quad \text{for} \quad t < r \tag{4.13}$$

To consider the contribution of $\mathscr{I}^+$ to symplectic product one needs to look at the asymptote of massive scalar field near $\mathscr{I}^+$. It can be shown that in this limit $\phi(x)$ falls as $\mathcal{O}(r^{-3/2})$ and substituting this in equation (4.10) the leading order of $w^r(\delta, \delta')$ is $\mathcal{O}(r^{-1})$. Note that $\sqrt{g}$ contains a factor proportional to r^2. Since the volume element on $\mathscr{I}^+$ is simply $\gamma_{z\bar{z}} du\, d^2z$ we see that in limit $r \to \infty$ the contribution vanishes. The sole contribution of massive scalar field to the symplectic product comes from the hyperboloid $\mathcal{H}^+$. To evaluate τ component of symplectic current one requires the Minkowski metric in the coordinates (ρ, τ) which is given by,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{\tau^2}{1+\rho^2} & 0 & 0 \\ 0 & 0 & \tau^2\rho^2\gamma_{11} & \tau^2\rho^2\gamma_{12} \\ 0 & 0 & \tau^2\rho^2\gamma_{21} & \tau^2\rho^2\gamma_{22} \end{pmatrix} \tag{4.14}$$

The determinant of $g_{\mu\nu}$ is,

$$det(g_{\mu\nu}) = -1\left[\frac{\tau^2}{1+\rho^2}\big(\tau^4\rho^4(\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21})\big)\right] = \frac{-\tau^2}{1+\rho^2}\big(\tau^4\rho^4\gamma\big). \tag{4.15}$$

Therefore $\sqrt{g} = \sqrt{|det(g_{\mu\nu})|} = \frac{\tau^3\rho^2\sqrt{\gamma}}{\sqrt{1+\rho^2}}$. So w^τ is given by,

$$w^\tau = -\sqrt{g}g^{\tau\tau}(\partial_\tau\delta\phi^*\delta'\phi + \partial_\tau\delta\phi\delta'\phi^* - \partial_\tau\delta'\phi^*\delta\phi - \partial_\tau\delta'\phi\delta\phi^*) \tag{4.16}$$

To evaluate this we need,

$$\delta\phi = \frac{e^{-i3\pi/4}\sqrt{m}}{2(2\pi\tau)^{3/2}}(\delta b(m\rho\hat{x})e^{-i\tau m} + i\delta d^*(m\rho\hat{x})e^{i\tau m}) \tag{4.17}$$

$$\begin{aligned}\partial_\tau\phi =& \frac{-3e^{-3i\pi/4}\sqrt{m}}{4(2\pi)^{3/2}\tau^{5/2}}(b(m\rho\hat{x})e^{-3i\tau m} + id^*(m\rho\hat{x})e^{i\tau m}) + \\ & \frac{e^{-i\pi/4}\sqrt{m}}{2(2\pi\tau)^{3/2}}(-imb(m\rho\hat{x})e^{-i\tau m} - md^*(m\rho\hat{x})e^{i\tau m})\end{aligned} \tag{4.18}$$

Substituting equation (4.18) in equation (4.16) and looking at first few terms we have

$$\begin{aligned}\partial_\tau\delta\phi^*\delta'\phi + \partial_\tau\delta\phi\delta'\phi^* = \frac{-3m}{8(2\pi)^3\tau^4}\Big[\delta b\delta' b^* + \delta b^*\delta' b + \delta d\delta' d^* + \delta d^*\delta' d + \\ (i\delta d^*\delta' b^* + i\delta b^*\delta' d^*)e^{2i\tau m} - i(\delta b\delta' d + \delta d\delta' b)e^{-2i\tau m}\Big] + \frac{m}{4(2\pi\tau)^3}\Big[im(\delta b^*\delta' b - \delta b\delta' b^*) \\ -e^{-2i\tau m}(m\delta b\delta' d + m\delta d\delta' b) - e^{2i\tau m}(m\delta d^*\delta' b^* + \delta b^*\delta' d^*) + im(\delta d^*\delta' d - \delta d\delta' d^*)\Big]\end{aligned} \tag{4.19}$$

Similarly one can calculate $\partial_\tau\delta'\phi^*\delta\phi + \partial_\tau\delta'\phi\delta\phi^*$ by interchanging δ and δ' in equation (4.19). Only the terms proportional to $\mathcal{O}(\tau^{-3})$ survives. After substituting the expression for $\sqrt{g}$ and $g^{\tau\tau} = -1$ in equation (4.16) the final form of w^τ is,

$$w^\tau = \frac{im^2\sqrt{\gamma}\rho^2(1+\rho^2)^{-1/2}}{2(2\pi)^3}\big[\delta' b\delta b^* + \delta d^*\delta' d - \delta b\delta' b^* - \delta d\delta' d^*\big] \tag{4.20}$$

For τ=constant surface, the unit normal vector is given by $n_\mu = \partial_\mu\tau = \delta^\tau_\mu$ and therefore $dS_\mu = d\rho\, d^2z\, \delta^\tau_\mu$. Finally the symplectic product is given by

$$\lim_{\tau\to\infty}\Omega_{\Sigma_\tau}(\delta, \delta') = \frac{im^2}{2(2\pi)^3}\int_{\mathcal{H}_+} d^3V\big[\delta' b\delta b^* + \delta d^*\delta' d - \delta b\delta' b^* - \delta d\delta' d^*\big], \tag{4.21}$$

where $d^3V = \sqrt{\gamma}\rho^2(1+\rho^2)^{-1/2}d\rho\, d^2z$ is the volume element on $\mathcal{H}^+$. The complex fields (b, d) are the free field data and form the space of solutions called as asymptotic phase space. They can be viewed as functions on manifold $\mathcal{H}^+$. This phase space is denoted by Γ^ϕ. The Poisson bracket between two free fields (b, b^*) is calculated using symplectic product. The calculation has been done in appendix (B). The results are as follows,

$$\{b(m\rho\hat{x}), b^*(m\rho'\hat{x'})\} = -\frac{2(2\pi)^3}{im^2}\delta(\rho-\rho')\delta^2(\hat{x}-\hat{x'}) \tag{4.22}$$

$$\{d(m\rho\hat{x}), d^*(m\rho'\hat{x'})\} = -\frac{2(2\pi)^3}{im^2}\delta(\rho-\rho')\delta^2(\hat{x}-\hat{x'}) \tag{4.23}$$

To convert this Poisson bracket in the momentum space substitute $\vec{p} = m\rho\hat{x}$ and $\vec{p'} = m\rho'\hat{x}'$ (Saddle point conditions). One would also need to transform Dirac delta function from $\delta(\rho-\rho')\delta^2(\hat{x}-\hat{x'})$ to $\delta^3(\vec{p}-\vec{p}')$. In order to accomplish this one needs to find the Jacobian of the transformation. From saddle point approximation we have,

$$\vec{p} = m\rho\hat{x} = \frac{m\rho}{1+z\bar{z}}(z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}), E = m\sqrt{1+\rho^2}. \tag{4.24}$$

One thing to tak note is that if we choose $(z, \bar{z})$ as the coordinates on the sphere then the metric becomes,

$$ds^2 = -d\tau^2 + \tau^2\left(\frac{d\rho^2}{1+\rho^2} + 2\rho^2\gamma_{z\bar{z}}dzd\bar{z}\right) \tag{4.25}$$

In this case $\sqrt{\gamma}$ is $\gamma_{z\bar{z}}$. Now getting back to the problem at hand , the Jacobian $|J|$ between the coordinate transformation $(\rho, \hat{x})$ to $\vec{p}$ is given by the inverse of Jacobian $|J'|$ between coordinate transformation $\vec{p}$ to $(\rho, \hat{x})$

$$J' = \begin{vmatrix} \frac{\partial p_1}{\partial \rho} & \frac{\partial p_2}{\partial \rho} & \frac{\partial p_3}{\partial \rho} \\ \frac{\partial p_1}{\partial z} & \frac{\partial p_2}{\partial z} & \frac{\partial p_3}{\partial z} \\ \frac{\partial p_1}{\partial \bar{z}} & \frac{\partial p_2}{\partial \bar{z}} & \frac{\partial p_3}{\partial \bar{z}} \end{vmatrix} \tag{4.26}$$

From equation (4.24) we can find the entries of this matrix. After substituting and finding the determinant of the matrix we get,

$$J' = im^3\rho^2\gamma_{z\bar{z}}, \qquad |J'| = m^3\rho^2\gamma_{z\bar{z}} \tag{4.27}$$

Now we have this property of Dirac delta function in momentum space,

$$\int d^3\vec{p}\,\delta^3(\vec{p}-\vec{p}') = 1 \tag{4.28}$$

$$\begin{aligned}
\int d^3\vec{p}\,\delta^3(\vec{p}-\vec{p}') &= \int |J'|d\rho\, d^2z\,\delta^3(\vec{p}-\vec{p}') \\
&= \int m^3\rho^2\gamma_{z\bar{z}}d\rho\, d^2z\,\delta^3(\vec{p}-\vec{p}') \\
&= \int m^3\rho^2\gamma_{z\bar{z}}\frac{\sqrt{1+\rho^2}}{\sqrt{1+\rho^2}}d\rho\, d^2z\,\delta^3(\vec{p}-\vec{p}') \\
&= \int m^2\rho^2\gamma_{z\bar{z}}\frac{E_p}{\sqrt{1+\rho^2}}d\rho\, d^2z\,\delta^3(\vec{p}-\vec{p}') \\
&= \int d^3V\, m^2E_p\delta^3(\vec{p}-\vec{p}') = 1
\end{aligned} \tag{4.29}$$

The delta functions on $\mathcal{H}^+$ is normalised as,

$$\int d^3V\,\delta(\rho-\rho')\delta^2(\hat{x}-\hat{x}') = 1 \tag{4.30}$$

Comparing equation (4.30) with equation (4.29) we see that,

$$m^2E_p\,\delta^3(\vec{p}-\vec{p}') = \delta(\rho-\rho')\delta^2(\hat{x}-\hat{x}') \tag{4.31}$$

Converting equation (4.23) to momentum space and substituting equation (4.33) in equation (4.23) one gets the poisson bracket as,

$$\{b(\vec{p}), b^*(\vec{p}')\} = i(2\pi)^3 2E_p\,\delta^3(\vec{p}-\vec{p}') \tag{4.32}$$

$$\{d(\vec{p}), d^*(\vec{p}')\} = i(2\pi)^3 2E_p\,\delta^3(\vec{p}-\vec{p}') \tag{4.33}$$

4.2 Asymptotic of massive QED

Consider a complex scalar field theory coupled to a gauge theory which has action,

$$\mathcal{S} = \int d^4x\sqrt{g}[-g^{\mu\nu}D_\mu\phi(x)D_\nu\phi^*(x) - m^2\phi(x)\phi^*(x) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] \tag{4.34}$$

Varying the action we get various field equations as,

$$\nabla^\nu\mathcal{F}_{\mu\nu} = \mathcal{J}_\mu, \qquad (-D_\mu D^\mu + m^2)\phi = 0. \tag{4.35}$$

$\mathcal{J}_\mu$ is the current of matter field and D_μ is the covariant derivative as defined in the section on massless QED. Much of the treatment of Maxwell field near $\mathscr{I}^+$ remains same as mentioned in chapter (3). The fall offs of Maxwell field near $\mathscr{I}^+$ was calculated in the section (3.1). The field $A_A(u,\hat{x})$ further satisfies the fall off,

$$A_A(u,\hat{x}) = A_A^\pm(\hat{x}) + \mathcal{O}(u^{-\epsilon}) \tag{4.36}$$

where $\pm$ denote the field at $u = \pm\infty$. This makes sure that fields at the boundary of u are finite and depends only on the unit vector on S^2.

Now the behavior of Maxwell fields near timelike infinity is studied. The transformation between coordinates (u,r) to (ρ,τ) are $u = \tau(\sqrt{1+\rho^2}-\rho)$ and $r = \tau\rho$. Using vector transformation law

$$\begin{aligned}\mathcal{A}_\tau &= \frac{\partial u}{\partial\tau}\mathcal{A}_u + \frac{\partial r}{\partial\tau}\mathcal{A}_r = (\sqrt{1+\rho^2}-\rho)\mathcal{A}_u + \rho\mathcal{A}_r\\ &= (\sqrt{1+\rho^2}-\rho)\frac{A_u(u,\hat{x})}{\tau\rho} + \rho\mathcal{A}_r(u,\hat{x})\end{aligned}$$

Writing $A_u(u,\hat{x}) = A_u(\infty,\hat{x}) + \mathcal{O}(u^{-\epsilon})$ and substituting u in terms of τ,ρ

$$\begin{aligned}\mathcal{A}_\tau =& \frac{(\sqrt{1+\rho^2}-\rho)}{\tau\rho}\left[A_u(\infty,\hat{x}) + \mathcal{O}((\tau(\sqrt{1+\rho^2}-\rho))^{-\epsilon})\right] + \mathcal{O}(\tau^{-2}\rho^{-1})\\ \mathcal{A}_\tau =& \frac{A_\tau(\rho,\hat{x})}{\tau} + \mathcal{O}(\tau^{-1-\epsilon}),\end{aligned} \tag{4.37}$$

where $A_\tau(\rho,\hat{x}) = \frac{(\sqrt{1+\rho^2}-\rho)}{\rho}A_u(\infty,\hat{x})$. The field A_u is evaluated at $u = +\infty$ since the massive particle coupled to gauge field will travel near $u = +\infty$. Similarly,

$$\begin{aligned}\mathcal{A}_\rho =& \frac{\partial u}{\partial\rho}\mathcal{A}_u + \frac{\partial r}{\partial\rho}\mathcal{A}_r\\ =& \tau\left(\frac{2\rho}{\sqrt{1+\rho^2}} - 1\right)\frac{1}{\tau\rho}[A_u(\infty,\hat{x}) + \mathcal{O}((\tau(\sqrt{1+\rho^2}-\rho))^{-\epsilon})] + \mathcal{O}(\tau^{-1}\rho^{-2})\\ \mathcal{A}_\rho =& A_\rho(\rho,\hat{x}) + \mathcal{O}(\tau^{-\epsilon}) && (4.38)\\ \mathcal{A}_A =& A_A^+(\hat{x}) + \mathcal{O}(\tau^{-\epsilon}) && (4.39)\end{aligned}$$

Using these fall-off conditions one can calculate,

$$\begin{aligned}D^\mu D_\mu\phi =& (\partial_\mu - ie\mathcal{A}_\mu)(\partial^\mu - ie\mathcal{A}^\mu)\\ =& \partial_\mu\partial^\mu\phi - ieg^{\mu\nu}\partial_\mu\mathcal{A}_\nu\phi - 2ieg^{\mu\nu}\mathcal{A}_\nu\partial_\mu\phi - e^2g^{\mu\nu}\mathcal{A}_\mu\mathcal{A}_-\nu\phi\\ =& \partial_\mu\partial^\mu\phi + 2ie\mathcal{A}_\tau\partial_\tau + \mathcal{O}(\tau^{-2}) = \partial_\mu\partial^\mu\phi + 2ie\frac{A_\tau}{\tau}\partial_\tau + \mathcal{O}(\tau^{-2})\end{aligned} \tag{4.40}$$

All other terms won't contribute to the leading order since they are being multiplied by inverse of metric tensor which other than $g^{\tau\tau}$ term has a $\frac{1}{\tau^2}$ term and therefore all other terms will have lower power than τ^{-1}. This extra term modifies the asymptotic form of the free massive field that was evaluated in equation (4.9) by introducing the phase term which is given by $e^{i\,ln(\tau)A_\tau(\rho,\hat{x})}$. If now symplectic current density w'^τ is calculated for this modified field by substituting it in equation (4.16) one finds that,

$$w'^\tau = w^\tau + \mathcal{O}(\tau^{-1}) \tag{4.41}$$

The covariant phase space symplectic density for this system of massive field coupled to gauge field is given by using Lagrangian in equation (4.34). The symplectic potential current density Θ^μ will be same as defined in equation (3.23) but with a slight modification that partial derivative ∂_ν is replaced by covariant derivative D_ν. Also there will be an extra contribution from the dynamical term of Maxwell field in the Lagrangian which is given by $\frac{\partial L}{\partial(\partial_\mu \mathcal{A}_\nu)}\delta\mathcal{A}_\nu = -\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu$. Therefore final expression is,

$$\Theta^\mu(\delta\phi, \delta\phi^*) = -\sqrt{g}g^{\mu\nu}(D_\nu\phi^*\,\delta\phi + D_\nu\phi\,\delta\phi^* + \delta\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu) \tag{4.42}$$

Substituting the above expression in equation (3.24) one gets,

$$w^\mu = -\sqrt{g}[D^\mu\delta\phi^*\delta'\phi + D^\mu\delta\phi\delta'\phi^* + \mathcal{F}^{\mu\nu}\delta'\mathcal{A}_\nu - \delta \leftrightarrow \delta'] \tag{4.43}$$

The symplectic product for a $t =$ constant Cauchy surface is given by

$$\Omega(\delta, \delta') = \lim_{t\to\infty}\int_{\Sigma_t} d\Sigma_t\, w^t, \tag{4.44}$$

where $dS_\mu = d\Sigma_t n_\mu$ with $n_\mu = \delta^t_\mu$. The symplectic current density w^t gets contribution from two parts in the limit $t \to \infty$ depending on the case of massless and massive particle. If one keeps $u = t-r$ constant in the limit $r \to \infty$ then one reaches null infinity and using vector transformation law from going to $(t, r, \hat{x})$ to $(u, t, \hat{x})$ which parameterize massless particle we get,

$$w^t = \frac{\partial t}{\partial r}w^r + \frac{\partial t}{\partial u}w^u = w^r + w^u \tag{4.45}$$

For massive scalar field near null infinity $w^r = \mathcal{O}(r^{-1})$ therefore the scalar field contribution to w^t in the above mentioned limit vanishes. Therefore in this limit we only need to look at the contribution of gauge field.

$$\begin{aligned} w^r =& -r^2\sqrt{\gamma}(\delta\mathcal{F}^{r\nu}\delta' A_\nu - \delta \leftrightarrow \delta') = -r^2\sqrt{\gamma}(\delta\mathcal{F}^{ru}\delta'\mathcal{A}_u - \delta\mathcal{F}^{rA}\delta'\mathcal{A}_A - \delta \leftrightarrow \delta') \\ =& -r^2\sqrt{\gamma}[g^{ru}g^{ru}(\partial_u\delta\mathcal{A}_r - \partial_r\delta\mathcal{A}_u)\delta'\mathcal{A}_u + g^{ru}g^{AB}(\partial_u\delta\mathcal{A}_B - \partial_B\delta\mathcal{A}_u)\delta'\mathcal{A}_A - \delta \leftrightarrow \delta'] \end{aligned} \tag{4.46}$$

Substituting the fall-off conditions of gauge fields we see that the first term in the above relation is proportional to $\mathcal{O}(r^{-1})$ which vanishes in the limit $r \to \infty$. Lets take a look at the second term,

$$w^r = -r^2\sqrt{\gamma}[-\frac{\gamma^{AB}}{r^2}(\partial_u\delta\mathcal{A}_B - \partial_B\delta\mathcal{A}_u)\delta'\mathcal{A}_A) - \delta \leftrightarrow \delta'] \tag{4.47}$$

After substituting fall off condition we have,

$$w^r = \sqrt{\gamma}\gamma^{AB}\partial_u\delta A_B\delta' A_A - \sqrt{\gamma}\gamma^{AB}\partial_u\delta' A_B\delta A_A \tag{4.48}$$

The term proportional to $\mathcal{A}_u$ drops out. For w^u we have,

$$\begin{aligned} w^u &= r^2\sqrt{\gamma}(\delta\mathcal{F}^{ur}\delta'\mathcal{A}_r + \delta\mathcal{F}^{uA}\delta'\mathcal{A}_A) - \delta \leftrightarrow \delta' \\ w^u &= r^2\sqrt{\gamma}[(g^{ru}g^{ur}\delta\mathcal{F}_{ru}\delta'\mathcal{A}_r) + g^{ur}g^{AB}\delta\mathcal{F}_{rB}\delta' A_A] \end{aligned} \tag{4.49}$$

Substituting $g^{AB} = \frac{\gamma^{AB}}{r^2}$ and fall-off conditions we see that first and second term are $\mathcal{O}(r^{-1})$ and $\mathcal{O}(r^{-2})$ respectively which means that w^u vanishes in the limit $r \to \infty$. Therefore the term that survives and gives contribution to w^t from massless sector near $\mathscr{I}^+$ is given by equation (4.48).

Now the second contribution to w^t comes from the massive scalar fields when $v = r/t$ is constant in the limit both r and t tends to infinity. In the coordinates $(\tau, \rho, z, \bar{z})$ using vector transformation law we have two contribution to w^t,

$$w^t = \sqrt{1+\rho^2}\, w^\tau + \frac{\rho\tau}{\sqrt{1+\rho^2}}\, w^\rho \tag{4.50}$$

Scalar field contribution to w^τ is given by equation (4.16) but now the field has an additional phase difference given by $e^{i\ln(\tau)A_\tau(\rho,\hat{x})}$. Substituting the modified field in equation (4.16) one finds that the contribution to the symplectic potential current density is still given by equation (4.20) and all the other contribution will be of the order of $\mathcal{O}(t^{-1})$ and higher as shown in equation (4.41) which vanishes as $t \to \infty$.

From calculations of w^ρ along the similar lines it is found that the leading contribution from the scalar field will be of the order $\mathcal{O}(t^{-2})$ which vanishes as $t \to \infty$. Similarly the contribution from the Maxwell field to w^τ and w^ρ also vanishes as $t \to \infty$ on the surface $\mathcal{H}^+$. To sum it up the symplectic potential current density w^t in the limit $t \to \infty$ gets contribution from the scalar field near time-like infinity and Maxwell field on the cauchy surface $\mathscr{I}^+$. Therefore one can write symplectic product in equation (4.44) as the sum of symplectic product of Maxwell field radiative phase space and free massive field asymptotic phase space.

$$\Omega(\delta, \delta') = \Omega_A(\delta, \delta') + \Omega_\phi(\delta, \delta') \tag{4.51}$$

where,

$$\Omega_A(\delta, \delta') = \int_{\mathscr{I}^+} d^2z\, du \gamma_{z\bar{z}}(\gamma^{z\bar{z}}\delta A_z \partial_u \delta' A_{\bar{z}} + \gamma^{z\bar{z}}\delta A_{\bar{z}} \partial_u \delta' A_z - \delta \leftrightarrow \delta') \tag{4.52}$$

$$\Omega_\phi(\delta, \delta') = \frac{im^2}{2(2\pi)^3} \int_{\mathcal{H}_+} d^3V \left[\delta' b \delta b^* + \delta d^* \delta' d - \delta b \delta' b^* - \delta d \delta' d^*\right] \tag{4.53}$$

The resultant asymptotic phase space is the direct product of the asymptotic phase space of massive scalar field and Maxwell field.

4.3 Asymptotic symmetries and charge

The asymptotic symmetries and charges defined on the boundary for Maxwell field at $\mathscr{I}^+$ is same as treated in sections (3.2) and section (3.3) on Massless QED. The only difference is now the matter fields instead of crossing $\mathscr{I}^+$ cross $\mathcal{H}^+$ so the focus will be on the matter fields and wherever necessary the results for Maxwell fields will be quoted. In Lorentz gauge, the theory has residual gauge invariance which means that infinitesimal transformation of field is given by,

$$\delta_{\tilde{\lambda}} \mathcal{A}_\mu = \partial_\mu \tilde{\lambda}, \qquad \delta_{\tilde{\lambda}} \phi = ie\tilde{\lambda}\phi. \tag{4.54}$$

The gauge parameter satisfies wave equation,

$$\partial_\mu \partial^\mu \tilde{\lambda} = 0 \tag{4.55}$$

The gauge parameter near null infinity follows the asymptotic conditions as mentioned in section (3.2) i.e $\hat{\lambda}(u, r, \hat{x}) = \lambda(\hat{x}) + \mathcal{O}(r^{-1})$. If one insists on gauge parameter preserving the fall-off of Maxwell fields near time like infinity i^+ then,

$$\tilde{\lambda}(\tau, \rho, \hat{x}) = \lambda_{\mathcal{H}}(\rho, \hat{x}) + \mathcal{O}(\tau^{-\epsilon}) \tag{4.56}$$

Using this one can see that for a gauge transformation $\mathcal{A}'_\mu = \mathcal{A}_\mu + \partial_\mu \lambda$ substituting the fall-off condition of fields given in equation (4.37), equation (4.38) and equation (4.39) fall-off of gauge parameter $\lambda(\tau, \rho, \hat{x})$ is justified. As evident from arguments, $\lambda_{\mathcal{H}}$ is the function on the hyperboloid $\mathcal{H}^+$. Since taking the limit $\rho \to \infty$ at $\tau =$ constant surface take us to the point $u = 0$ at $\mathscr{I}^+$, this requires that,

$$\lim_{\rho \to \infty} \lambda_{\mathcal{H}}(\rho, \hat{x}) = \lambda(\hat{x}) \tag{4.57}$$

Also the wave equation in equation (4.55) implies that $\lambda_{\mathcal{H}}$ satisfies laplace equation on $\mathcal{H}$,

$$\Delta \lambda_{\mathcal{H}}(\rho, \hat{x}) = 0 \tag{4.58}$$

where Δ is the Laplacian on $\mathcal{H}^+$. Instead of the limiting procedure between the two gauge parameteres defined on $\mathscr{I}^+$ and $\mathcal{H}^+$ one can look for the relation of the type,

$$\lambda_{\mathcal{H}}(\rho, \hat{x}) = \int_{S^2} d^2\hat{q}\, \mathcal{G}(\rho, \hat{x}; \hat{q}) \lambda(\hat{q}), \tag{4.59}$$

with the following properties,

$$\begin{gathered} \Delta \mathcal{G}(\rho, \hat{x}; \hat{q}) = 0 \\ \lim_{\rho \to \infty} \mathcal{G}(\rho, \hat{x}; \hat{q}) = \delta^2(\hat{x}, \hat{q}). \end{gathered} \tag{4.60}$$

$\mathcal{G}(\rho,\hat{x};\hat{q})$ gives the connection between function defined on the sphere at $\mathscr{I}^+$ and the function on the hyperboloid $\mathcal{H}^+$. If one thinks the boundary of $\mathcal{H}^+$ as a sphere S^2 the $\mathcal{G}$ can be interpreted as scalar bulk to boundary propagator which labels the point on the boundary as well as the points in the bulk which in this case is $\mathcal{H}^+$. Therefore one can say that the large gauge transformations of massive QED are also parametrized in term of a functions $\lambda(\hat{x})$ defined on S^2. Therefore in general sense for both massless and massive QED the asymptotic symmetries are generated by the functions $\lambda(\hat{x})$. The free data boundary fields are $\{A_A, b(\vec{p}), d(\vec{p})\}$. The infinitesimal large gauge transformations on these fields are given by using equation (4.54),

$$\delta_\lambda A_A(u,\hat{x}) = \partial_A\lambda(\hat{x}) \tag{4.61}$$

$$\delta_\lambda b(m\rho\hat{x}) = ie\lambda_{\mathcal{H}}\big(\rho,\hat{x}\big)b(m\rho\hat{x}) \tag{4.62}$$

$$\delta_\lambda d(m\rho\hat{x}) = -ie\lambda_{\mathcal{H}}\big(\rho,\hat{x}\big)d(m\rho\hat{x}). \tag{4.63}$$

Using saddle point approximation conditions from equation (4.8) we rewrite equation (4.62) and equation (4.63) in the momentum space as,

$$\delta_\lambda b(\vec{p}) = ie\lambda_{\mathcal{H}}\left(\frac{\vec{p}}{m}\right)b(\vec{p}) \tag{4.64}$$

$$\delta_\lambda d(\vec{p}) = -ie\lambda_{\mathcal{H}}\left(\frac{\vec{p}}{m}\right)d(\vec{p}) \tag{4.65}$$

where $\frac{\vec{p}}{m}$ denotes the point $(\rho,\hat{x}) = (\vec{p}/m,\hat{p})$ on $\mathcal{H}^+$. On this surface the generators of large gauge transformation i.e charge is given by using symplectic product of free scalar field in equation (4.21),

$$\delta Q^+ = \lim_{\tau\to\infty}\Omega_{\Sigma_\tau}(\delta,\delta'_\lambda) = \frac{im^2}{2(2\pi)^3}\int_{\mathcal{H}^+} d^3V\big[\delta'_\lambda b\delta b^* + \delta d^*\delta'_\lambda d - \delta b\delta'_\lambda b^* - \delta d\delta'_\lambda d^*\big] \tag{4.66}$$

Substituting equation (4.62) and equation (4.63) in the above equation we get,

$$\begin{aligned}\delta Q^+ =& \frac{im^2}{2(2\pi)^3}\int_{\mathcal{H}^+} d^3V[ie\lambda_{\mathcal{H}}b(m\rho\hat{x})\delta b^*(m\rho\hat{x}) - ie\lambda_{\mathcal{H}}\delta d^*(m\rho\hat{x})d(m\rho\hat{x}) + \\ & ie\lambda_{\mathcal{H}}\delta b(m\rho\hat{x})b^*(m\rho\hat{x}) - ie\lambda_{\mathcal{H}}\delta d(m\rho\hat{x})d^*(m\rho\hat{x})] \\ \delta Q^+ =& \frac{im^2}{2(2\pi)^3}\int_{\mathcal{H}^+} d^3V ie\lambda_{\mathcal{H}}\delta[b(m\rho\hat{x})b^*(m\rho\hat{x}) - d^*(m\rho\hat{x})d(m\rho\hat{x})]\end{aligned} \tag{4.67}$$

Therefore charges associated with large gauge transformation of massive scalar field is given by,

$$Q^+_{hard} = -\int_{\mathcal{H}^+} d^3V\lambda_{\mathcal{H}}(\rho,\hat{x})j^\tau(\rho,\hat{x}), \tag{4.68}$$

where $j^\tau(\rho,\hat{x})$ is the matter charge passing through $\mathcal{H}^+$ which is defined as,

$$j^\tau = \frac{em^2}{2(2\pi)^3}(b(m\rho\hat{x})b^*(m\rho\hat{x}) - d^*(m\rho\hat{x})d(m\rho\hat{x})) \tag{4.69}$$

The charges which generate large gauge transformations for Maxwell field at $\mathscr{I}^+$ is given by equation (3.40) with contribution of hard charge as zero because massive particle do not pass through $\mathscr{I}^+$ and hence there is no current. The total charge then is given by,

$$Q^+ = -\int_{\mathcal{H}_+} d^3V\lambda_{\mathcal{H}}(\rho,\hat{x})j^\tau(\rho,\hat{x}) + \int_{S^2} d^2z\lambda(z,\bar{z})\big(\partial_z[A_{\bar{z}}] + \partial_{\bar{z}}[A_z]\big) \tag{4.70}$$

$$Q^+ = Q^+_{hard} + Q^+_{soft} \tag{4.71}$$

where Q^+_{soft} is written in a different way by integrating out the u dependence. $[A_z]$ defined in equation (3.42) is the short hand for the difference of A_z field at $u=\pm\infty$. Here to proceed forward with the intent of showing equivalence of Ward identity to soft photon theorem one needs to make an assumption that $[F_{z\bar{z}}] = 0$ which implies that $D^z[A_z] = D^{\bar{z}}[A_{\bar{z}}]$, where D^z is the covariant derivative defined on the sphere. Using these conditions the soft charge becomes,

$$Q^+_{soft} = 2\int d^2z\lambda\partial_z[A_{\bar{z}}], \tag{4.72}$$

$$Q^+_{soft} = 2\int d^2z\lambda\partial_{\bar{z}}[A_z]. \tag{4.73}$$

The soft mode field $\partial_z N = [A_z]$ which was earlier stated in this fashion for massless case,

$$\partial_z N = \lim_{E_p \to 0} \frac{1}{2} \int_{-\infty}^{+\infty} du (e^{iE_p u} + e^{-iE_p u}) \partial_u A_z \tag{4.74}$$

can be defined differently this time,

$$[A_A(\hat{x})] = -i \lim_{E_s \to 0^+} E_s A_A(E_s, \hat{x}) \tag{4.75}$$

where $A_A(E, \hat{x}) = \int_{-\infty}^{\infty} A_A(u, \hat{x}) e^{iEu} du$ is the fourier tansform of the field $A_A(u, \hat{x})$.

Similarly one can define charges and asymptotic symmetries at $\mathscr{I}^-$ along the similar line of arguments made in this section.

4.4 Ward Identity

This section follows the procedure carried out in section (3.5). The commutator of various free fields at the boundary is given by multiplying Poisson bracket in equation (4.23) by i and therefore we have,

$$[\, b(m\rho\hat{x}), b^*(m\rho'\hat{x'})\,] = -\frac{2(2\pi)^3}{m^2} \delta(\rho - \rho') \delta^2(\hat{x} - \hat{x'}) \tag{4.76}$$

$$[\, d(m\rho\hat{x}), d^*(m\rho'\hat{x'})\,] = -\frac{2(2\pi)^3}{m^2} \delta(\rho - \rho') \delta^2(\hat{x} - \hat{x'}). \tag{4.77}$$

The commutator of free fields $b(\vec{p})$ and $d(\vec{p})$ with total charge gets contribution only from the *hard* charge.

$$\begin{aligned}
\left[\, b(m\rho'\hat{x}'),\, Q^+_{hard} \,\right] &= \int -d^3V \left[b(m\rho'\hat{x}'), \lambda_H(\rho, \hat{x}) j^\tau(\rho, \hat{x}) \right] \\
&= \int -d^3V\, \lambda_H(\rho, \hat{x}) [b(m\rho'\hat{x}'),\, b(m\rho\hat{x}) b^*(m\rho\hat{x})] \frac{em^2}{2(2\pi)^3} \\
&= e\lambda_H(\rho', \hat{x}') b(m\rho'\hat{x}')
\end{aligned}$$

$$\implies \left[\, b(\vec{p}),\, Q^+_{hard} \,\right] = e\lambda_H \left(\frac{\vec{p}}{m} \right) b(\vec{p}) \tag{4.78}$$

$$\left[\, b^*(\vec{p}),\, Q^+_{hard} \,\right] = -e\lambda_H \left(\frac{\vec{p}}{m} \right) b^*(\vec{p}), \tag{4.79}$$

Similarly for fields $d(\vec{p})$ and $d^*(\vec{p})$,

$$\left[\, d(\vec{p}),\, Q^+_{hard} \,\right] = -e\lambda_H \left(\frac{\vec{p}}{m} \right) d(\vec{p}) \tag{4.80}$$

$$\left[\, d^*(\vec{p}),\, Q^+_{hard} \,\right] = e\lambda_H \left(\frac{\vec{p}}{m} \right) d^*(\vec{p}) \tag{4.81}$$

In order to find the contribution of *soft* charge in terms of boundary fields one calculates $A_z(E, z, \bar{z})$ which is the Fourier transform of free field $A_z(u, z, \bar{z})$ at the boundary defined in equation (A.10). Therefore,

$$A_z(E, z, \bar{z}) = \frac{-i\sqrt{2}}{4\pi(1 + z\bar{z})} \int_0^\infty d\omega_q \left[a_+(\omega_q \hat{x}) \delta(\omega_q - E) - a_-^\dagger(\omega_q \hat{x}) \delta(\omega_q + E) \right] \tag{4.82}$$

For $E > 0$ only first term contributes and the expression becomes,

$$A_z(E, z, \bar{z}) = \frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi i} a_+(E, z, \bar{z}) \tag{4.83}$$

where $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$. Using equation (4.75) one can write $[A_z]$ as,

$$[A_z] = \lim_{E_p \to 0} -\frac{E_p \sqrt{\gamma_{z\bar{z}}}}{4\pi} a_+(E_p, z, \bar{z}) \tag{4.84}$$

Compare this with equation (3.74) which is the soft mode field for the massless case. Now substituting this expression in the soft charge given by equation (4.73) we have,

$$Q^+_{soft} = \lim_{E_p \to 0} \frac{-E_p}{2\pi} \int d^2 z \lambda(z,\bar{z}) \partial_{\bar{z}} \big(\sqrt{\gamma_{z\bar{z}}} a_+(E_p, z, \bar{z}) \big). \tag{4.85}$$

Before going on to derive the Ward identity lets define the asymptotic states. Unlike the case of massless QED it will be useful to work in momentum space. The Fock state on the boundary is defined as,

$$\langle out| = \langle \vec{p}_1, \vec{p}_2, \vec{p}_3, ...\vec{p}_n | = \langle 0 |\, b(\vec{p}_1)\, d(\vec{p}_2)\, b(\vec{p}_3) ... d\,(\vec{p}_n) \tag{4.86}$$

$$|in\rangle = |\, \vec{p}_1, \vec{p}_2, \vec{p}_3, ...\vec{p}_m \,\rangle = b^*(\vec{p}_1)\, d^*(\vec{p}_2)\, b^*(\vec{p}_3) ... d^*(\vec{p}_m)|0\rangle \tag{4.87}$$

The $|out\rangle$ and $|in\rangle$ comprise of n and m particles/ antiparticles respectively. To get particle or antiparticle from $|0\rangle$ one needs to act on the vacuum with $b^*(\vec{p})$ or $d^*(\vec{p})$. The RHS side of Ward identity in equation (3.68) is given by $\langle out|\mathcal{S}Q^-_{hard}|in\rangle - \langle out|Q^+_{hard}\mathcal{S}|in\rangle$. Therefore one has,

$$\langle out|\mathcal{S}Q^-_{hard}|in\rangle = \left[\sum e\lambda_{\mathcal{H}} \left(\frac{\vec{p}}{m} \right) - \sum e\lambda_{\mathcal{H}} \left(\frac{\vec{p}}{m} \right) \right] \langle out|\mathcal{S}|in\rangle \tag{4.88}$$

The first and second term is due to particles and antiparticles at $\mathscr{I}^-$. Similarly one can also evaluate,

$$\langle out|Q^+_{hard}\mathcal{S}|in\rangle = \left[\sum e\lambda_{\mathcal{H}} \left(\frac{\vec{p}}{m} \right) - \sum e\lambda_{\mathcal{H}} \left(\frac{\vec{p}}{m} \right) \right] \langle out|\mathcal{S}|in\rangle \tag{4.89}$$

The LHS of Ward identity in equation (3.68) using equation (4.85) is given by,

$$\begin{aligned} &\Big\langle out|Q^+_{soft}\mathcal{S}|in\Big\rangle - \Big\langle out|\mathcal{S}Q^-_{soft}|in\Big\rangle \\ &= \lim_{E_p \to 0} \frac{-E_p}{2\pi} \int d^2 z \lambda(z,\bar{z}) \partial_{\bar{z}} \big(\sqrt{\gamma_{z\bar{z}}} \langle out|a_+(E_p, z, \bar{z})\mathcal{S}|in\rangle \big) \end{aligned} \tag{4.90}$$

Finally one uses equation (4.90), equation 4.88 and equation (4.89) to bring the Ward identity in equation (3.68) to the form,

$$\begin{aligned} &\lim_{E_p \to 0} \frac{E_p}{2\pi} \int d^2 z \lambda(z,\bar{z}) \partial_{\bar{z}} \big(\sqrt{\gamma_{z\bar{z}}} \langle out|a_+(E_p, z, \bar{z})\mathcal{S}|in\rangle \big) \\ &\qquad = e \sum_i Q_i \lambda_{\mathcal{H}} \left(\frac{\vec{p}_i}{m} \right) \langle out|\mathcal{S}|in\rangle. \end{aligned} \tag{4.91}$$

The sum in RHS is over all the external particles with Q_i is $+1$ for outgoing particle at $\mathscr{I}^+$ and incoming antiparticle from $\mathscr{I}^-$ and $Q_i = -1$ for outgoing antiparticle at $\mathscr{I}^+$ and incoming particle from $\mathscr{I}^-$.

To derive Ward identity i.e equation (4.91) from soft photon theorem let us first establish relationship between gauge parameter defined on $\mathcal{H}^+$ and $\mathscr{I}^+$ by finding the expression of scalar bulk to boundary propogator $\mathcal{G}$. For this we consider soft photon theorem written as ,

$$\lim_{E_p \to 0} \left[E_p \langle out|a^{out}_+(\vec{q})\mathcal{S}|in\rangle \right] = e \sum_i \frac{Q_i p_i \cdot \epsilon^+}{p_k \cdot q/E_p} \langle out|\mathcal{S}|in\rangle. \tag{4.92}$$

i runs over all the particle in the external state, q is the momentum of soft photon and is parameterized by $(E_p, \hat{q})$, where $\hat{q}$ is the momentum unit vector on sphere which asymptotes to position unit vector on sphere given in terms of $(z, \bar{z})$ as defined earlier. To match this with Ward identity one needs to multiply the operator $(2\pi)^{-1} \int d^2 z \lambda(z,\bar{z}) \partial_{\bar{z}} (\sqrt{\gamma_{w\bar{w}}}$ on both the sides of the above equation. Therefore we have a resemblance with equation (4.91) where the LHS looks same but comparing the RHS of two equation give us the relation,

$$\lambda_{\mathcal{H}} \left(\frac{\vec{p}}{m} \right) = \frac{1}{2\pi} \int d^2 z \partial_{\bar{z}} \left(\sqrt{\gamma_{z\bar{z}}} \frac{p \cdot \epsilon^+(z,\bar{z})}{p \cdot q/E_p} \right) \lambda(z,\bar{z}) \tag{4.93}$$

The above equation has the form of equation (4.59) with the identification,

$$\mathcal{G}(\vec{p}/m; z, \bar{z}) = \frac{1}{2\pi}\partial_{\bar{z}}\bigg(\sqrt{\gamma_{z\bar{z}}}\frac{p.\epsilon^{+}(z,\bar{z})}{p.q/E_p}\bigg). \tag{4.94}$$

Now parameterizing the four momenta p^μ of the massive particle as,

$$p^\mu = (-m\sqrt{1+\rho^2}, m\rho\hat{x}) \tag{4.95}$$

This we get from the saddle point approximation conditions. To solve equation (4.94) one sees that,

$$\frac{p\cdot\epsilon^{+}(z,\bar{z})}{p\cdot q/E_p} = (1+z\bar{z})\partial_z \log\big[(1+z\bar{z})p\cdot q/E_p\big]$$

and $p.q = -m(\sqrt{1+\rho^2} - \rho\hat{x}.\hat{q})$. Therefore,

$$\mathcal{G}(\rho, \hat{x}; \hat{q}) = \frac{\sqrt{\gamma_{z\bar{z}}}}{4\pi}(\sqrt{1+\rho^2} - \rho\hat{q}\hat{x})^{-2} \tag{4.96}$$

This expression of $\mathcal{G}$ satisfies two properties, firstly $\Delta\mathcal{G} = 0$, where Δ is the Laplacian on $\mathcal{H}^+$ given by,

$$\Delta = (1+\rho^2)\partial_\rho^2 + \rho^{-1}(2+3\rho^2)\partial_\rho + \rho^{-2}(1+z\bar{z})^2\partial_w\partial_{\bar{w}}. \tag{4.97}$$

The partial derivatives will act on $\hat{x}$ which is given in terms of $w, \bar{w}$. Also one can calculate that,

$$\lim_{\rho\to\infty}\sqrt{\gamma_{w\bar{w}}}\frac{p.\epsilon^{+}(w,\bar{w})}{p.q/E_p} = (w-z)^{-1} \tag{4.98}$$

and therefore using $\partial_{\bar{w}}(w-z)^{-1} = 2\pi\delta^2(w-z)$ we have ,

$$\lim_{\rho\to\infty}\mathcal{G}(\rho, z, \bar{z}; w, \bar{w}) = \delta^2(z-w) \tag{4.99}$$

These two properties that $\mathcal{G}$ satisfies are same as mentioned in equation (4.60) for the requirement of scalar bulk to boundary propagator [9]. Using this one can see that soft photon theorem coincides with Ward identity. Note that this machinery was not required for the case of massless particles because everything was happening at null infinity but in massive particle case since the ward identity has a hard charge defined on the hyperboloid $\mathcal{H}_+$ we needed some suitable prescription that could take the information to the null infinity and this is given by $\mathcal{G}(\rho, \hat{x}; \hat{q})$.

Now suppose in equation (4.91) one substitute for gauge parameter $\lambda(z,\bar{z}) = \frac{1}{w_s - z}$, then LHS becomes,

$$\lim_{E_p\to 0}\frac{E_p}{2\pi}\int d^2z\partial_{\bar{z}}\bigg(\frac{1}{z-w_s}\bigg)\sqrt{\gamma_{z\bar{z}}}\langle out|a_+(E_p, z, \bar{z})\mathcal{S}|in\rangle \tag{4.100}$$

After solving the integral and relabelling w_s as z_s we get,

$$\lim_{E_p\to 0}E_p\sqrt{\gamma_{z\bar{z}}}\langle out|a_+(E_p, z, \bar{z})\mathcal{S}|in\rangle \tag{4.101}$$

Now coming to the RHS of equation (4.91), $\lambda_{\mathcal{H}}(\vec{p}/m)$ is given by equation (4.93),

$$\lambda_{\mathcal{H}}(\vec{p}/m) = \frac{1}{2\pi}\int d^2z\partial_{\bar{z}}\bigg(\sqrt{\gamma_{z\bar{z}}}\frac{p.\epsilon^{+}(z,\bar{z})}{p.q/E_p}\bigg)\frac{1}{w_s - z} \tag{4.102}$$

Integrating by parts and computing the integral we get after relabelling w_s as z_s,

$$\lambda_{\mathcal{H}}(\vec{p}/m) = \sqrt{\gamma_{z\bar{z}}}\frac{p.\epsilon^{+}(z_s,\bar{z}_s)}{p.q/E_p} \tag{4.103}$$

From equation (4.101) and equation (4.103) it is clear that Ward identity can be recasted in the form of soft photon theorem.

Chapter 5

Future directions

The primary focus of this report was to review some of the works of Strominger et al in [8] and Campiglia and Laddha in [9] that can be be primarily credited for the resurgence of interest in the study of asymptotic symmetries of QED and its relation to one of the universal formulae of quantum field theory. This equivalence was developed in parallel for gravitational theory in [6, 7] and non-abelain gauge theory in [5, 12]. Keeping in mind the tools and techniques reviewed in this report one can hope to understand the ideas presented in these papers. There still remains vast amount of literature that one can look at which explores new subleading and sub-subleading soft theorems both in context of gravity and QED [15–21]. The asymptotic symmetries that are the basis of most of these works are discussed in [22, 24, 25]. This equivalence has also been extended to theories living in higher dimensions [26–28], which is believed to be relevant in the context of string theory.

This report should serve as one of the preliminary steps in understanding flat space holography since one of the first steps in developing such principle is to look for the symmetries of quantum gravity S- matrix. In [6] it was established that this symmetry group is infinite dimensional and includes a certain subgroup (supertranslations) of BMS on past and future null infinity. In [22, 29, 30] it was proposed that this symmetry can be naturally extended to include the Virasoro group (superrotation) that acts on the sphere at null infinity referred to as celestial sphere $\mathcal{CS}^2$ in recent literature. These developments shed some light on the possibility that the holographic dual of four-dimensional flat- space quantum gravity might be realized as an exotic two-dimensional conformal field theory on $\mathcal{CS}^2$ [31–34].

One of the techniques reviewed in this report such as deriving conserved charges that generates large gauge transformation for scalar QED at null infinity using covariant phase space formulation will help in understanding asymptotic quantization and derivation of conserved charges in the context of gravity at null infinity and asymptotically de Sitter spacetimes.

Appendix A

Saddle point approximation of fields

A.1 Massive Scalar field near $\mathcal{H}^+$

The $\tau \to \infty$ asymptotic limit of massive scalar field is evaluated using saddle point approximation method. One start by finding the critical point of $x \cdot p$ in the coordinates $(\tau, \rho, \hat{x})$,

$$x \cdot p = -E_p \tau \sqrt{1+\rho^2} + \tau \rho\, \vec{p}.\hat{x} = \tau\, f(\vec{p}) \tag{A.1}$$

Differentiating $f(\vec{p})$ w.r.t to $\vec{p}$ we have,

$$\begin{aligned} \frac{\partial}{\partial p_i}(-\sqrt{p_j p_j + m^2}\sqrt{1+\rho^2} + \rho\, p_j \hat{x}_j) &= 0 \\ -\frac{p_i\sqrt{1+\rho^2}}{\sqrt{p_i p_i + m^2}} + \rho\, \hat{x}_i &= 0 \end{aligned}$$

Squaring both the sides and rearranging we have

$$\rho^2 m^2 = p^2 \implies \vec{p} = m\rho\hat{x} \tag{A.2}$$

The function $f(\vec{p})$ is taylor expanded around the critical point, therefore the function $f(\vec{p})$ in the saddle point limit is

$$f(m\rho\hat{x}) = -\sqrt{m^2\rho^2 + m^2}\sqrt{1+\rho^2} + m^2\rho^2 = -m(1+\rho^2) + m\rho^2 = -m$$

Since $f'(m\rho\hat{x}) = 0$ we need to evaluate second order term in the taylor series expansion which is

$$\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} f(\vec{p})\Big|_{\vec{p}=m\rho\hat{x}} = -\frac{\delta_{ij}\sqrt{1+\rho^2}}{E_p} + \frac{p_i p_j\sqrt{1+\rho^2}}{E_p^3}\Big|_{\vec{p}=m\rho\hat{x}} = -\frac{\delta_{ij}}{m} + \frac{\rho^2 \hat{x}_i \hat{x}_j}{m(1+\rho^2)} = P_{ij} \tag{A.3}$$

Also from straightforward calculation $\det(P_{ij}) = -\frac{1}{m^3(1+\rho^2)}$. Now $x \cdot p$ is expanded around the critical point $m\rho\hat{x}$,

$$x \cdot p = \tau\, f(m\rho\vec{x}) + \frac{\tau P_{ij}(p_i - m\rho\hat{x}_i)(p_j - m\rho\hat{x}_j)}{2} + \mathcal{O}(p^3) \tag{A.4}$$

Substituting equation (A.4) in the equation (4.1) the large τ behavior of scalar field is given by

$$\begin{aligned} \phi(x) = &\int \frac{d^3\vec{p}}{(2\pi)^3\, m\sqrt{1+\rho^2}} b(m\rho\hat{x}) e^{i\tau(f(m\rho\hat{x}) + \frac{P_{ij}}{2}(p_i - m\rho\hat{x}_i)(p_j - m\rho\hat{x}_j))} \\ &+ \int \frac{d^3\vec{p}}{(2\pi)^3\, m\sqrt{1+\rho^2}} d^*(m\rho\hat{x}) e^{-i\tau(f(m\rho\hat{x}) + \frac{P_{ij}}{2}(p_i - m\rho\hat{x})(p_j - m\rho\hat{x}))} \\ = &\int \frac{d^3\vec{p}}{(2\pi)^3\, m\sqrt{1+\rho^2}} b(m\rho\hat{x}) e^{-i\tau\, m} e^{i\tau\frac{P_{ij}}{2}p_i p_j}\left[1 - i\tau\frac{P_{ij}}{2}(m\rho\, p_i\hat{x}_j + m\rho\, p_j\hat{x}_i) + ..\right] \\ + &\int \frac{d^3\vec{p}}{(2\pi)^3\, m\sqrt{1+\rho^2}} d^*(m\rho\hat{x}) e^{i\tau\, m} e^{-i\tau\frac{P_{ij}}{2}p_i p_j}[1 - i\tau\frac{P_{ij}}{2}(m\rho\, p_i\hat{x}_j + m\rho\, p_j\hat{x}_i) + ..\Big] \end{aligned}$$

where the terms in the square bracket are kept only to first order in p. Using $\int d^3\vec{p}\, e^{-i\tau \frac{P_{ij}}{2} p_i p_j} = \sqrt{\frac{(2\pi)^3}{(i\tau)^3 det(P_{ij})}}$, we get

$$\phi(x) = \frac{e^{-3i\pi/4}\sqrt{m}}{2(2\pi\tau)^{3/2}} (b(m\rho\hat{x})e^{-i\tau m} + id^*(m\rho\hat{x})e^{i\tau m}) + \mathcal{O}(\tau^{-5/2}) \tag{A.5}$$

A.2 Maxwell Field near Null infinity

The saddle point approximation of electromagnetic field near $\mathscr{I}^+$ is given as follows, we consider the free gauge field expansion in terms plane wave basis and look at the aymptotes at $r \to \infty$ in $(u, r, z, \bar{z})$ coordinates.

$$\begin{aligned}
\mathcal{A}_\mu(x) &= \sum_{\alpha=\pm} \int \frac{d^3\, q}{(2\pi)^3\, 2E_p} \left[\epsilon_\mu^{*\alpha}(\vec{p}) a_\alpha(\vec{p}) e^{ip.x} + \epsilon_\mu^{\alpha}(\vec{p}) a_\alpha^\dagger(\vec{p}) e^{-ip.x} \right] \\
x.p &= -\, tE_p + r\vec{p}.\hat{x} = -uE_p - rE_p + r\vec{p}.\hat{x}, E_p = |\vec{p}| \\
&= -\, uE_p - rE_p(1 - \hat{p}.\hat{x}) \\
\mathcal{A}_\mu(x) &= \sum_{\alpha=\pm} \int \frac{d^3\, q}{(2\pi)^3\, 2E_p} \left[\epsilon_\mu^{*\alpha}(\vec{p}) a_\alpha(\vec{p}) e^{-iuE_p - irE_p(1-\hat{p}.\hat{x})} + \epsilon_\mu^{\alpha}(\vec{p}) a_\alpha^\dagger(\vec{p}) e^{iuE_p + irE_p(1-\hat{p}.\hat{x})} \right] \\
\mathcal{A}_\mu(x) &= \sum_{\alpha=\pm} \int \frac{d^3\, q}{(2\pi)^3\, 2E_p} \left[\epsilon_\mu^{*\alpha}(\vec{p}) a_\alpha(\vec{p}) e^{-iuE_p - irE_p(1-\cos\theta)} + \epsilon_\mu^{\alpha}(\vec{p}) a_\alpha^\dagger(\vec{p}) e^{iuE_p + irE_p(1-\cos\theta)} \right]
\end{aligned}$$

For saddle point approximation as $r \to \infty$, the critical point of $(1 - \cos\theta)$ is given by $\theta = 0, \pi$. $\theta = \pi$ will make exponential oscillate very fast as $r \to \infty$ and therefore will vanish due to Reimann-Lebesgue lemma. If instead we find the extremum of $(1 - \hat{x}.\hat{p})$ we get the condition that $\hat{p} = \frac{1}{\hat{x}}$. Therefore expanding around $\theta = 0$ we have,

$$\begin{aligned}
\mathcal{A}_\mu = \frac{1}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty dE_p E_p \epsilon_\mu^{*\alpha}(E_p\hat{x}) a_\alpha(E_p\hat{x}) e^{-iuE_p} \int_0^\pi d\theta\, \theta e^{-iE_p r\theta^2/2} + \\
\frac{1}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty dE_p E_p \epsilon_\mu^{\alpha}(E_p\hat{x}) a_\alpha^\dagger(E_p\hat{x}) e^{iuE_p} \int_0^\pi d\theta\, \theta e^{iE_p r\theta^2/2}
\end{aligned}$$

Substituting $\int_0^\pi d\theta\, \theta e^{-iE_p r\theta^2/2} = \frac{1}{iE_p r}$ as $r \to \infty$ we have,

$$\begin{aligned}
\mathcal{A}_\mu =& \frac{-i}{8\pi^2 r} \sum_{\alpha=\pm} \int_0^\infty dE_p \left[\epsilon_\mu^{*\alpha}(E_p\hat{x}) a_\alpha(E_p\hat{x}) e^{-iuE_p} - \epsilon_\mu^{\alpha}(E_p\hat{x}) a_\alpha^\dagger(E_p\hat{x}) e^{iuE_p} \right] \\
&+ \mathcal{O}\left(\frac{1}{r^2}\right).
\end{aligned} \tag{A.6}$$

Using tensor transformation law to find the $\mathcal{A}_z$ component of field we use $\mathcal{A}_z = \partial_z x^\mu \mathcal{A}_\mu$. The only vector component in $\mathcal{A}_\mu$ is polarisation vector $\epsilon_\mu^\alpha, \epsilon_\mu^{*\alpha}$, where α takes value $\pm$ for positive and negative helicity of electrons. So we calculate $\epsilon_z = \partial_z x^\mu \epsilon_\mu$ for both positive and negative helicity photons. The standard polarisation vector have components,

$$\begin{aligned}
\epsilon_\mu^+(\vec{p}) &= \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}) \\
\epsilon_\mu^-(\vec{p}) &= \frac{1}{\sqrt{2}}(z, 1, -i, -z)
\end{aligned} \tag{A.7}$$

The vector $\vec{x}$ in $(r, z, \bar{z})$ is given by,

$$\vec{x} = \frac{r}{1 + z\bar{z}}(z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) \tag{A.8}$$

From polarisation vector transformation law we find that,

$$\begin{aligned}
\partial_z x^\mu \epsilon_\mu^+(E_p\hat{x}) = \partial_z x^\mu \epsilon_\mu^{*-}(E_p\hat{x}) = 0 \\
\partial_z x^\mu \epsilon_\mu^{*+}(E_p\hat{x}) = \partial_z x^\mu \epsilon_\mu^-(E_p\hat{x}) = \frac{\sqrt{2}r}{1 + z\bar{z}}
\end{aligned} \tag{A.9}$$

Using equation (A.9) we find,

$$\begin{aligned}\mathcal{A}_z =&\frac{-i}{8\pi^2 r}\int_0^\infty dE_p\left[a_+(E_p\hat{x})\partial_z x^\mu \epsilon_\mu^{*+}(E_p\hat{x}) + a_-^\dagger(E_p\hat{x})\partial_z x^\mu \epsilon_\mu^{*-}(E_p\hat{x})\right]e^{-iuE_p}\\ &-\left[a_+(E_p\hat{x})\partial_z x^\mu \epsilon_\mu^{+}(E_p\hat{x}) + a_-^\dagger(E_p\hat{x})\partial_z x^\mu \epsilon_\mu^{-}(E_p\hat{x})\right]e^{iuE_p}\\ \mathcal{A}_z =&\frac{-i\sqrt{2}}{8\pi^2(1+z\bar{z})}\int_0^\infty dE_p\left[a_+(E_p\hat{x})e^{-iuE_p} - a_-^\dagger(E_p\hat{x})e^{iuE_p}\right] + \mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{A.10}$$

The leading component of $\mathcal{A}_z(u,z,\bar{z})$ is the free field data denoted as $A_z^{(0)}$ at the boundary ($\mathscr{I}^+$) from which we can find all other fields. The expression suggests that this field annihilates photons of positive helicity and creates photon of negative helicity at the boundary since the creation and annihilation operators are defined on sphere S^2. Similar field can be shown to exist at past null infinty $\mathscr{I}^-$ using the same exact method. For the $\mathcal{A}_u$ component of the field near null infinity $\epsilon_u = \partial_u x^\mu \epsilon_\mu = \epsilon_0$, as $x^0 = t = u + r$. Therefore,

$$\begin{aligned}\mathcal{A}_u =&\frac{-i}{8\pi^2 r}\sum_{\alpha=\pm}\int_0^\infty dE_p\left[\epsilon_0^{*\alpha}(E_p\hat{x})a_\alpha(E_p\hat{x})e^{-iuE_p} - \epsilon_0^{\alpha}(E_p\hat{x})a_\alpha^\dagger(E_p\hat{x})e^{iuE_p}\right]\\ &+\mathcal{O}\left(\frac{1}{r^2}\right)\end{aligned} \tag{A.11}$$

Fall-off of massless scalar fields near null inifinity can be calculated in exactly the same was as Maxwell field A_z. The saddle point is also similar and therefore for the scalar field defined in Minkowski space it fall off as $\mathcal{O}(1/r)$

Appendix B

Commutators using Symplectic product

In this section we review on how to get commutators from the symplectic product in field formalism [14]. Consider the radiative part of symplectic product of free Maxwell field on the cauchy surface at $\mathscr{I}^+$ given in equation (3.48),

$$\Omega_{radiative} = -2\int_{\mathscr{I}^+} d^2z\,du(\delta\partial_u\hat{A}_z\delta'\hat{A}_{\bar{z}} - \delta'\partial_u\hat{A}_z\delta\hat{A}_{\bar{z}}) \tag{B.1}$$

Comparing with standard form [14],

$$\Omega = \Omega_{AB}(\delta_1 A)^A(\delta_2 A)^B \tag{B.2}$$

we have

$$(\delta_1 A)^1 = \delta\partial_u\hat{A}_z, \qquad (\delta_1 A)^2 = \delta\hat{A}_{\bar{z}} \tag{B.3}$$

$$(\delta_2 A)^1 = \delta'\partial_u\hat{A}_z, \qquad (\delta_2 A)^2 = \delta'\hat{A} \tag{B.4}$$

$$\Omega_{12} = -2, \qquad \Omega_{21} = 2 \tag{B.5}$$

The poisson bracket of any two function $\{F, G\}$ is given by,

$$\{F, G\} = -\Omega^{AB}(\delta F)_A(\delta G)_B \tag{B.6}$$

Inverting Ω_{AB} we get,

$$\Omega^{12} = \frac{1}{2}, \qquad \Omega^{21} = -\frac{1}{2}$$

Therefore the poisson bracket $\{\partial_u\hat{A}_z, \hat{A}_{\bar{z}}\}$ is given by,

$$\begin{aligned}
\{\partial_u\hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{z}}(u',z',\bar{z}')\} &= -\,\Omega^{AB}(\delta\partial_u\hat{A}_z)_A(\delta\hat{A}_{\bar{z}})_B \\
&= -\,\Omega^{12}\frac{\delta(\partial_u\hat{A}_z(u,z,\bar{z}))}{\delta(\partial_{u''}\hat{A}_{z''}(u'',z'',\bar{z''}))}\frac{\delta(\hat{A}_{\bar{z}'}(u',z',\bar{z}'))}{\delta(\hat{A}_{\bar{z}''}(u'',z'',\bar{z}''))} \\
&= -\,\frac{1}{2}\delta(u-u'')\delta^2(z-z'')\delta(u'-u'')\delta^2(z'-z'') \\
&= = -\frac{1}{2}\delta(u-u')\delta^2(z-z')
\end{aligned}$$

To get the commutator in quantum mechanics we multiply poisson bracket by i. Therefore

$$[\partial_u\hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{z}}(u',z',\bar{z}')] = -\frac{i}{2}\delta(u-u')\delta^2(z-z') \tag{B.7}$$

Similary for the soft part of the symplectic product in equation (3.48),

$$\Omega_{soft} = 2\int_{\mathscr{I}^+} d^2z\,(\partial_z\delta\Phi\delta'\partial_{\bar{z}}N - \partial_z\delta'\Phi\delta\partial_{\bar{z}}N) \tag{B.8}$$

We can find the commutators of the boundary field using the method that was described earlier. The relevant commutator is,

$$[\partial_z\phi(z,\bar{z}),\partial_{\bar{w}}N(w,\bar{w})] = \frac{i}{2}\delta^2(z-w) \tag{B.9}$$

Integrating equation (B.7) with respect to u we get [5],

$$[\hat{A}_z(u,z,\bar{z}),\hat{A}_{\bar{z}}(u',z',\bar{z}')] = -\frac{i}{4}\Theta(u-u')\delta^2(z-z'), \tag{B.10}$$

where $\Theta(u) = \frac{1}{\pi i}\int\frac{d\omega}{\omega}e^{i\omega u}$ is a step function with $\Theta(u<0) = -1$ and $\Theta(u>0) = 1$. This definition of Θ can be verified by solving the contour integral. Now we calculate the commutator of conserved charge with the free field $A_z(u,z,\bar{z}')$ at the boundary. Since the charge also has contribution from the matter field i.e Q_{hard} , its commutator with the Maxwell field go to zero. So we would get only the contribution from the soft charge in the commutator. Therefore from equation (3.44),

$$\begin{aligned}
[\,Q^+,\,A_z(u,z',\bar{z}')\,] =&[\,Q^+_{soft},\,A'_z(u,z',\bar{z}')\,] = -2\int d^2z\partial_z\lambda[\,\partial_{\bar{z}}N,\,\hat{A}_z+\partial_z\phi\,]\\
[\,Q^+,\,A_z(u,z',\bar{z}')\,] =&i\partial_z\lambda(z,\bar{z}),
\end{aligned} \tag{B.11}$$

where we have used equation (B.9) and the fact that $[\,\partial_{\bar{z}}N,\,\hat{A}_z\,] = 0$. Rest all other commutators are trivial.

Now we calculate the commutators of massive scalar field using symplectic product at $\tau=$ constant Cauchy surface. The symplectic product is given by,

$$\lim_{\tau\to\infty}\Omega_{\Sigma_\tau}(\delta,\delta') = -\frac{im^2}{2(2\pi)^3}\int_{\mathcal{H}_+}d^3V\left[\delta b\delta' b^* + \delta d\delta' d^* - \delta' b\delta b^* - \delta d^*\delta' d\right] \tag{B.12}$$

Comparing it with the standard form,

$$\Omega = \Omega_{AB}(\delta_1\phi)^A(\delta_2\phi)^B \tag{B.13}$$

we have,

$$(\delta_1\phi)^1 =\delta b,\qquad (\delta_1\phi)^2 = \delta b^* \tag{B.14}$$

$$(\delta_2\phi)^1 =\delta' b,\qquad (\delta_2\phi)^2 = \delta' b^* \tag{B.15}$$

$$\Omega_{12} =\frac{-im^2}{2(2\pi)^3},\qquad \Omega_{21} = \frac{im^2}{2(2\pi)^3} \tag{B.16}$$

$$\Omega^{12} =\frac{2(2\pi)^3}{im^2},\qquad \Omega^{21} = \frac{-2(2\pi)^3}{im^2} \tag{B.17}$$

Therefore the poisson bracket $\{b(m\rho\hat{x}),b^*(m\rho'\hat{x}')\}$ is given by,

$$\{b(m\rho\hat{x}),b^*(m\rho'\hat{x}')\} = -\,\Omega^{12}(\delta b)_1(\delta b^*)_2 \tag{B.18}$$

$$= -\frac{2(2\pi)^3}{im^2}\frac{\delta b(m\rho\hat{x})}{\delta b(m\rho'\hat{x}')}\frac{\delta b^*(m\rho\hat{x})}{\delta b^*(m\rho'\hat{x}')} \tag{B.19}$$

$$\{b(m\rho\hat{x}),b^*(m\rho'\hat{x'})\} = -\frac{2(2\pi)^3}{im^2}\delta(\rho-\rho')\delta^2(\hat{x}-\hat{x'}) \tag{B.20}$$

Appendix C

Derivation of soft photon theorem

In this section we review the derivation of soft photon theorem [10]. For scalar QED we consider the scattering process of m incoming charged particles and n outgoing particles with the emission of one soft photon having polarization index μ. The process is depicted in the figure below. The Feynman rules for the propagators and vertex involved in scalar QED are given in figure (C.2).

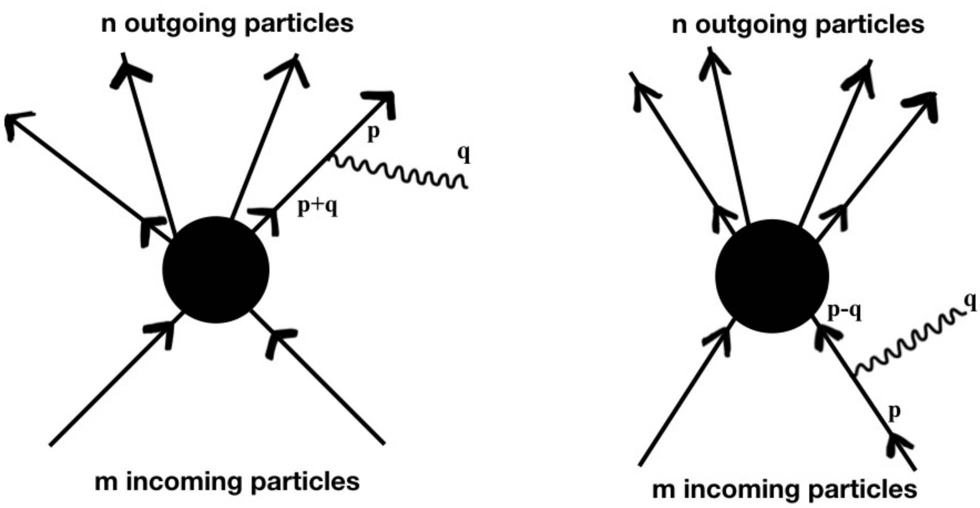

Figure C.1: Leading Feynman graph for emission of soft photon in an arbitrary process. The straight lines are particles in incoming and outgoing states and wavy lines are soft photon which is attached to external line.

Let the amplitude for scattering process of m incoming particles and n outgoing particles labeled by $\alpha \to \beta$ is denoted by $\mathcal{M}_{\alpha\beta}$. Given the Feynman rules the amplitude of scattering shown in equation (C.1) where the emission of soft photon takes place from outgoing external leg is,

$$\mathcal{M}^{\mu}_{\alpha\beta}(q) = -ie(p+q)^{\mu} + p^{\mu}\left[\frac{i}{(p+q)^2 + m^2 + i\epsilon}\right]\mathcal{M}_{\alpha\beta}. \tag{C.1}$$

Since the external legs are on-shell, therefore $p^2 = -m^2$. In the limit $q \to 0$ we have,

$$\mathcal{M}^{\mu}_{\alpha\beta}(q) = \lim_{q\to 0} \frac{ep^{\mu}}{p.q + i\epsilon}\mathcal{M}_{\alpha\beta} \tag{C.2}$$

If we consider soft photon emission from the incoming legs as shown in the right side of figure (C.1) then the amplitude becomes,

$$\mathcal{M}^{\mu}_{\alpha\beta}(q) = \lim_{q\to 0} \frac{ep^{\mu}}{-p.q + i\epsilon}\mathcal{M}_{\alpha\beta}. \tag{C.3}$$

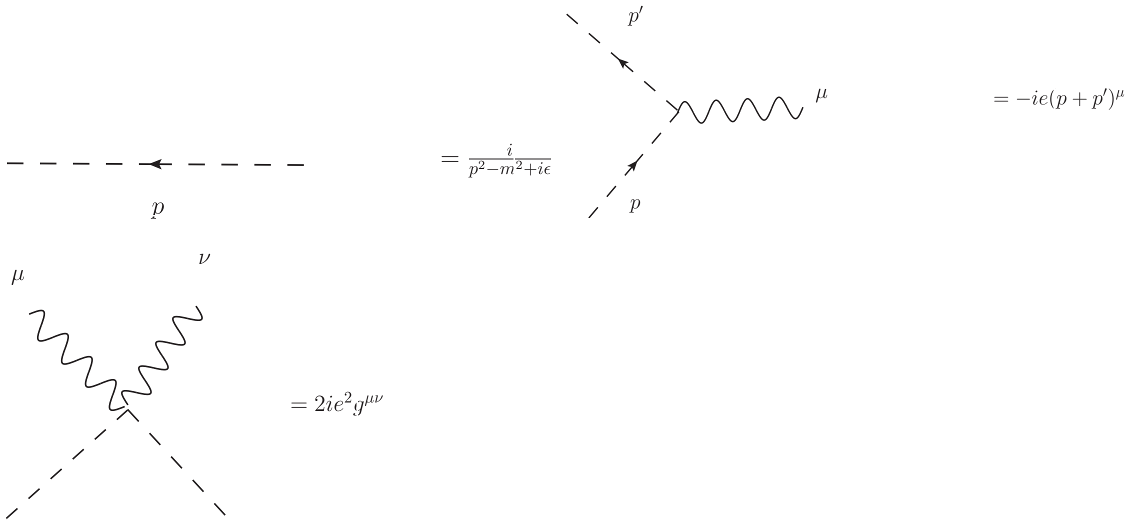

Figure C.2: Feynman rules for scalar QED

Since the soft photon can be emitted from any of the n external outgoing legs or m external incoming legs , the total amplitude can be calculated as sum of all the individual amplitudes of emission of one soft photon from an external leg for all the external legs. Therefore total amplitude is given by,

$$\mathcal{M}^{\mu}_{\beta\alpha} = \lim_{q\to 0}\left[\sum_{i=1}^{n} \frac{Q_i p_i^{\mu}}{p_i.q + i\epsilon} - \sum_{i=1}^{m} \frac{Q_i p_i^{\mu}}{p_i.q + i\epsilon}\right]\mathcal{M}_{\alpha\beta} \tag{C.4}$$

Q_i and p_i is the charge and four momentum of i^{th} particle and q is the four momentum of photon. To get the amplitude for the emission of photon with the definite helicity we contract equation (C.4) with positive helicity polarization vector e^{+}_{μ} as,

$$e^{+}_{\mu}\mathcal{M}^{\mu}_{\beta\alpha} = \mathcal{M}^{+}_{\beta\alpha} = \lim_{q\to 0}\left[\sum_{i=1}^{n} \frac{Q_i p_i^{\mu}.e^{+}_{\mu}}{p_i.q + i\epsilon} - \sum_{i=1}^{m} \frac{Q_i p_i^{\mu}.e^{+}_{\mu}}{p_i.q + i\epsilon}\right]\mathcal{M}_{\alpha\beta} \tag{C.5}$$

The contribution coming from the emission of photon from any of the internal lines are not considered because the internal propagators of particles are not on-shell and one simply cannot put $p^2 = -m^2$ and hence the mass term does not cancel. Therefore in the limit $q \to 0$ the contribution from the charged particle propagator is dropped out and we don't get any poles, therefore these type of diagrams can be ignored while considering the soft limit.

Appendix D

BMS group in 3+1 dimensions

Here the BMS group in $3+1$ dimension is reviewed [1, 2, 37, 38]. To define asymptotically flat spacetime one specify the fall off conditions of $g_{\mu\nu}$ in r and then specify the notion of asymptotic flatness which demands that as $r \to \infty$ the metric should become,

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \tag{D.1}$$

This is Minkowski metric whose boundary was constructed using conformal compactification in section (2). The aim is to study the asymptotic symmetries of these solutions of Einstien's equations which reduces to Minkowski spacetime as $r \to \infty$. The metric ansatz for the class of allowed metrics representing asymptotically flat spacetime is given by ,

$$ds^2 = \left(\frac{V}{r}e^{2\beta} + g_{AB}U^A U^B\right)du^2 - 2e^{2\beta}dudr - 2g_{BC}U^c dudx^B + g_{AB}dx^A dx^B \tag{D.2}$$

This metric is written in bondi gauge which is given by the conditions

$$g_{rr} = 0, \qquad g_{rA} = 0, \qquad det\big(g_{AB}\big) = r^4 det(\gamma_{AB}) \tag{D.3}$$

The third condition follows from the definition of $g_{AB} = r^2\gamma_{AB}$, where γ_{AB} is the unit metric on sphere. Apart from these gauge conditions, following boundary conditions are impose which are not too stringent and allows for possibility of some physical examples,

$$\begin{aligned}\lim_{r\to\infty} g_{uu} =& -1 + \mathcal{O}(r^{-1}),\ \lim_{r\to\infty} g_{ur} = -1 + \mathcal{O}(r^{-2}),\ g_{uA} = \mathcal{O}(1),\\ g_{AB} =& r^2\gamma_{AB} + \mathcal{O}(r)\end{aligned} \tag{D.4}$$

The above condition follows from fall-off of functions β, U^A and V as,

$$\beta = \mathcal{O}(r^{-2}),\ U^A = \mathcal{O}(r^{-2}),\ \frac{V}{r} = -1 + \mathcal{O}(r^{-1}) \tag{D.5}$$

To compute the isometries of the metric the Lie derivative of the metric $\mathcal{L}_\zeta g_{\mu\nu} = 0$, where ζ is the vector field that generates diffeomorphisms on the manifold. Under any coordinate transformation the form of metric should be preserved, therefore to find the diffeomorphisms that leave the form of asymptotically flat metric invariant we demand that under such transformations the bondi gauge conditions (D.4) and boundary conditions equation (D.5) are preserved. Requirement of such preservation translates to,

$$\mathcal{L}_\zeta g_{rr} = 0, \qquad \mathcal{L}_\zeta g_{rA} = 0, \qquad \mathcal{L}_\zeta det(g_{AB}) = 0 \tag{D.6}$$

The Lie derivative of metric tensor is defined as follows,

$$\mathcal{L}_\zeta g_{\mu\nu} = \partial_\sigma\zeta^\sigma g_{\mu\nu} + \partial_\mu\zeta^\sigma g_{\sigma\nu} + \partial_\nu\zeta^\sigma g_{\mu\sigma} \tag{D.7}$$

Therefore from the first condition in equation (D.6) we get,

$$\begin{aligned}\mathcal{L}_\zeta g_{rr} &= 0 = \partial_r\zeta^u\\ \implies \zeta^u &= f(u, x^A)\end{aligned} \tag{D.8}$$

From second condition [38],

$$\begin{aligned}\mathcal{L}_\zeta g_{rA} = \partial_r\zeta^B g_{BA} + \partial_A\zeta^u g_{ru} = 0\\ \partial_r\zeta^c = e^{2\beta}g^{AC}\partial_A\zeta^u\\ \zeta^A = Y^A(u,x^B) - \partial_B f(u,x^A)\int_r^\infty g^{AB}e^{2\beta}dr,\end{aligned} \tag{D.9}$$

where g_{ru} can be read off from the form of metric given in equation (D.2). Solving for third condition requires the use of expression for the variation of metric,

$$\begin{aligned}\delta det(g^{AB}) = det(g^{AB})g^{CD}\delta g_{CD}\\ \mathcal{L}_\zeta det(g^{AB}) = det(g^{AB})g^{CD}\mathcal{L}_\zeta g_{CD}\end{aligned} \tag{D.10}$$

For the inverse of metric $g_{AB} = r^2\gamma_{AB} + \mathcal{O}(r)$ lets work with $(z,\bar{z})$ coordinate, so that only $g^{z\bar{z}}$ is non zero. The determinant of g_{AB} is given by ,

$$\begin{aligned}det(g_{AB}) =& -(r^2\gamma_{z\bar{z}} + \mathcal{O}(r))(r^2\gamma_{z\bar{z}} + \mathcal{O}(r))\\ =& -(r^4\gamma_{z\bar{z}}\gamma_{z\bar{z}} + \mathcal{O}(r^3) + \mathcal{O}(r))\end{aligned} \tag{D.11}$$

Therefore g^{AB} is given by,

$$\begin{aligned}g^{AB} &= \frac{1}{-(r^4\gamma_{z\bar{z}}\gamma_{z\bar{z}} + \mathcal{O}(r^3) + \mathcal{O}(r))}\begin{pmatrix} 0 & -(r^2\gamma_{z\bar{z}} + \mathcal{O}(r)) \\ -(r^2\gamma_{z\bar{z}} + \mathcal{O}(r)) & 0\end{pmatrix}\\ \implies g^{z\bar{z}} &= \frac{1}{r^2\gamma_{z\bar{z}}} + \mathcal{O}\left(\frac{1}{r^3}\right) = \frac{\gamma^{z\bar{z}}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\end{aligned} \tag{D.12}$$

Therefore from the third condition in equation (D.4) we get,

$$\begin{aligned}g^{AB}\mathcal{L}_\zeta g_{AB} =&0\\ g^{AB}\zeta^\sigma\partial_\sigma g_{AB} + 2g^{AB}\partial_A\zeta^\sigma g_{\sigma B} =&0\end{aligned} \tag{D.13}$$

The first and second term in equation (D.13) is given by,

$$\begin{aligned}2g^{AB}\partial_A\zeta^\sigma g_{\sigma B} =& -2\partial_z f U^z g_{z\bar{z}}g^{z\bar{z}} - \partial_{\bar{z}} f U^{\bar{z}} g_{z\bar{z}}g^{z\bar{z}} + 2\partial_A\zeta^c g_{BC}g^{AB}\\ =& -2\partial_z f U^z - 2\partial_{\bar{z}} f U^{\bar{z}} + 2\partial_z\zeta^z + 2\partial_{\bar{z}}\zeta^{\bar{z}} + \mathcal{O}(r^{-3})\end{aligned} \tag{D.14}$$

$$g^{AB}\zeta^\sigma\partial_\sigma g_{AB} = \frac{4\zeta^r}{r} + 2\zeta^z\frac{\partial_z\gamma_{z\bar{z}}}{\gamma_{z\bar{z}}} + 2\zeta^{\bar{z}}\frac{\partial_{\bar{z}}\gamma_{z\bar{z}}}{\gamma_{z\bar{z}}} + \mathcal{O}\left(\frac{1}{r}\right) \tag{D.15}$$

Adding both the terms in the large r limit we get,

$$\begin{aligned}\frac{4\zeta^r}{r} =& 2\partial_z f U^z + 2\partial_{\bar{z}} f U^{\bar{z}} - 2D_z\partial^z - 2D_{\bar{z}}\zeta^{\bar{z}}\\ \implies \zeta^r =& -\frac{r}{2}[D_A\zeta^A - U^C\partial_C f]\end{aligned} \tag{D.16}$$

Therefore the components of vector fields are given by equation (D.8), equation (D.9) and equation (D.16). Preservation of boundary conditions(D.4) means that the Lie derivative of the metric component should fall-off like the metric component itself. In other words the terms which are not leading or sub-leading are set equal to zero. This gives relationship between various functions and further reduce the dependence of functions on the number of coordinates. The Lie derivative of g_{ur} is given by $\mathcal{L}_\zeta g_{ur} = \zeta^\sigma\partial_\sigma g_{ur} + \partial_u\zeta^\sigma g_{\sigma r} + \partial_r\zeta^\sigma g_{u\sigma}$. Solving this one gets,

$$\mathcal{L}_\zeta g_{ur} = [-\partial_u f - \zeta^r\partial_r - \zeta^u\partial_u - \zeta^A\partial_A + \frac{1}{2}(D_A\zeta^A - U^C\partial_c f) - \partial_D f U^D]e^{2\beta}$$

Substituting the fall-off of β and U^C from equation (D.5) we get,

$$-\partial_u f + \frac{1}{2}D_A\zeta^A + \mathcal{O}(r^{-2}) \tag{D.17}$$

Preservation of boundary condition (D.4) requires that terms of $\mathcal{O}(1)$ and $\mathcal{O}(r^{-1})$ are set equal to zero. Therefore we get,

$$-\partial_u f + \frac{1}{2} D_A Y^A(u, x^B) = 0 \tag{D.18}$$

The Lie derivative of g_{uA} is given by,

$$\begin{aligned}\mathcal{L}_\zeta g_{uA} =& \zeta^\sigma \partial_\sigma g_{uA} + \partial_u \zeta^\sigma g_{\sigma A} + \partial_A \zeta^\sigma g_{u\sigma} \\ =& -\zeta^r \partial_r (g_{AC} U^C) + \zeta^u \partial_u (g_{AC} U^C) - \zeta^B \partial_B (g_{AC} U^C) - \partial_u f g_{AC} U^C \\ &+ \partial_u \zeta^B g_{BA} + \partial_A f \left[\frac{V}{r} e^{2\beta} + g_{AB} U^A U^B \right] - \partial_A \zeta^r e^{2\beta} - \partial_A \zeta^B g_{BC} U^C \end{aligned} \tag{D.19}$$

After substitution of fall-off we set the terms of $\mathcal{O}(r^2)$ equal to zero therefore,

$$\partial_u Y^B = 0 \tag{D.20}$$

The Lie derivative of g_{AB} is given by the expression $\mathcal{L}_\zeta g_{AB} = \zeta^\sigma \partial_\sigma g_{AB} + \partial_A \zeta^C g_{\sigma B} + \partial_B \zeta^\sigma g_{A\sigma}$. The first term in the expression can be written as

$$\begin{aligned}\zeta^\sigma \partial_\sigma g_{AB} &= \zeta^r \partial_r [r^2 \gamma_{AB} + \mathcal{O}(r)] + \zeta^u \partial_u [r^2 \gamma_{AB} + \mathcal{O}(r)] + \zeta^c \partial_c [r^2 \gamma_{AB} + \mathcal{O}(r)] \\ &= 2r \zeta^r \gamma_{AB} + r^2 \zeta^C \partial_C \gamma_{AB} + r^2 \zeta^u \partial_u \gamma_{AB} + \mathcal{O}(1) \end{aligned} \tag{D.21}$$

The second term is given by,

$$\begin{aligned}\partial_A \zeta^\sigma g_{\sigma B} &= \partial_A \zeta^u g_{uB} + \partial_A \zeta^C g_{CB} = -(\partial_A f) g_{BC} U^C + \partial_A \zeta^C [r^2 \gamma_{CB} + \mathcal{O}(r)] \\ &= r^2 \gamma_{CB} \partial_A \zeta^C + \mathcal{O}(r) \end{aligned} \tag{D.22}$$

The third term in the expression is,

$$\partial_B \zeta^\sigma g_{A\sigma} = r^2 \gamma_{AC} \partial_B \zeta^C \tag{D.23}$$

Therefore adding these three terms and setting terms proportional to $\mathcal{O}(r^2)$ equal to zero we get,

$$\begin{aligned} r^2 \gamma_{AC} \partial_B \zeta^C + r^2 \gamma_{AB} \partial_A \zeta^c - r^2 [D_E \zeta^E - U^C \partial_c f] \gamma_{AB} + r^2 Y^C \partial_c \gamma_{AB} \\ \gamma_{AC} \partial_B Y^C + \gamma_{AB} \partial_A Y^C - D_E Y^E \gamma_{AB} + Y^C \partial_C \gamma_{AB} = 0 \end{aligned} \tag{D.24}$$

This is the conformal killing equation for Y^A with respect to the metric γ_{AB} on S^2 at null infinity. equation (D.18) and equation (D.20) suggests that Y^A is independent of u and f can be written as the sum of u independent part and term with linear dependence on u. Therefore final vector field that generates non trivial diffeomorphisms on asymptotically flat metric is given by,

$$\zeta^u = f = T(x^A) + \frac{u}{2} D_C Y^C \tag{D.25}$$

$$\zeta^A = Y^A(x^C) - \partial_B f \int_{r'}^{\infty} e^{2\beta} g^{AB} dr' \tag{D.26}$$

$$\zeta^r = -\frac{r}{2} (D_A \zeta^A - U^C \partial_C f) \tag{D.27}$$

This is the gauge parameter leaving invariant the form of BMS gauge in four dimensions. If one substitute the fall off of β, g^{AB} and U^C and take the limit $r \to \infty$, the gauge parameter at the boundary $\mathscr{I}^+$ becomes,

$$\zeta' = \left(T(x^A) + \frac{u}{2} D_C Y^C \right) \partial_u + Y^A \partial_A \tag{D.28}$$

ζ' is the gauge parameter defined on the boundary with functions T and Y^A are the function on S^2. The Lie bracket of two vector fields at the boundary is defined as,

$$[\, \zeta_1', \, \zeta_2' \,] = \zeta_1'^{\sigma} \partial_\sigma \zeta_2' - \zeta_2'^{\sigma} \partial_\sigma \zeta_1' \tag{D.29}$$

Substituting equation (D.28) in the above equation we see,

$$
\begin{aligned}
[\,\zeta_1',\,\zeta_2'\,] =& \zeta_1'^u\partial_u\zeta_2' - \zeta_2'^u\partial_u\zeta_1' + \zeta_1'^A\partial_A\zeta_2' - \zeta_2'^A\partial_A\zeta_1' \\
=& \Bigg[Y_1^C\partial_C T_2 - Y_2^C\partial_C T_1 + \frac{1}{2}(T_1 D_C Y_2^C - T_2 D_C Y_1^C) + \\
& \frac{1}{2}D_C[Y_1^E\partial_E Y_2^C - Y_2^E\partial_E Y_1^C]\Bigg]\partial_u + [Y_1^C\partial_C Y_2^A - Y_2^C\partial_C Y_1^A]\partial_A \\
=& \hat{f}\partial_u + \hat{Y}^A\partial_A
\end{aligned}
\tag{D.30}
$$

Therefore the Lie algebra is closed under Lie bracket. This is the symmetry algebra of BMS_4 group. If one sets $Y^A = 0$ then one can see from equation (D.30) the symmetry algebra of $\zeta_T' = T\partial_u$ is abelian subalegbra of bms_4 algebra. In this case ζ_T' generates angle dependent translations in the direction of u and are called *supertranslations*. The part $\frac{u}{2}D_C Y^C\partial_u + Y^A\partial_A$ is responsible for generating conformal transformations on null infinity. The BMS_4 is of two type depending on the space of functions under consideration.

Global BMS group: When the functions are restricted to the globally well defined transformations on S^2 this give rise to global conformal transformations and the associated group is isomorphic to $SL(2,C)/Z_2$ which as it turns out is isomorphic to proper, orthochronous Lorentz group. In this choice $T(\theta,\phi)$ can be expanded in the natural basis of spherical harmonics Y_{lm} which are smooth functions on S^2. The normal translations are the special case of supertranslations for l= 0 and l= 1 spherical harmonics for example, $T = a + a_x \sin\theta\cos\phi + a_y \sin\theta \sin +\phi + a_z \cos\phi$ form an invariant 4-dimensional translation group where a is time translation and (a_x, a_y, a_z) are space translations [39]. Therefore the resulting group become the semi-direct product of these smooth functions (supertranslations) and global conformal transformations which is isomorphic to Lorentz group.

Local BMS group Taking hint from 2 dimensional conformal field theory if the functions Y^A, T are allowed to be meromorphic functions with finite set of poles on the sphere then they can be developed in Laurent series if one works in stereographic coordinates $(z,\bar{z})$ defined in section (2). The general solution of conformal killing equation in (D.24) is $Y^z = Y(z)$ and $Y^{\bar{z}} = Y(\bar{z})$. Now if the standard basis vectors are chosen as, [23]

$$
\begin{aligned}
l_n &= -z^{n+1}\partial_z \\
\bar{l}_n &= -\bar{z}^{n+1}\partial_{\bar{z}}
\end{aligned}
\tag{D.31}
$$

These have the form of generators of local conformal transformation and if the generator of supertranslations are expanded as follows,

$$
T_{m,n} = z^m \bar{z}^n \tag{D.32}
$$

Therefore the local bms_4 Lie algebra is given by following relations,

$$
[\,l_m, l_n\,] = (m-n)l_{m+n}, \quad [\,\bar{l}_m, \bar{l}_n\,] = (m-n)\bar{l}_{m+n}, \quad [\,l_m, \bar{l}_m\,] = 0 \tag{D.33}
$$

This is basically the two copies of Witt algebra which manifest itself here because we allowed for mermorphic functions. The commutation relation of l_m with $T_{m,n}$ is given by,

$$
[\,l_n, T_{p,q}\,] = \left(\frac{n+1}{2} - p\right)T_{p+l,q}, \qquad [\,\bar{l}_n, T_{p,q}\,] = \left(\frac{n+1}{2} - q\right)T_{p,q+l} \tag{D.34}
$$

The local bms_4 algebra is the direct sum of supertranslations and local conformal translations which are also called superrotations [23]. For $n = -1, 0, 1$ and $p, q = 0, 1$ this becomes Poincare algebra.

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