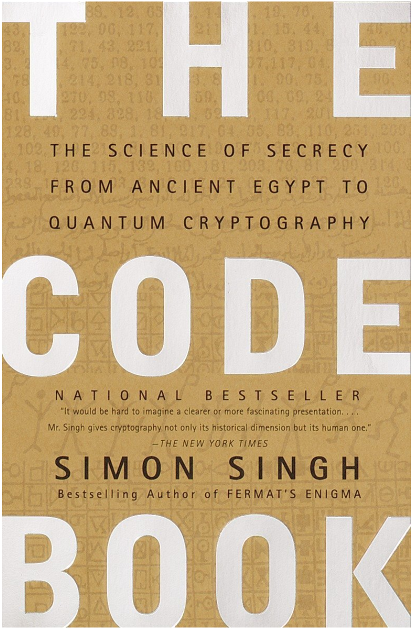

THE
THE SCIENCE OF SECRECY
FROM ANCIENT EGYPT TO
QUANTUM CRYPTOGRAPHY
CODE
NATIONAL BESTSELLER
"It would be hard to imagine a clearer or more fascinating presentation. . . .
Mr. Singh gives cryptography not only its historical dimension but its human one."
—THE NEW YORK TIMES
SIMON SINGH
Bestselling Author of FERMAT'S ENIGMA
BOOK

Image Source: Google

CSL 505

CRYPTOGRAPHY

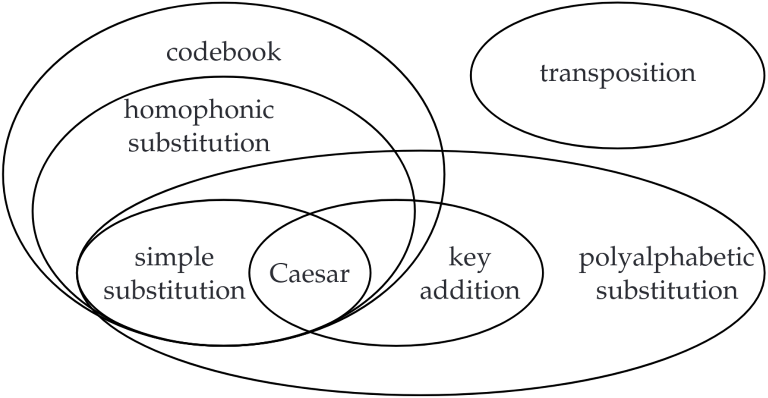
Lecture 2

Historical Ciphers (contd.)

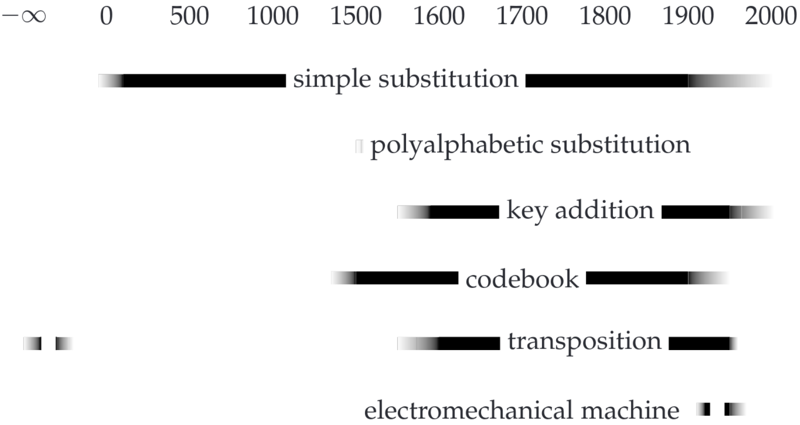
Instructor

Dr. Dhiman Saha

A Taxonomy of Basic Cryptosystems



Timeline of Crypto before Computers



$-\infty$ 0 500 1000 1500 1600 1700 1800 1900 2000

simple substitution

polyalphabetic substitution

key addition

codebook

transposition

electromechanical machine

Cryptographic Time Periods

antiquity	1500 BC – 100 AD
Arab civilization	800 – 1400
European Middle Ages	1000 – 1500
Renaissance	1450 – 1600
Baroque, salon cryptography	1600 – 1850
mechanical devices	1580 – 1950
electromechanical devices	1920 – 1950
computers	1943 – present
public key systems	1976 – present

Formal Notion Cryptosystem

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

Formal Notion

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

Formal Notion

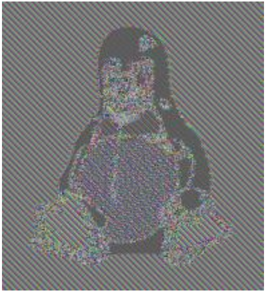
Cryptosystem

Definition

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where the following conditions are satisfied:

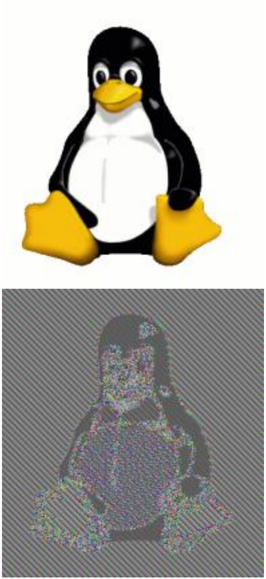
- $\mathcal{P}$ is a finite set of possible plaintexts
- $\mathcal{C}$ is a finite set of possible ciphertexts
- $\mathcal{K}$ the keyspace, is a finite set of possible keys
- For each $K \in \mathcal{K}$, there is an encryption rule $e_K \in \mathcal{E}$ and a corresponding decryption rule $d_K \in \mathcal{D}$.
- Each $e_K : \mathcal{P} \to \mathcal{C}$ and $d_K : \mathcal{C} \to \mathcal{P}$ such that $d_K(e_K(x)) = x$ for every plaintext element $x \in \mathcal{P}$.

- Each encryption function should be injective. Why?
- What is the nature of every encryption function if $\mathcal{P} = \mathcal{C}$?
- It is possible that $|\mathcal{C}| > |\mathcal{P}|$?
- d_K is definitely deterministic but what about e_K : Deterministic/Probabilistic?
- Notion of efficiency & security?



e.g. Deterministic Encryption

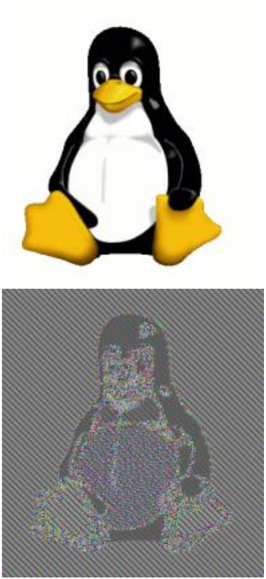
- Each encryption function should be injective. Why?
- What is the nature of every encryption function if $\mathcal{P} = \mathcal{C}$?
- It is possible that $|\mathcal{C}| > |\mathcal{P}|$?
- d_K is definitely deterministic but what about e_K : Deterministic/Probabilistic?
- Notion of efficiency & security?



e.g. Deterministic Encryption

Points to Ponder

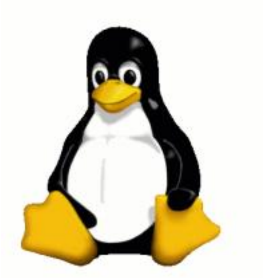
- Each encryption function should be injective. Why?
- What is the nature of every encryption function if $\mathcal{P} = \mathcal{C}$?
- It is possible that $\left|\mathcal{C}\right| > \left|\mathcal{P}\right|$?
- d_K is definitely deterministic but what about e_K : Deterministic/Probabilistic?
- Notion of efficiency & security?

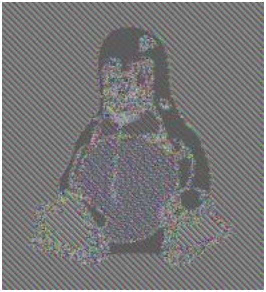


e.g. Deterministic Encryption

Points to Ponder

- Each encryption function should be injective. Why?
- What is the nature of every encryption function if $\mathcal{P} = \mathcal{C}$?
- It is possible that $|\mathcal{C}| > |\mathcal{P}|$?
- d_K is definitely deterministic but what about e_K : Deterministic/Probabilistic?
- Notion of efficiency & security?

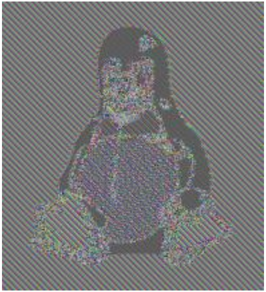




e.g. Deterministic Encryption

Points to Ponder

- Each encryption function should be injective. Why?
- What is the nature of every encryption function if $\mathcal{P} = \mathcal{C}$?
- It is possible that $\left|\mathcal{C}\right| > \left|\mathcal{P}\right|$?
- d_K is definitely deterministic but what about e_K : Deterministic/Probabilistic?
- Notion of efficiency & security?



e.g. Deterministic Encryption

Probabilistic Encryption — Indistinguishability

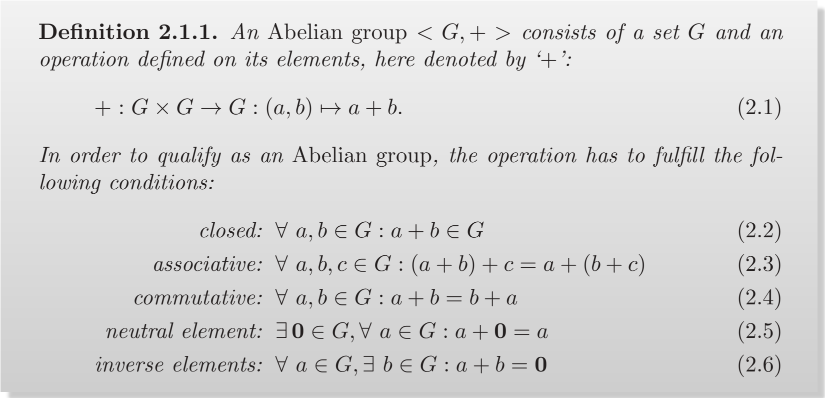
- Notion of Semantic Security and Randomized Encryption
- Captures the intuition that ciphertexts *shouldn't leak any information about plaintexts* as long as the key is secret.
- The indistinguishability game

- To be studied in detail in lecture on block ciphers

Math Recap

Algebraic Structures: Groups, Rings

Abelian Group



Definition 2.1.1. *An* Abelian group $< G, + >$ *consists of a set* G *and an operation defined on its elements, here denoted by '*$+$*':*

$$+ : G \times G \to G : (a, b) \mapsto a + b. \tag{2.1}$$

In order to qualify as an Abelian group, *the operation has to fulfill the following conditions:*

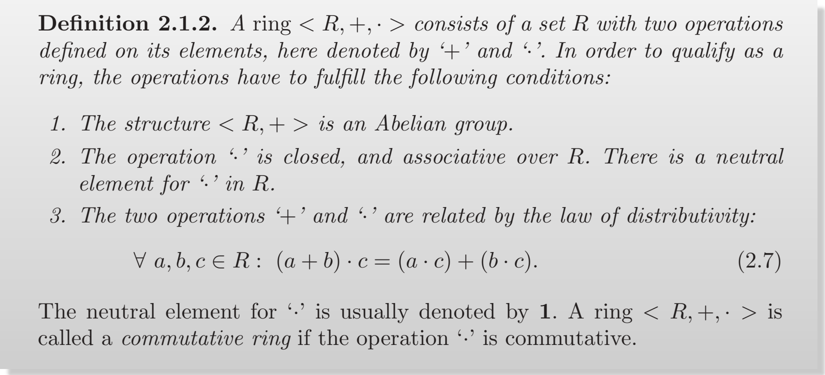
$$\begin{aligned} \textit{closed:}\quad & \forall\ a, b \in G : a + b \in G && (2.2)\\ \textit{associative:}\quad & \forall\ a, b, c \in G : (a + b) + c = a + (b + c) && (2.3)\\ \textit{commutative:}\quad & \forall\ a, b \in G : a + b = b + a && (2.4)\\ \textit{neutral element:}\quad & \exists \mathbf{0} \in G, \forall\ a \in G : a + \mathbf{0} = a && (2.5)\\ \textit{inverse elements:}\quad & \forall\ a \in G, \exists\ b \in G : a + b = \mathbf{0} && (2.6) \end{aligned}$$

Example

The set of integers with the operation 'addition': $< \mathbb{Z}, + >$

- What about $\mathbb{Z}_m$?

Math Recap — Ring



Definition 2.1.2. *A ring $< R, +, \cdot >$ consists of a set R with two operations defined on its elements, here denoted by '$+$' and '$\cdot$'. In order to qualify as a ring, the operations have to fulfill the following conditions:*

1. *The structure $< R, + >$ is an Abelian group.*
2. *The operation '$\cdot$' is closed, and associative over R. There is a neutral element for '$\cdot$' in R.*
3. *The two operations '$+$' and '$\cdot$' are related by the law of distributivity:*

$$\forall\, a, b, c \in R : \ (a + b) \cdot c = (a \cdot c) + (b \cdot c). \tag{2.7}$$

The neutral element for '$\cdot$' is usually denoted by $\mathbf{1}$. A ring $< R, +, \cdot >$ is called a *commutative ring* if the operation '$\cdot$' is commutative.

Example

The set of integers with the operation 'addition' and 'multiplication': $< \mathbb{Z}, +, \cdot >$

- What about $\mathbb{Z}_m$?

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then

$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$

- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

The Integer Ring: $\mathbb{Z}_m$

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then
$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$
- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

The Integer Ring: $\mathbb{Z}_m$

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then
$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$
- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

The Integer Ring: $\mathbb{Z}_m$

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then
$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$
- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

The Integer Ring: $\mathbb{Z}_m$

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then
$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$
- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

The Integer Ring: $\mathbb{Z}_m$

Informally

$\mathbb{Z}_m$ is the set of integers $\{0, 1, 2, \cdots, m-1\}$ in which we can add, subtract, multiply, and **sometimes** divide.

- **Sometimes** divide. Why?
- Recall, ring, by definition is not required to have multiplicative inverse for all elements,
- It exists only for some, say $a \in \mathbb{Z}_m$
- Then
$$\exists a^{-1} \in \mathbb{Z}_m : a \cdot a^{-1} \equiv 1 \mod m$$
- If inverse exists for a, then we can divide by a since $b/a \equiv b \cdot a^{-1} \mod m$

How do you know inverse exists for an element?

To invert or not to invert

- Elements relatively prime to m are invertible
- How to find?
- Hint: Greatest Common Divisor - gcd

$$\textit{Verify} \;\rightarrow\; gcd(a, m) = 1$$

Theorem

An element $a \in \mathbb{Z}_m$ is invertible if and only if $gcd(a, m) = 1$

- Check if 15,14 are invertible in $\mathbb{Z}_{26}$.
- If $a \in \mathbb{Z}_m$ is invertible, how find a^{-1}
- Euclidean GCD Algorithm

Affine Cipher

Further Generalization of Shift Cipher

Affine Cipher

- Recall Shift Cipher

Encryption

$$e(x) = (x + k) \mod 26$$

Decryption

$$d(x) = (x - k) \mod 26$$

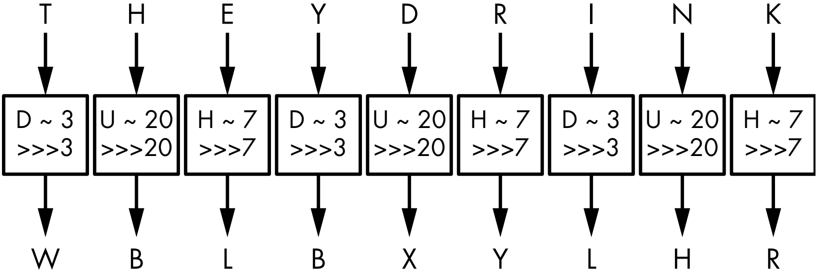
Definition (Affine Cipher)

Let $x, y, a, b \in \mathbb{Z}_{26}$
Encryption: $e_K(x) = y \equiv a \cdot x + b \mod 26$
Decryption: $d_K(y) = x \equiv a^{-1} \cdot (y - b) \mod 26$
with the key: K=(a,b) which has the restriction: $gcd(a, 26) = 1$

- "Affine" ?

Vigenere cipher

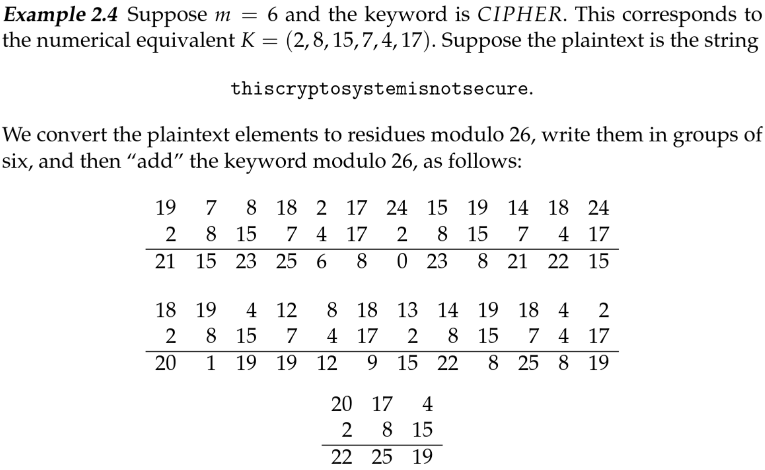
- By Giovan Battista Bellaso in 16th Century
- Used in American Civil War and World-War I
- Polyalphabetic Substitution



T	H	E	Y	D	R	I	N	K
D ~ 3 >>>3	U ~ 20 >>>20	H ~ 7 >>>7	D ~ 3 >>>3	U ~ 20 >>>20	H ~ 7 >>>7	D ~ 3 >>>3	U ~ 20 >>>20	H ~ 7 >>>7
W	B	L	B	X	Y	L	H	R

Key = 'DUH'

Another Example



Example 2.4 Suppose $m = 6$ and the keyword is *CIPHER*. This corresponds to the numerical equivalent $K = (2, 8, 15, 7, 4, 17)$. Suppose the plaintext is the string

`thiscryptosystemisnotsecure.`

We convert the plaintext elements to residues modulo 26, write them in groups of six, and then "add" the keyword modulo 26, as follows:

19	7	8	18	2	17	24	15	19	14	18	24
2	8	15	7	4	17	2	8	15	7	4	17
21	15	23	25	6	8	0	23	8	21	22	15

18	19	4	12	8	18	13	14	19	18	4	2
2	8	15	7	4	17	2	8	15	7	4	17
20	1	19	19	12	9	15	22	8	25	8	19

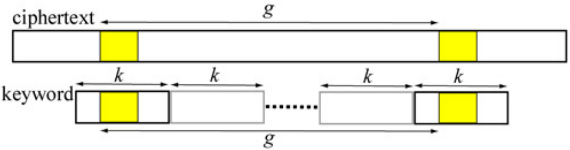
20	17	4
2	8	15
22	25	19

- Ciphertext → `VPXZGIAXIVWPUBTTMJPWIZITWZT`
- What is the size of the key-space?

Kasiski Test

Intuition

If a repeated substring in a plaintext is encrypted by the same substring in the keyword, then the ciphertext contains a repeated substring and the distance of the two occurences is a multiple of the keyword length.



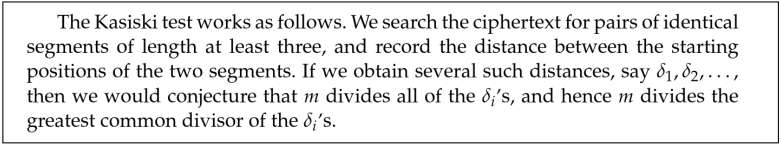
THEY DRINK THE TEA → WBLBXYLHRWBLWYH

- Note: Since the keyword of length k is repeated to fill the length of the ciphertext, the distance g is a multiple of the keyword length k.
- For example, if the distance is $g = 18$, since the factors of g are $2, 3, 6, 9$ and 18, one of them may be the length of the unknown keyword.

Image Source: `https://pages.mtu.edu/~shene/NSF-4/Tutorial/VIG/Vig-Kasiski.html`

Kasiski Test

- Friedrich W. Kasiski, a German military officer, published his book *Die Geheimschriften und die Dechiffrirkunst* (Cryptography and the Art of Decryption) in 1863



The Kasiski test works as follows. We search the ciphertext for pairs of identical segments of length at least three, and record the distance between the starting positions of the two segments. If we obtain several such distances, say $\delta_1, \delta_2, \ldots,$ then we would conjecture that m divides all of the δ_i's, and hence m divides the greatest common divisor of the δ_i's.

- Still difficult to break for short messages
- Good for short-lived messages

Trivia

The 19th-century cryptographer Auguste Kerckhoffs estimated that most encrypted wartime messages required confidentiality for only three to four hours

Hill Cipher

- Polyalphabetic; by Lester S. Hill. in 1929
- $\mathcal{P} = \mathcal{C} = (\mathbb{Z}_{26})^m$
- Let, $(x_1, x_2, \cdots, x_m) \in \mathcal{P}$, $K = (k_{i,j}) \in \mathcal{K}$ then
  $y = e_K(x) = (y_1, y_2, \cdots, y_m)$

$$(y_1, y_2, \cdots, y_m) = (x_1, x_2, \cdots, x_m) \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,m} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m,1} & k_{m,2} & \cdots & k_{m,m} \end{bmatrix}$$

- Note: every y_i is a linear combination of all x_i
- In matrix notation:
  Encryption $y = xK$ and Decryption $x = yK^{-1}$
- Invertibility of K

Permutation/Transposition Cipher

Definition

Let m be a positive integer. Let $\mathcal{P} = \mathcal{C} = (\mathbb{Z}_{26})^m$ and let $\mathcal{K}$ consist of all permutations of $1, \cdots, m$. For a key π, we define

$$e_\pi(x_1, x_2, \cdots, x_m) = x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(m)}$$

and

$$d_\pi(y_1, y_2, \cdots, y_m) = x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \cdots, x_{\pi^{-1}(m)}$$

where π^{-1} is the inverse permutation of π.

Example

Let $m = 6$ and a possible key is the permutation π:

x	1	2	3	4	5	6
$\pi(x)$	3	5	1	6	4	2

- This is a special case of Hill Cipher. How? △