Exercise: Robust and stochastic optimal control

In this exercise we will solve two non-standard optimal control problems variants. One of the promises of CasADi is that it supports going quickly from ideas on paper to a reasonably fast implementation. Let's check if it lives up to its expectations here.
We work with a Van der Pol system with two states x and one control u:

$$\frac{d\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{dt} = \begin{bmatrix} x_2 \\ -0.1(1-x_1^2)x_2 - x_1 + u \end{bmatrix} \tag{1}$$

We consider the following optimal control problem.

$$\begin{aligned} \underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad & \int_0^T \|x_1(t) - 3\|_2^2 dt \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)) \quad \forall t \in [0,T] \\ & -40 \leq u(t) \leq 40 \quad \forall t \in [0,T] \\ & -0.25 \leq x_1(t) \leq b(t) \quad \forall t \in [0,T] \\ & x(0) = \hat{x}, \end{aligned} \tag{2}$$

with $b(t) = 2 + 0.1\cos(10t)$, $T = 1$, $\hat{x} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}^T$.
A reference implementation using multiple-shooting is provided in `template`.

1 Stochastic optimal control

Tasks:

1. Run the `template`. You will notice that the upper path constraint on $x_1(t)$ is active a lot of the time. We consider now the case when $\hat{x}$ is not exactly known, but instead is described by a Gaussian distribution with expected value $E(\hat{x}) = \begin{bmatrix} 0.5 & 0 \end{bmatrix}^T$ and covariance $\Sigma_{\hat{x}} = \begin{bmatrix} 0.01^2 & 0 \\ 0 & 0.1^2 \end{bmatrix}$. This covariance propagates (in first-order approximation) through the system dynamics as the following matrix differential equation:

$$\dot{\Sigma}_{x(t)} = \frac{\partial f}{\partial x}(x(t), u(t))\Sigma_{x(t)} + \Sigma_{x(t)}\frac{\partial f}{\partial x}^T(x(t), u(t)) + Q(t),$$

with $f(\cdot,\cdot)$ the system dynamics and $Q(t)$ a white Gaussian noise injection term which we will take $0^{2\times 2}$ here.

Augment the multiple-shooting implementation with the covariance dynamics:

$$
\begin{aligned}
\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad & \int_0^T \|x_1(t) - 3\|_2^2 dt \\
\text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)) \quad \forall t \in [0, T] \\
& x(0) = \hat{x}, \\
& \dot{\Sigma}_{x(t)} = \frac{\partial f}{\partial x}(x(t), u(t))\Sigma_{x(t)} + \Sigma_{x(t)}\frac{\partial f}{\partial x}^T(x(t), u(t)) + Q(t) \quad \forall t \in [0, T] \\
& \Sigma_{x(0)} = \Sigma_{\hat{x}}, \\
& -40 \leq u(t) \leq 40 \quad \forall t \in [0, T] \\
& -0.25 \leq x_1(t) \leq b(t) \quad \forall t \in [0, T]
\end{aligned}
$$

The nominal solution trajectory $x^\star(t)$ should remain unchanged. Verify that you obtain a final state covariance `[0.0094851 0.0073140; 0.0073140 0.0058025]`.

The standard-deviation on $x_1(t)$ is given as $\sqrt{\begin{bmatrix}1 & 0\end{bmatrix} \Sigma_{x_1(t)} \begin{bmatrix}1 & 0\end{bmatrix}^T}$.

Plot $\pm$ 1 standard-deviation error bars on the solution trajectory.

2. Make the upper path constraint on $x_1(t)$ stochastically robust, ie make sure that the error bars stay below the upper bound. Verify that the resulting solution has an objective `8.5443900e+01`.

2 (extra) Robust optimal control

Tasks:

1. Let's head back to the base problem of the template.

   We consider the case where the system dynamics has a δ contribution:

$$
\frac{d\begin{bmatrix}x_1 \\ x_2\end{bmatrix}}{dt} = \begin{bmatrix} x_2 \\ -0.1(1 - x_1^2 + \delta)x_2 - x_1 + u + \delta \end{bmatrix} \tag{3}
$$

   δ is an adversial disturbance that, in each separate control interval can be realized as either -1 or $+1$.

   We'll need to consider a branching scenario tree of possibilities arising from a single control signal to be found. The path constraints should hold for each branch in the tree, and let's say that we want to optimize the worst case resultant objective term.

   To make the problem tractable, choose $N = 6$ and $\hat{x} = \begin{bmatrix}0.5 & 0\end{bmatrix}^T$.

   This problem is not trivial; there is a `hint` file available.