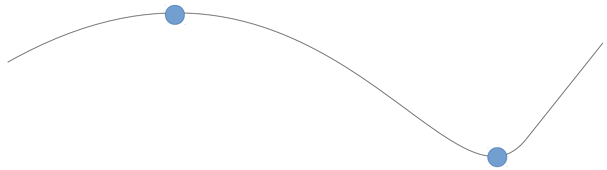
3. Nonlinear programming

Unconstrained NLP



$$\underset{x}{\text{minimize}} \quad f(x)$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}$$

Necessary condition:

$$\nabla f(x) = 0$$ Gradient

Sufficient condition: necessary condition + $\nabla^2 f(x) \succ 0$

Hessian

Unconstrained NLP

- Newton type method: apply rootfinding on $\nabla f(x)=0$
- Recall: $g(x)=0$ $\quad$ $\Delta x=-\left(\frac{\partial g}{\partial x}(x_k)\right)^{-1} g(x_k)$

For $g(x)=\nabla f(x)$

$$\Delta x=-\left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$$

Hessian

Quadratic convergence

Exact or BFGS or GN

Unconstrained NLP

- Newton type method: apply rootfinding on $\nabla f(x) = 0$
- Recall: $g(x) = 0$ $\quad \Delta x = -\left(\frac{\partial g}{\partial x}(x_k)\right)^{-1} g(x_k)$

For $g(x) = \nabla f(x)$

$$\Delta x = -\left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$$

Hessian

Quadratic convergence

$$\Delta x = -\alpha \nabla f(x_k)$$

Exact or BFGS or even a constant (steepest descent)

Caveat: must keep 'Hessian' positive definite (regularize) to go to minimum

Unconstrained NLP

- Newton type method: apply rootfinding on $\nabla f(x) = 0$

Interpretation 1

$$\Delta x = -\left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$$

Interpretation 2

- Minimum of quadratic approx.

$$f(x_k) + \frac{\partial f}{\partial x}(x_k)\Delta x + \frac{1}{2}\Delta x^T \nabla^2 f(x_k)\Delta x$$

Unconstrained NLP

- Newton type method: apply rootfinding on $\nabla f(x)=0$

Interpretation 1

$$\Delta x = -\left(\nabla^2 f_k\right)^{-1} \nabla f_k$$

Interpretation 2 $\nabla^2 f_k \succ 0$

- Minimum of quadratic approx.

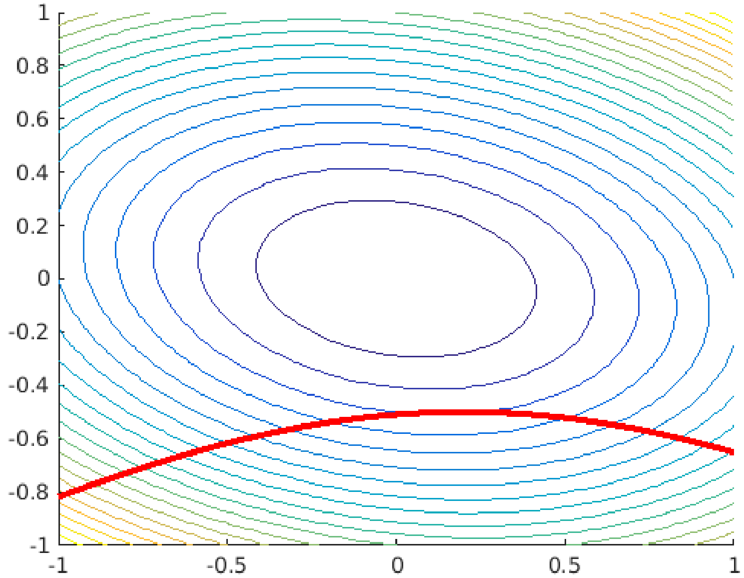
$$f_k + \frac{\partial f_k}{\partial x} \Delta x + \frac{1}{2} \Delta x^T \nabla^2 f_k \Delta x$$

Equality constrained NLP

$$\underset{x}{minimize} \quad f(x)$$

$$s.t. \quad g(x)=0$$

$$x \in \mathbb{R}^n, f:\mathbb{R}^n \rightarrow \mathbb{R},$$
$$g:\mathbb{R}^n \rightarrow \mathbb{R}$$



$$\nabla f \propto \nabla g$$

Gravitational pull is balanced by force normal to the constraint

$$\nabla f = -\lambda \nabla g$$

$$\nabla f + \lambda \nabla g = 0$$

Lagrange multiplier

Equality constrained NLP

$$\underset{x}{minimize} \quad f(x)$$

$$s.t. \quad g(x) = 0$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R},$$
$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Gravitational pull is balanced by forces normal to the constraints

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i = 0$$

Equality constrained NLP

$$\underset{x}{\mathrm{minimize}} \quad f(x)$$

$$s.t. \quad g(x) = 0$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R},$$
$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Necessary condition:

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i = 0$$
$$g = 0$$

Equality constrained NLP

$$\underset{x}{\text{minimize}} \quad f(x)$$

$$\text{s.t.} \quad g(x) = 0$$

$$x \in \mathbb{R}^n, f: \mathbb{R}^n \to \mathbb{R},$$
$$g: \mathbb{R}^n \to \mathbb{R}^m$$

Necessary condition:

$$\mathscr{L} = f + \lambda^T g$$

$$\nabla \mathscr{L} = 0$$
$$g = 0$$

Lagrangian

Equality constrained NLP

$$\underset{x}{minimize} \quad f(x)$$

$$s.t. \quad g(x)=0$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R},$$

$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Necessary condition:

$$\mathscr{L} = f + \lambda^T g$$

$$\nabla_x \mathscr{L} = 0$$

$$\nabla_\lambda \mathscr{L} = 0$$

Equality constrained NLP

$$\underset{x}{\text{minimize}} \quad f(x)$$

$$s.t. \quad g(x) = 0$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R},$$

$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathscr{L} = f + \lambda^T g$$

Necessary condition:

$$\nabla_x \mathscr{L} = 0$$

$$\nabla_\lambda \mathscr{L} = 0$$

apply rootfinding on $\nabla_{\begin{bmatrix} x \\ \lambda \end{bmatrix}} \mathscr{L}\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = 0$

$$\begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \mathscr{L}_k}{\partial x^2} & \frac{\partial^2 \mathscr{L}_k}{\partial x \partial \lambda} \\ \frac{\partial^2 \mathscr{L}_k}{\partial \lambda \partial x} & \frac{\partial^2 \mathscr{L}_k}{\partial \lambda^2} \end{bmatrix}^{-1} \begin{bmatrix} \nabla_x \mathscr{L}_k \\ \nabla_\lambda \mathscr{L}_k \end{bmatrix}$$

Equality constrained NLP

$$\underset{x}{\text{minimize}} \quad f(x)$$

$$s.t. \quad g(x) = 0$$

$$x \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R},$$
$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathscr{L} = f + \lambda^T g$$

Necessary condition:

$$\nabla_x \mathscr{L} = 0$$
$$\nabla_\lambda \mathscr{L} = 0$$

apply rootfinding on $\nabla_{\begin{bmatrix} x \\ \lambda \end{bmatrix}} \mathscr{L}\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = 0$

$$\begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \mathscr{L}_k}{\partial x^2} & \frac{\partial g_k}{\partial x}^T \\ \frac{\partial g_k}{\partial x} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla_x \mathscr{L}_k \\ g_k \end{bmatrix}$$

Equality constrained NLP

$$\underset{x}{\text{minimize}} \quad f(x)$$

$$s.t. \quad g(x)=0$$

$$x \in \mathbb{R}^n, f:\mathbb{R}^n \rightarrow \mathbb{R},$$
$$g:\mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathscr{L} = f + \lambda^T g$$

Necessary condition:

$$\nabla_x \mathscr{L} = 0$$
$$\nabla_\lambda \mathscr{L} = 0$$

apply rootfinding on $\nabla_{\begin{bmatrix} x \\ \lambda \end{bmatrix}} \mathscr{L}\left(\begin{bmatrix} x \\ \lambda \end{bmatrix}\right) = 0$

$$\begin{bmatrix} \Delta x \\ \lambda_{k+1} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \mathscr{L}_k}{\partial x^2} & \frac{\partial g_k}{\partial x}^T \\ \frac{\partial g_k}{\partial x} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla_x f_k \\ g_k \end{bmatrix}$$

Equality constrained NLP

$$\underset{\Delta x}{\text{minimize}} \quad f_k + \frac{\partial f_k}{\partial x}\Delta x + \frac{1}{2}\Delta x^T \nabla_x^2 \mathscr{L}_k \Delta x$$

$$s.t. \quad g_k + \frac{\partial g_k}{\partial x}\Delta x = 0$$

KKT matrix

Easy (convex) when:

$$\nabla_x^2 \mathscr{L}_k \succ 0$$

$$\begin{bmatrix} \Delta x \\ \lambda_{k+1} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 \mathscr{L}_k}{\partial x^2} & \frac{\partial g_k}{\partial x}^T \\ \frac{\partial g_k}{\partial x} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla_x f_k \\ g_k \end{bmatrix}$$

Needs to be full rank: constraints must be linearly independent (LICQ)

Inequality constrained NLP

$$\underset{x}{\text{minimize}} \quad f(x)$$

$$s.t. \quad g(x) = 0$$

$$h(x) \leq 0$$

$$x \in \mathbb{R}^n, f: \mathbb{R}^n \rightarrow \mathbb{R},$$
$$g: \mathbb{R}^n \rightarrow \mathbb{R}^m, h: \mathbb{R}^n \rightarrow \mathbb{R}^q$$

$$\underset{\Delta x}{\text{minimize}} \quad f_k + \frac{\partial f_k}{\partial x} \Delta x + \frac{1}{2} \Delta x^T \nabla_x \mathscr{L}_k^2 \Delta x$$

$$s.t. \quad g_k + \frac{\partial g_k}{\partial x} \Delta x = 0$$

$$h_k + \frac{\partial h_k}{\partial x} \Delta x \leq 0$$

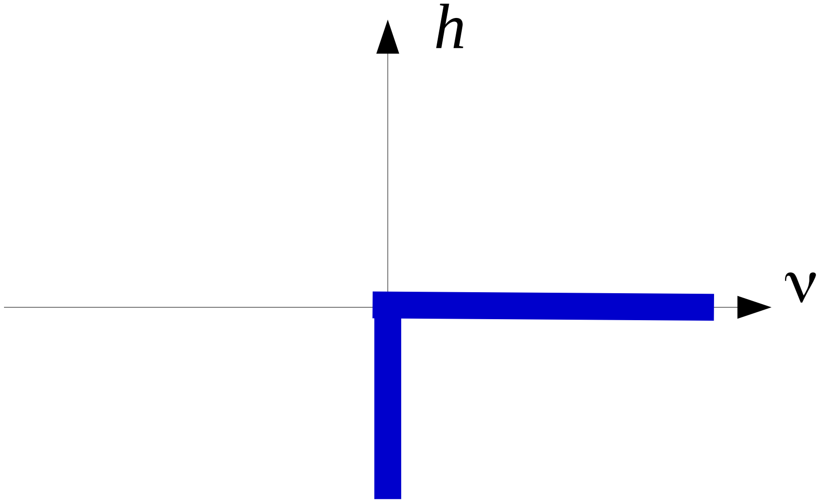
$$\begin{bmatrix} \Delta x \\ \lambda_{k+1} \\ \nu_{k+1} \end{bmatrix} \Updownarrow$$

Necessary condition:

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i + \sum_{i=1}^{q} \nu_i \nabla h_i = 0$$
$$g = 0$$
$$\nu \geq 0$$
$$h \leq 0$$
$$\nu_i h_i = 0$$

~~rootfinding~~

Inequality constrained NLP

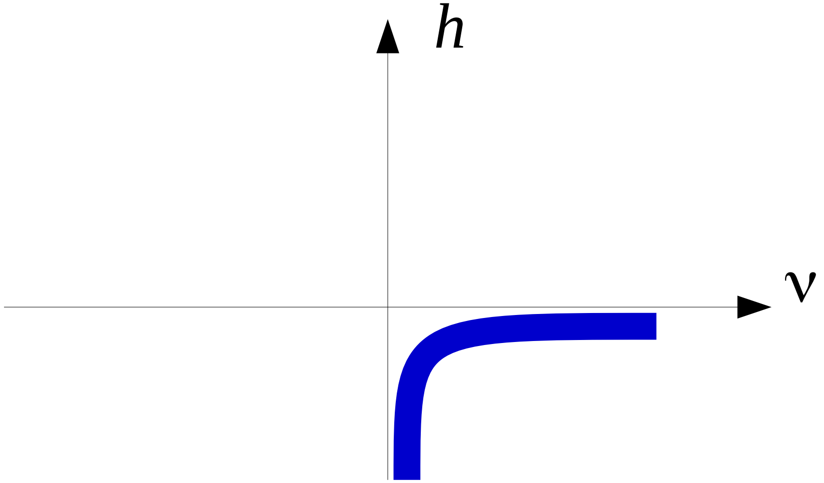


Necessary condition:

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i + \sum_{i=1}^{q} \nu_i \nabla h_i = 0$$
$$g = 0$$
$$\nu \geq 0$$
$$h \leq 0$$
$$\nu_i h_i = 0$$

Inequality constrained NLP

Interior point method (IP)



Necessary condition:

$$\nabla f + \sum_{i=1}^{m} \lambda_i \nabla g_i + \sum_{i=1}^{q} \nu_i \nabla h_i = 0$$
$$g = 0$$
$$\nu_i h_i = -\tau$$

Rootfinding!

Gradually decrease towards 0

NLP solving summary

$$\underset{x}{minimize} \quad f(x)$$

$$s.t. \quad g(x)=0$$

$$h(x)\leq 0$$

- Obtain $\begin{bmatrix}\Delta x\\ \Delta\lambda\\ \Delta\nu\end{bmatrix}$ from QP or IP-KKT

- Linesearch: choose t [0,1] such that $t\begin{bmatrix}\Delta x\\ \Delta\lambda\\ \Delta\nu\end{bmatrix}$ is a “good” step

Merit function

- Take a step, repeat

yacoda

Let's do some coding...

3. Nonlinear programming