Hands-on CasADi course on optimal control, Yacoda

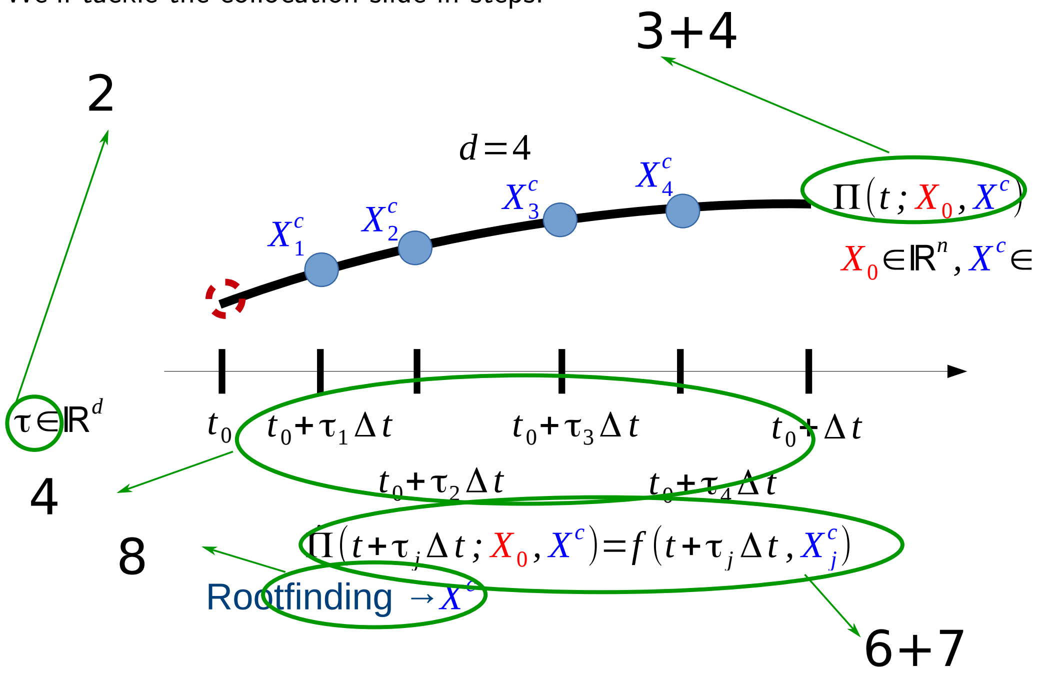
Exercise: Collocation integrator

In this exercise, we'll be integrating the following time-dependent ODE:

$$\frac{d\begin{bmatrix}x_1\\x_2\end{bmatrix}}{dt} = \begin{bmatrix}(1-x_2^2)x_1 - x_2 + t\\ x_1\end{bmatrix},$$

from $t_0 = 2$ to $t_f = 2.1$, with an initial value of $x_0 = \begin{bmatrix}1\\0.5\end{bmatrix}$.

We'll tackle the collocation slide in steps:



Tasks:

1. Implement a CasADi Function `f` that maps from time and state to the ODE's right-hand side. Create suitable `MX` symbols and use the following boilerplate:

   ```
   f = Function('f',list_of_input_symbols,list_of_output_expressions,...
                    list_of_input_labels,list_of_output_labels);
   ```

   Choose the lists such that `f` is represented as `f:(t,x[2])->(rhs[2]) MXFunction`

   Verify that you obtain `[2.25;1]` for the given initial value.

2. We will use a degree $d = 4$ `legendre` scheme for our collocation integrator. Make use of CasADi's `collocation_points` helper function to obtain values for the dimensionless collocation points τ. Use `help(collocation_points)` for help.

3. Locate and download the LagrangePolynomialEval function in the course material.

   Its mathematical prescription is given by:

$$p(\left[X_1,\ldots,X_N\right],\left[Y_1,\ldots,Y_N\right],x)=\sum_{i=1}^{N}\left(\prod_{\substack{1\leq j\leq N\\ j\neq i}}\frac{x-X_j}{X_i-X_j}\right)Y_i \tag{1}$$

   Plot the interpolating values for X=[0 0.5 1],Y=[7 1 3] and a range of x values between 0 and 1. Verify that the function performs an exact interpolation through each point (X_i, Y_i).

4. Scale the dimensionless collocation points τ_i to obtain a collocation grid of the integration interval $[t_0, t_f]$.

5. Next step is to construct Π as a CasADi Function with the following signature:

```
Pi:(t,X0[2],Xc[2x4])->(Pi[2]) MXFunction
```

   We desire that this Function interpolates exactly through (t_0, X_0) and also through the points constructed by the collocation grid and the helper states Xc defined on that grid.

   You'll need to introduce some MX symbols, and call LagrangePolynomialEval with three symbolic expressions as arguments. Verify that Pi evaluated at $t = t_0 + 0.05$,X0$= x_0$, and Xc=[x0+1 x0+2 x0+4 x0+5], yields [3.6501; 3.1501].

6. Construct a CasADi function of the time-derivative of the Π:

```
dot_Pi:(t,X0[2],Xc[2x4])->(dPi[2]) MXFunction
```

   Verify that it evaluates to [61.1122; 61.1122] for the same evaluation point.

7. Create a 8-by-1 expression g that represent the collocation residual (polynomial derivative should match right-hand side) for our system on the integration interval. Verify that, for X0$=$ x_0, and Xc=[x0+1 x0+2 x0+4 x0+5], this yields [107.130;103.137;35.726;16.509;...].

   Hint: To create a matrix expression out of individual parts formed in a for-loop, use the following idiom:

```
g = []
for j in range(d):
  g.append( ... )

g = vcat(g)
```

8. Finally, use a rootfinder to find the values of helper states X^c. Use X_0 as a parameter (p) of the rootfinder. Verify that you have an equal amount of unknowns and equations first. Note that the rootfinder expects a column vector of unknowns. You may need to flatten a matrix with vec(mat).

   Verify that your collocation integrator ends up at $x_f = [1.226454824197, 0.6113162319035]$.

9. Verify that this result is identical to the result from CasADi's built-in integrators:

```
ode = {'x':x,'t':t,'ode':rhs}
options = dict()
options["t0"] = t0
options["tf"] = tf
options["number_of_finite_elements"] = 1
options["interpolation_order"] = 4
options["collocation_scheme"] = 'legendre'
intg = integrator('intg','collocation',ode,options)
res=intg(x0=x0)
print(res["xf"])
```