NORTH-HOLLAND

Relationships Between Discrete-Time and Continuous-Time Algebraic Riccati Inequalities

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ABSTRACT

The bilinear transformation is used to establish a direct relationship between a discrete-time algebraic Riccati inequality (DARI) and an associated continuous-time algebraic Riccati inequality (CARI). It is shown that under mild conditions, the DARI is solvable if and only if the corresponding CARI is solvable. The relationship between the DARI and the CARI is then used to translate the general solvability conditions for a CARI given by Scherer into analogous conditions for the DARI. It is shown how such conditions can be applied to determine the solvability of a discrete-time H_∞ control problem whose solution set is characterized by two DARIs. © 1998 Elsevier Science Inc.

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1. INTRODUCTION

The algebraic Riccati equation has played a crucial role in the development of H_∞ control theory. More recently, the algebraic Riccati inequality has found its way into the characterization of solutions for H_∞ control problems (e.g., see [6–8, 10]). This gives rise to a need for determining the solvability of algebraic Riccati inequalities. In [3–7], Scherer has systematically considered the continuous-time algebraic Riccati inequality (CARI) and obtained a new reduction principle whereby the problem of the solvability of the CARI can be reduced to that of a linear continuous-time Lyapunov inequality. As discrete time H_∞ control theory is becoming fully developed, it is natural to consider the problem of the solvability of the discrete-time algebraic Riccati inequality (DARI).

Under the assumption that (A, B) is stabilizable and $R > 0$, it has been shown [1] that the DARI

$$A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA + Q - X \geqslant 0 \tag{1}$$

has a hermitian solution X such that

$$R + B^*XB > 0 \tag{2}$$

iff the corresponding discrete-time algebraic Riccati equation (DARE)

$$A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA + Q - X = 0 \tag{3}$$

has a unique hermitian solution $\tilde{X}$ such that $R + B^*\tilde{X}B > 0$ and $A - B(I + B^*\tilde{X}B)^{-1}B^*\tilde{X}A$ has all its eigenvalues on the closed unit circle. Furthermore, $X \leqslant \tilde{X}$ holds for any other hermitian solution X which satisfies (1) and (2). Hence the stabilizing solution $\tilde{X}$ to the DARE (3) is a *maximal* solution of the DARI (see [2, 9, 13]). In the case where the stabilizability assumption for (A, B) is not satisfied, knowledge about the solvability and the existence of solutions to the general DARI is far less complete.

One would expect the general solvability conditions for the CARI established in [3] to have counterparts in the discrete-time domain. The DARI is however more intricate than the CARI, and a direct analysis of the solvability of the DARI is bound to be very tedious. In the case of algebraic Riccati equations, one can use the bilinear transformation to carry results for the continuous-time domain over to the discrete-time domain. In the case of algebraic Riccati inequalities, we are not aware of any established connection between the DARI and the CARI. We note however that the bilinear transformation has been used in [12] to consider linear matrix inequalities.

In this paper, we will make use of the bilinear transformation to establish a direct relationship between the DARI and an associated CARI. We will use the relationship to deduce new solvability conditions for the DARI, and then show how such conditions can be applied to determine the solvability of a discrete-time H_∞ control problem whose solution set is characterized by two DARIs. To tailor our results for applications to H_∞ control problems, we will consider in the sequel the inequalities (1) and (2) with the inequality signs reversed.

The paper is organized as follows. In Section 2, the transformation between the discrete-time and continuous-time algebraic Riccati operators is defined and some necessary preliminary results are given. In Section 3, it is shown that under mild conditions, the DARI is solvable iff the corresponding CARI is solvable. The relationship between the DARI and the CARI is used in Section 4 to translate the solvability conditions for the CARI given in [3] into analogous conditions for the DARI. In Section 5, the solvability condition for the DARI is applied to a discrete-time H_∞ control problem. Section 6 contains some concluding remarks.

The following notation will be used. C denotes the field of complex numbers. C^0, C^-, and C^+ denote the $j\omega$ axis, the open left half plane, and the open right-half plane, respectively. D^0, D^-, and D^+ denote the unit circle, the open unit disk, and the region outside the unit circle, respectively. The hermitian conjugate of a matrix M is denoted by M^*. For an invertible M, we will write $(M^{-1})^*$ as M^{-*}. $M > 0$ ($\geqslant 0$) means the M is hermitian and positive definite (positive semidefinite). If $M - N > 0$ ($\geqslant 0$), we will write $M > N$ ($M \geqslant N$). For a square matrix M, its set of eigenvalues is denoted $\sigma(M)$.

2. PRELIMINARIES

Consider the following bilinear transformation between the discrete-time variable z and the continuous-time variable s:

$$s = \frac{z - \alpha}{z + \alpha}.$$

It maps the unit circle of the z-plane into the $j\omega$ axis of the s-plane. This relationship induces a transformation between discrete-time systems and continuous-time systems and between the DARE and CARE (continuous-time algebraic Riccati equation) arising from (e.g. H_∞) problems associated with such systems. In this paper, we will consider the bilinear transformation as a direct mapping between the discrete-time and continuous-time algebraic Riccati operators, stated as follows. Consider the DARE

$$A^*XA - (A^*XB + N)(R + B^*XB)^{-1}(B^*XA + N^*) + Q - X = 0 \quad (4)$$

where A, B, N, R, and Q are complex matrices of dimensions $n \times n$, $n \times m$, $n \times m$, $m \times m$, and $n \times n$, respectively, and R and Q are hermitian. Given the quintuple (A, B, Q, N, R) which characterizes the DARE (4), we define a discrete-time algebraic Riccati operator $\mathscr{D}(A, B, Q, N, R): C^{n\times n} \to C^{n\times n}$ which maps $X \in C^{n\times n}$ to the matrix

$$\begin{aligned}\mathscr{D}(A, B, Q, N, R)(X) \\ = A^*XA - (A^*XB + N)(R + B^*XB)^{-1}(B^*XA + N^*) + Q - X.\end{aligned} \quad (5)$$

Further, for $-\alpha \in D^0 \setminus \sigma(A)$, we define a transformation of the quintuple,

$$\mathscr{B}_\alpha : (A, B, Q, N, R) \mapsto \left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right),$$

by

$$\begin{aligned}\tilde{A} &= (A + \alpha I)^{-1}(A - \alpha I), \\ \tilde{B} &= \sqrt{2}(A + \alpha I)^{-1}B, \\ \tilde{Q} &= 2(A + \alpha I)^{-*}Q(A + \alpha I)^{-1}, \\ \tilde{N} &= (I + \alpha A^*)^{-1}(\sqrt{2}N - Q\tilde{B}), \\ \tilde{R} &= R - \frac{1}{\sqrt{2}}(N^*\tilde{B} + \tilde{B}^*N) + \tfrac{1}{2}\tilde{B}^*Q\tilde{B}.\end{aligned} \quad (6)$$

The transformation $\mathscr{B}_\alpha$ is well defined if $-\alpha \notin \sigma(A)$. We will also write

$$\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right) = \mathscr{B}_\alpha(A, B, Q, N, R) \quad (7)$$

Note that if R and Q are hermitian, then so are $\tilde{R}$ and $\tilde{Q}$. If $\tilde{R}$ is nonsingular, we can associate the quintuple $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R})$ with a CARE:

$$X\tilde{A} + \tilde{A}^*X + \tilde{Q} - (X\tilde{B} + \tilde{N})\tilde{R}^{-1}(\tilde{B}^*X + \tilde{N}^*) = 0. \tag{8}$$

Define the continuous-time algebraic Riccati operator $\mathscr{C}(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}): C^{n\times n} \to C^{n\times n}$ by

$$\begin{aligned}\mathscr{C}\big(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\big)(X)&\\ &= X\tilde{A} + \tilde{A}^*X + \tilde{Q} - (X\tilde{B} + \tilde{N})\tilde{R}^{-1}(\tilde{B}^*X + \tilde{N}^*).\end{aligned} \tag{9}$$

The transformation $\mathscr{B}_\alpha$ corresponds to a bilinear transformation of the set of eigenvalues of $\Delta(X) = A - B(R + B^*XB)^{-1}(B^*XA + N^*)$ associated with the DARE (4) into the set of eigenvalues of $\tilde{\Delta}(X) = \tilde{A} - \tilde{B}\tilde{R}^{-1}(\tilde{B}^*X + \tilde{N}^*)$ associated with the CARE (8), that is, if $z \in \sigma(\Delta(X))$, then $s = (z - \alpha)/(z + \alpha) \in \sigma(\tilde{\Delta}(X))$. Thus the transformation $\mathscr{B}_\alpha$ has the same system-theoretic interpretation as the bilinear transformation between discrete-time and continuous-time systems, although we have defined the transformation $\mathscr{B}_\alpha$ without reference to any explicit realization of a system. Under the condition $1 \notin \sigma(\tilde{A})$, the transformation $\mathscr{B}_\alpha$ has an inverse

$$\mathscr{B}_\alpha^{-1} : \big(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\big) \mapsto (A, B, Q, N, R)$$

given by

$$\begin{aligned}
A &= \alpha(I - \tilde{A})^{-1}(I + \tilde{A}),\\
B &= \sqrt{2}\,\alpha(I - \tilde{A})^{-1}\tilde{B},\\
Q &= 2(I - \tilde{A}^*)^{-1}\tilde{Q}(I - \tilde{A})^{-1},\\
N &= (I - \tilde{A}^*)^{-1}\big(\sqrt{2}\tilde{N} + \alpha^*\tilde{Q}B\big),\\
R &= \tilde{R} + \frac{\alpha^*}{\sqrt{2}}(\tilde{N}^*B + B^*\tilde{N}) + \tfrac{1}{2}B^*\tilde{Q}B.
\end{aligned} \tag{10}$$

We will also write

$$(A, B, Q, N, R) = \mathscr{B}_\alpha^{-1}\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right). \tag{11}$$

In the above notation, the DARE (4) and the CARE (8) can be represented as

$$\mathscr{D}(A, B, Q, N, R)(X) = 0, \tag{12}$$

$$\mathscr{C}\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right)(X) = 0. \tag{13}$$

It can be readily verified that (13) is equivalent to [2]

$$\mathscr{C}\left(\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{N}^*, \tilde{B}, \tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^*, 0, \tilde{R}\right)(X) = 0. \tag{14}$$

Clearly, the equivalence remains valid if the equal signs in (13) and (14) are replaced by the same inequality sign. The next lemma will be used to establish relationships between the discrete-time and continuous-time algebraic Riccati equations as well as inequalities [11, 13].

LEMMA 1. *Let* $-\alpha \in D^0 \setminus \sigma(A)$, $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}) = \mathscr{B}_\alpha(A, B, Q, N, R)$, *and*

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}}(A + \alpha I) & \frac{1}{\sqrt{2}}B \\ 0 & \alpha I \end{bmatrix}. \tag{15}$$

Then, for any $X \in C^{n \times n}$,

$$\begin{bmatrix} A^*XA + Q - X & A^*XB + N \\ B^*XA + N^* & R + B^*XB \end{bmatrix} = U^* \begin{bmatrix} X\tilde{A} + \tilde{A}^*X + \tilde{Q} & X\tilde{B} + \tilde{N} \\ \tilde{B}^*X + \tilde{N}^* & \tilde{R} \end{bmatrix} U. \tag{16}$$

Proof. The congruence relationship can be proved by substituting (15) and (6) into (16). ■

COROLLARY 2. *Let* $-\alpha \in D^0 \setminus \sigma(A)$ *and* $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}) = \mathscr{B}_\alpha(A, B, Q, N, R)$. *Then, for any* $X \in C^{n \times n}$,

$$\begin{bmatrix} A^*XA + Q - X & A^*XB + N \\ B^*XA + N^* & R + B^*XB \end{bmatrix} = \begin{bmatrix} L^* \\ V^* \end{bmatrix} S [L \quad V] \tag{17}$$

iff

$$\begin{bmatrix} X\tilde{A} + \tilde{A}^*X + \tilde{Q} & X\tilde{B} + \tilde{N} \\ \tilde{B}^*X + \tilde{N}^* & \tilde{R} \end{bmatrix} = \begin{bmatrix} \tilde{L}^* \\ \tilde{V}^* \end{bmatrix} S \begin{bmatrix} \tilde{L} & \tilde{V} \end{bmatrix}, \tag{18}$$

where L, V, $\tilde{L}$, *and* $\tilde{V}$ *are related by*

$$\tilde{L} = \sqrt{2} L (A + \alpha I)^{-1}, \qquad \tilde{V} = \alpha^* \left[V - L(A + \alpha I)^{-1} B \right], \tag{19}$$

$$L = \sqrt{2} \alpha \tilde{L} (I - \tilde{A})^{-1}, \qquad V = \alpha \left[\tilde{V} + \tilde{L}(I - \tilde{A})^{-1} \tilde{B} \right]. \tag{20}$$

Proof. The equivalence of (17) and (18) follows directly from Lemma 1. The relationships (19) and (20) can be obtained as

$$\begin{bmatrix} \tilde{L} & \tilde{V} \end{bmatrix} U = [L \quad V]. \qquad \blacksquare$$

Note that in Corollary 2, if V is square and invertible and $-\alpha \notin \sigma(A - BV^{-1}L)$, then $\tilde{V}$ is nonsingular; and if $\tilde{V}$ is invertible and $1 \notin \sigma(\tilde{A} - \tilde{B}\tilde{V}^{-1}\tilde{L})$, then V is nonsingular. The next result establishes a direct relationship between the DARE and the CARE that are related by (10), or equivalently by (6).

THEOREM 3. *Let* $-\alpha \in D^0 \setminus \sigma(A)$ *and*

$$\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R} \right) = \mathscr{B}_\alpha(A, B, Q, N, R). \tag{21}$$

For a hermitian matrix $X \in C^{n \times n}$, *the following two statements are equivalent*:

(i) *X is a solution to the DARE*

$$\mathscr{D}(A, B, Q, N, R)(X) = 0, \tag{22}$$

and $\Delta(X) = A - B(R + B^*XB)^{-1}(B^*XA + N^*)$ *has no eigenvalue at* $-\alpha$.

(ii) *X is a solution to the CARE*

$$\mathscr{C}\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right)(X) = 0, \tag{23}$$

and $\tilde{\Delta}(X) = \tilde{A} - \tilde{B}\tilde{R}^{-1}(\tilde{B}^* X + \tilde{N}^*)$ *has no eigenvalue at* 1.

Moreover, if either (i) *or* (ii) *holds,* $\tilde{R}$ *and* $R + B^* XB$ *have the same inertia.*

Proof. (i) $\Rightarrow$ (ii): Suppose X is a solution to (22) and $\Delta(X)$ has no eigenvalue at $-\alpha$. Let

$$F_X = -(R + B^* XB)^{-1}(B^* XA + N^*).$$

Then it can be readily verified that (17) holds with $L = -F_X$, $V = I$, and $S = R + B^* XB$. By Lemma 1, (18) holds with $\tilde{L}$ and $\tilde{V}$ given by

$$\tilde{L} = -\sqrt{2}\, F_X(A + \alpha I)^{-1}, \qquad \tilde{V} = \alpha^*\left[I + F_X(A + \alpha I)^{-1} B\right]. \tag{24}$$

Since A and $\Delta(X) = A + BF_X$ have no eigenvalue at $-\alpha$, $\tilde{V}$ is nonsingular. From (18),

$$\tilde{R} = \tilde{V}^*(R + B^* XB)\tilde{V}. \tag{25}$$

Hence, the nonsingularity of $R + B^* XB$ implies that $\tilde{R}$ is nonsingular. It then follows from (18) that X is a solution of the CARE (23). Moreover, making use of (10), (18), and (24), it can be verified by a direct calculation that

$$\tilde{\Delta}(X) = [\Delta(X) + \alpha I]^{-1}[\Delta(X) - \alpha I]. \tag{26}$$

Since $\Delta(X)$ has no eigenvalue at $-\alpha$, it follows that $\tilde{\Delta}(X)$ has no eigenvalue at 1.

(ii) $\Rightarrow$ (i): Let $\tilde{F}_X = \tilde{R}^{-1}(\tilde{B}^* X + \tilde{N}^*)$. Then (18) holds with $\tilde{L} = -\tilde{F}_X$, $\tilde{V} = I$, and $S = \tilde{R}$. By Lemma 1, (17) holds with $L = -\sqrt{2}\,\tilde{F}_X(I - \tilde{A})^{-1}$ and $V = \alpha[I - \tilde{F}_X(I - \tilde{A})^{-1}\tilde{B}]$, where V is nonsingular. The proof then proceeds in a way similar to that given above.

If one of statements (i) and (ii) is true, either (25) or the (2, 2) block of (17) holds, which implies that $\tilde{R}$ and $R + B^* XB$ have the same inertia. ■

Theorem 3 shows that if X solves the DARE (22) and if we transform the DARE into the CARE (23) by means of (21), then X is also a solution to the CARE, and vice versa. In the next section, we will make use of Lemma 1 and Theorem 3 to extend the relationship between the algebraic Riccati equations to the case of inequalities.

3. RELATIONSHIPS BETWEEN DARI AND CARI

In this section, we will establish the relationship between the DARI and the CARI that are related by (7). We will consider both the nonstrict and the strict case of the inequality

$$\mathscr{D}(A, B, Q, N, R)(X) \leqslant 0 \qquad (<0). \tag{27}$$

Clearly, (27) holds iff there exists $P \leqslant 0$ (<0) such that

$$\mathscr{D}(A, B, Q - P, N, R)(X) = 0. \tag{28}$$

The next two results (Theorem 4 for the strict inequality and Theorem 5 for the nonstrict inequality) show that if we transform the DARI into a CARI by means of (7), then the DARI has a solution X iff the associated CARI is solvable by the same X.

THEOREM 4. *Let* $-\alpha \in D^0 \setminus \sigma(A)$ *and* $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}) = \beta_\alpha(A, B, Q, N, R)$. *For any hermitian matrix* $X \in C^{n \times n}$, *the following two statements are equivalent*:

(i) $R + B^*XB < 0$, *and X satisfies the DARI*

$$\mathscr{D}(A, B, Q, N, R)(X) < 0. \tag{29}$$

(ii) $\tilde{R} < 0$, *and X satisfies the CARI*

$$\mathscr{C}\big(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\big)(X) < 0. \tag{30}$$

Proof. We first note that the two inequalities in condition (i) can be written equivalently in the form of a linear matrix inequality

$$\begin{bmatrix} A^*XA + Q - X & A^*XB + N \\ B^*XA + N^* & R + B^*XB \end{bmatrix} < 0. \tag{31}$$

By Lemma 1, (31) is equivalent to

$$\begin{bmatrix} X\tilde{A} + \tilde{A}^*X + \tilde{Q} & X\tilde{B} + \tilde{N} \\ \tilde{B}^*X + \tilde{N}^* & \tilde{R} \end{bmatrix} < 0, \tag{32}$$

which is equivalent to the two inequalities given in condition (ii). ■

The next theorem is the nonstrict version of Theorem 4. In the case of the nonstrict DARI [respectively CARI], an additional condition on the eigenvalues of $\Delta(X)$ [respectively $\tilde{\Delta}(X)$] is required to ensure that $\mathscr{B}_\alpha$ [respectively $\mathscr{B}_\alpha^{-1}$] transforms the DARI [respectively CARI] into a well-defined CARI [respectively DARI].

THEOREM 5. *Let* $-\alpha \in D^0 \setminus \sigma(A)$, $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}) = \mathscr{B}_\alpha(A, B, Q, N, R)$, *and* $X \in C^{n\times n}$ *be hermitian.*

(i) *Suppose* $\Delta(X) = A - B(R + B^*XB)^{-1}(B^*XA + N^*)$ *has no eigenvalue at* $-\alpha$. *Then*

$$R + B^*XB < 0 \quad \textit{and} \quad \mathscr{D}(A, B, Q, N, R)(X) \leqslant 0 \tag{33}$$

imply

$$\tilde{R} < 0 \quad \textit{and} \quad \mathscr{C}\big(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\big)(X) \leqslant 0. \tag{34}$$

(ii) *Suppose* $\tilde{\Delta}(X) = \tilde{A} - \tilde{B}\tilde{R}^{-1}(\tilde{B}^*X + \tilde{N}^*)$ *has no eigenvalue at* 1. *Then*, (34) $\Rightarrow$ (33).

Proof. (i): By the condition (33), there exists a hermitian matrix $P \leqslant 0$ such that

$$\mathscr{D}(A, B, Q - P, N, R)(X) = 0.$$

Let $(\tilde{A}, \tilde{B}, \tilde{Q}_P, \tilde{N}_P, \tilde{R}_P) = \mathscr{B}_\alpha(A, B, Q - P, N, R)$. Note that

$$\tilde{R}_P = \tilde{R} - \tfrac{1}{2}\tilde{B}^*P\tilde{B}. \tag{35}$$

By Theorem 3, $\tilde{R}_P$ has the same inertia as $R + B^*XB$. Since $R + B^*XB < 0$, we have that $\tilde{R}_P < 0$. It follows from (35) that $\tilde{R} < 0$. Now, noting that the equivalence of (31) and (32) remains valid if the strict inequalities are replaced by nonstrict inequalities, it follows that the nonstrict DARI given in (33) implies the nonstrict CARI given in (34), which completes part (i) of the proof.

Part (ii) of the theorem can be proved by an argument similar to that for (i). ∎

4. SOLVABILITY CONDITIONS FOR THE DARI

Given the pair (A, B), it is well known that a similarity transformation T can be found to transform the pair into a form which displays the uncontrollable modes on the unit circle and inside the unit circle, i.e.,

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} B_{11} \\ 0 \\ 0 \end{bmatrix}, \tag{36}$$

where the uncontrollable modes (if any) of (A_{11}, B_{11}) lie outside the unit circle, and A_{22} and A_{33} have eigenvalues only on the unit circle and inside the unit circle, respectively. We partition the transformed versions of Q, N, and X conformal with (36) as

$$\begin{aligned} T^*QT &= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}, \qquad T^*N = \begin{bmatrix} N_{11} \\ N_{21} \\ N_{31} \end{bmatrix}, \\ T^*XT &= \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix}. \end{aligned} \tag{37}$$

For any X, let

$$\Delta(X) = A - B(R + B^*XB)^{-1}(B^*XA + N^*).$$

Then, the transformed version of $\Delta(X)$ has a structure given by

$$T^{-1}\Delta(X)T = \begin{bmatrix} \Delta_{11}(X_{11}) & * & * \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \tag{38}$$

where

$$\Delta_{11}(X_{11}) = A_{11} - B_{11}(R + B_{11}^* X_{11} B_{11})^{-1}(B_{11}^* X_{11} A_{11} + N_{11}^*). \quad (39)$$

Further, let

$$G(X) = T^*[\mathscr{D}(A, B, Q, N, R)(X)]T = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^* & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix}, \quad (40)$$

where the partitioning is conformal with that given in (36). A direct calculation shows that the upper left 2×2 subblocks of $G(X)$ can be written as

$$\begin{aligned} G_{11}(X) &= \mathscr{D}(A_{11}, B_{11}, Q_{11}, N_{11}, R)(X), \\ G_{12}(X) &= \Delta_{11}^*(X_{11}) X_{12} A_{22} \\ &\quad + \Big[\Delta_{11}^*(X_{11}) X_{11} A_{12} - (A_{11}^* X_{11} B_{11} + N_{11}) \\ &\qquad \times (R + B_{11}^* X_{11} B_{11})^{-1} N_{21} + Q_{12}\Big] - X_{12}, \\ G_{22}(X) &= A_{22}^* X_{22} A_{22} - X_{22} \\ &\quad + \Big[A_{12}^* X_{11} A_{12} + A_{22}^* X_{12}^* A_{12} + A_{12}^* X_{12} A_{22} + Q_{22} \\ &\qquad + (A_{12}^* X_{11} + A_{22}^* X_{12}^* + N_{21}) B_{11}(R + B_{11}^* X_{11} B_{11})^{-1} B_{11}^* \\ &\qquad \times (X_{11} A_{12} + X_{12} A_{22} + N_{21}^*)\Big]. \end{aligned} \quad (41)$$

In order to derive the reduction principle for the DARI, we will first consider the subsystem

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad (42)$$

$$Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}, \qquad N_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}, \qquad X_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}. \quad (43)$$

Note that $\Delta_{11}(X_{11})$ and the upper left 2×2 subblock of $G(X)$ are completely defined by the matrices given in (42) and (43). Therefore, we can regard G_{11}, G_{12}, and G_{22} as functions of X_1 and write $G_{11}(X)$ as $G_{11}(X_1)$, etc. We will denote the upper 2×2 subblock of (38) by $\Delta_1(X_1)$, i.e.,

$$\begin{aligned}\Delta_1(X_1) &= \begin{bmatrix} \Delta_{11}(X_{11}) & * \\ 0 & A_{22} \end{bmatrix} \\ &= A_1 - B_1(R + B_1^* X_1 B_1)^{-1}(B_1^* X_1 A_1 + N_1^*). \end{aligned} \tag{44}$$

Clearly,

$$\sigma(\Delta_1(X_1)) = \sigma(\Delta_{11}(X_{11})) \cup \sigma(A_{22}). \tag{45}$$

The next lemma provides a necessary condition for the solvability of the upper left 2×2 subblock of the DARI. The lemma is stated for the case of the nonstrict inequality, with respective modifications for the case of the strict inequality indicated in brackets.

LEMMA 6. *If there exists a hermitian matrix X_1 satisfying*

$$R + B_1^* X_1 B_1 < 0 \tag{46}$$

and

$$\mathscr{D}(A_1, B_1, Q_1, N_1, R)(X_1) \leqslant 0 \qquad [< 0], \tag{47}$$

then there exists a hermitian matrix $\tilde{X}_1 \geqslant X_1$ $[\tilde{X}_1 > X_1]$ with a partitioning

$$\tilde{X}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^* & \tilde{X}_{22} \end{bmatrix}$$

satisfying

$$R + B_{11}^* \tilde{X}_{11} B_{11} < 0, \tag{48}$$

$$G_{11}(\tilde{X}_1) = \mathscr{D}(A_{11}, B_{11}, Q_{11}, N_1, R)(\tilde{X}_{11}) = 0, \tag{49}$$

$$G_{12}(\tilde{X}_1) = 0, \tag{50}$$

$$G_{22}(\tilde{X}_1) \leqslant 0 \qquad [< 0], \tag{51}$$

$$\sigma(\Delta_{11}(\tilde{X}_{11})) \subseteq D^+ \cup D^0 \qquad [\subseteq D^+].$$

A proof of Lemma 6 is given in the Appendix. We note that in the lemma, $\tilde{X}_{11}$ is the antistabilizing solution to the DARE (49) for the subsystem (A_{11}, B_{11}), and can be solved via a generalized eigenvalue problem associated with the matrix pencil [14]

$$W_{11}(z) = \begin{bmatrix} A_{11} - zI & 0 & B_{11} \\ -Q_{11} & I - zA_{11}^* & 0 \\ 0 & zB_{11}^* & R \end{bmatrix}. \tag{52}$$

We are now in a position to derive a new reduction principle for the DARI which is the discrete version of the reduction principle for the CARI given in [3]. The reduction principle described in the next theorem gives a necessary and sufficient condition for the solvability of the general DARI. The proof of the theorem relies on transforming the DARI to a CARI and then invoking the result of [3]. Note however that the result is a general version of the discrete counterpart of Scherer's result given in [3], and is independent of any restrictive assumptions that have to be made when characterizing the DARI in terms of a CARI.

THEOREM 7 (Reduction principle for DARI). *The DARI*

$$\mathscr{D}(A, B, Q, N, R)(X) \leqslant 0 \qquad [<0] \tag{53}$$

has a (positive semidefinite, or positive definite) hermitian solution X such that $R + B^*XB < 0$ iff there exists a (positive semidefinite, or positive definite) hermitian matrix

$$\tilde{X}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^* & \tilde{X}_{22} \end{bmatrix}$$

such that (48)–(51) *hold. Moreover, if one of the above equivalent conditions holds, then X_{11} (defined in* (37)) *satisfies*

$$X_{11} \leqslant \tilde{X}_{11} \qquad \left[X_{11} < \tilde{X}_{11}\right].$$

A proof of Theorem 7 can be found in the Appendix. We remark that in Theorem 7, X has the additional property of being positive definite or positive semidefinite iff $\tilde{X}_1$ has the same property, and this is independent of whether the strict or the nonstrict inequality is being considered. The next result relates the solvability of the DARI with an associated CARI.

COROLLARY 8. *Let* $-\alpha \in D^0 \setminus [\sigma(A) \cup \sigma(W_{11}(z))]$, *where* $W_{11}(z)$ *is given by* (52), *and let*

$$\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right) = \mathscr{B}_\alpha(A, B, Q, N, R).$$

Then there exists a (positive semidefinite, or positive definite) hermitian X *satisfying*

$$R + B^*XB < 0 \quad \text{and} \quad \mathscr{D}(A, B, Q, N, R)(X) \leqslant 0 \quad [<0] \tag{54}$$

iff $\tilde{R} < 0$ *and there exists a (positive semidefinite, or positive definite) hermitian solution* X *(not necessarily the same* X *as in* (54)) *satisfying*

$$\mathscr{C}\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right)(X) \leqslant 0 \qquad [<0]. \tag{55}$$

A proof of Corollary 8 is given in the Appendix. The corollary enables the problem of the solvability of the DARI (54) to be considered by a transformation into an auxiliary CARI which can then be studied using existing results for the CARI. However, Theorem 7 allows us to take a more direct approach to the problem of the solvability of the DARI. By Theorem 7, the solvability of the DARI (53) can be reduced to that of the DARE (49) for the subsystem (A_{11}, B_{11}), the linear matrix equation (50), and the discrete-time Lyapunov inequality (51). It is well known how to determine the solvability of the DARE (49) (with uncontrollable modes, if any, outside the unit circle only) and the linear matrix equation (50). Necessary and sufficient conditions together with a numerical procedure for solving the DARE without restrictive assumptions can be found in [15]. In the following, we will briefly discuss how to determine the solvability of the discrete-time Lyapunov inequality (51) involving an A_{22} which has eigenvalues on the unit circle only.

Consider the discrete-time Lyapunov inequality (DLI)

$$M^*YM + H - Y \leqslant 0 \qquad [<0] \tag{56}$$

where H is hermitian and M has eigenvalues on the unit circle only. Regarding the Lyapunov inequality as a special case of the algebraic Riccati inequality, we can make use of the transformations (6) and (10) to transform (56) into a continuous-time Lyapunov inequality and vice versa. For any $-\alpha \in D^0 \setminus \sigma(M)$, we may put

$$(\tilde{M}, 0, \tilde{H}, 0, I) = \mathscr{B}_\alpha(M, 0, H, 0, I). \tag{57}$$

If M has eigenvalues on the unit circle only, then clearly $\tilde{M}$ has eigenvalues on the $j\omega$ axis only. A direct application of Lemma 1 gives

LEMMA 9. *Let* $(\tilde{M}, \tilde{H})$ *and* (M, H) *be related by* (56). *Then, for any* Y,

$$M^*YM + H - Y = \tfrac{1}{2}(M^* + \alpha^* I)(Y\tilde{M} + \tilde{M}^*Y + \tilde{H})(M + \alpha I). \quad (58)$$

It follows from Lemma 9 that the discrete-time Lyapunov inequality (56) is equivalent to the continuous-time Lyapunov inequality

$$Y\tilde{M} + \tilde{M}^*Y + \tilde{H} \leqslant 0 \qquad [< 0]. \quad (59)$$

As the continuous-time Lyapunov inequality has been considered in [3–5] with established results, we will use the equivalence of (56) and (59) to carry the results for the continuous-time Lyapunov inequality over to the discrete-time case. The following lemma is a translation of Theorem 4 in [6] for the strict continuous-time Lyapunov inequality to the case of the strict discrete-time Lyapunov inequality.

LEMMA 10. *Suppose the matrix M has eigenvalues on the unit circle only. Then*:

(i) *The DLI*

$$M^*YM + H - Y < 0 \quad (60)$$

has a hermitian solution Y iff for any eigenvector y of M, the quadratic form $y^*Hy < 0$.

(ii) *If the DLI* (60) *has a hermitian solution Y, the solution can be chosen arbitrarily large, i.e., for any hermitian $\bar{Y}$, there exists a solution Y to the* (60) *satisfying $Y > \bar{Y}$.*

It we apply Lemma 10(i) to the case of the strict DARI given in Theorem 7, then the condition (51) can be written as an algebraic test, namely, that

$$y^*\Big[A_{12}^*\tilde{X}_{11}A_{12} + A_{22}^*\tilde{X}_{12}^*A_{12} + A_{12}^*\tilde{X}_{12}A_{22} + Q_{22} + \left(A_{12}^*\tilde{X}_{11} + A_{22}^*\tilde{X}_{12}^*\right)$$
$$\times B_{11}\left(R + B_{11}^*\tilde{X}_{11}B_{11}\right)^{-1}B_{11}^*\left(\tilde{X}_{11}A_{12} + \tilde{X}_{12}A_{22}\right)\Big]y < 0$$

should hold for any eigenvector y of A_{22}. Furthermore, an application of Lemma 10(ii) to the strict DARI given in Theorem 7 yields the following corollary.

COROLLARY 11. *In the notation of Theorem* 7, *if the strict DARI given in* (53) *has a solution* X *such that* $R + B^*XB < 0$, *then*:

(i) X *can be chosen positive definite iff* $\tilde{X}_{11} > 0$.
(ii) *For any* $X > 0$,

$$X^{-1} > T \operatorname{diag}\{\tilde{X}_{11}^{-1}, 0, 0\} T^* \tag{61}$$

Moreover, there exist a sequence of solutions $X(j)$, $j = 1, 2, \dots,$ *satisfying the strict DARI* (53) *and* $R + B^*X(j)B < 0$ *such that*

$$X(j)^{-1} \to T \operatorname{diag}\{\tilde{X}_{11}^{-1}, 0, 0\} T^* \tag{62}$$

A proof of Corollary 11 can be found in the Appendix.

5. AN APPLICATION

Algebraic Riccati inequalities play an important role in characterizing the solutions to H_∞ control problems. In this section, we will show how the solvability conditions derived in Section 4 can be applied to a discrete-time H_∞ control problem.

Consider a partitioned discrete-time system with state-space equations

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), \\ q(k) &= C_1 x(k) + D_{12} u(k), \\ y(k) &= C_2 x(k) + D_{21} w(k), \end{aligned} \tag{63}$$

where (A, B_2, C_2) is stabilizable and detectable. We will also make the following simplifying assumptions:

$$D_{12}^T [D_{12} \quad C_1] = [I \quad 0], \qquad D_{21}[D_{21}^T \quad B_1^T] = [I \quad 0].$$

Let

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix}$$

be the transfer function corresponding to the system given by (63). The discrete-time H_∞ suboptimal control problem is to find a stabilizing controller $K(z)$ such that

$$\|\mathscr{F}(P, K)\|_\infty < \gamma, \tag{64}$$

where $\mathscr{F}(P, K)$ denotes the linear functional transformation of P and K. The following result from [10] characterizes the solvability of the discrete-time H_∞ control problem in terms of two DARIs.

LEMMA 12. *There exists a proper real-rational stabilizing controller* $K(z)$ *with realization*

$$K(z) = D_K + C_K(zI - A_K)^{-1}B_K$$

satisfying (64) *if and only if there exists a pair of symmetric matrices* (U, V) *such that*

$$\begin{aligned}
&-\gamma I + C_1UC_1^T < 0, \qquad -\gamma I + B_1^TVB_1 < 0,\\
&AUA^T - AUC_1^T(-\gamma I + C_1UC_1^T)^{-1}C_1UA^T\\
&\qquad + \frac{1}{\gamma}(B_1B_1^T - \gamma^2B_2B_2^T) - U < 0,\\
&A^TVA - A^TVB_1(-\gamma I + B_1^TVB_1)^{-1}B_1^TVA\\
&\qquad + \frac{1}{\gamma}(C_1^TC_1 - \gamma^2C_2^TC_2) - V < 0,\\
&U > 0, \qquad V < 0, \qquad \max\{\sigma(U^{-1}V^{-1})\} \leqslant 1.
\end{aligned} \tag{65}$$

We need to define two similarity transformations before stating an algebraic test for the existence of (U, V). Let T_b and T_c be nonsingular matrices such that

$$T_b^{-1}AT_b = \begin{bmatrix} A_{11}^b & A_{12}^b & A_{13}^b \\ 0 & A_{22}^b & 0 \\ 0 & 0 & A_{33}^b \end{bmatrix}, \qquad T_b^{-1}B_1 = \begin{bmatrix} B_{11} \\ 0 \\ 0 \end{bmatrix}$$

and

$$T_c^{-1}A^TT_c = \begin{bmatrix} A_{11}^c & A_{12}^c & A_{13}^c \\ 0 & A_{22}^c & 0 \\ 0 & 0 & A_{33}^c \end{bmatrix}, \qquad T_c^{-1}C_1^T = \begin{bmatrix} C_{11} \\ 0 \\ 0 \end{bmatrix},$$

where the uncontrollable modes (if any) of (A_{11}^b, B_{11}) and (A_{11}^c, C_{11}) lie outside the unit circle, A_{22}^b and A_{22}^c have eigenvalues on the unit circle only, and A_{33}^b and A_{33}^c have eigenvalues inside the unit circle only. Partition the following matrices conformally with the above partitionings as

$$\frac{1}{\gamma}T_b^T(C_1^TC_1 - \gamma^2 C_2^TC_2)T_b = \begin{bmatrix} Q_{11}^b & Q_{12}^b & Q_{13}^b \\ (Q_{12}^b)^T & Q_{22}^b & Q_{23}^b \\ (Q_{13}^b)^T & (Q_{23}^b)^T & Q_{33}^b \end{bmatrix},$$

$$T_b^T V T_b = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix};$$

$$\frac{1}{\gamma}T_c^T(B_1^TB_1 - \gamma^2 B_2^TB_2)T_c = \begin{bmatrix} Q_{11}^c & Q_{12}^c & Q_{13}^c \\ (Q_{12}^c)^T & Q_{22}^c & Q_{23}^c \\ (Q_{13}^c)^T & (Q_{23}^c)^T & Q_{33}^c \end{bmatrix},$$

$$T_c^T U T_c = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^T & U_{22} & U_{23} \\ U_{13}^T & U_{23}^T & U_{33} \end{bmatrix}.$$

The next result shows how we can make use of Corollary 11 and Lemma 12 to transform the conditions given in (65) into a computable form which can be readily verified.

THEOREM 13. *The inequalities given in* (65) *hold if and only if there exist real matrices* $\tilde{U}_{11} > 0$, $\tilde{V}_{11} > 0$ *and* $\tilde{U}_{12}$, $\tilde{V}_{12}$ *such that*

$$-\gamma I + C_{11}^T\tilde{U}_{11}C_{11} < 0,$$
$$(A_{11}^c)^T\tilde{U}_{11}A_{11}^c - (A_{11}^c)^T\tilde{U}_{11}C_{11}(-\gamma I + C_{11}^T\tilde{U}_{11}C_{11})^{-1}C_{11}^T\tilde{U}_{11}A_{11}^c + Q_{11}^c - \tilde{U}_{11} = 0,$$
$$\sigma(\Delta_{\tilde{U}_{11}}) \subset D^+,$$
where $\Delta_{\tilde{U}_{11}} = A_{11}^c - C_{11}(-\gamma I + C_{11}^T\tilde{U}_{11}C_{11})^{-1}C_{11}^T\tilde{U}_{11}A_{11}^c$,
$$\Delta_{\tilde{U}_{11}}^T\tilde{U}_{12}A_{22}^c + (\Delta_{\tilde{U}_{11}}^T\tilde{U}_{11}A_{12}^c + Q_{12}^c) - \tilde{U}_{12} = 0,$$
$$y_c^*\{[(A_{12}^c)^T\tilde{U}_{11} + (A_{22}^c)^T\tilde{U}_{12}^T] \times C_{11}(-\gamma I + C_{11}^T\tilde{U}_{11}C_{11})^{-1}C_{11}^T(\tilde{U}_{11}A_{12}^c + \tilde{U}_{12}A_{22}^c) + Q_{22}^c + (A_{12}^c)^T\tilde{U}_{11}A_{12}^c + (A_{22}^c)^T\tilde{U}_{12}^TA_{12}^c + (A_{12}^c)^T\tilde{U}_{12}A_{22}^c\}y_c < 0$$

for any eigenvector y_c *of* A_{22}^c;

$$- \gamma I + B_{11}^T \tilde{V}_{11} B_{11} < 0,$$

$$(A_{11}^b)^T \tilde{V}_{11} A_{11}^b - (A_{11}^b)^T \tilde{V}_{11} B_{11}(-\gamma I + B_{11}^T \tilde{V}_{11} B_{11})^{-1} B_{11}^T \tilde{V}_{11} A_{11}^b + Q_{11}^b - \tilde{V}_{11} = 0,$$

$$\sigma(\Delta_{\tilde{V}_{11}}) \subset D^+,$$

$$\text{where } \Delta_{\tilde{V}_{11}} = A_{11}^b - B_{11}(-\gamma I + B_{11}^T \tilde{V}_{11} B_{11})^{-1} B_{11}^T \tilde{V}_{11} A_{11}^b,$$

$$\Delta_{\tilde{V}_{11}}^T \tilde{V}_{12} A_{22}^b + (\Delta_{\tilde{V}_{11}}^T \tilde{V}_{11} A_{12}^b + Q_{12}^b) - \tilde{V}_{12} = 0,$$

$$\begin{aligned} y_b^* \{ &[(A_{12}^b)^T \tilde{V}_{11} + (A_{22}^b)^T \tilde{V}_{12}^T] \\ &\times B_{11}(-\gamma I + B_{11}^T \tilde{V}_{11} B_{11})^{-1} B_{11}^T (\tilde{V}_{11} A_{12}^b + \tilde{V}_{12} A_{22}^b) \\ &+ Q_{22}^b + (A_{12}^b)^T \tilde{V}_{11} A_{12}^b + (A_{22}^b)^T \tilde{V}_{12}^T A_{12}^b + (A_{12}^b)^T \tilde{V}_{12} A_{22}^b \} y_b < 0 \end{aligned}$$

for any eigenvector y_b *of* A_{22}^b; *and*

$$\max\left\{ \sigma\left(T_b^T \begin{bmatrix} \tilde{V}_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T_b T_c^T \begin{bmatrix} \tilde{U}_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T_c \right) \right\} < 1.$$

Theorem 13 leads to a direct characterization of suboptimal solutions for the discrete-time H_∞ control problem (64). This can be regarded as the discrete-time counterpart of the result presented in [6].

6. CONCLUSION

In this paper, we have established a direct relationship between the DARI and CARI which allows us to transform a DARI into a CARI or vice versa while keeping the solution to the inequalities invariant. By means of the relationship, a new reduction principle for the DARI is derived which parallels the one given in [3] for the CARI. As a result, the problem of the solvability of the quadratic DARI is reduced to that of a DARE, a linear matrix equation, and a linear discrete-time Lyapunov inequality, all of which have known solutions. It is also shown that a DARI is solvable iff an associated CARI is solvable. Therefore, two methods are provided for determining the solvability of the DARI. An application of these results to the discrete-time H_∞ control problem is given.

APPENDIX

Proof of Lemma 6. Let X_1 be a hermitian matrix satisfying (46) and (47). Although $\tilde{X}_{11}$ is yet to be determined, we note here that for any $\tilde{X}_{11}$ satisfying the DARE (49), it is well established (e.g., see [11]) that the eigenvalues of $\Delta_{11}(\tilde{X}_{11})$ satisfy

$$\sigma\left(\Delta_{11}\left(\tilde{X}_{11}\right)\right) \subset \sigma\left(W_{11}(z)\right),$$

where $\sigma(W_{11}(z))$ denotes the generalized eigenvalues of the matrix pencil given in (52). Hence, we can choose $-\alpha \in D^0 \setminus [\sigma(A_1) \cup \sigma(\Delta_1(X_1)) \cup \sigma(W_{11}(z))]$ such that both $\Delta_1(X_1)$ and $\Delta_{11}(\tilde{X}_{11})$ have no eigenvalue at $-\alpha$. For such an α, define

$$\left(\tilde{A}_1, \tilde{B}_1, \tilde{Q}_1, \tilde{N}_1, \tilde{R}_1\right) = \mathscr{B}_\sigma(A_1, B_1, Q_1, N_1, R). \tag{66}$$

Then, Theorem 5 [Theorem 4] yields that $\tilde{R}_1 < 0$ and

$$\mathscr{C}\left(\tilde{A}_1, \tilde{B}_1, \tilde{Q}_1, \tilde{N}_1, \tilde{R}_1\right)(X_1) \leqslant 0 \qquad [< 0]. \tag{67}$$

Equation (67) is equivalent to

$$\mathscr{C}\left(-\tilde{A}_1 + \tilde{B}_1\tilde{R}_1^{-1}\tilde{N}_1^*, \tilde{B}_1\left(-\tilde{R}_1\right)^{-1/2}, -\tilde{Q}_1 + \tilde{N}_1\tilde{R}_1^{-1}\tilde{N}_1^*, 0, I\right)(X_1) \geqslant 0 \qquad [> 0]. \tag{68}$$

Note that after the above transformations, $\tilde{A}_1 - \tilde{B}_1\tilde{R}_1^{-1}\tilde{N}_1^*$ and $\tilde{B}_1(-\tilde{R}_1)^{-1/2}$ still retain the structures given in (42), so that we may write

$$\tilde{A}_1 - \tilde{B}_1\tilde{R}_1^{-1}\tilde{N}_1^* = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \qquad \tilde{B}_1\left(-\tilde{R}_1\right)^{-1/2} = \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}. \tag{69}$$

Since by construction all the uncontrollable modes (if any) of (A_{11}, B_{11}) lie outside the unit circle and A_{22} has eigenvalues only on the unit circle, the uncontrollable modes (if any) of $(\tilde{A}_{11}, \tilde{B}_{11})$ will lie in the open right complex plane, i.e., $(-\tilde{A}_{11}, \tilde{B}_{11})$ is stabilizable, and $\tilde{A}_{22}$ (or equivalently $-\tilde{A}_{22}$) has eigenvalues on the $j\omega$-axis only. Making use of Theorem 2 in [3], we deduce that there exists a hermitian matrix $\tilde{X}_1 \geqslant X_1$ $[\tilde{X}_1 > X_1]$ with partitioning

$$\tilde{X}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^* & \tilde{X}_{22} \end{bmatrix}$$

satisfying

$$\mathscr{C}\Big(-\tilde{A}_1 + \tilde{B}_1\tilde{R}_1^{-1}\tilde{N}_1^*, \tilde{B}_1\big(-\tilde{R}_1\big)^{-1/2}, -\tilde{Q}_1 + \tilde{N}_1\tilde{R}_1^{-1}\tilde{N}_1^*, 0, I\Big)\big(\tilde{X}_1\big) = \begin{bmatrix} 0 & 0 \\ 0 & -\tilde{P}_{22} \end{bmatrix} \tag{70}$$

such that

$$-\tilde{P}_{22} \geqslant 0 \qquad [> 0] \tag{71}$$

and

$$\sigma\big(-\tilde{\Delta}_{11}\big(\tilde{X}_{11}\big)\big) \subseteq C^- \cup C^0 \qquad [\subset C^-], \tag{72}$$

where $\tilde{\Delta}_{11}(\tilde{X}_{11})$ is the (1, 1) subblock of

$$\tilde{\Delta}_1\big(\tilde{X}_1\big) = \tilde{A}_1 - \tilde{B}_1\tilde{R}_1^{-1}\big(\tilde{B}_1^*\tilde{X}_1 + \tilde{N}_1^*\big) = \begin{bmatrix} \tilde{\Delta}_{11}\big(\tilde{X}_{11}\big) & * \\ 0 & \tilde{A}_{22} \end{bmatrix}.$$

It follows that

$$\sigma\big(\tilde{\Delta}_1\big(\tilde{X}_1\big)\big) = \sigma\big(\tilde{\Delta}_{11}\big(\tilde{X}_{11}\big)\big) \cup \sigma\big(\tilde{A}_{22}\big). \tag{73}$$

Obviously, (70)–(72) are equivalent to

$$\begin{aligned} &\mathscr{C}\big(\tilde{A}_1, \tilde{B}_1, \tilde{Q}_1, \tilde{N}_1, \tilde{R}_1\big) = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_{22} \end{bmatrix} = \tilde{P}_1, \\ &\tilde{P}_1 \leqslant 0, \quad \tilde{P}_{22} \leqslant 0 \qquad [< 0], \\ &\sigma\big(\tilde{\Delta}_{11}\big(\tilde{X}_{11}\big)\big) \subseteq C^+ \cup C^0 \qquad [\subset C^+]. \end{aligned} \tag{74}$$

From (66), in view of the structures of $\tilde{B}_1$ and $\tilde{P}_1$, we have that

$$\mathscr{B}_\alpha^{-1}\big(\tilde{A}_1, \tilde{B}_1, \tilde{Q}_1 - \tilde{P}_1, \tilde{N}_1, \tilde{R}_1\big) = (A_1, B_1, Q_1 - P_1, N_1, R)$$

for some

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix} \qquad \text{with} \quad P_{22} \leqslant 0 \quad [< 0].$$

Since A_1 has no eigenvalue at $-\alpha$, $\tilde{A}_1$ has no eigenvalue at 1 either. By a calculation similar to that indicated in the proof of Theorem 3 [see (25)], it can be shown that

$$\tilde{\Delta}_{11}(\tilde{X}_{11}) = \left[\Delta_{11}(\tilde{X}_{11}) + \alpha I\right]^{-1}\left[\Delta_{11}(\tilde{X}_{11}) - \alpha I\right]. \tag{75}$$

Since by construction $\Delta_{11}(\tilde{X}_{11})$ has no eigenvalue at $-\alpha$, it follows from (75) that $\tilde{\Delta}_{11}(\tilde{X}_{11})$ has no eigenvalue at 1. This, together with (73), implies that $\tilde{\Delta}_1(\tilde{X}_1)$ has no eigenvalue at 1. Therefore, by Theorem 3 and (74), we have that $\tilde{X}_1 \geqslant X_1$ $[\tilde{X}_1 > X_1]$ is such that

$$\begin{aligned} &R + B_1^* \tilde{X}_1 B_1 < 0, \\ &\mathscr{D}(A_1, B_1, Q_1, N_1, R)(\tilde{X}_1) = P_1 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix}, \\ &P_{22} \geqslant 0 \qquad [< 0], \\ &\sigma\big(\Delta_{11}(\tilde{X}_{11})\big) \subseteq D^+ \cup D^0 \qquad [\subseteq D^+]. \end{aligned} \tag{76}$$

This shows that $\tilde{X}_1$ satisfies (48)–(51). ■

Proof of Theorem 7. Necessity: Assume that the DARI (53) has a (positive semidefinite, or positive definite) hermitian solution X satisfying

$$R + B^* X B < 0.$$

Let X_1 be the upper left 2×2 subblock of X. It is obvious that X_1 is a (positive semidefinite, or positive definite) hermitian solution of the DARI (47) satisfying $R + B_1^* X_1 B_1 < 0$. Hence, the necessity part of the theorem follows directly from Lemma 6.

Sufficiency: In this part of the proof, we will use the notation already introduced in the proof of Lemma 6 as well as that defined in Section 4. Assume that (48)–(51) hold. Then, there exists a (positive semidefinite, or positive definite) hermitian matrix $\tilde{X}_1$ satisfying (76). Let $-\alpha \in D^0 \setminus [\sigma(A) \cup \sigma(\Delta_1(\tilde{X}_1))]$, and define $(\tilde{A}_1, \tilde{B}_1, \tilde{Q}_1 - \tilde{P}_1, \tilde{N}_1, \tilde{R}_1) = \mathscr{B}_\alpha(A_1, B_1, Q_1 - P_1, N_1, R)$. Theorem 3 yields that (74) holds true with $\tilde{R}_1 < 0$ and that $\tilde{\Delta}_1(\tilde{X}_1)$ has no eigenvalue at 1.

Now, define $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}) = \mathscr{B}_\alpha(A, B, Q, N, R)$. Note that $(\tilde{A}, \tilde{B})$ retains the structure given in (36), and because of the structure of $\tilde{B}$ and P_1, we have

$$\begin{aligned}\tilde{R} &= R - \frac{1}{\sqrt{2}}(N^*\tilde{B} + \tilde{B}^*N) + \tfrac{1}{2}\tilde{B}^*Q\tilde{B}\\ &= R - \frac{1}{\sqrt{2}}\left(N_1^*\tilde{B}_1 + \tilde{B}_1^*N_1\right) + \tfrac{1}{2}\tilde{B}_1^*(Q_1 - P_1)\tilde{B}_1 = \tilde{R}_1 < 0.\end{aligned}$$

In view of the structure of $(\tilde{A}, \tilde{B})$, we can partition the following matrices conformally with the partitioning of (36) and write

$$\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{N}^* = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & 0 \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix}, \qquad \tilde{B}(-\tilde{R})^{-1/2} = \begin{bmatrix} \tilde{B}_{11} \\ 0 \\ 0 \end{bmatrix}.$$

Clearly, $(-\tilde{A}_{11}, \tilde{B}_{11})$ is stabilizable in the continuous-time sense, $\sigma(-\tilde{A}_{22}) \subset C^0$, and $\sigma(-\tilde{A}_{33}) \subset C^+$. We also partition $\tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^*$ as

$$\tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^* = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{12}^* & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{13}^* & \tilde{Q}_{23}^* & \tilde{Q}_{33} \end{bmatrix}.$$

The upper left 2×2 subblock of the above matrices, i.e.,

$$\tilde{A}_1 - \tilde{B}_1\tilde{R}_1\tilde{N}_1^* = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \qquad \tilde{B}_1\left(-\tilde{R}_1\right)^{-1/2} = \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix},$$

$$\tilde{Q}_1 - \tilde{N}_1\tilde{R}_1^{-1}\tilde{N}_1^* = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12} & \tilde{Q}_{22} \end{bmatrix},$$

satisfy (70)–(72) and hence also the conditions of Theorem 1 in [3], based on which we deduce that there exists a (positive semidefinite, or positive definite) hermitian X such that

$$\mathscr{C}\left(-\tilde{A} + \tilde{B}\tilde{R}^{-1}\tilde{N}^*, \tilde{B}(-\tilde{R})^{-1/2}, \tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^*, 0, I\right)(X) = -\tilde{P} \geqslant 0 \quad [> 0], \tag{77}$$

or equivalently,

$$\mathscr{C}\left(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}\right)(X) = \tilde{P} \leqslant 0 \qquad [< 0].$$

Moreover, both $\tilde{A}$ and $\tilde{\Delta}(X) = \tilde{A} - \tilde{B}\tilde{R}^{-1}(\tilde{B}^*X + \tilde{N}^*)$ have no eigenvalue at 1. Hence, Theorem 5 [Theorem 4] gives that

$$R + B^*XB < 0 \quad \text{and} \quad \mathscr{D}(A, B, Q, N, R)(X) \leqslant 0 \quad [< 0]$$

This proves the sufficiency part of the theorem. ■

Proof of Corollary 8. Necessity: From Theorem 7, if the DARI (54) has a (positive semidefinite, or positive definite) hermitian solution, then there exists $\tilde{X}_1$ satisfying the conditions (48)–(51). Hence, the argument given in the sufficiency part of the proof of Theorem 7 gives that $\tilde{R} < 0$ and there exists a (positive semidefinite, or positive definite) hermitian X such that (77), or equivalently (55), holds true.

Sufficiency: This follows directly from Theorem 2 in [3] and our proof of the sufficiency part of Theorem 7. ■

Proof of Corollary 11.

(i) Suppose the strict DARI (53) has a solution X such that $R + B^*XB < 0$. By Theorem 7,

$$X > 0 \quad \text{iff} \quad \tilde{X}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^* & \tilde{X}_{22} \end{bmatrix} > 0.$$

Note that $\tilde{X}_{22}$ is a solution to the DLI (51). Since, by Lemma 10(ii), $\tilde{X}_{22}$ can be chosen arbitrarily large, we have that $\tilde{X}_1 > 0$ iff $\tilde{X}_{11} > 0$.

(ii) Since $T^*XT > 0$, it follows from (37) that

$$(T^*XT)^{-1} > \begin{bmatrix} X_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{78}$$

We also have from Lemma 6 that $X_{11} < \tilde{X}_{11}$, which together with (78) implies (61).

It remains to show that if the strict DARI (53) admits a solution satisfying $R + B^*XB < 0$, then it admits a sequence of solutions $X(j)$, $j = 1, 2, \ldots,$ approaching the limit (62). For this purpose, we will invoke the continuous-time version of this result given in [6]. Note that in the proof of Theorem 7, (74) holds in the strict case and $\tilde{R}_1 < 0$ for some α such that A_1 and $\Delta_1(\tilde{X}_1)$ have no eigenvalue at $-\alpha$. Making use of Theorem 7 in [6], there exist $X(j)$, $j = 1, 2, \ldots,$ satisfying the strict inequality in (77) such that (62) holds. Using the same argument following (77) in our proof of Theorem 7, it is clear that the sequence $X(j)$, $j = 1, 2, \ldots,$ satisfies the strict DARI (53) and $R + B^*X(j)B < 0$. ■

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