Chapter 4
The Heath-Jarrow-Morton Model

Role of a Model

- Interest-rate models are necessary for
  - pricing interest-rate derivatives (portfolios) and
  - hedging market risks.

- The basic building blocks of fixed-income pricing is the discount factors or zero-coupon bonds which, mathematically, is given by

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds} \right]$$

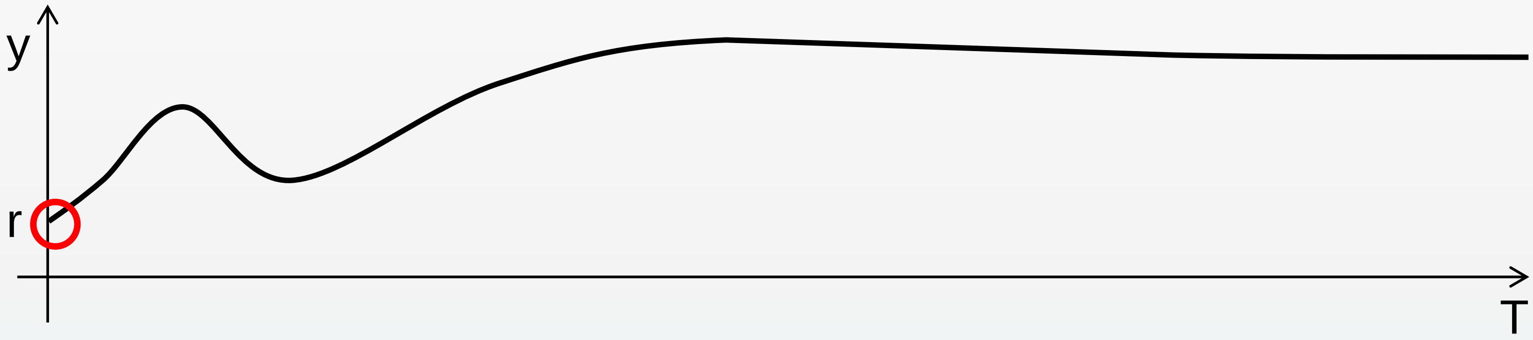
where r_t is the short rate.

Role of a Model

- The short rate had been a natural candidate of state variable for interest-rate models.
- In fact, it is the only state variable in many early models, like
  - Vasicek model (1977).
  - Cox-Ingersoll-Ross model (1984).
  - Black-Karasinski (1991).
- Early models are largely based on macro-economical arguments, and they are thus called *equilibrium models*.

Limitation of Short-rate Models

- Values of interest-rate security portfolios depend on the term structure of interest rates, not much on the short rate, the starting value of the term structure.



- Short-rate models lack sufficient capacity to describe the dynamic features of the entire yield curve.
- Short-rate models may be unable to calibrate to the market prices of benchmark instruments.
- Such models then cause mispricing and arbitrage.

Arbitrage-free Models

- The basic or underlying securities are zero-coupon bonds.
- To exclude arbitrage, an interest-rate model should, at least, price all zero-coupon bonds correctly.
- This can only be achieved by taking the prices of zero-coupon bonds of all maturities as inputs.
- The candidates for state variables include
  - Zero-coupon bonds;
  - Swap rates;
  - Forward rates.

Heath, Jarrow and Morton (HJM) model

- It turns out that, in most applications, the forward rates (either for continuous compounding or discrete compounding) are the best choice of state variables.
- With the continuous compounding forward rates, Heath, Jarrow and Morton (1992) developed an arbitrage framework for interest-rate modeling.
- The Heath-Jarrow-Morton (HJM) model became a framework for other specific arbitrage-free models.
- This chapter introduces the HJM model and its most famous and interesting special cases.

4.1 Lognormal Model: the Starting Point

The theoretical basis of this chapter starts from the usual assumption of lognormal asset dynamics for zero-coupon bonds of all maturities:

$$dP(t,T) = P(t,T)\left[\mu(t,T)dt + \mathbf{\Sigma}^{\mathrm{T}}(t,T)d\mathbf{W}_t\right], \tag{4.1}$$

under the physical measure, $\mathbb{P}$. Here, $\mu(t,T)$ is a scalar function of t and T, $\mathbf{\Sigma}(t,T)$ is a column vector,

$$\mathbf{\Sigma}(t,T) = \left(\Sigma_1(t,T), \Sigma_2(t,T), \ldots, \Sigma_n(t,T)\right)^{\mathrm{T}},$$

and $\mathbf{W}_t$ is an n-dimensional $\mathbb{P}$-Brownian motion,

$$\mathbf{W}_t = \left(W_1(t), W_2(t), \ldots, W_n(t)\right)^{\mathrm{T}}.$$

In principle, the coefficients in (4.1) can be estimated from time series data of zero-coupon bonds, yet it is not guaranteed that (4.1) with estimated drift and volatility functions can exclude arbitrage. For the time being, we assume that both $\mu(t,T)$ and $\mathbf{\Sigma}(t,T)$ are sufficiently regular deterministic functions on t, so that the SDE (4.1) admits a unique strong solution.

Looking for the Martingale Measure

- The purpose of a model like (4.1) is to price derivatives depending on (a portfolio of) $P(t,T), \forall T$ and $t \le T$. For this purpose, we need to find a martingale measure for zero-coupon bonds of all maturities. Similar to our discussions on the multiple-asset market, we define an $\mathcal{F}_t$-adaptive process, $\boldsymbol{\gamma}_t$, that satisfies the following equation

$$\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)\boldsymbol{\gamma}_t = \mu(t,T) - r_t.$$

Suppose that such a $\boldsymbol{\gamma}_t$ exists, is independent of T and satisfies the Novikov condition. We can define a measure, $\mathbb{Q}$, as

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t -\boldsymbol{\gamma}_s^{\mathrm{T}} d\mathbf{W}_s - \frac{1}{2}\left\|\boldsymbol{\gamma}_s\right\|^2 ds\right\}. \qquad (4.2)$$

Looking for the Martingale Measure, cont'd

Then, by the Cameron-Martin-Girsanov theorem, the process

$$\tilde{\mathbf{W}}_t = \mathbf{W}_t + \int_0^t \boldsymbol{\gamma}_s ds \qquad (4.3)$$

is a $\mathbb{Q}$-Brownian motion, and, in terms of $\tilde{\mathbf{W}}_t$, we can rewrite (4.1) as

$$\begin{aligned} dP(t,T) &= P(t,T)\left[r_t dt + \boldsymbol{\Sigma}^{\mathrm{T}}(t,T)\left(d\mathbf{W}_t + \boldsymbol{\gamma}_t dt \right) \right] \\ &= P(t,T)\left[r_t dt + \boldsymbol{\Sigma}^{\mathrm{T}}(t,T) d\tilde{\mathbf{W}}_t \right]. \end{aligned}$$

It then follows that the discounted prices of all maturities, $B_t^{-1}P(t,T)$, are $\mathbb{Q}$ -martingales.

The Existence of γ_t

- Now let us address the existence of such a $\boldsymbol{\gamma}_t$. Without loss of generality, we assume that the market of zero-coupon bonds is non-degenerate. That is, there exist at least n distinct zero-coupon bonds such that their volatility vectors constitute a non-singular matrix. Let $\{T_i\}_{i=1}^{n}$ be the maturities of the *n* bonds such that $T_i < T_{i+1}$, and let $\{\boldsymbol{\Sigma}(t,T_i)\}_{i=1}^{n}$ be the column vectors of their volatilities. By introducing matrices

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{\Sigma}^{\mathrm{T}}(t,T_1) \\ \boldsymbol{\Sigma}^{\mathrm{T}}(t,T_2) \\ \vdots \\ \boldsymbol{\Sigma}^{\mathrm{T}}(t,T_n) \end{pmatrix}, \quad \boldsymbol{\mu}_t = \begin{pmatrix} \mu(t,T_1) \\ \mu(t,T_2) \\ \vdots \\ \mu(t,T_n) \end{pmatrix}, \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{4.4}$$

we then define $\boldsymbol{\gamma}_t$ as a solution to the linear system

$$\mathbf{A}\boldsymbol{\gamma}_t = \boldsymbol{\mu}_t - r_t\mathbf{I}. \tag{4.5}$$

The Existence of γ_t

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This solution is unique provided that $\mathbf{A}$ is nonsingular. Such a solution, however, appears to depend on T_i, $1 \le i \le n$, the input maturities. With such a $\boldsymbol{\gamma}_t$, we can define a new measure, $\mathbb{Q}$, from (4.2), and under which the discounted prices of those n zero-coupon bonds, $B_t^{-1}P(t,T_i)$, $i = 1, \cdots, n$, are martingales.

Maturity Independence of γ_t

- Next, we will show that the solution to (4.5), γ_t, also satisfies

$$\mathbf{\Sigma}^{\mathrm{T}}(t,T)\boldsymbol{\gamma}_t = \mu(t,T) - r_t, \quad \text{for all } T \le T_n$$

- Because $\mathbf{A}$ is non-singular, for any T between 0 and T_n, there exists a unique vector, $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_n)$, such that

$$\left(\theta_1 P_1, \theta_2 P_2, \cdots, \theta_n P_n\right)\mathbf{A} = P(t,T)\mathbf{\Sigma}^{\mathrm{T}}(t,T)$$

where we have denoted $P_i = P(t,T_i)$.

- Consider the portfolio

$$V_t = P(t,T) - \sum_{i=1}^{n} \theta_i P(t,T_i).$$

Existence of γ_t , cont'd

- The value process of the portfolio is

$$
\begin{aligned}
dV_t &= dP(t,T) - \sum_{i=1}^{n} \theta_i dP(t,T_i) \\
&= \left(P(t,T)\mu(t,T) - \sum_{i=1}^{n} \theta_i P(t,T_i)\mu(t,T_i) \right) dt \\
&\qquad + \left(P(t,T)\boldsymbol{\Sigma}^{\mathrm{T}}(t,T) - \sum_{i=1}^{n} \theta_i P(t,T_i)\boldsymbol{\Sigma}^{\mathrm{T}}(t,T_i) \right) d\mathbf{W}_t \\
&= \underline{\left(P(t,T)\mu(t,T) - \sum_{i=1}^{n} \theta_i P(t,T_i)\mu(t,T_i) \right)} dt + 0 \times d\mathbf{W}_t
\end{aligned}
$$

- In the absence of arbitrage, the riskless portfolio must earn a return equal to the risk-free rate.

$$
dV_t = \underline{r_t V_t} dt + 0 d\mathbf{W}_t
$$

Existence of γ_t , cont'd

- It follows that

$$P(t,T)\mu(t,T)-\sum_{i=1}^{n}\theta_i P(t,T_i)\mu(t,T_i)=r_t\left(P(t,T)-\sum_{i=1}^{n}\theta_i P(t,T_i)\right). \quad (4.12)$$

By making use of (4.5), we can rewrite (4.12) as

$$\begin{aligned}\mu(t,T)-r_t &= \sum_{i=1}^{n}\theta_i \frac{P(t,T_i)}{P(t,T)}\left(\mu(t,T_i)-r_t\right)\\ &= \sum_{i=1}^{n}\theta_i \frac{P(t,T_i)}{P(t,T)}\boldsymbol{\Sigma}^{\mathrm{T}}(t,T_i)\boldsymbol{\gamma}_t\\ &= \boldsymbol{\Sigma}^{\mathrm{T}}(t,T)\boldsymbol{\gamma}_t,\end{aligned}$$

which is exactly (4.6).

We comment here that the components of $\boldsymbol{\gamma}_t$ are considered to be the market prices of risks for a zero-coupon bond, $P(t,T)$, $\forall T$.

4.2 The Heath-Jarrow-Morton Model

- Under the martingale measure, $\mathbb{Q}$, the price process of a zero-coupon bond becomes

$$dP(t,T) = P(t,T)\left[r_t dt + \mathbf{\Sigma}^{\mathrm{T}}(t,T) d\tilde{\mathbf{W}}_t \right].$$

- For the purpose of derivatives pricing, $\mathbf{\Sigma}(t,T)$ must satisfy at least the following additional conditions:

1) $P(t,t) = 1, \ \forall t$, and

2) $\mathbf{\Sigma}(t,t) = 0, \ \forall t$

3) $P(t,T) \downarrow T$

- The first two conditions are natural and easy to achieve, but the last condition, which ensures positive interest rates, is non-trivial.

Derivation of the Model

The specification of $\boldsymbol{\Sigma}(t,T)$ is a difficult job if we work directly with the process of $P(t,T)$. But the job will become quite amenable if we work with the process of forward rates. By Ito's lemma, there is

$$d\ln P(t,T) = \left(r_t - \tfrac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)\boldsymbol{\Sigma}(t,T)\right)dt + \boldsymbol{\Sigma}^{\mathrm{T}}(t,T)d\tilde{\mathbf{W}}_t. \qquad (4.14)$$

Assume, moreover, that $\boldsymbol{\Sigma}_T(t,T) = \partial\boldsymbol{\Sigma}(t,T)/\partial T$ exists and $\int_0^T \left\|\boldsymbol{\Sigma}_T(t,T)\right\|^2 dt < \infty$.

Derivation of the Model, cont'd

- By differentiating (4.14) with respect to T and recalling that

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}, \tag{4.15}$$

we obtain the process of forward rates under the $\mathbb{Q}$-measure,

$$df(t,T) = \mathbf{\Sigma}_T^{\mathrm{T}} \mathbf{\Sigma} dt - \mathbf{\Sigma}_T^{\mathrm{T}} d\tilde{\mathbf{W}}_t. \tag{4.16}$$

Here, the arguments about $\mathbf{\Sigma}$ are omitted for simplicity.

Derivation of the Model, cont'd

We consider $-\mathbf{\Sigma}_T(t,T)$ to be the volatility of the forward rate, and re-tag it as

$$\boldsymbol{\sigma}(t,T) = -\mathbf{\Sigma}_T(t,T). \tag{4.17}$$

By integrating the above equation with respect to T, we then obtain the volatility of the zero-coupon bond:

$$\mathbf{\Sigma}(t,T) = -\int_t^T \boldsymbol{\sigma}(t,s)ds. \tag{4.18}$$

Here, we have made use of the condition $\mathbf{\Sigma}(t,t) = 0$. Equation (4.18) fully describes the relationship between the volatility of a forward rate and the volatility of its corresponding zero-coupon bond.

Derivation of the Model, cont'd

In terms of $\boldsymbol{\sigma}(t,T)$, we can rewrite this forward rate process as

$$df(t,T) = \left(\boldsymbol{\sigma}^{\mathrm{T}}(t,T) \int_t^T \boldsymbol{\sigma}(t,s)ds \right) dt + \boldsymbol{\sigma}^{\mathrm{T}}(t,T) d\tilde{\mathbf{W}}_t . \qquad (4.19)$$

Equation (4.19) is the famous Heath-Jarrow-Morton (HJM) equation. An important feature of (4.19) is that the drift term of the forward-rate process under the martingale measure, $\mathbb{Q}$, is completely determined by its volatility.

A Milestone

- The HJM model lays down the foundation of arbitrage pricing in the context of fixed-income derivatives, and it is considered a milestone of financial derivative theory.
- Before 1992, fixed-income modeling was dominated by the so-called equilibrium models. Such models are based on macro economic arguments.
- The major limitation of equilibrium models is that these models do not naturally reproduce the market prices of basic instruments, zero-coupon bonds in particular, unless users go through a calibration procedure.

Arbitrage Pricing in Fixed Income Markets

- With arbitrage pricing models, the prices of the basic instruments are treated as model inputs rather than as outputs, so their prices are naturally reproduced.
- The arbitrage pricing models are rooted in the efficient market hypothesis, which states that in an efficient market, market prices of instruments do not induce any arbitrage opportunity.
- With an arbitrage model, derivative securities will be priced consistently with the basic instruments in the sense that no arbitrage opportunity would be induced.

Specification of Forward-rate Volatility

- There are two ways to specify the forward-rate volatility in the HJM model.
  - The first way is to estimate $\boldsymbol{\sigma}(t,T)$ (together with its dimension) directly from time series data on forward rates of various maturities. Note that $\boldsymbol{\sigma}^{\mathrm{T}}(t,T)\boldsymbol{\sigma}(t,T')$ should reflect the covariance between $f(t,T)$ and $f(t,T')$.
  - The second way is to specify $\boldsymbol{\sigma}(t,T)$ exogenously, using certain parametric functions of t and T.
- Different specifications of $\boldsymbol{\sigma}(t,T)$ generate different concrete models for applications.

Lognormal Forward-rate Model?

- To derive a model of positive interest rates, people have tried state dependent volatility of the form

$$\sigma(t,T) = \sigma_0(t,T) f^{\alpha}(t,T), \tag{4.135}$$

where $\sigma_0(t,T)$ is a deterministic function and α is a positive exponent. In the special case, $\alpha = 0$, we obtain a Gaussian model.

- Morton (1988) considered the "lognormal" model, corresponding to $\alpha = 1$, but only to see such a model blows up in finite time in the sense that a forward rate reaches infinity.
- One can imagine that similar results may apply to the case of $\alpha > 0$. Hence, volatility specification in the form of (4.136) is **denied**.

4.3 Special Cases of the HJM Model

- Since the publication of the HJM model in 1992, arbitrage pricing models have quickly acquired dominant status in fixed-income modeling.
- Theoretically, arbitrage pricing models can be generated from the HJM framework by making various specifications of forward-rate volatility.
- In this section, we study two specifications of the forward-rate volatility that, in terms of the short-rate dynamics, reproduce the popular one-factor models of Ho and Lee (1986) and Hull and White (1989), respectively.

4.3.1 The Ho-Lee Model

- The simplest specification of the HJM model is $\sigma = const$ for $n = 1$, corresponding to the forward-rate equation

$$df(t,T) = \sigma d\tilde{W}_t + \sigma^2 (T - t) dt \ .$$

By integrating the equation over $[0,t]$, we obtain

$$f(t,T) - f(0,T) = \sigma \tilde{W}_t + \frac{1}{2}\sigma^2 t(2T - t).$$

The Ho-Lee Model, cont'd

- By making $T = t$, we have the expression for the short rate:

$$r_t = f(t,t) = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma \tilde{W}_t \ .$$

In differential form, the last equation becomes

$$dr_t = \left(f_T(0,t) + \sigma^2 t \right) dt + \sigma d\tilde{W}_t. \qquad (4.26)$$

Equation (4.26) is interpreted as the continuous-time version of the so-called Ho-Lee model (1986), which was first developed in the context of binomial trees.

Drawbacks of the Ho-Lee Model

- Let us take a look at a basic feature of the Ho-Lee model. It is quite obvious to see that

$$
\begin{aligned}
E^{\mathbb{Q}}\left[r_t\right] &= f(0,t) + \frac{1}{2}\sigma^2 t^2, \\
Var\left(r_t\right) &= \sigma^2 t.
\end{aligned}
\tag{4.27}
$$

The two equations of (4.27) suggest that the short rate will fluctuate around a quadratic function of time with increasing variance. This feature is counter to common sense, and it has motivated alternative specifications to make the short rate behave more reasonably.

Bond Price Formula

- With the Ho-Lee model, we can obtain the following price formula of zero-coupon bonds:

$$P(t,T) = \exp\left\{-\int_t^T \left(f(0,s) + \sigma \tilde{W}_t + \frac{\sigma^2}{2} t\left(2s - t\right)\right) ds\right\}$$

$$= \exp\left\{-\left(\int_t^T f(0,s)ds + \sigma \tilde{W}_t\left(T-t\right) + \frac{\sigma^2}{2} tT\left(T-t\right)\right)\right\}$$

$$= \frac{P(0,T)}{P(0,t)} \exp\left\{-\left(\sigma \tilde{W}_t\left(T-t\right) + \frac{\sigma^2}{2} tT\left(T-t\right)\right)\right\}.$$

This formula can be used to, among other applications, price options on zero-coupon bonds.

4.3.2 The Hull-White Model (extended Vasicek Model)

- It has been empirically observed that forward-rate volatility decays with time-to-maturity, $T-t$. This motivates the following specification of the volatility:

$$\sigma(t,T)=\sigma e^{-\kappa(T-t)},\quad \kappa>0. \tag{4.28}$$

That is, the volatility decays exponentially as time goes forward. The corresponding HJM equation now reads,

$$\begin{aligned} df(t,T) &= \sigma e^{-\kappa(T-t)}d\tilde{W}_t+\left(\sigma e^{-\kappa(T-t)}\int_t^T \sigma e^{-\kappa(s-t)}ds\right)dt \\ &= \sigma e^{-\kappa(T-t)}d\tilde{W}_t+\sigma e^{-\kappa(T-t)}\frac{\sigma}{\kappa}\left[1-e^{-\kappa(T-t)}\right]dt \\ &= \sigma e^{-\kappa(T-t)}d\tilde{W}_t+\frac{\sigma^2}{\kappa}\left[e^{-\kappa(T-t)}-e^{-2\kappa(T-t)}\right]dt. \end{aligned}$$

Expression for the Short Rate

- Integrating the above equation over $(0,t)$ yields

$$f(t,T) = f(0,T) + \sigma\int_0^t e^{-\kappa(T-s)}d\tilde{W}_s + \frac{\sigma^2}{2\kappa^2}\left[\left(1-e^{-\kappa T}\right)^2 - \left(1-e^{-\kappa(T-t)}\right)^2\right].$$

By making $T = t$, we obtain the expression for the short rate:

$$r_t = f(0,t) + \sigma\int_0^t e^{-\kappa(t-s)}d\tilde{W}_s + \frac{\sigma^2}{2\kappa^2}\left(1-e^{-\kappa t}\right)^2. \qquad (4.29)$$

Advantages of the Hull-White Model

- Yet again, let us check the mean and variance of the short rate, which are

$$E^{\mathbb{Q}}\left[r_t\right]= f(0,t)+\frac{\sigma^2}{2\kappa^2}\left(1-e^{-\kappa t}\right)^2,$$

$$Var\left[r_t\right]=\sigma^2\int_0^t e^{-2\kappa(t-s)}ds=\sigma^2\frac{1}{2\kappa}\left(1-e^{-2\kappa t}\right)<\frac{\sigma^2}{2\kappa}.$$

The above equations suggest that both the mean and the variance of the short rate stay bounded, a very plausible feature for a short-rate model.

Mean Reversion Feature

- Next, let us study the differential form of (4.29). Denote

$$X_t = \sigma \int_0^t e^{-\kappa(t-s)} d\tilde{W}_s,$$

which satisfies

$$\begin{aligned} dX_t &= \sigma d\tilde{W}_t - \sigma\kappa \int_0^t e^{-\kappa(t-s)} d\tilde{W}_s dt \\ &= \sigma d\tilde{W}_t - \kappa X_t dt, \end{aligned} \tag{4.30}$$

and relates to the short rate as

$$X_t = r_t - f(0,t) - \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa t}\right)^2. \tag{4.31}$$

Mean Reversion Feature, cont'd

- By differentiating (4.29) and making use of (4.30), we then obtain

$$\begin{aligned} dr_t &= f_T(0,t)dt + \sigma d\tilde{W}_t - \kappa X_t dt + \frac{\sigma^2}{\kappa} e^{-\kappa t}\left(1 - e^{-\kappa t}\right)dt \\ &= \kappa\left(\theta_t - r_t\right)dt + \sigma d\tilde{W}_t, \end{aligned} \tag{4.32}$$

where

$$\theta_t \triangleq f(0,t) + \frac{1}{\kappa} f_T(0,t) + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-2\kappa t}\right). \tag{4.33}$$

Equation (4.32) is called the Hull-White model (1989) or sometimes the extended Vasicek model, because Vasicek (1977) was the first to adopt the formalism of (4.32) for short-rate modeling in an equilibrium approach. Note that when $\kappa \to 0$, the Hull-White model reduces to the Ho-Lee model (4.26).

Mean Reversion Feature, cont'd

- Let us highlight the so-called mean-reverting feature of the Hull-White model: when $r_t > \theta_t$, the drift is negative; when $r_t < \theta_t$, the drift turns positive. The drift term acts like a force that pushes the short rate toward its mean level, θ_t. The contribution of Hull and White is to identify the level of mean reversion, θ_t, displayed in (4.33), so that zero-coupon bond prices of all maturities are reproduced.

Bond Price Formula

- In terms of the short rate, we have the following formula for the zero-coupon bond price (Hull and White, 1989):

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{4.34}$$

where

$$B(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}, \tag{4.35}$$

and

$$\begin{aligned}\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,t)}{\partial t} \\ -\frac{\sigma^2}{4\kappa^3}\left(e^{-\kappa T} - e^{-\kappa t}\right)^2\left(e^{2\kappa t}-1\right).\end{aligned} \tag{4.36}$$

The proof is left as an exercise.

Advantages and Disadvantages

- Moreover, since the short rate is a Gaussian variable, the model can be implemented through a binomial tree.
- Therefore, it is a convenient choice of model to price path-dependent options.
- Under the Hull-White model, the short rate is a Gaussian random variable and can take a negative value (with a positive probability), which is unrealistic and is considered a major disadvantage of the model.
- In applications, the tree can also be truncated to avoid negative interest rates.

Short Rate Dynamics Under HJM

- The HJM model also implies the dynamics of the short rate. By integrating (4.16) form 0 to t , we obtain the expression of forward rates:

$$f(t,T) = f(0,T) - \int_0^t \boldsymbol{\sigma}^{\mathrm{T}}(s,T)\boldsymbol{\Sigma}(s,T)ds + \boldsymbol{\sigma}^{\mathrm{T}}(s,T)d\tilde{\mathbf{W}}_s. \qquad (4.20)$$

Then by setting $T = t$, we obtain an expression for the short rate:

$$r_t = f(t,t) = f(0,t) + \int_0^t \frac{1}{2}\frac{\partial\left\|\boldsymbol{\Sigma}(s,t)\right\|^2}{\partial t}ds + \boldsymbol{\sigma}^{\mathrm{T}}(s,t)d\tilde{\mathbf{W}}_s. \qquad (4.21)$$

- In differential form, (4.21) becomes

$$dr_t = \text{drift} + \left(\int_0^t \frac{\partial \boldsymbol{\sigma}^{\mathrm{T}}(s,t)}{\partial t} d\tilde{\mathbf{W}}_s \right) dt + \boldsymbol{\sigma}^{\mathrm{T}}(t,t) d\tilde{\mathbf{W}}_t. \quad (4.22)$$

By examining (4.22), we understand that, under the HJM framework, the short rate is in general a non-Markovian random variable, unless

$$\int_0^t \frac{\partial \boldsymbol{\sigma}^{\mathrm{T}}(s,t)}{\partial t} d\tilde{\mathbf{W}}_s \quad (4.23)$$

can be expressed as a function of some Markovian variables. Only in such a situation can we call the short-rate dynamics a Markovian variable. We devote all of Chapter 5 to Markovian short-rate models.

- For some subsequent applications, we rewrite (4.21) as

$$r_t = f(0,t) + \frac{\partial}{\partial t}\int_0^t \frac{1}{2}\left\|\mathbf{\Sigma}(s,t)\right\|^2 ds - \mathbf{\Sigma}^{\mathrm{T}}(s,t)\,d\tilde{\mathbf{W}}_s. \qquad (4.24)$$

Here, we have applied the following stochastic Fubini theorem (see Karatzas and Shreve, 1991) to exchange the order of differentiation and integration:

$$\frac{\partial}{\partial t}\int_0^t \theta(s,t)\,d\tilde{W}_s = \theta(t,t)\frac{d\tilde{W}_t}{dt} + \int_0^t \frac{\partial}{\partial t}\theta(s,t)\,d\tilde{W}_s,$$

and we have used the property $\mathbf{\Sigma}(t,t) = 0$.

HJM as A Necessary Condition

- The HJM model has been treated not only as a model in its own right, but also as a framework for fixed-income models that are deemed arbitrage free.
- The HJM equation is a necessary condition for no-arbitrage models, but not a sufficient one, because the interest rates can turn negative.
- For usual specifications of the volatility function, $\boldsymbol{\sigma}(t,T)$, the forward rate has a Gaussian distribution, so it can assume negative values with a positive possibility.
- An implicit condition on $\boldsymbol{\sigma}(t,T)$ that ensures positive forward rates is established in Yan and Glasserman (2001) (Themrem 5). But the condition does not suggest the prescription of $\boldsymbol{\sigma}(t,T)$.

A Gap between HJM and Positive-rate Models

- On the other hand, it can also be very difficult to identify the corresponding forward-rate volatility functions for some existing short-rate models, particularly the short-rate models that guarantee positive interest rates, including two major extensions to Hull-While model
  - Coss-Ross-Ingersoll Model
  - Black-Karasinski model

Beyond the Hull-White Model

- A Case Study with a two-factor model
- We can parameterize the forward-rate volatility to capture the stylized features of the forward-rate curves that are shaped by principal components.

- We consider a two-factor HJM model ($\eta = 2$) with the following forward-rate volatility components,

$$\sigma_1(T) = ae^{-k_1 T},$$
$$\sigma_2(T) = b(1 - 2e^{-k_2 T}), \tag{4.59}$$

where a, b, k_1 and k_2 are constants. To get a “flat” $\sigma_1(T)$ and a “tilted” $\sigma_2(T)$, we choose k_1 and k_2 such that $0 \le k_1 \ll 1$, $k_1 \ll k_2$.

The Covariance and Correlation

- The covariance between forward rates of two maturities, T and T', is calculated according to

$$c(T,T') = \sigma_1(T)\sigma_1(T') + \sigma_2(T)\sigma_2(T'). \tag{4.60}$$

The correlation between forward rates of two maturities is thus

$$\rho(T,T') = \frac{c(T,T')}{\sqrt{c(T,T)}\sqrt{c(T',T')}}. \tag{4.61}$$

- We consider the following choice of parameters

$$a = 0.008,\ b = 0.003,\ k_1 = 0.0,\ k_2 = 0.35.$$

Note that if we take $b = 0$, this two-factor model reduces to the Hull-White model, under which forward rates of all maturities are perfectly correlated.

The Covariance and Correlation, cont'd

- Taking $T = 0.25$ (3-month) and $T' = 30$, we have

$$c(0.25, 0.25) = 7.024 \times 10^{-5},$$
$$c(0.25, 30) = 5.651 \times 10^{-5},$$
$$c(30, 30) = 7.300 \times 10^{-5}.$$

The correlation coefficient between the 3-month and 30-year forward rates is

$$\rho(0.25, 30) = \frac{c(0.25, 30)}{\sqrt{c(0.25, 0.25)}\sqrt{c(30, 30)}} = 79\%,$$

which indicates a high level of correlation between the two rates.

The Covariance and Correlation, cont'd

- The eigenvalues and normalized eigenvectors of the covariance matrix, $c_{2\times2}$, are listed in Table 4.2.

Table 4.2. PCA analysis of short and long rates

$\lambda_1 = 1.281\times10^{-4}$	$\lambda_2 = 1.51\times10^{-5}$
$\mathbf{v}_1 = \begin{pmatrix} 0.6984 \\ 0.7157 \end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix} -0.7157 \\ 0.6984 \end{pmatrix}$
(for parallel shifts)	(for tilt moves)

The relative importance of the two modes is reflected by the weights

$$\frac{\lambda_1}{\lambda_1+\lambda_2} = 89.46\% \quad \text{and} \quad \frac{\lambda_2}{\lambda_1+\lambda_2} = 10.54\%,$$

which are consistent with the PCA of U.S. Treasury data by Litterman and Scheinkman (1991).

Magnitude of Short-rate Volatility

- Finally, we comment that the choices of the parameters correspond to reasonable short-rate volatility, which is calculated according to

$$\sigma_r = \sqrt{c(0,0)} = \sqrt{\sigma_1^2(0) + \sigma_2^2(0)} = \sqrt{a^2 + b^2} = 0.85\%.$$

Suppose the short-rate is $r(0) = 5\%$. Then, the percentage of volatility of the short rate is $\sigma_r / r(0) = 0.85\% / 5\% = 17\%$. The interval of one standard deviation is

$$\big((5 - 0.85)\%, (5 + 0.85)\%\big) = \big(4.15\%, 5.85\%\big),$$

into which the short rate will fall with 67% probability.

4.6 Monte Carlo Implementations

- We now consider the application of the HJM model to derivatives pricing. As a demonstration, we consider the pricing of a bond option that matures at T_0 with payoff

$$X_{T_0} = \left(\sum_{i=1}^{n} \Delta T \cdot c \cdot P_{T_0}^{T_i} + P_{T_0}^{T_n} - K \right)^+ .$$

Here, c is the coupon rate of the bond, K is the strike price of the option, and $T_i = T_0 + i\Delta T$ is the cash-flow date of the i^{th} coupon of the underlying bond. We call $T_n - T_0$, the life of the underlying bond beyond T_0, the tenor of the bond.

Bond Options

- The value of the option is given by

$$
\begin{aligned}
V_0 &= E^{\mathbb{Q}}\left[\frac{1}{B_{T_0}} \left(\sum_{i=1}^{n} \Delta T \cdot c \cdot P_{T_0}^{T_i} + P_{T_0}^{T_n} - K \right)^{+} \middle| \mathcal{F}_0 \right] \\
&= E^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} \Delta T c \cdot \frac{P_{T_0}^{T_i}}{B_{T_0}} + \frac{P_{T_0}^{T_n}}{B_{T_0}} - \frac{K}{B_{T_0}} \right)^{+} \middle| \mathcal{F}_0 \right],
\end{aligned}
\tag{4.62}
$$

where $\mathbb{Q}$ is the risk-neutral measure.

Price Formula for Zero-Coupon Bonds

- Finally, we note that, under the HJM model, the following price formula for zero-coupon bonds exists:

$$d\left(\frac{P(t,T)}{B_t}\right)=\frac{P(t,T)}{B_t}\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)d\tilde{\mathbf{W}}_t,\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)$$

$$d\ln\left(\frac{P(t,T)}{B_t}\right)=-\frac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)\boldsymbol{\Sigma}(t,T)dt+\boldsymbol{\Sigma}^{\mathrm{T}}(t,T)d\tilde{\mathbf{W}}_t,$$

- Solve it, we obtain

$$\frac{P(t,T)}{B_t}=\frac{P(0,T)}{B_0}\exp\left\{\int_0^t\left(-\frac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}(s,T)\boldsymbol{\Sigma}(s,T)\right)ds+\boldsymbol{\Sigma}^{\mathrm{T}}(s,T)d\tilde{\mathbf{W}}_s\right\},\qquad(4.25)$$

which will be used repeatedly for analyses as well as computations.

Formula for Zeros and MMA

- Based on (4.25), we have the following expression for the discounted value of zero-coupon bonds:

$$\frac{P_{T_0}^{T_i}}{B_{T_0}} = P_0^{T_i} \exp\left(\int_0^{T_0} -\frac{1}{2}\left\|\boldsymbol{\Sigma}(t,T_i)\right\|^2 dt + \boldsymbol{\Sigma}^{\mathrm{T}}(t,T_i)d\tilde{\mathbf{W}}_t\right), \qquad (4.63)$$

for $i = 0,1,\ldots,n$. Taking $i = 0$, in particular, we obtain the expression for the reciprocal of the money market account:

$$\frac{1}{B_{T_0}} = P_0^{T_0} \exp\left(\int_0^{T_0} -\frac{1}{2}\left\|\boldsymbol{\Sigma}(t,T_0)\right\|^2 dt + \boldsymbol{\Sigma}^{\mathrm{T}}(t,T_0)d\tilde{\mathbf{W}}_t\right). \qquad (4.64)$$

Expressions (4.63) and (4.64) allow us to calculate the option's payoff.

Monte-Carlo simulation algorithm

1. Construct the discount curve, P_0^T, $\forall T$, with the U.S. Treasury data at $t = 0$.

2. Simulate a number of Brownian paths and calculate the discounted prices according to the scheme

$$\frac{P_{t+\Delta t}^{T_i}}{B_{t+\Delta t}} = \frac{P_t^{T_i}}{B_t} \exp\left(-\frac{1}{2} \left\| \boldsymbol{\Sigma}(t, T_i) \right\|^2 \Delta t + \boldsymbol{\Sigma}^{\mathrm{T}}(t, T_i) \Delta \tilde{\mathbf{W}}_t \right) \tag{4.65}$$

   for $i = 0, 1, \ldots, n$, until $t + \Delta t$ equals T_0.

3. Average the payoffs:

$$\left(\sum_{i=1}^{n} \Delta T c \cdot \frac{P_{T_0}^{T_i}}{B_{T_0}} + \frac{P_{T_0}^{T_n}}{B_{T_0}} - \frac{K}{B_{T_0}} \right)^+ . \tag{4.66}$$

Swaptions

- We will use the Monte Carlo simulation method with the pricing of a set of options, which have the same strike price, $K=1$, and special coupon rates defined as

$$c=\left(P_0^{T_0}-P_0^{T_n}\right)\Big/\sum_{i=1}^{n}\Delta T P_0^{T_i}\ . \tag{4.67}$$

As we shall see later, such special bond options are equivalent to swaptions, the options on interest-rate swaps.

The Volatility of Zero-Coupon Bonds

- Let us take the two-factor model discussed in the last section, which has a forward-rate volatility prescribed in (4.59).
- The volatility for zero-coupon bond is obtained from an integration:

$$\boldsymbol{\Sigma}(t,T) = -\int_t^T \begin{pmatrix} a \\ b\left(1-2e^{-k_2 s}\right) \end{pmatrix} ds = -\begin{pmatrix} a\left(T-t\right) \\ b\left[\left(T-t\right)+\frac{2}{k_2}\left(e^{-k_2 T}-e^{-k_2 t}\right)\right] \end{pmatrix}.$$

One Factor vs. Two Factors

- We consider the following two sets of parameterizations:

    1. One-factor model: $a = 0.008544, b = 0, k_1 = 0.0, k_2 = 0.35.$

    2. Two factor model: $a = 0.008, b = 0.003, k_1 = 0.0, k_2 = 0.35.$

    Note that, for both models, $k_1 = 0$, and the one-factor model is exactly a Ho-Lee model. For the sake of comparison, we have chosen $a = 0.008544$ for the one-factor model so that the corresponding short-rate volatilities of the two models, $\sigma_r = \sqrt{a^2 + b^2}$, are identical.

- The size of each time step for the Monte Carlo simulation is $\Delta t = 0.25$.

The Yields of the Benchmark Bonds

- To construct the discount curve, we use the yield data of March 15, 2007, listed in Table 4.3, and go through the bootstrapping procedure described earlier.

Table 4.3. Yields at March 15, 2007

Maturity	Yield
3 month	4.89
6 month	4.88
2 year	4.56
3 year	4.47
5 year	4.44
10 year	4.52
30 year	4.69

Swaption and Caplet Prices

- **Table 4.4.** Prices of par-bond options in basis points

Bond option		Model	
Maturity, T_0	Tenor, $T_n - T_0$	Ho-Lee	2-factor
1	0.25	8.23	7.85
2	0.25	11.07	10.34
5	0.25	15.33	14.69
1	5	145.43	140.41
2	5	200.29	191.07
5	5	275.08	272.14
1	10	262.88	254.99
2	10	364.54	349.50
5	10	489.43	487.53

The options with tenor 0.25 are called caplets, while the other are called swaptions. For caplets, $\Delta T = 0.25$, while for swaptions, $\Delta T = 0.5$. One basis point corresponds to one cent for the notional of 100 dollars.

Comments on the Order of Prices

- The Ho-Lee model consistently produces higher prices.
- Explanations: under the one-factor model, the prices of all zero-coupon bonds are perfectly correlated, while under the two-factor model, they are not.
- Because a coupon bond is a portfolio of zero-coupon bonds, its volatility will be larger if all zero-coupon bonds move in the same direction, provided that the short-rate volatilities are the same.
- The larger volatility leads to a higher option premium.

Motivations for Alternative Pricing Methods

- Monte Carlo method is flexible and widely applicable, it suffers from slow convergence, and thus it is usually not the choice of market participants for whom pricing in real time is necessary.
- Fast pricing methods must be developed.
- An important device for speedy option pricing is by a proper change of measure.
- As a preparation, we first introduce forward contracts and the notion of forward prices.

Forward Prices

- Forward contract: a deal entered now to purchase an asset at a future time when both payment and delivery take place.
- What should be taken as the fair price for this transaction?
- We try arbitrage pricing.
- To ensure delivery, the seller must borrow money now and acquire certain units of the asset.

Arbitrage Pricing

- Denote
  - t --- the current time,
  - S_t --- the current price of the asset,
  - q --- the dividend yield
  - T --- the delivery time,
  - F --- the unknown fair transaction price.
- To be able to deliver one unit of the asset, the seller then does the following transactions:
  - Short $S_t e^{-q(T-t)} / P_t^T$ units of T-maturity ZCB, and
  - Long $e^{-q(T-t)}$ unit of the asset.
- Note that 1 and 2 are a set of zero-net transactions at time t .

Arbitrage Pricing, cont'd

- At the delivery time, T , the seller will deliver one unit of the asset to the buyer for the price of F , and thus ends up with the following profit/loss value,

$$V_T = F - \frac{S_t e^{-q(T-t)}}{P_t^T}.$$

- For the absence of arbitrage, there must be $V_T = 0$, giving the fair transaction price

$$F = \frac{S_t e^{-q(T-t)}}{P_t^T}.$$

- In an economy where there is no arbitrage, this price is fair and unique.

Forward Price for Dividend Paying Assets, cont'd

- Finally, we derive the fair transaction price when the asset pays discrete dividends. Assume that, prior to T, the asset pays cash dividend q_i at time $T_i \leq T$, $i = 1, \ldots, n$, such that $T_{i-1} < T_i$. In such a circumstance, the seller's strategy is to

  1. short q_i units of T_i-maturity zero-coupon bonds ($i \leq n$);

  2. short $\left(S_t - \sum_{t < T_i \leq T} q_i P_t^{T_i} \right) \Big/ P_t^T$ units of T-maturity zero-coupon bonds;

  3. long 1 unit of the asset.

Forward Price for Dividend Paying Assets, cont'd

- The proceeds from shorting are just enough to purchase one unit of the asset. Hence, this is still a set of zero-net transactions. After closing out all positions at time T , the seller ends up with the net value of

$$V_T = F - \left(S_t - \sum_{t<T_i\leq T} q_i P_t^{T_i} \right) \Big/ P_t^T .$$

Hence, the fair price for a transaction is

$$F = \left(S_t - \sum_{t<T_i\leq T} q_i P_t^{T_i} \right) \Big/ P_t^T . \tag{4.69}$$

Stripped-Dividend Prices

- For all three cases, we define

$$\hat{S}_t = \begin{cases} S_t, & \text{no dividend} \\ S_t \exp(-q(T-t)), & \text{dividend yield } q \\ (S_t - \sum_{t<T_i \le T} q_i P_t^{T_i}), & \text{discrete dividend } \{q_i\} \end{cases} \qquad (4.70)$$

- No that the stripped-dividend price has no jump!

Forward Price

Definition 4.1. The price of a stripped-dividend asset relative to the *T*-maturity zero-coupon bond,

$$F_t^T = \frac{\hat{S}_t}{P_t^T},$$

is called the forward price with delivery at time *T*.

Tradability of the Stripped-Dividend Assets

- Like the original asset itself, a stripped-dividend asset is also tradable (or replicable, in principle). Hence, its pricing process under the risk neutral measure $\mathbb{Q}$ is usually assumed to be

$$d\hat{S}_t = \hat{S}_t (r_t dt + \mathbf{\Sigma}_S^{\mathrm{T}} d\tilde{\mathbf{W}}_t).$$

The process for the original asset then follows.

Price Dynamics of the Original Asset Prices

- The process for the original asset then follows. In fact, for the asset paying a continuous dividend yield, the price process is

$$dS_t = S_t\left(\left(r_t - q\right)dt + \boldsymbol{\Sigma}_S^{\mathrm{T}} d\tilde{\mathbf{W}}_t\right), \tag{4.71}$$

while for the asset paying discrete dividends, the price process is

$$\begin{aligned} dS_t &= \left[r_t S_t - \sum q_i \delta(T_i - t)\right]dt \\ &\quad + \left[S_t \boldsymbol{\Sigma}_S^{\mathrm{T}} - \sum q_i \mathbf{1}_{t \le T_i} P_t^{T_i} \left(\boldsymbol{\Sigma}_S - \boldsymbol{\Sigma}\right)^{\mathrm{T}}\right] d\tilde{\mathbf{W}}_t, \end{aligned} \tag{4.72}$$

where $\delta(x)$ is the Dirac delta function and $\mathbf{1}_{t \le T_i}$ is the indicator function.

Options on Forward Prices

- Because the price of a zero-coupon bond equals par at maturity, a forward price equals its spot price at the delivery date, i.e. $F_T^T = \hat{S}_T = S_T$.
- As a result, any options written on S_T can equivalently be treated as an option on F_T^T .
- Next, we will try to price an option on F_T^T .
- Analogy: Heng Seng Index options vs. Heng Seng Futures Options.

Dynamics of Forward Prices

- As tradable assets, the price of a stripped-dividend asset and the price of a zero-coupon bond are assumed to be, respectively,

$$
\begin{aligned}
d\hat{S}_t &= \hat{S}_t\left(r_t dt + \mathbf{\Sigma}_S^{\mathrm{T}}(t) d\tilde{\mathbf{W}}_t\right), \\
dP_t^T &= P_t^T\left(r_t dt + \mathbf{\Sigma}^{\mathrm{T}}(t,T) d\tilde{\mathbf{W}}_t\right).
\end{aligned}
\tag{4.73}
$$

By the quotient rule, the forward price satisfies

$$
\begin{aligned}
d\left(\frac{\hat{S}_t}{P_t^T}\right) &= \frac{d\hat{S}_t}{P} - \frac{\hat{S}_t dP}{P^2} - \frac{d\hat{S}_t dP}{P^2} + \frac{\hat{S}_t (dP)^2}{P^3} \\
&= \frac{\hat{S}_t}{P}\left[r_t dt + \mathbf{\Sigma}_S^{\mathrm{T}} d\tilde{\mathbf{W}}_t - r_t dt - \mathbf{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_t - \mathbf{\Sigma}_S^{\mathrm{T}} \mathbf{\Sigma} dt + \mathbf{\Sigma}^{\mathrm{T}} \mathbf{\Sigma} dt\right] \\
&= \frac{\hat{S}_t}{P}\left(\mathbf{\Sigma}_S - \mathbf{\Sigma}\right)^{\mathrm{T}}\left[d\tilde{\mathbf{W}}_t - \mathbf{\Sigma} dt\right].
\end{aligned}
$$

◈ Define now a new measure, $\mathbb{Q}_T$, as

$$\left.\frac{d\mathbb{Q}_T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \mathbf{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_s - \tfrac{1}{2}\mathbf{\Sigma}^{\mathrm{T}}\mathbf{\Sigma}ds\right\} = \zeta_t. \tag{4.74}$$

Then, by the CMG theorem,

$$\hat{\mathbf{W}}_t = \tilde{\mathbf{W}}_t - \int_0^t \mathbf{\Sigma}(s,T)ds \tag{4.75}$$

is a $\mathbb{Q}_T$ -Brownian motion. It follows that F_t^T is also a lognormal $\mathbb{Q}_T$ -martingale such that it follows a driftless process,

$$dF_t^T = F_t^T \mathbf{\Sigma}_F^{\mathrm{T}} d\hat{\mathbf{W}}_t,$$

where

$$\mathbf{\Sigma}_F(t) = \mathbf{\Sigma}_S(t) - \mathbf{\Sigma}(t,T).$$

Forward Measure

- We call $\mathbb{Q}_T$ the forward measure with delivery at T, or simply the T-forward measure. According to the definition, there is a one-to-one correspondence between the T-maturity zero-coupon bond and the T-forward measure.
- Under the $\mathbb{Q}_T$, the price of any asset relative to P_t^T is a $\mathbb{Q}_T$-martingale.

Pricing Options on the Forward Prices

- Based on the understanding that the forward price is a $\mathbb{Q}_T$ -martingale, we can derive a general pricing principle of options on F_T^T . Define another $\mathbb{Q}_T$ -martingale as

$$N_t = E^{\mathbb{Q}_T}[X_T \mid \mathcal{F}_t]. \tag{4.77}$$

According to the martingale representation theorem, there exists a $\mathcal{F}_t$ -adaptive process, φ_t , such that

$$dN_t = \varphi_t dF_t^T. \tag{4.78}$$

Pricing Options on the Forward Prices, cont'd

- We now form a portfolio consisting of

  φ_t units of the underlying stripped-dividend asset, and

  $\psi_t = N_t - \varphi_t F_t^T$ units of the T - maturity zero-coupon bond.

Let $\hat{V}_t$ be the forward price of the portfolio at time t. By definition, $\hat{V}_t = N_t$ for all $t \le T$, and the spot price of the portfolio,

$$V_t = N_t P_t^T. \tag{4.79}$$

At the maturity of the option, there is $V_T = N_T = X_T$, meaning that the portfolio replicates the payoff of the option.

Pricing Options on the Forward Prices, cont'd

- In addition, we have

$$\begin{aligned} dV_t &= P_t^T dN_t + N_t dP_t^T + dN_t dP_t^T \\ &= P_t^T \varphi_t dF_t^T + \left(\varphi_t F_t^T + \psi_t\right) dP_t^T + \varphi_t dF_t^T dP_t^T \\ &= \varphi_t \left(P_t^T dF_t^T + F_t^T dP_t^T + dF_t^T dP_t^T\right) + \psi_t dP_t^T \\ &= \varphi_t d\hat{S}_t + \psi_t dP_t^T, \end{aligned}$$

which implies that the portfolio, (φ_t, ψ_t), is a self-financing one. In the absence of arbitrage, the value of the option should be nothing but that of the replicating portfolio. This yields the general price formula

$$V_t = P_t^T E^{\mathbb{Q}_T}[X_T \mid \mathcal{F}_t]$$

for the option.

Risk-Neutral Pricing

- Note that this is the second option price formula in addition to the one developed in Chapter 2,

$$V_t = B_t E^{\mathbb{Q}}[B_T^{-1} X_T \mid \mathcal{F}_t], \tag{4.82}$$

obtained under the risk-neutral measure, $\mathbb{Q}$. Both formulae are obtained through arbitrage arguments, so the values of the two formulae must be identical, or otherwise we have a big problem.

Identical Prices

- Mathematically, it remains interesting to verify that the two formulae give an identical price.
- In fact, we can derive one formula from the other by merely a change of measure.
- We know that under the risk-neutral measure, $\mathbb{Q}$, P_t^T follows

$$dP_t^T = P_t^T\left(r_t dt + \mathbf{\Sigma}^{\mathrm{T}}(t,T)d\tilde{\mathbf{W}}_t\right), \tag{4.83}$$

or

$$d\left(\frac{P_t^T}{B_t}\right) = \left(\frac{P_t^T}{B_t}\right)\mathbf{\Sigma}^{\mathrm{T}}(t,T)d\tilde{\mathbf{W}}_t, \tag{4.84}$$

where $\tilde{\mathbf{W}}_t$ is a $\mathbb{Q}$ - Brownian motion.

Identical Prices, cont'd

- By solving the equation, we obtain

$$\frac{P_T^T}{B_T} = \frac{P_t^T}{B_t} \exp\left\{ \int_t^T -\tfrac{1}{2} \mathbf{\Sigma}^{\mathrm{T}} \mathbf{\Sigma} ds + \mathbf{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_s \right\}. \quad (4.85)$$

From this equation, we can express B_t in terms of P_t^T :

$$\frac{B_t}{B_T} = P_t^T \exp\left\{ \int_t^T -\tfrac{1}{2} \mathbf{\Sigma}^{\mathrm{T}} \mathbf{\Sigma} ds + \mathbf{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_s \right\} = P_t^T \frac{\zeta_T}{\zeta_t}, \quad (4.86)$$

by making use of (4.74).

Identical Prices, cont'd

- Hence, starting from (4.82), we have

$$\begin{aligned} V_t &= E^{\mathbb{Q}}\left[\frac{B_t}{B_T} X_T \,\middle|\, \mathcal{F}_t\right] \\ &= E^{\mathbb{Q}}\left[P_t^T \frac{\zeta_T}{\zeta_t} X_T \,\middle|\, \mathcal{F}_t\right] \\ &= P_t^T E^{\mathbb{Q}_T}\left[X_T \mid \mathcal{F}_t\right]. \end{aligned} \tag{4.87}$$

This procedure can be reversed to derive (4.82) from (4.81).

Forward Rate as A Martingale

- **Lemma 4.1.** The forward rate $f(t,T)$ is a $\mathbb{Q}_T$ - martingale, and it satisfies

$$df(t,T) = \boldsymbol{\sigma}^{\mathrm{T}}(t,T)d\hat{\mathbf{W}}_t, \tag{4.88}$$

where $\hat{\mathbf{W}}_t$ is a $\mathbb{Q}_T$ -Brownian motion defined in (4.75).

Proof: Recall the HJM equation (4.16) for the forward rate. We have

$$df(t,T) = \boldsymbol{\sigma}^{\mathrm{T}}(t,T)\left(d\tilde{\mathbf{W}}_t - \boldsymbol{\Sigma}(t,T)dt\right), \tag{4.89}$$

where $\tilde{\mathbf{W}}_t$ is a Brownian motion under the risk-neutral measure, $\mathbb{Q}$. The conclusion then follows. □