4.9 Black's Formula for Call and Put Options

- Under stochastic interest rates, pricing of usual options should be done under proper former measures, using zero-coupon bonds as numeraires.

  The payoff of a call option on an asset, S_t, is

  S_t

$$V_T = \max(S_T - K, 0) \triangleq (S_T - K)^+. \tag{4.90}$$

- In terms of the forward price, $F_t^T = \hat{S}_t / P_t^T$, we also have

$$V_T = (F_T^T - K)^+. \tag{4.91}$$

  Under the T-forward measure, we know that the price is given by

$$V_t = P_t^T E^{\mathbb{Q}_T}\left[\left(F_T^T - K\right)^+ \middle| \mathcal{F}_t\right]. \tag{4.92}$$

Black's Formula for Call Options

- The good news here is that F_t^T is a lognormal martingale under $\mathbb{Q}_T$:

$$dF_t^T = F_t^T \boldsymbol{\Sigma}_F^{\mathrm{T}} d\hat{\mathbf{W}}_t, \tag{4.93}$$

where $\boldsymbol{\Sigma}_F$ is the difference between the volatilities of the asset and the T -maturity zero-coupon bond:

$$\boldsymbol{\Sigma}_F = \boldsymbol{\Sigma}_S - \boldsymbol{\Sigma}(t,T). \tag{4.94}$$

By repeating the procedure to derive the Black-Scholes formula, we obtain

$$V_t = \hat{S}_t N(d_1) - KP_t^T N(d_2), \tag{4.95}$$

Black's Formula for Call Options, cont'd

- where $N(\cdot)$ is the normal accumulative function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-y^2/2\right\} dy, \tag{4.96}$$

and

$$d_1 = \frac{\ln\left(\hat{S}_t \Big/ \left(P_t^T K\right) \right) + \frac{1}{2}\sigma_F^2 (T-t)}{\sigma_F \sqrt{T-t}},$$

$$d_2 = d_1 - \sigma_F \sqrt{T-t},$$

with

$$\sigma_F^2 = \frac{1}{T-t}\int_t^T \left\|\mathbf{\Sigma}_F\right\|^2 ds = \frac{1}{T-t}\int_t^T \left\|\mathbf{\Sigma}_S(s) - \mathbf{\Sigma}(s,T)\right\|^2 ds. \tag{4.97}$$

The Black's Volatility

- The price formula is called Black's formula, in recognition of a similar formula for futures options developed by Black (1976).
- For later reference, we call σ_F the Black's volatility of the option. Note that when the short rate becomes deterministic, $\Sigma(t,T)=0$, Black's formula reduces to the Black-Scholes formula.

Hedging Under Stochastic Interest Rates

- In addition to the option's price, Black's formula also offers a hedging strategy:
- We may say that the option can be replicated by a portfolio consisting of
  - $\varphi_t = N(d_1)$ units of the underlying asset, and
  - $\psi_t = -KN(d_2)$ units of T-maturity zero-coupon bonds.

Black's Formula for Put Options

- The price formula for put options can be derived through the *call-put parity*: for the same strike, K, the prices of a call option, a put option and a forward contract satisfy the relation

$$C(K) - P(K) = \hat{S}_t - P(t,T)K. \tag{4.98}$$

Thus, the formula for a put option follows:

$$\begin{aligned} P(K) &= C(K) - \hat{S}_t + P(t,T)K \\ &= KP_t^T\left(1 - N(d_2)\right) - \hat{S}_t\left(1 - N(d_1)\right) \\ &= KP_t^T N(-d_2) - \hat{S}_t N(-d_1). \end{aligned} \tag{4.99}$$

The hedging strategy is to short $N(-d_1)$ units of the underlying asset.

4.9.1 Equity Options under Stochastic Interest Rates

- To price either a call or a put option by Black's formula, we need to calculate Black's volatility of the forward price, (4.97).
- In applications, asset volatilities are often given in the form of a scalar instead of a vector, and asset correlations are given explicitly.
- In such a situation, the squared of Black's volatility, (4.97), takes a different form.

The Black's Volatility

- Assume that the local volatility of the underlying asset is a constant, σ_S , and the correlation between the asset and the zero-coupon bond is ρ .
- Then, Black's volatility of the forward price is

$$\begin{aligned}
\sigma_F^2 &= \frac{1}{T-t}\int_t^T \left\| \mathbf{\Sigma}_S(u) - \mathbf{\Sigma}(u,T) \right\|^2 du \\
&= \frac{1}{T-t}\int_t^T \left(\left\| \mathbf{\Sigma}_S(u) \right\|^2 - 2\mathbf{\Sigma}_S^T(u)\mathbf{\Sigma}(u,T) + \left\| \mathbf{\Sigma}(u,T) \right\|^2 \right) du \\
&= \frac{1}{T-t}\int_t^T \left(\sigma_S^2 - 2\rho\sigma_S \left\| \mathbf{\Sigma}(u,T) \right\| + \left\| \mathbf{\Sigma}(u,T) \right\|^2 \right) du.
\end{aligned}$$

- Ex: under Hull-White Model, $\Sigma(u,T) = -\frac{\sigma_0}{\kappa}\left[1 - e^{-\kappa(T-u)}\right]$

Pricing Bond Options under the Hull-White Model

- **Example 4.2.** Consider the pricing of a zero-coupon bond option with payoff $V_T = (P(T,\tau) - K)^+$ for $\tau > T$. Let us take $T = 2, \tau = 5$ and $K = P(0,\tau)/P(0,T)$. The initial term structure of interest rates is given by

$$f(0,T) = 0.02 + 0.002T. \qquad (4.102)$$

We will price the option first under the Hull-White model, with a forward rate volatility of $\sigma(t,T) = \sigma_0 e^{-\alpha(T-t)}$, $\sigma_0 = 0.005$, α=0.1. .

The Example, cont'd

- To calculate the strike, we need the price of zero-coupon bonds:

$$P(0,T) = \exp\left\{-\int_0^T f(0,u)du\right\}$$
$$= \exp\left\{-\int_0^T (0.02 + 0.002u)du\right\} \qquad (4.103)$$
$$= \exp\left\{-\left(0.02T + 0.001 \cdot T^2\right)\right\}.$$

With the above function, we obtain

$$P(0,2) = 0.9570,$$
$$P(0,5) = 0.8825, \qquad (4.104)$$

and the value of the strike,

$$K = F_0^2 = P(0,5) \big/ P(0,2) = 0.9222. \qquad (4.105)$$

Pricing Under the Hull-White Model

The volatility of a zero-coupon bond is then

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)du = \frac{\sigma_0}{\alpha}\left[e^{-\alpha(T-t)} - 1\right]. \tag{4.111}$$

It follows that,

$$\begin{aligned}\Sigma(t,\tau) - \Sigma(t,T) &= \frac{\sigma_0}{\alpha}\left[e^{-\alpha(\tau-t)} - e^{-\alpha(T-t)}\right] \\ &= \frac{\sigma_0}{\alpha} e^{\alpha t}\left(e^{-\alpha\tau} - e^{-\alpha T}\right),\end{aligned} \tag{4.112}$$

Pricing Under the Hull-White Model, cont'd

$$\begin{aligned}\sigma_F^2 &= \frac{1}{T}\frac{\sigma_0^2}{\alpha^2}\left(e^{-\alpha\tau} - e^{-\alpha T}\right)^2 \int_0^T e^{2\alpha t}dt \\ &= \frac{1}{T}\frac{\sigma_0^2}{\alpha^2}\left(e^{-\alpha\tau} - e^{-\alpha T}\right)^2 \frac{1}{2\alpha}\left(e^{2\alpha T} - 1\right) \\ &= \frac{1}{2T}\frac{\sigma_0^2}{\alpha^3}\left(e^{-\alpha(\tau - T)} - 1\right)^2\left(1 - e^{-2\alpha T}\right).\end{aligned} \tag{4.113}$$

By inserting Black's volatility into Black's formula, we obtain

$$\begin{aligned}\sigma_F^2 &= 0.00013841, \\ d_1 = -d_2 &= 0.0083,\end{aligned} \tag{4.114}$$

and finally

$$V_0 = 0.0059, \tag{4.115}$$

The Example, cont'd

- For the Ho-Lee model, the volatility of a zero-coupon bond is $\Sigma(t,T) = -\sigma_0(T-t)$. Thus,

$$\Sigma(t,\tau) - \Sigma(t,T) = -\sigma_0(\tau - T)\,. \tag{4.106}$$

Black's volatility is then

$$\begin{aligned}\sigma_F^2 &= \frac{1}{T}\int_0^T \left(\Sigma(t,\tau) - \Sigma(t,T)\right)^2 dt \\ &= (\sigma_0)^2(\tau - T)^2 \\ &= (0.005)^2 \times 9 \\ &= 0.000225.\end{aligned} \tag{4.107}$$

The Example, cont'd

- It follows that

$$\begin{aligned} d_1 &= 0.0106, \\ d_2 &= -d_1 = -0.0106, \end{aligned} \tag{4.108}$$

and the value of the option is

$$\begin{aligned} V_0 &= P(0,5)N(d_1) - P(0,2)KN(d_2) \\ &= P(0,5)\left(1 - 2N(d_2)\right) = 0.0075, \end{aligned} \tag{4.109}$$

or 75 basis points.

4.9.2 Options on Coupon Bonds

- Options on coupon bonds actually belong to the first generation of fixed-income derivatives.
- In section 4.6, we have studied the pricing of coupon bonds using Monte Carlo simulations.
- Here, we introduce an approximate analytical method for pricing options on coupon bonds.
- The payoffs of call options on coupon bonds take the form

$$V_T = \left(\sum_{i=1}^{N} \Delta T c P(T_0, T_i) + P(T_0, T_N) - K \right)^+, \qquad (4.116)$$

where T_0 is the maturity of the option, and $T_i = T_0 + i\Delta T$.

Options on Coupon Bonds, cont'd

- Let $\tilde{B}_t^c$ denote the price of the coupon bond with coupons before T_0 stripped. Then, the T_0- forward price of the coupon bond is

$$
\begin{aligned}
F_t^{T_0} &= \frac{\tilde{B}_t^c}{P(t,T_0)} \\
&= \sum_{i=1}^{N} \Delta T c \frac{P(t,T_i)}{P(t,T_0)} + \frac{P(t,T_N)}{P(t,T_0)} \\
&= \sum_{i=1}^{N} \Delta T c \frac{P(0,T_i)}{P(0,T_0)} M_i(t) + \frac{P(0,T_N)}{P(0,T_0)} M_N(t) \qquad (4.117) \\
&= \frac{\tilde{B}_0^c}{P(0,T_0)} \left(\sum_{i=1}^{N-1} \frac{\Delta T c P(0,T_i)}{\tilde{B}_0^c} M_i(t) + \frac{(1+\Delta T c) P(0,T_N)}{\tilde{B}_0^c} M_N(t) \right) \\
&= F_0^{T_0} \sum_{i=1}^{N} \omega_i M_i(t).
\end{aligned}
$$

Forward Price as A Sum of Martingales

- Here,

$$M_i(t) = \exp\left\{\int_0^t -\tfrac{1}{2}\left\|\mathbf{\Sigma}(s,T_i) - \mathbf{\Sigma}(s,T_0)\right\|^2 ds + \left(\mathbf{\Sigma}(s,T_i) - \mathbf{\Sigma}(s,T_0)\right)^{\mathrm{T}} d\hat{\mathbf{W}}_s\right\} \tag{4.118}$$

is a martingale under the T_0 - forward measure, and

$$\omega_i = \begin{cases} \Delta T c P(0,T_i) / \tilde{B}_0^c, & i < N, \\ (1+\Delta T c) P(0,T_N) / \tilde{B}_0^c, & i = N. \end{cases} \tag{4.119}$$

Lognormal Approximation

- To price the option, we approximate the process of $F_t^{T_0}$ by a geometric Brownian motion through moment matching:

$$\sum_{i=1}^{N} \omega_i M_i(T_0) \approx \exp\left\{-\tfrac{1}{2}\sigma_B^2 T_0 + \sigma_B \sqrt{T_0} \cdot \varepsilon\right\}, \qquad (4.120)$$

where $\varepsilon \sim N(0,1)$ under $\mathbb{Q}_{T_0}$, and

$$\begin{aligned}
\sigma_B^2 &= \frac{1}{T_0} \ln E^{\mathbb{Q}_{T_0}}\left[\left(\sum_{i=1}^{N} \omega_i M_i(T_0)\right)^2\right] \\
&= \frac{1}{T_0} \ln\left(\sum_{i,j} \omega_i \omega_j E^{Q_{T_0}}\left[M_i(T_0) M_j(T_0)\right]\right) \qquad (4.121) \\
&= \frac{1}{T_0} \ln\left(\sum_{i,j} \omega_i \omega_j e^{s_i s_j \rho_{ij} T_0}\right),
\end{aligned}$$

Lognormal Approximation, cont'd

- with

$$s_i^2 = \frac{1}{T_0}\int_0^{T_0} \left\| \boldsymbol{\Sigma}(t,T_i) - \boldsymbol{\Sigma}(t,T_0) \right\|^2 dt,$$

$$\rho_{ij} = \frac{1}{T_0 s_i s_j}\int_0^{T_0} \left(\boldsymbol{\Sigma}(t,T_i) - \boldsymbol{\Sigma}(t,T_0)\right)^{\mathrm{T}} \left(\boldsymbol{\Sigma}(t,T_j) - \boldsymbol{\Sigma}(t,T_0)\right) dt. \tag{4.122}$$

Note that when $\left| s_i s_j \rho_{ij} T_0 \right| \ll 1$ for all i and j, we can approximate σ_B as

$$\sigma_B \approx \sqrt{\sum_{ij} \omega_i \omega_j s_i s_j \rho_{ij}}\,. \tag{4.123}$$

Black's Formula Again

$$
\begin{aligned}
V_0 &= P(0,T_0)E^{\mathbb{Q}_{T_0}}\left[\left(F_{T_0}^{T}-K\right)^+ \middle| \mathcal{F}_0\right] \\
&\approx P(0,T_0)E^{\mathbb{Q}_{T_0}}\left[\left(F_0^{T_0}e^{-\frac{1}{2}\sigma_B^2 T_0+\sigma_B\sqrt{T_0}\cdot\varepsilon}-K\right)^+ \middle| \mathcal{F}_0\right] \\
&= P(0,T_0)\left[F_0^{T_0}N(d_1)-KN(d_2)\right] \\
&= B_0^c N(d_1)-KP(0,T_0)N(d_2),
\end{aligned}
\tag{4.124}
$$

where

$$
d_1=\frac{\ln\left(F_0^{T_0}/K\right)+\frac{1}{2}\sigma_B^2 T_0}{\sigma_B\sqrt{T_0}}=\frac{\ln\left(B_0^c/(KP(0,T_0))\right)+\frac{1}{2}\sigma_B^2 T_0}{\sigma_B\sqrt{T_0}},
$$

$$
d_2=d_1-\sigma_B\sqrt{T_0}.
$$

A Common Practice

- Although appearing rough, the above approximation often works quite well in the marketplace.
- In fact, it has become a common practice to approximate an addition of lognormal random variables by a single lognormal random variable, as such an approximation seems to be rather accurate when the original random variables are reasonably correlated.

An Example Revisited

- **Example 4.3.** To demonstrate the accuracy of Black's formula for options on coupon bonds, we re-price the options introduced in Section 4.6, and compare the results with those from Monte Carlo simulations. The results are listed in Table 4.5. One can see that the prices from Black's formula and from Monte Carlo simulations are fairly close. In fact, the root mean squared difference for all price pairs is very close to 1%. Such closeness really does support very positively the lognormal approximation of the price distribution of coupon bonds.

Repricing Swaptions of Section 4.6

- **Table 4.5.** Option prices by Black's formula and by the Monte Carlo simulation method.

Par bond option		Ho-Lee model		2-factor model	
Maturity, T_0	Tenor $T_n - T_0$	Black	MC	Black	MC
1	0.25	8.12	8.23	7.67	7.85
2	0.25	11.01	11.07	10.31	10.34
5	0.25	15.30	15.33	14.77	14.69
1	5	147.84	145.43	139.40	140.41
2	5	200.20	200.29	190.95	191.07
5	5	276.62	275.08	271.28	272.14
1	10	264.69	262.88	253.35	254.99
2	10	357.57	364.54	345.94	349.50
5	10	492.64	489.43	486.42	487.53

4.10 Numeraires and Changes of Measure

- A major achievement so far in this chapter is to take zero-coupon bonds as numeraires and price options under their corresponding forward measures.
- Mathematically, this is merely a technique of changing the numeraire asset, followed by taking the expectation of the option payoffs under the martingale measures of the numeraire assets.
- In this section, we discuss this technique in a general context.

Numeraires and Measures

- Let $\mathbb{Q}_A$ be the martingale measure associated with reference asset A_t, meaning that, for any traded asset V_t, its price relative to that of asset A_t,

$$\frac{V_t}{A_t}, \tag{4.125}$$

is a $\mathbb{Q}_A$-martingale. Consider another asset, B_t, and its associated martingale measure, $\mathbb{Q}_B$. According to the one-price principle, the value of any traded asset at time t, V_t, satisfies

$$V_t = A_t E_t^{\mathbb{Q}_A}\left[A_T^{-1} V_T\right] = B_t E_t^{\mathbb{Q}_B}\left[B_T^{-1} V_T\right]. \tag{4.126}$$

Finding the Radon-Nikodym Derivative

- From the above equation, we obtain

$$E_t^{\mathbb{Q}_B}\left[B_T^{-1}V_T\right]=\frac{A_t}{B_t}E_t^{\mathbb{Q}_A}\left[\frac{B_T}{A_T}\left(B_T^{-1}V_T\right)\right]. \tag{4.127}$$

Let ζ be the Radon-Nikodym derivative between $\mathbb{Q}_B$ and $\mathbb{Q}_A$:

$$\left.\frac{d\mathbb{Q}_B}{d\mathbb{Q}_A}\right|_{\mathcal{F}_t}=\zeta_t. \tag{4.128}$$

Then,

$$E_t^{\mathbb{Q}_B}\left[B_T^{-1}V_T\right]=\zeta_t^{-1}E_t^{\mathbb{Q}_A}\left[\zeta_T(B_T^{-1}V_T)\right]. \tag{4.129}$$

Finding the Radon-Nikodym Derivative, cont'd

- Subtracting (4.127) from (4.129), we obtain

$$0 = E_t^{\mathbb{Q}_A}\left[\left(\frac{\zeta_T}{\zeta_t} - \frac{B_T}{A_T}\Big/\frac{B_t}{A_t}\right)(B_T^{-1}V_T)\right]. \tag{4.130}$$

Note that (4.130) holds for prices of any tradable assets, so we can argue that

$$\zeta_t = \left.\frac{d\mathbb{Q}_B}{d\mathbb{Q}_A}\right|_{\mathcal{F}_t} = \frac{B_t}{A_t}\Big/\frac{B_0}{A_0} \quad \text{a.s.} \tag{4.131}$$

From the Risk-Neutral to Forward Measure

- **Example 4.4.** We have already established the following correspondence between a numeraire and its martingale measure:

$$B_t = \exp\left(\int_0^t r_s ds\right) \leftrightarrow \mathbb{Q},$$
$$P_t^T = P_0^T \exp\left\{\int_0^t (r_s - \tfrac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma})ds + \boldsymbol{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_s\right\} \leftrightarrow \mathbb{Q}_T. \tag{4.132}$$

According to the general formula (4.132), the Radon-Nikodym derivative of $\mathbb{Q}_T$ with respect to $\mathbb{Q}$ is

$$\left.\frac{d\mathbb{Q}_T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{P_t^T}{P_0^T} \bigg/ \frac{B_t}{B_0}$$
$$= \exp\left\{\int_0^t -\tfrac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma} ds + \boldsymbol{\Sigma}^{\mathrm{T}} d\tilde{\mathbf{W}}_s\right\}, \tag{4.133}$$

which is exactly (4.74). □

The Focuses of This Chapter

- Derivatives pricing under the HJM model is another focus of this chapter.
  - As a Markovian model in forward rates, the HJM model can be conveniently implemented through Monte Carlo simulation methods.
- We highlight the use of forward measures, a powerful device for pricing interest-rate derivatives.
  - Pricing under an appropriate forward measure can make both pricing and hedging transparent.
  - Pricing measures are not unique.
  - It is up to us to choose numeraire assets and their corresponding martingale measures.