Chapter 3
Interest Rates and the Associated Securities

Rates and Instruments

- The term “interest rate” encompasses an entire class of *rates of return* that are associated with various fixed-income instruments.
- The term most frequently referred to
  - short rate for saving account,
  - rates for money market accounts,
  - U.S. Treasury yields,
  - zero-coupon yields,
  - forward rates and
  - swap rates.
- We will introduce interest rates with associated securities

3.1 Interest Rates and Fixed-Income Instruments

3.1.1 Short Rate and Money Market Accounts

- The short rate is associated with a savings account in a bank.
- The short rate at time t is conventionally denoted as r_t.
- Interest on a savings account is accrued daily, using the actual/365 convention.
- Let B_t denote the account balance at time (or date) t, and let $\Delta t = 1$ day $= 1/365$ year. Then, the new balance the next day at $t + \Delta t$ is

$$B_{t+\Delta t} = B_t \left(1 + r_t \Delta t\right). \tag{3.1}$$

Continuous Compounding

- Because $\Delta t \ll 1$, daily compounding is very well approximated by *continuous compounding*: in the limit of $\Delta t \to 0$, equation (3.1) becomes

$$dB_t = r_t B_t dt. \qquad (3.2)$$

- Because r_t is applied to $(t, t+dt)$, an infinitesimal interval of time, it is also called the instantaneous interest rate.
- Solve the equation we obtain the balance at a later time, t:

$$B_t = B_0 e^{\int_0^t r_s ds}. \qquad (3.3)$$

The risk-free security

- In the real world, B_t is not known in advance due to the stochastic nature of the short rate.
- Nonetheless, the deposit in the savings account is considered a risk-free security, and its return is used as a benchmark to measure the profits and losses of other investments.

3.1.2 Term Rates and Certificates of Deposit

- *Term rates* are associated to certificates of deposit (CD). A CD is a deposit that is committed for a fixed period of time, and the interest rate applied to the CD is called a term rate. For retail customers, the available terms are typically
  - 1 month,
  - 3 months,
  - 6 months and
  - 1 year.
- Usually, the longer the term, the higher the term rate, as investors are awarded a higher premium for committing their money for a longer period of time.

LIBOR Rates

- LIBOR is a set of reference interest rates at which banks lend unsecured loans to other banks in the London wholesale money market.
- The LIBOR rates are benchmark rates for certificates of deposit (CDs).

LIBOR Rates

- As of 03-02-2020,

EUR | USD | GBP | JPY | CHF

USD	03-02-2020	02-28-2020	02-27-2020	02-26-2020	02-25-2020
USD LIBOR - overnight	1.57463 %	1.56775 %	1.57400 %	1.57150 %	1.56900 %
USD LIBOR - 1 week	1.56125 %	1.56800 %	1.58500 %	1.58250 %	1.57925 %
USD LIBOR - 2 weeks	-	-	-	-	-
USD LIBOR - 1 month	1.35575 %	1.51525 %	1.58113 %	1.60338 %	1.61263 %
USD LIBOR - 2 months	1.30488 %	1.50263 %	1.59738 %	1.61900 %	1.63563 %
USD LIBOR - 3 months	1.25375 %	1.46275 %	1.58038 %	1.61325 %	1.63763 %
USD LIBOR - 4 months	-	-	-	-	-
USD LIBOR - 5 months	-	-	-	-	-
USD LIBOR - 6 months	1.19838 %	1.39725 %	1.53325 %	1.59025 %	1.62863 %
USD LIBOR - 7 months	-	-	-	-	-
USD LIBOR - 8 months	-	-	-	-	-
USD LIBOR - 9 months	-	-	-	-	-
USD LIBOR - 10 months	-	-	-	-	-
USD LIBOR - 11 months	-	-	-	-	-
USD LIBOR - 12 months	1.15388 %	1.38150 %	1.53725 %	1.61013 %	1.64575 %

Source: www.global-rates.com. The day-count convention for USD and Euro is actual/360.

Simple Compounding for CDs

- The interest payments of CDs use simple compounding.
- Let $r_{t,\Delta t}$ be the interest rate for the term Δt and I_t be the value of the deposit at time t. Then, the balance at the maturity of the CD is

$$I_{t+\Delta t} = I_t\left(1 + r_{t,\Delta t}\Delta t\right). \qquad (3.1)$$

- Suppose that a CD is rolled over n times. Then, the terminal balance at time $t+n\Delta t$ is

$$I_{t+n\Delta t} = I_t \cdot \prod_{i=1}^{n}\left(1 + r_{t+(i-1)\Delta t,\Delta t}\Delta t\right). \qquad (3.2)$$

Compounding Frequency

- We call $\omega = 1/\Delta t$ the compounding frequency, which is the number of compoundings per year. For example, when $\Delta t = 3$ months or 0.25 year, we have $\omega = 1/\Delta t = 4$, corresponding to the so-called quarterly compounding. By the way, a savings account is compound daily, corresponding to $\omega = 365$.

Effective Annual Yields (EAY)

- Different term rates mean different rates of return. One way to compare CDs of different terms is to check their effective annual yields (EAY), defined as the dollar-value return over a year for a \$1 initial investment:

$$\text{effective annual yield} = \left(1 + r_{t,\Delta t}\Delta t\right)^{1/\Delta t} - 1. \tag{3.1}$$

- Should interest rates stay constant over the investment horizon, then a higher EAY gives a higher return in value.

Money market rates

- Money market account accrues according to actual/360 convention. If a deposit is made for n days, then

$$1 \quad \rightarrow \quad 1 + \frac{r_1 + r_1 + \cdots r_n}{360}$$

- This convention applies to calculate interest payment for repurchasing agreements (repo)

3.1.3 Bonds and Bond Markets

- A bond is a financial contract that promises to pay a stream of cash flows over a certain time horizon.

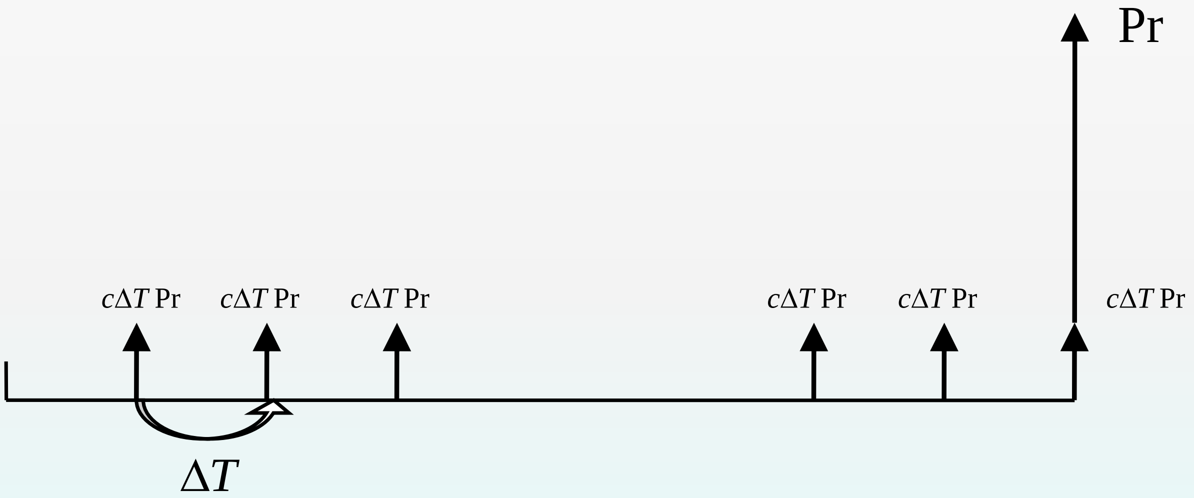


Fig. 3.1. Cash flows of a coupon bond

- We call bonds with such a cash-flow pattern bullet bonds or straight bonds.

Starting with Coupon Bonds

- Three aspects: In May 2010 the U.S. Treasury sold a bond with
  - a coupon rate of $2\,{}^{1}/_{8}\%$ and
  - a maturity date of May 31, 2015
  - a payment frequency of two a year, six months apart
- This bond is called “$2\,{}^{1}/_{8}$s of May 31, 2015”

Coupon rate

Coupon frequency, “s” is for “semi-annual”

maturity

Cash Flow of the Bond

- The unit for bond purchasing is \$1,000.
- Suppose that an investor purchases \$1m face value of the bond, i.e., 1,000 units.
- The the coupon payment is calculated according to

$$\frac{1}{2} \times 2\frac{1}{8}\% \times \$1{,}000{,}000 = \$10{,}625$$

Year fraction

Coupon rate

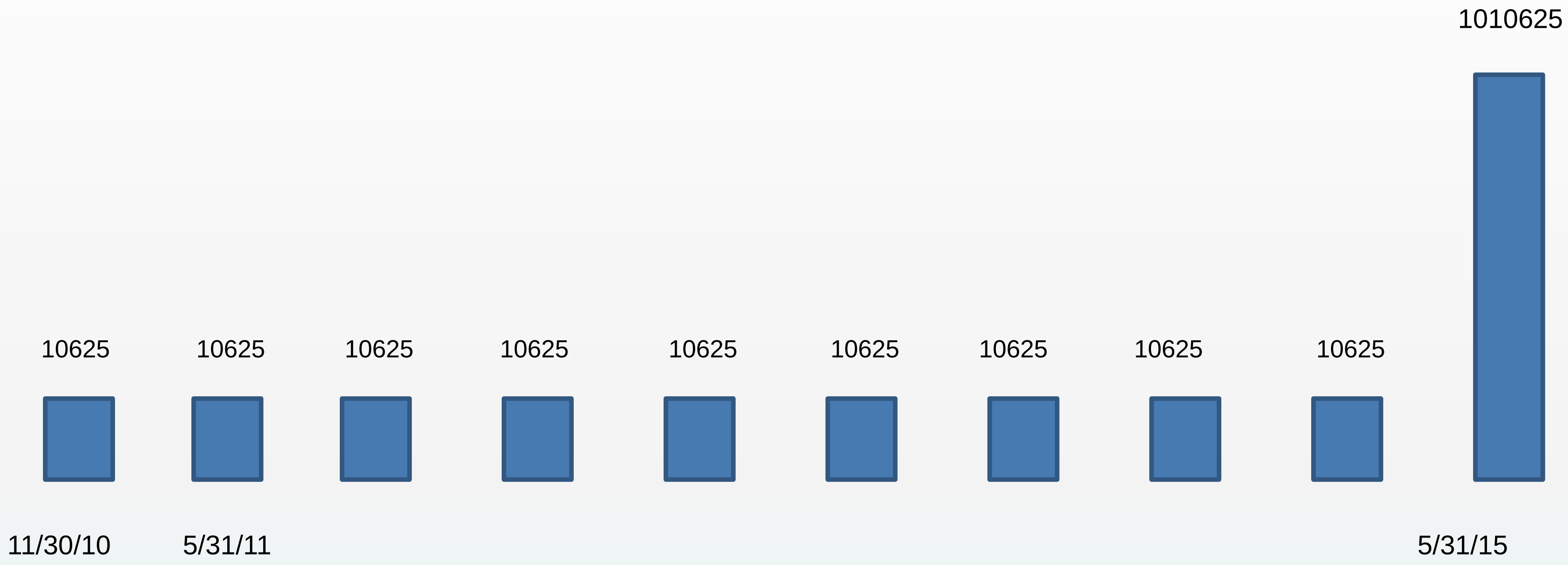
Face value

Cash Flow of the Bond, cont'd

Table 1.1: Cash Flows of the U.S. $2^1/_8$ from books of May 31, 2015

Date	Coupon Payment	Principal Payment
"11/30/2010"	\$10, 625	
"5/31/2011"	\$10, 625	
"11/30/2011"	\$10, 625	
"5/31/2012"	\$10, 625	
"11/30/2012"	\$10, 625	
"5/31/2013"	\$10, 625	
"11/30/2013"	\$10, 625	
"5/31/2014"	\$10, 625	
"11/30/2014"	\$10, 625	
"5/31/2015"	\$10, 625	\$1, 000, 000

Cash-flow Chart



Government Bonds

- US Treasury
- Exchange Fund Bills & Notes Fixings (Hong Kong)

Type of Bonds

- Government or non-government
- Domestic or foreign
- Domestic currency or foreign currency

U.S. Bond Market

- The U.S. domestic bond markets mainly consist of three sectors:
  1. **Government bonds**: bonds issued by the Treasury Department of the U.S. government and local governments;
  2. **Agency bonds:** bonds issued by certain agencies of the U.S. government and guaranteed or sponsored by the U.S. government;
  3. **Corporate bonds**: bonds issued by U.S. corporations.
- The U.S. Treasury Department is the largest single issuer of debt in the world.

Treasury Securities

- Treasury securities include
  - Treasury bills,
  - Treasury notes,
  - Treasury bonds and
  - Treasury Inflation-Protected Securities (TIPS).

Maturities of the Treasury Issues

- Upon issuance, the maturities of
  - Treasury bills: four weeks (one month), to 13 weeks (three months) and 26 weeks (six months).
  - Treasury notes: 2, 3, 5 and 10 years
  - Treasury bonds: 30 years
- TIPS are issued with maturities
  - five years, 10 years and 20 years.
  - Thirty-year TIPS have also been issued occasionally.
- TIPS are issued with fixed coupon rates. The principal of a TIPS is adjusted semiannually according to the inflation rate.

Cycle of Auction

- Four-week bills: every Tuesday,
- 13-week and 26-week bills: every Monday.
- 2-year and 5-year notes: the end of each month,
- 3-year and 10-year notes on the fifteenth of February, May, August and November.
- The 30-year bonds are auctioned in February and August.

Cycle of Auction, cont'd

- TIPS are auctioned in different cycles.
  - 5-year TIPS are generally auctioned in the last week of April;
  - 10-year TIPS are generally auctioned in the second week of January and July; and
  - 20-year TIPS are generally auctioned in the last week of January.
- The 10-year bonds have dominated the liquidity at issuance.

Importance of Treasury Market

- The U.S. Treasury Department is the largest single issuer of debt in the world. Bonds in this sector are deemed riskless as they are guaranteed by the full faith of the U.S. government. This sector is distinguished by the large volume of total debt and the large size of any single issue.
- In terms of notional value, the Treasury bond sector is not the largest Treasuries sector, and it trails behind the agency bond sector (The Federal National Mortgage Association (Fannie Mac) and The Federal Home Loan Mortgage Corporation (Freddy Mac)).

3.1.4 Quotation and Interest Accrual

- The price of zero-coupon bonds and the price of coupon bonds are quoted using different conventions.
  - The former is quoted using a discount yield, which translates conveniently into a dollar price.
  - The latter is quoted as a percentage of the principal value, using a price tick of one-32nd of a percentage point.

Table 3.1. Quotes for U.S. Treasuries as of 3/7/2008

U.S. Treasuries

Bills

	Maturity date	Discount/Yield	Discount/Yield change
3-Month	06/05/2008	1.42 / 1.44	-0.01 / .087
6-Month	09/04/2008	1.51 / 1.54	0.03 / -.031

Notes/Bonds

	Coupon	Maturity date	Current price/Yield	Price/Yield change
2-Year	2	02/28/2010	100-29¾ / 1.52	-0-00¾ / .012
3-Year	4.75	03/31/2011	103-21¾ /1.42	-0-02 / .018
5-Year	2.75	02/28/2013	~~101-16 / 2.43~~	0-06 / -.040
10-Year	3.5	02/15/2018	99-23+ / 3.53	0-14 / -.053
30-Year	4.375	02/15/2038	97-08½ / 4.54	0-08+ / -.017

Source: http://www.bloomberg.com/markets/rates/index.html

- The dollar value of a Treasury bill is calculated using the discount yield according to the formula

$$V = \Pr\left(1 - \frac{\tau}{360} Y_d\right), \tag{3.1}$$

where τ is the number of days remaining to maturity. Suppose, for instance, that the six-month Treasury bill has a time to maturity of $\tau = 100$ days. Then, its price is

$$P = 100 \times \left(1 - \frac{100}{360} \times 1.51\%\right) = \$99.5806.$$

- Note that the discount yield is a quoting mechanism rather than a good measure of returns on an investment in a Treasury bill.

From Quotation to Prices

- A "+" stands for 0.5, or half a tick. The dollar values of the five Treasury notes and bonds are calculated as follows:
- Examples.

$$
\begin{aligned}
100\text{-}29\tfrac{3}{4} &= 100 + 29.75/32 = 100.9297, \\
103 - 21\tfrac{1}{4} &= 103 + 21.25/32 = 103.6641, \\
101\text{-}16 &= 101 + 16/32 = 101.5, \\
99\text{-}23{+} &= 99 + 23.5/32 = 99.7344, \\
97\text{-}08\tfrac{1}{2} &= 97 + 8.5/32 = 97.2656.
\end{aligned}
\tag{3.8}
$$

Clean Price and Full Price

- The above dollar prices are not the prices for transactions, however. For transactions, we add the interest values accrued since the last coupon payment, which are calculated as follows.

Clean Price and Full Price

Consider a coupon bond with coupon rate c and coupon dates $\{T_i\}$. At t, a moment between two consecutive coupon dates, T_j and T_{j+1} (e.g., $T_j < t \leq T_{j+1}$), the accrued interest is

$$AI(t, T_j) = \Delta T \cdot c \cdot \Pr \cdot q,$$

where

$$q = \frac{t - T_j}{T_{j+1} - T_j}. \tag{3.1}$$

i.e., the fraction of time elapsed since the last coupon date over the current coupon period, then

$$\text{Transaction price} = \text{quote price} + AI(t, T_j).$$

An Example

Example 3.1: Consider a 10-year Treasury note maturing on February 15, 2018. On March 7, 2008, the bond quote is 99-23+ ($=99.7344$). We need to compute the accrued interest and then the dirty price.

The coupon dates are February 15 and August 15. There are 182 days between the coupon dates, and on March 7, twenty-one days have elapsed since February 15, the last coupon date. Hence,

$$q = 21/182 = 0.115385,$$

$$AI = 0.5 \times 3.5\% \times 100 \times 0.115385 = 0.2019.$$

Then, the full price is

$$B^c = 99.7344 + 0.2019 = 99.9363. \quad \square \tag{3.10}$$

3.2.1 Yield to Maturity

Denote the full price of a bullet bond as B^c. Suppose that all cash flows are discounted by a uniform rate, y, then y satisfy the following equation

$$B^c = \Pr\cdot\left(\sum_{i=1}^{n}\frac{c\Delta T}{(1+y\Delta T)^i}+\frac{1}{(1+y\Delta T)^n}\right), \qquad (3.1)$$

where n is the number of coupons.

This discount rate, which can be easily solved by a trial-and-error procedure using (3.1), is defined to be the *yield to maturity* (YTM), as well as the *internal rate of return* (IRR) of the bond, and it is often simply called the *bond yield*.

3.2.1 Yield to Maturity

As the function of the yield, the formula for a general time, $t \le T$, is

$$B_t^c = \Pr\cdot\left(\sum_{i;i\Delta T>t}^{n}\frac{c\Delta T}{\left(1+y\Delta T\right)^{(i\Delta T-t)/\Delta T}}+\frac{1}{\left(1+y\Delta T\right)^{(n\Delta T-t)/\Delta T}}\right). \qquad (3.1)$$

Assuming that $t\in(T_j,T_{j+1}]$, and introducing

$$q=\frac{t-T_j}{T_{j+1}-T_j}=\frac{t-T_j}{\Delta T},$$

we then can write

$$t=T_j+\Delta Tq=(j+q)\Delta T, \text{ and } i\Delta T-t=(i-j-q)\Delta T, \ \forall i.$$

Bond Price Formula

It follows that

$$
\begin{aligned}
B_t^c &= \Pr\cdot\left(\sum_{i=j+1}^{n}\frac{c\Delta T}{\left(1+y\Delta T\right)^{i-j-q}}+\frac{1}{\left(1+y\Delta T\right)^{n-j-q}}\right) \\
&= \Pr\cdot\left(1+y\Delta T\right)^{q}\left(\sum_{i=1}^{n-j}\frac{c\Delta T}{\left(1+y\Delta T\right)^{i}}+\frac{1}{\left(1+y\Delta T\right)^{n-j}}\right).
\end{aligned}
\qquad (3.1)
$$

Given the bond price at any time, t, the bond yield is implied by (3.1)

One-to-One Price–Yield Relationship

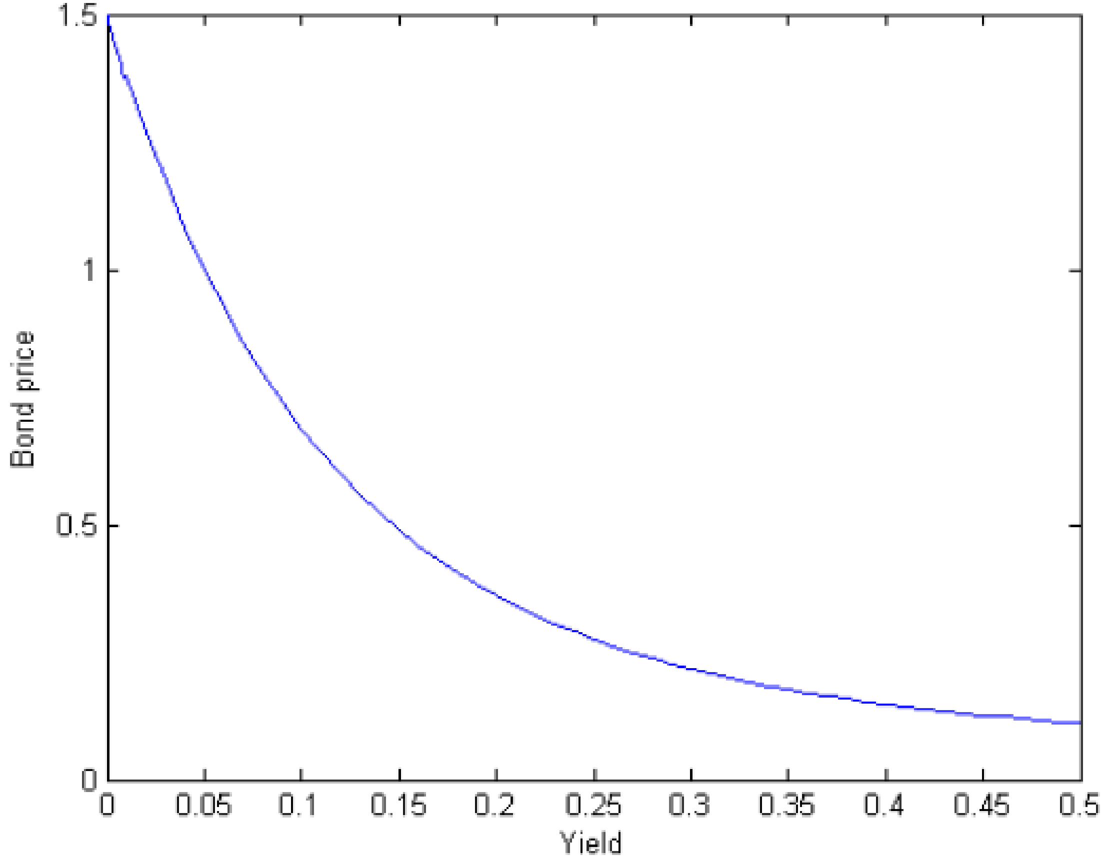


Fig. 3.2. The price–yield relationship for a 10-year bond with $c = 0.05$, $\Delta T = 0.5$

Bond Equivalent Yields

The price–yield relationship of a zero-coupon bond simplifies to

$$P = \Pr \times \left(1 + y\Delta T\right)^{-\frac{T-t}{\Delta T}}.$$

To derive an approximate value of the yield, we consider the following approximation of the "return on the investment",

$$\frac{\Pr - P}{P} = \left(1 + y\Delta T\right)^{\frac{T-t}{\Delta T}} - 1 \approx y \times (T - t), \qquad (3.1)$$

using the Taylor expansion. Equation (3.1) gives rise to an approximate yield-to-maturity:

$$y \approx \frac{1}{(T-t)} \frac{\Pr - P}{P},$$

which is also called the bond equivalent yield.

Bond Equivalent Yields, cont'd

Note that for Treasury zero-coupon bonds, the year has 365 days, meaning that

$$T - t \approx \frac{\tau}{365}, \tag{3.1}$$

where τ is the number of days to maturity.

3.2.2 Par Bonds, Par Yields and the Par Yield Curve

The summation in equation (3.12) can be worked out so that

$$\begin{aligned} B^c &= \Delta T \cdot c \cdot \Pr \sum_{i=1}^{n} (1+y\Delta T)^{-i} + \Pr (1+y\Delta T)^{-n} \\ &= \Pr \left[1 - \left(1-\frac{c}{y}\right)\left(1-\frac{1}{(1+y\Delta T)^n}\right)\right]. \end{aligned} \qquad (3.17)$$

From the above expression, we can tell when the price is smaller, equal to or larger than the principal value.

1. When $c < y$, $B^c < \Pr$. In such a case, we say that the bond is sold at discount (of the par value).

2. When the coupon rate is $c = y$, then $B^c = \Pr$, i.e., the bond price equals the par value of the bond. In such a case, we call the bond a *par bond,* and the corresponding coupon rate a *par yield*.

3. When $c > y$, $B^c > \Pr$. In such a case, we call the bond a premium bond (it is traded at a premium to par).

3.2.3 On-the-run U.S. Treasury Securities

- In the U.S. Treasury market, newly issued bills and notes/bonds are called on-the-run Treasury securities.
- Traditionally, the on-the-run issues enjoy higher liquidity and are thus treated as benchmarks.
- Table 3.1 provides the closing price quotes of the on-the-run issues for March 7, 2008.
- As can be seen in the table, the on-the-run issues have maturities of three months, six months, two years, three years, five years, 10 years and 30 years.

Yield Curves

- A Yield Curve is obtained by connecting the YTMs of the on-the-run Treasury securities
- A yield curve provides a rough idea of the level of yields for various maturities.

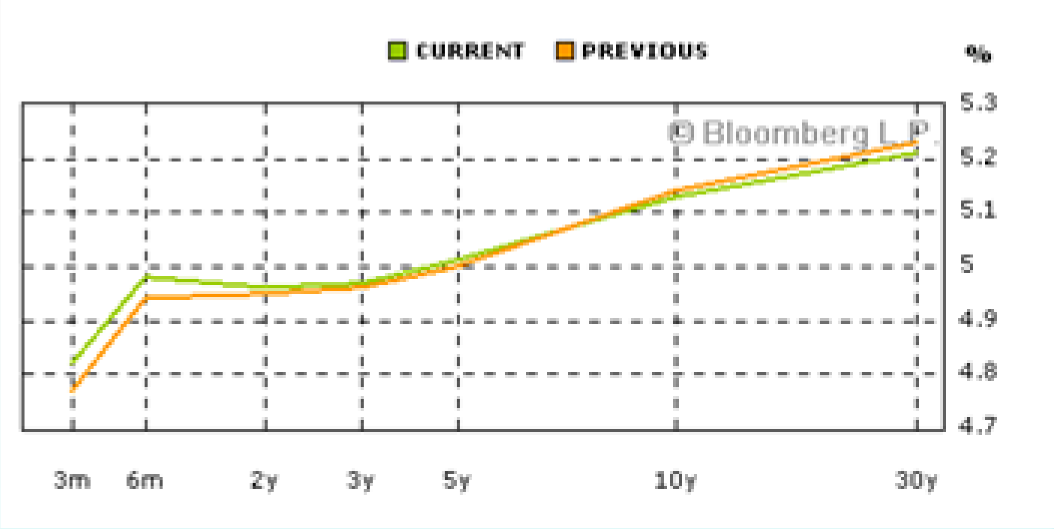


Fig. 3.3. The U.S. Treasury yield curve for April 28, 2006 (orange) and May 1, 2006 (green)

Discount Factors and Discount Curve

- The **discount factor**, *d(t,T)*, with $t \le T$, is the time *t* value of $1 cash flow to occur at a later time *T.*
- In other words, *d(t,T)* is the factor to present value a future cash flow.
- The collection of discount factors is called a discount curve.
- With the discount curve one can price the upcoming deterministic cash flows of any securities.
- Moreover, with the discount curve one can mark to market any bond portfolios.

Future and Present Values

Due to the (mostly) positive interest rates, the notional value of I_t today will become I_{t+T} $(>I_t)$ T years later. In terms of discount factors, we can define the present value of a future cash flow, I_{t+T}, to be

$$I_t = I_{t+T}d(t,t+T). \tag{3.1}$$

Conversely, we can rewrite (3.1) as

$$I_{t+T} = \frac{I_t}{d(t,t+T)}, \tag{3.2}$$

and call I_{t+T} the future value of I_t.

Discount Factors and Zero-coupon Bonds

- A zero-coupon bond is a special case of coupon bonds, with coupon rate $c = 0$.
- The time-*t* price of a zero-coupon bond maturing at time into a par value of one dollar is denoted as $P(t,T)$ or P_t^T.
- When the face value of the zero-coupon is $1, then

$$P(t,T) = d(t,T).$$

3.3 Zero-Coupon Yields

- The yield of a zero-coupon bond is called a zero-coupon yield.
- In terms of its yield for compounding frequency $\omega = 1/\Delta t$, the price of the zero-coupon bond for \$1 notional is

$$P_t^T = \frac{1}{\left(1 + y\Delta t\right)^{(T-t)/\Delta t}}, \tag{3.18}$$

where the time to maturity, $T - t$, does not have to be a multiple of Δt.

- We will call $\left\{P_t^T\right\}_{T \geq t}$ the discount curve.

Continuous Compounding

In continuous time finance, it is often favorable to work with continuous compounding, i.e., by letting the term $\Delta t \to 0$ in (3.18). At this limit, we have

$$P_t^T = e^{-y\times(T-t)}.$$

Given P_t^T, the corresponding zero-coupon yield can be calculated from the last equation:

$$y_{T-t} = -\frac{1}{T-t}\ln P_t^T.$$

Coupon Bonds as Portfolios

- We can express the price of the coupon bond in terms of those of zero-coupon bonds:

$$B^c(0) = \sum_{i=1}^{n} c \cdot \Delta T \, \mathrm{Pr} \cdot P_0^{i\Delta T} + \mathrm{Pr} \cdot P_0^{n\Delta T}. \tag{3.19}$$

- Any portfolio of coupon bonds can be treated as a portfolio of zero-coupon bonds.
- With the discount curve, one can price any bond portfolio with deterministic cash flows.

How Do We Obtain the Discount Factors?

- The Treasury zero-coupon bonds are actually traded in the U.S.
- But their liquidity is not high enough compared with that of coupon bonds.
- There is also noise in the trade prices (time differences, bid-ask spreads, liquidity issues and etc.)
- In addition, there are two categories of zero-coupon bonds, with different prices for the same maturities.
- So the zero-coupon prices cannot be taken literally as the discount factors.

Extracting the Discount Curves

- The market practice is to use a single discount curve for all cash flows.
- The information of the discount curve is contained in the price/yields of the on-the-run issues.
- We need technology as well as assumptions to extract the information.

3.3.2 Bootstrapping the Zero-Coupon Yields

- To determine a discount curve from the yields of several on-the-run Treasury issues is a grossly under-determined problem.
- One must parameterize the zero-coupon yield curve (to fill the gap of information).
- The simplest parameterization that is financially acceptable is to assume piece-wise constant functional forms for the zero-coupon yield curve.
- Under such a parameterization, the zero-coupon yield curve can be derived sequentially.
- Such a procedure is often called bootstrapping in finance.

Bootstrapping

Let $\left\{B_i^c, T_i\right\}_{i=1}^{7}$ be the prices and maturities of the seven on-the-run issues. We assume that the zero-coupon yield for maturities between $\left(T_{i-1}, T_i\right]$ is a constant, y_i, with $i = 1\ldots7$ and $T_0 = 0$. The determination of the YTMs is done sequentially. Because the first two on-the-run issues are zero-coupon bonds, we first back out y_1 and y_2, the zero yields for $\left(0, T_1\right]$ and $\left(T_1, T_2\right]$, using formula (3.18). This will require a root finding procedure. Once y_2 is found, we proceed to determining y_3 from the following equation

$$B_3^c = \sum_{i\Delta T \le T_2} \frac{c_3 \Delta T}{\left(1 + y_2 \Delta T\right)^i} + \sum_{i\Delta T > T_2} \frac{c_3 \Delta T}{\left(1 + y_3 \Delta T\right)^i} + \frac{1}{\left(1 + y_3 \Delta T\right)^{T_3/\Delta T}}. \quad (3.20)$$

Bootstrapping

Here, we have used y_2 as the discount rate for all cash flows between $\left(T_0, T_2\right]$, and y_3 is the only unknown for the equation that again can be determined through a root-finding procedure. This procedure can continue all the way to $i=7$. The entire zero-coupon yield curve for maturity $T \leq 30$ so-determined is displayed in Figure 3.4.

Bootstrapping – Constant Interpolation

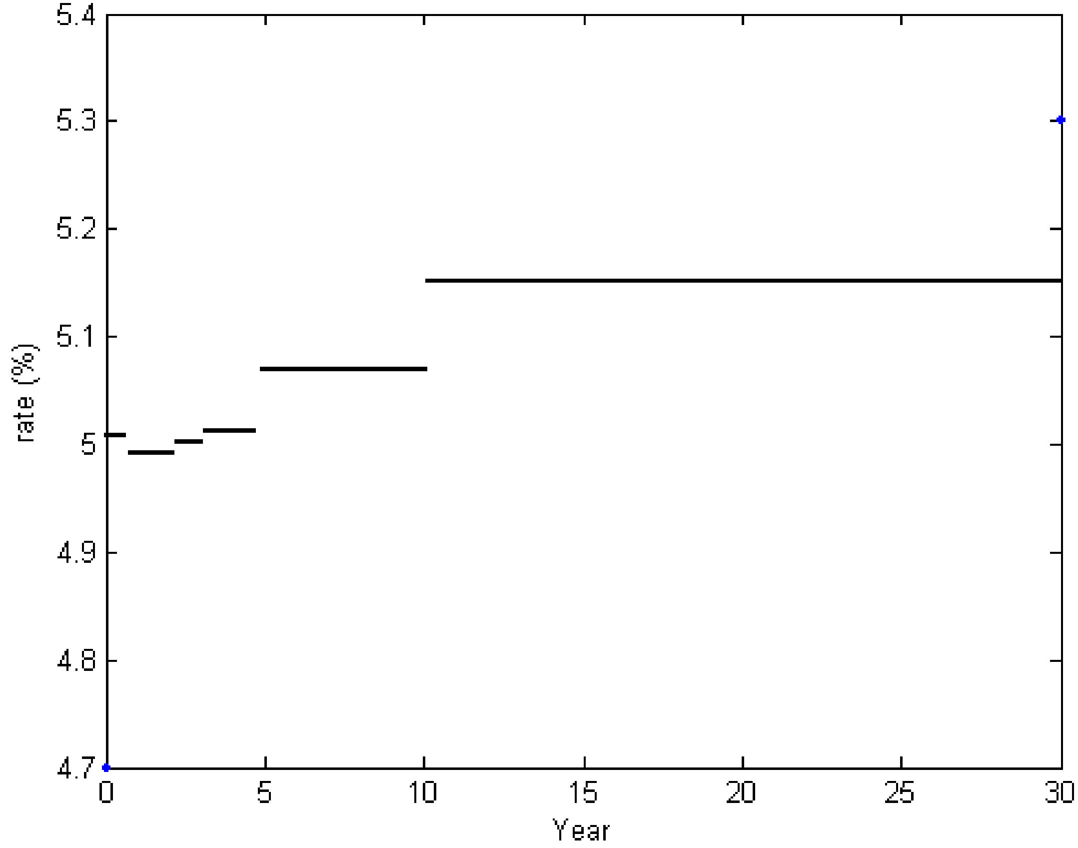


Fig. 3.4. The zero-coupon yield curve of U.S. Treasuries on May 1, 2006.

Bootstrapping – Linear Extrapolation

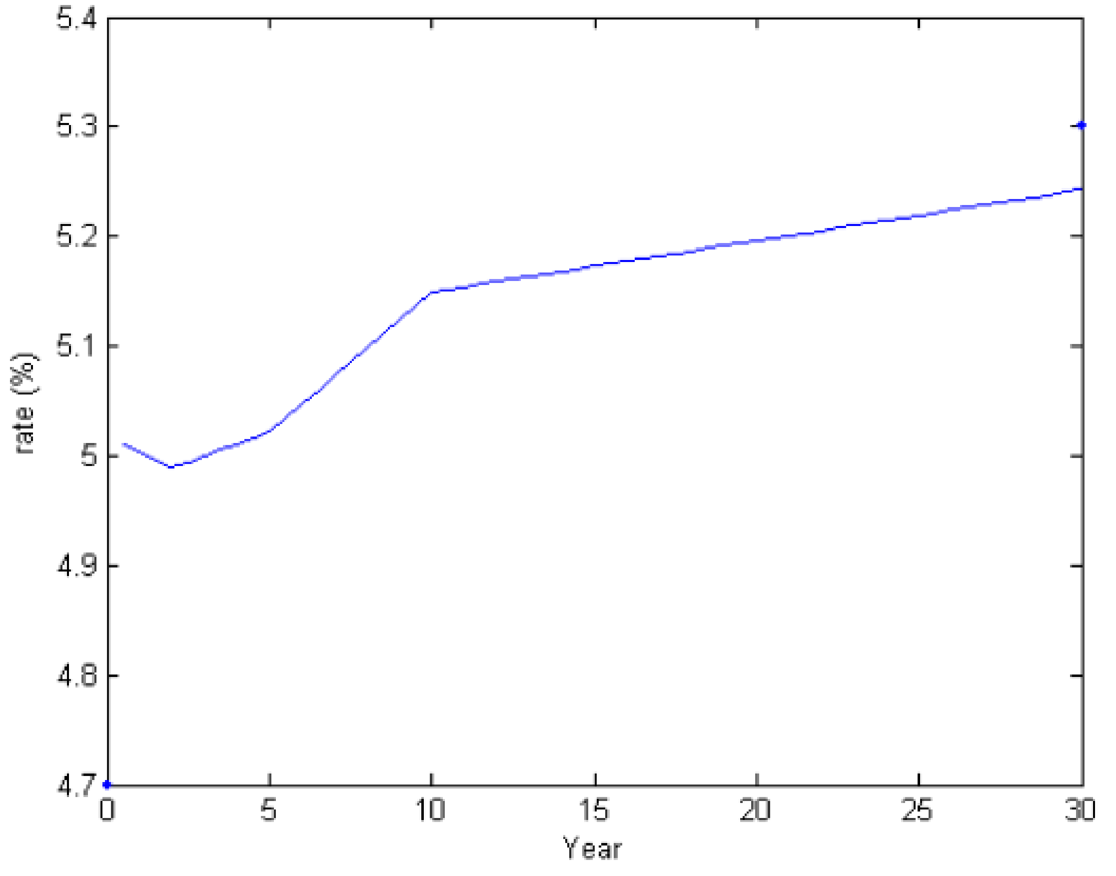
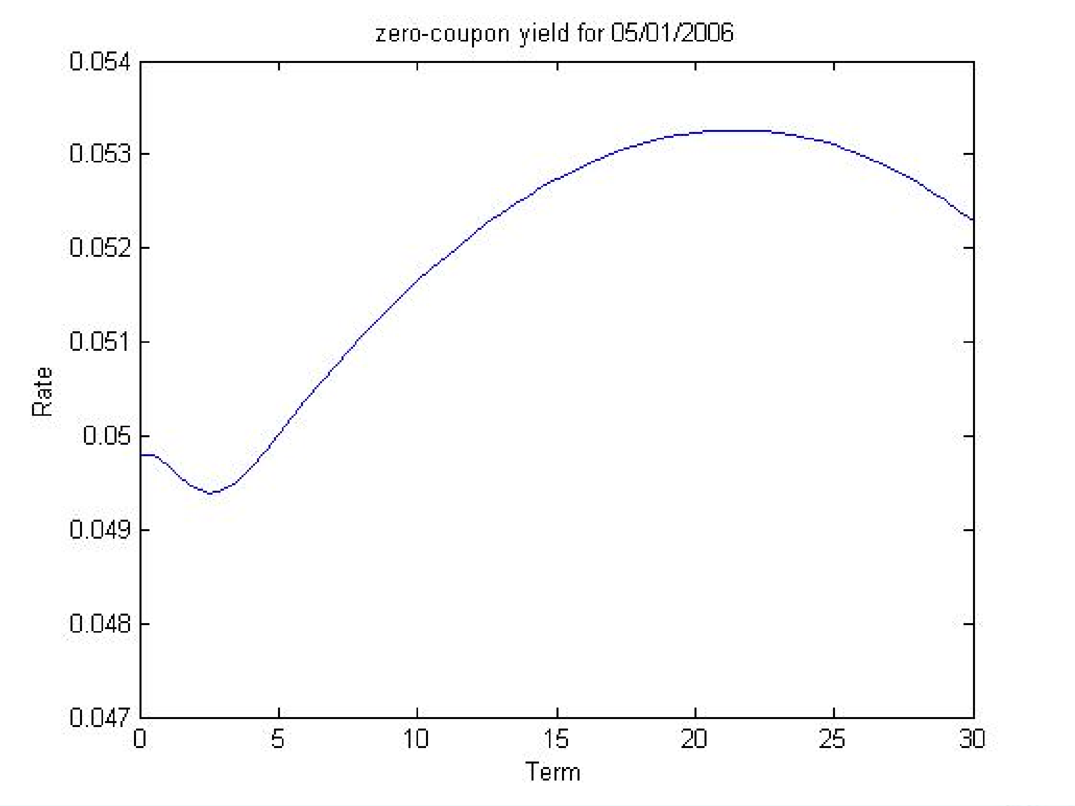
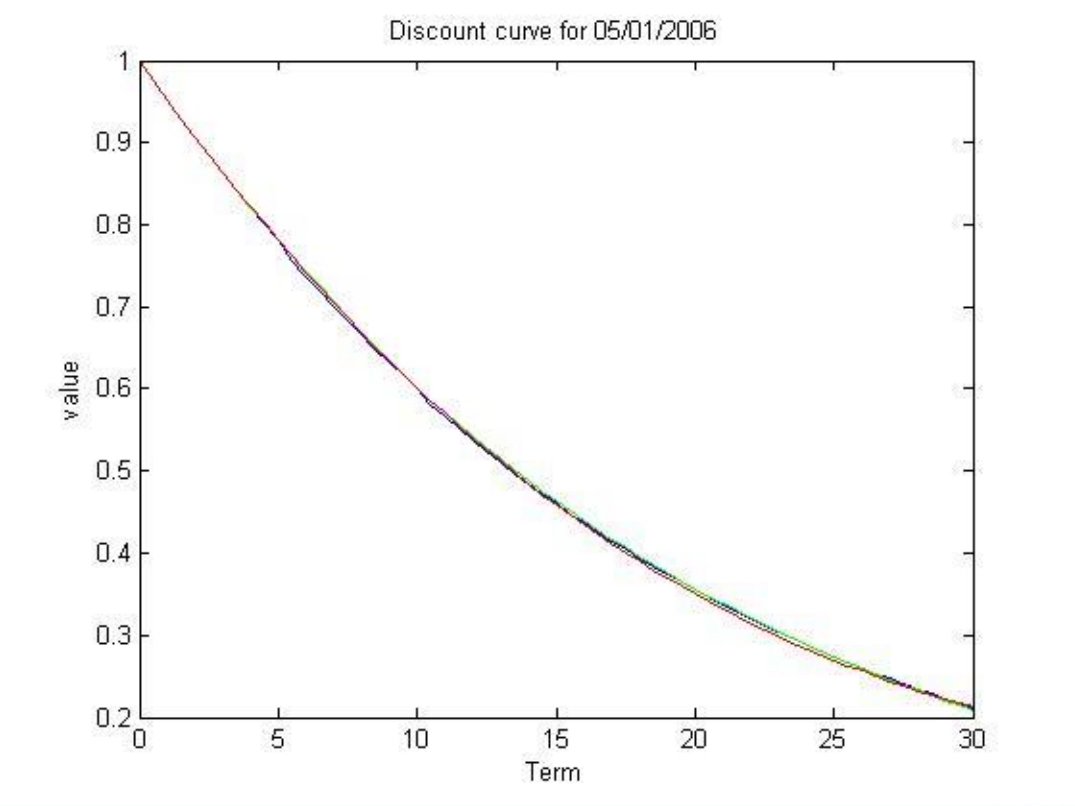


Fig. 3.4. The zero-coupon yield curve of U.S. Treasuries on May 1, 2006.

Bootstrapping – Spline Extrapolation



Three Discount Curves



3.4 Forward Rates and Forward-Rate Agreements

- A forward rate is a fair lending rate for a future period.
- The forward rate is applied in forward-rate agreements, a type of contracts between two parties to lend and borrow a certain amount of money for some future period of time.
- The forward-rate agreement is so structured that neither party needs to make an upfront payment. This is equivalent to saying that, as a financial instrument, the value of the contract is zero when the agreement is signed or entered.
- This fair rate can be determined through arbitrage arguments.

Forward Rate Agreements (FRA)

- An financial contract to pay/receive the difference between a *fixed* interest rate and the *realized* interest rate, LIBOR in particular, applied to certain notional value.
- Initially the value of the FRA is zero.
- At maturity, the P&L to the receiver is

$$\text{P\&L}(T) = d(T, T+\Delta T)\left[\$1m \times \Delta T \times (f_0 - f_T)\right]$$

where

f_0 - is the forward rate observed at t=0;

$f_T = \frac{1}{\Delta T}\left(\frac{1}{P(T, T+\Delta T)} - 1\right)$, the ΔT -term CD rate

Determination of f_0

- Suppose that A receives fixed rate (and short the FRA).
- A can offset the FRA by the following zero-net transactions
  - Short $\frac{d(0,T)}{d(0,T+\Delta T)}$ - unit of $(T+\Delta T)$ - unit zero-coupon bond (ZCB)
  - Long one unit of T-maturity ZCB

Determination of f_0

- At maturity *T*, the total P&L is

$$d(T,T+\Delta T)\Delta T\left(f_0-f_T\right)+d(T,T)-\tfrac{d(0,T)}{d(0,T+\Delta T)}d(T,T+\Delta T)$$

$$=d(T,T+\Delta T)\left[\Delta T\left(f_0-f_T\right)+\frac{d(T,T)}{d(T,T+\Delta T)}-\frac{d(0,T)}{d(0,T+\Delta T)}\right]$$

$$=d(T,T+\Delta T)\left[\left(\left(1+\Delta Tf_0\right)-\frac{d(0,T)}{d(0,T+\Delta T)}\right)\right.$$

$$\left.-\left(\left(1+\Delta Tf_T\right)-\frac{d(T,T)}{d(T,T+\Delta T)}\right)\right]$$

$$=d(T,T+\Delta T)\left(\left(1+\Delta Tf_0\right)-\frac{d(0,T)}{d(0,T+\Delta T)}\right)$$

Determination of f_0

- In order to avoid arbitrage, the total P&L must be zero, which gives rise to

$$f_0 = \frac{1}{\Delta T}\left(\frac{P(0,T)}{P(0,T+\Delta T)} - 1\right)$$

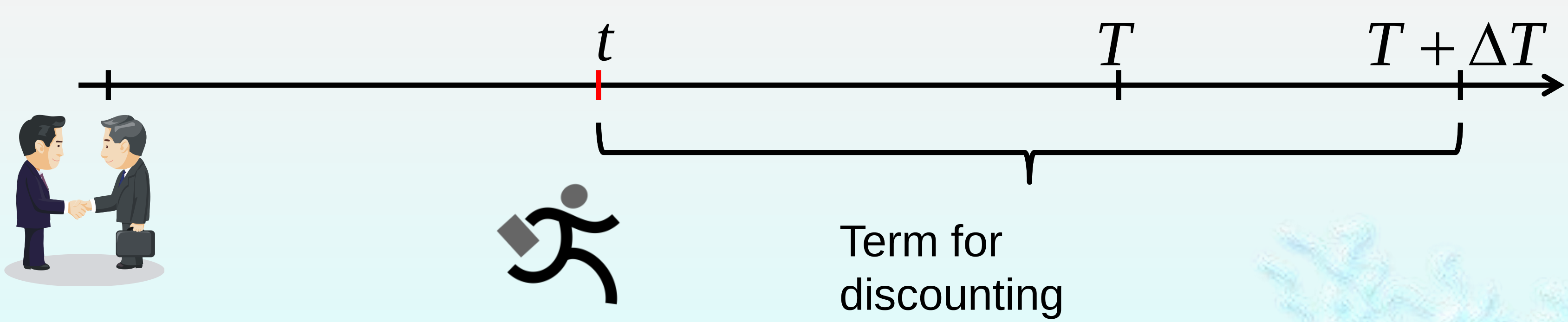
- The same arguments apply to any time $t \leq T$, such that the forward rate for $(T, T+\Delta T)$ is

$$f_t = \frac{1}{\Delta T}\left(\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right)$$

Marking-to-Market Value of FRA

- Let A receives fixed and pays LIBOR.
- Let f_t be the forward rate at a later time *t*, then the marking-to-market value of the FRA to the party who receives fixed and pays float is

$$P\,\&\,L \text{ to A} = d(t, T+\Delta T)\left[\$1m \times \Delta T \times (f_0 - f_t)\right]$$



Examples

- An FRA between A and B
  - A pays B 3M LIBOR and receives 5% from B
  - Maturity: 1Y
  - Notional: $1m

P&L at Maturity

- Scenario 1
  - 1Y later, 3M LIBOR is 5.5%, then

$$P\,\&\,L \text{ to A}=\frac{\$1m\times 0.25\times(5\%-5.5\%)}{1+5.5\%/4}=-\$1,233.46$$

- Scenario 2
  - 1Y later, 3M LIBOR is 4.8%, then

$$P\,\&\,L \text{ to A}=\frac{\$1m\times 0.25\times(5\%-4.8\%)}{1+4.8\%/4}=\$494.07$$

P&L upon Early Closeout

- Suppose 3 months later,
  - the 3m LIBOR for the period becomes 5.5%,
  - The 1Y LIBOR rate is 5.25% so the discount factor is

$$d(1) = \left(1 + 5.25\%\right)^{-1}$$

- So

$$\begin{aligned} P\,\&\,L \text{ to A} &= d(1) \times \$1m \times 0.25 \times (5\% - 5.5\%) \\ &= \frac{\$1m \times 0.25 \times (5\% - 5.5\%)}{1 + 5.25\%} \\ &= -\$1187.65 \end{aligned}$$

Forward-rate – Price Relationship

- Let $T - t = n\Delta T + \varepsilon$, with $0 < \varepsilon < \Delta T$. Then

$$
\begin{aligned}
P(t, T+\Delta T) &= \frac{P(t,T)}{1+\Delta T f(t,T,\Delta T)} \\
&= \frac{P(t,T-\Delta T)}{\left(1+\Delta T f(t,T-\Delta T,\Delta T)\right)\left(1+\Delta T f(t,T,\Delta T)\right)} \\
&= \cdots \\
&= \frac{P(t,t+\varepsilon)}{\prod_{k=0}^{n}\left(1+\Delta T f(t,T-k\Delta T,\Delta T)\right)}
\end{aligned}
$$

Forward-rate – Price Relationship, cont'd

- Let $T = n\Delta T$, then

$$
\begin{aligned}
P(t,T+\Delta T) &= \frac{P(t,T)}{1+\Delta T f(t,T,\Delta T)} \\
&= \frac{P(t,T-\Delta T)}{\left(1+\Delta T f(t,T-\Delta T,\Delta T)\right)\left(1+\Delta T f(t,T,\Delta T)\right)} \\
&= \cdots \\
&= \frac{P(t,t)}{\prod_{k=0}^{n}\left(1+\Delta T f(t,T-k\Delta T,\Delta T)\right)} \\
&= \prod_{k=0}^{n}\left(1+\Delta T f(t,T-k\Delta T,\Delta T)\right)^{-1}
\end{aligned}
$$

Instantaneous Forward Rates

Consider now the limiting case, $\Delta T \to 0$, for the forward rate. There is

$$\begin{aligned} f(t,T) &\triangleq \lim_{\Delta T \to 0} f(t;T,\Delta T) \\ &= \lim_{\Delta T \to 0} \frac{1}{\Delta T}\left(\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right) \\ &= \frac{-1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} \\ &= -\frac{\partial \ln P(t,T)}{\partial T}. \end{aligned} \tag{3.1}$$

We call $f(t,T)$ an *instantaneous forward rate*.

Zero-Coupon Bond Prices and Forward Rates

According to (3.23), we can express the price of a T -maturity zero-coupon bond in terms of $f(t,s), \; t \leq s \leq T$:

$$P(t,T) = e^{-\int_t^T f(t,s)ds}.$$

Note that once we have a model for the forward rates, we also have a model for the discount bonds. Because of the above relationship, the instantaneous forward rates are treated as potential candidates of state variables for interest-rate modeling.

3.5. Swaps and Swap Rates

- A swap is a contract to exchange interest payments out of a notional principal.
- Most interest-rate swap contracts exchange (floating) LIBOR rate payments for fixed-rate payments, and only the net payment is made.
- The party who swaps a fixed-rate payment for a floating-rate payment (to pay the fixed rate and to receive the floating rate) is said to hold a payer's swap, while the counter party is said to hold a receiver's swap.
- A payer's swap allows its holder to lock in a long-term yield (While an FRA or Eurodollar futures contract allows its holder to lock in a short rate in the future).

Example of Interest Rate Swaps

- Parties A & B agree, on May 28, 2010, to enter into an *interest rate swap* with the following terms. Starting in two business days, on June 2, 2010,
  - party A agrees to pay a fixed rate of 1.235% on a notional amount of $100 million to party B for two years,
  - party B, in return, agrees to pay *three-month LIBOR* (*London Interbank Offered Rate)* on this same notional to Party A.

INTEREST RATE SWAPS, cont'd

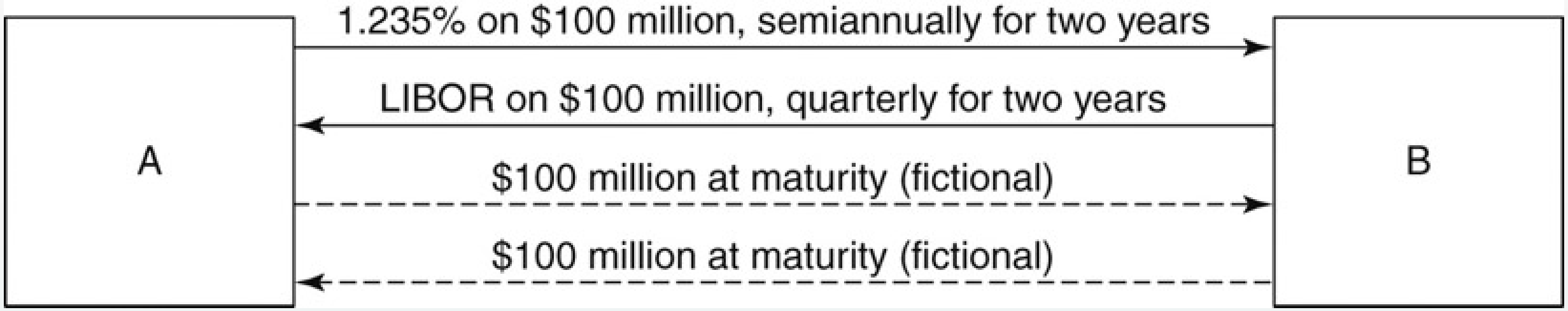
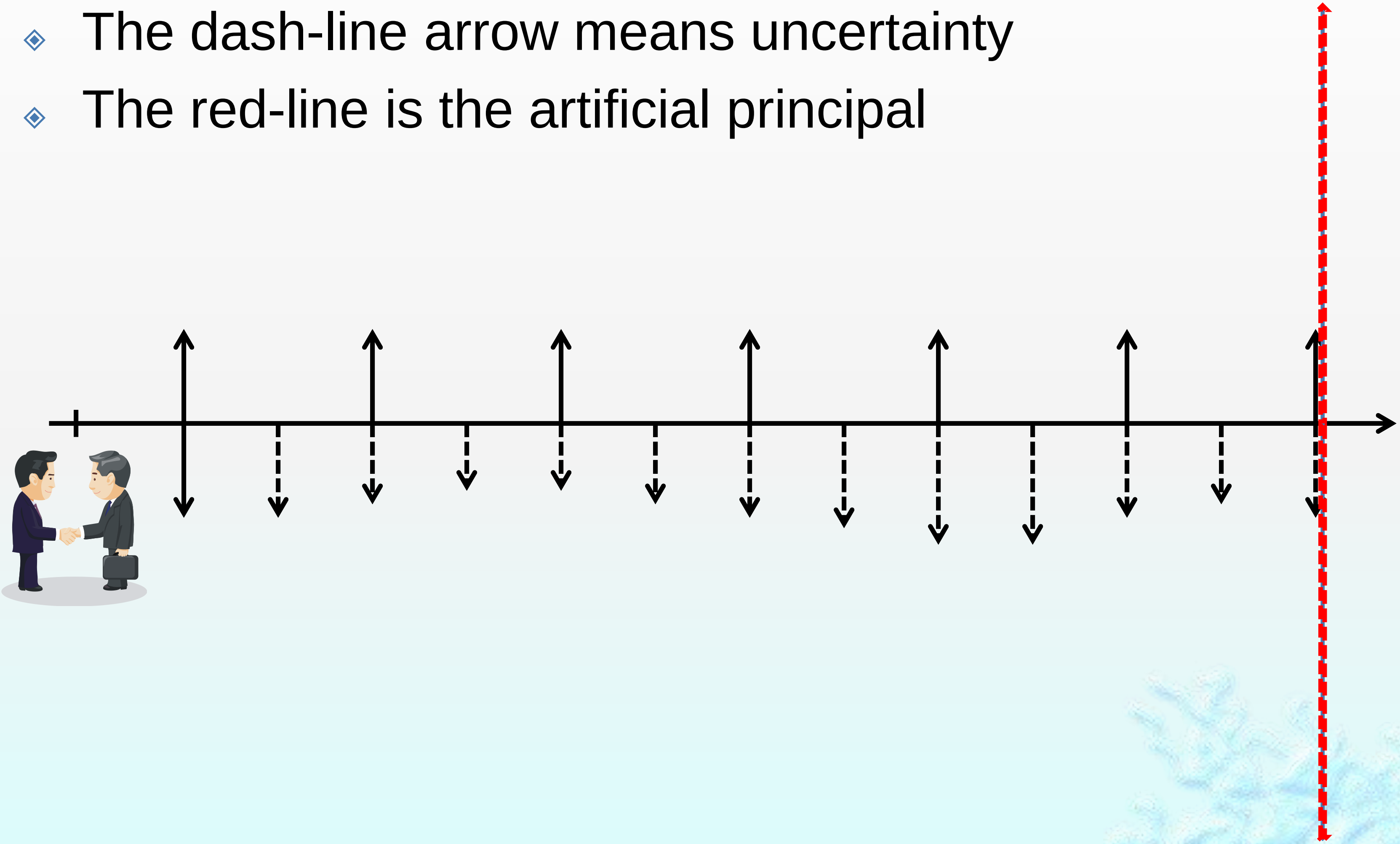


Figure 2.1: An Example of an Interest Rate Swap

Cash-Flow Pattern

- The dash-line arrow means uncertainty
- The red-line is the artificial principal



Cash Flow Calculation

- Fixed periodic payment from the fixed-leg

$$\frac{1.235\%}{2} \times \$100{,}000{,}000 = \$617{,}500$$

- Assume the current 3-mo LIBOR is 0.75%, then the next floating-leg payment is

$$\frac{0.75\%}{4} \times \$100{,}000{,}000 = \$187{,}500$$

- The second floating-leg payment is not yet known.

Pricing of Swaps

- The cash flows are generated from
  - A fixed-rate bond with semiannual payments
  - A floating-rate bond with quarterly payments
- To price the swap, we may present value the cash flows.
- The fixed-leg is about a usual bond pricing.
- With the principal added, the floating leg is a floating-rate bond, often called a floating-rate note (FRN).

Pricing of Floating Leg

- Proposition: Initially and right after a payment, the FRN is worth par.

  Proof:
- You start with \$100m, save it in three-month certificate of deposits (CDs).
- Three month later, you collect and pay the interest, and save the remaining \$100m to another CD.
- Repeat this transaction until the maturity of the swap.

Pricing of Swaps

- So the price of a swap is the difference between a coupon bond and its par value
- At initiation, all swap have zero value, meaning the values of the fixed legs are also par.
- After adding the principal payment, the fixed leg is a par bond.
- The swap rate is a par yield!!!

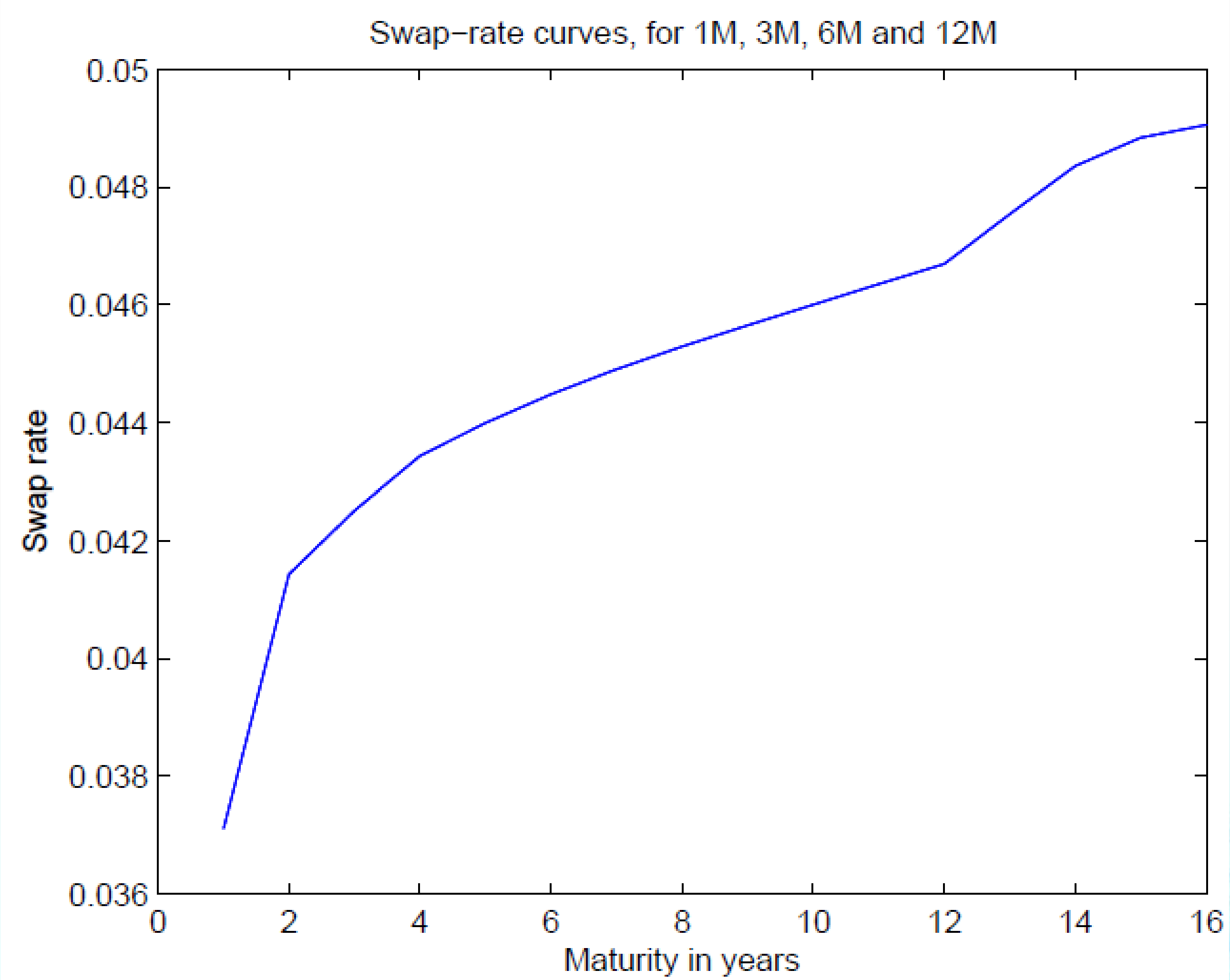
$$s(0;0,T) = \frac{1 - d(0,T)}{\sum_{i=1}^{2T} \frac{1}{2} d(0, \frac{i}{2})}$$

Swap Rates and Discount Factors

TABLE 2.1 Discount Factors, Spot Rates, and Forward Rates Implied by Par USD Swap Rates as of May 28, 2010

Term in Years	Swap Rate	Discount Factor	Spot Rate	Forward Rate
0.5	.705%	.996489	.705%	.705%
1.0	.875%	.991306	.875%	1.046%
1.5	1.043%	.984494	1.045%	1.384%
2.0	1.235%	.975616	1.238%	1.820%
2.5	1.445%	.964519	1.450%	2.301%

Swap-rate Curves for Various Tenors



Extracting Discount Curve from Par-yield Curve

- By definition, a T-maturity par yield, $s(0;0,T)$, satisfies

$$1 = s(0;0,T)\sum_{i=1}^{2T} \Delta T d(0, i/2) + d(0,T).$$

- Given the discount curve, we obtain yields as

$$s(0;0,T) = \frac{1 - d(0,T)}{\sum_{i=1}^{2T} \Delta T d(0, i/2)}$$

Discount Curve vs. Par-yield Curve

- Given the par-yield curve, we can solve for the discount curve as follows:

$$d(0,T) = \frac{1 - s(0;0,T)\sum_{i=1}^{2T-1} \Delta T d(0, i/2)}{1 + \Delta T \times s(0;0,T)}.$$

- So $d(0,\frac{i}{2}),\ i = 1,\cdots,2T$ are solved inductively.

MtM Swaps

- A trader long a payer's swap at time 0 for swap rate $s(0,T)$
- At a later time t, the value of the swap becomes

$$MtM = \left(s(t;t,T) - s(0;0,T)\right) \sum_{i=2t+1}^{2T} \tfrac{1}{2} d(t, \tfrac{i}{2})$$

- [Example]

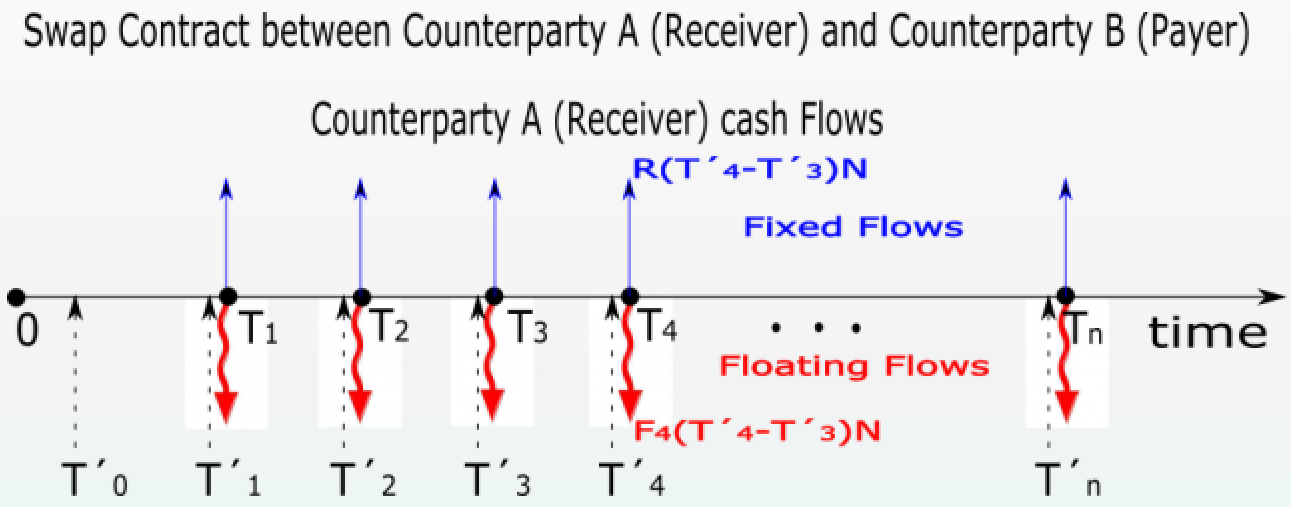
Why Swaps?

- People trade swaps for three purposes.
  - To lock in long-term low interest rate to borrow or high interest rate to receive.
  - To speculate.
  - To hedge other interest-rate sensitive securities/ portfolios.

Overnight Index Swaps

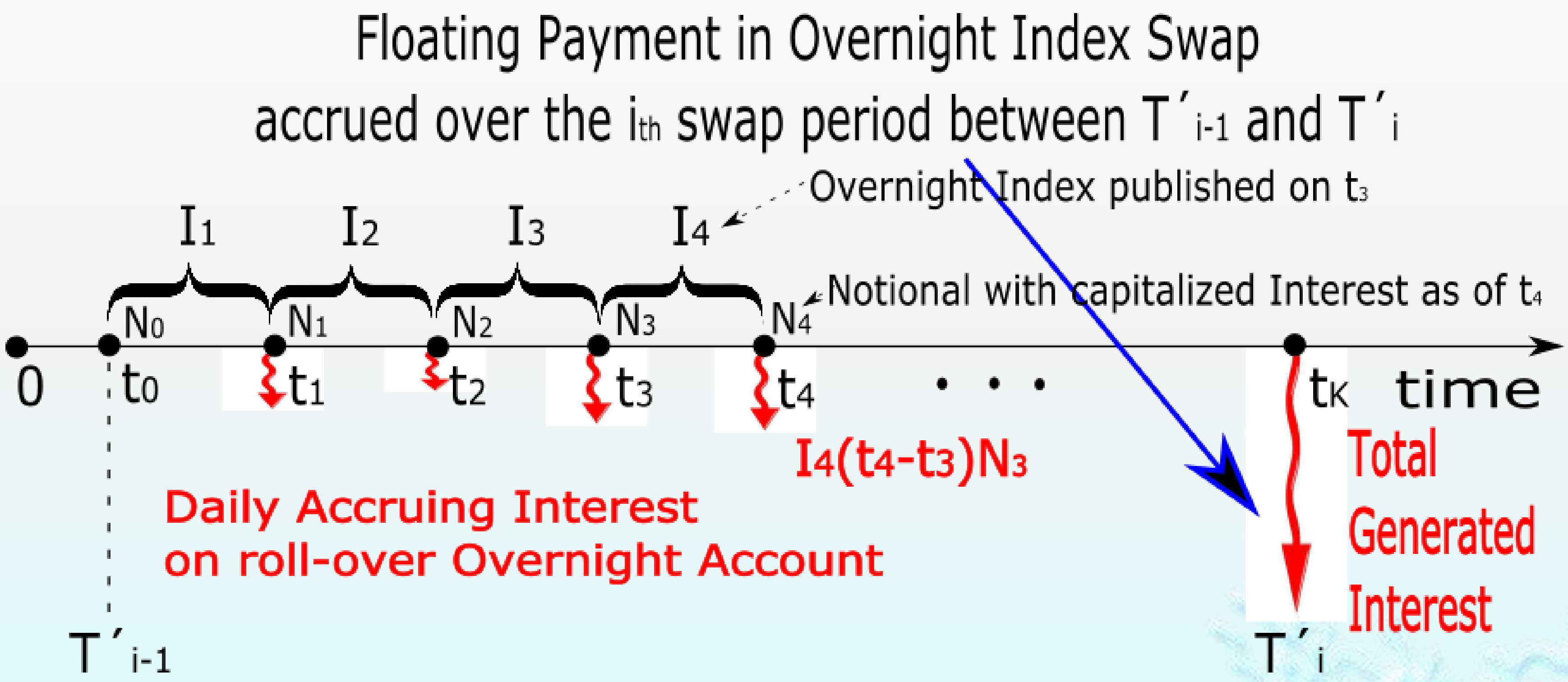
- An **overnight indexed swap** (**OIS**) is an interest rate swap over some fixed term where the periodic floating payment is generally based on a return calculated from a daily compound interest investment.
- Note that the swap term is not over-night; it is the reference rate that is an overnight rate.
- The Overnight Index Swap market has grown significantly in importance during the financial turmoil of the last decade.
- Examples of overnight rates are SOFR (USD), €STR (Euros) and SONIA (GBP).

Cash Flows of an OIS



- Here T_i' differs from T_i if the latter is a holiday.

Overnight Interest Rate I_j



Floating Payment Calculations

- The floating rate over a period (T_{i-1}, T_i) is simply

$$F_i = \frac{\prod_{j=1}^{n}(1 + I_j \Delta t) - 1}{T_i - T_{i-1}}$$

  where n is the number of days between T_{i-1} and T_i.
- The floating-leg payment at T_i is

$$F_i(T_i - T_{i-1}) = \prod_{j=1}^{n}(1 + I_j \Delta t) - 1$$

Pricing of the Fair Swap Rate

- The value of an OIS is zero initially.
- We add the fictitious principal to both legs.
- The floating lag is priced at PAR, as was argued earlier.
- So the fixed is also priced at PAR.
- The swap rate is then given by

$$s(0;0,T) = \frac{1 - d(0,T)}{\sum_{i=1}^{2T} \Delta T d(0, i/2)},$$

- The same as that of the vanilla swap!

Basis Swaps

- A basis swap exchange floating-rate payments indexed to LIBOR of different tenors, e.g. 3m against 6m.
- The basis is the difference between the swap rates of the two tenors.
- Suppose the 3m- and 6m-tenor swap rates are 1% and 1.25%, respectively, then the basis spread is 0.25%.
- In the 3m to 6m basis swap,
  - A pays B 3m LIBOR+0.25%; and
  - B pays A 6m LIBOR.
- When there is no counterparty default risk, the basis should be zero.
- This was not the case during the 2007-08 financial crisis.