3.5 Yield-Based Bond Risk Management

- In the bond market, bond prices change unpredictably on a daily basis.
- The changes in bond prices can be interpreted as the consequence of unpredictable changes in yields.
- Therefore, dealing with yield risks is the same as dealing with price risks.
- It has become a commonplace that yields to maturities of all bonds from the same issuer are highly correlated: yields (of different maturities) often move in the same direction with comparable magnitudes.
- This reality makes it possible to hedge against the price change of one bond with another.

DV01

- Dollar value per basis point, DV01 for short, is the first measure of price sensitivity, defined as

$$DV01 = -\Delta B^c(y) = -B^c(y + 0.01\%) + B^c(y)$$

- where y is the yield of the bond.
- DV01 describes the change in absolute value of bonds for one bps change in yield.

Percentage Change

- For percentage change, $\Delta B^c / B^c$, we take the following approach:

$$\frac{\Delta B^c}{B^c} = \frac{1}{B^c}\frac{\Delta B^c}{\Delta y}\Delta y \approx \frac{1}{B^c}\frac{dB^c}{dy}\Delta y$$

where

$$\frac{dB_t^c}{dy} = -\frac{\Pr}{1+y\Delta T}\left(\sum_{i;T_i>t}^{n} \Delta T \cdot c\left(1+y\Delta T\right)^{-(T_i-t)/\Delta T}\left(T_i - t\right) + \left(1+y\Delta T\right)^{-(T_n-t)/\Delta T}\left(T_n - t\right)\right).$$

Modified Duration

- Define the modified duration to be

$$D_{\text{mod}} = \frac{1}{1+y\Delta T}\frac{\Pr}{B^c}\left(\sum_{i;T_i>t}^{n} \Delta T \cdot c\left(1+y\Delta T\right)^{-(T_i-t)/\Delta T}\left(T_i-t\right) + \left(1+y\Delta T\right)^{-(T_n-t)/\Delta T}\left(T_n-t\right)\right)$$

- Then, in terms of the modified duration, we have

$$\frac{dB_t^c}{B_t^c} = -D_{\text{mod}}\, dy.$$

- In discrete approximation, it becomes

$$\frac{\Delta B^c}{B^c} = -D_{\text{mod}}\Delta y$$

Macaulay Duration

- It has been observed that the prices of long-maturity bonds are more sensitive to change in yields than are the prices of short-maturity bonds, and the impact of yield changes on bond prices seems proportional to the cash flow dates of the bonds.
- Macaulay (1938) introduced the weighted average of the cash flow dates as a measure of price sensitivity with respect to the bond yield:

$$D_{mac} = \frac{\Pr}{B_t^c}\left[\sum_{i,T_i>t}^{n} \Delta T \cdot c(1+y\Delta T)^{-(T_i-t)/\Delta T}\left(T_i-t\right)+(1+y\Delta T)^{-(T_n-t)/\Delta T}\left(T_n-t\right)\right].$$

- which is called the Macaulay duration.

Modified vs. MaCaulay Duration

- The modified and. MaCaulay durations are linked by

$$D_{\text{mod}} = \frac{D_{mac}}{1 + y\Delta T},$$

- and they are fairly close.
- For a zero-coupon bond, the duration is simply its maturity.

DV01 vs. the Modified Duration

- From DV01 we can derive duration, and vice versa, since

$$DV01 = -\Delta B^c = D_{\text{mod}} B^c \Delta y = \frac{B^c}{10,000} D_{\text{mod}}$$

or

$$D_{\text{mod}} = \frac{10,000}{B^c} DV01$$

Formula of Modified Duration

- By using the succinct bond price formula, we can obtain the following formula for the modified duration:

$$D_{\text{mod}} = \frac{\Pr}{B^c}\left[\frac{c}{y^2}\left(1-\frac{1}{(1+y\Delta T)^n}\right)+\left(1-\frac{c}{y}\right)\frac{n\Delta T}{(1+y\Delta T)^{n+1}}\right].$$

Example 3.2. Given a 30-year Treasury yielding 5% and trading at par, we can calculate the modified duration using (3.28) and obtain $D_{\text{mod}} = 15.45$ years. This means that a one basis point variation in the yield will cause a change of 15.45 cents per 100 dollars. □

Two Special Cases

- The modified duration for par bond: for $c = y$ and $B^c = \Pr$, we have

$$D_{\text{mod}} = \frac{1}{y}\left[1-\left(1+y\Delta T\right)^{-n}\right].$$

- The modified duration for zero-coupon bond: when $c = 0$,

$$D_{\text{mod}} = \frac{n\Delta T}{1+y\Delta T} = \frac{T}{1+y\Delta T}$$

- Or

$$D_{\text{m}ac} = T$$

Change of Bond Prices

- From a mathematical point of view, the estimation of price changes using duration is equivalent to estimating functional value using a linear approximation.
- The accuracy of such an approximation becomes poorer as the change of the yield becomes larger.
- To see that, we consider the second-order expansion of the bond price change with respect to the yield:

$$\Delta B_t^c = \frac{dB_t^c}{dy}\Delta y + \frac{1}{2}\frac{d^2 B_t^c}{dy^2}\left(\Delta y\right)^2. \qquad (3.29)$$

Convexity Measure

We also introduce the convexity measure:

$$C = \frac{1}{B_t^c}\frac{d^2 B_t^c}{dy^2} = \frac{\Pr}{B_t^c}\left(\sum_{i;T_i>t}^{n} \Delta T \cdot c\left(T_i - t\right)\left(T_{i+1} - t\right)\left(1 + y\Delta T\right)^{-\frac{T_i - t}{\Delta T}-2} + \left(T_n - t\right)\left(T_{n+1} - t\right)\left(1 + y\Delta T\right)^{-\frac{T_n - t}{\Delta T}-2}\right). \tag{3.30}$$

Then, we can rewrite (3.29) as

$$\frac{\Delta B_t^c}{B_t^c} = -D_{\text{mod}}\Delta y + \frac{1}{2}C\Delta y^2. \tag{3.31}$$

Note that C captures the convexity or curvature of the bond-price curve.

Duration and Convexity Measures

Given the duration measure, D_{mod}, and the convexity measure, C, we can readily calculate the percentage change in the price with (3.31). When $\Delta y^2 \ll \Delta y$, we can neglect the second-order term in (3.31) and thus return to (3.26). However, for relatively large Δy, the inclusion of the convexity term in (3.31) will be necessary to produce a good approximation of the percentage change. The curves of the exact price, the linear approximation and the quadratic approximation are shown in Figure 3.5.

Linear and Quadratic Approximations

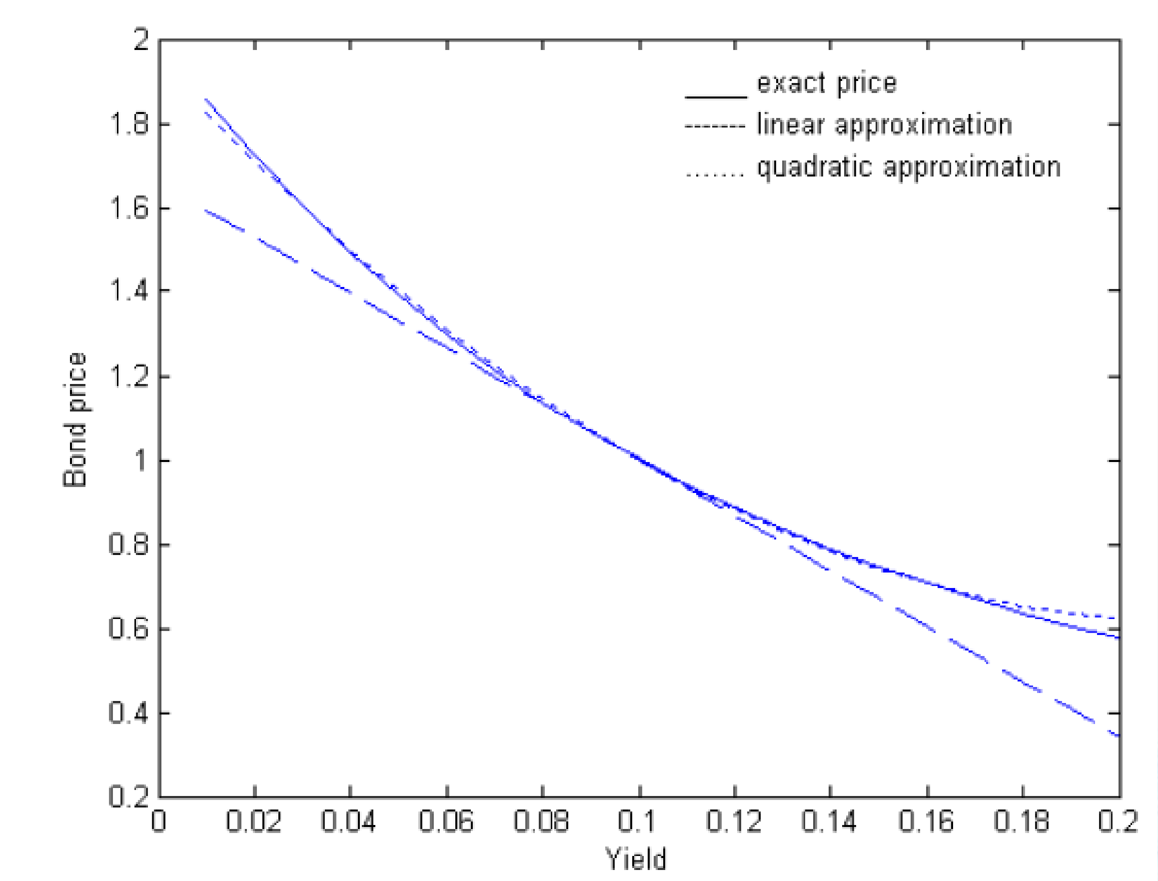

Fig. 3.5. Linear and quadratic approximations of bond prices

Central Difference Scheme for Derivatives

- If bond prices are given in discrete form, the first and the second derivatives of the bond price as a function can be approximated by central differencing

$$\frac{dB^c}{dy} \approx \frac{B^c(y+\Delta y) - B^c(y-\Delta y)}{2\Delta y}$$

$$\frac{d^2B^c}{dy^2} \approx \frac{B^c(y+\Delta y) - 2B^c(y) + B^c(y-\Delta y)}{(\Delta y)^2}$$

3.5.2 Portfolio Risk Management

Consider a portfolio of N instruments, with n_i units and price B_i^c for the i^{th} instrument. Then, the absolute change in the portfolio value upon a parallel yield shift is given by

$$dV = \sum_i n_i dB_i^c = \sum_i n_i B_i^c \cdot \left(-D_{\text{mod}}^i dy + \frac{1}{2} C^i dy^2 \right) \cdot$$

The percentage change is then

$$\frac{dV}{V} = -\left(\sum_i x_i D_{\text{mod}}^i \right) dy + \frac{1}{2} \left(\sum_i x_i C^i \right) dy^2, \qquad (3.1)$$

where $x_i = n_i B_i^c / V$ is the percentage of the value in the i^{th} instrument. Equation (3.1) indicates that the duration and convexity of a portfolio are the weighted average of the duration and convexity of its components, respectively.

Duration vs. Convexity

- In classical risk management, a portfolio manager can limit his/her exposure to interest-rate risk by reducing the duration while increasing the convexity of the portfolio.
- A portfolio with very small duration is called a duration-neutral portfolio (Duration neutral is the same as DV01 neutral, when $V \neq 0$).
- Practically, interest-rate futures and interest-rate swaps are often used as hedging instruments for duration management.
- To avoid possible losses in case of large yield moves, the manager usually will not tolerate negative net convexity.

Which One Do You Prefer?

- A: a 9-year zero-coupon bond
- B: a portfolio of 2- and 30-year zero-coupon bond with weights 0.75 and 0.25.
- Suppose the current yield curve is flat at 5%.
- Note that the MaCaulay duration of both portfolios are equal to 9.
- Check the Convexities:

$$C_A = {9 \times 9.5} \Big/ {\left(1 + {5\%}/{2}\right)^2} = 81.38$$

$$C_B = 0.75 \times {2 \times 2.5} \Big/ {\left(1 + {5\%}/{2}\right)^2} + 0.25 \times {30 \times 30.5} \Big/ {\left(1 + {5\%}/{2}\right)^2} = 221.30$$

- Your choice?

Limitation of Duration-based Hedging

- The basic premise of the duration and convexity technology is that the yield curves shift in parallel, either upward or downward by the same amount.
- This is, however, a very crude assumption about the yield curve movement, as, in reality, points in a yield curve do not often shift by the same amount and sometimes they do not even move in the same direction.
- For a more elaborate model of yield curve dynamics, we will have to resort to stochastic calculus in a multi-factor setting.

Chapter 2
Fundamental Theorems for Derivatives Pricing in Complete Markets

Two Theorems & Two Lemmas

From a mathematical point of view, asset pricing in a complete market is based on the following lemmas and theorems.

- Ito's Lemma
- The Cameron-Martin-Girsanov Theorem
- The martingale representation Theorem
- Lemma for martingale representation

Lemma 1.6 (Ito's Lemma). Let f be a deterministic twice continuous differentiable function, and let $\mathbf{X}_t$ be an n-factor Ito's process. Then, $Y_t = f(\mathbf{X}_t, t)$ is also an n-factor Ito's process, and it satisfies

$$dY_t = \left(\frac{\partial f}{\partial t} + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial X_j} + \frac{1}{2} \sum_{i,j=1}^{n} \boldsymbol{\sigma}_i^{\mathrm{T}}(t) \boldsymbol{\sigma}_j(t) \frac{\partial^2 f}{\partial X_i \partial X_j} \right) dt + \sum_{j=1}^{n} \frac{\partial f}{\partial X_j} \boldsymbol{\sigma}_j^{\mathrm{T}}(t) d\mathbf{W}_t.$$

Alternative Expression

- Bi-variate Ito's lemma: let X_t and Y_t be two Ito's processes and $f(t, X_t, Y_t) \in C^{1,2,2}\left(R^+, R, R\right)$, then the dynamics of f is

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX_t + \frac{\partial f}{\partial Y} dY_t$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial X^2} \left(dX_t\right)^2 + 2\frac{\partial^2 f}{\partial X \partial Y} dX_t dY_t + \frac{\partial^2 f}{\partial Y^2} \left(dY_t\right)^2 \right)$$

Product and Quotient Rules

- Let $f = X_t Y_t$, then

$$df = Y_t dX_t + X_t dY_t + dX_t dY_t$$

- Let $f = X_t / Y_t$, then

$$d\left(\frac{X_t}{Y_t}\right) = \frac{dX_t}{Y_t} - \frac{X_t dY_t}{Y_t^2} - \frac{dX_t dY_t}{Y_t^2} + \frac{X_t (dY_t)^2}{Y_t^3}$$

The Change of Measure Theorem

Theorem 2.1 (The Cameron-Martin-Girsanov Theorem). Let W_t be a $\mathbb{P}$-Brownian motion and γ_t be an $\mathcal{F}_t$-adaptive process satisfying the Novikov condition,

$$E^{\mathbb{P}}[\exp(\tfrac{1}{2}\int_0^T \gamma_t^2 dt)] < \infty. \tag{0.1}$$

Define a new measure, $\mathbb{Q}$, as

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(-\int_0^t \gamma_s dW_s + \tfrac{1}{2}\gamma_s^2 ds). \tag{0.2}$$

Then, $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds \tag{0.3}$$

is a $\mathbb{Q}$-Brownian motion.

The Change of Measure Theorem, cont'd

Theorem 2.3 (The Cameron-Martin-Girsanov Theorem). Let $\mathbf{W}_t = (W_1(t), W_2(t), \ldots, W_n(t))^{\mathrm{T}}$ be an n-dimensional $\mathbb{P}$-Brownian motion, and let $\boldsymbol{\gamma}_t = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))^{\mathrm{T}}$ be an n-dimensional $\mathcal{F}_t$-adaptive process such that

$$E^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \left\|\boldsymbol{\gamma}_t\right\|^2 dt\right)\right] < \infty.$$

Define a new measure, $\mathbb{Q}$, with a Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t -\boldsymbol{\gamma}_s^{\mathrm{T}} d\mathbf{W}_s - \frac{1}{2}\int_0^t \left\|\boldsymbol{\gamma}_s\right\|^2 ds\right). \qquad (0.1)$$

Then, $\mathbb{Q}$ is equivalent to $\mathbb{P}$, and

$$\tilde{\mathbf{W}}_t = \mathbf{W}_t + \int_0^t \boldsymbol{\gamma}_s ds$$

is an n-dimensional $\mathbb{Q}$-Brownian motion.

The Martingale Representation Theorem

- The martingale representation theorem plays a critical role in the so-called "martingale approach" to derivatives pricing.
- This theorem has two important consequences.
  - First, it leads to a general principle for derivatives pricing.
  - Second, it implies a replication or hedging strategy of a derivative using its underlying security.

Martingale as an Ito's Process

We first present a simple version of the theorem based on a single Brownian filtration, $\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)$. We begin with a martingale process, M_t, such that

$$dM_t = \sigma_t dW_t, \quad (0.1)$$

and we call σ_t the volatility of M_t.

Theorem 2.2 (The Martingale Representation Theorem). Suppose that N_t is a $\mathbb{Q}$-martingale process that is adaptive to $\mathcal{F}_t$ and satisfies $E^{\mathbb{Q}}\left[N_T^2\right] < \infty$ for some T. If the volatility of M_t is non-zero almost surely, then there exists a unique $\mathcal{F}_t$-adaptive process, φ_t, such that $E\left[\int_0^T \varphi_t^2 \sigma_t^2 dt\right] < \infty$, and

$$N_t = N_0 + \int_0^t \varphi_s dM_s, \ t \le T, \tag{2.40}$$

or, in differential form,

$$dN_t = \varphi_t dM_t. \tag{2.41}$$

Lemma. Suppose that N_t is a $\mathbb{Q}$-martingale process that is adaptive to $\mathcal{F}_t$ and satisfies $E^{\mathbb{Q}}\left[N_T^2\right] < \infty$ for some T, then there is

$$N_t = N_0 + \int_0^t \varphi_s dM_s,\ t \le T,$$

where φ_t is a $\mathcal{F}_t$-adaptive process given by

$$\varphi_t = \frac{d\langle N_t, M_t\rangle}{dt} \Bigg/ \frac{d\langle M_t, M_t\rangle}{dt} \quad \square$$

Theorem 2.4 (The Martingale Representation Theorem). Let $\mathbf{W}_t$ be an n-dimensional Brownian motion and suppose that $\mathbf{M}_t$ is an n-dimensional $\mathbb{Q}$-martingale process, $\mathbf{M}_t = \left(M_1(t), M_2(t), \ldots M_n(t)\right)^{\mathrm{T}}$, such that

$$dM_i(t) = \sum_{j=1}^{n} a_{ij}(t)\,dW_j(t).$$

Let $\mathbf{A} = (a_{ij})$ be a non-singular matrix.

If N_t is any one-dimensional $\mathbb{Q}$-martingale with $E^{\mathbb{Q}}\left[N_t^2\right]<\infty$, there exists an n-dimensional $\mathcal{F}_t$-adaptive process, $\mathbf{\Phi}_t=(\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t))^{\mathrm{T}}$, such that

$$\int_0^t\left(\sum_j a_{ij}^2(s)\varphi_j^2(s)ds\right)<\infty,\ \forall i, \qquad (0.1)$$

and

$$\begin{aligned}N_t &= N_0+\sum_{j=1}^n\int_0^t\varphi_j(s)dM_j(s)\\ &\triangleq N_0+\int_0^t\mathbf{\Phi}(s)\cdot d\mathbf{M}(s).\end{aligned}$$

Lemma. Suppose that N_t is a $\mathbb{Q}$-martingale process that is adaptive to $\mathcal{F}_t$ and satisfies $E^{\mathbb{Q}}\left[N_T^2\right] < \infty$ for some T, then there is

$$N_t = N_0 + \int_0^t \mathbf{\Phi}_s \cdot d\mathbf{M}_s,\ t \le T,$$

where φ_t is a $\mathcal{F}_t$-adaptive process given by

$$\varphi_i(t) = \frac{d\left\langle N_t, M_i(t)\right\rangle}{dt} \Bigg/ \frac{d\left\langle M_i(t)_t, M_i(t)\right\rangle}{dt}, \quad i = 1, \cdots, N. \ \square$$

2.4 A Complete Market with Two Securities

We consider the first "complete market" in continuous time, which consists of a money market account and a risky security. The price processes for the two securities, B_t and S_t, are assumed to be

$$dB_t = r_t B_t dt,\ B_0 = 1,$$
$$dS_t = S_t\left(\mu_t dt + \sigma_t dW_t\right),\ S_0 = S_0.$$

Here, the volatility of the risky asset is $\sigma_t \neq 0$ almost surely, and the short rate, r_t, can be stochastic.

Denote the discounted price of the risky asset as $Z_t = B_t^{-1} S_t$, which can be shown to follow the process

$$\begin{aligned} dZ_t &= Z_t\left((\mu_t - r_t)dt + \sigma_t dW_t\right) \\ &= Z_t \sigma_t d\left(W_t + \int_0^t \frac{(\mu_s - r_s)}{\sigma_s} ds\right). \end{aligned} \tag{2.42}$$

By introducing

$$\gamma_t = \frac{\mu_t - r_t}{\sigma_t}, \tag{2.43}$$

which is $\mathcal{F}_t$-adaptive, and by defining a new measure, $\mathbb{Q}$, according to (2.36), we have

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds,$$

which is a $\mathbb{Q}$-Brownian motion. In terms of $\tilde{W}_t$, Z_t satisfies

$$dZ_t = \sigma_t Z_t d\tilde{W}_t, \tag{2.44}$$

which is a lognormal $\mathbb{Q}$-martingale. Recall that in the binomial model for option pricing, we also derived the martingale measure for the underlying security.

2.5 Replicating and Pricing of Contingent Claims

Let X_T be a contingent claim (or option) with payoff day or maturity T. The claim is an $\mathcal{F}_T$-adaptive function whose value depends on $\{S_t,\ 0 \le t \le T\}$. Define first a $\mathbb{Q}$-martingale with the discounted payoff:

$$N_t = E^{\mathbb{Q}}(B_T^{-1} X_T \mid \mathcal{F}_t).$$

Without loss of generality, we assume that $E^{\mathbb{Q}}\left[N_t^2\right] < \infty$.

According to the martingale representation theorem, there exists an $\mathcal{F}_t$-adaptive function, φ_t, such that

$$dN_t = \varphi_t dZ_t, \tag{0.1}$$

where Z_t, defined in the last section, is the discounted price of S_t. Next, we define

$$\psi_t = N_t - \varphi_t Z_t. \tag{0.2}$$

Consider now the portfolio with φ_t units of the stock and ψ_t units of the money market account, denoted as (φ_t, ψ_t). According to the definition of ψ_t, the discount value of the replication portfolio is

$$V_t = \varphi_t Z_t + \psi_t = N_t.$$

This portfolio has two important properties.

First, at time T , when the option matures,

$$\tilde{V}_T = N_T = B_T^{-1} X_T, \tag{0.1}$$

which suggests that the (discounted) value of the portfolio equals that of the option.

Second, the replicating portfolio is a *self-financing* one, meaning that it can track the asset allocation, (φ_t, ψ_t), without the need for either capital infusion or capital withdrawal.

In fact, based on (2.45) and (2.47), we have

$$d\tilde{V}_t = dN_t = \varphi_t dZ_t. \tag{2.49}$$

In terms of the spot value, B_t and S_t, (2.49) becomes

$$\begin{aligned} dV_t &= d\left(\tilde{V}_t B_t\right) \\ &= B_t d\tilde{V}_t + \tilde{V}_t dB_t \\ &= B_t \varphi_t dZ_t + \left(\varphi_t Z_t + \psi_t\right) dB_t \\ &= \varphi_t \left(B_t dZ_t + Z_t dB_t\right) + \psi_t dB_t \\ &= \varphi_t d\left(B_t Z_t\right) + \psi_t dB_t \\ &= \varphi_t dS_t + \psi_t dB_t. \end{aligned} \tag{2.50}$$

A direct consequence of the above equation is the equality

$$\begin{aligned} V_{t+dt} &= V_t + dV_t \\ \varphi_{t+dt} S_{t+dt} + \psi_{t+dt} B_{t+dt} &= \varphi_t S_t + \psi_t B_t + \varphi_t dS_t + \psi_t dB_t \\ &= \varphi_t S_{t+dt} + \psi_t B_{t+dt} = V_{t+dt-} \end{aligned} \qquad (2.51)$$

which says that the values of the portfolio before and after rebalancing at time $t+dt$ are equal, and no cash flow is generated when we update the asset allocation from (φ_t, ψ_t) to $(\varphi_{t+dt}, \psi_{t+dt})$.

A direct consequence of the above equation is the equality

$$\begin{aligned}\varphi_{t+dt}S_{t+dt}+\psi_{t+dt}B_{t+dt} &= \varphi_t S_t+\psi_t B_t+\varphi_t dS_t+\psi_t dB_t \\ &= \varphi_t S_{t+dt}+\psi_t B_{t+dt},\end{aligned} \tag{0.1}$$

which says that the values of the portfolio before and after rebalancing at time $t+dt$ are equal, and no cash flow is generated when we update the asset allocation from (φ_t,ψ_t) to $(\varphi_{t+dt},\psi_{t+dt})$.

By the dominance principle, the value of the contingent claim is nothing but the value of the replication portfolio:

$$V_t = B_t \tilde{V}_t = B_t N_t = B_t E^{\mathbb{Q}}\left[B_T^{-1} X_T \mid \mathcal{F}_t\right] = E^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} X_T \mid \mathcal{F}_t\right]. \quad (2.1)$$

By already knowing the pricing measure, $\mathbb{Q}$, we have thus established a general pricing principle or pricing formula for contingent claims.

An important implication of the above pricing approach is hedging. A contingent claim can be replicated dynamically by its underlying risky asset and the money market account, or it can be dynamically hedged by the underlying security. The hedge ratio is φ_t, about which we know only its existence and its uniqueness based on the martingale representation theorem. Fortunately, Ito's lemma tells us more about φ_t.

Assume, for simplicity, that the short rate is a deterministic function and the value function, V_t , has continuous second-order partial derivatives. Then, by Ito's lemma, we have the following expression of the price process of the contingent claim:

$$dV_t = \frac{\partial V_t}{\partial S_t} dS_t + \left(\frac{\partial V_t}{\partial t} + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} \right) dt. \tag{2.1}$$

Through comparison with equation (2.50), we obtain

$$\varphi_t = \frac{\partial V_t}{\partial S_t}, \tag{2.2}$$

due to the uniqueness of φ_t . Hence, the hedge ratio is just the rate of change of the contingent claim with respect to the underlying asset.

We can also produce interesting results by comparing the drift terms of (2.50) and (2.53), yielding

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} = r_t B_t \psi_t. \tag{2.55}$$

According to (2.47),

$$\begin{aligned} B_t \psi_t &= B_t\left(\tilde{V}_t - \varphi_t Z_t\right) \\ &= V_t - \varphi_t S_t \\ &= V_t - S_t \frac{\partial V_t}{\partial S_t}. \end{aligned} \tag{2.56}$$

By substituting the right-hand side of (2.56) into (2.55) and rearranging the terms, we end up with

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + r_t S_t \frac{\partial V_t}{\partial S_t} - r_t V_t = 0, \qquad (2.57)$$

which is the celebrated Black-Scholes-Merton equation (Black and Scholes, 1973; Merton, 1973). To evaluate V_t, we may solve (2.57) with the terminal condition

$$V(S_T, T) = X_T. \qquad (2.58)$$

We emphasize here that (2.57) is valid when the short rate is deterministic.

Consider the pricing of a European call option on an asset, S_t, which has the payoff

$$C_T = \left(S_T - K\right)^+ \tag{2.72}$$

at time T. Assume that the short rate is a constant, $r_t = r$. According to the Black-Scholes-Merton equation (2.57) and the terminal condition (2.58), the value of the call option satisfies

$$\begin{cases} \dfrac{\partial C_t}{\partial t} + \dfrac{1}{2}\sigma_t^2 S^2 \dfrac{\partial^2 C_t}{\partial S^2} + rS\dfrac{\partial C_t}{\partial S} - rC_t = 0 \\ C_T = \left(S - K\right)^+ \end{cases}. \tag{2.73}$$

By solving this terminal-value problem of the partial differential equation (PDE), we can obtain the price of the option.

Black-Scholes Formula by Expectation

Alternatively, we can derive the formula for the call options by working out the expectation in (2.71) directly. We write

$$\begin{aligned} C_t &= e^{-r(T-t)} E_t^{\mathbb{Q}}\left[\left(S_T - K\right)^+ \right] \\ &= e^{-r(T-t)} \left(E_t^{\mathbb{Q}}\left[S_T 1_{S_T > K} \right] - K E_t^{\mathbb{Q}}\left[1_{S_T > K} \right] \right). \end{aligned} \tag{2.74}$$

Since

$$S_T = S_t \exp\left((r - \tfrac{1}{2}\bar{\sigma}^2)\tau + \bar{\sigma}\sqrt{\tau} \cdot \varepsilon \right), \ \varepsilon \sim N(0,1), \tag{2.75}$$

where $\tau = T - t$, and $\bar{\sigma}$ is the mean volatility,

$$\bar{\sigma} = \sqrt{\frac{1}{\tau}\int_0^\tau \sigma_s^2 ds}, \tag{2.76}$$

we have

$$E_t^{\mathbb{Q}}\left[1_{S_T>K}\right]=\text{Prob}\left(\varepsilon>-\frac{\ln S_t/K+(r-\frac{1}{2}\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}\right)=N(d_2), \tag{2.77}$$

with

$$d_2=\frac{\ln S_t/K+(r-\frac{1}{2}\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}. \tag{2.78}$$

Meanwhile,

$$E_t^{\mathbb{Q}}\left[S_T 1_{S_T>K}\right] = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} S_t \exp\left((r-\tfrac{1}{2}\bar{\sigma}^2)\tau + \bar{\sigma}\sqrt{\tau}x - \tfrac{1}{2}x^2\right)dx$$
$$= \frac{S_t e^{r\tau}}{\sqrt{2\pi}}\int_{-d_2-\bar{\sigma}\sqrt{\tau}}^{\infty} \exp\left(-\tfrac{1}{2}y^2\right)dy \qquad (2.79)$$
$$= S_t e^{r\tau} N(d_1),$$

where

$$d_1 = d_2 + \bar{\sigma}\sqrt{\tau}. \qquad (2.80)$$

By inserting (2.77) and (2.79) into (2.74), we arrive at the celebrated Black-Scholes formula:

$$C_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2). \tag{2.81}$$

By direct verification, we can show that the hedge ratio, φ_t , is

$$\frac{\partial C_t}{\partial S_t} = N(d_1). \tag{2.82}$$

Next, we proceed to derive the formula for a put option, which has the payoff function

$$P_T = \left(K - S_T\right)^+ . \tag{2.83}$$

Instead of pricing the put option by taking the expectation, we make use of the so-called *call-put parity*: a long call and a short put are equivalent to a forward contract, provided that they have the same strike. As a formula, it is

$$C_t - P_t = S_t - e^{-r(T-t)}K. \tag{2.84}$$

Equality (2.84) implies the formula for the put option:

$$\begin{aligned} P_t &= C_t - \left(S_t - e^{-r(T-t)}K\right) \\ &= S_t\left(N(d_1)-1\right) - e^{-r(T-t)}K\left(N(d_2)-1\right) \\ &= e^{-r(T-t)}KN(-d_2) - S_tN(-d_1). \end{aligned} \quad (2.85)$$

The hedge ratio, φ_t, can analogously be derived to be

$$\frac{\partial P_t}{\partial S_t} = -N(-d_1). \quad (2.86)$$

The negative number means that the hedger sells short the underlying asset.