Chapter 3
Interest Rates and the Associated Securities

Rates and Instruments

- The term “interest rate” encompasses an entire class of *rates of return* that are associated with various fixed-income instruments.
- The term most frequently referred to
  - short rate for saving account,
  - rates for money market accounts,
  - U.S. Treasury yields,
  - zero-coupon yields,
  - forward rates and
  - swap rates.
- We will introduce interest rates with associated securities

3.1 Interest Rates and Fixed-Income Instruments

3.1.1 Short Rate and Money Market Accounts

- The short rate is associated with a savings account in a bank.
- The short rate at time t is conventionally denoted as r_t.
- Interest on a savings account is accrued daily, using the actual/365 convention.
- Let B_t denote the account balance at time (or date) t, and let $\Delta t = 1$ day $= 1/365$ year. Then, the new balance the next day at $t + \Delta t$ is

$$B_{t+\Delta t} = B_t \left(1 + r_t \Delta t\right). \tag{3.1}$$

Continuous Compounding

- Because $\Delta t \ll 1$, daily compounding is very well approximated by *continuous compounding*: in the limit of $\Delta t \to 0$, equation (3.1) becomes

$$dB_t = r_t B_t dt. \tag{3.2}$$

- Because r_t is applied to $(t, t+dt)$, an infinitesimal interval of time, it is also called the instantaneous interest rate.
- Solve the equation we obtain the balance at a later time, t:

$$B_t = B_0 e^{\int_0^t r_s ds}. \tag{3.3}$$

The risk-free security

- In the real world, B_t is not known in advance due to the stochastic nature of the short rate.
- Nonetheless, the deposit in the savings account is considered a risk-free security, and its return is used as a benchmark to measure the profits and losses of other investments.

3.1.2 Term Rates and Certificates of Deposit

- *Term rates* are associated to certificates of deposit (CD). A CD is a deposit that is committed for a fixed period of time, and the interest rate applied to the CD is called a term rate. For retail customers, the available terms are typically
  - 1 month,
  - 3 months,
  - 6 months and
  - 1 year.
- Usually, the longer the term, the higher the term rate, as investors are awarded a higher premium for committing their money for a longer period of time.

LIBOR Rates

- LIBOR is a set of reference interest rates at which banks lend unsecured loans to other banks in the London wholesale money market.
- The LIBOR rates are benchmark rates for certificates of deposit (CDs).

LIBOR Rates

- As of 03-02-2020,

EUR USD GBP JPY CHF

USD	03-02-2020	02-28-2020	02-27-2020	02-26-2020	02-25-2020
USD LIBOR - overnight	1.57463 %	1.56775 %	1.57400 %	1.57150 %	1.56900 %
USD LIBOR - 1 week	1.56125 %	1.56800 %	1.58500 %	1.58250 %	1.57925 %
USD LIBOR - 2 weeks	-	-	-	-	-
USD LIBOR - 1 month	1.35575 %	1.51525 %	1.58113 %	1.60338 %	1.61263 %
USD LIBOR - 2 months	1.30488 %	1.50263 %	1.59738 %	1.61900 %	1.63563 %
USD LIBOR - 3 months	1.25375 %	1.46275 %	1.58038 %	1.61325 %	1.63763 %
USD LIBOR - 4 months	-	-	-	-	-
USD LIBOR - 5 months	-	-	-	-	-
USD LIBOR - 6 months	1.19838 %	1.39725 %	1.53325 %	1.59025 %	1.62863 %
USD LIBOR - 7 months	-	-	-	-	-
USD LIBOR - 8 months	-	-	-	-	-
USD LIBOR - 9 months	-	-	-	-	-
USD LIBOR - 10 months	-	-	-	-	-
USD LIBOR - 11 months	-	-	-	-	-
USD LIBOR - 12 months	1.15388 %	1.38150 %	1.53725 %	1.61013 %	1.64575 %

Source: www.global-rates.com. The day-count convention for USD and Euro is actual/360.

Simple Compounding for CDs

- The interest payments of CDs use simple compounding.
- Let $r_{t,\Delta t}$ be the interest rate for the term Δt and I_t be the value of the deposit at time t. Then, the balance at the maturity of the CD is

$$I_{t+\Delta t} = I_t\left(1+r_{t,\Delta t}\Delta t\right). \tag{3.1}$$

- Suppose that a CD is rolled over n times. Then, the terminal balance at time $t+n\Delta t$ is

$$I_{t+n\Delta t} = I_t \cdot \prod_{i=1}^{n}(1+r_{t+(i-1)\Delta t,\Delta t}\Delta t). \tag{3.2}$$

Compounding Frequency

- We call $\omega = 1/\Delta t$ the compounding frequency, which is the number of compoundings per year. For example, when $\Delta t = 3$ months or 0.25 year, we have $\omega = 1/\Delta t = 4$, corresponding to the so-called quarterly compounding. By the way, a savings account is compound daily, corresponding to $\omega = 365$.

Effective Annual Yields (EAY)

- Different term rates mean different rates of return. One way to compare CDs of different terms is to check their effective annual yields (EAY), defined as the dollar-value return over a year for a $1 initial investment:

$$\text{effective annual yield} = \left(1 + r_{t,\Delta t}\Delta t\right)^{1/\Delta t} - 1. \qquad (3.1)$$

- Should interest rates stay constant over the investment horizon, then a higher EAY gives a higher return in value.

Money market rates

- Money market account accrues according to actual/360 convention. If a deposit is made for n days, then

$$1 \quad \rightarrow \quad 1 + \frac{r_1 + r_1 + \cdots r_n}{360}$$

- This convention applies to calculate interest payment for repurchasing agreements (repo)

3.1.3 Bonds and Bond Markets

- A bond is a financial contract that promises to pay a stream of cash flows over a certain time horizon.

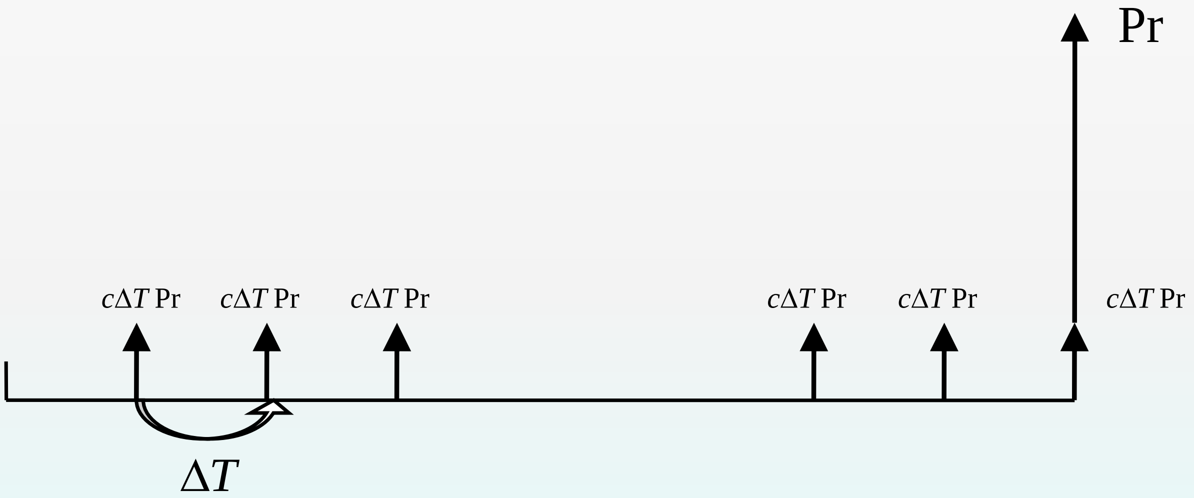


Fig. 3.1. Cash flows of a coupon bond

- We call bonds with such a cash-flow pattern bullet bonds or straight bonds.

Starting with Coupon Bonds

- Three aspects: In May 2010 the U.S. Treasury sold a bond with
  - a coupon rate of 2 $^{1}/_{8}$% and
  - a maturity date of May 31, 2015
  - a payment frequency of two a year, six months apart
- This bond is called "2 $^{1}/_{8}$s of May 31, 2015"

Coupon rate

Coupon frequency, "s" is for "semi-annual"

maturity

Cash Flow of the Bond

- The unit for bond purchasing is \$1,000.
- Suppose that an investor purchases \$1m face value of the bond, i.e., 1,000 units.
- The the coupon payment is calculated according to

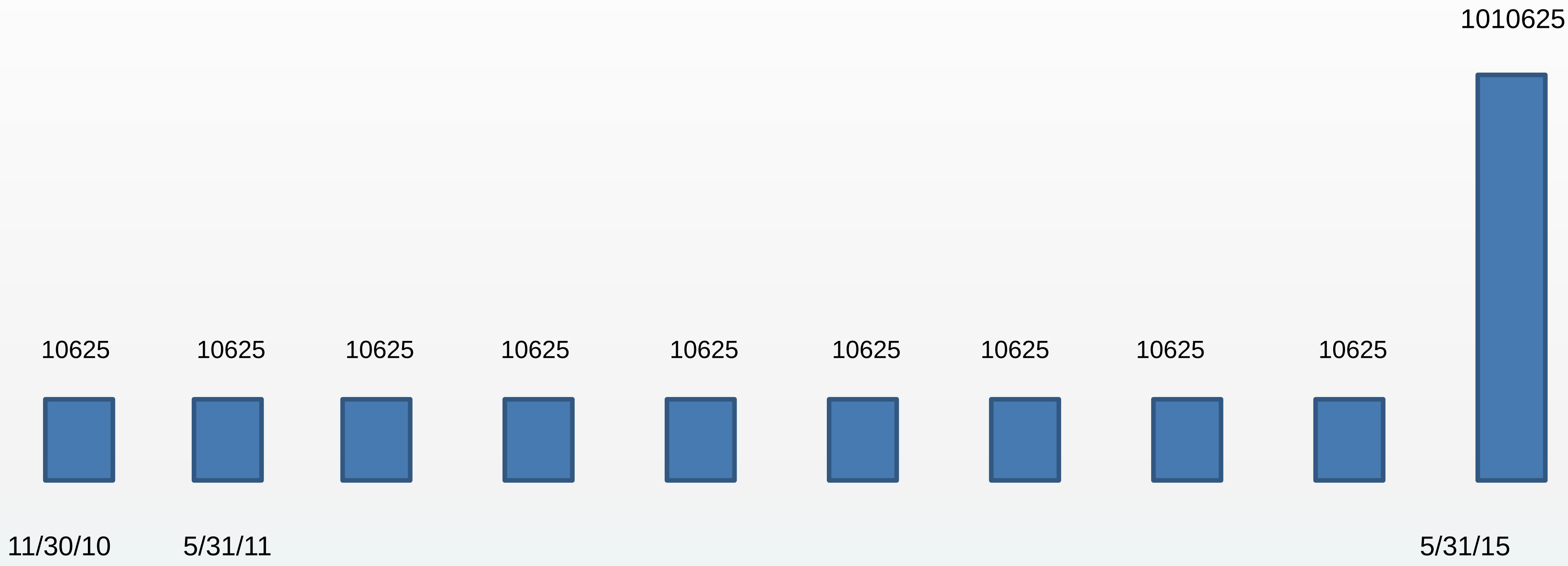
$$\underbrace{\frac{1}{2}}_{\text{Year fraction}} \times \underbrace{2\frac{1}{8}\%}_{\text{Coupon rate}} \times \underbrace{\$1{,}000{,}000}_{\text{Face value}} = \$10{,}625$$

Cash Flow of the Bond, cont'd

Table 1.1: Cash Flows of the U.S. $2^1/_8$ from books of May 31, 2015

Date	Coupon Payment	Principal Payment
"11/30/2010"	\$10, 625	
"5/31/2011"	\$10, 625	
"11/30/2011"	\$10, 625	
"5/31/2012"	\$10, 625	
"11/30/2012"	\$10, 625	
"5/31/2013"	\$10, 625	
"11/30/2013"	\$10, 625	
"5/31/2014"	\$10, 625	
"11/30/2014"	\$10, 625	
"5/31/2015"	\$10, 625	\$1, 000, 000

Cash-flow Chart



Government Bonds

- US Treasury
- Exchange Fund Bills & Notes Fixings (Hong Kong)

Type of Bonds

- Government or non-government
- Domestic or foreign
- Domestic currency or foreign currency

U.S. Bond Market

- The U.S. domestic bond markets mainly consist of three sectors:
    1. **Government bonds**: bonds issued by the Treasury Department of the U.S. government and local governments;
    2. **Agency bonds:** bonds issued by certain agencies of the U.S. government and guaranteed or sponsored by the U.S. government;
    3. **Corporate bonds**: bonds issued by U.S. corporations.
- The U.S. Treasury Department is the largest single issuer of debt in the world.

Treasury Securities

- Treasury securities include
  - Treasury bills,
  - Treasury notes,
  - Treasury bonds and
  - Treasury Inflation-Protected Securities (TIPS).

Maturities of the Treasury Issues

- Upon issuance, the maturities of
  - Treasury bills: four weeks (one month), to 13 weeks (three months) and 26 weeks (six months).
  - Treasury notes: 2, 3, 5 and 10 years
  - Treasury bonds: 30 years
- TIPS are issued with maturities
  - five years, 10 years and 20 years.
  - Thirty-year TIPS have also been issued occasionally.
- TIPS are issued with fixed coupon rates. The principal of a TIPS is adjusted semiannually according to the inflation rate.

Cycle of Auction

- Four-week bills: every Tuesday,
- 13-week and 26-week bills: every Monday.
- 2-year and 5-year notes: the end of each month,
- 3-year and 10-year notes on the fifteenth of February, May, August and November.
- The 30-year bonds are auctioned in February and August.

Cycle of Auction, cont'd

- TIPS are auctioned in different cycles.
  - 5-year TIPS are generally auctioned in the last week of April;
  - 10-year TIPS are generally auctioned in the second week of January and July; and
  - 20-year TIPS are generally auctioned in the last week of January.
- The 10-year bonds have dominated the liquidity at issuance.

Importance of Treasury Market

- The U.S. Treasury Department is the largest single issuer of debt in the world. Bonds in this sector are deemed riskless as they are guaranteed by the full faith of the U.S. government. This sector is distinguished by the large volume of total debt and the large size of any single issue.
- In terms of notional value, the Treasury bond sector is not the largest Treasuries sector, and it trails behind the agency bond sector (The Federal National Mortgage Association (Fannie Mac) and The Federal Home Loan Mortgage Corporation (Freddy Mac)).

3.1.4 Quotation and Interest Accrual

- The price of zero-coupon bonds and the price of coupon bonds are quoted using different conventions.
  - The former is quoted using a discount yield, which translates conveniently into a dollar price.
  - The latter is quoted as a percentage of the principal value, using a price tick of one-32nd of a percentage point.

Table 3.1. Quotes for U.S. Treasuries as of 3/7/2008

U.S. Treasuries

Bills

	Maturity date	Discount/Yield	Discount/Yield change
3-Month	06/05/2008	1.42 / 1.44	-0.01 / .087
6-Month	09/04/2008	1.51 / 1.54	0.03 / -.031

Notes/Bonds

	Coupon	Maturity date	Current price/Yield	Price/Yield change
2-Year	2	02/28/2010	100-29¾ / 1.52	-0-00¾ / .012
3-Year	4.75	03/31/2011	103-21¾ /1.42	-0-02 / .018
5-Year	2.75	02/28/2013	~~101-16 / 2.43~~	0-06 / -.040
10-Year	3.5	02/15/2018	99-23+ / 3.53	0-14 / -.053
30-Year	4.375	02/15/2038	97-08½ / 4.54	0-08+ / -.017

Source: http://www.bloomberg.com/markets/rates/index.html

- The dollar value of a Treasury bill is calculated using the discount yield according to the formula

$$V = \Pr\left(1 - \frac{\tau}{360} Y_d\right), \tag{3.1}$$

where τ is the number of days remaining to maturity. Suppose, for instance, that the six-month Treasury bill has a time to maturity of $\tau = 100$ days. Then, its price is

$$P = 100 \times \left(1 - \frac{100}{360} \times 1.51\%\right) = \$99.5806.$$

- Note that the discount yield is a quoting mechanism rather than a good measure of returns on an investment in a Treasury bill.

From Quotation to Prices

- A "+" stands for 0.5, or half a tick. The dollar values of the five Treasury notes and bonds are calculated as follows:
- Examples.

$$
\begin{aligned}
100\text{-}29\tfrac{3}{4} &= 100 + 29.75/32 = 100.9297, \\
103 - 21\tfrac{1}{4} &= 103 + 21.25/32 = 103.6641, \\
101\text{-}16 &= 101 + 16/32 = 101.5, \\
99\text{-}23+ &= 99 + 23.5/32 = 99.7344, \\
97\text{-}08\tfrac{1}{2} &= 97 + 8.5/32 = 97.2656.
\end{aligned}
\tag{3.8}
$$

Clean Price and Full Price

- The above dollar prices are not the prices for transactions, however. For transactions, we add the interest values accrued since the last coupon payment, which are calculated as follows.

Clean Price and Full Price

Consider a coupon bond with coupon rate c and coupon dates $\{T_i\}$. At t, a moment between two consecutive coupon dates, T_j and T_{j+1} (e.g., $T_j < t \leq T_{j+1}$), the accrued interest is

$$AI(t, T_j) = \Delta T \cdot c \cdot \Pr \cdot q,$$

where

$$q = \frac{t - T_j}{T_{j+1} - T_j}. \tag{3.1}$$

i.e., the fraction of time elapsed since the last coupon date over the current coupon period, then

$$\text{Transaction price} = \text{quote price} + AI(t, T_j).$$

An Example

Example 3.1: Consider a 10-year Treasury note maturing on February 15, 2018. On March 7, 2008, the bond quote is 99-23+ (=99.7344). We need to compute the accrued interest and then the dirty price.

The coupon dates are February 15 and August 15. There are 182 days between the coupon dates, and on March 7, twenty-one days have elapsed since February 15, the last coupon date. Hence,

$$q = 21/182 = 0.115385,$$

$$AI = 0.5 \times 3.5\% \times 100 \times 0.115385 = 0.2019.$$

Then, the full price is

$$B^c = 99.7344 + 0.2019 = 99.9363. \quad \square \qquad (3.10)$$

3.2.1 Yield to Maturity

Denote the full price of a bullet bond as B^c. Suppose that all cash flows are discounted by a uniform rate, y, then y satisfy the following equation

$$B^c = \Pr \cdot \left(\sum_{i=1}^{n} \frac{c\Delta T}{\left(1 + y\Delta T\right)^i} + \frac{1}{\left(1 + y\Delta T\right)^n} \right), \tag{3.1}$$

where n is the number of coupons.

This discount rate, which can be easily solved by a trial-and-error procedure using (3.1), is defined to be the *yield to maturity* (YTM), as well as the *internal rate of return* (IRR) of the bond, and it is often simply called the *bond yield*.

3.2.1 Yield to Maturity

As the function of the yield, the formula for a general time, $t \le T$, is

$$B_t^c = \Pr \cdot \left(\sum_{i;i\Delta T > t}^{n} \frac{c\Delta T}{\left(1 + y\Delta T\right)^{(i\Delta T - t)/\Delta T}} + \frac{1}{\left(1 + y\Delta T\right)^{(n\Delta T - t)/\Delta T}} \right). \qquad (3.1)$$

Assuming that $t \in (T_j, T_{j+1}]$, and introducing

$$q = \frac{t - T_j}{T_{j+1} - T_j} = \frac{t - T_j}{\Delta T},$$

we then can write

$$t = T_j + \Delta T q = (j + q)\Delta T, \text{ and } i\Delta T - t = (i - j - q)\Delta T, \ \forall i.$$

Bond Price Formula

It follows that

$$
\begin{aligned}
B_t^c &= \Pr\cdot\left(\sum_{i=j+1}^{n}\frac{c\Delta T}{\left(1+y\Delta T\right)^{i-j-q}}+\frac{1}{\left(1+y\Delta T\right)^{n-j-q}}\right) \\
&= \Pr\cdot\left(1+y\Delta T\right)^{q}\left(\sum_{i=1}^{n-j}\frac{c\Delta T}{\left(1+y\Delta T\right)^{i}}+\frac{1}{\left(1+y\Delta T\right)^{n-j}}\right).
\end{aligned}
\tag{3.1}
$$

Given the bond price at any time, t, the bond yield is implied by (3.1)

One-to-One Price–Yield Relationship

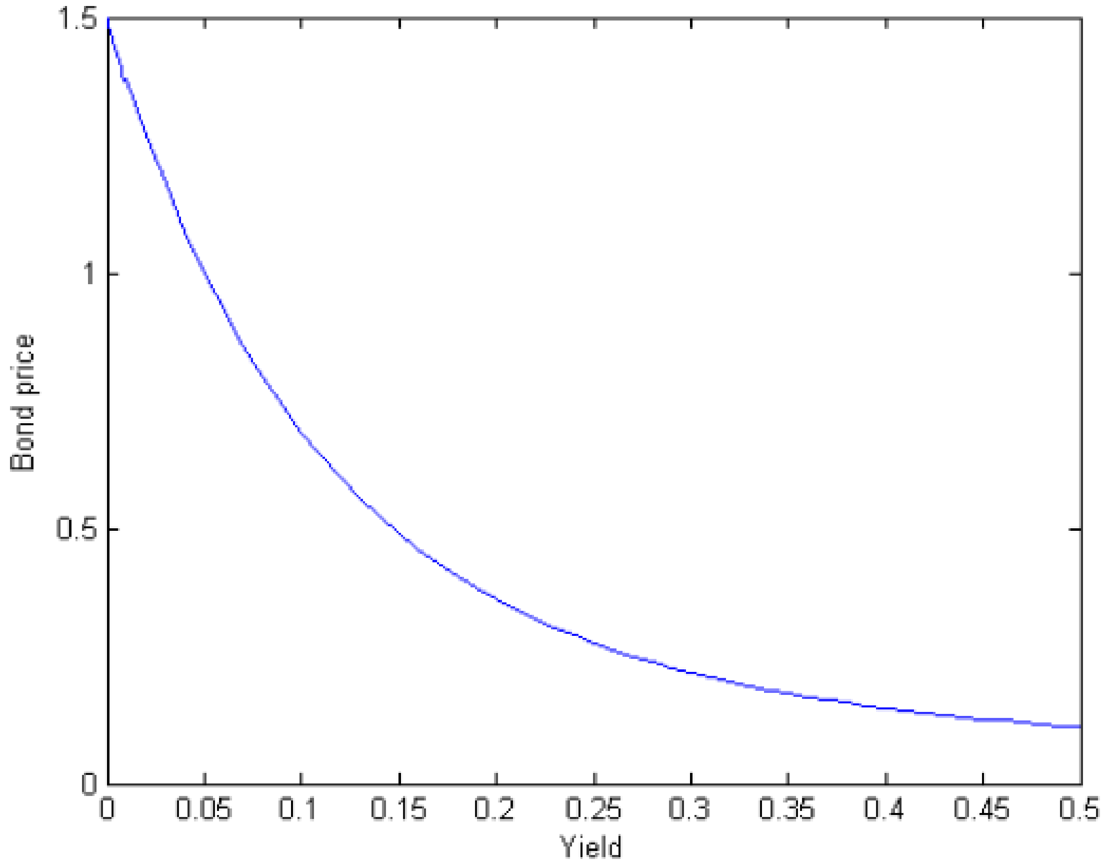


Fig. 3.2. The price–yield relationship for a 10-year bond with $c = 0.05,\ \Delta T = 0.5$

Bond Equivalent Yields

The price–yield relationship of a zero-coupon bond simplifies to

$$P = \Pr \times \left(1 + y\Delta T\right)^{-\frac{T-t}{\Delta T}}.$$

To derive an approximate value of the yield, we consider the following approximation of the "return on the investment",

$$\frac{\Pr - P}{P} = \left(1 + y\Delta T\right)^{\frac{T-t}{\Delta T}} - 1 \approx y \times (T - t), \qquad (3.1)$$

using the Taylor expansion. Equation (3.1) gives rise to an approximate yield-to-maturity:

$$y \approx \frac{1}{(T-t)} \frac{\Pr - P}{P},$$

which is also called the bond equivalent yield.

Bond Equivalent Yields, cont'd

Note that for Treasury zero-coupon bonds, the year has 365 days, meaning that

$$T - t \approx \frac{\tau}{365}, \tag{3.1}$$

where τ is the number of days to maturity.

3.2.2 Par Bonds, Par Yields and the Par Yield Curve

The summation in equation (3.12) can be worked out so that

$$
\begin{aligned}
B^c &= \Delta T \cdot c \cdot \Pr \sum_{i=1}^{n} \left(1 + y\Delta T\right)^{-i} + \Pr\left(1 + y\Delta T\right)^{-n} \\
&= \Pr\left[1 - \left(1 - \frac{c}{y}\right)\left(1 - \frac{1}{\left(1 + y\Delta T\right)^{n}}\right)\right].
\end{aligned}
\tag{3.17}
$$

From the above expression, we can tell when the price is smaller, equal to or larger than the principal value.

1. When $c < y$, $B^c < \Pr$. In such a case, we say that the bond is sold at discount (of the par value).

2. When the coupon rate is $c = y$, then $B^c = \Pr$, i.e., the bond price equals the par value of the bond. In such a case, we call the bond a *par bond,* and the corresponding coupon rate a *par yield*.

3. When $c > y$, $B^c > \Pr$. In such a case, we call the bond a premium bond (it is traded at a premium to par).

$$B^c = \left[\underbrace{\sum_{i=1}^{n} \frac{\delta T c}{(1+y\delta T)^i}} + \frac{1}{(1+y\delta T)^n} \right] \times Pr$$

$$z = \frac{1}{1+y\delta T}$$

$$B^c = Pr \times \left[\delta T c \sum_{i=1}^{n} z^i + z^n \right]$$

$$\sum_{i=1}^{n} z^i = z + z^2 + \cdots + z^n$$

$$= z \underbrace{\left(1 + z + \cdots + z^{n-1} \right)}_{\text{Von Neumann series}}$$

$$1 + z + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$$

$$B^c = Pr \times \left[\delta T \cdot c \cdot z \cdot \frac{1-z^n}{1-z} + z^n \right]$$

$$= Pr \times \left[\delta T \cdot c \cdot \frac{1}{(1+y\delta T)} \frac{1-z^n}{1-\frac{1}{(1+y\delta T)}} + z^n \right]$$

$$= Pr \times \left[\delta T \cdot c \, \frac{1-z^n}{1+y\delta T-1} + z^n \right]$$

$$= P_r \times \left[\frac{c}{y} \left(1 - z^n\right) + z^n \right]$$

$$= P_r \times \left[\frac{c}{y} + \left(1 - \frac{c}{y}\right) z^n \right]$$

$$= P_r \times \left[1 + \left(\frac{c}{y} - 1\right) + \left(1 - \frac{c}{y}\right) z^n \right]$$

$$= P_r \times \left[1 + \left(\frac{c}{y} - 1\right)\left(1 - z^n\right) \right]$$

$$= P_r \times \left[1 + \left(\frac{c}{y} - 1\right)\left(1 - \frac{1}{(1 + y\sigma T)^n}\right) \right] \#$$