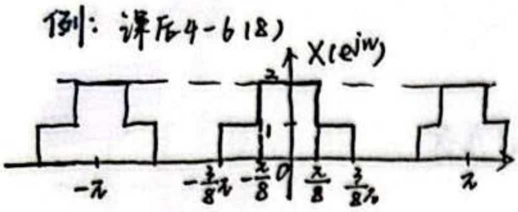
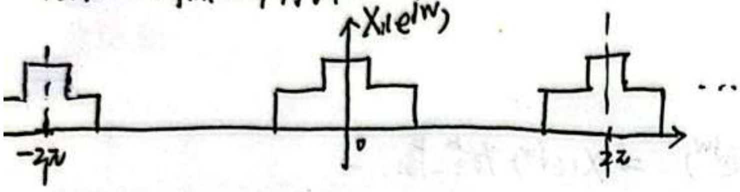
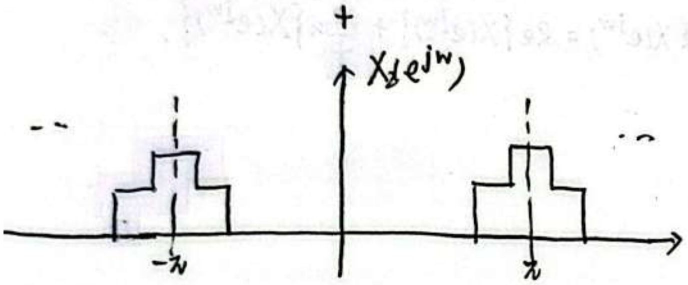
例：课后4-6(18)



法1：拆成2个序列：



+



易得：

$$X_2(e^{jw}) = X_1(e^{j(w-\pi)})$$

$$\therefore x_2[n] = e^{j\pi n}x_1[n] = (-1)^n x_1[n]$$

$$\therefore x[n] = [1+(-1)^n]\cdot x_1[n]$$

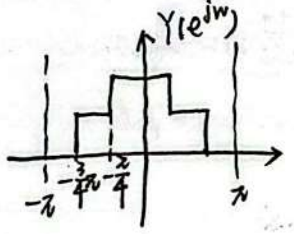
求 $x_1[n]$：叠加.

$$x_1[n] = \frac{\sin(\frac{\pi}{8}n)}{\pi n} + \frac{\sin(\frac{3\pi}{8}n)}{\pi n}$$

$$\therefore x[n] = [1+(-1)^n]\cdot\frac{\sin(\frac{\pi}{8}n)+\sin(\frac{3\pi}{8}n)}{\pi n}$$ ✓

法2：

先求 ? $\xrightarrow{F}$



由时域扩展，$X(e^{jw}) = Y(e^{j\cdot 2w})$

又 $\because y[n] = \frac{\sin(\frac{\pi}{4}n)+\sin(\frac{3}{4}\pi n)}{\pi n}$

$$\Rightarrow x[n] = y_{(2)}[n] = \begin{cases} \frac{2[\sin(\frac{\pi}{8}n)+\sin(\frac{3}{8}\pi n)]}{\pi n}, & n\text{为偶} \\ 0, & n\text{为奇数} \end{cases}$$ ✓

两种方法得到的结果一致.

⑦频域微分性质

若 $x[n] \xrightarrow{F} X(e^{jw})$，那么 $nx[n] \xrightarrow{F} j\frac{dX(e^{jw})}{dw}$

证明：$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$ $\quad\therefore \frac{dX(e^{jw})}{dw} = \frac{d(\sum_{-\infty}^{+\infty} x[n]e^{-jwn})}{dw} = \sum_{-\infty}^{+\infty}(-jnx[n])e^{-jwn}$

$\therefore -jnx[n] \xrightarrow{F} \frac{d(X(e^{jw}))}{dw}$ $\quad\therefore nx[n] \xrightarrow{F} j\cdot\frac{d(X(e^{jw}))}{dw}$

推论：$x[n]$ 实偶 $\underset{F^{-1}}{\overset{F}{\rightleftarrows}}$ $X(e^{jw})$ 实偶

$x[n]$ 实奇 $\xrightarrow{F}$ $X(e^{jw})$ 虚奇 ；$X(e^{jw})$ 实奇 $\xrightarrow{F^{-1}}$ $x[n]$ 虚奇

推导： $x[n]$ 实偶，有 $x[n] = x^*[n]$

$$\downarrow F \quad \downarrow F$$

$$X(e^{jw}) = \underbrace{X^*(e^{-jw}) = X^*(e^{jw})}_{\text{由偶}\to\text{偶}} \Rightarrow X(e^{jw}) \text{ 为实}$$

$x[n]$ 实奇， $x[n] = x^*[n]$

$$\downarrow F \quad \downarrow F$$

$$X(e^{jw}) = X^*(e^{-jw}) = -X^*(e^{jw}) \Rightarrow X(e^{jw}) \text{ 为纯虚.}$$

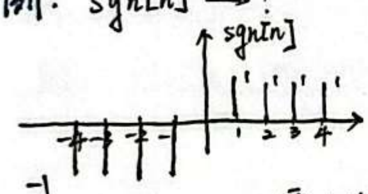
补充定理：若 $x[n]$ 是实函数， $x[n] \xrightarrow{F} X(e^{jw})$，设 $X(e^{jw}) = Re\{X(e^{jw})\} + jIm\{X(e^{jw})\}$，

那么 <u>$Re\{X(e^{jw})\}$ 是偶函数，$Im\{X(e^{jw})\}$ 是奇函数</u>

定理2： $x[n] \xrightarrow{F} X(e^{jw}) = |X(e^{jw})| e^{j\theta(w)}$

那么幅度谱 $|X(e^{jw})|$ 为实偶，相位 $\theta(w)$ 为奇.

例： $sgn[n] \xrightarrow{F}$?



解： $sgn[n] = u[n] - u[-n]$

$$u[n] \xrightarrow{F} \frac{1}{1-e^{-jw}} + \pi \sum_{-\infty}^{+\infty} \delta(w-2k\pi)$$

∴要计算虚部.

$$\because \frac{1}{1-e^{-jw}} = \frac{1}{1-\cos w + j\sin w} = \frac{1-\cos w - j\sin w}{(1-\cos w)^2 + (\sin w)^2}$$

$$\therefore sgn[n] = 2 \cdot \frac{-j\sin w}{2-2\cos w} = \frac{-j\sin w}{1-\cos w}$$

可以进一步化简，原式 $= \dfrac{-j \cdot 2\sin\frac{w}{2}\cos\frac{w}{2}}{2\sin^2\frac{w}{2}} = -\dfrac{j}{\tan\frac{w}{2}}$

⑥时域扩展：

若 $x[n] \xrightarrow{F} X(e^{jw})$，则有 $x_{(k)}[n] \xrightarrow{F} X(e^{jwk})$

其中 $x_{(k)}[n]$ 是 $x[n]$ 的时域扩展，

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], & \text{当 } k \mid n \text{ 时} \\ 0, & \text{当 } k \nmid n \text{ 时} \end{cases}$$

（证明简单，略）

Tips：若 $X(e^{jw})$ 以 2π 为周期 $\Rightarrow$ $X(e^{jwk})$ 以 $\frac{2\pi}{k}$ 为周期.

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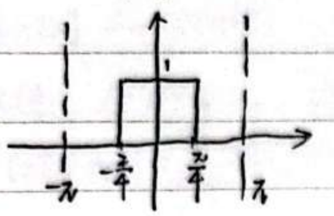
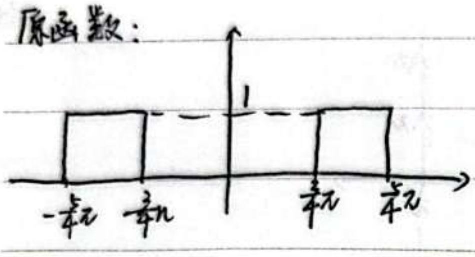
△法二：平移



$\xrightarrow{F^{-1}} \frac{\sin(\frac{\pi}{4}n)}{\pi n}$

原函数：



∴右移π.

∴ $e^{j\pi n}\cdot\frac{\sin(\frac{\pi}{4}n)}{n}$

又有：$e^{j\pi n}=(-1)^n=\cos(\pi n)=\underline{e^{-j\pi n}}$ （本题中左/右移π等价）

∴得到：$(-1)^n\cdot\frac{\sin(\frac{\pi}{4}n)}{n}$

·证明两种方式得到的结果一样：即 $\delta[n]-\frac{\sin(\frac{3}{4}\pi n)}{\pi n}=(-1)^n\cdot\frac{\sin(\frac{\pi}{4}n)}{n}$

① 当 $n=0$ 时，左边 $=1-\frac{3}{4}=\frac{1}{4}$，右式 $=\frac{1}{4}$ ✓

② 当 $n\neq 0$ 时，$\sin(\frac{3}{4}\pi n)=\sin(\pi n-\frac{1}{4}\pi n)=\underbrace{\sin(\pi n)\cos(\frac{\pi}{4}n)}_{=0}-\cos(\pi n)\sin(\frac{\pi}{4}n)$

又∵ $\cos(\pi n)=(-1)^n$ ∴原式 $=(-1)^n\sin(\frac{\pi}{4}n)$

④ 时域翻转

若 $x[n]\xrightarrow{F}X(e^{jw})$，那么 $x[-n]\xrightarrow{F}X(e^{-jw})$

性质：偶 $(x[n]) \underset{F^{-1}}{\overset{F}{\rightleftarrows}}$ 偶 $(X(e^{jw}))$

奇 $(x[n]) \underset{F^{-1}}{\overset{F}{\rightleftarrows}}$ 奇 $(X(e^{jw}))$

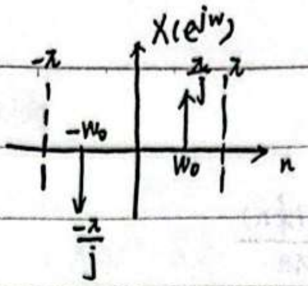
⑤ 共轭和共轭对称性 （注：$(a+jb)^*=a-jb$）

若 $x[n]\xrightarrow{F}X^*(e^{jw})$，则 $x^*[n]\xrightarrow{F}X^*(e^{-jw})$

推导：$F[x^*[n]]=\sum_{n=-\infty}^{+\infty}x^*[n]e^{-jwn}=\left[\sum_{n=-\infty}^{+\infty}x[n]e^{jwn}\right]^*$

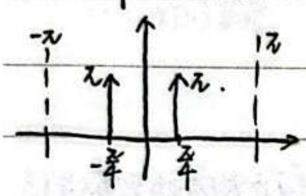
（用到性质：$(ab)^*=a^*b^*$，$(a+b)^*=a^*+b^*$，$(e^{jwn})^*=e^{-jwn}$）

$\sin(W_0 n) = \dfrac{e^{jW_0 n} - e^{-jW_0 n}}{2j} \xrightarrow{F}$

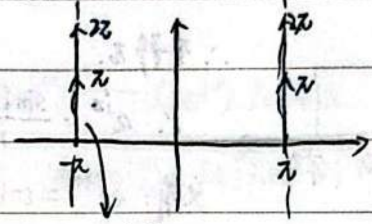


上面限制了 $0<W_0<\pi$，但应当可以进一步推广：

① $\cos(\frac{\pi}{4}n) \xrightarrow{F} ?$



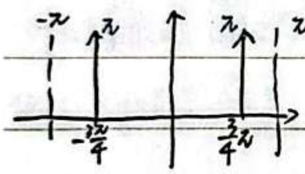
⑤ $\cos(\pi n) \xrightarrow{F}$



△会叠加

② $\cos(\frac{5}{4}\pi n) \xrightarrow{F} ?$

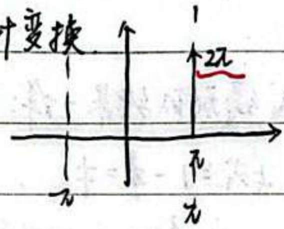
由周期性，$\cos(\frac{5}{4}\pi n) = \cos(-\frac{3}{4}\pi n) = \cos(\frac{3}{4}\pi n)$



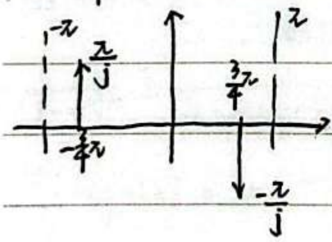
实际上，$\boxed{\cos(\pi n) = (-1)^n} = e^{j\pi n} = e^{-j\pi n}$

由1的傅立叶变换

$\cos(\pi n) \xrightarrow{F}$



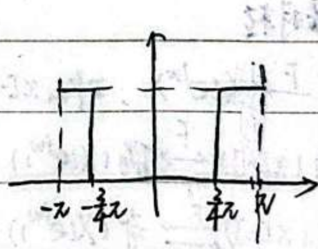
③ $\sin(\frac{3}{4}\pi n) \xrightarrow{F}$



$\therefore (-1)^n \xrightarrow{F} 2\pi \sum_{-\infty}^{+\infty} \delta(w-(2k+1)\pi)$

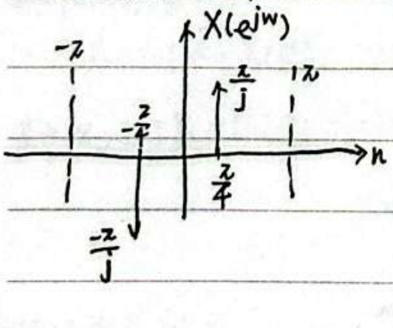
△例题：

$? \xrightarrow{F}$

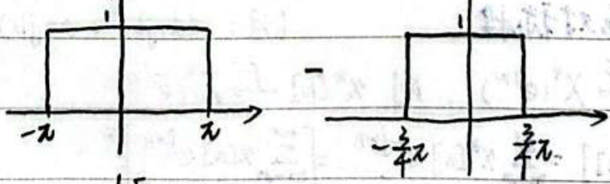


④ $\sin(\frac{9}{4}\pi n) \xrightarrow{F} ?$

由周期性，$\sin(\frac{9}{4}\pi n) = \sin(\frac{\pi}{4}n)$



△法1：线性变换



$\downarrow F$

$\delta[n] \quad - \quad \dfrac{\sin(\frac{3}{4}\pi n)}{\pi n}$

($T=2\pi$，∴函数一直是1)

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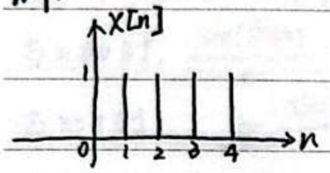
② 时域平移

若 $x[n] \xrightarrow{F} X(e^{jw})$，则 $x[n-n_0] \xrightarrow{F} e^{-jwn_0} X(e^{jw})$

证明：令 $n' = n-n_0$，那么 $n = n_0+n'$

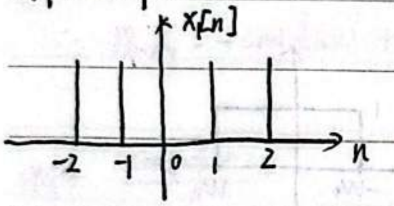
$$F[x[n-n_0]] = \sum_{n=-\infty}^{+\infty} x[n-n_0] e^{-jwn} = \sum_{n=-\infty}^{+\infty} x[n'] e^{-jwn'} \cdot e^{-jwn_0} = e^{-jwn_0} X(e^{jw})$$

例：



$\xrightarrow{F}$?

解：先算：



$N_1 = 2$

$$\xrightarrow{F} \frac{\sin(\frac{5}{2}w)}{\sin(\frac{1}{2}w)} = X_1(e^{jw})$$

$$x[n] = x_1[n-2] \xrightarrow{F} X_1(e^{jw}) \cdot e^{-j2w} = \frac{\sin(\frac{5}{2}w)}{\sin(\frac{1}{2}w)} \cdot e^{-j2w}$$

③ 频域平移

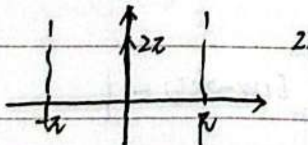
若 $x[n] \xrightarrow{F} X(e^{jw})$，则 $e^{jw_0 n} x[n] \xrightarrow{F} X(e^{j(w-w_0)})$

例：若 $0 < w_0 < \pi$.

$\cos(w_0 n) \xrightarrow{F}$?，$\sin(w_0 n) \xrightarrow{F}$?

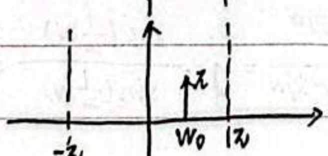
解：$\cos(w_0 n) = \dfrac{e^{jw_0 n} + e^{-jw_0 n}}{2}$

由于 $1 \xrightarrow{F}$



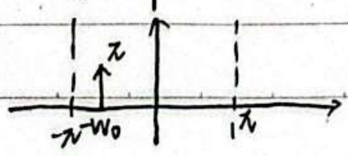
$2\pi \sum_{k=-\infty}^{+\infty} \delta(w-2k\pi)$

$\therefore \frac{1}{2} e^{jw_0 n} \xrightarrow{F}$



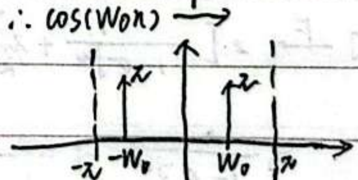
$\pi \sum_{k=-\infty}^{+\infty} \delta(w-w_0-2k\pi)$

$\frac{1}{2} e^{-jw_0 n} \xrightarrow{F}$



$\pi \sum_{k=-\infty}^{+\infty} \delta(w+w_0-2k\pi)$

$\therefore \cos(w_0 n) \xrightarrow{F}$



$$= \pi \sum_{k=-\infty}^{+\infty} \left[\delta(w-w_0-2k\pi) + \delta(w+w_0-2k\pi)\right]$$

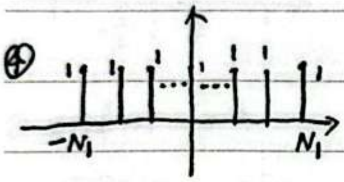
离散/连续的傅里叶变换公式汇总.

离散　　　　连续

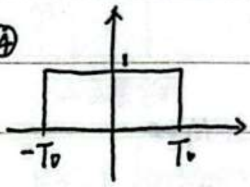
① $a^n u[n] \xrightarrow{F} \frac{1}{1-ae^{-jw}}$ $(|a|<1)$　　① $e^{-at}u(t) \xrightarrow{F} \frac{1}{a+jw}$ $(a>0)$

② $\delta[n] \xrightarrow{F} 1$　　② $\delta(t) \xrightarrow{F} 1$

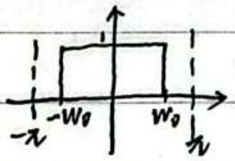
③ $1 \xrightarrow{F} 2\pi\sum_{k=-\infty}^{+\infty}\delta(w-2k\pi)$　　③ $1 \xrightarrow{F} 2\pi\delta(w)$

④ 

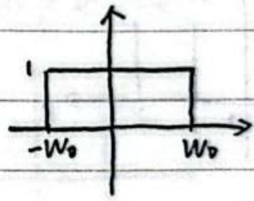
$\xrightarrow{F} \frac{\sin[(N_1+\frac{1}{2})w]}{\sin(\frac{1}{2}w)}$

④ 

$\xrightarrow{F} 2T_0\,Sa(T_0w)$

⑤ $\frac{\sin(W_0n)}{\pi n} \xrightarrow{F}$ 

$(0<W_0<\pi)$

⑤ $\frac{\sin(W_0t)}{\pi t} \xrightarrow{F}$ 

⑥ $u[n] \xrightarrow{F} \frac{1}{1-e^{-jw}}+\pi\sum_{k=-\infty}^{+\infty}\delta(w-2k\pi)$　　⑥ $u(t) \xrightarrow{F} \frac{1}{jw}+\pi\delta(w)$

离散傅里叶变换性质

① 线性: $x_1[n] \xrightarrow{F} X_1(e^{jw})$, $x_2[n] \xrightarrow{F} X_2(e^{jw})$

$\Rightarrow ax_1[n]+bx_2[n] \xrightarrow{F} aX_1(e^{jw})+bX_2(e^{jw})$

例: 计算 $sgn[n]=\begin{cases}1, & n>0\\ 0, & n=0\\ -1, & n<0\end{cases}$ 的DFT.

解: $sgn[n]=2u[n]-1-\delta[n]$

$\therefore sgn[n] \xrightarrow{F} 2\left[\frac{1}{1-e^{-jw}}+\pi\sum_{k=-\infty}^{+\infty}\delta(w-2k\pi)\right]-2\pi\sum_{k=-\infty}^{+\infty}\delta(w-2k\pi)-1$

$=\frac{2}{1-e^{-jw}}-1=\frac{e^{-jw}+1}{1-e^{-jw}}=\frac{e^{\frac{1}{2}jw}+e^{-\frac{1}{2}jw}}{e^{\frac{1}{2}jw}-e^{-\frac{1}{2}jw}}=\frac{1}{j}\cdot\frac{\cos(\frac{1}{2}w)}{\sin(\frac{1}{2}w)}=\frac{-j}{\tan(\frac{1}{2}w)}$

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(因此，我们已可以处理 $-\pi \le w \le \pi$)。

例3：$\frac{\sin(\frac{5}{4}\pi n)}{n} \xrightarrow{F} ?$

解：利用周期性，$\sin(x+2k\pi) = \sin x$

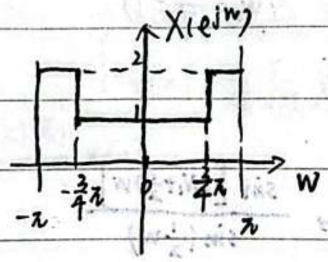
$\therefore \sin(\frac{5}{4}\pi n) = \sin(\frac{5}{4}\pi n - 2\pi n) = -\sin(\frac{3}{4}\pi n)$

当 $n \neq 0$ 时，$\frac{\sin(\frac{5}{4}\pi n)}{n} = -\frac{\sin(\frac{3}{4}\pi n)}{n}$

当 $n = 0$ 时，$\lim_{n\to+\infty}\frac{\sin(\frac{5}{4}\pi n)}{n} = \frac{5}{4}$，而 $\lim_{n\to+\infty} -\frac{\sin(\frac{3}{4}\pi n)}{n} = -\frac{3}{4}$

(利用 $\lim_{n\to+\infty}\frac{\sin(W_0 n)}{n} = \frac{W_0}{\pi}$)

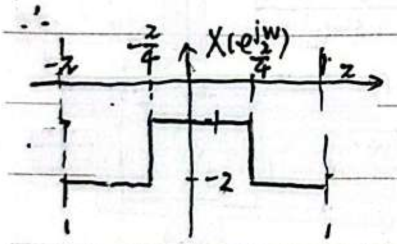
$\therefore$ 原式 $= -\sin(\frac{3}{4}\pi n) + 2\delta[n] \xrightarrow{F}$



例4：$\frac{\sin(-\frac{7}{4}\pi n)}{n}$

解：$\frac{\sin(-\frac{7}{4}\pi n)}{n} = \frac{\sin(\frac{1}{4}\pi n)}{n} - 2\delta[n]$

$\therefore$



例5：$\frac{\sin(\pi n)}{\pi n} = \delta[n] \neq 0 \xrightarrow{F} 1$

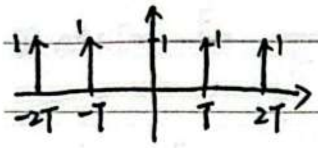
$\frac{\sin(3\pi n)}{\pi n} = 3\delta[n] \xrightarrow{F} 3$

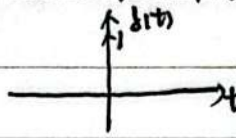
6. $u[n] \xrightarrow{F} \frac{1}{1-e^{-jw}} + \pi\sum_{k=-\infty}^{+\infty}\delta(w-2k\pi)$

从左→右：

第三章中我们讲过，$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$，求傅里叶变换，方法如下（周期傅里叶变换）

① 求一个周期傅里叶变换





$\xrightarrow{F} X(jw)=1$

② $a_k=\frac{1}{T}X(jkW_0)=\frac{1}{T}$

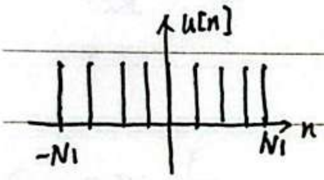
③ $p(t)=\sum_{-\infty}^{+\infty}a_k e^{jkW_0t}=\frac{1}{T}\sum_{-\infty}^{+\infty}e^{jkW_0t}$

设 $T=2\pi$，$p(t)=\frac{1}{2\pi}\sum_{-\infty}^{+\infty}e^{jkt}$.　$(W_0=\frac{2\pi}{T}=1)$

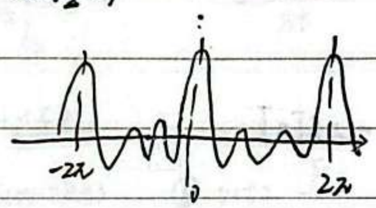
将 t 代换为 $-W$，$k\to n$

$\therefore 2\pi p(-W)=\sum_{-\infty}^{+\infty}e^{-jWn}$.

4. $u[n+N_1]-u[n-N_1-1]\xrightarrow{F}\frac{\sin\left[(N_1+\frac{1}{2})W\right]}{\sin(\frac{1}{2}W)}$



$\xrightarrow{F}$



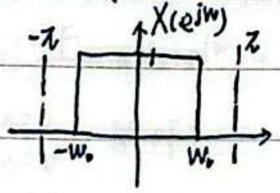
解：$X(e^{jw})=\sum_{n=-N_1}^{N_1}e^{-jwn}=1+2\cos(W_0)+2\cos(2W_0)+\cdots+2\cos(N_1W_0)=\frac{\sin[(N_1+\frac{1}{2})W_0]}{\sin(\frac{1}{2}W)}$

证明：将 $\sin(\frac{1}{2}W)$ 乘到左边　$2\cos(nW_0)\sin(\frac{1}{2}W)=\sin(\frac{1}{2}+n)W_0-\sin(n-\frac{1}{2})W_0$.

$\therefore$ 左式 $=\cancel{\sin(\frac{1}{2}W)}+\cancel{\sin(\frac{3}{2}W)}-\cancel{\sin(\frac{1}{2}W)}\ \cdots\cdots+\sin[(N_1+\frac{1}{2})W_0]-\cancel{\sin[(N_1-\frac{1}{2})W_0]}=\sin[(N_1+\frac{1}{2})W_0]$

（易证以 2π 为周期）

5. $X[n]=\frac{\sin(W_0n)}{\pi n}\xrightarrow{F}$

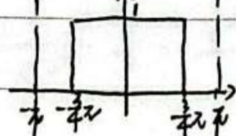


$(0<W_0<\pi)$

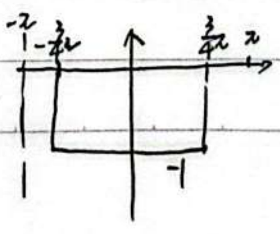
解：右→左．$x[n]=\frac{1}{2\pi}\int_{-W_0}^{W_0}X(e^{jw})e^{jwn}dw$

$=\frac{1}{2\pi}\cdot\frac{1}{jn}e^{jwn}\Big|_{-W_0}^{W_0}=\frac{\sin(W_0n)}{\pi n}$

例1：$\frac{\sin(\frac{3}{4}\pi n)}{\pi n}\xrightarrow{F}$



例2：$\frac{\sin(-\frac{3}{4}\pi n)}{\pi n}=-\frac{\sin(\frac{3}{4}\pi n)}{\pi n}\xrightarrow{F}$



例：求 $(n+1)a^n u[n] \xrightarrow{F} ?$

解：∵ $a^n u[n] \xrightarrow{F} \frac{1}{1-ae^{-jw}}$ 由频域微分性质

∴ $na^n u[n] \xrightarrow{F} j\cdot\frac{d(\frac{1}{1-ae^{-jw}})}{dw} = \frac{ae^{-jw}}{(1-ae^{-jw})^2}$

∴ $(n+1)a^n u[n] = na^n u[n] + a^n u[n] = \frac{1}{1-ae^{-jw}} + \frac{ae^{-jw}}{(1-ae^{-jw})^2} = \frac{1}{(1-ae^{-jw})^2}$

易证：$(n+2)(n+1)a^n u[n] \xrightarrow{F} \frac{1}{(1-ae^{-jw})^3}$

△推论：$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n] \xrightarrow{F} \frac{1}{(1-ae^{-jw})^r}$

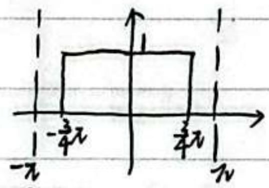
⑧帕斯瓦尔定理

若 $x[n] \xrightarrow{F} X(e^{jw})$，$\sum_{-\infty}^{+\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(e^{jw})|^2 dw$

回顾连续：若 $x(t) \xrightarrow{F} X(jw)$，则 $\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(jw)|^2dw$

例：计算 $\sum_{n=-\infty}^{+\infty}\left[\frac{\sin(\frac{3}{4}\pi n)}{\pi n}\right]^2$

解：$x[n] = \frac{\sin(\frac{3}{4}\pi n)}{\pi n} \xrightarrow{F}$



∴ $\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{jw})|^2dw = \frac{1}{2\pi}\cdot\frac{3}{2}\pi = \frac{3}{4}$

例2：$x[n] \xrightarrow{F} X(e^{jw})$

① $x[n]=0,\ n>0$， ③ $Im[X(e^{jw})] = \sin(w)-\sin(2w)$

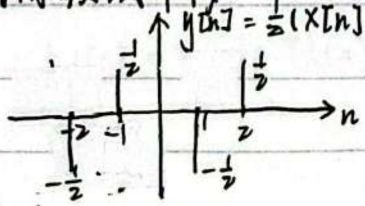
② $x[0]>0$ ④ $\int_{-\pi}^{\pi}|X(e^{jw})|^2dw = 6\pi$

求 $x[n]$.

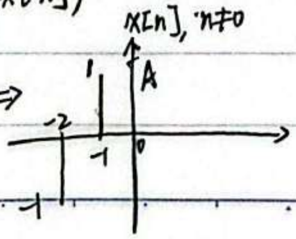
解：利用 $x_o[n] = \frac{x[n]-x[-n]}{2} \xrightarrow{F} jIm\{X(e^{jw})\} = j\cdot[\sin(w)-\sin(2w)] = j\frac{e^{jw}-e^{-jw}}{2j} - j\frac{e^{j2w}-e^{-j2w}}{2j}$

$\frac{1}{2}(x[n]-x[-n]) \xrightarrow{F} \frac{1}{2}e^{jw} - \frac{1}{2}e^{-jw} - \frac{1}{2}e^{j2w} + \frac{1}{2}e^{-j2w}$

利用频域平移： 再由(4)，$\sum_{-\infty}^{+\infty}|x[n]|^2 = 3$



⇒



∴ $1+1+A^2=3$ $A^2=1$ $A>0$

∴ $A=1$