

**Problem 1.5.10:** Determine the equilibrium temperature distribution inside a circle ( $r \leq r_0$ ) if the boundary is fixed at a temperature  $T_0$ .

In order to find the equilibrium temperature distribution  $u(r, \theta)$  inside a circle we can use the 2D Polar Laplace Equation (No sources/sinks) at equilibrium ( $\nabla^2 u = 0$ ):

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Since the boundary of the circle is at a fixed (constant) temperature  $T_0$ , the temperature inside the circle does not depend on the angular coordinate  $\theta$ , which means that the equilibrium temperature distribution  $u(r)$  will also not depend on  $\theta$  ( $\frac{\partial u}{\partial \theta} = 0$ ). With this in mind, we can simplify the Laplace equation and integrate twice to find the general solution to the simplified ODE:

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) &= 0 \\ \frac{d}{dr} \left( r \frac{du}{dr} \right) &= 0 \\ r \frac{du}{dr} &= C_1 \\ \frac{du}{dr} &= \frac{C_1}{r} \\ u(r) &= C_1 \ln(r) + C_2 \end{aligned}$$

Now it's time to apply the boundary conditions. In order for the temperature to be defined at the center of the circle ( $r = 0$ ),  $C_1 = 0$  must be true so that the term  $C_1 \ln(r)$  goes away (because  $\ln(0)$  is undefined). So with  $C_1 = 0$  we can simplify the equation to get:  $u(r) = C_2$ . In addition, the temperature at the boundary of the circle  $r = r_0$  is a constant  $T_0$  so we know that  $u(r_0) = T_0$ . But if  $u(r) = C_2$  and  $u(r_0) = T_0$ , then that means that  $C_2 = T_0$ , which yields a final equilibrium temperature distribution of  $u(r) = T_0$ .

**Problem 1.5.11:** Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \quad a < r < b,$$

subject to  $u(r, 0) = f(r)$ ,  $\frac{\partial u}{\partial r}(a, t) = \beta$ , and  $\frac{\partial u}{\partial r}(b, t) = 1$ . For what value(s) of  $\beta$  does an equilibrium temperature distribution exist?

In order to find the value(s) of  $\beta$  for which an equilibrium temperature distribution exists, we need to follow a similar procedure as the previous problem (1.5.10) (First simplify PDE with given information, next integrate twice to find the general solution to the ODE, then plug in boundary conditions):

$$\begin{aligned} \frac{k}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) &= 0 \\ \frac{d}{dr} \left( r \frac{du}{dr} \right) &= 0 \\ r \frac{du}{dr} &= C_1 \\ \frac{du}{dr} &= \frac{C_1}{r} \end{aligned}$$

$$u(r) = C_1 \ln(r) + C_2$$

Which is the same general solution as the previous problem (1.5.10), however, this time our boundary conditions are different. We know that  $\frac{\partial u}{\partial r}(a) = \beta$  and  $\frac{\partial u}{\partial r}(b) = 1$ , so we can plug these boundary conditions into  $\frac{\partial u}{\partial r}(r)$  and solve for  $\beta$  to find where the equilibrium temperature distribution exists:

$$u'(a) = \frac{C_1}{a} = \beta$$

$$u'(b) = \frac{C_1}{b} = 1$$

$$C_1 = a\beta$$

$$\frac{a\beta}{b} = 1$$

$$a\beta = b$$

$$\beta = \frac{b}{a}$$

So, the equilibrium temperature distribution only exists when  $\beta = \frac{b}{a}$ .

**Problem 2.2.1:** Show that any linear combination of linear operators is a linear operator.

An operator  $L$  is linear if it satisfies both the sum and scalar multiplication definitions of a linear operator. To satisfy both definitions of a linear operator, the superposition principle:

$$L(au + bv) = aL(u) + bL(v)$$

must hold for any arbitrary scalars  $a$  and  $b$ , and any arbitrary vectors  $u$  and  $v$  in the operator's domain. Let  $L_1$  and  $L_2$  be two linear operators; we want to show that their sum  $L_1 + L_2$  is also a linear operator. By definition, the sum of two linear operators is:

$$(L_1 + L_2)(u) = L_1(u) + L_2(u)$$

for any vector  $u$  in the domain. But we also need to show that the product of an arbitrary scalar and a linear operator is also linear. By definition, the product of a scalar and a linear operator is:

$$(cL)(u) = cL(u)$$

for any vector  $u$  in the domain and an arbitrary real number  $c$ . For example: let  $V$  be a vector space, let  $a, b \in \mathbb{R}$ , and let  $u, v \in V$ . So we can plug in our arbitrary linear combination into the superposition principle from before to get:

$$(L_1 + L_2)(au + bv) = L_1(au + bv) + L_2(au + bv)$$

Next we can apply the linearity property, rearrange the terms, factor out  $a$  and  $b$ , and then use the definition of the sum of operators again to prove that the definition is satisfied.

$$\begin{aligned} &= (aL_1(u) + bL_1(v)) + (aL_2(u) + bL_2(v)) \\ &= aL_1(u) + aL_2(u) + bL_1(v) + bL_2(v) \\ &= a(L_1(u) + L_2(u)) + b(L_1(v) + L_2(v)) \\ &= a(L_1 + L_2)(u) + b(L_1 + L_2)(v) \\ &= a(L_1 + L_2)(u) + b(L_1 + L_2)(v) = aL(u) + bL(v) = L(au + bv) \end{aligned}$$

Which is equal to the superposition principle from the beginning, so both the sum and multiplication definitions of a linear operator hold with our arbitrary linear combination. Since both definitions hold with our arbitrary linear combination, then any finite combination of linear operators is also a linear operator.

**Problem 2.2.2:** Show that

(a)  $L(u) = \frac{\partial}{\partial x} [K_0(x) \frac{\partial u}{\partial x}]$  is a linear operator

In order to show that  $L$  is a linear operator, the superposition principle from earlier must hold for an arbitrary linear combination. We can use a similar setup to the previous problem (2.2.1), except this time we need to apply it to the PDE and show that  $L$  is linear.

$$\begin{aligned} L(au + bv) &= aL(u) + bL(v) \\ &= \frac{\partial}{\partial x} \left[ K_0(x) \frac{\partial u}{\partial x} \right] \\ L(au + bv) &= \frac{\partial}{\partial x} [K_0(x)(au_x + bv_x)] \\ &= \frac{\partial}{\partial x} [aK_0(x)u_x + bK_0(x)v_x] \\ &= a \frac{\partial}{\partial x} [K_0(x)u_x] + b \frac{\partial}{\partial x} [K_0(x)v_x] \\ &= a \frac{\partial}{\partial x} \left[ K_0(x) \frac{\partial u}{\partial x} \right] + b \frac{\partial}{\partial x} \left[ K_0(x) \frac{\partial v}{\partial x} \right] \\ L(u) &= \frac{\partial}{\partial x} \left[ K_0(x) \frac{\partial u}{\partial x} \right] \\ L(au + bv) &= aL(u) + bL(v) \end{aligned}$$

Which is equal to the superposition principle on the first line, verifying that  $L$  is a linear operator

(b) and usually  $L(u) = \frac{\partial}{\partial x} [K_0(x, u) \frac{\partial u}{\partial x}]$  is not a linear operator.

In order to show that  $L$  is not a linear operator, the superposition principle must be contradicted by any arbitrary linear combination  $(au + bv)$ . Once again, we can use a similar setup to the previous problems (2.2.1 and 2.2.2a), except this time we need to apply it to the PDE and show that  $L$  is not linear.

$$\begin{aligned} L(au + bv) &= aL(u) + bL(v) \\ &= \frac{\partial}{\partial x} \left[ K_0(x, u) \frac{\partial u}{\partial x} \right] \\ u_x &= \frac{\partial u}{\partial x}, v_x = \frac{\partial v}{\partial x} \\ L(au + bv) &= \frac{\partial}{\partial x} [K_0(x, au + bv)(au_x + bv_x)] \\ &= \frac{\partial K_0}{\partial x}(x, au + bv)(au_x + bv_x) + \frac{\partial K_0}{\partial u}(x, au + bv)(au_x + bv_x)^2 + K_0(x, au + bv)(au_{xx} + bv_{xx}) \\ &\neq a \left( \frac{\partial}{\partial x} \left[ K_0(x, u) \frac{\partial u}{\partial x} \right] \right) + b \left( \frac{\partial}{\partial x} \left[ K_0(x, v) \frac{\partial v}{\partial x} \right] \right) = aL(u) + bL(v) \end{aligned}$$

This is not equal to the superposition principle because  $K_0(x, au + bv)$  and its derivatives are not linear combinations of  $K_0(x, u)$  and  $K_0(x, v)$ . The extra terms involving  $\frac{\partial K_0}{\partial u}$  and the squared partial derivative  $(au_x + bv_x)^2$  produce nonlinear combinations that contradict the definitions of a linear operator. Therefore,  $L$  is not a linear operator.

**Problem 2.2.3:** Show that  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$  is linear if  $Q = \alpha(x, t)u + \beta(x, t)$  and, in addition, homogeneous if  $\beta(x, t) = 0$ .

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t) \\ L(u) &= \beta(x, t) = u_t - ku_{xx} - \alpha(x, t)u \\ L(au + bv) &= (au + bv)_t - k(au + bv)_{xx} - \alpha(x, t)(au + bv) \\ &= au_t + bv_t - k(au_{xx} + bv_{xx}) - \alpha au - \alpha bv \\ &= a(u_t - ku_{xx} - \alpha u) + b(v_t - kv_{xx} - \alpha v) \\ L(au + bv) &= aL(u) + bL(v)\end{aligned}$$

Which satisfies the superposition principle, therefore  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t)$  is a linear PDE. Usually, this PDE is non-homogeneous because it has the form  $L(u) = \beta(x, t)$ . However, when  $\beta(x, t) = 0$  then  $L(u) = 0$  and the linear PDE becomes homogeneous because  $u_t - ku_{xx} - \alpha(x, t)u = 0$ .

**Problem 2.2.4:** In this exercise we derive superposition principles for non-homogeneous problems.

(a) Consider  $L(u) = f$ . If  $u_p$  is a particular solution,  $L(u_p) = f$ , and if  $u_1$  and  $u_2$  are homogeneous solutions,  $L(u_i) = 0$ , show that  $u = u_p + c_1u_1 + c_2u_2$  is another particular solution.

$$\begin{aligned}L(u_p) &= f \\ L(u_1) &= L(u_2) = 0 \\ u &= u_p + c_1u_1 + c_2u_2 \\ L(u) &= L(u_p) + c_1L(u_1) + c_2L(u_2) \\ &= f + c_1(0) + c_2(0) = f = L(u)\end{aligned}$$

Thus  $u$  is also a particular solution because it's a linear combination of the pre-existing particular solution  $u_p$ .

(b) If  $L(u) = f_1 + f_2$ , where  $u_{pi}$  is a particular solution corresponding to  $f_i$ , what is a particular solution for  $f_1 + f_2$ ?

$$\begin{aligned}L(u) &= f_1 + f_2 \\ L(u_{p1}) &= f_1, L(u_{p2}) = f_2 \\ u_p &= u_{p1} + u_{p2} \\ L(u_p) &= L(u_{p1} + u_{p2}) \\ &= L(u_{p1}) + L(u_{p2}) = f_1 + f_2 = L(u)\end{aligned}$$

Thus the particular solution for  $f_1 + f_2$  is  $u_p = u_{p1} + u_{p2}$ .

**Problem 2.3.1 b,d:** For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b)  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$

$$\begin{aligned} u(x, t) &= X(x)T(t) \\ u_t &= k u_{xx} - v_0 u_x \\ X(x)T'(t) &= k X''(x)T(t) - v_0 X'(x)T(t) \\ \frac{T'(t)}{T(t)} &= k \frac{X''(x)}{X(x)} - v_0 \frac{X'(x)}{X(x)} = \lambda \end{aligned}$$

The left side only depends on  $t$  and the right side only depends on  $x$  but both are equal to the same separation constant  $\lambda$ . We can separate both sides to get the following two ODEs:

$$\begin{aligned} T'(t) - \lambda T(t) &= 0 \\ kX''(x) - v_0 X'(x) - \lambda X(x) &= 0 \end{aligned}$$

(d)  $\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r})$

$$\begin{aligned} u(r, t) &= R(r)T(t) \\ u_t &= \frac{k}{r^2} \frac{\partial}{\partial r} (r^2 u_r) \\ R(r)T'(t) &= \frac{k}{r^2} (r^2 R'(r))' T(t) \\ \frac{T'(t)}{T(t)} &= \frac{k}{r^2} \frac{(r^2 R'(r))'}{R(r)} = \lambda \end{aligned}$$

The left side only depends on  $t$  and the right side only depends on  $r$  but both are equal to the same separation constant  $\lambda$ . We can separate both sides to get the following two ODEs:

$$\begin{aligned} T'(t) - \lambda T(t) &= 0 \\ \frac{k}{r^2} (r^2 R'(r))' - \lambda R(r) &= 0 \end{aligned}$$

**Problem 2.3.2 a,e:** Consider the differential equation

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0,$$

where  $\phi$  is a function of  $x$  only. Determine the eigenvalues  $\lambda$  (and corresponding eigenfunctions) if  $\phi$  satisfies the following boundary conditions. Analyze three cases ( $\lambda > 0$ ,  $\lambda = 0$ , and  $\lambda < 0$ ). You may assume that the eigenvalues are real.

(a)  $\phi(0) = 0$  and  $\phi(\pi) = 0$

Case 1:  $\lambda > 0$

$$\lambda = \mu^2, \mu > 0$$

$$\phi(x) = A\cos(\mu x) + B\sin(\mu x)$$

$$\phi(0) = 0 = A$$

$$\phi(x) = B\sin(\mu x)$$

$$\phi(\pi) = 0 = B\sin(\mu\pi)$$

For a nontrivial solution, we need  $B \neq 0$ , so  $\sin(\mu\pi) = 0$ . To find the eigenvalue  $\lambda$  and the corresponding eigenfunction, assume  $n \in \mathbb{N}$  such that  $n = 1, 2, 3, \dots$ , then:

$$\mu\pi = n\pi$$

$$\mu = n$$

$$\lambda_n = \mu^2 = n^2$$

$$\phi_n(x) = B\sin(nx) = 0$$

Case 2:  $\lambda = 0$

$$\phi''(x) = 0$$

$$\phi(x) = Ax + B$$

$$\phi(0) = 0 = B$$

$$\phi(\pi) = 0 = A\pi$$

$$A = 0$$

$$\phi(x) = 0$$

So only the trivial solution  $\phi(x) = 0$  exists and there is no nontrivial eigenfunction for  $\lambda = 0$ .

Case 3:  $\lambda < 0$

$$\lambda = -\mu^2, \mu > 0$$

$$\phi(x) = Ce^{\mu x} + De^{-\mu x}$$

$$\phi(0) = 0 = C + D$$

$$= C(e^{\mu x} - e^{-\mu x}) = 2C\sinh(\mu x)$$

$$\phi(\pi) = 0 = 2C\sinh(\mu\pi)$$

However  $\sinh(\mu\pi) \neq 0$  for  $\mu > 0$ , so  $C = 0$ . This means that only the trivial solution exists and there is no nontrivial eigenfunction for  $\lambda < 0$ . Taking all three of these cases into account, there is only one nontrivial eigenvalue  $\lambda_n = n^2$  and eigenfunction  $\phi_n(x) = B\sin(nx)$  (where  $B \in \mathbb{R}$  and  $n \in \mathbb{N}$ ) for the case when  $\lambda > 0$ . There are no nontrivial solutions for  $\lambda \leq 0$ .

(e)  $\frac{d\phi}{dx}(0) = 0$  and  $\phi(L) = 0$

Case 1:  $\lambda > 0$

$$\lambda = \mu^2, \mu > 0$$

$$\phi(x) = A\cos(\mu x) + B\sin(\mu x)$$

$$\phi'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

$$\phi'(0) = 0 = B\mu$$

$$B = 0$$

$$\phi(x) = A\cos(\mu x)$$

$$\phi(L) = 0 = A \cos(\mu L)$$

For a nontrivial solution, we need  $A \neq 0$ , so  $\cos(\mu L) = 0$ . To find the eigenvalue  $\lambda$  and the corresponding eigenfunction, assume  $n \in \mathbb{N}$  such that  $n = 1, 2, 3, \dots$ , then:

$$\begin{aligned}\mu L &= \frac{(2n+1)\pi}{2} \\ \mu &= \frac{(2n+1)\pi}{2L} \\ \lambda_n = \mu^2 &= \left(\frac{(2n+1)\pi}{2L}\right)^2 \\ \phi_n(x) &= \cos\left(\frac{(2n+1)\pi}{2L}x\right)\end{aligned}$$

Case 2:  $\lambda = 0$

$$\begin{aligned}\phi''(x) &= 0 \\ \phi(x) &= Ax + B \\ \phi'(x) &= A \\ \phi'(0) &= 0 = A \\ \phi(x) &= B \\ \phi(L) &= 0 = B\end{aligned}$$

So only the trivial solution  $\phi(x) = 0$  exists and there is no nontrivial eigenfunction for  $\lambda = 0$ .

Case 3:  $\lambda < 0$

$$\begin{aligned}\lambda &= -\mu^2, \mu > 0 \\ \phi(x) &= Ce^{\mu x} + De^{-\mu x} \\ \phi'(x) &= C\mu e^{\mu x} - D\mu e^{-\mu x} \\ \phi'(0) &= 0 = \mu(C - D) \\ C - D &= 0 \\ C &= D \\ \phi(x) &= 2C \cosh(\mu x) \\ \phi(L) &= 0 = 2C \cosh(\mu L)\end{aligned}$$

However  $\cosh(\mu x) > 0$  for all  $x$ , so  $C = 0$ . This means that only the trivial solution exists and there is no nontrivial eigenfunction for  $\lambda < 0$ . Taking all three of these cases into account, there is only one nontrivial eigenvalue  $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$  and eigenfunction  $\phi_n(x) = \cos\left(\frac{(2n+1)\pi}{2L}x\right)$  for the case when  $\lambda > 0$ . There are no nontrivial solutions for  $\lambda \leq 0$ .

**Problem 2.3.3 a,c,d:** Consider the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions  $u(0, t) = 0$  and  $u(L, t) = 0$ . Solve the initial value problem if the temperature is initially

(a)  $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$

$$\begin{aligned} u_t &= k u_{xx} \\ u(x, 0) &= f(x) = 6 \sin\left(\frac{9\pi x}{L}\right) \\ u(x, t) &= X(x)T(t) \\ X(x)T'(t) &= k X''(x)T(t) \\ \frac{1}{k} \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -\lambda \\ T'(t) &= -\lambda k T(t) \\ X''(x) + \lambda X(x) &= 0 \end{aligned}$$

Case  $\lambda > 0$ :

$$\begin{aligned} \lambda &= \mu^2 \\ X(x) &= A \cos(\mu x) + B \sin(\mu x) \\ X(0) &= 0 = A \\ X(x) &= B \sin(\mu x) \\ X(L) &= 0 = B \sin(\mu L) \end{aligned}$$

For a nontrivial solution, we need  $B \neq 0$ , so  $\sin(\mu L) = 0$ , assume  $n \in \mathbb{N}$  such that  $n = 1, 2, 3, \dots$ , then:

$$\begin{aligned} \mu L &= n\pi \\ \mu &= \frac{n\pi}{L} \\ X_n(x) &= \sin\left(\frac{n\pi x}{L}\right), \lambda_n = \mu^2 = \left(\frac{n\pi}{L}\right)^2 \\ T_n(t) &= a_n e^{-k\lambda_n t} = a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u_n(x, t) &= a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u(x, 0) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \\ f(x) &= 6 \sin\left(\frac{9\pi x}{L}\right) \\ a_9 &= 6, \quad a_n = 0 \quad (n \neq 9) \end{aligned}$$

$$u(x, t) = 6 \sin\left(\frac{9\pi x}{L}\right) e^{-k\left(\frac{9\pi}{L}\right)^2 t}$$

(c)  $u(x, 0) = 2 \cos\left(\frac{3\pi x}{L}\right)$

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x, 0) = f(x) = 2 \cos\left(\frac{3\pi x}{L}\right)$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L 2 \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{4n}{\pi} \frac{1 + (-1)^n}{n^2 - 9} \quad (n \neq 3) \end{aligned}$$

There is a removable singular point at  $n = 3$ , which means that  $a_3 = 0$  and only the even  $n$  terms remain. Let  $m \in \mathbb{N}$  such that  $n = 2m$ , then:

$$\begin{aligned} a_{2m} &= \frac{16m}{\pi(4m^2 - 9)} \\ u(x, t) &= \sum_{m=1}^{\infty} \left( \frac{16m}{\pi(4m^2 - 9)} \right) \sin\left(\frac{2m\pi x}{L}\right) e^{-k\left(\frac{2m\pi}{L}\right)^2 t} \end{aligned}$$

(d)  $u(x, 0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$u(x, 0) = f(x) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$$

Compute two integrals for both intervals and let  $\alpha = \frac{n\pi}{L}$ :

$$\int_0^{\frac{L}{2}} (1) \sin(\alpha x) dx = \frac{1 - \cos(\alpha \frac{L}{2})}{\alpha}$$

$$\int_{\frac{L}{2}}^L (2) \sin(\alpha x) dx = \frac{2(\cos(\alpha \frac{L}{2}) - \cos(\alpha L))}{\alpha}$$

$$\begin{aligned} a_n &= \frac{2}{L} \frac{1}{\alpha} [1 + \cos(\frac{n\pi}{2}) - 2(-1)^n] \\ &= \frac{2}{n\pi} [1 + \cos(\frac{n\pi}{2}) - 2(-1)^n] \end{aligned}$$

If  $n$  is odd then  $\cos(\frac{n\pi}{2}) = 0$  and  $(-1)^n = -1$ , so  $a_n = \frac{6}{n\pi}$ . If  $n$  is even then  $\cos(\frac{n\pi}{2}) = -1$  and  $(-1)^n = 1$ , so  $a_n = -\frac{4}{n\pi}$ . Then:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{6}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t} + \sum_{n=1}^{\infty} \left(-\frac{4}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t}$$

**Problem 2.3.4 a-b:** Consider

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions  $u(0, t) = 0$ ,  $u(L, t) = 0$  and  $u(x, 0) = f(x)$ .

(a) What is the total heat energy in the rod as a function of time?

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t} \\ a_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ e(x, t) &= \int_0^L c\rho u(x, t) dx \\ &= c\rho \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx \\ &= c\rho \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \left(\frac{L}{n\pi} (1 - (-1)^n)\right) \end{aligned}$$

When  $n$  is even  $1 - (-1)^n = 0$ , and when  $n$  is odd  $1 - (-1)^n = 2$  so the total heat energy can be simplified to:

$$e(x, t) = c\rho \sum_{n=1}^{\infty} a_n \left(\frac{2L}{n\pi}\right) e^{-k(\frac{n\pi}{L})^2 t}$$

(b) What is the flow of heat energy out of the rod at  $x = 0$ ? at  $x = L$ ?

At  $x = 0$ :

$$\phi(0, t) = -k \frac{\partial u}{\partial x}(0, t)$$

At  $x = L$ :

$$\phi(L, t) = k \frac{\partial u}{\partial x}(L, t)$$