Problem Similar to 2.3.5: Evaluate (be careful if $n = m \neq 0$ and if $n = m = 0$)

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx,$$

and

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx,$$

for $n > 0$, $m > 0$.

In order to evaluate the above integrals, we need the following two trig identities:

$$sin(a)sin(b) = \frac{1}{2}[cos(a-b) - cos(a+b)]$$

$$cos(a)cos(b) = \frac{1}{2}[cos(a+b) + cos(a-b)]$$

First, in order to evaluate the Orthogonality of Sines integral we need to plug in the $sin(a)sin(b)$ identity:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \frac{1}{2}\int_{-L}^{L} (cos(\frac{(n-m)\pi x}{L}) - cos(\frac{(n+m)\pi x}{L}))dx$$

Let $k \in \mathbb{Z}$ such that $k \neq 0$, then:

$$\int_{-L}^{L} cos(\frac{k\pi x}{L})dx = \frac{L}{k\pi} sin(\frac{k\pi x}{L})\Big|_{-L}^{L}$$

$$= \frac{L}{k\pi}(sin(k\pi) - sin(-k\pi)) = 0$$

This is because $sin(k\pi) = 0$. If $k = 0$ then the integral becomes:

$$\int_{-L}^{L} cos(0)dx = \int_{-L}^{L} 1dx = x\Big|_{-L}^{L} = 2L$$

With this in mind, if $n \neq m$ then both $n - m$ and $n + m$ are nonzero integers and the resulting integral is 0. If $n = m$ then the resulting integral is:

$$\frac{1}{2}\int_{-L}^{L} (cos(0) - cos(\frac{2n\pi x}{L}))dx = \frac{1}{2}\int_{-L}^{L} 1dx = \frac{1}{2}(2L) = L$$

So the final Orthogonality of Sines integral for $n,\ m > 0$ is:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

Next, in order to evaluate the Orthogonality of Cosines integral we need to plug in the $cos(a)cos(b)$ identity:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \frac{1}{2}\int_{-L}^{L} (cos(\frac{(n+m)\pi x}{L}) + cos(\frac{(n-m)\pi x}{L}))dx$$

Just like before, when $k \neq 0$ the resulting integral will be 0, and when $k = 0$ the resulting integral will be $2L$. So if $n \neq m$ then both $n \pm m$ are nonzero and the resulting integral is 0. If $n = m \neq 0$ then the resulting integral will be L, and if $n = m = 0$ then the resulting integral will be $2L$. So the final Orthogonality of Cosines integral for $n,\ m > 0$ is:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

Problem 2.4.3: Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

subject to $\phi(0) = \phi(2\pi)$, and $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi)$. *Note: This is an ODE, not a PDE so there is no time dependence here.* You must consider whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$.

Case 1: $\lambda > 0$

$$\lambda = \mu^2,\ \mu > 0$$

$$\phi(x) = Acos(\mu x) + Bsin(\mu x)$$

$$\phi'(x) = -A\mu sin(\mu x) + B\mu cos(\mu x)$$

$$\phi(0) = \phi(2\pi) = A = Acos(2\pi\mu) + Bsin(2\pi\mu)$$

$$\phi'(0) = \phi'(2\pi) = B\mu = -A\mu sin(2\pi\mu) + B\mu cos(2\pi\mu)$$

So $cos(2\pi\mu) = 1$ and $2\pi\mu = 2\pi n$ for $n \in \mathbb{N}$. This means that $\mu = n$ and the resulting positive eigenvalues are:

$$\lambda_n = \mu^2 = n^2$$

For each n any constants A and B give a periodic solution, so the eigenspace is 2D and spanned by the eigenfunctions $cos(nx)$ and $sin(nx)$. This also means that the eigenvalue λ_n has a multiplicity of 2.
Case 2: $\lambda = 0$

$$\phi''(x) = 0$$

$$\phi(x) = Ax + B$$

$$\phi(0) = \phi(2\pi) = B = A(2\pi) + B$$

$$A(2\pi) = 0$$

$$A = 0$$

$$\phi(x) = B$$

Thus $\lambda = 0$ is an eigenvalue with a 1D eigenspace spanned by the constant function $\phi(x) = B$.
Case 3: $\lambda < 0$

$$\lambda = -\mu^2,\ \mu > 0$$

$$\phi(x) = Ce^{\mu x} + De^{-\mu x}$$

$$\phi(0) = \phi(2\pi) = C + D = Ce^{2\pi\mu} + De^{-2\pi\mu x}$$

Which only holds if $C = D = 0$ because $e^{2\pi\mu} \neq 1$. This results in the trivial solution and no negative eigenvalues. Taking all three cases into account, the eigenvalues and corresponding eigenfunctions are $\lambda_0 = 0$ with $\phi_0(x) = 1$ (constant function), and $\lambda_n = n^2$ (multiplicity 2) with $\phi_n^{(1)}(x) = cos(nx)$ and $\phi_n^{(2)}(x) = sin(nx)$.

Problem 2.4.6: Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) directly from the equilibrium problem (see Section 1.4):

Time-independent (equilibrium) heat equation:

$$\frac{d^2u}{dx^2} = 0$$

$$\frac{du}{dx} = C_1$$

$$u(x) = C_1x + C_2$$

BCs like in Section 1.4 but periodic for a thin circular ring:

$$u(0) = T_1 = u(2\pi) = T_2,\ u'(0) = u'(2\pi)$$

Plug the BCs into the temperature equation $u(x)$:

$$u(0) = T_1 = C_2 = u(2\pi) = T_2 = C_1(2\pi) + C_2$$

$$C_2 = C_1(2\pi) + C_2$$

$$C_1(2\pi) = 0$$

$$C_1 = 0$$

$$\frac{du}{dx} = u'(x) = C_1$$

$$u'(0) = u'(2\pi) = C_1 = 0$$

$$u(x) = C_2 = T_1$$

So the equilibrium temperature distribution for a thin circular ring is $u(x) = T_1$, where T_1 is the initial temperature of the ring. Due to the circular shape of the ring, both ends are connected and must share the same temperature at $x = 0$ and $x = 2\pi$ which is why $u(0) = u(2\pi)$. If the ring is insulated, then the total heat is conserved and the equilibrium temperature distribution will be the average of the initial temperature distribution:

$$u(x) = C_2 = T_1 = \frac{1}{2\pi}\int_0^{2\pi} u(x,0)dx$$

(b) and by computing the limit as $t \to \infty$ of the time-dependent problem:

Time-dependent heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ x \in [0, 2\pi],\ t > 0,\ k > 0$$

Periodic BCs:

$$u(0,t) = u(2\pi,t),\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2\pi,t),\ u(x,0) = f(x)$$

We can use the eigenvalues and eigenfunctions from Problem 2.4.3 to get the temperature distribution as a Fourier series and its coefficients:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(nx)dx$$

By separation of variables, each series term evolves independently with exponential decay proportional to its eigenvalue, so the time-dependent temperature distribution becomes:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-kn^2t}(a_n cos(nx) + b_n sin(nx))$$

Now take the limit as $t \to \infty$. For every $n \geq 1$, the exponential term e^{-kn^2t} becomes 0 because $k > 0$, so all of the series terms go away leaving only the constant a_0:

$$\lim_{t\to\infty} u(x,t) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} u(x,0)dx$$

Which is equal to the average of the initial temperature distribution from part (a) of this problem.

Problem 2.5.1(a): Solve Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the following boundary conditions:

(a) $\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = f(x)$.

2D Cartesian Laplace's Equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Use separation of variables to get a system of ODEs:

$$u(x,y) = X(x)Y(y)$$
$$X''(x)Y(y) + X(x)Y''(y) = 0$$
$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$
$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$
$$X''(x) + \lambda X(x) = 0$$
$$Y''(y) - \lambda Y(y) = 0$$

Now plug the BCs into X(x) to find the eigenvalue λ:

$$X'(0) = 0 = X'(L)$$

Case 1: $\lambda > 0$

$$X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$
$$X'(x) = -A\sqrt{\lambda}sin(\sqrt{\lambda}x) + B\sqrt{\lambda}cos(\sqrt{\lambda}x)$$
$$X'(0) = 0 = B\sqrt{\lambda}$$

$$B = 0$$
$$X'(L) = 0 = -A\sqrt{\lambda}sin(\sqrt{\lambda}L)$$

For a nontrivial solution we need $A \neq 0$ which causes $sin(\sqrt{\lambda}L) = 0$. Let $n \in \mathbb{N}$ such that $\sqrt{\lambda}L = n\pi$, then the eigenvalue λ_n and the eigenfunction $X_n(x)$ are:

$$\lambda_n = (\frac{n\pi}{L})^2$$
$$X_n(x) = cos(\frac{n\pi x}{L})$$

Case 2: $\lambda = 0$

$$X''(x) = 0$$
$$X'(x) = A$$
$$X(x) = Ax + B$$
$$X'(0) = 0 = X'(L) = A$$
$$X(x) = B$$

So $\lambda = 0$ is an eigenvalue with a constant eigenfunction.
Case 3: $\lambda < 0$
Let $\lambda = -\mu^2$, with $\mu > 0$, then:

$$X(x) = Ccosh(\mu x) + Dsinh(\mu x)$$
$$X'(x) = C\mu sinh(\mu x) + D\mu cosh(\mu x)$$
$$X'(0) = 0 = D\mu$$
$$D = 0$$
$$X(x) = Ccosh(\mu x)$$
$$X'(L) = 0 = C\mu sinh(\mu L)$$

For a nontrivial solution we need $C \neq 0$ so $sinh(\mu L)$ must be equal to 0, but $sinh(\mu L) \neq 0$ for $\mu > 0$ and $L > 0$. So C must equal 0, yielding no eigenvalues and the trivial solution of $X(x) = 0$. Taking all three of these λ cases into account, we now have the eigenvalues and eigenfunctions required to solve the ODE dependent on y.

$$Y_n''(y) - \lambda_n Y_n(y) = 0$$

Case $n = 0$ ($\lambda_0 = 0$):

$$Y_0''(y) = 0$$
$$Y_0(y) = A_0 y + B_0$$

Case $n \geq 1$ ($\lambda_n = (\frac{n\pi}{L})^2 > 0$):

$$Y_n(y) = A_n sinh(\frac{n\pi y}{L}) + B_n cosh(\frac{n\pi y}{L})$$

Next apply the BCs to $Y_n(y)$:

$$u(x, 0) = 0 = X(x)Y(0)$$

For this to be true for all values of x, we need $Y_n(0) = 0$ for both cases of n:
Case $n = 0$:

$$Y_0(0) = 0 = B_0$$
$$Y_0(y) = A_0 y$$

Case $n \geq 1$:

$$Y_n(0) = 0 = B_n$$
$$Y_n(y) = A_n sinh(\frac{n\pi y}{L})$$

Now that we know $X_n(x)$ and $Y_n(y)$, we can construct the general solution and plug in the top BC $u(x, H) = f(x)$ to find the series coefficients and the final solution to the 2D Laplace's Equation:

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi x}{L}) sinh(\frac{n\pi y}{L})$$

$$u(x, H) = A_0 H + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi x}{L}) sinh(\frac{n\pi H}{L}) = f(x)$$

So $f(x)$ has a cosine Fourier series and coefficients of the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L})$$
$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)cos(\frac{n\pi x}{L})dx$$
$$A_0 H = a_0$$
$$A_0 = \frac{a_0}{H}$$
$$A_n sinh(\frac{n\pi H}{L}) = a_n$$
$$A_n = \frac{a_n}{sinh(\frac{n\pi H}{L})}$$

So the final solution to the 2D Laplace's Equation is:

$$u(x, y) = \frac{a_0}{H} y + \sum_{n=1}^{\infty} \frac{a_n}{sinh(\frac{n\pi H}{L})} cos(\frac{n\pi x}{L}) sinh(\frac{n\pi y}{L})$$

Problem 2.5.3(a): Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary conditions

(a) $u(a, \theta) = \ln 2 + 4\cos\theta$

The general solution for the 2D Polar Laplace's Equation for $r \geq a$ is:

$$u(r, \theta) = A_0 + B_0 ln(r) + \sum_{n=1}^{\infty}(A_n r^n + B_n r^{-n})cos(n\theta) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})sin(n\theta)$$

However, since we are interested in the region outside of a circle, we need to ensure that the solution is bounded as $r \to \infty$. This means that the r^n and $ln(r)$ terms must be excluded so that the solution is well-behaved for large values of r. So the general solution can be simplified to:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} B_n r^{-n} cos(n\theta) + \sum_{n=1}^{\infty} D_n r^{-n} sin(n\theta)$$

Now we need to apply the BC at $r = a$ which only has a constant and cosine terms so all of the coefficients go away except the terms when $n = 0$ or $n = 1$. This means the the BC has the form:

$$u(r, \theta) = A_0 + B_1 \frac{cos(\theta)}{r}$$

So we can plug in $r = a$ to find the remaining coefficients:

$$u(a, \theta) = A_0 + B_1 \frac{cos(\theta)}{a} = ln(2) + 4cos(\theta)$$

$$A_0 = ln(2)$$

$$B_1 \frac{cos(\theta)}{a} = 4cos(\theta)$$

$$\frac{B_1}{a} = 4$$

$$B_1 = 4a$$

So the final solution to the 2D Polar Laplace's Equation outside the circle is:

$$u(r, \theta) = ln(2) + 4\frac{a}{r}cos(\theta),\ \ (r \geq a)$$

Problem 2.5.5(a): Solve Laplace's equation inside the quarter circle of radius 1 ($0 \leq \theta \leq \pi/2$, $0 \leq r \leq 1$) subject to the boundary conditions

(a) $\frac{\partial u}{\partial \theta}(r, 0) = 0$, $u(r, \pi/2) = 0$, and $u(1, \theta) = f(\theta)$.

The general solution for the 2D Polar Laplace's Equation is:

$$u(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n cos(n\theta) + B_n r^n sin(n\theta))$$

Now apply the BCs:

$$\frac{\partial u}{\partial \theta}(r, \theta) = \sum_{n=1}^{\infty} r^n(-nA_n sin(n\theta) + nB_n cos(n\theta))$$

$$\frac{\partial u}{\partial \theta}(r, 0) = 0 = \sum_{n=1}^{\infty} nB_n r^n$$

$$B_n = 0$$

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n cos(n\theta)$$

$$u(r, \frac{\pi}{2}) = 0 = \sum_{n=0}^{\infty} A_n r^n cos(\frac{n\pi}{2})$$

For odd values of n, $cos(\frac{n\pi}{2}) = 0$, and for even values of n, $cos(\frac{n\pi}{2}) = \pm 1$. Let $k \in \mathbb{N}$ such that $n = 2k + 1$, then $A_{2k} = 0$ for every value of k and only the odd values of n survive.

$$u(r, \theta) = \sum_{k=0}^{\infty} A_{2k+1} r^{2k+1} cos((2k + 1)\theta)$$

$$u(1,\theta) = f(\theta) = \sum_{k=0}^{\infty} A_{2k+1} cos((2k+1)\theta)$$

The functions $cos((2k+1)\theta)$ are orthogonal on $[0, 2\pi]$, so their norm is:

$$\int_0^{\frac{\pi}{2}} cos^2((2k+1)\theta)d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + cos(2(2k+1)\theta))d\theta = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

So the coefficients are:

$$A_{2k+1} = \frac{1}{\parallel cos((2k+1)\theta) \parallel^2} \int_0^{\frac{\pi}{2}} f(\theta) cos((2k+1)\theta)d\theta$$

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(\theta) cos((2k+1)\theta)d\theta$$

So the final solution is:

$$u(r,\theta) = \sum_{k=0}^{\infty} (\frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(\theta) cos((2k+1)\theta)d\theta) r^{2k+1} cos((2k+1)\theta)$$

Problem 2.5.6(a): Solve Laplace's equation inside a semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions

(a) $u(r,\theta) = 0$ on the diameter and $u(a,\theta) = g(\theta)$.

2D Polar Laplace's Equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}(\frac{\partial u}{\partial r}) + \frac{1}{r^2}(\frac{\partial^2 u}{\partial \theta^2}) = 0$$

Use separation of variables to get a system of ODEs:

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

The BC $\Theta(0) = \Theta(\pi) = 0$ forces the eigenvalue $\lambda = n^2$ where $n \in \mathbb{N}$, which results in the eigenfunction:

$$\Theta_n(\theta) = sin(n\theta)$$

For the eigenvalue $\lambda = n^2$ the radial ODE will be in the form of the Euler Equation with a general solution of:

$$R_n(r) = C_1 r^n + C_2 r^{-n}$$

However, regularity at the origin ($r = 0$) requires $C_2 = 0$ so the general solution becomes:

$$R_n(r) = C_n r^n$$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n sin(n\theta)$$

Now plug in the BC at $r = a$ to find the coefficient b_n:

$$u(a,\theta) = g(\theta) = \sum_{n=1}^{\infty} A_n a^n sin(n\theta)$$

Let $b_n = A_n a^n$, then:

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(\theta) sin(n\theta) d\theta,\ A_n = \frac{b_n}{a^n}$$

Which results in the final solution to the 2D Polar Laplace's Equation:

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n (\frac{r}{a})^n sin(n\theta)$$

Problem 2.5.15(a): Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary conditions

(a) $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$, and $u(0,y) = f(y)$.

2D Cartesian Laplace's Equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Use separation of variables to get a system of ODEs:

$$u(x,y) = X(x)Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = -\lambda$$

$$Y''(y) + \lambda Y(y) = 0$$

$$X''(x) - \lambda X(x) = 0$$

Applying the BCs $Y'(0) = Y'(H) = 0$ yields the following eigenvalues and eigenfunctions (assuming that $n \in \mathbb{N}$):

$$\lambda_0 = 0,\ Y_0(y) = 1$$

$$\lambda_n = (\frac{n\pi}{H})^2,\ Y_n(y) = cos(\frac{n\pi y}{H})$$

Plugging in the λ_n eigenvalue into the x-equation yields an x-dependent general solution of:

$$X_n(x) = A_n e^{-(\frac{n\pi}{H})x} + B_n e^{(\frac{n\pi}{H})x}$$

However the solution needs to remain bounded as $x \to \infty$, which forces $B_n = 0$ so the solution for $n \geq 1$ is:

$$u_n(x,y) = A_n e^{-(\frac{n\pi}{H})x} cos(\frac{n\pi y}{H})$$

For $n = 0$:

$$X_0''(x) = 0$$

$$X_0'(x) = C_0$$
$$X_0(x) = C_0 x + D_0$$

But to ensure boundedness as $x \to \infty$, C_0 must equal 0, so the $n = 0$ term is just the constant D_0. Now plug in the BC at $x = 0$ to find the coefficients:

$$u(0, y) = f(y) = D_0 + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi y}{H})$$

$$D_0 = a_0 = \frac{1}{H} \int_0^H f(y) dy$$
$$A_n = a_n = \frac{2}{H} \int_0^H f(y) cos(\frac{n\pi y}{H}) dy$$

Therefore, the final solution to the 2D Cartesian Laplace's Equation for a semi-infinite strip is:

$$u(x, y) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{H})x} cos(\frac{n\pi y}{H})$$