RUBRIC:

Questions	Points	Score
Total		

Problem 1.3.1: Consider a one-dimensional rod, $0 \leq x \leq L$. Assume that the heat energy flowing out of the rod at $x = L$ is proportional to the temperature difference between the end temperature of the bar and the known external temperature. Derive (1.3.5); briefly, physically explain why $H > 0$.

$$1.3.5\text{: } -K_0(L)\tfrac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)]$$

The heat flux leaving the rod at $x = L$ in the positive x direction is proportional to the temperature gradient there, which yields this heat flux equation by Fourier's law:

$$\phi(L,t) = -K_0(L)\tfrac{\partial u}{\partial x}(L,t)$$

where $\phi(L,t)$ is the heat flux at $x = L$ and $K_0(L)$ is the thermal conductivity constant, which is negated with the minus sign to signify that heat flows from hot to cold. Additionally, the rate at which the rod loses heat to the external environment is proportional to the temperature difference between the end of the rod and the external temperature $u_B(t)$. This results in the following heat loss equation by Newton's law of cooling:

$$\phi(L,t) = H[u(L,t) - u_B(t)]$$

where $\phi(L,t)$ is the convective heat loss leaving the rod and H is the heat transfer constant. At the boundary $x = L$, these two rates must be equal so that the heat conduction from the end of the rod matches the heat that is lost to the external environment. By setting these two rates equal to each other we get:

$$-K_0(L)\tfrac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)]$$

which is equivalent to 1.3.5. In this equation, $H > 0$ because it is a positive proportionality constant that represents how strongly the external environment removes heat. If the end of the rod is hotter than the external temperature then there is a positive heat loss and $\phi(L,t) > 0$, which requires $H > 0$. If the external temperature is hotter than the end of the rod, then there is a negative heat loss (heat is flowing into the rod) and $\phi(L,t) < 0$, which still requires $H > 0$ otherwise the negatives would cancel out. Finally, if the external temperature is the same as the temperature at the end of the rod, then $\phi(L,t) = 0$ and no heat is transferred.

Problem 1.3.2: Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_{0-}, t) = u(x_{0+}, t)$$

and no heat energy is lost at $x = x_0$. What mathematical equation represents the later condition at $x = x_0$? Under what special condition is $\frac{\partial u}{\partial x}$ continuous at $x = x_0$?

Since no heat energy is lost at $x = x_0$, that means the heat flux of both rods are equal at $x = x_0$. So we can represent this condition by using Fourier's law and setting both fluxes equal to each other:

$$-K_1 \tfrac{\partial u}{\partial x}(x_0^-, t) = -K_2 \tfrac{\partial u}{\partial x}(x_0^+, t)$$

where K_i is the thermal conductivity constant for rod i. This equation ensures that the heat flux leaving the left rod is the same as the heat flux entering the right rod, so that there is no lost heat energy. In order for $\frac{\partial u}{\partial x}$ to be continuous, the two thermal conductivity constants must be equal ($K_1 = K_2$). Then the flux relation reduces to:

$$\tfrac{\partial u}{\partial x}(x_0^-, t) = \tfrac{\partial u}{\partial x}(x_0^+, t)$$

Otherwise if $K_1 \neq K_2$, then there will be a jump discontinuity at $x = x_0$ and $\frac{\partial u}{\partial x}$ will not be continuous.

Problem 1.4.1a,d, e-f: Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a) $Q = 0$, $u(0) = 0$, and $u(L) = T$

With constant thermal properties and no sources/sinks, the equilibrium heat equation (not dependent on time) is:

$$\tfrac{d^2u}{dx^2} = 0,\ 0 \leq x \leq L$$

By integrating twice we can find the temperature $u(x)$, then apply the boundary conditions (BCs) to solve for C_1 and C_2:

$$\begin{gathered} \tfrac{du}{dx} = C_1 \\ u(x) = C_1 x + C_2 \\ u(0) = 0 = C_2 \\ u(L) = T = C_1 L \\ C_1 = \tfrac{T}{L} \end{gathered}$$

Therefore, the equilibrium temperature distribution for this rod is:

$$u(x) = \tfrac{T}{L} x$$

(d) $Q = 0$, $u(0) = T$, and $\frac{\partial u}{\partial x}(L) = \alpha$

Follow same procedure as 1.4.1a but with different BCs:

$$\begin{gathered}
\frac{d^2u}{dx^2} = 0,\ 0 \le x \le L \\
\frac{du}{dx} = C_1 \\
u(x) = C_1 x + C_2 \\
\frac{du}{dx}(L) = \alpha = C_1 \\
u(0) = T = C_2
\end{gathered}$$

Therefore, the equilibrium temperature distribution for this rod is:

$$u(x) = \alpha x + T$$

(e) $\frac{Q}{K_0} = 1$, $u(0) = T_1$, and $u(L) = T_2$

With constant thermal properties, the equilibrium heat equation (not dependent on time) is:

$$\begin{gathered}
0 = K_0 \frac{\partial^2 u}{\partial x^2} + Q \\
\frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0}
\end{gathered}$$

In this case, $\frac{Q}{K_0} = 1$ so $\frac{\partial^2 u}{\partial x^2} = -1$. Then integrate twice to find the temperature $u(x)$, and apply the BCs to solve for C_1 and C_2:

$$\begin{gathered}
\frac{\partial u}{\partial x} = -x + C_1 \\
u(x) = -\frac{1}{2}x^2 + C_1 x + C_2 \\
u(0) = T_1 = C_2 \\
u(L) = T_2 \\
-\frac{1}{2}L^2 + C_1 L + T_1 = T_2 \\
\frac{T_2 - T_1 + \frac{1}{2}L^2}{L} = \frac{T_2 - T_1}{L} + \frac{L}{2} = C_1
\end{gathered}$$

Therefore, the equilibrium temperature distribution for this rod is:

$$u(x) = -\frac{1}{2}x^2 + \left(\frac{T_2 - T_1}{L} + \frac{L}{2}\right)x + T_1$$

(f) $\frac{Q}{K_0} = x^2$, $u(0) = T$, and $\frac{\partial u}{\partial x}(L) = 0$

With constant thermal properties, the equilibrium heat equation (not dependent on time) is:

$$0 = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$
$$\frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0}$$

In this case, $\frac{Q}{K_0} = x^2$ so $\frac{\partial^2 u}{\partial x^2} = -x^2$. Then integrate twice to find the temperature $u(x)$, and apply the BCs to solve for C_1 and C_2:

$$\frac{\partial u}{\partial x} = -\frac{x^3}{3} + C_1$$
$$u(x) = -\frac{x^4}{12} + C_1 x + C_2$$
$$u(0) = T = C_2$$
$$\frac{\partial u}{\partial x}(L) = 0$$
$$-\frac{L^3}{3} + C_1 = 0$$
$$\frac{L^3}{3} = C_1$$

Therefore, the equilibrium temperature distribution for this rod is:

$$u(x) = -\frac{x^4}{12} + \frac{L^3}{3} x + T$$

Problem 1.4.2a-b: Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $\frac{Q}{K_0} = x$ of thermal energy subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

(a) Determine the heat energy generated per unit time inside the entire rod.

If $Q(x)$ is a source of heat generated per unit length and it is given that $\frac{Q}{K_0} = x$, then:

$$Q(x) = K_0 x$$

The total heat generated per unit time over the entire rod is the integral of $Q(x)$ over $0 \leq x \leq L$:

$$\int_0^L Q(x) dx = \int_0^L K_0 x dx = K_0 \frac{L^2}{2}$$

Therefore, the rod generates $K_0 \frac{L^2}{2}$ units of heat energy per unit time.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$ (remember, this is at equilibrium).

With constant thermal properties, the equilibrium heat equation (not dependent on time) is:

$$0 = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$
$$\frac{\partial^2 u}{\partial x^2} = -\frac{Q}{K_0}$$

If $Q(x)$ is a source of heat and it is given that $\frac{Q}{K_0} = x$, then:

$$\frac{\partial^2 u}{\partial x^2} = -x$$

Then integrate twice to find the temperature $u(x)$, and apply the BCs to solve for C_1 and C_2:

$$\frac{du}{dx} = -\frac{x^2}{2} + C_1$$
$$u(x) = -\frac{x^3}{6} + C_1 x + C_2$$
$$u(0) = 0 = C_2$$
$$u(L) = 0$$
$$-\frac{L^3}{6} + C_1 L = 0$$
$$\frac{L^2}{6} = C_1$$

Therefore, the equilibrium temperature distribution for this rod is:

$$u(x) = -\frac{x^3}{6} + \frac{L^2}{6} x$$

To find the heat flux at the ends of the rod, we can use the 1D Fourier's law equation and the derivative of the equilibrium temperature distribution:

$$q(x) = -K_0 \frac{du}{dx}$$
$$\frac{du}{dx} = -\frac{x^2}{2} + \frac{L^2}{6}$$

At $x = 0$ and $x = L$:

$$\frac{du}{dx}|_{x=0} = 0 + \frac{L^2}{6} = \frac{L^2}{6}$$
$$q(0) = -K_0 \frac{L^2}{6} = -\frac{K_0 L^2}{6}$$
$$\frac{du}{dx}|_{x=L} = -\frac{L^2}{2} + \frac{L^2}{6} = -\frac{L^2}{3}$$
$$q(L) = -K_0 (-\frac{L^2}{3}) = -\frac{K_0 L^2}{3}$$

Therefore, the heat flux is $-\frac{K_0 L^2}{6}$ at the left end of the rod ($x = 0$) and $-\frac{K_0 L^2}{3}$ at the right end of the rod ($x = L$).

Problem 1.4.3: Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho - 1$, $K_0 = 1$) with a constant source ($Q = 1$), whereas for the other $1 < x < 2$, there are no sources ($Q = 0$, $c\rho = 2$, $K_0 = 2$ with $u(0) = 0 = u(2)$.

In order to find the equilibrium temperature distribution for this rod we must first find the equilibrium heat equations for both regions.
Region 1 ($0 < x < 1$):

$$\begin{gathered} \frac{d^2u_1}{dx^2} + \frac{Q_1}{K_1} = 0 \\ \frac{d^2u_1}{dx^2} + \frac{1}{1} = 0 \\ \frac{d^2u_1}{dx^2} = -1 \end{gathered}$$

Region 2 ($1 < x < 2$):

$$\begin{gathered} \frac{d^2u_2}{dx^2} + \frac{Q_2}{K_2} = 0 \\ \frac{d^2u_2}{dx^2} + 0 = 0 \\ \frac{d^2u_2}{dx^2} = 0 \end{gathered}$$

Then integrate both regions and apply the BCs to find the equilibrium heat distributions.
Region 1:

$$\begin{gathered} \frac{d^2u_1}{dx^2} = -1 \\ \frac{du_1}{dx} = -x + C_1 \\ u_1(x) = -\frac{x^2}{2} + C_1x + C_2 \\ u(0) = 0 \\ u_1(0) = 0 = C_2 \\ u_1(x) = -\frac{x^2}{2} + C_1x \end{gathered}$$

Region 2:

$$\begin{gathered} \frac{d^2u_2}{dx^2} = 0 \\ \frac{du_2}{dx} = C_3 \\ u_2(x) = C_3x + C_4 \\ u(2) = 0 \\ u_2(2) = 0 = 2C_3 + C_4 \\ -2C_3 = C_4 \\ u_2(x) = C_3x - 2C_3 = C_3(x-2) \end{gathered}$$

Next we can apply continuity to the temperature and heat flux at $x = 1$ because the two materials are in perfect thermal contact to solve for C_1 and C_3.
Temperature continuity:

$$\begin{gathered} u_1(1) = u_2(1) \\ -\frac{1^2}{2} + C_1(1) = C_3(1-2) \\ -\frac{1}{2} + C_1 = -C_3 \\ C_1 + C_3 = \frac{1}{2} \end{gathered}$$

Heat flux continuity:

$$\begin{gathered} K_1 \frac{du_1}{dx}(1) = K_2 \frac{du_2}{dx}(1) \\ (1)(-1 + C_1) = 2C_3 \\ C_1 - 1 = 2C_3 \\ C_1 - 2C_3 = 1 \end{gathered}$$

Solve for constants C_1 and C_3:

$$\begin{gathered} C_3 = \frac{1}{2} - C_1 \\ C_1 - 2(\frac{1}{2} - C_1) = 1 \\ C_1 - 1 + 2C_1 = 1 \\ 3C_1 - 1 = 1 \\ 3C_1 = 2 \\ C_1 = \frac{2}{3} \\ C_3 = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6} \end{gathered}$$

Now we can finally write the equilibrium temperature distributions for both regions:

$$\begin{gathered} u_1(x) = -\frac{x^2}{2} + \frac{2}{3}x \\ u_2(x) = -\frac{1}{6}(x-2) = -\frac{x}{6} + \frac{1}{3} \end{gathered}$$

Problem 1.4.5: Consider a one-dimensional rod $0 \leq x \leq L$ of known length and constant thermal properties without sources or sinks. Suppose that the temperature is an *unknown* constant T at $x = L$. Determine T if we known (in the steady state) for the temperature and the heat flow at $x = 0$.

For a 1D rod with no sources/sinks and constant thermal properties we have the following equilibrium heat equation and distribution:

$$\frac{d^2u}{dx^2} = 0$$
$$u(x) = C_1 x + C_2$$

With the given BCs at $x = 0$ and $x = L$, we know:

$$u(0) = u_o$$
$$q(0) = -K_0 \frac{du}{dx}|_{x=0} = q_0$$
$$u(L) = T$$

where T is the unknown temperature at the right end of the rod. Next we must solve for the constants C_1 and C_2 using the temperature and heat flux equations to determine the temperature T.

$$u(0) = C_1(0) + C_2 = u_0$$
$$u_0 = C_2$$
$$q(0) = -K_0 \frac{du}{dx}(0)$$
$$-K_0 C_1 = q_0$$
$$-\frac{q_0}{K_0} = C_1$$
$$u(L) = C_1 L + C_2 = -\frac{q_0}{K_0} L + u_0$$
$$T = u(L) = u_0 - \frac{q_0}{K_0} L$$

Problem 1.4.7a-b: For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

(a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, and $\frac{\partial u}{\partial x}(L,T) = \beta$

In order to determine this equilibrium heat distribution we can follow the same basic procedure from the above problems. First find the equilibrium heat equation, then integrate twice and apply BCs to solve for the equilibrium heat distribution. The only difference is this time we need to check conditions for the equilibrium's existence.
Equilibrium heat equation:

$$\frac{d^2u}{dx^2} + 1 = 0$$
$$\frac{d^2u}{dx^2} = -1$$

Integrate twice:

$$\frac{du}{dx} = -x + C_1$$
$$u(x) = -\frac{x^2}{2} + C_1 x + C_2$$

Apply BCs:

$$\frac{du}{dx}(0) = 1$$
$$\frac{du}{dx}(L) = \beta$$

At $x = 0$:

$$\frac{du}{dx}(0) = 1 = -(0) + C_1$$
$$1 = C_1$$

At $x = L$:

$$\frac{du}{dx}(L) = \beta = -(L) + C_1$$
$$1 - L = \beta$$

Equilibrium temperature distribution:

$$u(x) = -\frac{x^2}{2} + x + C_2$$

where C_2 is an arbitrary constant. In this case, the equilibrium temperature distribution only exists if $\beta = 1 - L$ because the net heat flow must be balanced. The rod has a uniform internal heat source that is constantly generating heat, so for an equilibrium to exist, the heat leaving the rod must be equal to the heat that is being generated in the rod. This requires the derivative at $x = L$ to satisfy $\beta = 1 - L$, otherwise the heat in the rod would accumulate indefinitely and there would be no equilibrium.

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, and $\frac{\partial u}{\partial x}(L,T) = \beta$

Repeat the procedure for the previous problem but with different BCs:
Equilibrium heat equation:

$$\frac{d^2 u}{dx^2} = 0$$

Integrate twice:

$$\frac{du}{dx} = C_1$$
$$u(x) = C_1 x + C_2$$

Apply BCs:

$$\frac{du}{dx}(0) = 1 = C_1$$
$$\frac{du}{dx}(L) = \beta = C_1$$

In this case, the equilibrium only exists if $\beta = 1$ and the equilibrium temperature distribution is:

$$u(x) = x + C_2$$

where C_2 is an arbitrary constant. This case is different than the previous distribution because there are no internal sources/sinks so the temperature gradient must be constant throughout the entire rod and the heat flux must be equal at both ends of the rod. If $\beta \neq 1$ at $x = L$, there would be a non-constant temperature gradient throughout the rod, therefore preventing equilibrium.

Problem 1.4.10: Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, and $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

We can represent the energy density of the rod as follows:

$$e(x,t) = c\rho u(x,t)$$

Then the total energy in the rod as a function of time is:

$$E(t) = \int_0^L c\rho u(x,t)dx = c\rho \int_0^L u(x,t)dx$$

Next, differentiate with respect to time:

$$\frac{dE}{dt} = c\rho \int_0^L \frac{\partial u}{\partial t}(x,t)dx$$
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$$
$$\frac{dE}{dt} = c\rho \int_0^L (\frac{\partial^2 u}{\partial x^2} + 4)dx$$

Then, simplify the integrals and apply BCs:

$$\frac{dE}{dt} = c\rho([\frac{\partial u}{\partial x}(x,t)]_0^L + 4L)$$
$$\frac{dE}{dt} = c\rho([(6-5) + 4L) = c\rho(1+4L)$$

Next, integrate in time:

$$E(t) = E(0) + c\rho(1+4L)t$$

where

$$E(0) = c\rho \int_0^L f(x)dx$$

This yields the final equation, which represents the total thermal energy in the 1D rod:

$$E(t) = c\rho \int_0^L f(x)dx + c\rho(1+4L)t$$