HOMEWORK 1 **NAME:** Josh Derrow

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Questions	Points	Score
Total		

Problem 1.2.1(a): Briefly explain the minus sign in conservation law (1.2.3) or (1.2.5) if $Q = 0$.

Conservation Law: $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$
In this case, $Q = 0$ (no sources/sinks) so: $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$ where $e(x,t)$ is the internal energy density and $\phi(x,t)$ is the heat flux. $\frac{\partial \phi}{\partial x}$ represents how the heat flux changes with respect to x and the law is kept consistent with the minus sign in the front because it ensures that the heat flowing out of a certain region reduces the total energy in the system ($-\frac{\partial \phi}{\partial x} < 0$). On the other hand, if heat is flowing into a region, then $-\frac{\partial \phi}{\partial x} > 0$, which positively contributes to the internal energy density and represents the region getting warmer.

Problem 1.2.4: Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

There are no sources/sinks, so $Q = 0$.
If $e(x,t)$ is the internal energy density of the rod and $A(x)$ is the variable cross-sectional area of the rod, then the total thermal energy of the rod from $x = a$ to $x = b$ is: $\int_a^b e(x,t)A(x)dx$.
Assume $\phi(x,t)$ is the heat flux flowing in and out of the rod, then the net heat flow from $x = a$ to $x = b$ is: $[A\phi](b,t) - [A\phi](a,t) = \int_a^b \frac{\partial}{\partial x}(A\phi)dx$.
Then $\frac{d}{dt}\int_a^b eAdx = -\int_a^b \frac{\partial}{\partial x}(A\phi)dx$ by the law of conservation.
$A(x)$ does not depend on time t.
Then $\int_a^b A\frac{\partial e}{\partial t}dx = -\int_a^b \frac{\partial}{\partial x}(A\phi)dx$
$\Rightarrow A\frac{\partial e}{\partial t} = -\frac{\partial}{\partial x}(A\phi)$
Then divide both sides by A : $\frac{\partial e}{\partial t} = -\frac{1}{A}\frac{\partial}{\partial x}(A\phi)$
Heat Equation: $\frac{\partial e}{\partial t} = -\frac{1}{A(x)}\frac{\partial}{\partial x}(A(x)\phi(x,t))$

Problem 1.2.5: Derive the diffusion equation for a chemical pollutant.

(a) Consider the total amount of the chemical in a thin region between x and $x + \Delta x$.

Assume:

- area A is constant
- diffusivity constant D
- $c(x,t)$ is the concentration of the pollutant (mass of pollutant per volume)
- $S(x,t)$ is a source/sink of the pollutant
- $J(x,t) = -D\frac{\partial c}{\partial x}$ is the diffusion flux by Fick's first law

Total mass of the pollutant in the region: $\int_x^{x+\Delta x} c(\xi,t)Ad\xi \approx c(x,t)A\Delta x$ (for small Δx).
Then $\frac{d}{dt}(cA\Delta x) = [J(x,t)A] - [J(x+\Delta x,t)A] + S(x,t)A\Delta x$ by the law of conservation.
A and Δx do not depend on time so: $\frac{\partial c}{\partial t}A\Delta x = -(J(x+\Delta x,t) - J(x,t))A + SA\Delta x$
Dividing both sides by $A\Delta x$ yields: $\frac{\partial c}{\partial t} = -\frac{J(x+\Delta x,t)-J(x,t)}{\Delta x} + S(x,t)$
Then, take the limit as $\Delta x \to 0$ (limit definition of the derivative) to get the conservation equation: $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} + S(x,t)$
Substitute into the conservation equation: $\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x}(-D\frac{\partial c}{\partial x}) + S = \frac{\partial}{\partial x}(D\frac{\partial c}{\partial x}) + S$
If D is constant, then the diffusion equation is: $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + S(x,t)$

(b) Consider the total amount of the chemical between $x = a$ and $x = b$.

Assume:

- area A is constant
- diffusivity constant D
- $c(x,t)$ is the concentration of the pollutant (mass of pollutant per volume)
- $S(x,t)$ is a source/sink of the pollutant
- $J(x,t) = -D\frac{\partial c}{\partial x}$ is the diffusion flux by Fick's first law

Total mass of the pollutant in the region: $\int_a^b c(x,t)Adx$
Then $\frac{d}{dt}\int_a^b cAdx = AJ(a,t) - AJ(b,t) + \int_a^b SAdx = -\int_a^b \frac{\partial}{\partial x}(AJ)dx + \int_a^b SAdx$ by the law of conservation.
A does not depend on time so: $\int_a^b A\frac{\partial c}{\partial t}dx = -\int_a^b A\frac{\partial J}{\partial x}dx + \int_a^b SAdx$
Dividing both sides by A and use FTC to get the conservation equation: $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} + S(x,t)$
Substitute into the conservation equation: $\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x}(-D\frac{\partial c}{\partial x}) + S = \frac{\partial}{\partial x}(D\frac{\partial c}{\partial x}) + S$
If D is constant, then the diffusion equation is: $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + S(x,t)$

Problem 1.2.6: Derive an equation for the concentration $u(x,t)$ of a chemical pollutant if the chemical is produced due to a chemical reaction at a rate of $\alpha u\,(\beta - u)$ per unit volume.

Assume:

- diffusivity constant D
- $u(x,t)$ is the concentration of the pollutant (mass of pollutant per volume)
- $R(u) = \alpha u(\beta - u)$ is the source of the chemical reaction
- $J(x,t) = -D\frac{\partial u}{\partial x}$ is the diffusion flux by Fick's first law

If we consider a slice $(x, x+\Delta x)$ of the region, then $\frac{\partial}{\partial t}(u\Delta x) = J(x) - J(x+\Delta x) + R(u)\Delta x$
Divide by Δx and take the limit as $\Delta x \to 0$: $\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x} + R(u)$ (conservation equation)
Substitute into the conservation equation: $-\frac{\partial J}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$
Then the reaction-diffusion equation is: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \alpha u(\beta - u)$

Problem 1.2.9: If $u(x,t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

Assume:

- $u(x,t)$ is the temperature of the rod
- area A is constant
- c is the heat capacity
- ρ is the density of the rod
- the rod extends from $x = 0$ to $x = L$

Then the energy density $e(x,t) = c\rho u(x,t)$
Then the total thermal energy is: $\int_0^L e(x,t)Adx = \int_0^L c\rho u(x,t)Adx = c\rho A\int_0^L u(x,t)dx$

Problem 1.2.10: Consider a thin, one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

Assume:

- $u(x,t)$ is the temperature of the rod
- area A is constant
- P is the perimeter of the rod's lateral surface
- w(x, t) is the heat flux
- c is the heat capacity
- ρ is the density of the rod
- k is the thermal conductivity

If we consider a slice $(x, x+\Delta x)$ of the region, then the volume is: $A\Delta x$
The total energy in the slice is: $c\rho u(x,t)A\Delta x$
Then the rate of change of the energy is: $\frac{\partial}{\partial t}(c\rho uA\Delta x)$
The lateral heat flux $\phi(x,t) = -k\frac{\partial u}{\partial x}$ and the total flux through the cross-section is: $A\phi$
Then the net flux out of the slice is: $[A\phi(x+\Delta x,t) - A\phi(x,t)] \approx A\frac{\partial \phi}{\partial x}\Delta x$
Because the energy leaving the slice reduces the total internal energy, the contribution is: $-\frac{\partial}{\partial x}(A\phi)\Delta x$
The lateral area of the slice is: $P\Delta x$, and the heat lost per time is: $w(x,t)P\Delta x$
This also reduces energy from the slice so it's contribution is: $-wP\Delta x$
The conservation equation for the slice is: $c\rho A\frac{\partial u}{\partial t}\Delta x = -\frac{\partial}{\partial x}(A\phi)\Delta x - wP\Delta x$
Divide by $A\Delta x : c\rho\frac{\partial u}{\partial t} = -\frac{1}{A}\frac{\partial}{\partial x}(A\phi) - \frac{P}{A}w$
For a constant area A: $c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} - \frac{P}{A}w$
Substitute Fourier's law $(\phi = -k\frac{\partial u}{\partial x}) : c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial u}{\partial x}) - \frac{P}{A}w$
Simplify for constant k and the pde for the temperature is: $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} - \frac{P}{c\rho A}w(x,t)$ where $\alpha = \frac{k}{c\rho}$

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$ Derive

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}\left[u(x,t) - \gamma(x,t)\right]h(x),$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

Energy balance for a slice $(x, x+\Delta x)$: $c\rho A\frac{\partial u}{\partial t}\Delta x = -\frac{\partial}{\partial x}(A\phi)\Delta x - Pw(x,t)\Delta x$
Divide by $A\Delta x$ and take the limit as $\Delta x \to 0$: $c\rho\frac{\partial u}{\partial t} = -\frac{1}{A}\frac{\partial}{\partial x}(A\phi) - \frac{P}{A}w(x,t)$
For constant A: $c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} - \frac{P}{A}w(x,t)$
Substitute Fourier's law $(\phi = -K_0\frac{\partial u}{\partial x})$: $-\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x})$
Proportional lateral heat loss: $w(x,t) = h(x)(u(x,t) - \gamma(x,t))$ with $h(x) > 0$
Substitute into the energy balance for the final equation: $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K_0\frac{\partial u}{\partial x}) - \frac{P}{A}h(x)(u(x,t) - \gamma(x,t))$