

Problem 1.5.10: Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at a temperature T_0 .

In order to find the equilibrium temperature distribution $u(r, \theta)$ inside a circle we can use the 2D Polar Laplace Equation (No sources/sinks) at equilibrium ($\nabla^2 u = 0$):

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Since the boundary of the circle is at a fixed (constant) temperature T_0 , the temperature inside the circle does not depend on the angular coordinate θ , which means that the equilibrium temperature distribution $u(r)$ will also not depend on θ ($\frac{\partial u}{\partial \theta} = 0$). With this in mind, we can simplify the Laplace equation and integrate twice to find the general solution to the simplified ODE:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$\frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$r \frac{du}{dr} = C_1$$

$$\frac{du}{dr} = \frac{C_1}{r}$$

$$u(r) = C_1 \ln(r) + C_2$$

Now it's time to apply the boundary conditions. In order for the temperature to be defined at the center of the circle ($r = 0$), $C_1 = 0$ must be true so that the term $C_1 \ln(r)$ goes away (because $\ln(0)$ is undefined). So with $C_1 = 0$ we can simplify the equation to get: $u(r) = C_2$. In addition, the temperature at the boundary of the circle $r = r_0$ is a constant T_0 so we know that $u(r_0) = T_0$. But if $u(r) = C_2$ and $u(r_0) = T_0$, then that means that $C_2 = T_0$, which yields a final equilibrium temperature distribution of $u(r) = T_0$.

Problem 1.5.11: Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad a < r < b,$$

subject to $u(r, 0) = f(r)$, $\frac{\partial u}{\partial r}(a, t) = \beta$, and $\frac{\partial u}{\partial r}(b, t) = 1$. For what value(s) of β does an equilibrium temperature distribution exist?

In order to find the value(s) of β for which an equilibrium temperature distribution exists, we need to follow a similar procedure as the previous problem (1.5.10)(First simplify PDE with given information, next integrate twice to find the general solution to the ODE, then plug in boundary conditions):

$$\frac{k}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$\frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$r \frac{du}{dr} = C_1$$

$$\frac{du}{dr} = \frac{C_1}{r}$$

$$u(r) = C_1 \ln(r) + C_2$$

Which is the same general solution as the previous problem (1.5.10), however, this time our boundary conditions are different. We know that $\frac{\partial u}{\partial r}(a) = \beta$ and $\frac{\partial u}{\partial r}(b) = 1$, so we can plug these boundary conditions into $\frac{\partial u}{\partial r}(r)$ and solve for β to find where the equilibrium temperature distribution exists:

$$\begin{aligned} u'(a) &= \frac{C_1}{a} = \beta \\ u'(b) &= \frac{C_1}{b} = 1 \\ C_1 &= a\beta \\ \frac{a\beta}{b} &= 1 \\ a\beta &= b \\ \beta &= \frac{b}{a} \end{aligned}$$

So, the equilibrium temperature distribution only exists when $\beta = \frac{b}{a}$.

Problem 2.2.1: Show that any linear combination of linear operators is a linear operator.

An operator L is linear if it satisfies both the sum and scalar multiplication definitions of a linear operator. To satisfy both definitions of a linear operator, the superposition principle:

$$L(au + bv) = aL(u) + bL(v)$$

must hold for any arbitrary scalars a and b , and any arbitrary vectors u and v in the operator's domain. Let L_1 and L_2 be two linear operators; we want to show that their sum $L_1 + L_2$ is also a linear operator. By definition, the sum of two linear operators is:

$$(L_1 + L_2)(u) = L_1(u) + L_2(u)$$

for any vector u in the domain. But we also need to show that the product of an arbitrary scalar and a linear operator is also linear. By definition, the product of a scalar and a linear operator is:

$$(cL)(u) = cL(u)$$

for any vector u in the domain and an arbitrary real number c . For example: let V be a vector space, let $a, b \in \mathbb{R}$, and let $u, v \in V$. So we can plug in our arbitrary linear combination into the superposition principle from before to get:

$$(L_1 + L_2)(au + bv) = L_1(au + bv) + L_2(au + bv)$$

Next we can apply the linearity property, rearrange the terms, factor out a and b , and then use the definition of the sum of operators again to prove that the definition is satisfied.

$$\begin{aligned} &= (aL_1(u) + bL_1(v)) + (aL_2(u) + bL_2(v)) \\ &= aL_1(u) + aL_2(u) + bL_1(v) + bL_2(v) \\ &= a(L_1(u) + L_2(u)) + b(L_1(v) + L_2(v)) \\ &= a(L_1 + L_2)(u) + b(L_1 + L_2)(v) \\ &= a(L_1 + L_2)(u) + b(L_1 + L_2)(v) = aL(u) + bL(v) = L(au + bv) \end{aligned}$$

Which is equal to the superposition principle from the beginning, so both the sum and multiplication definitions of a linear operator hold with our arbitrary linear combination. Since both definitions hold with our arbitrary linear combination, then any finite combination of linear operators is also a linear operator.

Problem 2.2.2: Show that

(a) $L(u) = \frac{\partial}{\partial x} [K_0(x) \frac{\partial u}{\partial x}]$ is a linear operator

In order to show that L is a linear operator, the superposition principle from earlier must hold for an arbitrary linear combination. We can use a similar setup to the previous problem (2.2.1), except this time we need to apply it to the PDE and show that L is linear.

$$L(au + bv) = aL(u) + bL(v)$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left[K_0(x) \frac{\partial u}{\partial x} \right] \\ L(au + bv) &= \frac{\partial}{\partial x} [K_0(x)(au_x + bv_x)] \\ &= \frac{\partial}{\partial x} [aK_0(x)u_x + bK_0(x)v_x] \\ &= a \frac{\partial}{\partial x} [K_0(x)u_x] + b \frac{\partial}{\partial x} [K_0(x)v_x] \\ &= a \frac{\partial}{\partial x} \left[K_0(x) \frac{\partial u}{\partial x} \right] + b \frac{\partial}{\partial x} \left[K_0(x) \frac{\partial v}{\partial x} \right] \\ L(u) &= \frac{\partial}{\partial x} \left[K_0(x) \frac{\partial u}{\partial x} \right] \\ L(au + bv) &= aL(u) + bL(v) \end{aligned}$$

Which is equal to the superposition principle on the first line, verifying that L is a linear operator

(b) and usually $L(u) = \frac{\partial}{\partial x} [K_0(x, u) \frac{\partial u}{\partial x}]$ is not a linear operator.

In order to show that L is not a linear operator, the superposition principle must be contradicted by any arbitrary linear combination $(au + bv)$. Once again, we can use a similar setup to the previous problems (2.2.1 and 2.2.2a), except this time we need to apply it to the PDE and show that L is not linear.

$$L(au + bv) = aL(u) + bL(v)$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left[K_0(x, u) \frac{\partial u}{\partial x} \right] \\ u_x &= \frac{\partial u}{\partial x}, v_x = \frac{\partial v}{\partial x} \\ L(au + bv) &= \frac{\partial}{\partial x} [K_0(x, au + bv)(au_x + bv_x)] \\ &= \frac{\partial K_0}{\partial x}(x, au + bv)(au_x + bv_x) + \frac{\partial K_0}{\partial u}(x, au + bv)(au_x + bv_x)^2 + K_0(x, au + bv)(au_{xx} + bv_{xx}) \\ &\neq a \left(\frac{\partial}{\partial x} \left[K_0(x, u) \frac{\partial u}{\partial x} \right] \right) + b \left(\frac{\partial}{\partial x} \left[K_0(x, v) \frac{\partial v}{\partial x} \right] \right) = aL(u) + bL(v) \end{aligned}$$

This is not equal to the superposition principle because $K_0(x, au + bv)$ and its derivatives are not linear combinations of $K_0(x, u)$ and $K_0(x, v)$. The extra terms involving $\frac{\partial K_0}{\partial u}$ and the squared partial derivative $(au_x + bv_x)^2$ produce nonlinear combinations that contradict the definitions of a linear operator. Therefore, L is not a linear operator.

Problem 2.2.3: Show that $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$ is linear if $Q = \alpha(x, t)u + \beta(x, t)$ and, in addition, homogeneous if $\beta(x, t) = 0$.

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t) \\ L(u) &= \beta(x, t) = u_t - ku_{xx} - \alpha(x, t)u \\ L(au + bv) &= (au + bv)_t - k(au + bv)_{xx} - \alpha(x, t)(au + bv) \\ &= aut + bv_t - k(au_{xx} + bv_{xx}) - \alpha au - \alpha bv \\ &= a(u_t - ku_{xx} - \alpha u) + b(v_t - kv_{xx} - \alpha v) \\ L(au + bv) &= aL(u) + bL(v)\end{aligned}$$

Which satisfies the superposition principle, therefore $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t)$ is a linear PDE. Usually, this PDE is non-homogeneous because it has the form $L(u) = \beta(x, t)$. However, when $\beta(x, t) = 0$ then $L(u) = 0$ and the linear PDE becomes homogeneous because $u_t - ku_{xx} - \alpha(x, t)u = 0$.

Problem 2.2.4: In this exercise we derive superposition principles for non-homogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

$$\begin{aligned}L(u_p) &= f \\ L(u_1) &= L(u_2) = 0 \\ u &= u_p + c_1u_1 + c_2u_2 \\ L(u) &= L(u_p) + c_1L(u_1) + c_2L(u_2) \\ &= f + c_1(0) + c_2(0) = f = L(u)\end{aligned}$$

Thus u is also a particular solution because it's a linear combination of the pre-existing particular solution u_p .

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i , what is a particular solution for $f_1 + f_2$?

$$\begin{aligned}L(u) &= f_1 + f_2 \\ L(u_{p1}) &= f_1, L(u_{p2}) = f_2 \\ u_p &= u_{p1} + u_{p2} \\ L(u_p) &= L(u_{p1} + u_{p2}) \\ &= L(u_{p1}) + L(u_{p2}) = f_1 + f_2 = L(u)\end{aligned}$$

Thus the particular solution for $f_1 + f_2$ is $u_p = u_{p1} + u_{p2}$.

Problem 2.3.1 b,d: For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$

$$\begin{aligned} u(x, t) &= X(x)T(t) \\ u_t &= ku_{xx} - v_0 u_x \\ X(x)T'(t) &= kX''(x)T(t) - v_0 X'(x)T(t) \\ \frac{T'(t)}{T(t)} &= k \frac{X''(x)}{X(x)} - v_0 \frac{X'(x)}{X(x)} = \lambda \end{aligned}$$

The left side only depends on t and the right side only depends on x but both are equal to the same separation constant λ . We can separate both sides to get the following two ODEs:

$$\begin{aligned} T'(t) - \lambda T(t) &= 0 \\ kX''(x) - v_0 X'(x) - \lambda X(x) &= 0 \end{aligned}$$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$

$$\begin{aligned} u(r, t) &= R(r)T(t) \\ u_t &= \frac{k}{r^2} \frac{\partial}{\partial r} (r^2 u_r) \\ R(r)T'(t) &= \frac{k}{r^2} (r^2 R'(r))' T(t) \\ \frac{T'(t)}{T(t)} &= \frac{k}{r^2} \frac{(r^2 R'(r))'}{R(r)} = \lambda \end{aligned}$$

The left side only depends on t and the right side only depends on r but both are equal to the same separation constant λ . We can separate both sides to get the following two ODEs:

$$\begin{aligned} T'(t) - \lambda T(t) &= 0 \\ \frac{k}{r^2} (r^2 R'(r))' - \lambda R(r) &= 0 \end{aligned}$$

Problem 2.3.2 a,e: Consider the differential equation

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0,$$

where ϕ is a function of x only. Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, and $\lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

Case 1: $\lambda > 0$

$$\lambda = \mu^2, \mu > 0$$

$$\phi(x) = A\cos(\mu x) + B\sin(\mu x)$$

$$\phi(0) = 0 = A$$

$$\phi(x) = B\sin(\mu x)$$

$$\phi(\pi) = 0 = B\sin(\mu\pi)$$

For a nontrivial solution, we need $B \neq 0$, so $\sin(\mu\pi) = 0$. To find the eigenvalue λ and the corresponding eigenfunction, assume $n \in \mathbb{N}$ such that $n = 1, 2, 3, \dots$, then:

$$\mu\pi = n\pi$$

$$\mu = n$$

$$\lambda_n = \mu^2 = n^2$$

$$\phi_n(x) = B\sin(nx) = 0$$

Case 2: $\lambda = 0$

$$\phi''(x) = 0$$

$$\phi(x) = Ax + B$$

$$\phi(0) = 0 = B$$

$$\phi(\pi) = 0 = A\pi$$

$$A = 0$$

$$\phi(x) = 0$$

So only the trivial solution $\phi(x) = 0$ exists and there is no nontrivial eigenfunction for $\lambda = 0$.

Case 3: $\lambda < 0$

$$\lambda = -\mu^2, \mu > 0$$

$$\phi(x) = Ce^{\mu x} + De^{-\mu x}$$

$$\phi(0) = 0 = C + D$$

$$= C(e^{\mu x} - e^{-\mu x}) = 2C\sinh(\mu x)$$

$$\phi(\pi) = 0 = 2C\sinh(\mu\pi)$$

However $\sinh(\mu\pi) \neq 0$ for $\mu > 0$, so $C = 0$. This means that only the trivial solution exists and there is no nontrivial eigenfunction for $\lambda < 0$. Taking all three of these cases into account, there is only one nontrivial eigenvalue $\lambda_n = n^2$ and eigenfunction $\phi_n(x) = B\sin(nx)$ (where $B \in \mathbb{R}$ and $n \in \mathbb{N}$) for the case when $\lambda > 0$. There are no nontrivial solutions for $\lambda \leq 0$.

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

Case 1: $\lambda > 0$

$$\lambda = \mu^2, \mu > 0$$

$$\phi(x) = A\cos(\mu x) + B\sin(\mu x)$$

$$\phi'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x)$$

$$\phi'(0) = 0 = B\mu$$

$$B = 0$$

$$\phi(x) = A\cos(\mu x)$$

$$\phi(L) = 0 = A \cos(\mu L)$$

For a nontrivial solution, we need $A \neq 0$, so $\cos(\mu L) = 0$. To find the eigenvalue λ and the corresponding eigenfunction, assume $n \in \mathbb{N}$ such that $n = 1, 2, 3, \dots$, then:

$$\begin{aligned}\mu L &= \frac{(2n+1)\pi}{2} \\ \mu &= \frac{(2n+1)\pi}{2L} \\ \lambda_n &= \mu^2 = \left(\frac{(2n+1)\pi}{2L}\right)^2 \\ \phi_n(x) &= \cos\left(\frac{(2n+1)\pi}{2L}x\right)\end{aligned}$$

Case 2: $\lambda = 0$

$$\begin{aligned}\phi''(x) &= 0 \\ \phi(x) &= Ax + B \\ \phi'(x) &= A \\ \phi'(0) &= 0 = A \\ \phi(x) &= B \\ \phi(L) &= 0 = B\end{aligned}$$

So only the trivial solution $\phi(x) = 0$ exists and there is no nontrivial eigenfunction for $\lambda = 0$.

Case 3: $\lambda < 0$

$$\begin{aligned}\lambda &= -\mu^2, \mu > 0 \\ \phi(x) &= Ce^{\mu x} + De^{-\mu x} \\ \phi'(x) &= C\mu e^{\mu x} - D\mu e^{-\mu x} \\ \phi'(0) &= 0 = \mu(C - D) \\ C - D &= 0 \\ C &= D \\ \phi(x) &= 2C \cosh(\mu x) \\ \phi(L) &= 0 = 2C \cosh(\mu L)\end{aligned}$$

However $\cosh(\mu x) > 0$ for all x , so $C = 0$. This means that only the trivial solution exists and there is no nontrivial eigenfunction for $\lambda < 0$. Taking all three of these cases into account, there is only one nontrivial eigenvalue $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$ and eigenfunction $\phi_n(x) = \cos\left(\frac{(2n+1)\pi}{2L}x\right)$ for the case when $\lambda > 0$. There are no nontrivial solutions for $\lambda \leq 0$.

Problem 2.3.3 a,c,d: Consider the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Solve the initial value problem if the temperature is initially

(a) $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$

$$\begin{aligned} u_t &= ku_{xx} \\ u(x, 0) = f(x) &= 6 \sin\left(\frac{9\pi x}{L}\right) \\ u(x, t) &= X(x)T(t) \\ X(x)T'(t) &= kX''(x)T(t) \\ \frac{1}{k} \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = -\lambda \\ T'(t) &= -\lambda k T(t) \\ X''(x) + \lambda X(x) &= 0 \end{aligned}$$

Case $\lambda > 0$:

$$\lambda = \mu^2$$

$$\begin{aligned} X(x) &= A \cos(\mu x) + B \sin(\mu x) \\ X(0) &= 0 = A \\ X(x) &= B \sin(\mu x) \\ X(L) &= 0 = B \sin(\mu L) \end{aligned}$$

For a nontrivial solution, we need $B \neq 0$, so $\sin(\mu L) = 0$, assume $n \in \mathbb{N}$ such that $n = 1, 2, 3, \dots$, then:

$$\begin{aligned} \mu L &= n\pi \\ \mu &= \frac{n\pi}{L} \\ X_n(x) &= \sin\left(\frac{n\pi x}{L}\right), \lambda_n = \mu^2 = \left(\frac{n\pi}{L}\right)^2 \\ T_n(t) &= a_n e^{-k\lambda_n t} = a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u_n(x, t) &= a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u(x, 0) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \\ f(x) &= 6 \sin\left(\frac{9\pi x}{L}\right) \\ a_9 &= 6, \quad a_n = 0 \quad (n \neq 9) \end{aligned}$$

$$u(x, t) = 6 \sin\left(\frac{9\pi x}{L}\right) e^{-k\left(\frac{9\pi}{L}\right)^2 t}$$

(c) $u(x, 0) = 2 \cos\left(\frac{3\pi x}{L}\right)$

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ u(x, 0) &= f(x) = 2 \cos\left(\frac{3\pi x}{L}\right) \\ a_n &= \frac{2}{L} \int_0^L 2 \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{4n}{\pi} \frac{1 + (-1)^n}{n^2 - 9} \quad (n \neq 3) \end{aligned}$$

There is a removable singular point at $n = 3$, which means that $a_3 = 0$ and only the even n terms remain. Let $m \in \mathbb{N}$ such that $n = 2m$, then:

$$\begin{aligned} a_{2m} &= \frac{16m}{\pi(4m^2 - 9)} \\ u(x, t) &= \sum_{m=1}^{\infty} \left(\frac{16m}{\pi(4m^2 - 9)} \right) \sin\left(\frac{2m\pi x}{L}\right) e^{-k\left(\frac{2m\pi}{L}\right)^2 t} \end{aligned}$$

(d) $u(x, 0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ a_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ u(x, 0) &= f(x) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases} \end{aligned}$$

Compute two integrals for both intervals and let $\alpha = \frac{n\pi}{L}$:

$$\begin{aligned} \int_0^{\frac{L}{2}} (1) \sin(\alpha x) dx &= \frac{1 - \cos(\alpha \frac{L}{2})}{\alpha} \\ \int_{\frac{L}{2}}^L (2) \sin(\alpha x) dx &= \frac{2(\cos(\alpha \frac{L}{2}) - \cos(\alpha L))}{\alpha} \\ a_n &= \frac{2}{L} \frac{1}{\alpha} [1 + \cos(\frac{n\pi}{2})] - 2(-1)^n \\ &= \frac{2}{n\pi} [1 + \cos(\frac{n\pi}{2})] - 2(-1)^n \end{aligned}$$

If n is odd then $\cos(\frac{n\pi}{2}) = 0$ and $(-1)^n = -1$, so $a_n = \frac{6}{n\pi}$. If n is even then $\cos(\frac{n\pi}{2}) = -1$ and $(-1)^n = 1$, so $a_n = -\frac{4}{n\pi}$. Then:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{6}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t} + \sum_{n=1}^{\infty} \left(-\frac{4}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t}$$

Problem 2.3.4 a-b: Consider

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$ and $u(x, 0) = f(x)$.

(a) What is the total heat energy in the rod as a function of time?

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(\frac{n\pi}{L})^2 t} \\ a_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ e(x, t) &= \int_0^L c\rho u(x, t) dx \\ &= c\rho \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx \\ &= c\rho \sum_{n=1}^{\infty} a_n e^{-k(\frac{n\pi}{L})^2 t} \left(\frac{L}{n\pi}(1 - (-1)^n)\right) \end{aligned}$$

When n is even $1 - (-1)^n = 0$, and when n is odd $1 - (-1)^n = 2$ so the total heat energy can be simplified to:

$$e(x, t) = c\rho \sum_{n=1}^{\infty} a_n \left(\frac{2L}{n\pi}\right) e^{-k(\frac{n\pi}{L})^2 t}$$

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

At $x = 0$:

$$\phi(0, t) = -k \frac{\partial u}{\partial x}(0, t)$$

At $x = L$:

$$\phi(L, t) = k \frac{\partial u}{\partial x}(L, t)$$