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25. $p_n = \frac{1}{3}p_{n-1}$, $p_1 = \frac{1}{3}$

$q_n = 4q_{n-1} - \frac{11}{9}q_{n-2}$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{9}$

a) Assume: $p_{n-1} = \left(\frac{1}{3}\right)^{n-1}$

Then: $p_n = \frac{1}{3}\cdot\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n \stackrel{\checkmark}{=} \left(\frac{1}{3}\right)^n$

Assume: $q_{n-1} = \left(\frac{1}{3}\right)^{n-1}$, $q_{n-2} = \left(\frac{1}{3}\right)^{n-2}$

Then: $q_n = 4\left(\frac{1}{3}\right)^{n-1} - \frac{11}{9}\left(\frac{1}{3}\right)^{n-2} = \left(\frac{1}{3}\right)^{n-2}\left[4\cdot\frac{1}{3} - \frac{11}{9}\right]$

$= \left(\frac{1}{3}\right)^{n-2}\left[\frac{4}{3} - \frac{11}{9}\right] = \left(\frac{1}{3}\right)^{n-2}\left[\frac{12-11}{9}\right]$

$= \left(\frac{1}{3}\right)^{n-2}\cdot\frac{1}{9} = \left(\frac{1}{3}\right)^n \stackrel{\checkmark}{=} \left(\frac{1}{3}\right)^n$

b) $p_{30} = 4.85694\times10^{-15}$ Error: 1.57772×10^{-30}

$\approx \left(\frac{1}{3}\right)^{30}$ (Low Error → Good approximation)

$q_{30} = -1.05458\times10^{-1}$ Error: 1.05458×10^{-1}

(High Error → Bad approximation)

c) $q_n = \frac{33a-9b}{10}\left(\frac{1}{3}\right)^n + \frac{9b-3a}{110}\left(\frac{11}{3}\right)^n$

$q_n = A\left(\frac{1}{3}\right)^n + B\left(\frac{11}{3}\right)^n$

$r^2 - 4r + \frac{11}{9} = 0$

$r = \frac{4 \pm \sqrt{16 - 4\cdot\frac{11}{9}}}{2} = \frac{4 \pm \sqrt{\frac{144-44}{9}}}{2} = \frac{4 \pm \sqrt{\frac{100}{9}}}{2} = \frac{4 \pm \frac{10}{3}}{2}$

$r_1 = \frac{11}{3}$, $r_2 = \frac{1}{3}$

$q_1 = A\left(\frac{1}{3}\right)^1 + B\left(\frac{11}{3}\right)^1 = \frac{A}{3} + \frac{11B}{3} = \frac{1}{3} \rightarrow A + 11B = 1$

$q_2 = A\left(\frac{1}{3}\right)^2 + B\left(\frac{11}{3}\right)^2 = \frac{A}{9} + \frac{121B}{9} = \frac{1}{9} \rightarrow A + 121B = 1$

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25c. $(A+121B)-(A+11B) = 110B = 0 \to B=0$

$A + 11(0) = 1 \to A = 1$

$q_n = A\left(\frac{1}{3}\right)^n + B\left(\frac{11}{3}\right)^n$

$q_n = 1\left(\frac{1}{3}\right)^n + 0\left(\frac{11}{3}\right)^n = \left(\frac{1}{3}\right)^n$

The recurrence relation q is bad because $\left(\frac{11}{3}\right)^n$ grows exponentially fast. This unstable second term dominates quickly leading to huge errors.

27. $f(x) = 273000x^4 - 277490x^3 - 228731x^2 + 256181x - 31234$

a) Bisection: $x_1 = -0.9700000000$
$f(x) = 0$ $\quad x_2 = 0.1428571428$
$x_3 = 0.7666666666$
$x_4 = 1.0769230769$

b) Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = -1$ (Starting Value 1)

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = -1 - \dfrac{f(-1)}{f'(-1)} = -0.9716359149$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = -0.9716359149 - \dfrac{f(-0.97\ldots)}{f'(-0.97\ldots)} = -0.97000$

3 steps to find with $x_1 = -1$

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27b. $x_1 = 0$ (Starting value 2)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0.1219216101$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.1219216101 - \frac{f(0.12\ldots)}{f'(0.12\ldots)}$

$= 0.1421448811$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.1421448811 - \frac{f(0.14\ldots)}{f'(0.14\ldots)}$

$= 0.1428562451$

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.1428562451 - \frac{f(0.14\ldots)}{f'(0.14\ldots)}$

$= 0.1428571428$

5 steps to find with $x_1 = 0$

$x_1 = 0.5$ (starting value 3)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.9992358634$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.9992358634 - \frac{f(0.99\ldots)}{f'(0.99\ldots)} = 1.1442302919$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.1442302919 - \frac{f(1.14\ldots)}{f'(1.14\ldots)} = 1.0915360971$

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.0915360971 - \frac{f(1.09\ldots)}{f'(1.09\ldots)} = 1.0778467411$

$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.0778467411 - \frac{f(1.07\ldots)}{f'(1.07\ldots)} = 1.0769271298$

$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = 1.0769271298 - \frac{f(1.07\ldots)}{f'(1.07\ldots)} = 1.076923077$

7 steps to find with $x_1 = 0.5$

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27b. $x_1 = 0.75$ (Starting Value 4)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{f(0.75)}{f'(0.75)} = 0.7664391571$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.7664391571 - \frac{f(0.76\ldots)}{f'(0.76\ldots)}$

$= 0.7666666127$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.7666666127 - \frac{f(0.76\ldots)}{f'(0.76\ldots)}$

$= 0.7666666666$

4 Steps to find with $x_1 = 0.75$

27c. $x = -0.9700000000$
6 Steps with $x_0 = -2$, $x_1 = -1$

$x = 0.1428571428$
8 Steps with $x_0 = -1$, $x_1 = 0$

$x = 0.7666666666$
11 Steps with $x_0 = 0$, $x_1 = 0.5$

$x = 1.0769230769$
10 Steps with $x_0 = 1$, $x_1 = 2$

28. $f(x) = x^3 - x^2$ has 2 roots: $(0,0)$ and $(1,0)$. Finding $(1,0)$ via bisection would be fine because it crosses the x-axis, however $f(x)$ does not cross the x-axis at $(0,0)$ so the method of bisection would not work because it would fail the IVT. ($f(x) < 0$ on both sides of $(0,0)$)

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29. $f(x) = \frac{1}{x} - c = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - c}{-\frac{1}{x_n^2}} = x_n + \left(\frac{\frac{1}{x_n} - c}{\frac{1}{x_n^2}}\right) = x_n + (1 - cx_n)x_n$$

$$= x_n(2 - cx_n)$$

$c = 7,\ x_0 = 0.1 \qquad x = \frac{1}{7} \approx 0.1428571428$

$$x_1 = x_0(2 - 7x_0) = 0.1(2 - 7(0.1)) = 0.13$$
$$x_2 = x_1(2 - 7x_1) = 0.1417$$
$$x_3 = x_2(2 - 7x_2) = 0.14284777$$
$$x_4 = x_3(2 - 7x_3) = 0.1428571422$$
$$x_5 = x_4(2 - 7x_4) = 0.1428571428 \approx \frac{1}{7}$$

5 Steps with $c = 7$ and $x_0 = 0.1$