Math 440 – P&A

Suppose a function and its derivative are piecewise smooth. Prove that the Fourier Series of the function can be differentiated term-by-term if the Fourier Series of the function is continuous.

Let $f(x)$ be a function on $[-L,L]$ and assume $f(x)$ & $f'(x)$ are piecewise smooth. Then the Fourier Series of $f(x)$ is given by:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Let c_n & d_n be the Fourier Series coefficients of $f'(x)$:

$$c_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$d_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Now, integrate c_n & d_n by parts:

$$c_n = \frac{1}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\right]\Big|_{-L}^{L} + \frac{n\pi}{L^2}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{(-1)^n}{L}\left(f(L) - f(-L)\right) + \frac{n\pi}{L} b_n$$

Since $\cos\left(\frac{n\pi L}{L}\right) = \cos(n\pi) = (-1)^n$ & $\int f(x)\sin(x)\,dx = L b_n$

Math 440 - P&A

$$d_n = \frac{1}{L}\left[f(x)\sin\left(\frac{n\pi x}{L}\right)\right]\Big|_{-L}^{L} - \frac{n\pi}{L^2}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{n\pi}{L}a_n$$

Since $\sin(\pm n\pi) = 0$. Then the resulting coefficients are:

$$c_n = \frac{n\pi}{L}b_n$$

$$d_n = -\frac{n\pi}{L}a_n$$

Which means that the Fourier series of $f'(x)$ is given by:

$$f'(x) \sim \sum_{n=1}^{\infty}\left(c_n\cos\left(\frac{n\pi x}{L}\right) + d_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L}a_n\sin\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}b_n\cos\left(\frac{n\pi x}{L}\right)\right)$$

Which is equal to the term-by-term derivative of the Fourier Series of $f(x)$:

$$= \frac{d}{dx}\left(a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)\right)$$

Additionally, since $f'(x)$ is piecewise smooth, its Fourier Series converges to $\frac{1}{2}(f'(x^+) + f'(x^-))$ and to $f'(x)$ at points where $f'(x)$ is continuous. Therefore, the Fourier Series of $f(x)$ can be differentiated term-by-term, and the derivative series represents $f'(x)$.