Problem 1.5.10: Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at a temperature T_0.

Problem 1.5.11: Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad a < r < b,$$

subject to $u(r,0) = f(r)$, $\frac{\partial u}{\partial r}(a,t) = \beta$, and $\frac{\partial u}{\partial r}(b,t) = 1$. For what value(s) of β does an equilibrium temperature distribution exist?

Problem 2.2.1: Show that any linear combination of linear operators is a linear operator.

Problem 2.2.2: Show that

(a) $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator

(b) and usually $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.

Problem 2.2.3: Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$ is linear if $Q = \alpha(x,t)u + \beta(x,t)$ and, in addition, homogeneous if $\beta(x,t) = 0$.

Problem 2.2.4: In this exercise we derive superposition principles for non homogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

Problem 2.3.1 b,d: For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$

Problem 2.3.2 a,e: Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

where ϕ is a function of x only. Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, and $\lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

Problem 2.3.3 a,c,d: Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$

(c) $u(x,0) = 2\cos\left(\frac{3\pi x}{L}\right)$

(d) $u(x,0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

Problem 2.3.4 a-b: Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ and $u(x,0) = f(x)$.

(a) What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out to the rod at $x = 0$? at $x = L$?