Problem 4.4.3(b): Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

Determine the solution (by separation of variables) for a string with fixed ends and with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$

The solution $u(x,t)$ of the wave equation with dampening is given by the Fourier sine series of u:

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$b_n(t) = \frac{2}{L} \int_0^L u(x,t) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n'(0) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Now we must plug the series into the wave equation PDE and find the angular frequency ω and the damping rate γ. Once these two values are known we can determine the damped frequency Ω:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

$$\rho_0 b_n''(t) + \beta b_n'(t) + T_0 \left(\frac{n\pi}{L}\right)^2 b_n(t) = 0$$

$$\omega_n^2 = \frac{T_0}{\rho_0} \left(\frac{n\pi}{L}\right)^2$$

$$\gamma = \frac{\beta}{2\rho_0}$$

$$\Omega_n = \sqrt{\omega_n^2 - \gamma^2}$$

Now we must plug these values back into the series coefficient formula:

$$b_n(t) = e^{-\gamma t}\Big[b_n(0)\cos(\Omega_n t) + \frac{b_n'(0) + \gamma b_n(0)}{\Omega_n} \sin(\Omega_n t)\Big]$$

So the final solution to the dampened wave equation is:

$$u(x,t) \sim \sum_{n=1}^{\infty} e^{-\gamma t}\Big[b_n(0)\cos(\Omega_n t) + \frac{b_n'(0) + \gamma b_n(0)}{\Omega_n} \sin(\Omega_n t)\Big] \sin\left(\frac{n\pi x}{L}\right)$$

Problem 4.4.7: If a vibrating string satisfying (4.4.1)-(4.4.3) is initially at rest ($g(x) = 0$), show that

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right],$$

where $F(x)$ is the odd-periodic extension of $f(x)$.

4.4.1 - Wave equation PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

4.4.2 - Boundary condition 1:

$$u(0,t) = 0 = u(L,t)$$

4.4.3 - Initial conditions:

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

First we must find the odd extension F of f which is given by the following piecewise function on $[-L, L]$:

$$F(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x \le 0 \end{cases}$$

Now we will apply the initial conditions at $t = 0$:

$$u(x,0) = \frac{1}{2}\Big(F(x) + F(x)\Big) = F(x) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = -\frac{c}{2}F'(x) + \frac{c}{2}F'(x) = 0$$

Now we will apply the boundary conditions at $x = 0$ and $x = L$:

$$u(0,t) = \frac{1}{2}\Big(F(-ct) + F(ct)\Big) = \frac{1}{2}\Big(-F(ct) + F(ct)\Big) = 0$$

$$u(L,t) = \frac{1}{2}\Big(F(L-ct) + F(L+ct)\Big) = \frac{1}{2}\Big(F(L-ct) - F(L-ct)\Big) = 0$$

Therefore $u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$, which satisfies the wave equation PDE, the fixed-end boundary conditions, and the initial condition on $[0, L]$.