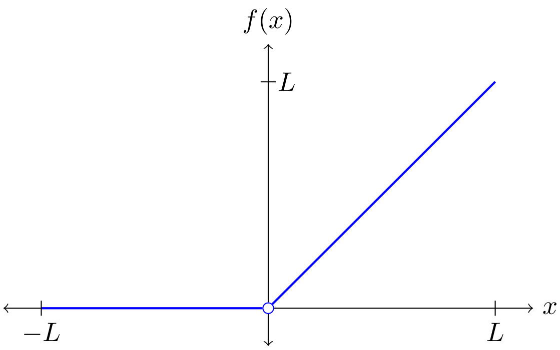
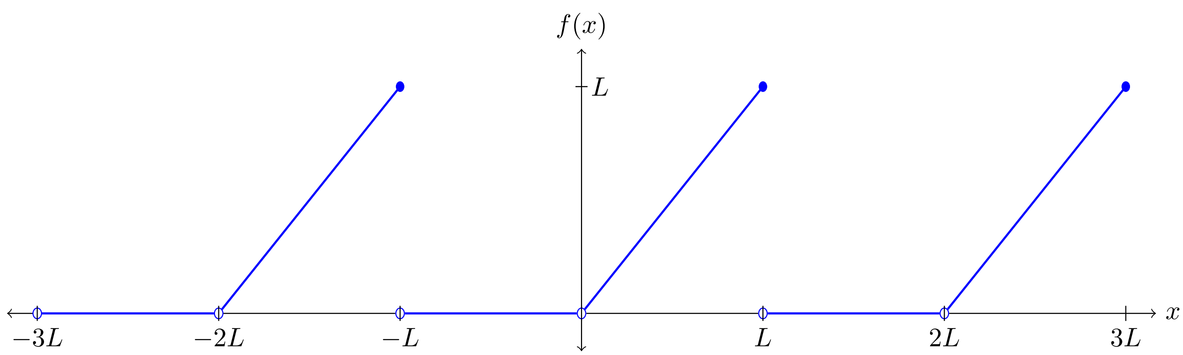
**Problem 3.2.2(d):** For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \leq x \leq L$) and determine the Fourier coefficients where

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$$

Normal graph of $f(x)$:




Periodic extension for $f(x)$:




Which can be represented by the Fourier series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

With the coefficients:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

In order to determine the coefficients of the Fourier series, we need to substitute $f(x) = x$ for $0 < x < L$ from the original piecewise function into each coefficient and then evaluate the integrals (reverse power rule for $a_0$, and integration by parts for $a_n$ and $b_n$).

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$= \frac{1}{2L}\int_{0}^{L} xdx$$

$$= \frac{1}{2L}\frac{x^2}{2}\Big|_0^L$$

$$= \frac{1}{2L}\Big(\frac{L^2}{2}\Big)$$

$$= \frac{L^2}{4L}$$

$$= \frac{L}{4}$$

Let $k = \frac{n\pi}{L}$, then:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(kx)dx$$

$$= \frac{1}{L}\int_{0}^{L} x\cos(kx)dx$$

$$u = x, \quad du = dx, \quad v = \frac{\sin(kx)}{k}, \quad dv = \cos(kx)dx$$

$$a_n = \frac{1}{L}\Big(\frac{x\sin(kx)}{k}\Big|_0^L - \int_0^L \frac{\sin(kx)}{k}dx\Big)$$

$$= \frac{1}{L}\Big(\frac{Lsin(kL)}{k} + \frac{\cos(kL)-1}{k^2}\Big)$$

$$= \frac{sin(kL)}{k} + \frac{\cos(kL)-1}{Lk^2}$$

But $kL = \frac{n\pi L}{L} = n\pi$, and we know that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so we can simplify the coefficient $a_n$ to:

$$a_n = \frac{(-1)^n - 1}{Lk^2}$$

$$= \frac{(-1)^n - 1}{L(\frac{n\pi}{L})^2}$$

$$= \frac{(-1)^n - 1}{n^2\pi^2}L$$

$$= \begin{cases} 0, & n \quad even \\ -\frac{2L}{n^2\pi^2}, & n \quad odd \end{cases}$$

Now we can repeat this process to find $b_n$ (still assuming that $k = \frac{n\pi}{L}$):

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(kx)dx$$

$$= \frac{1}{L}\int_{0}^{L} x\sin(kx)dx$$

$$u = x, \quad du = dx, \quad v = -\frac{\cos(kx)}{k}, \quad dv = \sin(kx)dx$$

$$\begin{aligned} b_n &= \frac{1}{L}\Big( -\frac{x\cos(kx)}{k}\Big|_0^L + \int_0^L \frac{\cos(kx)}{k} dx\Big) \\ &= \frac{1}{L}\Big( -\frac{L\cos(kL)}{k} + \frac{\sin(kL)}{k^2}\Big) \\ &= -\frac{\cos(kL)}{k} + \frac{\sin(kL)}{Lk^2} \end{aligned}$$

But once again $kL = \frac{n\pi L}{L} = n\pi$, so $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, and we can simplify the coefficient $b_n$ to:

$$\begin{aligned} b_n &= -\frac{(-1)^n}{k} \\ &= -\frac{(-1)^n}{\frac{n\pi}{L}} \\ &= -\frac{L(-1)^n}{n\pi} \\ &= \frac{L}{n\pi}(-1)^{n+1} \end{aligned}$$

So the final Fourier coefficients are:

$$a_0 = \frac{L}{4}$$

$$a_n = \frac{(-1)^n - 1}{n^2\pi^2} L = \begin{cases} 0, & n \quad even \\ -\frac{2L}{n^2\pi^2}, & n \quad odd \end{cases}$$

$$b_n = \frac{L}{n\pi}(-1)^{n+1}$$

**Problem 3.2.3:** Show that the Fourier series operation is linear. That is, show that the Fourier series of $c_1 f(x) + c_2 g(x)$ is the sum of $c_1$ times the Fourier series of $f(x)$ and $c_2$ times the series of $g(x)$.

Let $f$ and $g$ be piecewise smooth functions on $[-L, L]$, and let $c_1, c_2 \in \mathbb{R}$. Then the function $h$ can be defined as an arbitrary linear combination of $f$ and $g$:

$$h(x) = c_1 f(x) + c_2 g(x)$$

Recall that the Fourier series of a function $f$ is given by:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \Big( a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big)$$

With the coefficients:

$$a_0[f] = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n[f] = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)dx$$

$$b_n[f] = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)dx$$

To show that the Fourier series operation is linear, we must plug in the arbitrary linear combination function $h$ and find the Fourier series coefficients $a_0[h]$, $a_n[h]$, and $b_n[h]$:

$$a_0[h] = \frac{1}{2L}\int_{-L}^{L} h(x)dx$$

$$= \frac{1}{2L}\int_{-L}^{L} (c_1 f(x) + c_2 g(x))dx$$

$$= c_1 \frac{1}{2L}\int_{-L}^{L} f(x)dx + c_2 \frac{1}{2L}\int_{-L}^{L} g(x)dx$$

$$= c_1 a_0[f] + c_2 a_0[g]$$

$$a_n[h] = \frac{1}{L}\int_{-L}^{L} h(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} (c_1 f(x) + c_2 g(x))\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= c_1 \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + c_2 \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= c_1 a_n[f] + c_2 a_n[g]$$

$$b_n[h] = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} (c_1 f(x) + c_2 g(x))\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= c_1 \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + c_2 \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= c_1 b_n[f] + c_2 b_n[g]$$

Now plug in the linear combination coefficients of $h$ into the formula for the Fourier series operator $S$ to confirm that it's a linear operation:

$$S[h](x) = a_0[h] + \sum_{n=1}^{\infty}\left(a_n[h]\cos\left(\frac{n\pi x}{L}\right) + b_n[h]\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= c_1 a_0[f] + c_2 a_0[g] + \sum_{n=1}^{\infty}\left((c_1 a_n[f] + c_2 a_n[g])\cos\left(\frac{n\pi x}{L}\right) + (c_1 b_n[f] + c_2 b_n[g])\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= c_1\left(a_0[f] + \sum_{n=1}^{\infty}\left(a_n[f]\cos\left(\frac{n\pi x}{L}\right) + b_n[f]\sin\left(\frac{n\pi x}{L}\right)\right)\right) + c_2\left(a_0[g] + \sum_{n=1}^{\infty}\left(a_n[g]\cos\left(\frac{n\pi x}{L}\right) + b_n[g]\sin\left(\frac{n\pi x}{L}\right)\right)\right)$$

$$= c_1 S[f](x) + c_2 S[g](x)$$

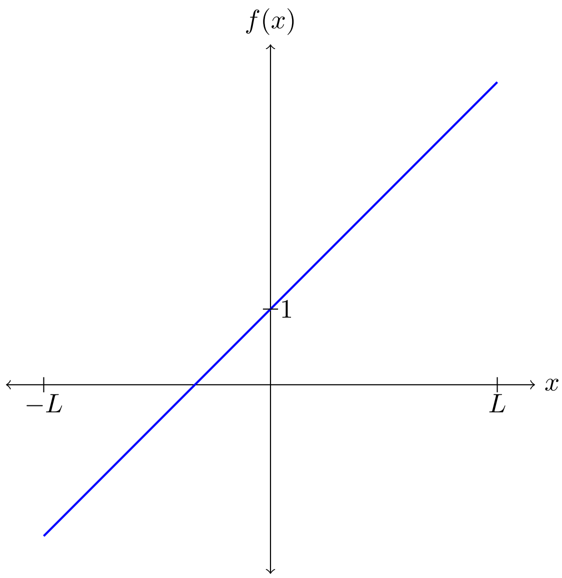
Which is $c_1$ times the Fourier series of $f(x)$ and $c_2$ times the series of $g(x)$. Therefore, the Fourier series operator $S$ is linear:

$$S[h](x) = S[c_1 f + c_2 g] = c_1 S[f] + c_2 S[g]$$

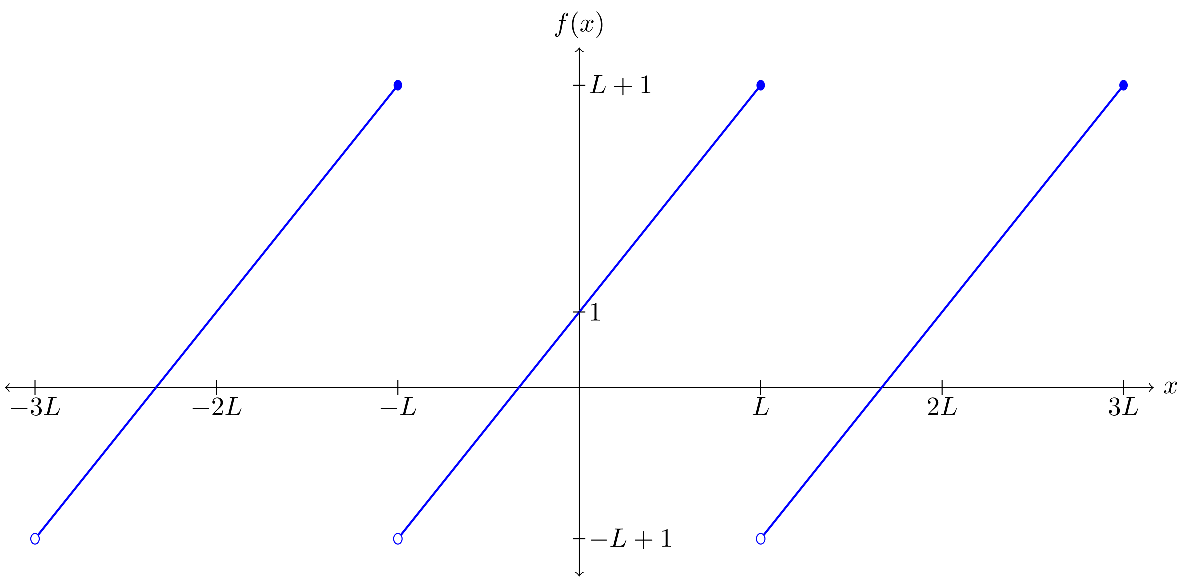
**Problem 3.3.1(b):** For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series:

$f(x) = 1 + x$

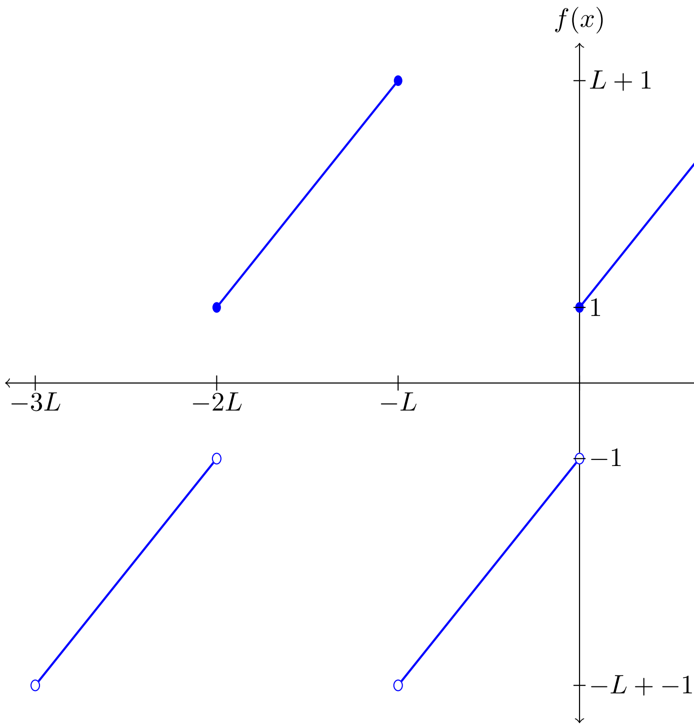
Normal graph of $f(x)$:



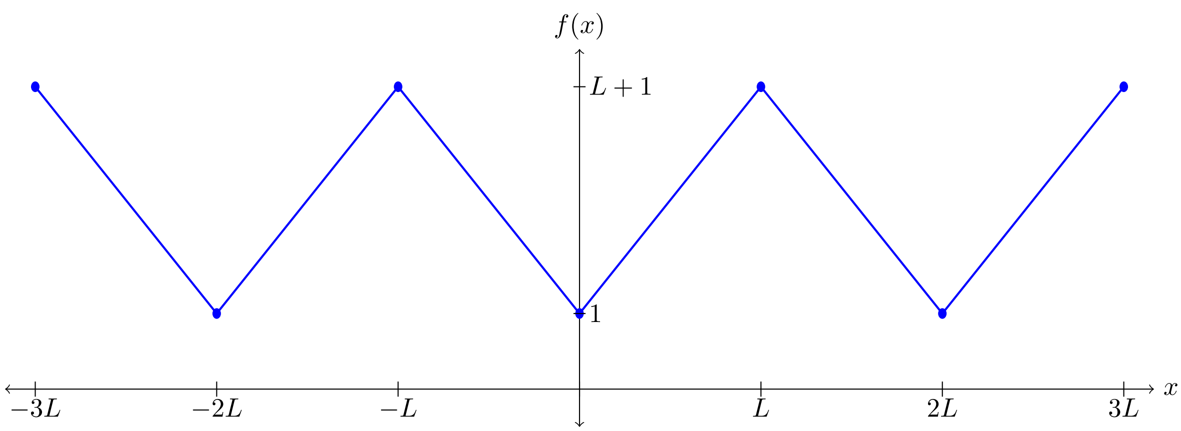

Fourier series - Periodic extension for $f(x)$:



Fourier sine series - Odd extension for $f(x)$:



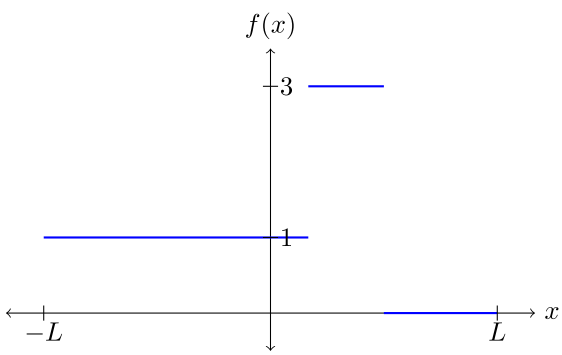

Fourier cosine series - Even extension for $f(x)$:



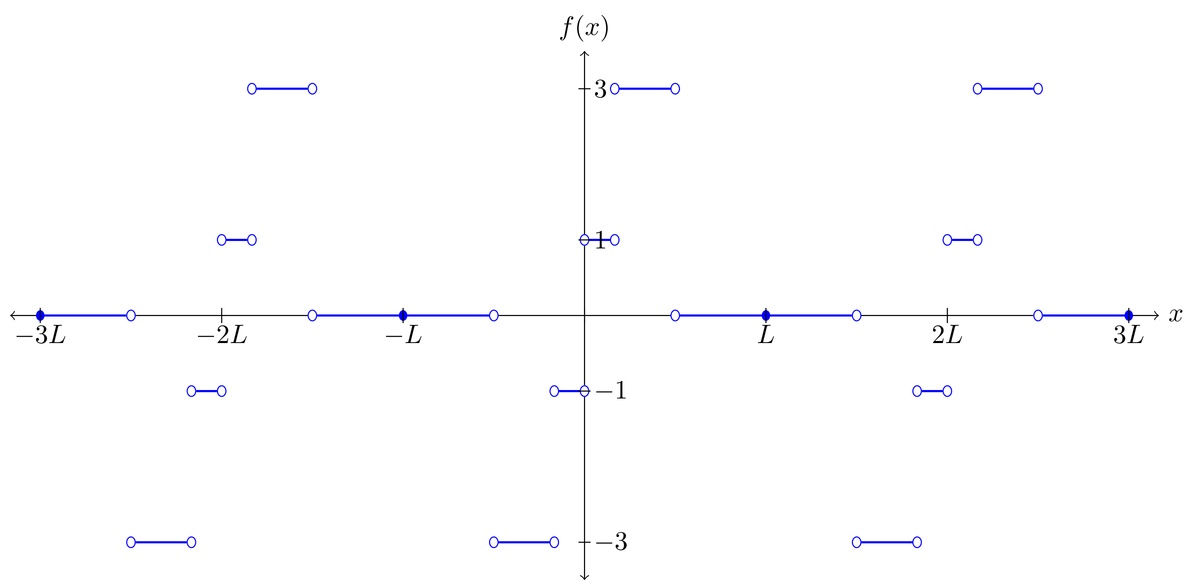
**Problem 3.3.2(b):** For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients:

$$f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$$

Normal graph of $f(x)$:




Fourier sine series - Odd extension for $f(x)$:




Which can be represented by the Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

With the coefficient (only one for the Fourier sine series):

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

In order to find $b_n$, we need to substitute $f(x) = 1$ for $0 < x < \frac{L}{6}$ and $f(x) = 3$ for $\frac{L}{6} < x < \frac{L}{2}$ from the original piecewise function into the formula for $b_n$ and then evaluate the integral. Only the first two subintervals of $f$ contribute to the integral because the third subinterval is $f(x) = 0$ for $\frac{L}{2} < x < L$. With this in mind, and assuming that $k = \frac{n\pi}{L}$, then:

$$b_n = \frac{2}{L}\Big(\int_0^{\frac{L}{6}} 1 \cdot \sin(kx)dx + \int_{\frac{L}{6}}^{\frac{L}{2}} 3 \cdot \sin(kx)dx\Big)$$

$$= \frac{2}{L}\Big(\frac{1-\cos\left(\frac{kL}{6}\right)}{k} + 3\frac{\cos\left(\frac{kL}{6}\right)-\cos\left(\frac{kL}{2}\right)}{k}\Big)$$

$$= \frac{2}{L}\Big(\frac{1-\cos\left(\frac{kL}{6}\right)+3\cos\left(\frac{kL}{6}\right)-3\cos\left(\frac{kL}{2}\right)}{k}\Big)$$

$$= \frac{2}{L}\Big(\frac{1+2\cos\left(\frac{kL}{6}\right)-3\cos\left(\frac{kL}{2}\right)}{k}\Big)$$

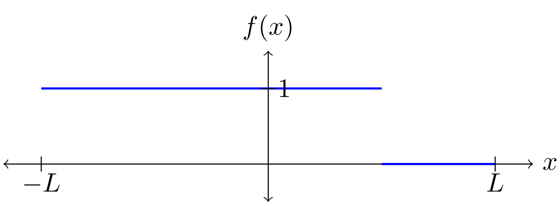
But once again $kL = \frac{n\pi L}{L} = n\pi$, so we can simplify the final coefficient $b_n$ to:

$$b_n = \frac{2}{n\pi}\Big(1 + 2\cos\left(\frac{n\pi}{6}\right) - 3\cos\left(\frac{n\pi}{2}\right)\Big)$$

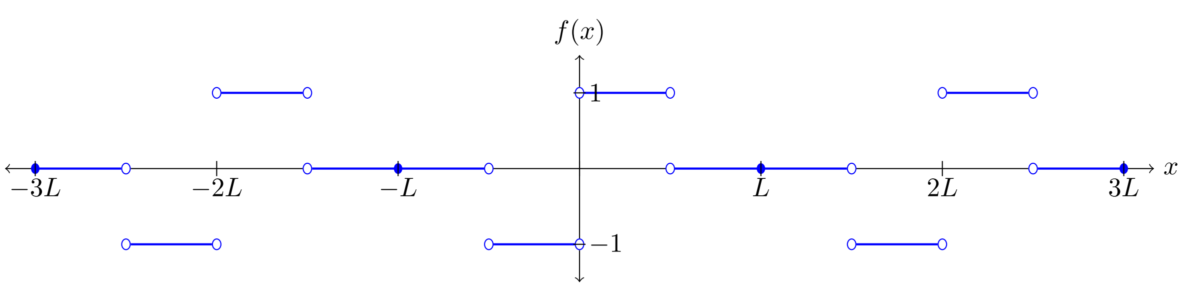
**Problem 3.3.3(b):** For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier sine series:

**(b)** $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

Normal graph of $f(x)$:




Fourier sine series - Odd extension for $f(x)$:



Which can be represented by the Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

With the coefficient (only one for the Fourier sine series):

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_0^{\frac{L}{2}} 1\cdot\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right) \end{aligned}$$

So the complete Fourier sine series is:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Now we will find the partial sums ($S_n(x)$) for the first five terms, to do this we must plug in the first five values of $n$:

| $n$ | $\cos\left(\frac{n\pi}{2}\right)$ | $b_n$ |
|---|---|---|
| 1 | 0 | $\frac{2}{\pi}$ |
| 2 | $-1$ | $\frac{2}{\pi}$ |
| 3 | 0 | $\frac{2}{3\pi}$ |
| 4 | 1 | 0 |
| 5 | 0 | $\frac{2}{5\pi}$ |

$$S_1(x) = \frac{2}{\pi}\sin\left(\frac{\pi x}{L}\right)$$

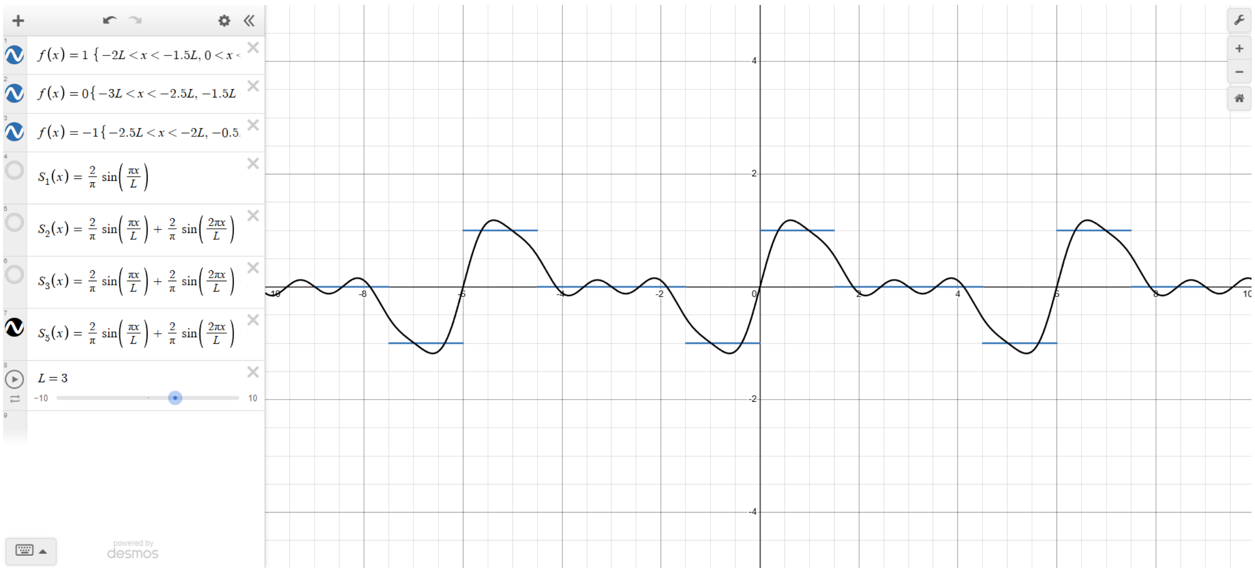
$$S_2(x) = \frac{2}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{2}{\pi}\sin\left(\frac{2\pi x}{L}\right)$$

$$S_3(x) = \frac{2}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{2}{\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi x}{L}\right)$$

$$S_4(x) = S_3(x)$$

$$S_5(x) = \frac{2}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{2}{\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \frac{2}{5\pi}\sin\left(\frac{5\pi x}{L}\right)$$

Below is the graph of the partial sum $S_5(x)$ compared to the Fourier sine series on Desmos:

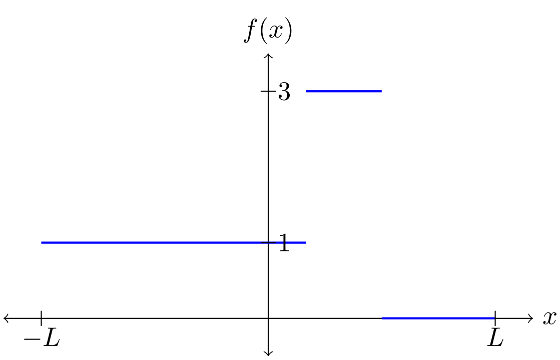



Which is notably not entirely accurate due to a relatively low number of terms, but it is still a fairly decent approximation of the Fourier sine series of $f(x)$.

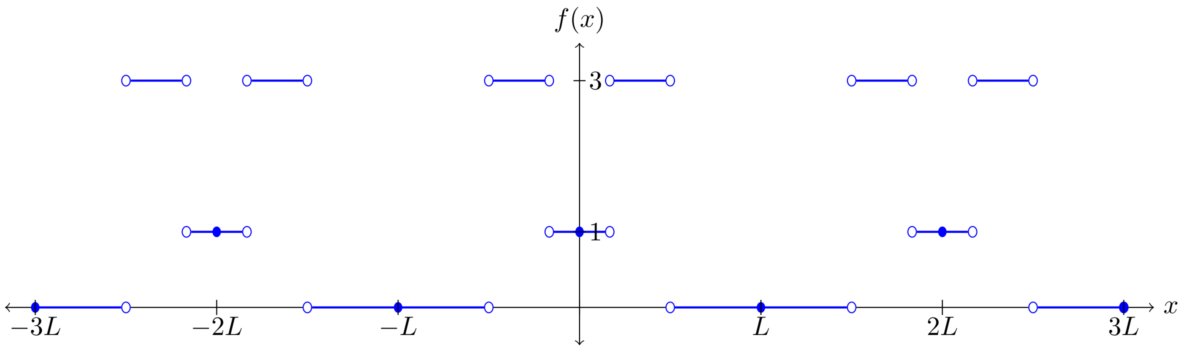
**Problem 3.3.5(b):** For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

$$f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$$

Normal graph of $f(x)$:



Fourier cosine series - Even extension for $f(x)$:




Which can be represented by the Fourier cosine series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$\begin{aligned} a_0 &= \frac{2}{2L}\int_0^L f(x)dx \\ &= \frac{1}{L}\int_0^L f(x)dx \end{aligned}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

In order to find $a_0$ and $a_n$, we once again need to substitute $f(x) = 1$ for $0 < x < \frac{L}{6}$ and $f(x) = 3$ for $\frac{L}{6} < x < \frac{L}{2}$ from the original piecewise function into the coefficient formulas and then evaluate the integrals. Like before, only the first two subintervals of $f$ contribute to the integral because the third subinterval is $f(x) = 0$ for $\frac{L}{2} < x < L$. With this in mind, and still assuming that $k = \frac{n\pi}{L}$, then:

$$\begin{aligned} a_0 &= \frac{1}{L}\Big(\int_0^{\frac{L}{6}} 1dx + \int_{\frac{L}{6}}^{\frac{L}{2}} 3dx\Big) \\ &= \frac{1}{L}\Big(\frac{L}{6} + 3\Big(\frac{L}{2} - \frac{L}{6}\Big)\Big) \\ &= \frac{1}{L}\Big(\frac{L}{6} + 3\frac{L}{3}\Big) \\ &= \frac{1}{L}\Big(\frac{7L}{6}\Big) \\ &= \frac{7}{6} \end{aligned}$$

$$a_n = \frac{2}{L}\Big(\int_0^{\frac{L}{6}} 1 \cdot \cos(kx)dx + \int_{\frac{L}{6}}^{\frac{L}{2}} 3 \cdot \cos(kx)dx\Big)$$

$$= \frac{2}{L}\Big(\frac{\sin\left(\frac{kL}{6}\right) + 3\Big(\sin\left(\frac{kL}{2}\right) - \sin\left(\frac{kL}{6}\right)\Big)}{k}\Big)$$

$$= \frac{2}{L}\Big(\frac{3\sin\left(\frac{kL}{2}\right) - 2\sin\left(\frac{kL}{6}\right)}{k}\Big)$$

But once again $kL = \frac{n\pi L}{L} = n\pi$, so we can simplify the coefficient $a_n$ to:

$$a_n = \frac{2}{n\pi}\Big(3\sin\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{6}\right)\Big)$$

So the final Fourier cosine coefficients are:

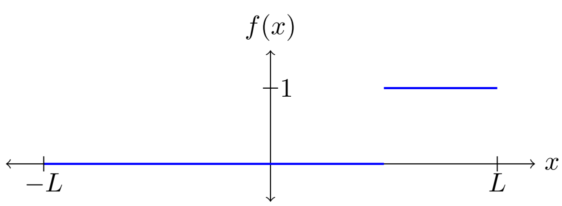
$$a_0 = \frac{7}{6}$$

$$a_n = \frac{2}{n\pi}\Big(3\sin\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{6}\right)\Big)$$

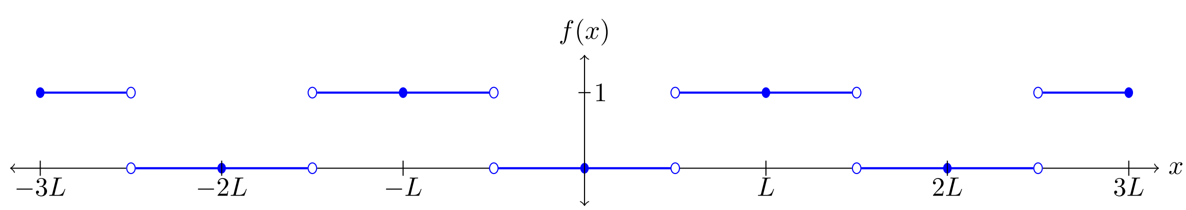
**Problem 3.3.6(b):** For the following function, sketch the Fourier cosine series of $f(x)$. Also roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

**(b)** $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$

Normal graph of $f(x)$:




Fourier cosine series - Even extension for $f(x)$:



Which can be represented by the Fourier cosine series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$\begin{aligned}
a_0 &= \frac{2}{2L}\int_0^L f(x)dx \\
&= \frac{1}{L}\int_0^L f(x)dx \\
&= \frac{1}{L}\int_{\frac{L}{2}}^L 1dx \\
&= \frac{1}{L}\left(L - \frac{L}{2}\right) \\
&= \frac{1}{L}\left(\frac{L}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\int_{\frac{L}{2}}^L 1\cdot\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\left(\frac{L}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)\right) \\
&= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

So the complete Fourier cosine series is:

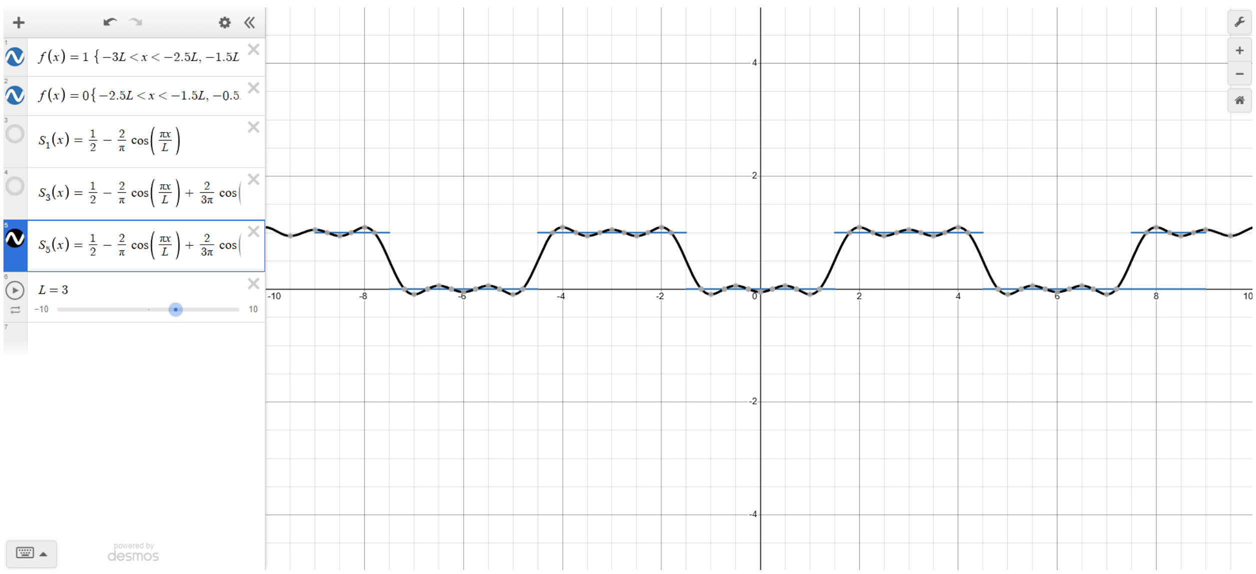
$$f(x) \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Now we will find the partial sums ($S_n(x)$) for the first five terms, to do this we must plug in the first five values of $n$:

| $n$ | $\sin\left(\frac{n\pi}{2}\right)$ | $a_n$ |
|---|---|---|
| 1 | 1 | $-\frac{2}{\pi}$ |
| 2 | 0 | 0 |
| 3 | $-1$ | $\frac{2}{3\pi}$ |
| 4 | 0 | 0 |
| 5 | 1 | $-\frac{2}{5\pi}$ |

$$S_1(x) = \frac{1}{2} - \frac{2}{\pi}\cos\left(\frac{\pi x}{L}\right)$$
$$S_2(x) = S_1(x)$$
$$S_3(x) = \frac{1}{2} - \frac{2}{\pi}\cos\left(\frac{\pi x}{L}\right) + \frac{2}{3\pi}\cos\left(\frac{3\pi x}{L}\right)$$
$$S_4(x) = S_3(x)$$
$$S_5(x) = \frac{1}{2} - \frac{2}{\pi}\cos\left(\frac{\pi x}{L}\right) + \frac{2}{3\pi}\cos\left(\frac{3\pi x}{L}\right) - \frac{2}{5\pi}\cos\left(\frac{5\pi x}{L}\right)$$

Below is the graph of the partial sum $S_5(x)$ compared to the Fourier cosine series on Desmos:




Which is also not entirely accurate due to a relatively low number of terms, but this partial sum is a pretty good approximation of the Fourier sine series of $f(x)$.

**Problem 3.3.7:** Show that $e^x$ is the sum of an even and an odd function.

A function $f(x)$ can always be written as the even portion plus the odd portion:

$$f(x) = f_{even}(x) + f_{odd}(x)$$

where the even and odd portions are:

$$f_{even}(x) = \frac{f(x) + f(-x)}{2}$$

$$f_{odd}(x) = \frac{f(x) - f(-x)}{2}$$

Now we need to plug in $f(x) = e^x$ to determine the specific even and odd portions:

$$f(-x) = e^{-x}$$

$$f_{even}(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$
$$f_{odd}(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Therefore, $e^x$ is the sum of the even function $\cosh(x)$ (hyperbolic cosine) and the odd function $\sinh(x)$ (hyperbolic sine):

$$f(x) = e^x = \cosh(x) + \sinh(x)$$

**Problem 3.3.8(a-c):**

**(a)** Determine the formulas for the even extension of any function $f(x)$. Compare to the formula for the even part of $f(x)$.

An even extension $F$ only flips the function $f$ over the y-axis so it basically mirrors the right half of the original function forming a symmetric graph. The formula for even extensions can be expressed as the following:

$$\begin{aligned} F(x) &= f(|x|) \\ &= \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases} \end{aligned}$$

On the other hand, the formula for the even portion of $f$ is:

$$f_{even}(x) = \frac{f(x) + f(-x)}{2}$$

The key difference is that the even extension $F$ uses only the values of $f$ that are on the right (positive) side of the interval while the even portion of the function $f$ averages the two values on both sides of the interval.

**(b)** Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

An odd extension $F$ flips the function $f$ over the x and y-axis so it basically rotates the right half of the original function by 180° forming a asymmetric graph. The formula for odd extensions can be expressed as the following:

$$F(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

On the other hand, the formula for the odd portion of $f$ is:

$$f_{odd}(x) = \frac{f(x) - f(-x)}{2}$$

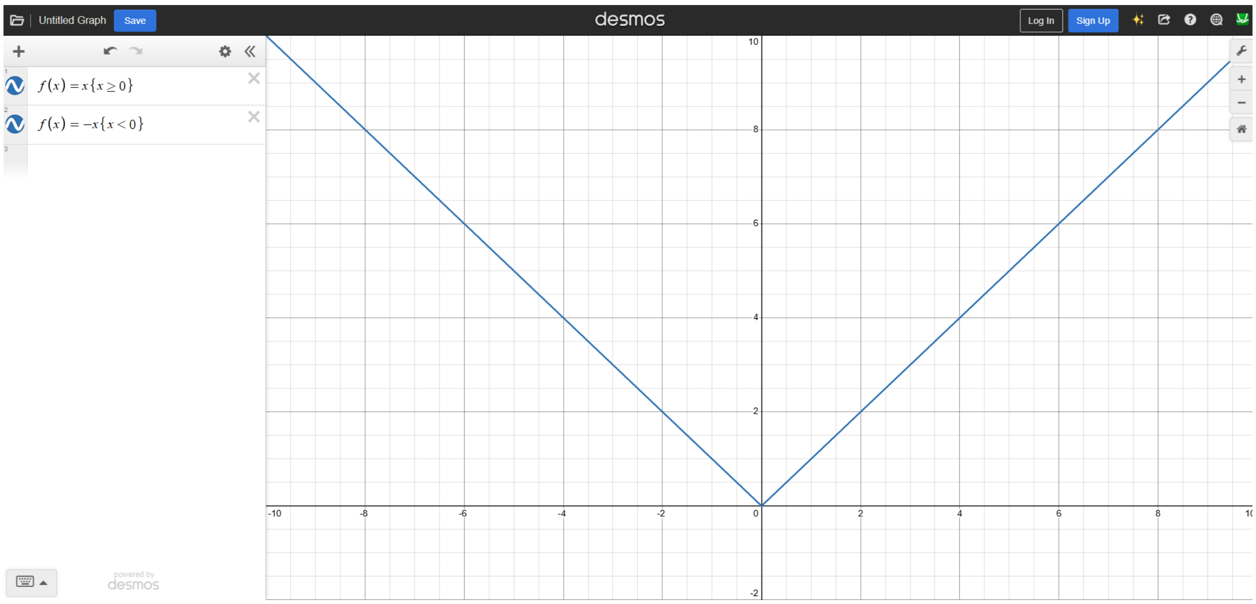
The key difference is that the odd extension $F$ creates an odd function by reflecting and flipping the positive side while the odd portion of the function $f$ extracts only the odd-symmetric portion that already exists on both sides of the origin.

**(c)** Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0 \end{cases}$$

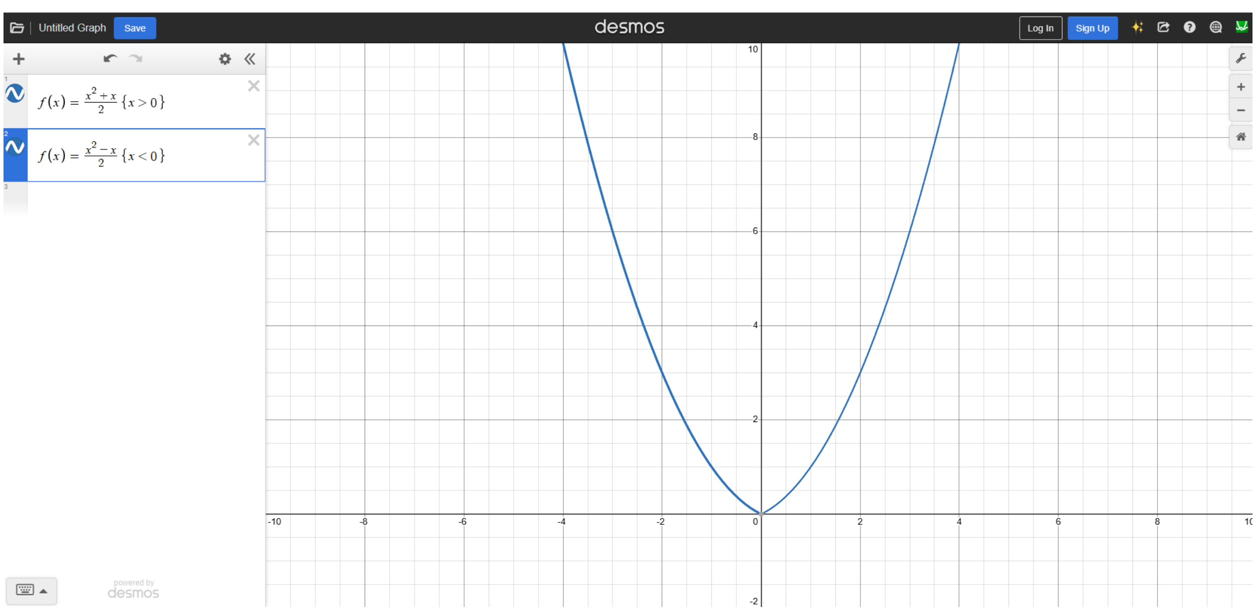
Even extension of $f(x)$:

$$F_{evenext}(x) = f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



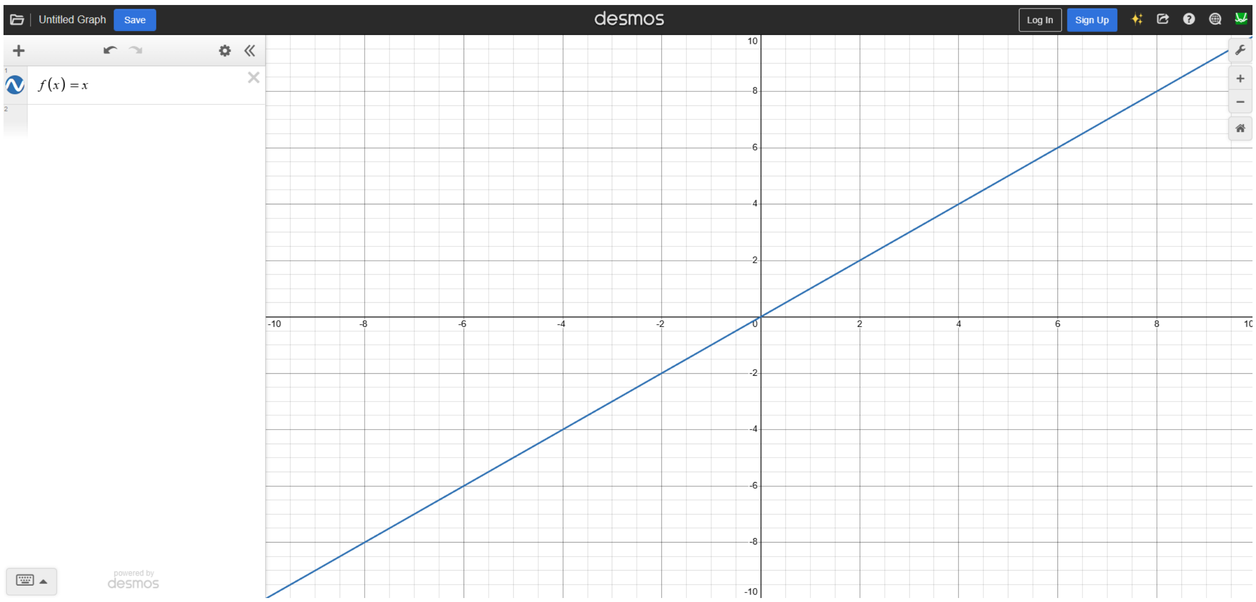

Even part of $f(x)$:

$$f_{even}(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} \frac{x^2+x}{2} & x > 0 \\ \frac{x^2-x}{2} & x < 0 \end{cases}$$



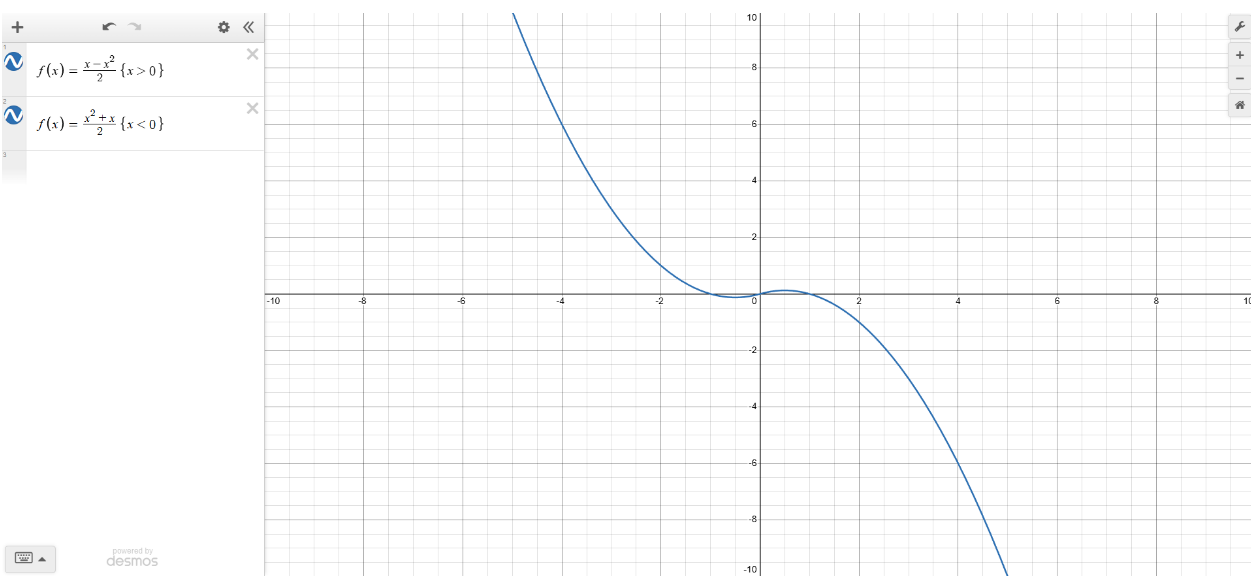
Odd extension of $f(x)$:

$$F_{oddext}(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases} = \begin{cases} x & x \geq 0 \\ -(-x) & x < 0 \end{cases} = x$$




Odd part of $f(x)$:

$$f_{odd}(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} \frac{x-x^2}{2} & x > 0 \\ \frac{x^2+x}{2} & x < 0 \end{cases}$$



**Problem 3.4.4(b):** Suppose that $f(x)$ and $df/dx$ are piecewise smooth. Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

The Fourier cosine series is given by:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

In order to prove that the Fourier cosine series of $f(x)$ can be differentiated term-by-term, we first need to differentiate $f(x)$ to find the resulting derivative series of $f'(x)$:

$$f'(x) = \frac{d}{dx}\Big[f(x)\Big] = \frac{d}{dx}\Big[a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\Big]$$

$$= \sum_{n=1}^{\infty} a_n \frac{d}{dx}\Big[\cos\left(\frac{n\pi x}{L}\right)\Big]$$

$$= \sum_{n=1}^{\infty} a_n\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L}a_n\right)\sin\left(\frac{n\pi x}{L}\right)$$

Now we can use integration by parts on $a_n$ to determine the new derivative series coefficient $b_n$:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$u = f(x), \quad du = f'(x)dx, \quad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right), \quad dv = \cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \Big[\frac{L}{n\pi}f(x)\sin\left(\frac{n\pi x}{L}\right)\Big]_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$-\frac{n\pi}{L}a_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

If we define $b_n$ as the following, then we can plug it into the series formula to get the final derivative series:

$$b_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right) = -\frac{n\pi}{L}a_n$$

$$f'(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Which is exactly the Fourier sine series of $f'(x)$. Since $f(x)$ is continuous and both $f(x)$ and $f'(x)$ are piecewise smooth, the Fourier sine series is guaranteed to converge to $f'(x)$ at points where $f'(x)$ is continuous, and the average of the left and right limits at points where there is a jump discontinuity. Since all three of these conditions (continuous, piecewise smooth, and converges to $f'(x)$) are met, we can conclude that the Fourier cosine series of $f(x)$ can be differentiated term-by-term.

**Problem 3.4.8:** Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$ and $u(x,0) = f(x)$. Solve in the following way. Look for the solution as a Fourier cosine series. Assume that $u$ and $\partial u/\partial x$ are continuous and that $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Justify all differentiations of infinite series.

From the given heat equation and boundary conditions, the Fourier cosine series of $u$ is given by:

$$u(x,t) \sim a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

With the coefficients (note that the initial $t = 0$ coefficients come from $f(x)$):

$$a_0(t) = \frac{1}{L}\int_0^L u(x,t)dx$$

$$a_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_0(0) = \frac{1}{L}\int_0^L f(x)dx$$

$$a_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Now we need to differentiate the series term-by-term and then plug the result into the heat equation PDE to find the solution series. We are able to perform this differentiation because the function $u$ is continuous and the partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$ are piecewise smooth.

$$\frac{\partial u}{\partial t} = a_0'(t) + \sum_{n=1}^{\infty} a_n'(t)\cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\frac{\partial^2}{\partial x^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n(t)\Big(-\left(\frac{n\pi}{L}\right)^2\Big)\cos\left(\frac{n\pi x}{L}\right)$$

$$a_n'(t) = -k\Big(\frac{n\pi}{L}\Big)^2 a_n(t)$$

$$a_0'(t) = 0$$

Finally, we can solve the simple exponential decay ODEs for the coefficients and determine the final Fourier cosine series of $u$:

$$a_0(t) = a_0(0)$$

$$a_n(t) = a_n(0)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

So the final Fourier cosine series of $u$is given by:

$$u(x,t) \sim a_0(0) + \sum_{n=1}^{\infty} a_n(0)e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

**Problem 3.5.2(a-b):**

**(a)** Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of $x^2$.

***3.3.11.** If $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$, what are the even and odd parts of $f(x)$?

**3.3.12.** Given a sketch of $f(x)$, describe a procedure to sketch the even and odd parts of $f(x)$.

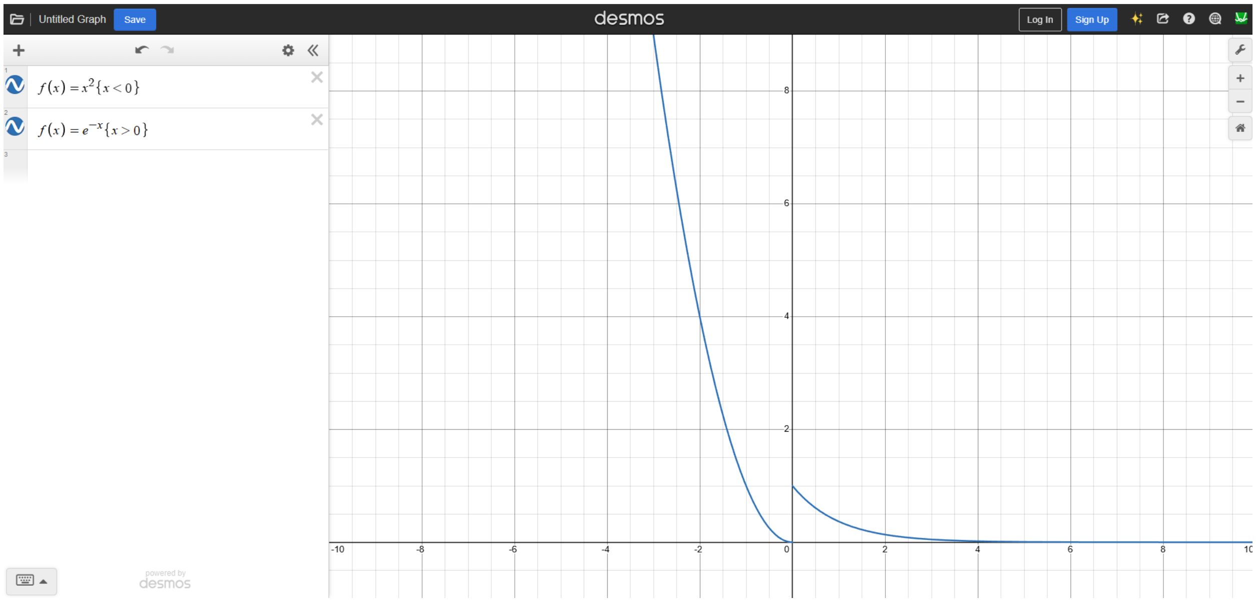
Finding the even and odd portions of $f(x)$:

$$f(x) = f_{even}(x) + f_{odd}(x)$$

$$f_{even}(x) = \frac{f(x) + f(-x)}{2} = \begin{cases} \frac{e^{-x}+x^2}{2} & x > 0 \\ \frac{x^2+e^x}{2} & x < 0 \end{cases}$$

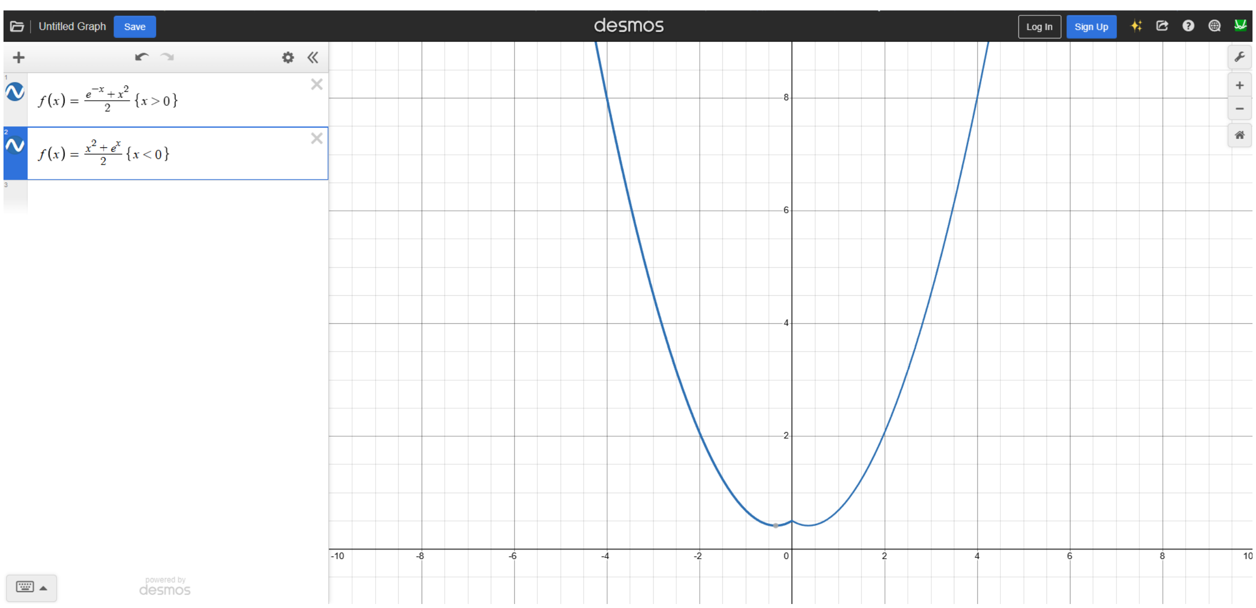
$$f_{odd}(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} \frac{e^{-x}-x^2}{2} & x > 0 \\ \frac{x^2-e^x}{2} & x < 0 \end{cases}$$

Sketch of $f(x)$ on Desmos:




To sketch the even and odd portions of $f(x)$, you just need to plug in a bunch of x values into the piecewise functions and then plot the resulting function values on a graph (keeping the possible discontinuities in mind). Below are the sketches of the even and odd portions of $f(x)$ on Desmos:

Even:



Odd:


Now we want to find the Fourier cosine series of $x^2$. Since $x^2$ is even, its Fourier cosine series on $[-L, L]$ is the same as its normal Fourier series and there are only cosine terms. The Fourier cosine series of $x^2$ is given by:

$$x^2 \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

With the coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2L}\int_{-L}^{L} x^2 dx \\
&= \frac{1}{L}\int_{0}^{L} x^2 dx \\
&= \frac{1}{L}\cdot\frac{L^3}{3} \\
&= \frac{L^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

We can use integration by parts twice to solve for $a_n$, assuming that $k = \frac{n\pi}{L}$:

$$a_n = \frac{2}{L}\int_{0}^{L} x^2\cos(kx)dx$$

$$u = x^2, \quad du = 2xdx, \quad v = \frac{\sin(kx)}{k}, \quad dv = \cos(kx)dx$$

$$a_n = \frac{2}{L} \cdot \Big( \Big[ \frac{x^2 \sin(kx)}{k} \Big]_0^L - \frac{2}{k} \int_0^L x \sin(kx)dx \Big)$$

$$= \frac{2}{L} \Big( - \frac{2}{k} \int_0^L x \sin(kx)dx \Big)$$

$$u = x, \quad du = dx, \quad v = -\frac{\cos(kx)}{k}, \quad dv = \sin(kx)dx$$

$$a_n = \frac{2}{L} \Big( - \frac{2}{k} \Big( \Big[ - \frac{x \cos(kx)}{k} \Big]_0^L + \frac{1}{k} \int_0^L \cos(kx)dx \Big) \Big)$$

$$= \frac{2}{L} \Big( - \frac{2}{k} \Big( - \frac{L \cos(kL)}{k} + \frac{1}{k} \Big[ \frac{\sin(kx)}{k} \Big]_0^L \Big) \Big)$$

$$= \frac{2}{L} \Big( - \frac{2}{k} \Big( - \frac{L(-1)^n}{k} \Big) \Big)$$

$$= \frac{2}{L} \Big( \frac{2L(-1)^n}{k^2} \Big)$$

$$= \frac{4(-1)^n}{k^2}$$

$$= \frac{4(-1)^n}{\left(\frac{n\pi}{L}\right)^2}$$

$$= \frac{4L^2(-1)^n}{n^2\pi^2}$$

So the final Fourier cosine series of $x^2$ on $[-L, L]$ is given by:

$$x^2 \sim \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2\pi^2} \cos\Big(\frac{n\pi x}{L}\Big)$$

**(b)** From part (a), determine the Fourier sine series of $x^3$.

Now we want to find the Fourier sine series of $x^3$. Since $x^3$ is odd, its Fourier sine series on $[-L, L]$ has only sine terms. The Fourier sine series of $x^3$ is given by:

$$x^3 \sim \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

With the coefficient:

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big) dx$$

$$= \frac{1}{L} \int_{-L}^{L} x^3 \sin\Big(\frac{n\pi x}{L}\Big) dx$$

$$= \frac{2}{L} \int_0^L x^3 \sin\Big(\frac{n\pi x}{L}\Big) dx$$

We can use integration by parts three times to solve for $b_n$ (steps not shown for brevity), assuming that $k = \frac{n\pi}{L}$:

$$\begin{aligned} b_n &= \frac{2}{L}\Big((-1)^n \frac{L^4(6-n^2\pi^2)}{n^3\pi^3}\Big) \\ &= 2L^3(-1)^n \frac{6-n^2\pi^2}{n^3\pi^3} \end{aligned}$$

So the final Fourier sine series of $x^3$ on $[-L, L]$ is given by:

$$x^3 \sim \sum_{n=1}^{\infty} 2L^3(-1)^n \Big(\frac{6-n^2\pi^2}{n^3\pi^3}\Big) \sin\Big(\frac{n\pi x}{L}\Big)$$